

## Skyrmions in a truncated BPS theory

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**ABSTRACT:** Recently, it has been shown that (4+1)-dimensional Yang-Mills theory may be written as a (3+1)-dimensional BPS Skyrme model, in which the Skyrme field is coupled to an infinite tower of vector mesons. Truncating this tower to a single vector meson yields an extension of the standard Skyrme model to a theory of pions coupled to the  $\rho$  meson, with the significant simplification that no additional free parameters are introduced. The present paper is concerned with this truncated theory and results are presented for Skyrmions with baryon numbers one to four. The approach involves the use of an extended version of the Atiyah-Manton construction, in which the Skyrme field is approximated by the holonomy of a Yang-Mills instanton. It is found that the coupling to the  $\rho$  meson significantly reduces Skyrmion binding energies, to produce an improved comparison with the experimental data on nuclei. A truncation that includes both a vector and an axial vector meson is also investigated, providing a model of pions, the  $\rho$  meson and the  $a_1$  meson. Binding energies are further reduced by the inclusion of this additional meson, shifting the Skyrmion energies a little closer to those of nuclei. Fixing the energy unit by equating the energy of the baryon number four Skyrmion to the  $\text{He}^4$  mass, yields masses for all lower baryon numbers that are within 20 MeV of the experimental values, which is an error that is four times smaller than in the standard Skyrme model.

**KEYWORDS:** Solitons Monopoles and Instantons, Sigma Models

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**1 Introduction**

Skyrmions are topological solitons that describe baryons within a nonlinear theory of pions [1]. It is an ambitious goal to accurately capture the properties of nuclei in terms of Skyrmions, given that in the standard Skyrme model (with massless pions) the only parameters of the theory correspond to energy and length units.

There are several aspects of nuclei that are reproduced remarkably well by the Skyrme model (for a review see [2, 3]), but there is only limited success regarding the important issue of nuclear masses. A main problem is that Skyrmions are too tightly bound in comparison to the experimental data for nuclei. For a large range of nuclei, binding energies are fairly constant at around 8 MeV per nucleon, which is of the order of 1% of the mass of the nucleon. However, in the Skyrme model binding energies per Skyrmion are more like 10% of the mass of a single Skyrmion, even for baryon numbers as low as four, and can rise to almost double this for much larger baryon numbers [4, 5]. Introducing a pion mass into the Skyrme model improves the situation slightly, but there is only a significant effect for larger baryon numbers, where there is a dramatic change in the qualitative form of Skyrmions [6–8].

Recently, it has been shown that (4+1)-dimensional Yang-Mills theory may be written as a (3+1)-dimensional BPS Skyrme model, in which the Skyrme field is coupled to an infinite tower of vector mesons [9]. This is clearly relevant to the above issue, since in a BPS Skyrme theory all binding energies vanish. If the BPS Skyrme theory is truncated by neglecting all the vector mesons then the standard Skyrme model is recovered. This suggests that a truncation in which only a small number of vector mesons are included should lower the binding energies of Skyrmions, in comparison to the standard Skyrme model. The purpose of the present paper is to investigate this issue. Skyrmions are first studied in the simplest example of the truncated theory, where only a single vector meson survives the truncation. Physically, this describes a nonlinear theory of pions coupled to the  $\rho$  meson.

Skyrme models including the  $\rho$  meson have been the subject of considerable study in the past [10–14] but there are difficulties because of the large number of coupling constants that need to be determined. A significant advantage of the truncated BPS theory is that all parameters are uniquely determined once the energy and length units are fixed, so the standard Skyrme model is extended without the introduction of any additional unknown parameters. This is a simplification that is also shared by the holographic model of Sakai and Sugimoto [15], in which a string theory derivation yields a similar extension of the standard Skyrme model to include an infinite tower of vector mesons. Indeed the theory of Sakai and Sugimoto provided the inspiration for the construction of the BPS Skyrme model, but the latter has an additional mathematical advantage in that its solutions are given by self-dual Yang-Mills instantons.

