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Supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM and pure spinors

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Abstract: We study supersymmetric Wilson loop operators in four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. We show that the contour of a supersymmetric Wilson loop is either an orbit of some conformal transformation of the space-time (case I), or an arbitrary contour in the subspace where local superalgebra generator is a pure spinor (case II). In the more interesting case II we find and classify all pairs (Q, W) of the supercharges and the corresponding operators modulo the action of the global symmetry group.

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Contents

1 Introduction

The four-dimensional maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM) is a fascinating model which exhibits rich but rigid mathematical structure. Thanks to the AdS/CFT correspondence [1–4] the theory has been in focus of theoretical research for the past decade. Many interesting results including those based on integrability [5, 6] suggest that $\mathcal{N} = 4$ SYM may have an exact solution in the large N limit at least in the supersymmetric sector. This motivates our interest in studying the supersymmetric sector to identify non-local gauge invariant observables.

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The $\mathcal{N} = 4$ SYM is a superconformal theory. The fermionic subspace of its superconformal algebra is generated by Poincare supercharges Q_α and special conformal supercharges S^α. In the scope of the present work we call an operator supersymmetric if there exists at least one non-zero linear combination of Q_α and S^α that annihilates the operator.

In this paper we are interested in one-dimensional non-local operators. Familiar examples of such operators are 't Hooft and Wilson loop operators. Presently we focus on supersymmetric Wilson loop operators, which are obtained from the ordinary Wilson loops by coupling them to the scalars of the $\mathcal{N} = 4$ SYM [7]. We consider the theory on the Euclidean space-time $\mathbb{R}^4_{\text{spt}}$.

A number of such supersymmetric Wilson loops have been found and analyzed previously, see e.g. [8–16]. All supersymmetric Wilson loops that have been studied previously are captured by two classes: the loops of arbitrary shape on $\mathbb{R}^4_{\text{spt}}$ found by Zarembo in [16] and the loops of arbitrary shape on a three-sphere $S^3 \subset \mathbb{R}^4_{\text{spt}}$ found by Drukker-Giombi-Ricci-Trancanelli (DGRT) in [12]. Zarembo's loops on $\mathbb{R}^4_{\text{spt}}$ are the same Wilson loops which appear in topological Langlands twist of $\mathcal{N} = 4$ SYM [17]; they have trivial expectation value. The string dual surfaces to these loops were described in [18]. The most familiar example of the loops in DGRT class is the 1/2 BPS circular loop coupled to one of the scalars; this Wilson loop can be computed exactly by Gaussian matrix model [8, 9, 19] and the results agree with the string dual computation. The subset of DGRT loops restricted to S^2 was also recently studied in great details and a connection between this sector of $\mathcal{N} = 4$ SYM and two-dimensional Yang-Mill on S^2 was established [14, 15, 20–26]. It has not been clear whether these two classes capture all possible supersymmetric Wilson loops.

In this note we give a systematic answer to this question. We find all possible Wilson loop operators W that are invariant at least under one superconformal symmetry Q. Moreover, we classify the interesting subclass of pairs (Q, W) modulo equivalence under the action of the superconformal group of the $\mathcal{N} = 4$ SYM.

We find new supersymmetric Wilson loops which has not been identified before. In many cases the new operators involve complex couplings to the scalar fields that clearly distinguishes them from the previously studied cases. In certain cases the new operators could be related to the previously known ones by a complexified conformal transformation. However, unless we define the theory on the complexified space-time, and stay in the framework of the conventional theory formulated in the real Euclidean space, the novel operators are not equivalent to the known ones.

The crucial ingredient in our construction are the ten-dimensional pure spinors. Their relevance is not so surprising given that the four-dimensional $\mathcal{N} = 4$ SYM is a dimensional reduction of the ten-dimensional $\mathcal{N} = 1$ SYM, where pure spinors appear naturally [27–29]. The space-time dependent spinor ε that parametrizes the superconformal transformations of $\mathcal{N} = 4$ SYM, can be viewed as a reduction of a chiral ten-dimensional spinor.

Locally, at a point x of the space-time, Wilson loop operator can be locally described by the tangent vector to the curve and the scalar couplings at x. We combine this data into ten-dimenensional vector $v(x)$. If we want to find supersymmetric Wilson loops with respect to a supersymmetry generated by a given spinor $\varepsilon(x)$, we get a certain system of equations on $v(x)$. This system of equations might be of different types depending on $\varepsilon(x)$. If $\varepsilon(x)$ is not a pure spinor, then the system has the unique solution, so that the tangent to the curve and the scalar couplings at x are completely fixed. Namely, the tangent to the curve and the scalar couplings could be combined into a ten-dimensional vector $v(x)$. This vector, projectively, is precisely the ten-dimensional vector constructed in the canonical way as the bilinear in $\varepsilon(x)$. The curves, resulting in this way from a generic supersymmetry parameter $\varepsilon(x)$, are nothing else but the orbits of the conformal transformation generated by Q_ε^2. If we ask for the orbits to be compact, then modulo conformal equivalence, the only resulting compact curves are the (p, q) Lissajous figures where $\frac{p}{q} \in \mathbb{Q}$ is the rational ratio of two eigenvalues of the $\mathfrak{so}(4)$ matrix which represents the action of Q^2 on the space-time $\mathbb{R}^4_{\text{spt}}$.

If $\varepsilon(x)$ is pure then there are more solutions for the vector $v(x)$ (which tangent to the curve at x and scalar couplings described together by the ten-dimensional vector $v(x)$. More precisely, a pure spinor $\varepsilon(x)$ defines ten-dimensional almost complex structure $J(x)$, and then the supersymmetry condition of the Wilson loop at x translates to the condition that $v(x)$ is anti-holomorphic vector with respect to $J(x)$. On the subspace Σ of the space-time where $\varepsilon(x)$ is pure there is richer space of solutions for supersymmetric Wilson loops. Generically, for any curve sitting inside Σ one can find scalar couplings to make supersymmetric Wilson loop.

The supersymmetry spinor $\varepsilon(x)$ of $\mathcal{N} = 4$ SYM can be extended to the $AdS_5 \times S^5$ space where it plays the role of the supersymmetry spinor of the IIB String Theory. Similarly the space Σ where $\varepsilon(x)$ is pure can be extended to the the subspace $\Sigma_\mathbb{C}$ in $AdS_5 \times S^5$. The pure spinor defines an almost complex structure $J \in \text{End}(T_{\Sigma_\mathbb{C}} + N_{\Sigma_\mathbb{C}})$, where T and N stand respectively for the tangent and normal bundles of $\Sigma_\mathbb{C} \subset AdS_5 \times S^5$.

We conjecture that for a Wilson loop operator with the contour in Σ the classical dual string worldsheet lies on $\Sigma_\mathbb{C}$ and is pseudo-holomorphic with respect to J. This is supported by the fact that the J-pseudo-holomorphic solution is necessarily supersymmetric. Thus the results [12, 18] developed earlier for the string duals of Zarembo loops [16] and DGRT loops [12] are the particular examples of this general picture.

The structure of the paper is as follows. In section 2 we summarize our conventions on $\mathcal{N} = 4$ SYM and superconformal transformations in Euclidean space-time. In section 3 we give general construction of supersymmetric Wilson operators and relate that to pure spinors. In section 4 we find the pure-spinor-surfaces Σ construct the supersymmetric Wilson loop operators. The next section 5 deals with classification of the pairs (Q, W) related to pure spinors modulo equivalence under the action of the superconformal group of the $\mathcal{N} = 4$ SYM.

2 Conventions

We consider the Euclidean space-time $\mathbb{R}^4_{\text{spt}}$ equipped with the standard flat unit metric.

We take the action of the $\mathcal{N} = 4$ SYM gauge theory with gauge group G on $\mathbb{R}^4_{\text{spt}}$ to be

$$\begin{aligned} S = -\frac{1}{2g_{\text{YM}}^2} \int d^4x \, \text{Tr} \Bigg(\frac{1}{2} \Big(F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi_A D^\mu \Phi^A + \frac{1}{2}[\Phi_A \Phi_B][\Phi^A \Phi^B] \Big) - \\ - \Psi \Gamma^\mu D_\mu \Psi - \Psi \Gamma^A [\Phi_A \Psi] \Bigg). \end{aligned} \tag{2.1}$$

The indexes $\mu, \nu = 1, \ldots, 4$ label the directions in the space-time, the indices $A, B = 5 \ldots 10$ label the directions in the target space of scalars. We often combine indexes μ, ν with A, B into ten-dimensional indexes $N, M = 1 \ldots 10$, and the gauge field A_μ and the scalar fields Φ_A into $A_M := (A_\mu, \Phi_A)$. That could be interpreted about as dimensional reduction of the gauge field of $d = 10$ $\mathcal{N} = 1$ SYM. All fields take value in the Lie algebra of the gauge group G, the conventions for the covariant derivative and the curvature are $D_\mu = \partial_\mu + A_\mu$ and $F_{\mu\nu} = [D_\mu, D_\nu]$.

The fermionic fields Ψ are sixteen-component spinors obtained by dimensional reduction from the chiral spin representation of $\mathrm{Spin}(10, \mathbb{R})$ which we call S^+. The chiral spin representation of $\mathrm{Spin}(10, \mathbb{R})$ dual to S^+ is called S^-. The matrices $\Gamma^M : S^+ \to S^-$ are the 16×16 matrices which are the chiral blocks of the 32×32 ten-dimensional Dirac gamma-matrices γ_μ. We use conventions where

$$\gamma_M : \begin{pmatrix} S^+ \\ S^- \end{pmatrix} \to \begin{pmatrix} S^+ \\ S^- \end{pmatrix}, \quad \gamma_M = \begin{pmatrix} 0 & \Gamma^*_M \\ \Gamma_M & 0 \end{pmatrix}, \quad \Gamma_M = \Gamma^T_M, \quad \{\gamma_M, \gamma_N\} = \delta_{MN}. \tag{2.2}$$

The explicit form of Γ^M can be found in appendix A. In ten dimensions there is no need for complex or Dirac-like conjugation to write down a fermionic bilinear like $\Psi\Gamma^M\Psi$ which is literally

$$\sum_{\alpha,\beta=1}^{16} \Psi^\alpha \Gamma^M_{\alpha\beta} \Psi^\beta \,. \tag{2.3}$$

We use the indexes $\alpha, \beta = 1 \ldots 16$ to denote the sixteen components of S^+ spinors such as Ψ_α. Since we consider the theory as dimensionally reduced from Euclidean space $\mathbb{R}^{10}$ rather than Minkowski space $\mathbb{R}^{9,1}$, we do not require Ψ to be real. However, in the path integral we integrate only over Ψ but not over their complex conjugates. This is consistent because complex conjugate to Ψ never appears in the action or anywhere else.

We consider the superconformal transformations

$$\begin{aligned} \delta A_M &= \varepsilon \Gamma_M \Psi, \\ \delta \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \varepsilon + \Phi_A \Gamma^{\mu A} \nabla_\mu \varepsilon, \end{aligned} \tag{2.4}$$

where spinor $\varepsilon(x)$ is a parameter. We treat the spinor $\varepsilon(x)$ as a bosonic parameter of the fermionic supersymmmetry transformation. It transforms in the same spin representation as Ψ, i.e. in S^+. The $\mathcal{N} = 4$ SYM action (2.1) is invariant under (2.4) if $\varepsilon(x)$ is a conformal Killing spinor (twistor spinor) [30].

By definition, a conformal Killing spinor $\varepsilon(x)$ is a solution of the twistor equation (see [31, 31, 32] for a review on conformal Killing spinors)

$$D_\mu \varepsilon = \frac{1}{4} \Gamma_\mu \not{D} \varepsilon. \tag{2.5}$$

We use the notation $\tilde{\varepsilon} = \frac{1}{4} \not{D} \varepsilon$, so the conformal Killing spinor equation is $D_\mu \varepsilon = \Gamma_\mu \tilde{\varepsilon}$. The solutions on $\mathbb{R}^4$ are parametrized by two constant spinors, which we call $\varepsilon_s \in S^+$ and $\varepsilon_c \in S^-$

$$\varepsilon(x) = \varepsilon_s + x^\mu \Gamma_\mu \varepsilon_c \,. \tag{2.6}$$

In total there are $16+16=32$ complex generators of superconformal symmetries. The spinor ε_s generates the usual supersymmetries associated with 16 supercharges which are customarily called Q_α, and the spinor ε_c generates the remaining special conformal supersymmetries associated with 16 supercharges which are customarily called S^α. The supersymmetry transformation Q_ε is given by $Q_\varepsilon = \varepsilon_s^\alpha Q_\alpha + \varepsilon_\beta^c S^\beta$.

3 Supersymmetric Wilson loops and pure spinors

For a closed contour $\gamma : S^1 \to \mathbb{R}^4_{\text{spt}}$ and a representation R of the gauge group, a Wilson loop operator $W_R(\gamma)$ is the trace in the representation R of the path ordered integral of the gauge field along the contour γ

$$W_R(\gamma) = \mathrm{Tr}_R\, \mathrm{Pexp} \oint_\gamma A_\mu dx^\mu. \tag{3.1}$$

A natural generalization of (3.1) for a theory with adjoint scalars is obtained by coupling to the scalar fields Φ_A [7, 33]

$$\begin{aligned} W_R(\hat{\gamma}) &= \mathrm{Tr}_R\, \mathrm{Pexp} \oint_\gamma A_M v^M ds \\ &= \mathrm{Tr}_R\, \mathrm{Pexp} \oint_\gamma (A_\mu v^\mu + \Phi_A v^A) ds\,, \end{aligned} \tag{3.2}$$

where the generalized contour $\hat{\gamma} = (x^\mu(s), v^A(s))$ is now defined by specifying the four-dimensional tangent velocity vector $v^\mu(s) = dx^\mu/ds$ and a six-vector of scalar couplings $v^A(s)$. To make the usual sense of the contour in the real space-time the tangent vector $v^\mu(s)$ must be real. At the same time the scalar couplings $v^A(s)$ generically could be complex. Our notation v^A is related to the common notation θ^A, used in the literature on the subject [7, 12, 16], via $v^A = i\theta^A$. A local operator, say $\frac{1}{J!}\mathrm{Tr}_R(\Phi_5 + i\Phi_6)^J$ is also captured by the generic definition (3.2). It corresponds to contour γ which is point in $\mathbb{R}^4_{\text{spt}}$ but a unit interval in $\mathbb{R}^6_{\text{scl}}$ such that $\int_\gamma v^A = (1, i, 0, 0, 0, 0)$ and taking J-th term in Taylor series of the exponent expansion.

For the Wilson loop (3.2) to be supersymetric $v^M A_M$ must be invariant under (2.4), that is

$$v^M(x)\varepsilon(x)\Gamma_M \Psi = 0 \tag{3.3}$$

has to vanish for any Ψ at each point on the contour $\gamma(s)$. This implies

$$v^M(x)\Gamma_M \varepsilon(x) = 0\ . \tag{3.4}$$

To find all possible solutions of (3.4), we first consider the problem locally a point x. We assume that we have a generic spinor ε, and we want to identify the space of directions $L \subset \mathbb{R}^{10} \otimes \mathbb{C}$ which annihilate ε under the Clifford action:

$$v^M \Gamma_M \varepsilon = 0, \quad v \in L. \tag{3.5}$$

At this moment we allow v^M to be complex and consider possible reality conditions later.

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For any ε we can canonically construct a bilinear vector $u^M(\varepsilon)$ as

$$u^M = \varepsilon\Gamma^M\varepsilon. \tag{3.6}$$

Now, depending on whether $u^M = 0$ or $u^M \neq 0$ we have two distinct cases. If $u^M = 0$ the spinor ε is *pure spinor*, and if $u^M \neq 0$ the spinor ε is *not pure spinor*. This requires some clarification which will be given below.

First consider the generic case when $u^M(x) \neq 0$ (ε is not pure spinor). It is a simple exercise to show that in ten dimensions $v^M = \lambda u^M, \lambda \in \mathbb{C}$ is the only solution to (3.5) unless $u^M = 0$. The fact that $v^M = u^M$ is a solution follows from the following identity for the ten-dimensional gamma-matrices

$$\Gamma^M_{\alpha(\beta}\Gamma^M_{\gamma\delta)} = 0 \ . \tag{3.7}$$

(This Fierz identity is used to establish supersymmetry of $d = 10$ $\mathcal{N} = 1$ SYM.) The proof of the uniqueness of the solution $v^M \sim u^M$ for $u^M \neq 0$ can be found in appendix A.

Second consider the case when $u^M(x) = 0$ (ε is pure spinor). In the ten-dimensional space the equation

$$\varepsilon\Gamma^M\varepsilon = 0 \tag{3.8}$$

is equivalent to saying that ε is a *pure spinor* [27]. Generically, a spinor ε for $\mathrm{Spin}(\mathbb{R}^{2n})$ is called *pure* if it is annihilated by half of gamma-matrices: there exists a half-dimensional subspace $L \subset \mathbb{R}^{2n} \otimes \mathbb{C}$ such that

$$v^M\Gamma_M\varepsilon = 0 \Leftrightarrow v \in L \ . \tag{3.9}$$

A pure spinor ε defines a complex structure on the vector space $\mathbb{R}^{2n}\otimes\mathbb{C}$ by saying that L is the space of anti-holomorphic vectors $L = V^{(0,1)}$. In general, a complex structure on vector space $\mathbb{R}^{2n}$ can be defined as a $2n \times 2n$ antisymmetric matrix J such that $J^2 = -1$. Under action by J, the complexified vector space $\mathbb{R}^{2n} \otimes \mathbb{C}$ splits as $\mathbb{R}^{2n} \otimes \mathbb{C} = V^{(1,0)} + V^{(0,1)}$, where holomorphic $V^{(1,0)}$ is the $+i$-eigenspace of J and anti-holomorphic $V^{(0,1)}$ is the $-i$-eigenspace of J.

