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Nekrasov functions and exact Bohr-Sommerfeld integrals

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ABSTRACT: In the case of $SU(2)$, associated by the AGT relation to the $2d$ Liouville theory, the Seiberg-Witten prepotential is constructed from the Bohr-Sommerfeld periods of $1d$ sine-Gordon model. If the same construction is literally applied to monodromies of exact wave functions, the prepotential turns into the one-parametric Nekrasov prepotential $\mathcal{F}(a, \epsilon_1)$ with the other epsilon parameter vanishing, $\epsilon_2 = 0$, and ϵ_1 playing the role of the Planck constant in the sine-Gordon Schrödinger equation, $\hbar = \epsilon_1$. This seems to be in accordance with the recent claim in [1] and poses a problem of describing the full Nekrasov function as a seemingly straightforward double-parametric quantization of sine-Gordon model. This also provides a new link between the Liouville and sine-Gordon theories.

KEYWORDS: Supersymmetric gauge theory, Integrable Hierarchies, Topological Field Theories

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1 Introduction

The AGT conjecture [2], which is now explicitly checked and even proved in various particular cases and limits [3]–[22], provides a new prominent role for the Nekrasov functions [23]. Originally they appeared in description of regularized integrals over moduli spaces of ADHM instantons [24, 25], but now it is getting clear that they provide a clever generalization of hypergeometric series [8], and thus can serve as a new class of special functions, closely related to matrix model τ -functions [26–35]. This is a dramatic extension of the original role of the Nekrasov functions and this means that they should be thoroughly investigated within the general context of group and integrability theory, without any references to particular constructions like moduli spaces and graviphoton backgrounds. There are several directions in which such study can be performed. In the present paper we consider a possible way to embed the Nekrasov functions into the context of Seiberg-Witten (SW) theory [36]–[48], as suggested by N.Nekrasov and S.Shatashvili in [1]. We concentrate on the case of SU(2) gauge group, where Seiberg-Witten theory [36] and its relation to quantum mechanical integrable systems [38] looks especially simple. This allows one to formulate the claim of [1] (as we understand it) in a very clear and transparent way, what makes it understandable to non-experts in integrability theory.

According to [38], the SW prepotential [37] for the pure gauge SU(2) $\mathcal{N} = 2$ SUSY theory is defined by the 1d sine-Gordon quantum model

$$S = \int \left(\frac{1}{2} \dot{\phi}^2 - \Lambda^2 \cos \phi \right) dt \tag{1.1}$$

in the following way: construct the Bohr-Sommerfeld periods

$$\Pi^{(0)}(C) = \oint_C \sqrt{2(E - \Lambda^2 \cos \phi)} d\phi \tag{1.2}$$

for two complementary contours $C = A$ and $C = B$, encircling the two turning points $\pm\phi_0$, $E = \gamma \cos \phi_0$, $\gamma \equiv \Lambda^2$. Then, the SW prepotential $\mathcal{F}^{(0)}(a)$ is defined from a pair of equations

$$\begin{aligned} a &= \Pi^{(0)}(A), \\ \frac{\partial \mathcal{F}^{(0)}(a)}{\partial a} &= \Pi^{(0)}(B) \end{aligned} \tag{1.3}$$

after excluding E . In the case of $SU(2)$ with a single modulus a , there is no consistency condition for these equations to be resolvable, however, it is slightly non-trivial that this construction is directly generalized to higher-rank groups [39, 40].

The Bohr-Sommerfeld (BS) integrals are known to describe the quasiclassical approximation $E^{(0)}$ to the eigenvalues E of the Shrödinger equation

$$\left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial \phi^2} + \Lambda^2 \cos \phi \right) \Psi(\phi) = E \Psi(\phi) \tag{1.4}$$

by solving the equation

$$\Pi^{(0)}(A) = 2\pi\hbar \left(n + \frac{1}{2} \right) \tag{1.5}$$

with respect to E . The exact eigenvalues E are defined from a similar equation [52–56]

$$\Pi(A) = 2\pi\hbar \left(n + \frac{1}{2} \right), \tag{1.6}$$

where the exact BS periods are

$$\Pi(C) = \oint_C P(\phi) d\phi \tag{1.7}$$

and $P(\phi)$ is an exact solution to the Shrödinger equation (1.4),

$$\Psi(\phi) = \exp \left(\frac{i}{\hbar} \int^\phi P(\phi) d\phi \right) \tag{1.8}$$

One can define the exact (quantized) prepotential $\mathcal{F}(a|\hbar)$ from the same system (1.3)

$$\begin{cases} a = \Pi(A), \\ \frac{\partial \mathcal{F}(a|\hbar)}{\partial a} = \Pi(B) \end{cases} \tag{1.9}$$

with $\Pi^{(0)}$ substituted by the exact (quantized) periods Π . Again, in the case of $SU(2)$ there is no problem of consistency (resolvability) of this system.¹