The work of Atiyah and Manton [16, 17] has shown that Skyrmions in the standard Skyrme model are well-approximated by the holonomy of Yang-Mills instantons. In the BPS Skyrme model this approximation becomes exact, therefore it should provide a good approximation in the truncated BPS theory, being at least as accurate as in the standard Skyrme model, if not better. This is the approach adopted here, to calculate the energies of Skyrmions with baryon numbers one to four, without the need to resort to computationally intensive full field numerical simulations. It is found that the coupling to the  $\rho$  meson significantly reduces Skyrmion binding energies, to less than half their values in the standard Skyrme model. Although Skyrmion binding energies are still too large in comparison with nuclei, this is certainly a considerable improvement.

The truncation that retains both a vector and an axial vector meson is also investigated, providing a model of pions, the  $\rho$  meson and the  $a_1$  meson. In this theory Skyrmion binding energies are further reduced, shifting the Skyrmion energies a little closer to those of nuclei. Fixing the energy unit by equating the energy of the baryon number four Skyrmion to the  $\text{He}^4$  mass, yields masses for all lower baryon numbers that are within 20 MeV of the experimental values.

The following section provides a brief review of the derivation of the BPS Skyrme model, as described in [9]. The later sections are concerned with truncations of the BPS theory and the computation of Skyrmion energies in these theories.

## 2 Skyrme from Yang-Mills

The starting point is to consider  $\text{SU}(2)$  Yang-Mills theory in  $(4+1)$ -dimensions. As this paper is only concerned with static fields then the theory may be defined by its static energy

$$E = -\frac{1}{8} \int \text{Tr}(F_{IJ}F_{IJ}) d^4x, \quad (2.1)$$

where  $x_I$ , with  $I = 1, \dots, 4$ , denote the spatial coordinates in four-dimensional Euclidean space and  $F_{IJ} = \partial_I A_J - \partial_J A_I + [A_I, A_J]$  are the components of the  $\text{su}(2)$ -valued field strength.

There is a lower bound on the energy

$$E \geq 2\pi^2 |N|, \quad (2.2)$$

in terms of the instanton number of the gauge field

$$N = -\frac{1}{16\pi^2} \int \text{Tr}(F_{IJ} {}^*F_{IJ}) d^4x, \tag{2.3}$$

where  ${}^*F_{IJ} = \frac{1}{2}\varepsilon_{IJKL}F_{KL}$  is the dual field strength. This is a BPS theory, in that the lower bound is attained by self-dual instantons,  ${}^*F_{IJ} = F_{IJ}$ , for which there is an  $8N$ -dimensional moduli space.

For notational convenience let  $z = x_4$  and denote the three remaining spatial coordinates by  $x_i$  with  $i = 1, 2, 3$ . A Skyrme theory in three-dimensional space is obtained by performing a dimensional deconstruction in the  $z$ -direction. Explicitly, this involves expanding all components of the gauge potential  $A_I$  in terms of a complete set of orthonormal basis functions  $\psi_n(z)$ , with  $n$  a non-negative integer. These are taken to be Hermite functions

$$\psi_n(z) = \frac{(-1)^n}{\sqrt{n! 2^n \sqrt{\pi}}} e^{\frac{1}{2}z^2} \frac{d^n}{dz^n} e^{-z^2}. \tag{2.4}$$

A key step is to transform to the gauge  $A_z = 0$ , in which the remaining components have an expansion of the form

$$A_i = -\partial_i U U^{-1} \psi_+(z) + \sum_{n=0}^{\infty} V_i^n(\mathbf{x}) \psi_n(z), \tag{2.5}$$

where  $U$  is the holonomy

$$U(\mathbf{x}) = \mathcal{P} \exp \int_{-\infty}^{\infty} A_z(\mathbf{x}, z) dz. \tag{2.6}$$

The kink function  $\psi_+(z)$  that appears in (2.5) is obtained from the integral of the first basis function  $\psi_0(z)$  as

$$\psi_+(z) = \frac{1}{\sqrt{2\pi^{\frac{1}{4}}}} \int_{-\infty}^z \psi_0(\xi) d\xi = \frac{1}{2} + \frac{1}{2} \text{erf}(z/\sqrt{2}), \tag{2.7}$$

where  $\text{erf}(z)$  the usual error function, and the constant of integration has been chosen so that  $\psi_+(-\infty) = 0$  and  $\psi_+(\infty) = 1$ .