Therefore, whenever $\varepsilon\Gamma^M\varepsilon = 0$, the solutions to the local supersymmetry equation (3.5) are the *anti-holomorphic* vectors v^M with respect to the complex structure J_ε. In our case $V^{(0,1)}_\varepsilon$ is a five-dimensional complex vector space.

Now we can return back to the Wilson loop (3.2) and describe the operators invariant under a superconformal generator Q_ε. At a generic point in the space-time x where $u^M(\varepsilon(x)) \neq 0$, locally, the only supersymmetric Wilson loop is

$$\mathrm{Pexp}\int_\gamma (A_\mu u^\mu + \Phi_A u^A)\frac{ds}{(u^\mu u_\mu)^{1/2}} \ . \tag{3.10}$$

The tangent to the contour γ, specified by $x^\mu(s)$, must be aligned with $u^\mu(x)$. In order for the contour γ to be in $\mathbb{R}^4_{\mathrm{spt}}$ the vector u^μ must be projectively real, i.e. there is $\lambda \in \mathbb{C}^*$ such that $\frac{dx^\mu}{ds} = \lambda u^\mu$ is real.

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The vector field $u^\mu(\varepsilon(x))$ has simple geometrical interpretation. It is the vector field of the infinitesimal conformal transformation generated by $Q^2_{(\varepsilon_s,\varepsilon_c)}$. One can check (see e.g. [19]) that the action of Q^2_ε on any field ϕ of the theory is represented as

$$Q^2_\varepsilon \phi(x) = (-L_u - G_{u^M A_M} - R - \Omega)\phi(x) \tag{3.11}$$

where L_u is the Lie derivative in the direction of u, the symbol $G_{u^M A_M}$ denotes gauge transformation, the symbol R is the R-symmetry transformation and the symbol Ω is a local scale transformations acting on fields according to their conformal dimensions.

In points x where $u^\mu(x) = 0$ but $u^M \neq 0$ the supersymmetric Wilson loop reduces to a local operator

$$W_R(x^\mu, u) = \mathrm{Tr}_R \exp(\lambda u^A \Phi_A(x^\mu)), \quad \lambda \in \mathbb{C}^*. \tag{3.12}$$

The most interesting case is when u^M vanishes on some subspace $\Sigma_\varepsilon \subset \mathbb{R}^4_{\mathrm{spt}}$

$$\Sigma_\varepsilon = \{x \in \mathbb{R}^4_{\mathrm{spt}} | u^M(\varepsilon(x)) = 0\} \ . \tag{3.13}$$

The spinor $\varepsilon(x)$ is pure everywhere on Σ_ε. Locally at a given point x, the tangent v^μ and the scalar couplings v^A of supersymmetric Wilson loop must be components of an anti-holomorphic vector $v^M \in V^{(0,1)}_{\varepsilon(x)}$.

To find all Wilson loop operators in this class, for each Q_ε we find pure-spinor-surface Σ_ε and the bundle of anti-holomorphic vectors $V^{(0,1)}_\varepsilon \to \Sigma_\varepsilon$. For each contour γ such that the tangent vector v^μ is a projection to T_{Σ_ε} of some section v of $V^{(0,1)}$ we can associate supersymmetric Wilson loop.

We remark that we do not require any integrability condition for the almost complex structure along Σ as it was not needed to establish supersymmetry of the Wilson loop operators. Unless explicitly stated otherwise, by complex structure we always mean an almost complex structure.

3.1 Pure spinors in $AdS_5 \times S^5$

The conformal Killing spinor (2.6) can be extended from the boundary $\mathbb{R}^4_{\mathrm{spt}}$ of AdS_5 into the bulk of $AdS_5 \times S^5$

$$\varepsilon_{\mathrm{AdS}}(x^M) = \frac{1}{\sqrt{z}}\left(\varepsilon_s + x^M \Gamma_M \varepsilon_c\right), \tag{3.14}$$

where it becomes the supersymmetry transformation parameter for the theory in the bulk, see e.g. [12, 34]. The explicit formula (3.14) is presented in the vielbein for the spin bundle over $AdS_5 \times S^5$ associated canonically to the coordinates (x^μ, y^A) on $AdS_5 \times S^5$ in which metric has the form

$$ds^2 = y^2 dx^\mu dx_\mu + \frac{dy^A dy_A}{y^2}. \tag{3.15}$$

The coordinates y^A are related to coordinates z^A as $y^A = z^A/z^2$ where in coordinates $x^M = (x^\mu, z^A)$ the same metric (3.15) is

$$ds^2 = G_{MN} dx^M dx^N = \frac{dx^\mu dx_\mu + dz^A dz_A}{z^2}. \tag{3.16}$$

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The subspace $\Sigma \subset \mathbb{R}^4_{\text{spt}}$ where $\varepsilon(x)$ is pure can also be extended to $\Sigma_{\mathbb{C}} \subset AdS_5 \times S^5$. Then ε_{AdS} defines an almost complex structure J on $\Sigma_{\mathbb{C}}$, more precisely J is a section of $\text{End}(T_{\Sigma_{\mathbb{C}}} + N_{\Sigma_{\mathbb{C}}})$ such that it is compatible with the metric and that $J^2 = -1$. We conjecture that the classical stringy world-sheet dual to the supersymmetric Wilson loop operator with contour living on Σ will be given by a pseudo-holomorphic surface in $\Sigma_{\mathbb{C}}$. In support of this idea we show that such a solution would satisfy the κ-symmetry condition in the bulk i.e. will be supersymmetric.

We choose the coordinates on the stringy world-sheet such that the induced metric is flat $g_{\alpha\beta} = \delta_{\alpha\beta}$. In this notations the pseudo-anti-holomorphic surface is given by

$$V_\alpha^M = \partial_\alpha X^M - \epsilon_{\alpha\beta} J_N^M \partial_\beta X^N = 0\ . \tag{3.17}$$

This condition guarantees that the corresponding profile is supersymmetric i.e. it satisfies the κ-symmetry condition

$$(\epsilon_{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \Gamma_{MN} - i\delta_{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN})\varepsilon_{AdS_5} = 0\ . \tag{3.18}$$

Following [12] we prove (3.18) by showing that

$$\partial_\alpha X^M (\delta_M^N + iJ_M^N)\Gamma_N \varepsilon_{\text{AdS}} = 0\,, \tag{3.19}$$

is satisfied (the κ-symmetry condition can by obtained from (3.19) by multiplying it by $\partial_\alpha X^M \Gamma_M$). The latter is obvious because the vector $\partial_\alpha X^M(\delta_M^N + iJ_M^N)$ is anti-holomorphic i.e. it is an $-i$-eigenvalue of the pseudo-complex structure J. Therefore it annihilates the spinor ε_{AdS_5} according to the *definition* of J.

We remark that this result for the specific cases of the strings dual to Zarembo's loops [16] and DGRT's loops [12] was obtained in [12, 18]. However, there the pseudo-holomorphic structure J appeared as an extra input, not directly related to ε, and (3.19) was established with help of the explicit form of J and ε_{AdS}. We cosntruct J canonically starting from an arbitrary superconformal symmetry parameter ε at points where ε is pure.

In addition, one can easily see that the supersymmetry implies that the world-sheet is psedo-holomorphic provided that it lies in $\Sigma_{\mathbb{C}}$. To show that one can multiply (3.18) by ε_{AdS_5} from the right and use that $(V_\alpha^N)^2 = 0$ implies $V_\alpha^N = 0$. We do not have a general argument why the world-sheet dual to Wilson loop in Σ must sit inside $\Sigma_{\mathbb{C}}$, but that seems to be a reasonable conjecture.

The pseudo-holomorphic surface is always calibrated by some calibration form $P[J]$ as follows from the following inequality

$$\int d^2\sigma G_{MN} V_\alpha^M V_\alpha^N \geq 0, \tag{3.20}$$

and hence

$$S_{\text{string}} \geq \int P[J], \quad J_{[MN]} = G_{ML} J_N^L. \tag{3.21}$$

In general the calibration form J is not closed, therefore we cannot immediately compute the classical action as a functional of the boundary conditions.

4 Pure-spinor surfaces Σ

In this section we will find explicitly all superconformal generators Q_ε that admit a non-trivial pure-spinor-surface Σ_ε. We call Σ_ε non-trivial if it has at least one component of positive dimension.

We pick any connected component of positive dimension of Σ_ε and call it Σ_ε in what follows.[1]

We choose any point in Σ_ε to be an origin of the coordinate system in $\mathbb{R}^4_{\rm spt}$. In this coordinate system the conformal Killing spinor ε has the form

$$\varepsilon(x) = \varepsilon_s + x^\mu \Gamma_\mu \varepsilon_c \,, \tag{4.1}$$

where $\varepsilon_s = \varepsilon|_{x=0}$ is pure. Our goal is to find for which ε_c there is a nontrivial pure spinor surface Σ_ε (3.13) and what shape Σ_ε has. From the definition of Σ_ε and (3.8) it follows that Σ is an intersection of 10 quadric hypersurfaces in $\mathbb{R}^4_{\rm spt}$. Potentially Σ can have a complicated shape. It turns out that it is easier first to solve a more generic problem in ten dimensions. For that reason we formally continue the conformal Killing spinor (4.1) from $\mathbb{R}^4_{\rm spt}$ to $\mathbb{R}^{10}$ by replacing $x^\mu\Gamma_\mu$ by $x^M\Gamma_M$. We have seen in the previous section that the extended spinor $\varepsilon(x)$ in ten dimensions (3.14) plays the role of the supersymmetry parameter of string theory in $AdS_5 \times S^5$.

4.1 Form notations and pure spinor constraints

We start by introducing the subsurface $\Sigma_{\mathbb{C}} \subset \mathbb{R}^{10}$ where the spinor is pure

$$\Sigma_{\mathbb{C}} = \{x \in \mathbb{R}^{10} | u^M(x) = 0\}. \tag{4.2}$$

If we find $\Sigma_{\mathbb{C}} \in \mathbb{R}^{10}$ then we get Σ simply by intersecting $\Sigma_{\mathbb{C}}$ with the space-time $\mathbb{R}^4_{\rm spt} \subset \mathbb{R}^{10}$.

To solve the pure spinor equations (3.8) it is convenient to identify the Spin(10) spinor representation $S \simeq \mathbb{C}^{32}$ with the space of anti-holomorphic $(0,p)$ forms, $p = 0,\ldots,5$, on the vector space $\mathbb{C}^5 \simeq \mathbb{R}^{10}$.

The spinor $\varepsilon(x)$ is pure at the origin. We use it to define a complex structure on the vector space $\mathbb{R}^{10}$, so in the following we assume $\mathbb{R}^{10} \simeq \mathbb{C}^5$ where the isomorphism is defined by the pure spinor ε_s.

Given a pure spinor ε_s, the spinor representation S of Spin(10) can be constructed as a Fock space using action of the gamma-matrices. As was explained around formula (3.9) we use the conventions such that the spinor ε_s is annihilated by the anti-holomorphic vectors $v^{\bar I}$. In the following, we use the indices $I, \bar I = 1\ldots 5$ to denote the holomorphic and anti-holomorphic coordinates $x^I, x^{\bar I}$ on $\mathbb{C}^5 \simeq \mathbb{R}^{10}$. (Note that if $x^I, x^{\bar I}$ are coordinates of a point in the original real space $\mathbb{R}^{10} \simeq \mathbb{C}^5$ then $x^{\bar I}$ is a complex conjugate of x^I. However, on the complexified space $\mathbb{R}^{10} \otimes \mathbb{C} = \mathbb{C}^{10}$ we use coordinates $x^I, x^{\bar I}$ as indendent.) From our definition of the complex structure

$$v^{\bar I}\gamma_{\bar I}\, \varepsilon_s = 0 \quad \text{for any} \quad v \in V^{(0,1)} \tag{4.3}$$

[1] Actually we will see later that Σ_ε is always connected.

we get that ε_s is annihilated by matrices $\gamma_{\bar{I}}, \bar{I} = 1 \ldots 5$.

Let us fix our notations more precisely. The 32×32 Dirac gamma-matrices representing the Clifford algebra on the space $\mathbb{R}^{10}$ satisfy the canonical anticommutation relations

$$\{\gamma_M, \gamma_N\} = 2g_{MN}\,, \tag{4.4}$$

where $g_{MN} = \delta_{MN}$ is the standard unit metric on $\mathbb{R}^{10}$.

Given the complex structure J on $\mathbb{R}^{10}$ compatible with the metric g_{MN}, we get a Hermitian metric $g_{I\bar{J}}$ on the complexified space $\mathbb{C}^{10} = \mathbb{R}^{10} \otimes \mathbb{C}$ and then a structure of the Clifford algebra on $\mathbb{C}^{10}$. If $(x^I, x^{\bar{I}})$ are the coordinates on $\mathbb{C}^{10}$, the corresponding basis elements of Clifford algebra are represented by the matrices $\gamma_I, \gamma_{\bar{I}}$. Moreover, since $g_{IJ} = g_{\bar{I}\bar{J}} = 0$ we have

$$\{\gamma_I, \gamma_{\bar{J}}\} = 2g_{I\bar{J}}\,, \quad \{\gamma_I, \gamma_J\} = 0\,, \quad \{\gamma_{\bar{I}}, \gamma_{\bar{J}}\} = 0\,. \tag{4.5}$$

We can use the inverse metric to raise indexes and then define gamma-matrices with the upper index

$$\gamma^I = g^{I\bar{J}}\gamma_{\bar{J}}\,, \quad \gamma^{\bar{I}} = g^{\bar{I}J}\gamma_J\,, \tag{4.6}$$

where $g^{I\bar{K}}g_{\bar{K}J} = \delta^I_J$. Then

$$\{\gamma^I, \gamma_J\} = 2\delta^I_J\,, \quad \{\gamma^{\bar{I}}, \gamma_{\bar{J}}\} = 2\delta^{\bar{I}}_{\bar{J}}\,, \quad \{\gamma^{\bar{I}}, \gamma_J\} = \{\gamma^I, \gamma_{\bar{J}}\} = 0. \tag{4.7}$$

The construction of the spin representation S as a Fock space is straightforward. We define the *vacuum state* $|\varepsilon_s\rangle$ as a state annihilated by all anti-holomorphic vectors in $V^{(0,1)} \subset \mathbb{C}^{10}$ under the Clifford action (compare with (4.3))

$$v^{\bar{I}}\gamma_{\bar{I}}|\varepsilon_s\rangle = 0 \quad \text{for all } (0,1) \text{ vectors } v. \tag{4.8}$$

It will be more convenient to use the p-forms instead of p-vectors in what follows and we use the Hermitian metric $g_{I\bar{J}}$ to identify $V^{(0,1)}$ with the space of holomorphic one-forms $V^*_{(1,0)}$. Then

$$v_I\gamma^I|\varepsilon_s\rangle = 0 \quad \text{for all } (1,0) \text{ forms } v. \tag{4.9}$$

We call γ^I the lowering operators and $\gamma^{\bar{I}}$ the raising operators. The Fock space as a vector space is spanned on the states (with $n = 5$ in our case)

$$\gamma^{\bar{I}_1\cdots\bar{I}_k}|\varepsilon_s\rangle\,, \quad I_1 < I_2 < \cdots < I_k\,, \quad k \le n. \tag{4.10}$$

Let ρ_p denote an antisymmetric $(0,p)$-form

$$\rho_p = \sum_{\bar{I}_1<\bar{I}_2<\cdots<\bar{I}_p} \rho_{\bar{I}_1\ldots\bar{I}_p}\gamma^{\bar{I}_1\ldots\bar{I}_p}. \tag{4.11}$$

Then an arbitrary spinor ε as a state in Fock space can be written as

$$\varepsilon = \sum_{p=0}^{n} \rho_p|\varepsilon_s\rangle. \tag{4.12}$$

The space of anti-holomorphic forms $\oplus_p V^*_{(0,p)}$ is isomorphic to the spin representation space S. There is a natural $\mathbb{Z}_2$ grading on S that is compatible with the action of the generators γ_{MN} of Spin$(2n)$. This $\mathbb{Z}_2$ grading defines the chiral decomposition $S = S^+ \oplus S^-$. The space S^+ of spinors of positive chirality is the space of forms of even degree p and the space S^- of spinors of negative chirality is the space of forms of odd degree p.

