

On factorizations in perturbative quantum gravity

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ABSTRACT: Some features of Einstein gravity are most easily understood from string theory but are not manifest at the level of the usual Lagrangian formulation. One example is the factorization of gravity amplitudes into gauge theory amplitudes. Based on the recently constructed ‘double field theory’ and a geometrical frame-like formalism developed by Siegel, we provide a framework of perturbative Einstein gravity coupled to a 2-form and a dilaton in which, as a consequence of T-duality, the Feynman rules factorize to all orders in perturbation theory. We thereby establish the precise relation between the field variables in different formulations and discuss the Lagrangian that, when written in terms of these variables, makes a left-right factorization manifest.

KEYWORDS: Models of Quantum Gravity, Gauge Symmetry, Space-Time Symmetries, Global Symmetries

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1 Introduction

The usual perturbative approach of covariant quantum gravity on flat space starts with the Einstein-Hilbert theory and expands the Riemannian metric g_{ij} around a constant background, $g_{ij} = \delta_{ij} + h_{ij}$. The diffeomorphism invariance then translates into a gauge symmetry of the fluctuation h_{ij} , which to lowest order reads $\delta h_{ij} = \partial_i \xi_j + \partial_j \xi_i$. From this point of view, the problem of formulating the corresponding quantum field theory is conceptually not much different from Yang-Mills theory. Technically, however, gravity appears to be much more complicated than Yang-Mills theory in that the Einstein-Hilbert action is non-polynomial in the fluctuation h_{ij} (and, of course, physically it is also very different in that it is non-renormalizable).

In recent years it has become clear, however, that the amplitudes reveal a hidden simplicity that is obscured at the level of the Lagrangian and the corresponding Feynman rules; see [1–3] and references therein. For instance, string theory exhibits the so-called KLT relations which imply a factorization of closed string or graviton amplitudes into open string or gauge theory amplitudes [4]. It turns out that similar KLT relations also hold in the corresponding (low-energy) field theories, i.e., in Einstein gravity and its supersymmetric extensions. These relations have been instrumental in the recent UV-finiteness proofs for $\mathcal{N} = 8$ supergravity at higher loops; see [5] for a review.

Given these intriguing simplifications, it is natural to ask whether there is a way to make these properties, at least to some extent, manifest at the level of the Lagrangian. Specifically, one has the freedom to perform field redefinitions of h_{ij} , and one may expect that there is a field-basis that is better adapted to the features inherited from string theory. In fact, in closed string field theory, for instance, a non-linear and non-polynomial field redefinition is required in order to connect the ‘string variables’ to the ‘Einstein variables’ h_{ij} [6].

Early attempts to render the KLT relations manifest upon field redefinitions (and non-linear generalizations of the de Donder gauge-fixing condition) are due to Bern and Grant [7]; see [8, 9] for more recent results. The idea is to factorize the metric fluctuation into two ‘gauge vectors’,

$$h_{ij} \rightarrow A_i \bar{A}_j, \tag{1.1}$$

and then to require that the Feynman rules factorize into ‘left-handed’ parts depending only on A and ‘right-handed’ parts depending only on \bar{A} . Put differently, one can think of the first index of h_{ij} as left-handed or unbarred and the second index as right-handed or barred, and then demand that the Lagrangian contains only like-wise index contractions. This requires a field redefinition which reads to lowest order [7]

$$h_{ij} \rightarrow h_{ij} + \frac{1}{2} h_i^k h_{kj} + \dots \tag{1.2}$$

Of course, since h_{ij} is actually symmetric, this assignment of left and right indices may sound somewhat obscure, in particular, this requirement is not very strong. One purpose of this note is to introduce field redefinitions that establish consistent left-right factorization to all orders in perturbation theory for the low-energy theory of the bosonic string, i.e., for Einstein gravity coupled to a 2-form and a dilaton. The relevant formulation allows to combine the metric and 2-form fluctuations into a non-symmetric field e_{ij} , which is the natural variable in string theory and which leads to an unambiguous assignment of left-right indices.

To this end, we use the recently formulated ‘double field theory’ and its T-duality invariance [10–13] (see also [14, 15] and [16] for a review). Specifically, in this theory the space-time coordinates are doubled in such a way that the T-duality group $O(D, D)$ (with D denoting the space-time dimension) acts naturally, and it is equivalent to the standard low-energy action when the dependence on the new coordinates is dropped. The theory requires a constraint that eliminates half of the coordinates, and for the purposes of this paper we may thus think of the new coordinates as purely auxiliary objects; in particular, we do not require the coordinates to be compact. We introduce this formulation in section 2 and review how the $O(D, D)$ invariance of the full non-linear background independent action is, in fact, equivalent to consistent left-right index contractions (in a sense to be made precise below). When expanded around a constant background, however, this left-right factorization is no longer manifest in the sense required above, but the field redefinition that relates to the basis in string field theory and that should therefore restore this property is known to all orders [6, 11]. Here, we prove that in this field basis the $O(D, D)$ invariance indeed implies left-right factorization to all orders in perturbation theory, thereby solving the problem stated above.

Apart from the last technical step, this result is already largely contained in the existing literature on double field theory, but it can actually be cast into a somewhat more geometrical language, using a powerful formalism based on enlarged frame fields introduced by Siegel [17, 18]. The frame field e_A^M is subject to global $O(D, D)$ transformations, acting in the fundamental representation indicated by indices $M, N, \dots = 1, \dots, 2D$, and to local

$GL(D) \times GL(D)$ tangent space transformations, corresponding to the flat index $A = (a, \bar{a})$.¹ Siegel also introduces a perturbative expansion, in which the fluctuation h carries only flat indices,

$$e_A^M = \langle e_A^M \rangle - h_A^B \langle e_B^M \rangle, \tag{1.3}$$

where $\langle e_A^M \rangle$ denotes the constant background. The tangent space symmetry can be gauge-fixed such that the only independent variable is the off-diagonal component $h_{a\bar{b}}$ [17].

Recently, this frame formalism has been related in detail to the double field theory and thereby to the conventional variables in Einstein gravity [13, 15]. Based on this, we present in section 3 as the main results of this note the precise relation between the perturbations in Einstein gravity and string field theory on the one hand and the perturbations in Siegel’s frame formalism on the other. Remarkably, we find that the string field theory variable, here denoted by e_{ij} , can be identified with the frame-like variable $h_{a\bar{b}}$ to all orders,

$$e_{ij} = \langle e_i^a \rangle \langle e_j^{\bar{b}} \rangle h_{a\bar{b}}. \tag{1.4}$$

Here, the two (independent) background vielbeins can be rotated into Kronecker symbols by means of background $GL(D) \times GL(D)$ transformations. This formulation provides therefore a significant technical simplification in that the field redefinition that establishes the left-right factorization need not be carried out explicitly, but rather is implicitly incorporated by use of the frame-like variable. It has already been pointed out by Siegel that this formalism allows us to make certain features inherited from string theory manifest in conventional field theory [17]. Here, this will be investigated explicitly, in particular we discuss the Lagrangian formulation that makes the left-right factorization manifest, as displayed in eq. (3.44) below.

2 T-duality and redefinition of Einstein variables

In this section we review the double field theory and its $O(D, D)$ invariance. Next, using this duality invariance, we prove that in the field-basis suggested by string field theory left-right factorization is realized to all orders.