In the three-dimensional theory the fields  $V_i^n$  correspond to a tower of vector mesons and  $U$  is the Skyrme field, encoding the pion degrees of freedom. As discussed by Atiyah and Manton [16, 17], the Skyrme field defined by the instanton holonomy (2.6) captures all the topological information of the instanton, in that the instanton number is equal to the baryon number of the Skyrme field. Explicitly, it is easy to show that  $N = B$ , where

$$B = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} \text{Tr}(R_i R_j R_k) d^3x, \tag{2.8}$$

is the topological charge that is identified with baryon number, and the above formula allows its calculation in terms of the  $\text{su}(2)$ -valued currents  $R_i = \partial_i U U^{-1}$  of the Skyrme field.

A truncated theory can be defined by including only the first  $K$  vector mesons and substituting the truncated expansion

$$A_i = -\partial_i U U^{-1} \psi_+(z) + \sum_{n=0}^{K-1} V_i^n(\mathbf{x}) \psi_n(z), \quad (2.9)$$

into the Yang-Mills energy (2.1). Performing the integration over  $z$  yields a three-dimensional theory with an energy that will be denoted by  $E^{(K)}$ . The simplest example is to neglect all the vector mesons, which reproduces the standard Skyrme model

$$E^{(0)} = \int \left( -\frac{c_1}{2} \text{Tr}(R_i R_i) - \frac{c_2}{16} \text{Tr}([R_i, R_j]^2) \right) d^3x, \quad (2.10)$$

where the constants are given by

$$c_1 = \frac{1}{4\sqrt{\pi}} = 0.141, \quad c_2 = \int_{-\infty}^{\infty} 2\psi_+^2 (\psi_+ - 1)^2 dz = 0.198. \quad (2.11)$$

This is the standard Skyrme model in dimensionless units, but it is not in standard Skyrme units because the constants  $c_1$  and  $c_2$  are not equal to unity. In these units the Faddeev-Bogomolny energy bound [18] becomes

$$E^{(0)} \geq 12\pi^2 \sqrt{c_1 c_2} |B| = 2.005 \pi^2 |B|. \quad (2.12)$$

This bound is very close to the Yang-Mills derived energy bound (2.2)

$$E^{(K)} \geq 2\pi^2 |B|, \quad (2.13)$$

which is valid for all non-negative integer  $K$ , including  $K = 0$  and the limit  $K \rightarrow \infty$ .

The fact that the Yang-Mills BPS bound (2.13) is within  $\frac{1}{4}\%$  of the Faddeev-Bogomolny bound (2.12) is an indication that the choice of basis functions  $\psi_n(z)$  is close to optimal. An ideal choice would result in the two bounds being identical, and it is easy to show that this occurs only if the kink function is  $\psi_+(z) = \frac{1}{2} + \frac{1}{2} \tanh(z)$ , up to an arbitrary rescaling of  $z$ . However, there is no suitable infinite set of complete basis functions such that the first basis function is proportional to the derivative of this kink function, hence the ideal choice is unattainable.