If n is odd, then the representation S^+ and S^- are dual to each other, which means that there is a natural Spin$(2n)$-invariant pairing between S^+ and S^-. If $\rho \in S^+$ and $\sigma \in S^-$, in the conventional spin index notations the pairing is simply $\rho^\alpha \sigma_\alpha$. The same contraction in Fock space representation (4.12) is

$$(\rho, \sigma) := (R[\rho] \wedge \sigma)_{\text{top}}. \tag{4.13}$$

Here $|_{\text{top}}$ stands for picking up the coefficient of the top degree form normalized by some fixed element in $V^*_{(0,n)}$, and $R[\rho]$ denotes the *reverse order operation* on S^+, see e.g. [35, 36]

$$\begin{aligned} R[\rho_p] &= \rho_p \qquad &&\text{for} \quad p = 4k, 4k+1, \\ R[\rho_p] &= -\rho_p \qquad &&\text{for} \quad p = 4k+2, 4k+3. \end{aligned} \tag{4.14}$$

For $n = 5$ the pairing between spinor $\rho = \rho_0 + \rho_2 + \rho_4 \in S^+$ and spinor $\sigma = \sigma_1 + \sigma_3 + \sigma_5 \in S^-$ is

$$(\rho, \sigma) = (\rho_0 \wedge \sigma_5 - \rho_2 \wedge \sigma_3 + \rho_4 \wedge \sigma_1). \tag{4.15}$$

At the next step we rewrite the pure spinor condition for a spinor $\varepsilon \in S^+$

$$\varepsilon = (\rho_0 + \rho_2 + \rho_4)|\varepsilon_s\rangle\,, \tag{4.16}$$

in terms of the constraints on the forms ρ_0, ρ_2, ρ_4. In general, given a vector space $V = \mathbb{R}^{2n}$, and a complex structure on V, a pure spinor is a vacuum state in the spin representation constructed as a Fock space. In other words, $\varepsilon \in S$ is a pure spinor if it is annihilated by a half-dimensional isotropic subspace $L \subset V_{\mathbb{C}}$ with $L \cap \bar{L} = 0$.[2] A choice of $L \subset V_{\mathbb{C}}$ defines a complex structure on V by declaring L to be the space of anti-holomorphic vectors $L = V^{(0,1)}$. To summarize, the space of complex structures on V is isomorphic to the space of equivalence classes of pure spinors ε modulo rescaling $\varepsilon \sim \lambda\varepsilon, \varepsilon \in \mathbb{C}^*$.

As we already mentioned above, if $n = 5$ a spinor ε is a pure if and only if

$$\varepsilon \Gamma_M \varepsilon = 0, \quad M = 1, \dots, 10. \tag{4.17}$$

Now we rewrite (4.17) using the form notation (4.16) and (4.13)

$$\begin{aligned} \varepsilon v_{\bar{I}} \gamma^{\bar{I}} \varepsilon &= 0, \quad v \in V^*_{0,1} \\ \varepsilon v^{\bar{I}} \gamma_{\bar{I}} \varepsilon &= 0, \quad v \in V^{0,1}. \end{aligned} \tag{4.18}$$

To simplify notations in the calculation we notice that for any spinor $\varepsilon = \rho|\varepsilon_s\rangle$, where ρ is a polyform, we have

$$v_{\bar{I}} \gamma^{\bar{I}} \varepsilon = (v \wedge \rho)|\varepsilon_s\rangle\,, \tag{4.19}$$

[2] Isotropic means that $g(L, L) = 0$ i.e. $g_{IJ} = g_{\bar{I}\bar{J}} = 0$.

where $v \wedge \rho$ denotes the usual external product of the antisymmetric forms v and ρ. Similarly, using (4.6) we also have

$$v^{\bar{I}}\gamma_{\bar{I}}\varepsilon = (2i_v\rho)|\varepsilon_s\rangle\,, \tag{4.20}$$

where $i_v\rho$ denotes a contraction of the vector v and a polyform ρ.

We want to express the condition that a spinor is pure spinor as a constraint on ρ. After contracting (4.19) with $\langle\varepsilon|$ we get (first equation of (4.18))

$$\rho_0 \wedge v \wedge \rho_4 - \rho_2 \wedge v \wedge \rho_2 + \rho_4 \wedge v \wedge \rho_0 = 0\,, \tag{4.21}$$

for any anti-holomorphic one-form v, which means that if ρ is pure then

$$\rho_0\rho_4 = \frac{1}{2}\rho_2 \wedge \rho_2. \tag{4.22}$$

Similarly, the second equation of (4.18) implies that if ρ is pure then

$$\rho_2 \wedge i_v\rho_4 = \rho_4 \wedge i_v\rho_2 \quad \text{for any vector } v. \tag{4.23}$$

Since $0 = i_v(\rho_2 \wedge \rho_4) = i_v\rho_2 \wedge \rho_4 + \rho_4 \wedge i_v\rho_2$, we get that (4.23) is equivalent to

$$i_v\rho_2 \wedge \rho_4 = 0 \quad \text{for any vector } v. \tag{4.24}$$

Notice that if $\rho_0 \neq 0$, the condition (4.22) implies (4.23). Indeed, it is easy to check that $\rho_2 \wedge \rho_2 \wedge i_v\rho_2$ vanishes identically in five dimensions for any two-form ρ_2 and vector v.

Another way to derive the pure spinor constraint (4.22) is to notice that all pure spinors ρ with $\rho_0 \neq 0$, modulo rescalings $\rho \to \lambda\rho, \lambda \in \mathbb{C}^*$ are in the Spin(10) orbit of the vacuum spinor $|\varepsilon_s\rangle$. The Spin(10) acts on S^+ as

$$|\varepsilon_s\rangle \mapsto \exp(\omega_{\bar{I}\bar{J}}\gamma^{\bar{I}\bar{J}})|\varepsilon_s\rangle\ . \tag{4.25}$$

(We write only $(0,2)$ components $\omega_{\bar{I}\bar{J}}$ of all Spin(10) generators, because $|\varepsilon_s\rangle$ is annihilated by holomorphic generators γ^I). Then

$$|\varepsilon_s\rangle \mapsto (1 + \omega_{\bar{I}\bar{J}}\gamma^{\bar{I}\bar{J}} + \frac{1}{2}\omega_{\bar{I}\bar{J}}\omega_{\bar{K}\bar{L}}\gamma_{\bar{I}\bar{J}}\gamma_{\bar{K}\bar{L}})|\varepsilon_s\rangle, \tag{4.26}$$

which can be rewritten as

$$|\varepsilon_s\rangle \mapsto (1 + \omega_2 + \frac{1}{2}\omega_2 \wedge \omega_2)|\varepsilon_s\rangle. \tag{4.27}$$

Here ω_2 is a two-form $\omega_2 = \omega_{\bar{I}\bar{J}}\gamma^{\bar{I}\bar{J}}$. Hence, all pure spinors with $\rho_0 \neq 0$ can be parametrized by a scale factor $\tilde{\rho}_0 \in \mathbb{C}$ and a two-form $\omega_2 \in \Lambda^2(\mathbb{C}^5)$. (This is a well-known local parametrization of pure spinors in ten dimensions used in [37, 38]). In the $\rho = \rho_0 + \rho_2 + \rho_4$ is expressed in terms of $\tilde{\rho}_0$ and ω as

$$\rho_0 = \tilde{\rho}_0\,, \quad \rho_2 = \tilde{\rho}_0\omega_2\,, \quad \rho_4 = \frac{1}{2}\tilde{\rho}_0\omega_2 \wedge \omega_2. \tag{4.28}$$

The quadratic constraints (4.22) are satisfied.

4.2 Pure spinor surface in $\mathbb{R}^{10}$

Now we are ready to rewrite the conformal Killing spinor (4.1) in the form notations on $\mathbb{C}^5$ and solve the pure spinor constraint (4.22) and (4.24).

We use the Fock space representation of S^- to identify the superconformal generator ε_c with three anti-holomorphic forms $\mathtt{v}, \mathtt{m}, \mathtt{w}$, where $\mathtt{v}$ is a $(0,1)$-form, $\mathtt{m}$ is a $(0,3)$ and $\mathtt{w}$ is a $(0,5)$-form on $\mathbb{C}^5$ (clearly, the total number of components matches as $5+10+1=16$). More explicitly

$$\varepsilon_c = \left(\mathtt{v}_{\bar{I}}\gamma^{\bar{I}} + \frac{1}{3!}\mathtt{m}_{\bar{I}_1\bar{I}_2\bar{I}_3}\gamma^{\bar{I}_1\bar{I}_2\bar{I}_3} + \frac{1}{5!}\mathtt{w}_{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4\bar{I}_5}\gamma^{\bar{I}_1\bar{I}_2\bar{I}_3\bar{I}_4\bar{I}_5} \right)\varepsilon_s \,. \tag{4.29}$$

A conformal Killing spinor (2.6) formally extended to $\mathbb{R}^{10} = \mathbb{C}^5$ is then

$$\begin{aligned} \varepsilon(x) &= \varepsilon_s + (\xi_{\bar{J}}\gamma^{\bar{J}} + x^{\bar{I}}\gamma_{\bar{I}})\varepsilon_c \\ &= ((1+2i_x\mathtt{v}) + (\xi\wedge\mathtt{v} + 2i_x\mathtt{m}) + (\xi\wedge\mathtt{m} + 2i_x\mathtt{w}))|\varepsilon_s\rangle, \end{aligned} \tag{4.30}$$

where we introduced the $(0,1)$ one-form $\xi_{\bar{I}} = g_{\bar{I}J}x^J$.

If $x \in \mathbb{R}^{10}$, so the coordinates x^M are real, then x^I and $x^{\bar{I}}$ are complex conjugate to each other. In this case the $(0,1)$ form $\xi_{\bar{I}}$ and the $(0,1)$ vector $x^{\bar{I}}$ are related through complex conjugation. More generally, one can treat $\xi_{\bar{I}}$ and $x^{\bar{I}}$ as independent, which corresponds to taking complex x^M.

Recall that we defined $\Sigma_{\mathbb{C}} \subset \mathbb{R}^{10}$ as a set of points where the spinor $\varepsilon(x)$ (4.30) is pure. Clearly, the point $x^M = 0$ is always in $\Sigma_{\mathbb{C}}$. We say that $\Sigma_{\mathbb{C}}$ is non-trivial if $x^M = 0$ belongs to a component of positive dimension.

We call a $(0,3)$ form $\mathtt{m}$ totally decomposable if there exist three $(0,1)$-forms μ_1, μ_2, μ_3 such that $\mathtt{m} = \mu_1\wedge\mu_2\wedge\mu_3$.

Now we formulate the key result of this section.

Proposition. Given a pure spinor ε_s, a pure spinor hypersurface $\Sigma_{\mathbb{C}} \subset \mathbb{R}^{10}$ is non-trivial if and only if ε_c in parametrization of (4.29) satisfies $\mathtt{w} = 0$ and $\mathtt{m}$ is totally decomposable. In this case the hypersurface $\Sigma_{\mathbb{C}}$ is described by the equation

$$(\xi + 2(\xi, x)\mathtt{v})\wedge\mathtt{m} = 0\,, \tag{4.31}$$

where the complex coordinates $(x^{\bar{I}}, \xi_{\bar{I}} = g_{\bar{I}J}x^J)$ are defined by the complex structure on $\mathbb{R}^{10}$ associated to the pure spinor ε_s.

We delegate the proof that the non-trivial $\Sigma_{\mathbb{C}}$ requires $\mathtt{w} = 0$ and $\mathtt{m}$ to be decomposable to the appendix B. Here we just show that if both conditions are satisfied $\Sigma_{\mathbb{C}}$ is given by (4.31).

For the spinor (4.29) the quadratic pure spinor constraint (4.24) with $v = x$ takes the form

$$0 = i_x\rho_2\wedge\rho_4 = i_x(\xi\wedge\mathtt{v})\wedge(\xi\wedge\mathtt{m}) = (x,\xi)\mathtt{v}\wedge\xi\wedge\mathtt{m}\,. \tag{4.32}$$

For a real non-zero x the pairing $(x,\xi) = \frac{1}{2}|x|^2$ is also non-zero. Therefore $\mathtt{v}\wedge\xi\wedge\mathtt{m}$ must vanish and consequently

$$0 = i_x(\xi\wedge\mathtt{v}\wedge\mathtt{m}) = i_x\xi\ \mathtt{v}\wedge\mathtt{m} - i_x\mathtt{v}\ \xi\wedge\mathtt{m} + \xi\wedge\mathtt{v}\wedge i_x\mathtt{m}\,. \tag{4.33}$$

Now we proceed with the constraint (4.22)

$$(1 + 2i_x\mathtt{v}) \wedge (\xi \wedge \mathtt{m}) = \frac{1}{2}(\xi \wedge \mathtt{v} + 2i_x\mathtt{m})^2 \ . \tag{4.34}$$

First we expand both sides

$$\xi \wedge \mathtt{m} + 2i_x\mathtt{v} \wedge \xi \wedge \mathtt{m} = \frac{1}{2}(\xi \wedge \mathtt{v})^2 + 2\xi \wedge \mathtt{v} \wedge i_x\mathtt{m} + 2(i_x\mathtt{m})^2 \,, \tag{4.35}$$

and notice that $(\xi \wedge \mathtt{v})^2 = 0$ and also $(i_x\mathtt{m})^2 = 0$ because we assume that $\mathtt{m}$ is totally decomposable.

Together with (4.33) the equation (4.35) reduces to

$$(\xi + 2(x,\xi)\mathtt{v}) \wedge \mathtt{m} = 0 \ . \tag{4.36}$$

Since (4.36) imply $\xi \wedge \mathtt{v} \wedge \mathtt{m} = 0$ we conclude that if $\mathtt{w} = 0$ and $\mathtt{m}$ is totally decomposable the pure spinor constraints (4.22), (4.24) are equivalent to (4.36).

Now let us solve the equation (4.36) for $\Sigma_{\mathbb{C}}$. There are only two topologically distinct cases: $\mathtt{m} = 0$ and $\mathtt{m} \neq 0$. If $\mathtt{m} = 0$ the equation (4.31) for $\Sigma_{\mathbb{C}}$ is trivial and $\Sigma_{\mathbb{C}} = \mathbb{R}^{10}$, $\Sigma = \mathbb{R}^4\_\mathrm{spt}$. If $\mathtt{m} = \mu_1 \wedge \mu_2 \wedge \mu_3 \neq 0$ then it is convenient to choose an orthonormal coordinate system $z_1, \ldots, z_5$ in $\mathbb{C}^5 \cong \mathbb{R}^{10}$ such that

$$\mathtt{m} = \mu \ \overline{dz_1} \wedge \overline{dz_2} \wedge \overline{dz_3} \tag{4.37}$$

with $\mu \in \mathbb{C}^*$.

Orthonormality of the chosen coordinate system implies that $g_{I\bar{J}} = g_{\bar{I}J} = \frac{1}{2}$. In this coordinates the equation for $\xi_{\bar{I}}$ is

$$\begin{aligned} \xi_{\bar{4}} &= -|\xi|^2 \mathtt{v}_{\bar{4}} \,, \\ \xi_{\bar{5}} &= -|\xi|^2 \mathtt{v}_{\bar{5}} \,, \end{aligned} \tag{4.38}$$

where $|\xi|^2 = 2g^{I\bar{J}}\overline{\xi_{\bar{I}}}\xi_{\bar{J}} = x^2$.

If $\mathtt{v}_{\bar{4}} = \mathtt{v}_{\bar{5}} = 0$ then $\Sigma_{\mathbb{C}}$ is a complex three-plane $\Sigma_{\mathbb{C}} = \mathbb{C}^3$ defined by $z^4 = z^5 = 0$. Otherwise, $\Sigma_{\mathbb{C}}$ is a real six-dimensional sphere $\Sigma_{\mathbb{C}} = S^6$ defined by the equations (4.38). For illustration, consider an example when $z_1, \ldots, z_5$ are related to the original coordinates $x_1, \ldots, x_{10}$ on $\mathbb{R}^{10}$ in the simplest way[3]

$$z^I = x^{2I-1} - ix^I. \tag{4.39}$$

Then the equations (4.38) can be written in real notations as follows

$$x^a + x^2\mathtt{v}^a = 0 \,, \quad a = 7 \ldots 10, \tag{4.40}$$

and $x^2 = x^M x_M$. The sphere S^6 is located inside the $\mathbb{R}^7$ spanned by first six directions in $\mathbb{R}^{10}$ and the vector $\mathtt{v}^a$.

[3]Notice that we have chosen the simplest relation just to illustrate the idea. In general the relation between the original basis x^M and the complex basis z^I that diagonalize $\mathtt{m}$ could be different.

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Now, depending on the relative orientation of the space-time $\mathbb{R}^4_{\text{spt}} \subset \mathbb{R}^{10}$ and $\Sigma_{\mathbb{C}} \subset \mathbb{R}^{10}$, we obtain various pure spinor hypersurfaces $\Sigma = \Sigma_{\mathbb{C}} \bigcap \mathbb{R}^4_{\text{spt}}$. In the example above Σ is just a point $x^M = 0$ i.e. it is trivial, but in general $\Sigma = S^n$ with $n = 1, \dots, 3$ or $\Sigma = \mathbb{R}^n$ with $n = 1, \dots, 4$.

Let us summarize possible cases for Σ:

1. If $\mathtt{m} = 0$ then $\Sigma_{\mathbb{C}} = \mathbb{R}^{10}$. Then automatically $\Sigma = \mathbb{R}^4_{\text{spt}}$.

2. If $\mathtt{m} \neq 0$ but $\mathtt{v} \wedge \mathtt{m} = 0$ then $\Sigma_{\mathbb{C}} = \mathbb{R}^6$. Then Σ is $\mathbb{R}^n$, where $n = 1, \dots, 4$ depending on the relative orientation of $\mathbb{R}^4_{\text{spt}}$ and $\Sigma_{\mathbb{C}}$.