2.1 Double field theory and Einstein variables

We start from the standard low-energy action for bosonic string theory, i.e., with Einstein gravity coupled to a 2-form b_{ij} and a scalar dilaton ϕ ,

$$S = \int dx \sqrt{g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right], \tag{2.1}$$

where $H_{ijk} = 3\partial_{[i} b_{jk]}$. The double field theory extension of this action is written in terms of a variable that combines the metric and b -field into a ‘non-symmetric metric’, $\mathcal{E}_{ij} = g_{ij} + b_{ij}$, and a dilaton d , which is a density rather than a scalar and defined by $\sqrt{g} e^{-2\phi} = e^{-2d}$ [12],

$$S = \int dx d\tilde{x} e^{-2d} \left[-\frac{1}{4} g^{ik} g^{jl} \mathcal{D}^p \mathcal{E}_{kl} \mathcal{D}_p \mathcal{E}_{ij} + \frac{1}{4} g^{kl} (\mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \bar{\mathcal{D}}^j \mathcal{E}_{ki} \bar{\mathcal{D}}^i \mathcal{E}_{lj}) \right. \\ \left. + (\mathcal{D}^i d \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\mathcal{D}}^i d \mathcal{D}^j \mathcal{E}_{ji}) + 4\mathcal{D}^i d \mathcal{D}_i d \right], \tag{2.2}$$

¹For an alternative formulation with local $GL(D)$ symmetry see [19].

where all indices are raised with g^{ij} , which is the inverse of $g_{ij} = \mathcal{E}_{(ij)}$. Here, all fields depend on the ‘doubled’ coordinates $X \equiv (x, \tilde{x})$, with derivatives defined by

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} - \mathcal{E}_{ik} \frac{\partial}{\partial \tilde{x}_k}, \quad \bar{\mathcal{D}}_i = \frac{\partial}{\partial x^i} + \mathcal{E}_{ki} \frac{\partial}{\partial \tilde{x}_k}. \quad (2.3)$$

The consistency of the action requires the constraint

$$\mathcal{D}^i A \mathcal{D}_i B - \bar{\mathcal{D}}^i A \bar{\mathcal{D}}_i B = 0 \quad (2.4)$$

for arbitrary fields and parameters A, B , which implies that locally the fields depend only on half of the coordinates. When the fields are assumed to be independent of \tilde{x} , i.e., $\tilde{\partial} = 0$, (2.2) is equivalent to (2.1) [12].

The crucial property of the double field theory action (2.2) that will be used below is its global $O(D, D)$ invariance under

$$\mathcal{E}'(X') = (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1}, \quad d'(X') = d(X), \quad (2.5)$$

where the coordinates transform in the fundamental representation of $O(D, D)$,

$$X' = \begin{pmatrix} \tilde{x}' \\ x' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ x \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D, D). \quad (2.6)$$

This invariance is not manifest, but can be reduced to consistent index contractions as follows [12]. The transformation behavior of the metric g and the calligraphic derivatives of \mathcal{E} and d is governed by the matrices \mathcal{M} and $\bar{\mathcal{M}}$ defined by

$$\mathcal{M}(X) = d^t - \mathcal{E}(X)c^t, \quad \bar{\mathcal{M}}(X) = d^t + \mathcal{E}^t(X)c^t. \quad (2.7)$$

More precisely, the (inverse) metric transforms as

$$g'^{ij} = (\bar{\mathcal{M}}^{-1})_p^i g'^{pq} (\bar{\mathcal{M}}^{-1})_q^j, \quad g^{ij} = (\mathcal{M}^{-1})_p^i g'^{pq} (\mathcal{M}^{-1})_q^j, \quad (2.8)$$

and the calligraphic derivatives as

$$\mathcal{D}_i = \mathcal{M}_i^k \mathcal{D}'_k, \quad \bar{\mathcal{D}}_i = \bar{\mathcal{M}}_i^k \bar{\mathcal{D}}'_k. \quad (2.9)$$

Moreover, despite the non-linear form of (2.5), it can be checked that \mathcal{E} , acted on by a calligraphic derivative, transforms as

$$\mathcal{D}_i \mathcal{E}_{jk} = \mathcal{M}_i^q \mathcal{M}_j^p \bar{\mathcal{M}}_k^\ell \mathcal{D}'_q \mathcal{E}'_{p\ell}, \quad \bar{\mathcal{D}}_i \mathcal{E}_{jk} = \bar{\mathcal{M}}_i^q \mathcal{M}_j^p \bar{\mathcal{M}}_k^\ell \bar{\mathcal{D}}'_q \mathcal{E}'_{p\ell}. \quad (2.10)$$

Therefore, in this formalism there are two types of indices, unbarred and barred, corresponding to a transformation with \mathcal{M} or $\bar{\mathcal{M}}$ under $O(D, D)$. In particular, \mathcal{E}_{ij} acted on by a calligraphic derivative can be viewed as an object for which the first index is unbarred and the second index is barred, while the index of an (un-)barred calligraphic derivative is (un-)barred. Finally, from (2.8) we infer that the indices of the (inverse) metric can be thought of as either both unbarred or both barred. The $O(D, D)$ invariance of (2.2) then

follows from the fact that the action has only like-wise index contractions, which one may easily confirm by inspection.

Thus, the double field theory formulation of the low-energy action exhibits already a left-right factorization that it reminiscent to the requirement stated in the introduction. However, once we consider the fluctuations around a flat background, this factorization does not translate into a corresponding factorization in terms of the fluctuation. To see this, we decompose \mathcal{E} according to

$$\mathcal{E}_{ij} = E_{ij} + \check{e}_{ij} , \tag{2.11}$$

where we denoted the fluctuation by \check{e}_{ij} (which is the sum of the usual metric fluctuation h_{ij} and the fluctuation of the 2-form) in order to distinguish it from the string field theory variable e_{ij} to be discussed below. Moreover, $E_{ij} = G_{ij} + B_{ij}$ encodes the constant background metric and B-field. If one computes, for instance, the inverse metric g^{ij} , it will contain arbitrary higher powers of $\check{e}_{(ij)}$ that generally mix left- and right-indices. Next, we discuss the field redefinition to e_{ij} and show that it restores the required left-right factorization.

2.2 Left-right factorization and string theory variables

The full non-linear field redefinition that relates the fluctuation \check{e}_{ij} to e_{ij} can be written in closed form as [6, 11, 12],

$$\check{e}_{ij} = F_i^k(e) e_{kj} , \quad F = \left(1 - \frac{1}{2} e G^{-1} \right)^{-1} , \tag{2.12}$$

where we used matrix notation. If the b -field is set to zero, this agrees to lowest order with (1.2), but we note that the non-linear extension differs from the field redefinition proposed in [7].