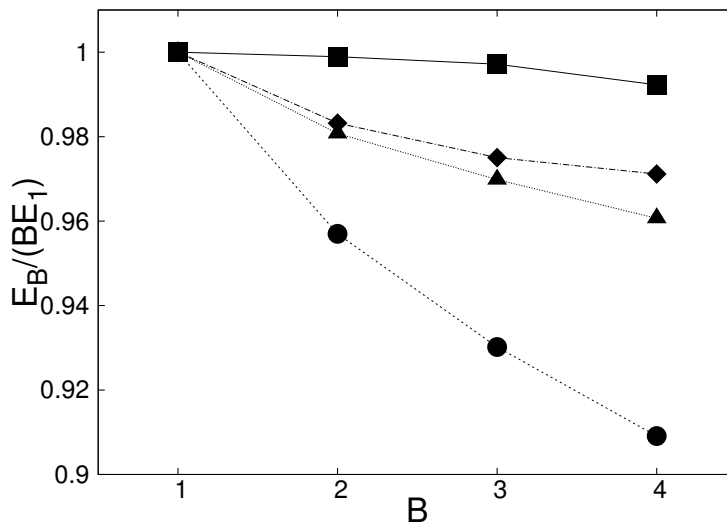
Including the infinite tower of vector mesons extends the standard Skyrme model to a BPS Skyrme model, since it is simply equivalent to Yang-Mills theory with one extra dimension. Self-dual instantons attain the energy bound  $E^{(\infty)} = 2\pi^2 |B|$  and the Skyrme field of the BPS Skyrme model is given exactly by the holonomy of an instanton. This provides an explanation of the Atiyah-Manton construction [16, 17], of approximate solutions of the standard Skyrme model in terms of instanton holonomies, since it is a truncation of an exact equivalence.

### 3 Including the $\rho$ meson

In the standard Skyrme model, minimal energy Skyrmions with baryon numbers one to four have spherical, axial, tetrahedral and cubic symmetry respectively [2]. The energies

$B$	$E_B^{(0)}/(2\pi^2 B)$	$E_B^{(1)}/(2\pi^2 B)$	$E_B^{(2)}/(2\pi^2 B)$
1	1.235	1.071	1.048
2	1.182	1.050	1.030
3	1.149	1.038	1.021
4	1.123	1.029	1.017

**Table 1.** The ratio of the energy of the charge  $B$  Skyrmion to the energy bound  $2\pi^2 B$ , for  $1 \leq B \leq 4$ , in the standard Skyrme model (second column), the theory including a vector meson (third column) and the theory including both a vector and an axial vector meson (fourth column).



**Figure 1.** The energy per baryon, in units of the single baryon energy, for baryon numbers one to four. Squares denote the experimental data. Circles are the energies in the standard Skyrme model. Triangles are the energies in the theory including a vector meson. Diamonds are the energies in the theory including both a vector and an axial vector meson.

of these Skyrmions, using the dimensionless units defined by (2.10), are presented in the second column of table 1, as ratios to the Yang-Mills BPS energy bound  $2\pi^2 B$ . The energies per baryon are plotted as the circles in figure 1, in units of the single baryon energy, which removes any dependence on the units of the theory and hence the parameters of the Skyrme model. For comparison, the associated experimental data on the masses of these nuclei are plotted as the squares in figure 1. This clearly illustrates the point made earlier, that Skyrmions are too tightly bound in the Skyrme model compared to nuclei.

For each of these baryon numbers there is a unique instanton (up to position, orientation and scale) whose holonomy yields a Skyrme field with the correct symmetry [16, 17, 19]. In each case, minimizing over the scale of the instanton yields an approximate Skyrmion with an energy that is only around 1% above that of the true Skyrmion. In this section the instanton approximation is applied to calculate the energies of Skyrmions in the truncated theory including a vector meson.