3. If $\mathtt{m} \neq 0$ and $\mathtt{v} \wedge \mathtt{m} \neq 0$ then $\Sigma_{\mathbb{C}} = S^6$. Then Σ is S^n, where $n = 1, \dots, 3$ depending on the relative orientation of $\mathbb{R}^4_{\text{spt}}$ and $\Sigma_{\mathbb{C}}$.

The third case could be related to the second one by a suitable conformal transformation as explained in section 5.2.

4.3 Complex structure on the pure spinor hypersurface

We have just shown that for a suitable choice of spinors $(\varepsilon_s, \varepsilon_c)$ the supersymmetry spinor $\varepsilon(x)$ is pure on a hypersurface $\Sigma_{\mathbb{C}} \subset \mathbb{R}^{10}$. If $\Sigma_{\mathbb{C}}$ is non-trivial then $\Sigma_{\mathbb{C}}$ is either $\mathbb{R}^{10}$, $\mathbb{R}^6$ or S^6.

In the previous section we used the pure spinor $\varepsilon_s = \varepsilon|_{x=0}$ to define complex structure on $\mathbb{R}^{10}$ as on the vector space (not as on the manifold $\mathbb{R}^{10}$). It was merely a technical trick that helped us to find $\Sigma_{\mathbb{C}}$. Now, when this is done, we will find an almost complex structure $J(x) \in \text{End}(\mathbb{R}^{10}, \mathbb{R}^{10})$ at each point x on $\Sigma_{\mathbb{C}}$ defined by $\varepsilon(x)$. The complex structure at the origin $J(x = 0)$ coincides with the base complex structure on $\mathbb{R}^{10}$ defined by ε_s and used in the previous section.

This complex structure $J(x)$ or, more precisely, the space of anti-holomorphic vectors $V_x^{(0,1)}$ at each point x defines locally the space of allowed supersymmetric combinations of the contour directions v^μ and the scalar couplings v^A of the Wilson loop (3.2).

Let $Z^M_{\bar{I}}$ where $M = 1 \dots 10$, $I, \bar{I} = 1 \dots 5$ be x-dependent 10×5 basis matrix of $V_x^{(0,1)}$. Similarly, let Z^M_I be the basis matrix of $V_x^{(1,0)}$, so

$$x^M = Z^M_I x^I + Z^M_{\bar{I}} x^{\bar{I}}. \tag{4.41}$$

The matrix $Z^M_{\bar{I}}$ defines the anti-holomorphic vector space $V_x^{(0,1)}$ associated with the pure spinor $\varepsilon(x)$ at a given point $x \in \Sigma_{\mathbb{C}}$

$$Z^M_{\bar{I}}(x) \gamma_M \varepsilon(x) = 0. \tag{4.42}$$

We can normalize Z^M_I as

$$\delta_{MN} Z^M_I Z^N_J = 0 \,, \quad \delta_{MN} Z^M_I Z^N_{\bar{J}} = g_{I\bar{J}}. \tag{4.43}$$

We assume for now that $\mathtt{v} \wedge \mathtt{m} = 0$ which means that $\Sigma_{\mathbb{C}}$ is either the total space $\mathbb{R}^{10}$ if $\mathtt{m}$ vanishes, or a six-plane $\mathbb{R}^6 \subset \mathbb{R}^{10}$ if $\mathtt{m}$ is a non-zero decomposable three-form. The supersymmetry spinor $\varepsilon(x)$ (4.30) is explicitly given by

$$\varepsilon(x) = (1 + 2x^{\bar{I}}\mathtt{v}_{\bar{I}}) \left(1 + \frac{1}{2}\alpha_{\bar{I}\bar{J}}\gamma^{\bar{I}\bar{J}}\right)\varepsilon_s \tag{4.44}$$

where $(1 + 2x^{\bar{I}}\mathtt{v}_{\bar{I}})$ is a scalar multiplier and the $(0,2)$ form $\alpha_{\bar{I}\bar{J}}$ is

$$\alpha_{\bar{I}\bar{J}}(x) = \frac{\xi_{\bar{I}}v_{\bar{J}} - \xi_{\bar{J}}v_{\bar{I}} + 2x^{\bar{K}}\mathtt{m}_{\bar{K}\bar{I}\bar{J}}}{1 + 2x^{\bar{I}}\mathtt{v}_{\bar{I}}}. \tag{4.45}$$

To find $Z^M_{\bar{I}}(x)$ we start with (4.42) at $x = 0$

$$\hat{Z}^M_{\bar{I}}\gamma_M\varepsilon_s \equiv \gamma_{\bar{I}}\varepsilon_s = 0, \tag{4.46}$$

where $\hat{Z}^M_{\bar{I}} = Z^M_{\bar{I}}(x = 0)$, and multiply it by $(1 + \frac{1}{2}\alpha_{\bar{I}\bar{J}}\gamma^{\bar{I}\bar{J}})$ from the left. Then we use the anticommutation relations to move this factor to the right and also the fact that $\alpha \wedge \alpha = 0$ to express ε_s through $\varepsilon(x)$. As a result we get

$$Z^M_{\bar{I}}(x) = \hat{Z}^M_{\bar{I}} + 2\hat{Z}^M_K g^{K\bar{J}}\alpha_{\bar{J}\bar{I}}(x). \tag{4.47}$$

5 Classification of the SO(5, 1) × SO(6) orbits in the space of superconformal charges

In section 4 we found the conditions on a pair of spinors $(\varepsilon_s, \varepsilon_c)$ such that the conformal Killing spinor $\varepsilon(x)$ (2.5) is pure on a nontrivial hypersurface $\Sigma \in \mathbb{R}^4$. The Wilson loop operator (3.2) on Σ is supersymmetric with respect to $\varepsilon(x)$ if v^M is anti-holomorphic at each point $x \in \Sigma$ with respect to the complex structure $J(x)$. That means $v^M(X) = Z^M_{\bar{I}}(x)v^{\bar{I}}$ for a suitable $v^{\bar{I}}$. To find a real contour in $\mathbb{R}^4_{\rm spt}$ one is compelled to choose $v^{\bar{I}}$ such that v^μ is real and $v^\mu = dx^\mu/ds$ for some contour $\gamma : x^\mu(s) \subset \Sigma$. Using the matrix $Z^M_{\bar{I}}$, introduced in the previous section, one can construct all possible supersymmetric Wilson loop operators on Σ.

It is clear nevertheless that this description is not unique in a sense that different operators can be related to each other by the action of the global symmetry group. For example if we start with some pair $(\varepsilon_s, \varepsilon_c)$, that leads to a nontrivial Σ, we can always move the origin of the coordinate system and obtain a new pair $(\varepsilon'_s, \varepsilon'_c)$. Therefore the same contour on Σ and hence the corresponding Wilson loop operator will be described twice, once as corresponding to $(\varepsilon_s, \varepsilon_c)$ and another time as corresponding to $(\varepsilon'_s, \varepsilon'_c)$. To avoid double-counting we should factorize the space of suppersymmetric Wilson operators by the shifts in $\mathbb{R}^4_{\rm spt}$, and in general by the total global bosonic symmetry group of the theory $\mathrm{SO}(5,1) \times \mathrm{SO}(6)$, where
$SO(5,1)$ is the conformal group of one-point compactification of $\mathbb{R}^4_{\rm spt}$ and SO(6) is the R-symmetry group.

Let us notice that partially we have already fixed the "conformal gauge" by requiring that ε_s is pure i.e. the origin of coordinate system belongs to Σ. Clearly this is not enough as other symmetries including shifts along Σ and conformal transformations still have to be gauged away. Ultimately, we would like to find the space of all equivalence classes of pairs (Q_ε, W) modulo the global symmetry.

In this paper we consider only an interesting subclass of the pairs (Q_ε, W) the pure-spinor case of this problem, i.e. the pairs when the contour of W is located on a pure-spinor surface Σ and the couplings on W are defined by anti-holomorphic vectors.

The problem of finding equivalence classes in the other, not pure-spinor case, when contour of W is just an orbit of conformal transformation generated by Q, is left for the future. As we have mentioned in the introduction, if we require that the orbits are compact, there are no other curves except simple generalization of circle known as $\frac{p}{q}$ Lissajous figure. For x^μ being coordinates on $\mathbb{R}^4$, take the orbit $x_1 + ix_2 = r_1 e^{ipt}, x_3 + ix_4 = r_2 e^{iqt}$ corresponding to generator of the SO(2)⊕SO(2) rotations of the 12-plane and the 34-plane, such that $\frac{p}{q} \in \mathbb{Q}$.

The bosonic global symmetry group of the $\mathcal{N} = 4$ SYM on $\mathbb{R}^4_{\text{spt}}$ is the product of the conformal group $\mathrm{SO}(5,1)$ of the four-dimensional Euclidean space $\mathrm{SO}(5,1)$ and the R-symmetry group SO(6).

Actually, to classify pairs (Q, W) in a meaningful way we should say more precisely that Q denotes one-dimensional fermionic subspace of the superconformal algebra. In other words, if Q_ε is a symmetry of W then so obivously is a $Q_{\lambda\varepsilon}, \lambda \in \mathbb{C}^*$. When we represent Q by a pair of spinors $(\varepsilon_s, \varepsilon_c)$ we actually consider equivalence classes under the action of $\mathrm{SO}(5,1)\times\mathrm{SO}(6)\times\mathbb{C}^*$ on this space, where $\mathbb{C}^*$ acts by a simple rescaling $(\varepsilon_s, \varepsilon_c) \to (\lambda\varepsilon_s, \lambda\varepsilon_c)$, $\lambda \in \mathbb{C}^*$.

It is convenient to represent the action of $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ on the space of pairs $(\varepsilon_s, \varepsilon_c)$ using the spinor representation of the $\mathrm{SO}(11,1)$ group acting on the 64 component spinor that is built of $(\varepsilon_s,\, \varepsilon_c)$.

Before we proceed with further details let us explain how the conformal group $\mathrm{SO}(5,1)$ acts on the $(\varepsilon_s,\, \varepsilon_c)$. First we compactify $\mathbb{R}^4_{\text{spt}}$ into S^4. The group $\mathrm{SO}(5,1)$ acts on S^4 as follows. Let $(1,2,3,4,11,12)$ be the set of indexes in the space $\mathbb{R}^{5,1}$ where acts $\mathrm{SO}(5,1)$ canonically. Let us consider the $SO(5,1)$-invariant cone

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_{11}^2 - X_{12}^2 = 0. \tag{5.1}$$

For any point on this cone and $X_{12} \neq 0$ the five dimensional vector $n_i = X_i/X_{12}$ has unit norm and therefore parametrizes unit S^4 within $\mathbb{R}^5$. The action of the conformal group $\mathrm{SO}(5,1)$ on S^4 is the action on $\vec{n}$ induced from the canonical action of $\mathrm{SO}(5,1)$ on $(X_1, \ldots, X_{11}, X_{12})$.

For example the generators K_μ of the special conformal transformation on $\mathbb{R}^4$ are related to the generators of $\mathrm{SO}(5,1)$ as follows

$$K_\mu = -R_{11,\mu} + R_{12,\mu}\ . \tag{5.2}$$

To check this we perform a special conformal transformation (5.2) parametrized by vector $b_\mu = \frac{v_\mu}{2}$. Without loss of generality we can choose v_μ to be along the direction X_1. Then the action of $\mathrm{SO}(5,1)$ is restricted on the directions $\{X_1, X_{11}, X_{12}\}$. The corresponding generator

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.3}$$

can be exponentiated as follows

$$e^{bK} = \begin{pmatrix} 1+\frac{b^2}{2} & b & -\frac{b^2}{2} \\ b_1 & 1 & -b_1 \\ \frac{b^2}{2} & b & 1-\frac{b^2}{2} \end{pmatrix} . \tag{5.4}$$

This matrix generates the transformation

$$\begin{aligned} n_1 &\to \frac{n_1 + b(1-n_5)}{(1+\frac{b^2}{2}) + bn_1 - \frac{b^2}{2}n_5} , \\ n_\mu &\to \frac{n_\mu}{(1+\frac{b^2}{2}) + bn_1 - \frac{b^2}{2}n_5} , \qquad \mu \neq 1 , \\ n_5 &\to \frac{(1-\frac{b^2}{2})n_5 + bn_1 + \frac{b^2}{2}}{(1+\frac{b^2}{2}) + bn_1 - \frac{b^2}{2}n_5} . \end{aligned} \tag{5.5}$$

Using the relation between the unit vector n_μ, n_5 on $S^4 \subset \mathbb{R}^5$ and the stereographic projective coordinates x_μ on $\mathbb{R}^4$

$$\begin{aligned} x_\mu &= n_\mu \frac{2}{1+n_5} , \\ n_\mu &= \frac{x_\mu}{1+\frac{x^2}{4}} , \quad n_5 = r\frac{1-\frac{x^2}{4}}{1+\frac{x^2}{4}} , \end{aligned} \tag{5.6}$$

we get the usual formula for the special conformal transformations

$$x_\mu \to \frac{x_\mu + v_\mu x^2}{1 + 2v_\mu x_\mu + v^2 x^2} . \tag{5.7}$$

At the next step we want to find the action of the conformal group SO(5, 1) on the conformal Killing spinor on $\mathbb{R}^4_{\text{spt}}$ (2.6). It is defined as follows. If u^μ is a vector field generating a conformal transformations, then $\varepsilon(x)$ transforms as

$$\delta\varepsilon = L_u \varepsilon - \frac{1}{2}\lambda\varepsilon \tag{5.8}$$

where $L_u\varepsilon = u^\mu \partial_\mu \varepsilon + \frac{1}{4}\partial_\mu u_\nu \Gamma^{\mu\nu}\varepsilon$ is the Lie derivative acting on ε and $\lambda = \frac{1}{4}\partial_\mu u^\mu$ is the conformal scaling factor. This formula follows from the fact, that the conformal Killing spinor ε under conformal rescaling of metric $g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}$ transforms as $\varepsilon \to e^{\Omega/2}\varepsilon$. To find the the action of the conformal group SO(5, 1) on the pair $(\varepsilon_s, \varepsilon_c)$ one may find vector field u^μ that corresponds to a generator $R_{mn} \in so(5, 1)$, and then find $(\delta\varepsilon_s, \delta\varepsilon_c)$ through (5.8).

As an example, consider the case of the special conformal transformation $-R_{\mu,11} + R_{\mu,12}$. For an infinitesimal v^μ the corresponding vector field is $u^\mu = v^\mu x^2 - 2x^\mu(xv)$ as follows from (5.7). Then (5.8) implies that $\delta\varepsilon_s = 0$ and $\delta\varepsilon_c = v^\mu \Gamma_\mu \varepsilon_s$. This infinitesimal transformation can be easily integrated for a finite v^μ

$$\begin{pmatrix} \varepsilon_s \\ \varepsilon_c \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ \Gamma_\mu v^\mu & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_s \\ \varepsilon_c \end{pmatrix}. \tag{5.9}$$

In the case of translation of x^μ by v^μ one obviously gets

$$\begin{pmatrix} \varepsilon_s \\ \varepsilon_c \end{pmatrix} \to \begin{pmatrix} 1 & \Gamma_\mu v^\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_s \\ \varepsilon_c \end{pmatrix}, \tag{5.10}$$

The dilatations by factor e^Ω are represented as

$$\begin{pmatrix} \varepsilon_s \\ \varepsilon_c \end{pmatrix} \to \begin{pmatrix} e^{\Omega/2} & 0 \\ 0 & e^{-\Omega/2} \end{pmatrix} \begin{pmatrix} \varepsilon_s \\ \varepsilon_c \end{pmatrix}, \tag{5.11}$$

and, finally, the space-time SO(4) rotations and the SO(6) R-symmetry transformations are represented as

$$\begin{pmatrix} \varepsilon_s \\ \varepsilon_c \end{pmatrix} \to \exp \begin{pmatrix} \frac{1}{4} R_{MN} \Gamma^*_{[M} \Gamma_{N]} & 0 \\ 0 & \frac{1}{4} R_{MN} \Gamma_{[M} \Gamma^*_{N]} \end{pmatrix} \begin{pmatrix} \varepsilon_s \\ \varepsilon_c \end{pmatrix}. \tag{5.12}$$

The spin representation (5.9)–(5.12) of $\mathrm{SO}(5,1) \times \mathrm{SO}(6)$ can can be embedded into the Clifford algebra of $\mathbb{R}^{11,1}$ represented by the following 64×64 gamma-matrices

$$\begin{aligned} &\hat\gamma_M = \begin{pmatrix} \gamma_M & 0 \\ 0 & -\gamma_M \end{pmatrix}, \qquad\qquad M = 1 \dots 10\,, \\ &\hat\gamma_{11} = \begin{pmatrix} 0 & 1_{32\times 32} \\ 1_{32\times 32} & 0 \end{pmatrix}, \quad \hat\gamma_{12} = \begin{pmatrix} 0 & -1_{32\times 32} \\ 1_{32\times 32} & 0 \end{pmatrix}. \end{aligned} \tag{5.13}$$

Then the $\mathrm{SO}(11,1)$ chirality operator is

$$\hat\gamma_{13} = -i\hat\gamma_1 \hat\gamma_2 \dots \hat\gamma_{12} = \begin{pmatrix} 1_{16\times 16} & 0 & 0 & 0 \\ 0 & -1_{16\times 16} & 0 & 0 \\ 0 & 0 & -1_{16\times 16} & 0 \\ 0 & 0 & 0 & 1_{16\times 16} \end{pmatrix}. \tag{5.14}$$

Therefore the spinor

$$\varepsilon = \begin{pmatrix} \varepsilon_s \\ 0 \\ 0 \\ \varepsilon_c \end{pmatrix}, \tag{5.15}$$

is a $\mathrm{SO}(11,1)$ Weyl spinor of positive chirality, while the ε_s and ε_c from (5.15) are the $\mathrm{SO}(10)$ chiral Weyl spinors of opposite chiralities. One can check that the action by the conformal $\mathrm{SO}(5,1)$ group and the $\mathrm{SO}(6)$ group on the conformal Killing spinor $\varepsilon(x) = \varepsilon_s + \gamma_\mu x^\mu \varepsilon_c$ is represented precisely in the same way as $\mathrm{SO}(5,1) \times \mathrm{SO}(6) \subset \mathrm{SO}(11,1)$ action on (5.15).