The claim of [1] is that this $\mathcal{F}(a)$ is the $\epsilon_2 = 0$ limit of the Nekrasov function,

$$\mathcal{F}(a|\epsilon_1) = \lim_{\epsilon_2 \rightarrow 0} \left\{ \epsilon_1 \epsilon_2 \log Z(a, \epsilon_1, \epsilon_2) \right\} \tag{1.10}$$

so that ϵ_1 plays the role of the Planck constant \hbar in (1.4). The SW prepotential *per se* is [23]

$$\mathcal{F}^{(0)}(a) = \mathcal{F}(a|\epsilon_1 = 0) \tag{1.11}$$

¹ Note that, in variance with the quasiclassical pdx , the exact Pdx is not a SW differential: its a -derivative is not holomorphic on the spectral Riemann surface, moreover, there is no smooth spectral surface anymore at all.

Nota bene: the deformation $\mathcal{F}^{(0)} \rightarrow \mathcal{F}$ is different from old Nekrasov’s quantization [23] of the SW prepotential, in the direction $\epsilon_2 = -\epsilon_1$, which is AGT-related to conformal models with integer central charge $c = \text{rank}$, i.e. with $c = 1$ in the $SU(2)/\text{Virasoro}$ case. The latter deformation, associated with a background of self-dual graviphoton, is a full topological partition function lifting $\mathcal{F}^{(0)}$ from zero to arbitrary genus [15]. It is a τ -function which is known to play a nice role in combinatorics of symmetric groups [49–51], but still lacks any nice description in the simple terms of the sine-Gordon system (1.1). Such a description is now found for the alternative deformation in the direction of ϵ_1 , while $\epsilon_2 = 0$. The two-parameter (ϵ_1, ϵ_2) deformation of the SW prepotential, providing the full Nekrasov function $Z(a, \epsilon_1, \epsilon_2)$ should be related to a further, double-loop (elliptic or p -adic?), quantization of the same sine-Gordon system.

Our goal in this letter is to demonstrate that (1.9) and (1.10) are, indeed, correct by calculating the first orders of \hbar expansion of the exact BS periods and comparing them with the ϵ_1 -expansion of the Nekrasov prepotential $\mathcal{F}(a|\epsilon_1)$ in the simplest case of $SU(2)$ theory. We begin in section 2 with extracting $\mathcal{F}(a|\epsilon_1)$ from the Nekrasov functions. We actually prefer to use the already established AGT relation to describe $\mathcal{F}(a|\epsilon_1)$ in terms of the Shapovalov matrix for the Virasoro algebra: this is a nice and clear representation, which can be also used for other purposes. Then in section 3 we remind the old WKB construction [52–56] of the exact (quantized) BS periods, which provides exact eigenvalues of the Shrödinger equation, i.e. describe corrections to the quasiclassical BS quantization rule, which we realize by action of differential operators. In section 4 this construction is applied to the case of sine-Gordon potential $V(\phi) = \Lambda^2 \cos \phi$. The simplest way to calculate the BS periods is with the help of the Picard-Fucks equations [42–44, 57], and corrections are also obtained by action of peculiar differential operators. Prepotential, defined from these corrected periods by the SW rule, (1.9) does indeed coincide with $\mathcal{F}(a|\epsilon_1)$ from section 2, provided one identifies $\hbar = \epsilon_1$. The last section 5 contains a short summary and discussion. Accurate proofs and numerous generalizations are left beyond this letter to make presentation as clear as possible, they are relatively straightforward and will be considered elsewhere.

2 One parametric prepotential $\mathcal{F}(a|\epsilon_1)$ and Shapovalov matrix

The Nekrasov partition function for the $SU(2)$ pure gauge theory possesses the nice group-theoretical description

$$Z_{SU(2)}^{\text{inst}}(a, \epsilon_1, \epsilon_2) = \sum_{n=0}^{\infty} \frac{\Lambda^{4n}}{(\epsilon_1 \epsilon_2)^{2n}} Q_{\Delta}^{-1}([1^n], [1^n]) \tag{2.1}$$

where the Shapovalov matrix Q is defined for a generic (non-degenerate) Verma module of the Virasoro algebra with the central charge c and the highest weight state V_{Δ} . Its elements are labeled by pairs of Young diagrams and given by

$$Q_{\Delta}(Y, Y') = \langle L_{-Y} V_{\Delta} | L_{-Y'} V_{\Delta} \rangle = \langle V_{\Delta} | L_Y L_{-Y'} V_{\Delta} \rangle \tag{2.2}$$

L_{-Y} is an ordered monomial made from negative Virasoro operators, while the dimension and central charge are given by the AGT rule

$$\Delta = \frac{1}{\epsilon_1 \epsilon_2} \left(a^2 - \frac{\epsilon^2}{4} \right), \quad c = 1 - \frac{6\epsilon^2}{\epsilon_1 \epsilon_2}, \quad \epsilon = \epsilon_1 + \epsilon_2 \quad (2.3)$$