Expanding the double field theory action (2.2) to cubic order in terms of e_{ij} , one arrives at the action that has been derived in [10] from closed string field theory [12], and whose quadratic piece we display,

$$S^{(2)} = \int dx d\tilde{x} \left[\frac{1}{4} e_{ij} \square e^{ij} + \frac{1}{4} (\bar{D}^j e_{ij})^2 + \frac{1}{4} (D^i e_{ij})^2 - 2 d D^i \bar{D}^j e_{ij} - 4 d \square d \right] . \tag{2.13}$$

This action exhibits only consistent left-right index contractions and a corresponding T-duality property in the following sense. Instead of the matrices (2.7), the transformation rules are governed in this background-*dependent* formulation by [10, 20]

$$M = d^t - E c^t , \quad \bar{M} = d^t + E^t c^t , \tag{2.14}$$

where the constant background E_{ij} rather than \mathcal{E}_{ij} enters. Similarly, the calligraphic derivatives (2.3) are replaced by D_i and \bar{D}_i depending only on E , while index contractions are done with G^{ij} . We now require that the background transforms under $O(D, D)$ as

$$E' = (aE + b)(cE + d)^{-1} , \tag{2.15}$$

which implies for the background metric

$$G'^{-1} = \bar{M}^t G^{-1} \bar{M} = M^t G^{-1} M . \quad (2.16)$$

This is the analogue of (2.8) and thus the indices on G^{-1} can again be thought of as being either both unbarred or both barred. (Here, we have written the primed variables in terms of the unprimed ones, because it is this form that will be used below.) Moreover, we require that d is invariant, and we prove below that e_{ij} transforms according to

$$e_{ij} = M_i^k \bar{M}_j^l e'_{kl} . \quad (2.17)$$

Thus, the left index transforms with M and the right index with \bar{M} , and since the action has only consistent left-right index contractions, it follows that the action has the T-duality property

$$S[E', e', d'] = S[E, e, d] . \quad (2.18)$$

The logic can now be turned around in order to find a prescription that guarantees left-right factorization to arbitrary orders. If we start from the full non-linear action (2.2) and use the non-linear field redefinition (2.12), we can in principle expand the action in terms of e_{ij} to any desired order. The original action is $O(D, D)$ invariant and, therefore, the resulting action has the T-duality property (2.18). Since the action is then written only in terms of objects that transform ‘covariantly’ with M or \bar{M} , it follows that all terms with inconsistent left-right index contractions cancel out, leaving an action with left-right factorization.²

In order to complete the above proof we have to show that e_{ij} transforms according to (2.17) to all orders. In [10] this has been verified to lowest order, while the validity of the non-linear field redefinition (2.12) has been confirmed by inspection of the gauge symmetries in [11, 12]. Here we complete the existing literature by showing that the original $O(D, D)$ transformation (2.5), together with the form of the field redefinition (2.12), indeed implies the simple transformation rule (2.17) to all orders.

We first determine the transformation behavior of \check{e} from (2.5),

$$\begin{aligned} E + \check{e} &\rightarrow (aE + b + a\check{e})(cE + d + c\check{e})^{-1} \\ &= (aE + b + a\check{e})(cE + d)^{-1} (1 - c\check{e}(cE + d)^{-1} + (c\check{e}(cE + d)^{-1})^2 \mp \dots) \\ &= E' + (a - E'c)\check{e}(\bar{M}^t)^{-1} (1 - c\check{e}(\bar{M}^t)^{-1} + (c\check{e}(\bar{M}^t)^{-1})^2 \mp \dots) \\ &= E' + (a - E'c)\check{e}(\bar{M}^t)^{-1} (1 + c\check{e}(\bar{M}^t)^{-1})^{-1} \\ &= E' + (a - E'c)\check{e} (\bar{M}^t + c\check{e})^{-1} . \end{aligned} \quad (2.19)$$

Here we have used the following matrix identity for general X and Y ,

$$(X + Y)^{-1} = X^{-1} (1 - YX^{-1} + (YX^{-1})^2 \mp \dots) , \quad (2.20)$$

²This is to be contrasted with expressions that contain both covariant *and* non-covariant objects, as eq. (2.27) to be discussed below. In this case, the non-covariant transformation of one term can cancel against the non-covariant transformation of another term.

together with (2.14). Moreover, we have used (2.15) in order to identify the transformed background. The final expression in (2.19) can be further simplified using eq. (4.13) from [10], which implies

$$E' = -M^{-1}(b^t - Ea^t). \quad (2.21)$$

We also have to use the group properties of $O(D, D)$, which require in particular

$$b^t c + d^t a = \mathbf{1}, \quad a^t c + c^t a = 0. \quad (2.22)$$

We then find

$$\begin{aligned} a - E'c &= a + M^{-1}(b^t - Ea^t)c \\ &= a + M^{-1}(\mathbf{1} - d^t a + Ec^t a) \\ &= a + M^{-1}(\mathbf{1} - Ma) \\ &= M^{-1}. \end{aligned} \quad (2.23)$$

Using this in (2.19) we can read off the transformation behavior of \check{e} ,

$$\check{e}' = M^{-1} \check{e} (\bar{M}^t + c\check{e})^{-1} = M^{-1} \check{e} (\mathcal{M}^t)^{-1}, \quad (2.24)$$

where we used in the last equation

$$\bar{M}^t + c\check{e} = cE + d + c\check{e} = c\mathcal{E} + d \equiv \mathcal{M}^t. \quad (2.25)$$

Thus, curiously, the left index of \check{e}_{ij} transforms with M , but the right index with the background-independent \bar{M} defined in (2.7).

Next, we determine from this result the transformation behavior of e_{ij} according to (2.12) in order to verify (2.17), which we write here in matrix notation as

$$e' = M^{-1} e (\bar{M}^t)^{-1}. \quad (2.26)$$

To prove this we show that the defining relation (2.12) between e and \check{e} , which we write as

$$\check{e} - \frac{1}{2} e G^{-1} \check{e} = e, \quad (2.27)$$

is invariant under (2.16), (2.24) and (2.26). Replacing in (2.27) all objects by primed objects we get

$$M^{-1} \check{e}' (\bar{M}^t)^{-1} - \frac{1}{2} M^{-1} e' (\bar{M}^t)^{-1} M^t G^{-1} M M^{-1} \check{e}' (\bar{M}^t)^{-1} = M^{-1} e' (\bar{M}^t)^{-1}. \quad (2.28)$$

Multiplying from the left with M and from the right with \bar{M}^t we obtain

$$\check{e} - \frac{1}{2} e (\bar{M}^t)^{-1} M^t G^{-1} \check{e} = e (\bar{M}^t)^{-1} \bar{M}^t. \quad (2.29)$$

There are two contributions which are ‘non-invariant’: the terms on the left-hand side which do not cancel because one matrix is M and the other \bar{M} ; and the terms on the

right-hand side which do not cancel because one matrix is the background-dependent \bar{M} and the other the background-independent $\bar{\mathcal{M}}$. We compute the failure of covariance in each case. First,

$$\begin{aligned} (\bar{M}^t)^{-1} M^t &= (\bar{M}^t)^{-1} (\bar{M}^t + (M^t - \bar{M}^t)) \\ &= \mathbf{1} + (\bar{M}^t)^{-1} (d - cE^t - d - cE) \\ &= \mathbf{1} - 2(\bar{M}^t)^{-1} cG . \end{aligned} \tag{2.30}$$

Second,

$$(\bar{M}^t)^{-1} \bar{\mathcal{M}}^t = (cE + d)^{-1} (cE + d + c\check{e}) = \mathbf{1} + (\bar{M}^t)^{-1} c\check{e} . \tag{2.31}$$

Inserting these into (2.29) one infers that the non-invariant terms cancel each other, giving back (2.27). This completes the proof that e transforms according to (2.26).