We denote positive and negative chiral representations of SO(11, 1) as $S^+_{11,1}$ and $S^-_{11,1}$. Now recall that the Weyl representations of SO(10), which we called S^+ and S^-, are related by complex conjugation. Namely, if $\varepsilon_s^*, \varepsilon_c^*$ denote complex conjugates to $\varepsilon_s, \varepsilon_c$, then ε_s^* transforms in S^- and ε_c^* transforms in S^+. Another important observation is that the pair $(\varepsilon_s^*, \varepsilon_c^*)$ transforms under $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ in the same way as the SO(11, 1) Weyl spinor of negative chirality

$$\tilde{\varepsilon} = \begin{pmatrix} 0 \\ \varepsilon_s^* \\ \varepsilon_c^* \\ 0 \end{pmatrix} . \tag{5.16}$$

To classify the supercharges modulo $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ we construct the $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ invariants in the spin space $S^+_{11,1}\oplus S^-_{11,1}$ by contracting a twelve-dimensional spinor with a Dirac conjugated one. Thus for any pair of spinors $(\varepsilon_1, \varepsilon_2)\in S^+_{11,1}\oplus S^-_{11,1}$ we can construct a bilinear

$$\rho_{i_1\dots i_p j_1\dots j_q} = \bar{\varepsilon}_1 \hat{\gamma}_{i_1\dots i_p}\hat{\gamma}_{j_1\dots j_q}\varepsilon_2\,, \quad i_1\dots i_p = 1\dots 4, 11, 12\,, \quad j_1\dots j_q = 5\dots 10, \tag{5.17}$$

which is a p-forms in $\mathbb{R}^{5,1}$ under SO(5, 1) and a q-form in $\mathbb{R}^6$ under SO(6). Here $\bar{\varepsilon}_1$ stands for the Dirac conjugated spinor

$$\bar{\varepsilon}_1 = \varepsilon_1^* \hat{\gamma}_{12}. \tag{5.18}$$

The (p,q) form in (5.17) is generically nonzero if ε_1 and ε_2 have the same chirality for odd $p+q$, and opposite chirality for even $p+q$. We use the spinors (5.15) and (5.16) to construct the non-trivial $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ (p,q)-forms either as

$$\rho_{i_1\dots i_p j_1\dots j_q} = \tilde{\varepsilon}^* \hat{\gamma}_0 \hat{\gamma}_{i_1\dots i_p}\hat{\gamma}_{j_1\dots j_q}\varepsilon \quad \text{for even } p+q\,, \tag{5.19}$$

or as

$$\rho_{i_1\dots i_p j_1\dots j_q} = \varepsilon^* \hat{\gamma}_0 \hat{\gamma}_{i_1\dots i_p}\hat{\gamma}_{j_1\dots j_q}\varepsilon \quad \text{for odd } p+q\ . \tag{5.20}$$

The forms of even degree are holomorphic in $\varepsilon_s, \varepsilon_c$, while the forms of odd degree depend on $\varepsilon_s, \varepsilon_c$ and their complex conjugates. Now one can easily construct a bilinear in $\rho_{p,q}$ invariants by contracting the i and j indexes. We use (a,b) notation to denote the standard metric pairing of the (p,q)-forms a and b as

$$(a,b) := \frac{1}{p!q!} a_{i_1\dots i_p j_1\dots j_q} b_{i'_1\dots i'_p j'_1\dots j'_q} g^{i_1 i'_1}\dots g^{i_p i'_p} g^{j_1 j'_1}\dots g^{j_q j'_q}\ . \tag{5.21}$$

Clearly, not all resulting invariants will be independent and our job will be to identify the complete set of the independent ones that parametrize the space of supercharges uniquely. It turns out that to built independent invariants it is enough to consider (5.21) with either p or q equal to zero. We introduce the following concise notations for the contraction of the $(p=n, q=0)$ form ρ with itself

$$I^n_p = (\rho,\rho)\,, \quad \tilde{I}^n_p = (\rho,\rho^*)\,, \tag{5.22}$$

and similarly $I^n_q, \tilde{I}^n_q$ for the invariants built out of the $(p=0, q=n)$ form.

In the rest of this section we proceed with a systematic consideration of all cases when Σ is non-trivial, namely $\mathtt{m}=0$ (when $\Sigma_{\mathbb{C}}=\mathbb{R}^{10}$) and when $\mathtt{m}\neq 0$ (when $\Sigma_{\mathbb{C}}=S^6$ or $\Sigma_{\mathbb{C}}=\mathbb{R}^6$).

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5.1 The case $\mathtt{m} = 0$, $\Sigma_{\mathbb{C}} = \mathbb{R}^{10}$, $\Sigma = \mathbb{R}^4_{\text{spt}}$

We start with the case when the 3-form $\mathtt{m}$ (4.29) vanishes and the supersymmetry spinor $\varepsilon(x)$ is pure everywhere in the space-time $\Sigma = \mathbb{R}^4_{\text{spt}}$. In this case the pair $(\varepsilon_s, \varepsilon_c)$ is parametrized by 30 real parameters where 20 parameters parametrize a pure spinor ε_s modulo C^* action, and 10 parameters $\mathtt{v}^M$ define ε_c via (4.29). Out of 30 parameters only 2 combinations are invariant under the transformation of the global symmetry group. In principle we can write down a general 30-parameter dependent spinor ε and calculate the invariants using (5.21). But this strategy is not very practical because in order to write the unique ε explicitly we would have to express 30 parameters through just two.

It is much easier to use geometrical intuition to cast the pair $(\varepsilon_s, \varepsilon_c)$ to the simplest possible form in the first place. Let us start by choosing the simplest possible form for a *generic* pure spinor ε_s. As was discussed in section (3), a pure spinor can be characterized by the complex structure J^M_N, or, after lowering one index, by 10×10 antisymmetric matrix J_{MN}. Its 4×4 space-time block $J_{\mu\nu}$ can be thought of as an element in the $so(4)$ algebra. After applying an appropriate rotation of $\mathbb{R}^4_{\text{spt}}$, this 4×4 block can be transformed to a canonical form parametrized by two numbers α, β (the non-zero components can not be larger than 1 to ensure $J^2 = -1_{10\times 10}$)

$$J_{\mu\nu} = \begin{pmatrix} 0 & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & 0 & 0 & \\ 0 & 0 & 0 & -\sin(\beta) \\ 0 & 0 & \sin(\beta) & 0 \end{pmatrix} . \tag{5.23}$$

The rest of J^M_N can be transformed to the canonical form below by an appropriate SO(6) transformation

$$J = \begin{pmatrix} 0 & -\sin(\alpha) & 0 & 0 & \cos(\alpha) & 0 & 0 & 0 & 0 & 0 \\ \sin(\alpha) & 0 & 0 & 0 & 0 & \cos(\alpha) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin(\beta) & 0 & 0 & \cos(\beta) & 0 & 0 & 0 \\ 0 & 0 & \sin(\beta) & 0 & 0 & 0 & 0 & \cos(\beta) & 0 & 0 \\ -\cos(\alpha) & 0 & 0 & 0 & 0 & \sin(\alpha) & 0 & 0 & 0 & 0 \\ 0 & -\cos(\alpha) & 0 & 0 & -\sin(\alpha) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos(\beta) & 0 & 0 & 0 & 0 & \sin(\beta) & 0 & 0 \\ 0 & 0 & 0 & -\cos(\beta) & 0 & 0 & -\sin(\beta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \tag{5.24}$$

To understand why this is always possible, let's take J^M_N with $J_{\mu\nu}$ given by (5.23) and act by it on the unit vector in the direction 1, and then choose the projection of the resulting vector on the orthogonal compliment to $\mathbb{R}^4_{\text{spt}}$ to be the direction 5. Then we do the same with the direction 2 and call the resulting direction 6. Notice that the directions 5 and 6 are orthogonal to each other because of $J^2 = -1_{10\times 10}$. Similarly acting by J on 3 and 4 gives 7 and 8 respectively. Eventually the remaining directions $9, 10$ must transform into each other.

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What we achieve at this point, using $\mathrm{SO}(4)\times\mathrm{SO}(6)$ symmetry, is the parametrization of the projective pure spinor ε_s by only two parameters instead of 20

$$\varepsilon_s = \left(\cos\frac{\alpha+\beta}{2}, -i\sin\frac{\alpha-\beta}{2}, 0, 0, i\cos\frac{\alpha-\beta}{2}, \sin\frac{\alpha+\beta}{2}, 0,0,0,0,0,0,0,0,0,0\right) . \tag{5.25}$$

At the next step, we reduce ten components $\mathtt{v}^M$ parameterizing ε_c to just three components. First of all, the first four components $\mathtt{v}_1,\ldots,\mathtt{v}_4$ can be set to zero because they correspond to the special conformal transformation of the space-time (see (5.9)). As a result we are left with six parameters $\mathtt{v}_5,\ldots,\mathtt{v}_{10}$.

The projective spinor ε_s (5.25) is invariant under $\mathrm{U}(1)^3$ as evident from (5.24). The first U(1) simultaneously rotates the 1-2 and 5-6 planes, the second U(1) simultaneously rotates the 3-4 and 7-8 planes, and the third one U(1) rotates the 9-10 plane. This symmetry is enough to kill $\mathtt{v}_6,\mathtt{v}_8,\mathtt{v}_{10}$ leaving $\mathtt{v}^M=(0,0,0,0,\mathtt{v}_\alpha,0,\mathtt{v}_\beta,0,\mathtt{v}_n,0)$.

We summarize that the conformal supercharges Q "of the type $\mathtt{m}=0$" can be parametrized modulo the action of $\mathrm{SO}(4)\times\mathrm{SO}(6)$ symmetry by two angles α,β (which determine the complex structure J at the origin $x=0$) and three non-negative real numbers $\mathtt{v}_\alpha,\mathtt{v}_\beta,\mathtt{v}_n$. Still, not all five parameters $\alpha,\beta,\mathtt{v}_\alpha,\mathtt{v}_\beta,\mathtt{v}_n$ are independent. Now we will use the $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ invariants (5.22) to find which points in the five-dimensional parametric space $(\alpha,\beta,\mathtt{v}_\alpha,\mathtt{v}_\beta,\mathtt{v}_n)$ are related to each other by a $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ transformation.

There are just two independent invariants (under $SO(5,1)\times\mathrm{SO}(6)\times\mathbb{C}^*$)

$$\begin{aligned} I_1 &= \frac{\tilde I_p^6}{I_p^1} = \frac{\mathtt{v}_n^2}{\mathtt{v}^2}\cos^2\alpha\cos^2\beta\,, \\ I_2 &= \frac{I_p^5}{I_p^1} = \frac{\mathtt{v}_n^2}{\mathtt{v}^2}\sin^2\alpha\sin^2\beta + \frac{\mathtt{v}_\alpha^2}{\mathtt{v}^2}\sin^2\beta + \frac{\mathtt{v}_\beta^2}{\mathtt{v}^2}\sin^2\alpha\,, \end{aligned} \tag{5.26}$$

where $\mathtt{v}^2=\mathtt{v}_\alpha^2+\mathtt{v}_\beta^2+\mathtt{v}_n^2$. They uniquely parametrize the conformal (projective) Killing spinor $\varepsilon(x)$ up to the action of the global symmetry group.

Now, we would like to pick a canonical representative for the pair $(\varepsilon_s,\varepsilon_c)$ in each $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ orbit in the space of conformal killing spinors to write down concrete formulas for the Wilson loop operators.

To choose such a representative, we analyze the allowed range of values for I_1, I_2 and parametrize it in a convenient way. To find the allowed range, we fix I_1 and vary I_2 with respect to $\alpha,\beta,\mathtt{v}_\alpha^2/\mathtt{v}^2,\mathtt{v}_\beta^2/\mathtt{v}^2$. A simple calculation reveals that for the given I_1 the maximal value of I_2 is achieved when two of three terms in I_2 vanish. In general the invariants belong to the interval $0\le I_1,I_2\le 1$ and satisfy

$$\sqrt{I_1}+\sqrt{I_2}\le 1\ . \tag{5.27}$$

The same range of allowed values could be parametrized by two parameters $\alpha,\mathtt{v}_\beta^2/\mathtt{v}^2$ keeping $\beta=\mathtt{v}_\beta=0$. Indeed, in this case we introduce $t_1=\mathtt{v}_\beta^2/\mathtt{v}^2, t_2=\cos^2\beta$ with $0\le t_1,t_2\le 1$ and

$$\begin{aligned} I_1 &= t_1t_2\,, \\ I_2 &= (1-t_1)(1-t_2)\ . \end{aligned} \tag{5.28}$$

Clearly I_1, I_2 from (5.28) cover the same allowed range of values (5.27).

The conclusion is that for any allowed values of I_1, I_2 there exists a point on the $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ orbit such that $\alpha = 0, \mathtt{v}_\beta = 0$. The space of nonequivalent pure conformal Killing spinors is parametrized by angle β and the cosine of the angle between the vector v and the α-plane $\mathtt{v}_\alpha/\mathtt{v}$, keeping $\alpha = 0$ and $\mathtt{v}_\beta = 0$.

5.1.1 Complex structure and Wilson loop operators

Now we give concrete formula for the Wilson loop operator in this case. The most general supersymmetric coupling at point x is given by $\varphi^{\bar{I}} Z^M_{\bar{I}}(x) A_M$ where $A_M = (A_\mu, \Phi_A)$, $Z^M_{\bar{I}}(x)$ satisfies (4.42) and $\varphi^{\bar{I}}(x)$ are five arbitrary complex numbers. For the contour $x^\mu(s)$ to be a real contour in $\mathbb{R}^4_{\mathrm{spt}}$ we ask $v^\mu = \dot{x}^\mu$ to be real. In general the matrix $Z^M_{\bar{I}}(x)$ can be found with help of (4.47). To make the connection between $\varphi^{\bar{I}}$ and dx^μ more obvious it is preferable to make a transformation $Z^M_{\bar{I}} \to \tilde{Z}^M_{\bar{I}}(x) = Z^M_{\bar{J}}(x) U^{\bar{J}}_{\bar{I}}(x)$ bringing it to the form

$$\tilde{Z}^M_{\bar{I}} = \begin{pmatrix} I_{5\times5} \\ -i\Theta \end{pmatrix} . \tag{5.29}$$

The new matrix $\tilde{Z}$ is still a matrix of the antiholomorphic vectors. Therefore the supersymmetric coupling takes the following simple form (here φ is an arbitrary complex number)

$$A_M \tilde{Z}^M_{\bar{I}} \begin{pmatrix} \dot{x}_1 \\ \cdots \\ \dot{x}_4 \\ \varphi \end{pmatrix} . \tag{5.30}$$

To find $\tilde{Z}(x)$ we start with the matrix $\hat{Z}$ at the origin which satisfies (4.43) with $g_{MN} = \frac{1}{2}\delta_{MN}$ and is annihilated by $(\mathbb{I}_{10\times10} + iJ)$ with J given by (5.24)

$$(\hat{Z}^M_{\bar{I}})^T = \frac{1}{2}\begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} & 0 & 0 & -i\cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} & 0 & 0 & 0 & 0 \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} & 0 & 0 & \sin\frac{\alpha}{2} & -i\cos\frac{\alpha}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\frac{\beta}{2} & i\sin\frac{\beta}{2} & 0 & 0 & -i\cos\frac{\beta}{2} & -\sin\frac{\beta}{2} & 0 & 0 \\ 0 & 0 & -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} & 0 & 0 & \sin\frac{\beta}{2} & -i\cos\frac{\beta}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & i \end{pmatrix} . \tag{5.31}$$

The corresponding holomorphic coordinates x^I on $\mathbb{R}^{10}$ are

$$\begin{aligned} x^{I=1} &= \cos\frac{\alpha}{2}x_1 - \sin\frac{\alpha}{2}x_6 + i\left(\sin\frac{\alpha}{2}x_2 - \cos\frac{\alpha}{2}x_5\right) , \\ x^{I=2} &= \cos\frac{\alpha}{2}x_2 + \sin\frac{\alpha}{2}x_5 - i\left(\sin\frac{\alpha}{2}x_1 + \cos\frac{\alpha}{2}x_6\right) , \\ x^{I=3} &= \cos\frac{\beta}{2}x_3 - \sin\frac{\beta}{2}x_8 + i\left(\sin\frac{\beta}{2}x_4 - \cos\frac{\beta}{2}x_7\right) , \\ x^{I=4} &= \cos\frac{\beta}{2}x_4 + \sin\frac{\beta}{2}x_7 - i\left(\sin\frac{\beta}{2}x_3 + \cos\frac{\beta}{2}x_8\right) , \\ x^{I=5} &= x_9 + ix_{10} . \end{aligned} \tag{5.32}$$