To include a single vector meson the truncation (2.9) is performed at level  $K = 1$ . For notational convenience write  $V_i^0 = V_i$ . Substituting (2.9) into the Yang-Mills energy (2.1)

and performing the integration over  $z$  yields an extension of the standard Skyrme model to an energy of the form

$$E^{(1)} = E^{(0)} + E_V + E_{SV}. \quad (3.1)$$

Here  $E_V$  is the vector meson energy

$$E_V = \int -\text{Tr} \left\{ \frac{1}{8} (\partial_i V_j - \partial_j V_i)^2 + \frac{1}{4} m^2 V_i^2 + c_3 (\partial_i V_j - \partial_j V_i) [V_i, V_j] + c_4 [V_i, V_j]^2 \right\} d^3 x, \quad (3.2)$$

with dimensionless mass  $m = \frac{1}{\sqrt{2}}$  and constants

$$c_3 = \int_{-\infty}^{\infty} \frac{1}{4} \psi_0^3 dz = \frac{1}{2\sqrt{6}\pi^{\frac{1}{4}}} = 0.153, \quad c_4 = \int_{-\infty}^{\infty} \frac{1}{8} \psi_0^4 dz = \frac{1}{8} \sqrt{\frac{1}{2\pi}} = 0.050. \quad (3.3)$$

The interaction energy between the Skyrme field and the vector meson is

$$\begin{aligned} E_{SV} = \int -\text{Tr} \left\{ c_5 ([R_i, V_j] - [R_j, V_i])^2 - c_6 [R_i, R_j] (\partial_i V_j - \partial_j V_i) - c_7 [R_i, R_j] [V_i, V_j] \right. \\ \left. + \frac{1}{2} c_6 [R_i, R_j] ([R_i, V_j] - [R_j, V_i]) - \frac{1}{8} ([R_i, V_j] - [R_j, V_i]) (\partial_i V_j - \partial_j V_i) \right. \\ \left. - \frac{1}{2} c_3 ([R_i, V_j] - [R_j, V_i]) [V_i, V_j] \right\} d^3 x, \quad (3.4) \end{aligned}$$

where the constants are

$$\begin{aligned} c_5 = \int_{-\infty}^{\infty} \frac{1}{8} \psi_+^2 \psi_0^2 dz = 0.038, \quad c_6 = \int_{-\infty}^{\infty} \frac{1}{4} \psi_+ (1 - \psi_+) \psi_0 dz = \frac{\pi^{1/4}}{12\sqrt{2}} = 0.078, \\ c_7 = \int_{-\infty}^{\infty} \frac{1}{4} \psi_+ (1 - \psi_+) \psi_0^2 dz = 0.049. \quad (3.5) \end{aligned}$$

The vector mesons  $V_i^n$  that appear in the expansion (2.5) do not have a definite parity, but an additional gauge transformation yields an expansion in terms of parity eigenstates and reveals that even values of  $n$  correspond to vector mesons and odd values of  $n$  to axial vector mesons [9]. This means that  $V_i = V_i^0$  should be identified with the lightest vector meson, namely the  $\rho$  meson. Numerically it is more convenient to work in the gauge presented above, rather than the gauge in which parity is manifest.

It seems a reasonable assumption, at least for baryon numbers up to four, that the symmetry of the Skyrmion in the theory extended by the inclusion of a small number of vector mesons is the same as in the standard Skyrme theory. This is based on the highly symmetric form for these Skyrmions and the fact that as further vector fields are included the theory flows to a BPS theory in which all points in the instanton moduli space produce Skyrme fields with equal energy  $2\pi^2 B$ . With this assumption, the appropriate instanton is known [16, 17, 19] and all that remains is to determine the energy minimizing scale. This is an easy task once all the contributions to the energy have been computed at any given scale, since the behaviour of each term under a rescaling is easily determined.

Given the fields  $A_I(\mathbf{x}, z)$  of an appropriate instanton, a numerical gauge transformation is performed to arrive at the gauge  $A_z = 0$ . By comparison with the expansion (2.5) the

currents of the Skyrme field are then given by  $R_i(\mathbf{x}) = -A_i(\mathbf{x}, \infty)$ . The required vector mesons  $V_i^n$  are then extracted as the integrals

$$V_i^n(\mathbf{x}) = \int_{-\infty}^{\infty} \left( A_i(\mathbf{x}, z) + R_i(\mathbf{x})\psi_+(z) \right) \psi_n(z) dz. \quad (3.6)$$

These integrals are performed numerically by mapping  $z \in (-\infty, \infty)$  to the finite interval  $Z \in (-1, 1)$  via the transformation  $z = \tan(Z\pi/2)$  and using an equally spaced grid in the  $Z$  coordinate containing (at least) 400 grid points. The same procedure is used in performing the numerical gauge transformation to  $A_z = 0$ .