3 $GL(D) \times GL(D)$ covariant frame formulation

In this section we reformulate the above results using the frame formalism developed by Siegel. We first briefly review the aspects of this formalism that are relevant for our subsequent analysis; for a more detailed account we refer to the original literature [17] or to the recent papers [13, 15]. Then we turn to the perturbative expansion about a constant background and prove that the natural frame-like variable can be identified with the string field theory variable e_{ij} above.

3.1 Siegels frame formalism

This formalism is based on the frame field $e_A{}^M$ that is a $2D \times 2D$ matrix,

$$e_A{}^M = \begin{pmatrix} e_{ai} & e_a{}^i \\ e_{\bar{a}i} & e_{\bar{a}}{}^i \end{pmatrix} . \tag{3.1}$$

Here, the index M is a fundamental $O(D, D)$ index and splits according to $M = (i, \bar{i})$; A is the $GL(D) \times GL(D)$ index and splits according to $A = (a, \bar{a})$. Thus, the frame field transforms under global $O(D, D)$ and under local $GL(D) \times GL(D)$ transformations. Moreover, the frame field transforms under a gauge symmetry with a parameter $\xi^M = (\tilde{\xi}_i, \xi^i)$ that combines the usual diffeomorphism parameter ξ^i and the b -field gauge parameter $\tilde{\xi}_i$ into a fundamental $O(D, D)$ vector,

$$\delta_\xi e_A{}^M = \xi^N \partial_N e_A{}^M + (\partial^M \xi_N - \partial_N \xi^M) e_A{}^N , \tag{3.2}$$

where $\partial_M = (\tilde{\partial}^i, \partial_i)$, and indices M, N, \dots are raised and lowered with the $O(D, D)$ invariant metric η_{MN} .

Next, we define a space-time dependent tangent space metric from η_{MN} using the frame field $e_A{}^M$,

$$\mathcal{G}_{AB} = e_A{}^M e_B{}^N \eta_{MN} . \tag{3.3}$$

This metric will be used to raise and lower flat indices. In order for the frame field to describe the same degrees of freedom as the massless sector of closed string theory, it needs to satisfy the $GL(D) \times GL(D)$ covariant constraint

$$\mathcal{G}_{a\bar{b}} = 0 \quad \Leftrightarrow \quad e_{(a}{}^i e_{\bar{b})i} = 0. \quad (3.4)$$

There are various ways to identify the conventional Einstein variables $\mathcal{E}_{ij} = g_{ij} + b_{ij}$ in this formalism. The most direct way is to gauge-fix the $GL(D) \times GL(D)$ symmetry by setting the vielbein components $e_a{}^i$ and $e_{\bar{a}}{}^i$ equal to the unit matrix (assuming that these vielbeins are invertible),

$$e_A{}^M = \begin{pmatrix} -\mathcal{E}_{ai} & \delta_a{}^i \\ \mathcal{E}_{i\bar{a}} & \delta_{\bar{a}}{}^i \end{pmatrix}. \quad (3.5)$$

In here, the constraint (3.4) is implemented by parametrizing the remaining components by a single matrix \mathcal{E}_{ij} . A $GL(D) \times GL(D)$ covariant definition that does not require a gauge fixing, and which will be more useful below, is to identify [13]

$$\mathcal{E}_{ij} = -e_i{}^a e_{aj} = e_{\bar{a}i} e_j{}^{\bar{a}}, \quad (3.6)$$

where $e_i{}^a$ is the inverse of $e_a{}^i$, and $e_i{}^{\bar{a}}$ is the inverse of $e_{\bar{a}}{}^i$. We stress that the vielbeins in (3.6) are independent, and the flat indices are not raised or lowered by means of an invariant tensor ('two-vierbein formalism' [18]). For the gauge choice (3.5) the definition (3.6) coincides with the previous definition of \mathcal{E}_{ij} , but (3.6) is more general in that it holds for arbitrary gauge choices. The metric g_{ij} can then be obtained from the tangent space metric (3.3) according to

$$g_{ij} = -\frac{1}{2} e_i{}^a e_j{}^b \mathcal{G}_{ab} = \frac{1}{2} e_i{}^{\bar{a}} e_j{}^{\bar{b}} \mathcal{G}_{\bar{a}\bar{b}}. \quad (3.7)$$

Next, we mention that one can introduce connections ω_A for the $GL(D) \times GL(D)$ tangent space symmetry in order to construct covariant derivatives $\nabla_A = e_A + \omega_A$, where $e_A = e_A{}^M \partial_M$ is the ordinary (but 'flattened') derivative. Without repeating details here, we record that the calligraphic derivatives of \mathcal{E} can be identified with $GL(D) \times GL(D)$ covariant derivatives of $e_A{}^M$ as follows [15],

$$\begin{aligned} e_a{}^i e_b{}^j e_{\bar{c}}{}^k \mathcal{D}_i \mathcal{E}_{jk} &= e_b{}^M \nabla_a e_{\bar{c}M} = -e_{\bar{c}}{}^M \nabla_a e_{bM}, \\ e_{\bar{a}}{}^i e_b{}^j e_{\bar{c}}{}^k \bar{\mathcal{D}}_i \mathcal{E}_{jk} &= e_b{}^M \nabla_{\bar{a}} e_{\bar{c}M} = -e_{\bar{c}}{}^M \nabla_{\bar{a}} e_{bM}. \end{aligned} \quad (3.8)$$

Even though here we have used covariant derivatives in order to make the full tangent space symmetry manifest, we note that the connections in (3.8) actually drop out as a consequence of (3.4), which will be used below. The double field theory action is then equivalent to [15]

$$\begin{aligned} S = \int dx d\tilde{x} e^{-2d} \left[\mathcal{G}^{ab} \mathcal{G}^{\bar{c}\bar{d}} \left(-\frac{1}{2} e_a{}^M \nabla^c e_{\bar{c}M} e_b{}^N \nabla_c e_{\bar{d}N} + \frac{1}{2} e_{\bar{c}}{}^M \nabla_a e_{cM} e_{\bar{d}}{}^N \nabla^c e_{bN} \right. \right. \\ \left. \left. - \frac{1}{2} e_a{}^M \nabla^{\bar{a}} e_{\bar{c}M} e_b{}^N \nabla_{\bar{d}} e_{\bar{a}N} - e_a{}^M \nabla_{\bar{c}} e_{\bar{d}M} \nabla_b d + e_{\bar{c}}{}^M \nabla_a e_{bM} \nabla_{\bar{d}} d \right) \right. \\ \left. - 2 \nabla^a d \nabla_a d \right]. \quad (3.9) \end{aligned}$$

Using the relations (3.7) and (3.8) between ‘tangent space’ and ‘world’ objects, it can be easily seen, upon converting indices with e_a^i and $e_{\bar{a}}^i$, that this action is equal to (2.2), up to an irrelevant overall factor.