Similarly the holomorphic vector $\mathtt{v}^I$ is built of $\mathtt{v}^M$ with all $\mathtt{v}^M = 0$ except for $\mathtt{v}_5 = \mathtt{v}_\alpha$ and $\mathtt{v}_9 = \mathtt{v}_n$. In fact the formulae above are too general because we can always put $\alpha = 0$. Now one can use the computer algebra to construct the 5×5 matrix $\alpha_{\bar{I}\bar{J}}$ using (4.45), calculate $Z^M_{\bar{I}}(x)$ using (4.47) and then transform it to the form (5.29) by multiplying it by an appropriate $U^{\bar{J}}_{\bar{I}}$. It is convenient to rearrange index M as follows

$$A_M = (A_1, \dots A_4, \Phi_5, \dots, \Phi_{10}) \to A_{\tilde{M}} = (A_1, \dots, A_4, \Phi_9, \Phi_5, \dots, \Phi_8, -\Phi_{10}) \,. \tag{5.33}$$

In this case the equation (4.47) that determines Z away from the origin obviously stays the same while the matrix (5.31) assumes a simpler form (remember that we put angle $\alpha = 0$)

$$\hat{Z}^{\tilde{M}}_{\bar{I}} = \frac{1}{2}\begin{pmatrix} \mathrm{z} \\ -i\bar{\mathrm{z}} \end{pmatrix}, \quad \mathrm{z} = \begin{pmatrix} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & \cos\frac{\beta}{2} & i\sin\frac{\beta}{2} & 0 \\ 0 & 0 & -i\sin\frac{\beta}{2} & \cos\frac{\beta}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{5.34}$$

With help of (4.47) and using that $g_{MN} = \frac{1}{2}\delta_{MN}$ the matrix Θ from (5.29) is given by

$$\Theta = (\bar{\mathrm{z}} + 4\mathrm{z}\alpha)(\mathrm{z} + 4\bar{\mathrm{z}}\alpha)^{-1} \,, \tag{5.35}$$

with α given by (4.45). In general the explicit expression for Θ can be calculated with help of computer algebra. Here we present a simple analytical derivation for the specific case $\beta = 0$. In this case $\Theta = (1 - A)(1 + A)^{-1}$ where matrix $A = 4\alpha$ has a specific structure $A_{ij} = a_i b_j - a_j b_i$. For any such matrix A with arbitrary vectors a_i, b_i the inverse matrix $(1 + A)^{-1}$ has a simple analytical form

$$((1 + A)^{-1})_{ij} = \delta_{ij} + \frac{-(a_i b_j - a_j b_i) + (ab)(a_i b_j + b_j a_i) - a^2 b_i b_j - b^2 a_i a_j}{1 - ((ab)^2 - a^2 b^2)} \,. \tag{5.36}$$

This immediately gives for Θ

$$\begin{gathered}\Theta^I_{\bar{J}} = \delta^{IJ} + 2\frac{(1 + \overline{x}\mathtt{v})(-x^I \mathtt{v}^J + \mathtt{v}^I x^J) + (x\mathtt{v})(x^I \mathtt{v}^J + x^J \mathtt{v}^I) - x^I x^J \mathtt{v}^2 - \mathtt{v}^I \mathtt{v}^J x^2}{(1 + \overline{x}\mathtt{v})^2 - (x\mathtt{v})^2 + x^2 \mathtt{v}^2} \,, \\ x^2 \equiv x^I x^I \,, \quad \mathtt{v}^2 \equiv \mathtt{v}^I \mathtt{v}^I \,, \quad x\mathtt{v} \equiv x^I \mathtt{v}^I \,, \quad \overline{x}\mathtt{v} \equiv \overline{x^I} \mathtt{v}^I \,.\end{gathered} \tag{5.37}$$

In the generic case $\beta \neq 0$, the expression (5.37) is not applicable anymore. Nevertheless, the explicit calculation reveals that all $\Theta^I_{\bar{J}}$ remain the same except for $\Theta^{A=3,4}_{\bar{I}}$. We present the expressions for these couplings below and notice that they coincide with (5.37) in the limit $\beta = 0$

$$\left(\Theta^{A=3}_{\bar{I}}\right)^T(x) = \frac{1}{\cos\beta}\begin{pmatrix} \frac{2\left(i\mathtt{v}_\alpha + \left(\mathtt{v}_\alpha^2 - \mathtt{v}_n^2\right)x_1\right)(x_3 + i\sin\beta x_4)}{1 - 2i\mathtt{v}_\alpha x_1 - x^2(\mathtt{v}_\alpha^2 - \mathtt{v}_n^2)} \\ \frac{2\left(\mathtt{v}_\alpha^2 - \mathtt{v}_n^2\right)x_2(x_3 + i\sin\beta x_4)}{1 - 2i\mathtt{v}_\alpha x_1 - x^2(\mathtt{v}_\alpha^2 - \mathtt{v}_n^2)} \\ \frac{1 - 2i\mathtt{v}_\alpha x_1 - (\mathtt{v}_\alpha^2 - \mathtt{v}_n^2)\left(x_1^2 + x_2^2 - x_3^2 - 2i\sin\beta x_3 x_4 + x_4^2\right)}{1 - 2i\mathtt{v}_\alpha x_1 - x^2(\mathtt{v}_\alpha^2 - \mathtt{v}_n^2)} \\ \frac{2\left(\mathtt{v}_\alpha^2 - \mathtt{v}_n^2\right)x_3 x_4 + i\left(1 - 2i\mathtt{v}_\alpha x_1 - (\mathtt{v}_\alpha^2 - \mathtt{v}_n^2)\left(x_1^2 + x_2^2 + x_3^2 - x_4^2\right)\right)\sin\beta}{1 - 2i\mathtt{v}_\alpha x_1 - x^2(\mathtt{v}_\alpha^2 - \mathtt{v}_n^2)} \\ -\frac{2\mathtt{v}_n(x_3 + i\sin\beta x_4)}{1 - 2i\mathtt{v}_\alpha x_1 - x^2(\mathtt{v}_\alpha^2 - \mathtt{v}_n^2)} \end{pmatrix}, \tag{5.38}$$

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$$\left(\Theta_{\bar{I}}^{A=4}\right)^T(x) = \frac{1}{\cos\beta}\begin{pmatrix} \frac{2\left(i\mathtt{v}_\alpha+\left(\mathtt{v}_\alpha^2-\mathtt{v}_n^2\right)x_1\right)(x_4-i\sin\beta x_3)}{1-2i\mathtt{v}_\alpha x_1-x^2(\mathtt{v}_\alpha^2-\mathtt{v}_n^2)} \\ \frac{2\left(\mathtt{v}_\alpha^2-\mathtt{v}_n^2\right)x_2(x_4-i\sin\beta x_3)}{1-2i\mathtt{v}_\alpha x_-x^2(\mathtt{v}_\alpha^2-\mathtt{v}_n^2)} \\ \frac{2\left(\mathtt{v}_\alpha^2-\mathtt{v}_n^2\right)x_3x_4-i\left(1-2i\mathtt{v}_\alpha x_1-(\mathtt{v}_\alpha^2-\mathtt{v}_n^2)\left(x_1^2+x_2^2-x_3^2+x_4^2\right)\right)\sin\beta}{1-2i\mathtt{v}_\alpha x_1-x^2(\mathtt{v}_\alpha^2-\mathtt{v}_n^2)} \\ \frac{1-2i\mathtt{v}_\alpha x_1-(\mathtt{v}_\alpha^2-\mathtt{v}_n^2)\left(x_1^2+x_2^2+x_3^2+2i\sin\beta x_3x_4-x_4^2\right)}{1-2i\mathtt{v}_\alpha x_1-x^2(\mathtt{v}_\alpha^2-\mathtt{v}_n^2)} \\ -\frac{2\mathtt{v}_n(x_4-i\sin\beta x_3)}{1-2i\mathtt{v}_\alpha x_1-x^2(\mathtt{v}_\alpha^2-\mathtt{v}_n^2)} \end{pmatrix}. \tag{5.39}$$

Finally, the supersymmetric Wilson loop, parametrized by an arbitrary contour γ in $\mathbb{R}^4_{\text{spt}}$ and a complex coupling $\varphi(s)$, is

$$W_R[\gamma(s),\varphi(s)] = \mathrm{Tr}_R\,\mathrm{Pexp}\oint_\gamma A_{\tilde{M}}\begin{pmatrix} \mathbb{I}_{5\times5} \\ -i\Theta(x)\end{pmatrix}\begin{pmatrix}\dot{x}_1 \\ \cdots \\ \dot{x}_4 \\ \varphi\end{pmatrix} ds\,, \tag{5.40}$$

$$A_{\tilde{M}} = (A_1,\ldots,A_4,\Phi_9,\Phi_5,\ldots,\Phi_8,-\Phi_{10})\,. \tag{5.41}$$

In the specific case $\beta = \mathtt{v}_\alpha = \mathtt{v}_n = 0$ the operator (5.40) becomes the supersymmetric Wilson loop on $\mathbb{R}^4$ discovered by Zarembo in [16].

5.2 The case $\mathtt{m} \neq 0$, $\Sigma_{\mathbb{C}} = \mathbb{R}^6$ or $\Sigma_{\mathbb{C}} = S^6$

Now we are ready to consider a more interesting case when $\mathtt{m} \neq 0$, and hence Σ is either a sphere S^n or a plane $\mathbb{R}^n$ in $\mathbb{R}^4$. First of all, if Σ is a sphere, we can always perform an appropriate special conformal transformation that turns Σ into a plane. Explicitly, such transformation amounts to a shift of $\mathtt{v}^\mu$ in such a way that $\mathtt{v}_{\bar{4}}, \mathtt{v}_{\bar{5}}$ in (4.38) vanish. Let us show that this is always possible. We assumed that Σ is non-trivial, hence there are other points in $\mathbb{R}^4_{\text{spt}}$ besides the origin that satisfy (4.38). Then we can choose the coordinates x^*_μ of one of those points to be the parameters of a special conformal transformation

$$\mathtt{v}_\mu \to \mathtt{v}_\mu + \frac{x^*_\mu}{|x^2|}\,. \tag{5.42}$$

Obviously such transformation kills $\mathtt{v}_{\bar{4}}, \mathtt{v}_{\bar{5}}$ from (4.38). From now on we therefore assume that Σ is a plane $\mathbb{R}^n$, $n = 1,2,3,4$. The dimension n depends on the mutual orientation within $\mathbb{R}^{10}$ of the space-time $\mathbb{R}^4_{\text{spt}}$ and the pure-spinor-surface $\Sigma_{\mathbb{C}} = \mathbb{R}^6$

Below we classify all possible scenarios.

5.3 $\Sigma = \mathbb{R}^1$

Perhaps the simplest scenario is when $\Sigma = \mathbb{R}^1$. In this case the pure spinor ε_s is unique up to a $\mathrm{SO}(5,1)\times\mathrm{SO}(6)$ rotation. The main difference with the $\mathtt{m} = 0$ case, where ε_s was parametrized by two angles α,β, comes from the fact that J transforms $\Sigma_{\mathbb{C}}$ (and its orthogonal compliment $\Sigma^{\perp}_{\mathbb{C}}$) into itself and this rigidly constraints J and hence ε_s. As always, we choose the directions $1\ldots4$ to be along the space-time $\mathbb{R}^4_{\text{spt}}$, and we choose the direction 1 be along $\Sigma = \mathbb{R}^1$. Then $\Sigma_{\mathbb{C}}$ includes the directions $1,5,7-10$ and its orthogonal compliment $\Sigma^{\perp}_{\mathbb{C}}$ includes the directions $2-4,6$.

We can always choose the coordinate x_5 to be along the J-image of x_1 and x_2 to be along the J-image of x_6. After an appropriate $\mathrm{SO}(4) \subset \mathrm{SO}(6)$ rotation of the $7-10$ directions the matrix J (and the corresponding spinor ε_s) acquires the form (5.24) with $\alpha = 0$ and $\beta = \pi/2$.

The corresponding complex coordinates x^I are

$$\begin{aligned} x^{I=1} &= x_1 + i x_5 \,, \\ x^{I=2} &= x_7 + i x_8 \,, \\ x^{I=3} &= x_{10} + i x_9 \,, \\ x^{I=4} &= x_2 + i x_6 \,, \\ x^{I=5} &= x_4 + i x_3 \,. \end{aligned} \tag{5.43}$$

Thus we chose $x^{I=1,2,3}$ to lie within $\Sigma_{\mathbb{C}}$ and $x^{I=4,5}$ to be orthogonal.

Since we assume that Σ is $\mathbb{R}^1$ rather than S^1, $\mathtt{v}_2 - i\mathtt{v}_6$ and $\mathtt{v}_3 - i\mathtt{v}_4$ must vanish. We can also kill $\mathtt{v}_1$ using a special conformal transformation along Σ.

There are two independent invariants that depend on six real parameters $\mathtt{v}_5, \mathtt{v}_7, \ldots, \mathtt{v}_{10}$ and one complex parameter μ from (4.37)

$$\begin{aligned} I_1 &= -\frac{I_q^5}{I_q^1 + I_p^q} = \frac{\mathtt{v}_5^2}{|\mu^2|} \,, \\ I_2 &= \frac{I_q^5 - I_q^1}{I_q^1 + I_p^1} = \frac{\mathtt{v}_7^2 + \mathtt{v}_8^2 + \mathtt{v}_9^2 + \mathtt{v}_{10}^2}{|\mu^2|} \,. \end{aligned} \tag{5.44}$$

There is a great deal of degeneracy in (5.44) as the two invariants depend on seven parameters. This partially can be explained by the fact that we did not fix all geometric symmetries of the setup. There are three U(1) symmetries which rotate the $7-8$, $9-10$ and both planes simultaneously. Moreover, there is dilatation that rescales all coordinates together with $\mathtt{v}^M$ and μ. These symmetries allow us to set $\mu = 1$ and to kill two components out of the four $\mathtt{v}_7, \ldots, \mathtt{v}_{10}$. Another parameter can be killed because of shifts along the subspace Σ: if we choose a different point along Σ to be an origin of the coordinate system then the original combination of parameters $\mathtt{v}_7, \ldots, \mathtt{v}_{10}$ will turn into a new one such that I_1, I_2 do not change. Since the only invariant quantities are (5.44) we can choose two variables $\mathtt{v}_5, \mathtt{v}_7$ to parametrize I_1, I_2 while taking $\mu_1 = \mathrm{Re}\mu = 1, \mu_2 = \mathrm{Im}\mu = 0$ and $\mathtt{v}_M = 0$ for all $M \neq 5, 7$.

5.3.1 Complex structure and supersymmetric Wilson loops

The complex structure at the origin is given by (5.24) with $\alpha = 0$ and $\beta = \pi/2$. The matrix $\hat{Z}_I^M$ (5.31) admits the form (5.29) with $\Theta = \mathbb{I}_{5\times 5}$ if we rearrange the index M as follows

$$A_M = (A_1, \ldots, A_4, \Phi_5, \ldots, \Phi_{10}) \to A_{\tilde{M}} = (A_1, \Phi_7, \Phi_{10}, A_2, A_4, \Phi_8, \Phi_5, \Phi_9, \Phi_6, A_3) \,. \tag{5.45}$$

The components of the three-form $\mathtt{m}_{\bar{K}\bar{I}\bar{J}}$ written in coordinates (5.43) are non-zero only if all three indexes are $1, 2$, or 3, and zero otherwise

$$\mathtt{m}_{\bar{K}\bar{I}\bar{J}} = \mu g_{\bar{K}K} g_{\bar{I}I} g_{\bar{J}J} \varepsilon^{KIJ}, \quad \bar{I}, \bar{J}, \bar{K} = 1, 2, 3 \,. \tag{5.46}$$

Here ε^{KIJ} is the absolutely antisymmetric tensor, $\varepsilon^{123} = 1$. As follows from the expression for $\alpha_{\bar{I}\bar{J}}$ (4.45) and formula for $Z^M_{\bar{I}}$ (4.47), only $Z^M_{\bar{I}}$ for $\bar{I} = 1, 2, 3$ change when we move along Σ, while $Z^M_{\bar{I}=4,5}$ remain the same. Note, that one can not add the couplings

$$Z^M_{\bar{I}=4} A_M = (A_2 - i\Phi_6)\,, \tag{5.47}$$
$$Z^M_{\bar{I}=5} A_M = (A_4 - iA_3)\,, \tag{5.48}$$

to the supersymmetric Wilson loop operator because this would require the contour γ to leave Σ.