The above scheme allows the construction of the Skyrme currents and vector mesons for any given point in three-dimensional space. This is implemented at all points in a spatial lattice containing  $101^3$  grid points with a lattice spacing  $\Delta x = 0.15$ . The energy  $E^{(1)}$  is then computed using the formulae (2.10), (3.2) and (3.4), where the spatial derivatives of the vector meson are approximated using fourth-order accurate finite differences. As a numerical check, the baryon number is computed using the same lattice, and is found to be equal to an integer to at least four decimal places for all baryon numbers considered.

The numerical results for  $E^{(1)}$  are presented in the third column of table 1, as ratios to the BPS bound, and are plotted as the triangles in figure 1. These results show that the Yang-Mills derived coupling to the  $\rho$  meson moves the Skyrmion energies much closer to the BPS bound and significantly reduces Skyrmion binding energies, to less than half their values in the standard Skyrme model, as is evident from figure 1. However, it is also clear that Skyrmion binding energies are still larger than those of nuclei, even with this considerable improvement.

#### 4 Including the $a_1$ meson

In this section the truncated theory at level  $K = 2$  is investigated, in which both a vector and an axial vector meson are coupled to the standard Skyrme model. The notation of the previous section  $V_i = V_i^0$  is retained and the first axial vector meson is denoted by  $W_i = V_i^1$ . Physically, this field is to be identified with the lightest axial vector meson, which is the  $a_1$  meson.

Substituting the level  $K = 2$  truncation (2.9) into the Yang-Mills energy (2.1) and integrating over  $z$  gives an extension of the energy of the previous section to

$$E^{(2)} = E^{(1)} + E_W + E_{SW} + E_{VW} + E_{SVW}. \quad (4.1)$$

In the above  $E_W$  is the axial vector meson energy

$$E_W = \int -\text{Tr} \left\{ \frac{1}{8} (\partial_i W_j - \partial_j W_i)^2 + \frac{1}{4} M^2 W_i^2 + \frac{3}{4} c_4 [W_i, W_j]^2 \right\} d^3x, \quad (4.2)$$

with dimensionless mass  $M = \sqrt{\frac{3}{2}}$ .

Note that the dimensionful masses of the particles in the theory depend upon the choice of energy and length units, which may be fixed in a variety of ways according to



which physical properties are deemed most desirable to reproduce. A theme of this paper has been to consider fundamental aspects that are independent of the choice of units, and another example is the ratio of the mass of the lightest axial vector meson to the mass of the lightest vector meson. From (3.2) and (4.2) this mass ratio is

$$\frac{M}{m} = \sqrt{3} = 1.73 \quad (4.3)$$

to be compared with the experimental result

$$\frac{m_{a_1}}{m_\rho} = \frac{1230 \text{ MeV}}{776 \text{ MeV}} = 1.59 \quad (4.4)$$

for the ratio of the  $a_1$  to  $\rho$  mass. Given that this ratio is completely determined in the theory, with no adjustable parameters, then an error of less than 9% is striking.