3.2 Perturbation theory in terms of frame-like variables

We next discuss the linearization around a constant background [17],

$$e_A^M = \langle e_A^M \rangle - h_A^B \langle e_B^M \rangle, \quad (3.10)$$

where we introduced a fluctuation h_{AB} with flat indices (that are raised and lowered with the background tangent space metric $\langle \mathcal{G}_{AB} \rangle$). This expansion is meant to be exact, i.e., all higher powers in h in the full theory will originate from taking the inverse of this expression.

Let us first examine the $GL(D) \times GL(D)$ gauge symmetries, whose infinitesimal form we define to be $\delta_\Lambda e_A^M = \Lambda_A^B e_B^M$, with the only non-trivial parameters Λ_a^b and $\Lambda_{\bar{a}}^{\bar{b}}$. As for the field, we expand also the gauge parameter into a background and a first-order part,

$$\Lambda_A^B = \bar{\Lambda}_A^B - \epsilon_A^B, \quad (3.11)$$

where $\bar{\Lambda}$ is constant and thus represents a *global* $GL(D) \times GL(D)$ symmetry.³ Acting on (3.10),

$$\delta_\Lambda (\langle e_A^M \rangle - h_A^B \langle e_B^M \rangle) = (\bar{\Lambda}_A^B - \epsilon_A^B) (\langle e_B^M \rangle - h_B^C \langle e_C^M \rangle), \quad (3.12)$$

we read off by comparing the orders

$$\begin{aligned} \delta_\Lambda \langle e_A^M \rangle &= \bar{\Lambda}_A^B \langle e_B^M \rangle, \\ \delta_\Lambda h_{AB} &= \epsilon_{AB} - \epsilon_A^C h_{CB} + \bar{\Lambda}_A^C h_{CB} + \bar{\Lambda}_B^C h_{AC}, \end{aligned} \quad (3.13)$$

where we used

$$\delta_\Lambda h_{AB} = \delta_\Lambda h_A^C \langle \mathcal{G}_{CB} \rangle + h_A^C \delta_\Lambda \langle \mathcal{G}_{CB} \rangle, \quad \delta_\Lambda \langle \mathcal{G}_{AB} \rangle = \bar{\Lambda}_{AB} + \bar{\Lambda}_{BA}. \quad (3.14)$$

We infer that to lowest order the fluctuations are subject to a shift symmetry, $\delta_\epsilon h_{ab} = \epsilon_{ab}$ and $\delta_\epsilon h_{\bar{a}\bar{b}} = \epsilon_{\bar{a}\bar{b}}$, and so it is natural to impose the gauge fixing condition

$$h_{ab} = h_{\bar{a}\bar{b}} = 0. \quad (3.15)$$

This is also meant to be an exact gauge fixing condition. The constraint (3.4) then implies

$$\begin{aligned} 0 &= e_a^M e_{\bar{b}}^N \eta_{MN} = (\langle e_a^M \rangle - h_a^{\bar{c}} \langle e_{\bar{c}}^M \rangle) (\langle e_{\bar{b}}^N \rangle - h_{\bar{b}}^d \langle e_d^N \rangle) \eta_{MN} \\ &= \langle \mathcal{G}_{\bar{a}\bar{b}} \rangle - h_{\bar{a}\bar{b}} - h_{\bar{b}\bar{a}} \quad \Rightarrow \quad h_{\bar{a}\bar{b}} + h_{\bar{b}\bar{a}} = 0. \end{aligned} \quad (3.16)$$

Here we used that the background tangent space metric satisfies the constraint

$$0 = \langle \mathcal{G}_{\bar{a}\bar{b}} \rangle = \langle e_a^M \rangle \langle e_{\bar{b}}^N \rangle \eta_{MN}. \quad (3.17)$$

We infer from (3.15) and (3.16) that $h_{\bar{a}\bar{b}}$ is the only independent component.

³One may compare this with a similar splitting in the conventional formulation of gravity. Here, after expansion about a flat background, the diffeomorphism symmetry gives rise to two types of symmetries: global Poincaré transformations and local gauge transformations $\delta h_{ij} = \partial_i \xi_j + \partial_j \xi_i$.

After having fixed the gauge symmetries spanned by ϵ_{AB} , let us briefly turn to the background $GL(D) \times GL(D)$ transformations parametrized by $\bar{\Lambda}_A{}^B$. In order to make contact with the formalism in section 2, we fix this global symmetry, in analogy to (3.5), by setting the background frame field equal to

$$\langle e_A{}^M \rangle = \begin{pmatrix} -E_{ai} & \delta_a{}^i \\ E_{i\bar{a}} & \delta_{\bar{a}}{}^i \end{pmatrix}. \quad (3.18)$$

We stress that the complete gauge fixing of the $GL(D) \times GL(D)$ symmetry consisting of (3.15) and (3.18) is inequivalent to the gauge fixing in (3.5), because here $e_a{}^i = \delta_a{}^i$, etc., holds only in the background.

Before imposing the condition (3.18), the fluctuations h_{AB} are inert under global $O(D, D)$ transformations, because they carry only flat indices. After the gauge fixing, however, compensating background $GL(D) \times GL(D)$ transformations are required, which will lead to non-trivial $O(D, D)$ transformations of h_{AB} . We write the finite form of the background $GL(D) \times GL(D)$ transformations generated by $\bar{\Lambda}$ in (3.13) as

$$\langle e_a{}^{M'} \rangle = (M^{-1})_a{}^b \langle e_b{}^M \rangle, \quad \langle e_{\bar{a}}{}^{M'} \rangle = (\bar{M}^{-1})_{\bar{a}}{}^{\bar{b}} \langle e_{\bar{b}}{}^M \rangle, \quad (3.19)$$

and thus for the independent fluctuation as

$$h'_{a\bar{b}} = (M^{-1})_a{}^c (\bar{M}^{-1})_{\bar{b}}{}^{\bar{d}} h_{c\bar{d}}. \quad (3.20)$$

If we require now that the matrices M and \bar{M} are determined in terms of the $O(D, D)$ matrix in such a way that the form of the background frame field (3.18) be preserved, we find (in complete analogy to the analysis of section 4.1 in [15])

$$M = d^t - E c^t, \quad \bar{M} = d^t + E^t c^t, \quad (3.21)$$

while E_{ij} transforms according to (2.15). These matrices coincide with (2.14), and thus we recovered the formalism of section 2, in which unbarred indices transform with M and barred indices transform with \bar{M} . More precisely, via the trivial background vielbeins $\langle e_a{}^i \rangle = \delta_a{}^i$ and $\langle e_{\bar{a}}{}^i \rangle = \delta_{\bar{a}}{}^i$ one can identify indices i, j, \dots with flat unbarred or barred indices that transform with (3.21). From (3.20) we conclude that $h_{a\bar{b}}$ transforms in the same way as e_{ij} under $O(D, D)$. Indeed, we will prove below that these two variables are identical for the background choice (3.18). In the following we present all results more generally, i.e., not assuming that the background takes the specific form (3.18), unless stated differently. The field e_{ij} is then related to $h_{a\bar{b}}$ via

$$e_{ij} = \langle e_i{}^a \rangle \langle e_j{}^{\bar{b}} \rangle h_{a\bar{b}}. \quad (3.22)$$

In particular, this variable transforms under $O(D, D)$ according to (2.17) in general.