From now on we can neglect $Z^{\tilde{M}}_{\bar{I}}$ for $\bar{I} = 4, 5$, and assume in what follows that index $\bar{I} = 1, 2, 3$. Similarly we do not need to worry about $\tilde{M} = 4, 5, 9, 10$, and the matrix $\tilde{Z}$ effectively becomes 6×3

$$\tilde{Z} = \begin{pmatrix} \mathbb{I}_{3\times 3} \\ -i\Theta \end{pmatrix} . \tag{5.49}$$

Let us define a three-dimensional vector $\tilde{\alpha}^{\bar{K}}$ dual to the 2-form $\alpha_{\bar{I}\bar{J}}$

$$\tilde{\alpha}^{\bar{K}} = 2\varepsilon^{\bar{I}\bar{J}\bar{K}}\alpha_{\bar{I}\bar{J}} = \frac{\varepsilon^{\bar{I}\bar{J}\bar{K}} x^I \mathrm{v}^J + \mu x^{\bar{K}}}{1 + \bar{x}\mathrm{v}} . \tag{5.50}$$

The matrix Θ is then given by

$$\Theta = (1 - 4\alpha)(1 + 4\alpha)^{-1}\,, \tag{5.51}$$
$$4\alpha^{IJ} = \varepsilon^{IJK}\alpha^{\bar{K}}. \tag{5.52}$$

This expression can be easily calculated analytically

$$\Theta^{\bar{I}}_{\bar{J}} = \frac{\delta^{\bar{I}\bar{J}}(1 - \tilde{\alpha}^2) + 2\tilde{\alpha}^{\bar{I}}\tilde{\alpha}^{\bar{J}} - 2\varepsilon^{\bar{I}\bar{J}\bar{K}}\alpha^{\bar{K}}}{1 + \tilde{\alpha}^2}, \tag{5.53}$$
$$\tilde{\alpha}^2 \equiv \tilde{\alpha}^{\bar{I}}\tilde{\alpha}^{\bar{I}} . \tag{5.54}$$

The resulting supersymmetric Wilson loop associated with Σ is

$$W_R[\gamma, \varphi_1, \varphi_2] = \mathrm{Tr}_R\, \mathrm{Pexp} \oint_\gamma ds \begin{pmatrix} A_1 & \Phi_7 & \Phi_{10} & \Phi_5 & \Phi_9 & \Phi_8 \end{pmatrix} \begin{pmatrix} \mathbb{I}_{3\times 3} \\ -i\Theta \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} . \tag{5.55}$$

Here contour $\gamma(s)$ is just a straight line $x_1(s)$ and $\varphi_{1,2}(s)$ are arbitrary complex functions of the contour parameter s. If $\varphi_1 = \varphi_2 = 0$ the operator (5.55) is defined through the vector

$$\Theta^{\tilde{A}}_1 = \begin{pmatrix} \frac{1 - 2i\mathrm{v}_5 x_1 + \left(1 - \mathrm{v}_5^2 - \mathrm{v}_7^2\right)x_1^2}{1 - 2i\mathrm{v}_5 x_1 + \left(1 - \mathrm{v}_5^2 + \mathrm{v}_7^2\right)x_1^2} \\ \frac{2\mathrm{v}_7 x_1 (1 - i\mathrm{v}_5 x_1)}{1 - 2i\mathrm{v}_5 x_1 + \left(1 - \mathrm{v}_5^2 + \mathrm{v}_7^2\right)x_1^2} \\ \frac{2\mathrm{v}_7 x_1^2}{1 - 2i\mathrm{v}_5 x_1 + \left(1 - \mathrm{v}_5^2 + \mathrm{v}_7^2\right)x_1^2} \end{pmatrix} . \tag{5.56}$$

If $\mathrm{v}_5 = 0$ this Wilson loop is the conformal transformation of the circular Wilson loop with zero expectation value from [16]. Let us notice here that the denominator $1 - 2i\mathrm{v}_5 x_1 +$

$\left(1 - \mathrm{v}_5^2 + \mathrm{v}_7^2\right) x_1^2$ never vanishes and hence the corresponding operator (5.55) is well defined for any smooth $\dot{\varphi}_1, \dot{\varphi}_2$.

Besides the Wilson loops described above, there are some supersymmetric Wilson loops associated with the vector field u^M (3.6)

$$u^M \cong (0, x_3 + ix_4, -x_2 + ix_6, -x_6 - ix_2, 0, -ix_3 + x_4, 0, 0, 0, 0)\,, \tag{5.57}$$

living outside of Σ. The components u^μ on $\mathbb{R}^4_{\text{spt}}$ ($x_6 = \cdots = x_{10} = 0$) should be real. Therefore $x_2 = 0$ and x_3/x_4 must be constant. The corresponding contour $\gamma : x^\mu(s)$ is a straight line

$$x^\mu(s) = (x_1, 0, k_3 s, k_4 s) \tag{5.58}$$

in the 3-4 plane while x_1 is some constant and $x_2 = 0$. The corresponding Wilson loop operator is a straight line in $\mathbb{R}^4_{\text{spt}}$ with the string fixed at the north pole of S^5 [7].

5.4 $\Sigma = \mathbb{R}^2$

The next scenario is $\Sigma \equiv \mathbb{R}^4_{\text{spt}} \bigcap \Sigma_{\mathbb{C}} = \mathbb{R}^2$. In this case J has the most general form (5.24) and the complex coordinates on $\mathbb{R}^{10}$ are

$$\begin{aligned}
x^{I=1} &= \cos\frac{\alpha}{2}x_1 - \sin\frac{\beta}{2}x_6 + i\left(\sin\frac{\alpha}{2}x_2 - \cos\frac{\alpha}{2}x_5\right),\\
x^{I=2} &= \cos\frac{\alpha}{2}x_2 + \sin\frac{\alpha}{2}x_5 - i\left(\sin\frac{\alpha}{2}x_1 + \cos\frac{\alpha}{2}x_6\right),\\
x^{I=3} &= x_9 + ix_{10}\,,\\
x^{I=4} &= \cos\frac{\beta}{2}x_3 - \sin\frac{\beta}{2}x_8 + i\left(\sin\frac{\beta}{2}x_4 - \cos\frac{\beta}{2}x_7\right),\\
x^{I=5} &= \cos\frac{\beta}{2}x_4 + \sin\frac{\beta}{2}x_7 - i\left(\sin\frac{\beta}{2}x_3 + \cos\frac{\beta}{2}x_8\right).
\end{aligned} \tag{5.59}$$

We rearranged x^I (compare with (5.32)) in such a way that $x^{\mu=1,2}$ parametrize Σ and $x^{I=1,2,3}$ parametrize $\Sigma_{\mathbb{C}}$.

As usual, $\mathrm{v}_3, \mathrm{v}_4, \mathrm{v}_7, \mathrm{v}_8$ vanish after a conformal transformation that makes Σ flat, and we kill $\mathrm{v}_1, \mathrm{v}_2$ by a special conformal transformation along Σ. The only non-zero parameters are $\alpha, \beta, \mu, \mathrm{v}_5, \mathrm{v}_6, \mathrm{v}_9, \mathrm{v}_{10}$. There are four independent real invariants which can be combined into two real and one complex variables

$$I_1 = -\frac{I_q^1}{I_q^1 + I_p^1}\frac{\mathrm{v}_5^2 + \mathrm{v}_6^2 + \mathrm{v}_9^2 + \mathrm{v}_{10}^2}{|\mu^2|}\,, \tag{5.60}$$

$$I_2 = \frac{I_q^6}{I_q^2} = \frac{((\mathrm{v}_{10} - i\mathrm{v}_9)\cos\alpha - \mu\sin\alpha)^2}{\mu^2}\,, \tag{5.61}$$

$$I_3 = -\frac{\tilde{I}_p^2}{I_q^1 + I_p^1} = \cos^2\beta\,. \tag{5.62}$$

The list of invariants is somewhat long but we still have symmetries to play with. First of all, we can rotate the 1-2 plane and the 5-6 plane to eliminate v_6, and the 9-10 plane to get rid of Imμ. Then the dilatation sets $\mu = 1$, leaving five non-trivial parameters $\alpha, \beta, \mathtt{v}_5, \mathtt{v}_9, \mathtt{v}_{10}$. It is not surprising that β is an invariant. After these geometrical symmetries are used up, the only transformation that could relate different β's is the shift along $\Sigma \subset \Sigma_{\mathbb{C}}$. Those shifts change complex structure in $\Sigma_{\mathbb{C}}$, but leave the orthogonal compliment to $\Sigma_{\mathbb{C}}$ invariant. Therefore β that governs the complex structure in $N_{\Sigma_{\mathbb{C}}}$ (unlike α that governs the complex structure in $T_{\Sigma_{\mathbb{C}}}$) is an invariant.

To completely fix the conformal gauge, we eliminate one of the four parameters $\mathtt{v}_5, \mathtt{v}_9, \mathtt{v}_{10}, \alpha$, using the simplified invariants:

$$\begin{aligned} I_1 &= \mathtt{v}_5^2 + \mathtt{v}_9^2 + \mathtt{v}_{10}^2\,, \\ \mathrm{Re}\ I_2^{1/2} &= \cos\alpha \mathtt{v}_{10} - \sin\alpha\,, \\ \mathrm{Im}\ I_2^{1/2} &= -\cos\alpha \mathtt{v}_9\ . \end{aligned} \tag{5.63}$$

One easy way to do that is to express $\mathtt{v}_9$ and $\mathtt{v}_{10}$ from the last two equations and substitute into the first equation. We get then

$$\mathtt{v}_5^2 = I_1 - \left(\frac{\mathrm{Im} I_2^{1/2}}{\cos\alpha}\right)^2 - \left(\frac{\mathrm{Re} I_2^{1/2} + \sin\alpha}{\cos\alpha}\right)^2 \tag{5.64}$$

For the generic values of the invariants I_1, I_2 such that $\mathtt{v}_5^2 > 0$, the r.h.s. of (5.64) is positive. But for α sufficiently close to $\pi/2$ the r.h.s. of (5.64) is negative, hence it must vanish at some intermediate value of α. At that point $\mathtt{v}_5 = 0$ and α can be expressed through I_1, I_2. In fact one can always choose α in such a way that $\mathtt{v}_5$ vanishes leaving $\alpha, \beta, \mathtt{v}_9, \mathtt{v}_{10}$ as the parameters, while setting $\mu = 1, \mathtt{v}_5 = \mathtt{v}_6 = 0$.

5.4.1 Complex structure and supersymmetric Wilson loops

Similarly to the previous case $\Sigma = \mathbb{R}^1$, we rearrange index M as follows

$$A_M = (A_1, \ldots, A_4, \Phi_5, \ldots, \Phi_{10}) \to A_{\tilde{M}} = (A_1, A_2, \Phi_9, A_3, A_4, \Phi_5, \Phi_6, \Phi_{10}, \Phi_7, \Phi_8), \tag{5.65}$$

to bring $\hat{Z}$ to the form (5.34) with z given by

$$\mathrm{z} = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} & 0 & 0 & 0 \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.66}$$

As in the previous case $\Sigma = \mathbb{R}^1$, the two couplings $Z^{\tilde{M}}_{\bar{I}=4,5} A_{\tilde{M}}$ are the same for all points on Σ. In general they can not be added to the supersymmetric Wilson loop operator because they require non-zero $\dot{x}^{3,4}$, and hence lead away from Σ (in the exceptional case $\beta = \pi/2$ one of the couplings becomes $\Phi_7 - i\Phi_8$ and can be added with arbitrary complex coefficient $\varphi(s)$). Therefore we neglect two last columns $\bar{I} = 4, 5$ and the rows $\tilde{M} = 4, 5, 9, 10$, similarly to the previous case, effectively reducing Z to the 6×3 size. The resulting matrix of the antiholomorphic vectors can be presented in the form (5.49) with Θ given by (5.35) with 3×3 matrix α given by (4.45) and the 3×3 matrix

$$\mathrm{z} = \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} & 0 \\ -i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{5.67}$$

In a particular case, when $\alpha = 0$, the matrix Θ is given by (5.53). Even in this case the explicit expression is too bulky to be written here.

The general supersymmetric Wilson operator associated with Σ is given by

$$W_R[\gamma,\varphi_1] = \mathrm{Tr}_R\,\mathrm{Pexp}\oint_\gamma ds \begin{pmatrix} A_1 & A_2 & \Phi_9 & \Phi_5 & \Phi_6 & \Phi_{10} \end{pmatrix} \begin{pmatrix} \mathbb{I}_{3\times 3} \\ -i\Theta \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \varphi_1 \end{pmatrix} , \tag{5.68}$$

with contour γ living on $\Sigma = \mathbb{R}^2$.

In the special case $\alpha = \mathrm{v}_M = 0$ the matrix Θ acquires a simple form

$$\Theta^{\bar{I}\bar{J}} = \frac{\delta^{\bar{I}\bar{J}}(1-\mathrm{x}^2) + 2\mathrm{x}^{\bar{I}}\mathrm{x}^{\bar{J}} - 2\varepsilon^{\bar{I}\bar{J}\bar{K}}\mathrm{x}^{\bar{K}}}{1+\mathrm{x}^2} , \tag{5.69}$$

$$\mathrm{x}^{\bar{I}} = (x_1, x_2, 0) . \tag{5.70}$$

These loops are related by a conformal transformation to the particular case of the DGRT loops on S^3 [12] when the contour is limited to the equator $S^2 \subset S^3$.

The vector field u^M (3.6)

$$\begin{aligned} u^M \cong\ & (0, 0, \cos\beta x_4 + \sin\beta x_7 - i x_8, -\cos\beta x_3 + i x_7 + \sin\beta x_8, \\ & 0, 0, -\sin\beta x_3 - i x_4 - \cos\beta x_8, i x_3 - \sin\beta x_4 + \cos\beta x_7, 0, 0) , \end{aligned} \tag{5.71}$$

gives rise to the suppersymmetric Wilson loops along the concentric circles in the 3-4 plane for any fixed x_1, x_2. The corresponding operators are the non-equator circular lines on S^4 with β playing the role of the latitude [19].

5.5 $\Sigma = \mathbb{R}^3$

In the case $\Sigma \equiv \mathbb{R}^4 \bigcap \Sigma_{\mathbb{C}} = \mathbb{R}^3$ one of the angles α, β must vanish. We choose $\beta = 0$ with the directions $1,2,3$ and $5,6,7$ to lie along $\Sigma_{\mathbb{C}}$. The corresponding complex structure is given by (5.24) and the holomorphic coordinates are

$$\begin{aligned} x^{I=1} &= \cos\frac{\alpha}{2}x_1 - \sin\frac{\beta}{2}x_6 + i\left(\sin\frac{\alpha}{2}x_2 - \cos\frac{\alpha}{2}x_5\right) , \\ x^{I=2} &= \cos\frac{\alpha}{2}x_2 + \sin\frac{\alpha}{2}x_5 - i\left(\sin\frac{\alpha}{2}x_1 + \cos\frac{\alpha}{2}x_6\right) , \\ x^{I=3} &= x_3 - i x_7 , \\ x^{I=4} &= x_4 - i x_8 , \\ x^{I=5} &= x_9 + i x_{10} , \end{aligned} \tag{5.72}$$

with $x_{1,2,3}$ parameterizing Σ.

As usually $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4, \mathrm{v}_8, \mathrm{v}_9, \mathrm{v}_{10}$ vanish after an appropriate special conformal transformation and we end up with μ and $\mathrm{v}_5, \mathrm{v}_6, \mathrm{v}_7$. There are two invariants

$$I_1 = -\frac{I_q^1}{I_q^1 + I_p^1} = \frac{\mathrm{v}_5^2 + \mathrm{v}_6^2 + \mathrm{v}_7^2}{|\mu^2|}, \tag{5.73}$$

$$I_2 = -\frac{I_q^5}{I_q^1 + I_p^1} = \frac{(\mathrm{Re}\,\mu\cos\alpha - \mathrm{v}_7\sin\alpha)^2}{|\mu^2|}. \tag{5.74}$$

The U(1) symmetry that rotates the 1-2 and 5-6 planes can be used to eliminate the phase of μ and then we use dilatation to set $\mu = 1$. It is also clear that we can always choose v_6^2 to be zero as it is always combined with v_5^2 in (5.73). It is clear then that the invariant I_1 is an arbitrary positive number when I_2 is any postive number in the range

$$0 \leq I_2 \leq 1 + I_1 . \tag{5.75}$$

One can cover exactly the same range by letting v_5 vanish, leaving v_7 and α as the only independent variables.

5.5.1 Complex structure and supersymmetric Wilson loops

This case is very similar to the previous one $\Sigma = \mathbb{R}^2$. After rearranging index M

$$A_M = (A_1, \ldots, A_4, \Phi_5, \ldots, \Phi_{10}) \to A_{\tilde{M}} = (A_1, \ldots, A_4, \Phi_9, \Phi_5, \ldots, \Phi_8, -\Phi_{10}). \tag{5.76}$$

the matrix $\hat{Z}$ acquires the form (5.34) with z given by (5.66). The last two columns $Z^M_{\bar{I}=4,5}$ are the same everywhere on Σ. One of the corresponding couplings $Z^M_{\bar{i}=5} A_M = \Phi_9 + i\Phi_{10}$ can be added to the supersymmetric Wilson operator with arbitrary complex coefficient although the other one $Z^M_{\bar{I}=5} A_M = A_4 - i\Phi_8$ requires non-zero $\dot{x}_4$ and therefore leads outside of Σ. Upon elementating two last columns and the $4, 5, 9, 10$ rows, the 6×3 matrix $\hat{Z}$ becomes of the form (5.34) with z given by (5.67).

The matrix of antiholomorphic vectors can be presented in the form (5.49) with Θ given by (5.35) with 3×3 matrix α given by (4.45) and the 3×3 matrix z (5.67). If $\alpha = 0$ the result simplifies and Θ is given by (5.53) but even in this case the explicit expression is too bulky to be written here.