The remaining terms in the energy expression (4.1) are rather cumbersome and are presented below. The interaction energy between the Skyrme field and the axial vector meson is

$$\begin{aligned} E_{\text{SW}} = \int -\text{Tr} \left\{ c_8 ([R_i, W_j] - [R_j, W_i])^2 - c_9 [R_i, R_j][W_i, W_j] \right. \\ \left. + c_{10} [R_i, R_j]([R_i, W_j] - [R_j, W_i]) - \frac{1}{8} ([R_i, W_j] - [R_j, W_i])(\partial_i W_j - \partial_j W_i) \right. \\ \left. - c_{11} ([R_i, W_j] - [R_j, W_i])[W_i, W_j] - c_{12} R_i W_i \right\} d^3x, \quad (4.5) \end{aligned}$$

where the constants are

$$\begin{aligned} c_8 &= \int_{-\infty}^{\infty} \frac{1}{8} \psi_+^2 \psi_1^2 dz = 0.047, & c_9 &= \int_{-\infty}^{\infty} \frac{1}{4} \psi_+(1-\psi_+) \psi_1^2 dz = 0.030, \\ c_{10} &= \int_{-\infty}^{\infty} \frac{1}{4} \psi_+(1-\psi_+) \psi_1 dz = 0.016, \\ c_{11} &= \int_{-\infty}^{\infty} \frac{1}{4} \psi_+ \psi_1^3 dz = \frac{11\sqrt{2}}{144\pi^{3/4}} = 0.046, & c_{12} &= \frac{1}{4\pi^{1/4}} = 0.188. \end{aligned} \quad (4.6)$$

Note that the last term in (4.5) is the familiar mixing between the Skyrme field and the lightest axial vector meson that arises in coupling the Skyrme model to vector mesons [11].

The interaction energy between the vector meson and the axial vector meson is

$$\begin{aligned} E_{\text{VW}} = \int -\text{Tr} \left\{ \frac{1}{2} c_4 ([V_i, W_j] - [V_j, W_i])^2 + c_4 [V_i, V_j][W_i, W_j] \right. \\ \left. + \frac{2}{3} c_3 ([V_i, W_j] - [V_j, W_i])(\partial_i W_j - \partial_j W_i) \right. \\ \left. + \frac{2}{3} c_3 ([W_i, W_j] - [W_j, W_i])(\partial_i V_j - \partial_j V_i) \right\} d^3x. \quad (4.7) \end{aligned}$$

$B$	Mass in MeV	
	Experiment	Theory
1	939	959
2	1876	1887
3	2809	2806
4	3727	3727

**Table 2.** For baryon numbers one to four the experimental values of the masses of nuclei are compared with the theoretical predictions in the truncated Skyrme model containing a vector and an axial vector meson.

Finally, there is an interaction energy coupling the Skyrme field to both the vector and axial vector mesons

$$\begin{aligned}
 E_{\text{SVW}} = \int -\text{Tr} \left\{ -\frac{6}{11} c_{11} [V_i, V_j] ([R_i, W_j] - [R_j, W_i]) - c_{13} ([R_i, V_j] - [R_j, V_i]) (\partial_i W_j - \partial_j W_i) \right. \\
 - c_{13} ([R_i, W_j] - [R_j, W_i]) (\partial_i V_j - \partial_j V_i) - \frac{1}{3} c_3 [W_i, W_j] ([R_i, V_j] - [R_j, V_i]) \\
 + c_{13} ([R_i, V_j] - [R_j, V_i]) ([R_i, W_j] - [R_j, W_i]) \\
 - \frac{6}{11} c_{11} ([R_i, V_j] - [R_j, V_i]) ([V_i, W_j] - [V_j, W_i]) \\
 \left. - \frac{1}{3} c_3 ([R_i, W_j] - [R_j, W_i]) ([V_i, W_j] - [V_j, W_i]) \right\} d^3x, \quad (4.8)
 \end{aligned}$$

where

$$c_{13} = \int_{-\infty}^{\infty} \frac{1}{4} \psi_+ \psi_0 \psi_1 dz = \frac{1}{4\sqrt{6\pi}} = 0.058. \quad (4.9)$$

Applying the numerical procedure described in the previous section, involving the same symmetric instantons (though with different energy minimizing scales) produces the energies presented in the final column of table 1 and plotted as the diamonds in figure 1.