Let us now turn to the gauge symmetries (3.2) parametrized by ξ^M . We assume from now on that the background is inert under gauge transformations, and thus we obtain for the gauge transformation of the fluctuation

$$\delta_\xi h_A{}^B \langle e_B{}^M \rangle = \xi^N \partial_N h_A{}^B \langle e_B{}^M \rangle - (\partial^M \xi_N - \partial_N \xi^M) (\langle e_A{}^N \rangle - h_A{}^B \langle e_B{}^N \rangle). \quad (3.23)$$

Multiplying with $\langle e_{CM} \rangle$, relabeling indices and rearranging terms we find

$$\delta_\xi h_{AB} = D_A \xi_B - D_B \xi_A + \xi^N \partial_N h_{AB} + (D_B \xi^C - D^C \xi_B) h_{AC}, \quad (3.24)$$

where we introduced gauge parameters and derivatives whose indices have been flattened with the background frame field,

$$D_A = \langle e_A^M \rangle \partial_M, \quad \xi_A = \langle e_A^M \rangle \xi_M. \quad (3.25)$$

The derivatives D_A are related to the derivatives in section 2 through

$$D_i = \langle e_i^a \rangle D_a, \quad \bar{D}_i = \langle e_i^{\bar{a}} \rangle D_{\bar{a}}, \quad (3.26)$$

i.e., they agree for the choice (3.18) of the background. Using that the background metric G_{ij} is related to the tangent space metric via

$$G_{ij} = -\frac{1}{2} \langle e_i^a \rangle \langle e_j^b \rangle \langle \mathcal{G}_{ab} \rangle = \frac{1}{2} \langle e_i^{\bar{a}} \rangle \langle e_j^{\bar{b}} \rangle \langle \mathcal{G}_{\bar{a}\bar{b}} \rangle, \quad (3.27)$$

we note that the constraint (2.4) translates into

$$D^a D_a + D^{\bar{a}} D_{\bar{a}} = 0, \quad (3.28)$$

when acting on fields and their products.

Next, we inspect the gauge variation of the gauge-fixed components,

$$\begin{aligned} \delta_\xi h_{ab} &= D_a \xi_b - D_b \xi_a + (D_b \xi^{\bar{c}} - D^{\bar{c}} \xi_b) h_{a\bar{c}}, \\ \delta_\xi h_{\bar{a}\bar{b}} &= D_{\bar{a}} \xi_{\bar{b}} - D_{\bar{b}} \xi_{\bar{a}} + (D_{\bar{b}} \xi^c - D^c \xi_{\bar{b}}) h_{\bar{a}c}, \end{aligned} \quad (3.29)$$

where we made use of the gauge fixing condition (3.15). Thus, the gauge condition is not invariant under ξ^M transformations. This requires compensating gauge transformations parametrized by ϵ_{AB} . Choosing ϵ_{ab} to be

$$\epsilon_{ab} = D_b \xi_a - D_a \xi_b - (D_b \xi^{\bar{c}} - D^{\bar{c}} \xi_b) h_{a\bar{c}}, \quad (3.30)$$

restores the gauge condition $h_{ab} = 0$. With this form of the compensating gauge transformation we can compute the complete gauge variation of $h_{\bar{a}\bar{b}}$ from (3.13) and (3.24),

$$\begin{aligned} \delta_\xi h_{\bar{a}\bar{b}} &= D_a \xi_{\bar{b}} - D_{\bar{b}} \xi_a + (\xi^c D_c + \xi^{\bar{c}} D_{\bar{c}}) h_{a\bar{b}} + (D_{\bar{b}} \xi^{\bar{c}} - D^{\bar{c}} \xi_{\bar{b}}) h_{a\bar{c}} - \epsilon_a^c h_{c\bar{b}} \\ &= D_a \xi_{\bar{b}} - D_{\bar{b}} \xi_a \\ &\quad + (\xi^c D_c + \xi^{\bar{c}} D_{\bar{c}}) h_{a\bar{b}} + (D_{\bar{b}} \xi^{\bar{c}} - D^{\bar{c}} \xi_{\bar{b}}) h_{a\bar{c}} + (D_a \xi^c - D^c \xi_a) h_{c\bar{b}} \\ &\quad + h_{a\bar{d}} (D^c \xi^{\bar{d}} - D^{\bar{d}} \xi^c) h_{c\bar{b}}, \end{aligned} \quad (3.31)$$

where we replaced ϵ in the second equation by (3.30).

In order to compare this result with the gauge transformation of the string field theory variable (3.22) as determined in [11], we introduce gauge parameters according to

$$\lambda_i = -\langle e_i^a \rangle \xi_a, \quad \bar{\lambda}_i = \langle e_i^{\bar{a}} \rangle \xi_{\bar{a}}, \quad (3.32)$$

and we assume that indices are contracted with the background metric G_{ij} in (3.27). We then obtain

$$\begin{aligned} \delta_\lambda e_{ij} &= D_i \bar{\lambda}_j + \bar{D}_j \lambda_i \\ &+ \frac{1}{2} (\lambda \cdot D + \bar{\lambda} \cdot \bar{D}) e_{ij} + \frac{1}{2} (\bar{D}_j \bar{\lambda}^k - \bar{D}^k \bar{\lambda}_j) e_{ik} + \frac{1}{2} (D_i \lambda^k - D^k \lambda_i) e_{kj} \\ &- \frac{1}{4} e_{ik} (D^l \bar{\lambda}^k + \bar{D}^k \lambda^l) e_{lj} , \end{aligned} \quad (3.33)$$

which agrees precisely with eq. (2.20) in [11].

So far we have seen that $h_{a\bar{b}}$ gives rise, via (3.22), to a variable that transforms under $O(D, D)$ with the matrices M and \bar{M} and that transforms under gauge transformations as required by the exact result (3.33). This shows that $h_{a\bar{b}}$ can be identified with the string field theory variable e_{ij} according to (3.22). In fact, while it is possible to perform field redefinitions that preserve the left-right index structure, e.g.

$$h_{a\bar{b}} \rightarrow h_{a\bar{b}} + \alpha h_a^{\bar{c}} h_{d\bar{c}} h^d_{\bar{b}} + \mathcal{O}(h^5) , \quad (3.34)$$

this would induce higher order terms in the gauge transformation (3.31) and thus be inconsistent with the form (3.33).