The general supersymmetric Wilson operator associated with Σ is given by

$$W_R[\gamma, \varphi] = \mathrm{Tr}_R\,\mathrm{Pexp} \oint_\gamma ds \begin{pmatrix} A_1 & A_2 & A_3 & \Phi_5 & \Phi_6 & \Phi_7 \end{pmatrix} \begin{pmatrix} \mathbb{I}_{3\times 3} \\ -i\Theta \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \\ + ds\varphi(\Phi_9 + i\Phi_{10}), \tag{5.77}$$

with contour γ living on $\Sigma = \mathbb{R}^3$.

In the special case $\alpha = \mathrm{v}_M = 0$ the matrix Θ acquires simple form (5.69) with $\mathrm{x}^{\bar{I}} = (x_1, x_2, x_3)$. These loops are related by a conformal transformation to the DGRT loops on S^3 [12].

The space-time part of the vector field u^M (3.6)

$$u^M = x_4(0,\ldots,0,1,i), \tag{5.78}$$

is zero on the boundary $x_5 = \cdots = x_{10} = 0$ and therefore there are no suppersymmetric Wilson loops besides those described above and the local operator $\Phi_9 + i\Phi_{10}$.

5.6 $\Sigma = \mathbb{R}^4$

The exotic case is when Σ coincides with the total space-time $\Sigma \equiv \mathbb{R}^4_{\rm spt} \bigcap \Sigma_{\mathbb{C}} = \mathbb{R}^4_{\rm spt}$. If we choose the directions $5,6$ to lie inside Σ and be defined in the same way as in the cases $\Sigma = \mathbb{R}^{2,3}$ above, the complex structure J will be given by (5.24) with some α and $\beta = \pi/2$. The remaining parameters $\alpha, \mu, {\rm v}_5, {\rm v}_6$ form the unique invariant

$$I_1 = \frac{I^1_q}{I^1_q + I^1_p} = \frac{{\rm v}_5^2 + {\rm v}_6^2}{|\mu^2|}\ . \tag{5.79}$$

Clearly we can set $\mu = 1$ as we did before, and also $\alpha = 0$ because I_1 is α-independent. It also follows from (5.79) that we can fix ${\rm v}_6 = 0$ leaving ${\rm v}_5$ to be the only non-trivial parameter.

5.6.1 Complex structure and supersymmetric Wilson loops

Since we fixed $\alpha = 0$ the appropriate choice of holomorphic coordinates on $\mathbb{R}^{10}$ with first three coordinates $x^{I=1,2,3}$ parametrizing $\Sigma_{\mathbb{C}}$ is

$$x^{I=1} = x_1 + ix_5\,, \tag{5.80}$$
$$x^{I=2} = x_2 + ix_6\,, \tag{5.81}$$
$$x^{I=3} = x_3 - ix_4\,, \tag{5.82}$$
$$x^{I=4} = x_7 + ix_8\,, \tag{5.83}$$
$$x^{I=5} = x_9 - ix_{10}\ . \tag{5.84}$$

We rearrange index M

$$A_M = (A_1,\ldots,A_4,\Phi_5,\ldots,\Phi_{10}) \to A_{\tilde M} = (A_1,A_2,A_3,\Phi_7,\Phi_9,\Phi_5,\Phi_6,-A_4,\Phi_8,-\Phi_{10})\,,$$

to bring $\hat Z$ to the form (5.34) with ${\rm z} = \mathbb{I}_{5\times5}$. As in the previous cases, we remove the last two columns, which correspond to the couplings $\Phi_7 - i\Phi_8$ and $\Phi_9 + i\Phi_{10}$ (these couplings should be added to the supersymmetric Wilson loop operator) and the rows $4,5,9,10$ from $\hat Z$ to reduce it (and consequently $\tilde Z$) to the form (5.49). The matrix Θ is given by (5.53) and the supersymmetric Wilson loop operator is

$$W_R[\gamma,\varphi_1,\varphi_2] = {\rm Tr}_R\,{\rm Pexp}\oint_\gamma ds\begin{pmatrix} A_1 & A_2 & A_3 & \Phi_5 & \Phi_6 & -A_4\end{pmatrix}\begin{pmatrix}\mathbb{I}_{3\times3}\\ -i\Theta\end{pmatrix}\begin{pmatrix}\dot x_1\\ \dot x_2\\ \varphi\end{pmatrix} + \\ ds\varphi_1(\Phi_7 - i\Phi_8) + ds\varphi_2(\Phi_9 + i\Phi_{10})\ . \tag{5.85}$$

By definition of the Wilson loop in $\mathbb{R}^4_{\rm spt}$ we need to choose contour γ and function φ such that the coefficients in front of all four $A_1, \ldots, A_4$ are real for all s. At each point on $\Sigma = \mathbb{R}^4_{\rm spt}$ only two out of four tangent directions would satisfy this requirement with an appropriately chosen φ. First of all, φ must be real $\varphi = \dot{x}_3$ to avoid multiplying A_3 by a complex number. Moreover it should be such that the coefficient in front of A_4 is real as well (we denote it by $\dot{x}_4$)

$$\sum_{\mu=1}^{3} \mathrm{Im}\left(i\Theta^3_\mu \dot{x}_\mu\right) = 0\,, \qquad (5.86)$$
$$\sum_{\mu=1}^{3} \mathrm{Re}\left(i\Theta^3_\mu \dot{x}_\mu\right) = \dot{x}_4\ .$$

The resulting two-dimensional vector space is quite complicated and we do not present the explicit expression for the vectors $\dot{x}^\mu(\dot{x}_1, \dot{x}_2)$ here.[4] We will call the contours that satisfy (5.86) *allowed* and from now on assume that $\gamma(s)$ is one of them. Since the space of allowed directions is two dimensional at each point the contour can be parametrized by an initial point and one real degree of freedom. We note that commutator of two generic non-collinear allowed vectors at a given point is not an allowed vector

$$\xi_1 = \dot{x}^\mu(1,0)\,, \qquad \xi_2 = \dot{x}^\mu(0,1)\,,$$
$$[\xi_1, \xi_2] \wedge \xi_1 \wedge \xi_2 \neq 0\ . \qquad (5.87)$$

Therefore the space of allowed directions at each point $x \in \Sigma = \mathbb{R}^4_{\rm spt}$ can not be thought of as a tangent-space to some two-dimensional submanifold in $\mathbb{R}^4_{\rm spt}$. Even more so, the commutators of the commutators would span the whole four-dimensional space

$$[\xi_1, [\xi_1, \xi_2]] \wedge [\xi_1, \xi_2] \wedge \xi_1 \wedge \xi_2 \neq 0\,, \qquad (5.88)$$

which means that the contour γ is not restricted to any particular submanifold in $\mathbb{R}^4_{\rm spt}$. In this sense γ is four-dimensional. Given that it is parametrized by only one real function (which chooses the angle on the allowed plane at each point) there is not enough degrees of freedom to ensure that γ is closed. Therefore our general predictions would be that the contour γ that locally ensures gauge symmetry is not closed and can not be used to construct a gauge-invariant Wilson loop. Nevertheless there could be some particular examples of closed γ which would be interesting to identify.

To demonstrate that the contour γ can have a non-trivial shape we consider a particular case $\mathtt{v}_5 = 0$ and notice that in this case both vectors ξ_1, ξ_2 have no projection on fourth direction if calculated at $x_4 = 0$. Therefore the contour γ will stay at the plane $x_4 = 0$ if the original point belongs to it. For such a contour the tangent vector can be described by

$$\dot{x}_3 = -2\frac{\dot{x}_1(x_2 - x_1 x_3) - \dot{x}_2(x_1 - x_2 x_3)}{1 - x_1^2 - x_2^2 + x_3^2}\ . \qquad (5.89)$$

[4]This vector space can be defined as a zero eigenspace of the projector matrix $\mathbb{I}_{4\times4} - P$, where the projector P is a properly normalized combination $\mathbb{I}_{4\times4} + J^2_{4\times4}$ with $J_{4\times4}$ being the 4×4 upper-left corner part of the complex structure matrix $J^M_N(x)$.

Similarly such a contour will stay at $x_2 = 0$ if the starting point is at $x_2 = 0$ as follows from (5.89). In this case the contour will stretch in the $x_1 - x_3$ plain and will be uniquely specified by the starting point. Let us introduce a complex coordinate $z = x_1 + ix_3$. Then the contour $z(s)$ will satisfy $\dot{z} = 1 - z^2$ with the solution

$$z = \tanh(s + s_0) \ . \tag{5.90}$$

Here s is a real parameter of the contour and s_0 is the complex number that specifies the starting point $x_1 + ix_3 = \tanh(s_0)$. This contour interpolates between the points $(x_1 = \pm 1, x_3 = 0)$ and the imaginary part of s_0 specifies the maximal value of $x_3 = \tanh \mathrm{Im} s_0$.

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A Proof that $v^M \cong u^M$ is the unique solution if u^M does not vanish

Here we show that $v_M = \lambda u_M$ with some complex non-zero λ and u^μ given by (3.6) is the unique solution to (3.5) if $u^M \neq 0$. First we notice that it follows from (3.7) that $u_M u^M = 0$, i.e. u^M is a light-like vector. In Euclidean signature it means that u is necessarily complex. Let $u'_M = \mathrm{Re}\, u_M$ and $u''_M = \mathrm{Im}\, u_M$, so $u_M = u'_M + iu''_M$. Since $u^2 = 0$ we get $(u')^2 = (u'')^2$ and $u'_M u''_M = 0$. That is, the u' and the u'' are two non-zero orthogonal vectors of equal length. The two-plane in $\mathbb{R}^{10}$ spanned by u' and u'' defines breakes SO(10) to SO(8)×SO(2). Let us make SO(10) transformation so that the basis vectors 9 and 10 are aligned with u' and u'' respectively. Now we take the following representation of the ten-dimensional chiral gamma-matrices Γ_M

$$\begin{aligned} \Gamma_M &= \begin{pmatrix} 0 & E_M^T \\ E_M & 0 \end{pmatrix} , \qquad\qquad M = 1 \dots 8 , \\ \Gamma_9 &= \begin{pmatrix} 1_{8\times 8} & 0 \\ 0 & -1_{8\times 8} \end{pmatrix} , \qquad \Gamma_{10} = i \begin{pmatrix} 1_{8\times 8} & 0 \\ 0 & 1_{8\times 8} \end{pmatrix} . \end{aligned} \tag{A.1}$$

Here E_M are the gamma-matrices for SO(8). They can be also thought of as the 8×8 matrices representing left multiplication in the octonion algebra (see e.g. appendix A in [19]). These matrices satisfy the standard anticommutation relations

$$E_M E_N^T + E_N E_M^T = 2\delta_{MN} \ . \tag{A.2}$$

Since we have chosen direction 9 to be aligned with u' and direction 10 to be aligned with u'' we get

$$(\Gamma_9 + i\Gamma_{10})\varepsilon = 0 \ . \tag{A.3}$$

Written explicitly this means

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon^u \\ \varepsilon^d \end{pmatrix} = 0 \,, \tag{A.4}$$

where we represented the spinor $\varepsilon \in S^+$ of SO(10) as $\mathbf{8_s} \oplus \mathbf{8_c}$ according to the breaking $\mathrm{SO}(8) \otimes \mathrm{SO}(2) \subset \mathrm{SO}(10)$. Clearly ε_d must vanish. The equation (3.5) then splits into two parts

$$(v_9 + iv_{10})\varepsilon^u = 0 \,, \tag{A.5}$$

$$v^i E_i \varepsilon^u = 0 \,, \quad i = 1 \dots 8 \ . \tag{A.6}$$

Let us show now that there is no 8-dimensional vector v^i which would solve the equation (A.6). First we assume that such a vector exists. Then we pick up any vector p^i such that $p_i v_i \neq 0$ and multiply (A.6) by $\varepsilon^u E_i^T p_i$ from the left. Using (A.2) we get

$$(p_i v_i)(\varepsilon^u)^2 = 0 \ . \tag{A.7}$$

Since $u^9 = u' = (\varepsilon^u)^2 = \sum_{\alpha=1}^{8} \varepsilon^u_\alpha \varepsilon^u_\alpha$ is non-zero we get a contradiction. Therefore $v_M \cong u_M$ is the only solution to (3.5) if u^M is not zero.

B Proof that a non-trivial pure spinor hypersurface requires zero $\mathtt{w}$ and decomposable $\mathtt{m}$

We will get the result in several steps. First we show that a nontrivial Σ requires $\mathtt{w} = 0$. We start with the equation (4.22)

$$(1 + 2i_x v) \wedge (\xi \wedge \mathtt{m} + 2i_x \mathtt{w}) = \frac{1}{2}(\xi \wedge \mathtt{v} + 2i_x \mathtt{m})^2 \ . \tag{B.1}$$

Our assumption is that Σ is a non-trivial smooth manifold passing through the origin. Therefore we can expand the (B.1)up to the linear level in x

$$\xi \wedge \mathtt{m} + 2i_x \mathtt{w} = 0 \ . \tag{B.2}$$

Here ξ and x are in the tangent space of Σ at the origin.

Now we multiply (B.2) by ξ to get

$$2\xi \wedge i_x \mathtt{w} = 0 \,, \tag{B.3}$$

and then rewrite it as

$$2\xi \wedge i_x \mathtt{w} = 2i_x \xi \mathtt{w} - 2i_x \xi \wedge \mathtt{w} = 2(x, \xi)\mathtt{w} = 0 \ . \tag{B.4}$$

Here we have used that $\mathtt{w}$ is a form of the top degree and hence $\xi \wedge \mathtt{w} = 0$ for any ξ. Now, for real $x^\mu \neq 0$ we have

$$2i_x\xi \equiv 2(x,\xi) = 2x^{\bar{I}} g_{\bar{I}J} x^J = x^M x_M > 0\,, \tag{B.5}$$

and therefore $\mathtt{w} = 0$.

The analysis of the constraint (4.24) in section (4.2) yielding $\mathtt{v} \wedge \xi \wedge \mathtt{m} = 0$ did not assume that $\mathtt{m}$ is decomposable. Using this we can rewrite the constraint (4.22) as (compare with (4.36))

$$\begin{aligned}(\xi + 2(x,\xi)\mathtt{v}) \wedge \mathtt{m} &= 2(i_x\mathtt{m}) \wedge (i_x\mathtt{m}) \\ &= 2i_x(\mathtt{m} \wedge i_x\mathtt{m})\ .\end{aligned} \tag{B.6}$$

If we multiply both sides by ξ we get zero in the left hand side implying

$$0 = 2(x,\xi)(\mathtt{m} \wedge i_x\mathtt{m})\,, \tag{B.7}$$

because the six form $\xi \wedge \mathtt{m} \wedge i_x\mathtt{m} = 0$ exceeds the dimension of the space. As a result we have

$$\mathtt{m} \wedge i_x\mathtt{m} = 0. \tag{B.8}$$

and (B.6) reduces to (4.36).

The equation (4.36) actually implies that $\mathtt{m}$ is decomposable if Σ is non-trivial. To show that, we introduce an antisymmetric matrix (bi-vector) $\hat{\mathtt{m}}$ as follows

$$\hat{\mathtt{m}}^{i_4 i_5} = \frac{1}{3!}\epsilon^{i_1 i_2 i_3 i_4 i_5}\mathtt{m}_{i_1 i_2 i_3}\,, \tag{B.9}$$

and reinterpret the equation (4.36) in a way that vector $\xi + 2(x,\xi)\mathtt{v}$ is a zero vector of matrix $\hat{\mathtt{m}}$. Clearly a non-zero antisymmetric matrix $\hat{\mathtt{m}}$ must have at least one zero vector although there could be three ones if $\mathtt{m}$ is decomposable. The simplest way to understand it is to bring $\hat{\mathtt{m}}$ to the canonical form by an appropriate SU(5) transformation

$$\hat{\mathtt{m}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu' & 0 & 0 \\ 0 & -\mu' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & -\mu & 0 \end{pmatrix} \tag{B.10}$$

in the new coordinate basis $z_1, \ldots, z_5$. If $\hat{\mathtt{m}}$ has only one zero vector (i.e. both μ and μ' are non-zero) the equation (4.36) requires vector $\xi + 2(x,\xi)\mathtt{v}$ to be aligned with the direction z_1 while the equation (B.8) which can be rewritten as

$$\hat{\mathtt{m}} \wedge \hat{\mathtt{m}} \wedge x = 0\,, \tag{B.11}$$

requires vector x to have zero projection on that direction. Therefore the contraction of x and $\xi + 2(x,\xi)\mathtt{v}$ would give zero

$$0 = i_x(\xi + 2(x,\xi)\mathtt{v}) = (x,\xi)(1 + 2(x,v))\ . \tag{B.12}$$

Hence for any non-zero x on Σ we have

$$1 + 2(x, v) = 0 \ . \tag{B.13}$$

This equation does not have solutions for x being arbitrary close to 0, and therefore there could be no nontrivial Σ passing through $x = 0$.

To resolve the contradiction we have to assume that $\mu' = 0$ and therefore $\mathtt{m}$ is decomposable

$$\mathtt{m} = \mu \, d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \ . \tag{B.14}$$

In this case the equation (4.32) is automatically satisfied for any x and the only remaining equation on Σ is (4.36).

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