The addition of the axial vector meson shifts the Skyrmion energies a little closer to the BPS bound and slightly decreases the binding energies. As the energy of the  $B = 1$  Skyrmion is less than 5% above the BPS bound then this provides an upper limit on the binding energy per baryon of 5% of the energy of the single baryon, which is a significant improvement on the standard Skyrme model.

For  $B > 1$  the slope of the curve joining the triangles in figure 1 is similar to the slope of the curve joining the squares that represent the data for nuclei. Hence a reasonable approximation to the masses of nuclei with  $B = 2, 3, 4$  can be obtained at the expense of overestimating the energy of the single baryon. In fact, it is wise to fix the energy unit by matching the energy of the  $B = 4$  Skyrmion to the mass of  $\text{He}^4$  since its ground state has zero spin and isospin, so there are no associated quantum corrections to the classical energy. Choosing the energy unit in this way allows the data in the final column of table 1 to be written in terms of the predicted physical masses for nuclei, which are presented in the final column of table 2. For comparison, the second column of table 2 displays the experimental values measured for nuclei. It can be seen that this gives a reasonable

approximation to the experimental data, particularly for baryon numbers greater than one, but even the single baryon mass is only 20 MeV above the true value. Note that a similar calculation in the standard Skyrme model gives an energy excess which is more than four times greater than this.

The Skyrmion energies presented in this paper are purely classical, but there are also quantum contributions associated with spin and isospin. These contributions can be calculated within a semiclassical rigid-body quantization, though the computations become significantly more involved with the inclusion of vector mesons. The calculation has been performed [9] for the single baryon in the theory including just one vector meson and the result is similar to that in the standard Skyrme model. The magnitude of the quantum corrections depends strongly upon the choice of energy and length units, but it is clear that such quantum contributions can only exacerbate the problem regarding binding energies. In fact, considerations of binding energies require that these quantum corrections must be small; since they vanish for  $B = 4$  such quantum corrections for the single nucleon must not be much greater than 8 MeV otherwise, even in a BPS theory, they would produce a binding energy for nuclei such as  $B = 4$  that exceeds the experimental value. It may therefore be judicious to fix energy and length units by considering this issue in more detail, though this will be left for a future investigation, as it would be more useful to perform an analysis once further results are available for larger baryon numbers. Note that these considerations will certainly require that the energy units for quantum corrections are much smaller than those determined by fixing to the mass of the delta resonance [20], but it has already been demonstrated that this is not a good approach to fixing the parameters of the Skyrme model, as the result it determines is an artefact of the rigid-body approximation [21].

## 5 Conclusion

Skyrmions have been investigated in an extension of the standard Skyrme model to include the  $\rho$  and  $a_1$  mesons, with all couplings and masses uniquely determined by the truncation of a BPS theory. The results are encouraging, with binding energies being dramatically reduced so that discrepancies between the values for nuclei and Skyrmions are reduced to around one quarter of those found in the standard Skyrme model.

Skyrmions have been approximated using self-dual instantons, with the assumption that for baryon numbers up to four the symmetry of the Skyrmion in the extended theory is the same as that in the standard Skyrme model. For  $B \leq 4$  these symmetries select a unique instanton, up to the obvious freedom associated with position, orientation and scale, but for  $B > 4$  symmetry alone is not sufficient to select the required instanton; except for the special cases  $B = 7$  and  $B = 17$ , where icosahedral symmetry does pin down the instanton [22, 23]. Therefore, to extend the results to larger baryon numbers, and also to check the assumed symmetries and accuracy of the instanton approximation, it will be necessary to perform full field numerical simulations of the extended model. This will be a computational challenge because there is a significant increase in both the number of degrees of freedom and the number of terms contributing to the energy, in comparison to the standard Skyrme model. However, this would certainly be a worthwhile avenue

for future research and would also allow Skyrmions to be studied in the extended theory including a pion mass term, which is known to be necessary when considering larger baryon numbers.

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