We close this section by verifying that $h_{a\bar{b}}$ is related to the Einstein variable \check{e} according to the field redefinition (2.12), which provides a direct proof for the above conclusion. In order to simplify the notation, we assume that the background takes the specific form (3.18), such that we can identify e_{ij} and $h_{a\bar{b}}$ directly. We use matrix notation and denote the matrix with components E_{ij} by E , the matrix with components e_{ai} by e_* , the matrix with components e_a^i by e^* and the matrix with components $h_a^{\bar{b}}$ by h . The expansions

$$e_{ai} = -E_{ai} - h_a^{\bar{b}} E_{i\bar{b}} , \quad e_a^i = \delta_a^i - h_a^{\bar{b}} \delta_{\bar{b}}^i , \quad (3.35)$$

then read

$$e_* = -E - h E^t , \quad e^* = 1 - h , \quad (3.36)$$

while \mathcal{E} becomes according to (3.6)

$$\mathcal{E} = -(e^*)^{-1} e_* . \quad (3.37)$$

Inserting here the expansions (3.36), we obtain

$$\begin{aligned} \mathcal{E} &= (1 - h)^{-1} (E + h E^t) \\ &= (1 + h + h^2 + h^3 + \dots) (E + h E^t) \\ &= E + h (E + E^t) + h^2 (E + E^t) + h^3 (E + E^t) + \dots \\ &= E + 2hG + 2h^2G + 2h^3G + \dots , \end{aligned} \quad (3.38)$$

where we used $G = \frac{1}{2}(E + E^t)$. Comparison with $\mathcal{E} = E + \check{e}$ then implies

$$\check{e} = 2(1 + h + h^2 + \dots) hG = 2(1 - h)^{-1} hG . \quad (3.39)$$

The identification (3.22) relates e_{ij} to $h_a^{\bar{c}}\langle\mathcal{G}_{\bar{c}\bar{b}}\rangle$, which yields with (3.27)

$$e = 2hG. \tag{3.40}$$

Inserting this into (3.39) finally implies

$$\check{e} = \left(1 - \frac{1}{2}eG^{-1}\right)^{-1} e, \tag{3.41}$$

exactly as required by (2.12). Thus, we recovered the field redefinition between Einstein and string theory variables from the frame formalism.

3.3 The action in terms of frame-like variables

We turn now to the expansion of the action (3.9) in terms of the frame-like variable $h_{a\bar{b}}$. Moreover, we introduce e^{-d} as a fundamental variable with a corresponding fluctuation,

$$\Phi^2 \equiv e^{-2d} \quad \Phi = 1 + \varphi. \tag{3.42}$$

This implies that the new field φ is related to d via

$$\varphi = -d + \frac{1}{2}d^2 + \dots. \tag{3.43}$$

The Lagrangian corresponding to (3.9) then reads in terms of Φ ,⁴

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}\Phi^2 \mathcal{G}^{ab} \mathcal{G}^{\bar{c}\bar{d}} \left(\mathcal{G}^{cd} e_a^M \nabla_c e_{\bar{c}M} e_b^N \nabla_d e_{\bar{d}N} - \mathcal{G}^{cd} e_{\bar{c}}^M \nabla_a e_{cM} e_{\bar{d}}^N \nabla_d e_{bN} \right. \\ & \left. + \mathcal{G}^{\bar{a}\bar{b}} e_a^M \nabla_{\bar{a}} e_{\bar{c}M} e_b^N \nabla_{\bar{d}} e_{\bar{b}N} \right) \\ & + \Phi \mathcal{G}^{ab} \mathcal{G}^{\bar{c}\bar{d}} \left(e_a^M \nabla_{\bar{c}} e_{\bar{d}M} \nabla_b \Phi - e_{\bar{c}}^M \nabla_a e_{bM} \nabla_{\bar{d}} \Phi \right) - 2\nabla^a \Phi \nabla_a \Phi. \end{aligned} \tag{3.44}$$

Next, we work out the various expressions in here, using that the connections inside the covariant derivatives drop out [15]. For instance, we find

$$\begin{aligned} e_a^M \nabla_b e_{\bar{c}M} &= (\langle e_a^M \rangle - h_a^{\bar{b}} \langle e_{\bar{b}}^M \rangle) (\langle e_b^N \rangle - h_b^{\bar{d}} \langle e_{\bar{d}}^N \rangle) \partial_N (-h_{\bar{c}}^d \langle e_{dM} \rangle) \\ &= D_b h_{a\bar{c}} - h_{b\bar{d}} D^{\bar{d}} h_{a\bar{c}}, \end{aligned} \tag{3.45}$$

where the constraint (3.17) has been used. Similarly,

$$\begin{aligned} e_{\bar{c}}^M \nabla_a e_{bM} &= -D_a h_{b\bar{c}} + h_{a\bar{d}} D^{\bar{d}} h_{b\bar{c}}, & e_a^M \nabla_{\bar{b}} e_{\bar{c}M} &= D_{\bar{b}} h_{a\bar{c}} + h_{d\bar{b}} D^d h_{a\bar{c}}, \\ \nabla_a \Phi &= D_a \varphi - h_{a\bar{b}} D^{\bar{b}} \varphi, & \nabla_{\bar{a}} \Phi &= D_{\bar{a}} \varphi + h_{b\bar{a}} D^b \varphi. \end{aligned} \tag{3.46}$$

Inserting this into (3.44), we obtain an action which manifestly preserves the left-right structure. Moreover, the choice of field basis employed here is such that non-polynomial

⁴Using identities like (3.8) and (3.28) the action could be rewritten in various ways, for instance such that it becomes manifest, at the cost of extra terms, that the barred and unbarred indices enter on the same footing.

couplings originate exclusively from taking the inverses of \mathcal{G}_{AB} . This tangent space metric is given in terms of the fluctuation by

$$\mathcal{G}_{ab} = \langle \mathcal{G}_{ab} \rangle + h_a^{\bar{c}} h_{b\bar{c}}, \quad \mathcal{G}_{\bar{a}\bar{b}} = \langle \mathcal{G}_{\bar{a}\bar{b}} \rangle + h_{\bar{a}}^c h_{c\bar{b}}, \quad (3.47)$$

which are exact relations. Its inverse can be written in closed form using matrix notation where \mathcal{G} is the matrix with components \mathcal{G}_{ab} , $\bar{\mathcal{G}}$ is the matrix with components $\mathcal{G}_{\bar{a}\bar{b}}$ and, as above, h is the matrix with components $h_a^{\bar{b}}$,

$$\mathcal{G}^{-1} = \langle \mathcal{G} \rangle^{-1} \sum_{n=0}^{\infty} (-1)^n (h \langle \bar{\mathcal{G}} \rangle h^t \langle \mathcal{G} \rangle^{-1})^n, \quad (3.48)$$

and similarly for $\mathcal{G}^{\bar{a}\bar{b}}$. Restoring explicit index notation, this reads

$$\begin{aligned} \mathcal{G}^{ab} &= \langle \mathcal{G}^{ab} \rangle - h_a^{\bar{c}} h^{b\bar{c}} + h_a^{\bar{c}} h^{d\bar{c}} h_{d\bar{e}} h^{e\bar{b}} \mp \dots, \\ \mathcal{G}^{\bar{a}\bar{b}} &= \langle \mathcal{G}^{\bar{a}\bar{b}} \rangle - h^{\bar{c}a} h_{c\bar{b}} + h^{\bar{c}a} h_c^{\bar{d}} h_{e\bar{d}} h^{e\bar{b}} \mp \dots. \end{aligned} \quad (3.49)$$

Using (3.45), (3.46) and (3.49) it is, in principle, straightforward to read off the n -point couplings from (3.44) to any desired order. By virtue of using the frame-like variable $h_{a\bar{b}}$, the left-right factorization is manifest without further field redefinitions.

As an illustration, we display the quadratic and cubic Lagrangians and verify their equivalence with the results in the literature. The free Lagrangian reads

$$\mathcal{L}^{(2)} = -\frac{1}{2} D^a h^{b\bar{c}} D_a h_{b\bar{c}} + \frac{1}{2} D^a h^{b\bar{c}} D_b h_{a\bar{c}} - \frac{1}{2} D^{\bar{a}} h^{b\bar{c}} D_{\bar{c}} h_{b\bar{a}} + 2 D_{\bar{b}} h^{a\bar{b}} D_a \varphi - 2 D^a \varphi D_a \varphi. \quad (3.50)$$

This is equivalent to (2.13), using the identifications (3.27) and (3.43). The cubic Lagrangian reads

$$\begin{aligned} \mathcal{L}^{(3)} &= h_{a\bar{b}} \left(D^a h^{c\bar{d}} D^{\bar{b}} h_{c\bar{d}} - D^c h^{a\bar{d}} D^{\bar{b}} h_{c\bar{d}} - D^a h_{c\bar{d}} D^{\bar{d}} h^{c\bar{b}} \right) \\ &\quad + \varphi \left(D^a h^{b\bar{c}} D_b h_{a\bar{c}} - D^a h^{b\bar{c}} D_a h_{b\bar{c}} - D^{\bar{a}} h^{b\bar{c}} D_{\bar{c}} h_{b\bar{a}} + D_{\bar{b}} h^{a\bar{b}} D_a \varphi + D_a h^{a\bar{b}} D_{\bar{b}} \varphi \right) \\ &\quad + h_{a\bar{b}} \left(D^a h^{c\bar{b}} D_c \varphi + D_c h^{c\bar{b}} D^a \varphi - D_{\bar{c}} h^{a\bar{c}} D^{\bar{b}} \varphi - D^{\bar{b}} h^{a\bar{c}} D_{\bar{c}} \varphi + 4 D^a \varphi D^{\bar{b}} \varphi \right). \end{aligned} \quad (3.51)$$

Next, we compare this with the cubic action in [10]. The $D\varphi$ terms in the second line can be rewritten as $D(\varphi^2)$ and then partially integrated. Moreover, we partially integrate in the third line in order to move first derivatives away from φ , after which the Lagrangian is equivalent to

$$\begin{aligned} \mathcal{L}^{(3)'} &= h_{a\bar{b}} \left(D^a h^{c\bar{d}} D^{\bar{b}} h_{c\bar{d}} - D^c h^{a\bar{d}} D^{\bar{b}} h_{c\bar{d}} - D^a h_{c\bar{d}} D^{\bar{d}} h^{c\bar{b}} \right) \\ &\quad - \varphi \left(D^a h^{b\bar{c}} D_a h_{b\bar{c}} + (D^a h_{a\bar{b}})^2 - (D^{\bar{b}} h_{a\bar{b}})^2 + 2 h_{a\bar{b}} (D^a D_c h^{c\bar{b}} - D^{\bar{b}} D_{\bar{c}} h^{a\bar{c}}) \right) \\ &\quad - \varphi^2 D_a D_{\bar{b}} h^{a\bar{b}} + 4 h_{a\bar{b}} D^a \varphi D^{\bar{b}} \varphi. \end{aligned} \quad (3.52)$$

Now we have to rewrite this in terms of d , using the non-linear relation (3.43). Specifically, performing this field redefinition, the quadratic Lagrangian (3.50) gives a contribution to

the cubic couplings, which finally read, up to total derivatives,

$$\begin{aligned} \mathcal{L}^{(3)''} = & h_{a\bar{b}} \left(D^a h^{c\bar{d}} D^{\bar{b}} h_{c\bar{d}} - D^c h^{a\bar{d}} D^{\bar{b}} h_{c\bar{d}} - D^a h_{c\bar{d}} D^{\bar{d}} h^{c\bar{b}} \right) \\ & + d \left(D^a h^{b\bar{c}} D_a h_{b\bar{c}} + (D^a h_{a\bar{b}})^2 - (D^{\bar{b}} h_{a\bar{b}})^2 + 2h_{a\bar{b}} (D^a D_c h^{c\bar{b}} - D^{\bar{b}} D_c h^{a\bar{c}}) \right) \\ & - 4h_{a\bar{b}} d D^a D^{\bar{b}} d - 2d^2 \square d, \end{aligned} \quad (3.53)$$

where $\square = D^a D_a = -D^{\bar{a}} D_{\bar{a}}$. Using the identifications (3.27) and (3.43), this coincides with the cubic couplings given in eq. (3.25) of [10].

We close this section with a brief discussion of possible gauge-fixing terms to be added to the action, which is necessary in order to obtain an invertible propagator and thus to derive the Feynman rules. For the free theory, a natural choice of gauge conditions is given by [17]

$$f_a := D^{\bar{b}} h_{a\bar{b}} - D_a \varphi = 0, \quad f_{\bar{a}} := -D^b h_{b\bar{a}} - D_{\bar{a}} \varphi = 0, \quad (3.54)$$

which to lowest order reduces to the usual de Donder gauge-fixing condition if the b -field, the (scalar) dilaton ϕ and the tilde derivatives $\tilde{\partial}$ are set to zero. The gauge-fixed quadratic action then reads [17]

$$\mathcal{L}_{\text{g.f.}}^{(2)} = \mathcal{L}^{(2)} + \frac{1}{2} (f^a f_a - f^{\bar{a}} f_{\bar{a}}) = \frac{1}{2} h^{a\bar{b}} \square h_{a\bar{b}} + \varphi \square \varphi. \quad (3.55)$$

The gauge conditions (3.54) can be taken to be exact, in which case they translate via the field redefinition (2.12) into a non-linear extension of the de Donder gauge-fixing condition for the usual metric fluctuation, but any non-linear generalization of (3.54) that is $GL(D) \times GL(D)$ covariant would also be consistent with the required left-right factorization to all orders.

4 Summary and outlook

In this note we have discussed the perturbative expansion of the double field theory formulation of the low-energy gravity action of closed string theory around a flat background. When expressed in the field basis natural for string field theory, this Lagrangian exhibits a left-right factorization to all orders in perturbation theory by virtue of its T-duality invariance. Moreover, we established the precise relation between the perturbation theory in this formulation and in the frame-like formalism of Siegel. In particular, we showed that the string field theory variable coincides precisely with a frame-like fluctuation in Siegel's formalism. This allows for significant technical simplifications when expanding the action since the field redefinition relating Einstein to string theory variables is already encoded in the frame-like fluctuation. Finally, this relationship might be useful for a more geometrical understanding of string field theory.

The manifest left-right factorization at the level of the Lagrangian discussed here elevates to a corresponding factorization of the Feynman rules: each Feynman graph contributing to an amplitude will factorize into left- and right-handed parts. This formulation

exhibits, therefore, properties inherited from string theory and provides the first step towards the goal to render, for instance, the KLT relations more manifest. The accomplishment of this program requires, however, further work because the left- and right-handed factors of the action in terms of $h_{a\bar{b}}$, which is still non-polynomial, remain to be matched with Yang-Mills theory. In order to achieve this, one procedure could be, as in [9], to bring both the gravity and Yang-Mills action to cubic order upon introducing auxiliary fields. Such a set of auxiliary fields is not unique, but one might hope that the present formulation will eventually suggest a natural choice. We should stress, however, that the field basis discussed here may still be redefined in a way consistent with the factorization property, as in (3.34), and while it is natural to expect that the string field theory basis should exhibit ‘stringy’ features most directly, it remains to be seen which form is more practicable for applications. We will leave these questions for future research.

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