$Q_\Delta(Y, Y')$ has a block form, it does not vanish only when the two diagrams have the same size (number of boxes), $|Y| = |Y'|$. Eq. (2.1) can be obtained as the large-mass limit of the four-fundamentals AGT formula [2, 5], see [6, 14] or, alternatively, as the large- M limit of the adjoint AGT formula, associated with the toric 1-point function [22]

$$Z_{\text{adj}}^{\text{inst}}(a, M, \epsilon_1, \epsilon_2) = \sum_{Y, Y'} x^{|Y|} Q_\Delta^{-1}(Y, Y') \langle L_{-Y} V_\Delta | L_{-Y'} V_\Delta(0) V_{\Delta_{\text{ext}}}(1) \rangle \quad (2.4)$$

with $\Delta_{\text{ext}} = (M^2 - \frac{\epsilon^2}{4})/(\epsilon_1 \epsilon_2)$. This formula (but not its large- M limit) was recently considered in [18].

In the limit of $\epsilon_2 \rightarrow 0$ both the dimension Δ and the central charge c tend to infinity, however, the singularities are nicely combined and exponentiated, so that [23]

$$Z_{\text{SU}(2)}(a, \epsilon_1, \epsilon_2) = \exp \left(\frac{\mathcal{F}(a, \epsilon_1, \epsilon_2)}{\epsilon_1 \epsilon_2} \right) \quad (2.5)$$

where $\mathcal{F}(a, \epsilon_1, \epsilon_2)$ remains finite when $\epsilon_1 \rightarrow 0$ or $\epsilon_2 \rightarrow 0$.

Substituting explicit expressions for

$$\begin{aligned} Q_\Delta^{-1}(1, 1) &= \frac{1}{2\Delta} \quad - \text{ does not depend on } c, \\ Q_\Delta^{-1}(11, 11) &= \frac{8\Delta + c}{4\Delta(16\Delta^2 + 2c\Delta - 10\Delta + c)}, \\ Q_\Delta^{-1}(111, 111) &= \frac{24\Delta^2 + 11c\Delta + c^2 - 26\Delta + 8c}{24\Delta(16\Delta^2 + 2c\Delta - 10\Delta + c)(3\Delta^2 + c\Delta - 7\Delta + c + 2)}, \\ &\dots \end{aligned} \quad (2.6)$$

one easily obtains for the first terms of the Λ -expansion of the instanton part of $\mathcal{F}(a|\epsilon_1) \equiv \mathcal{F}(a, \epsilon_1, \epsilon_2 = 0) = \mathcal{F}^{\text{class}}(a|\epsilon_1) + \mathcal{F}^{\text{pert}}(a|\epsilon_1) + \mathcal{F}^{\text{inst}}(a|\epsilon_1)$:

$$\begin{aligned} \mathcal{F}^{\text{inst}}(a|\epsilon_1) &= \frac{\Lambda^4}{2\tilde{\Delta}} + \frac{\Lambda^8(10\tilde{\Delta} + 6\epsilon_1^2)}{16\tilde{\Delta}^3(8\tilde{\Delta} - 6\epsilon_1^2)} + \dots \\ &= \left(\frac{\Lambda^4}{2a^2} + \frac{5\Lambda^8}{64a^6} + \dots \right) + \epsilon_1^2 \left(\frac{\Lambda^4}{8a^4} + \frac{21\Lambda^8}{128a^8} + \dots \right) + O(\epsilon_1^4) \end{aligned} \quad (2.7)$$

where $\tilde{\Delta}$ denotes the rescaled $\Delta \rightarrow \epsilon_1 \epsilon_2 \Delta$.

It would also be interesting to describe $\mathcal{F}^{\text{inst}}(a|\epsilon_1)$ by taking the $\epsilon_2 \rightarrow 0$ limit of coherent state [6, 14], which provides an alternative description of the pure gauge theory

$$Z_{\text{SU}(2)}^{\text{inst}}(a, \epsilon_1, \epsilon_2) = \langle \Lambda^2, \Delta | \Lambda^2, \Delta \rangle \quad (2.8)$$

which satisfies

$$\begin{aligned}
 L_0|\Lambda^2, \Delta \rangle &= \Delta|\Lambda^2, \Delta \rangle, \\
 L_1|\Lambda^2, \Delta \rangle &= \Lambda^2|\Lambda^2, \Delta \rangle, \\
 L_{k \geq 2}|\Lambda^2, \Delta \rangle &= 0
 \end{aligned}
 \tag{2.9}$$

According to [1, 23], the perturbative contribution to $\mathcal{F}(a|\epsilon_1)$ is defined from its a -derivative,

$$\frac{\partial \mathcal{F}^{\text{pert}}}{\partial a} = -2\epsilon_1 \log \left(\frac{\Gamma(1+z)}{\Gamma(1-z)} \right)
 \tag{2.10}$$

where $z = 2a/\epsilon_1$. Making use of large- z asymptotics of the Γ -function,

$$\log \Gamma(z+1) = \log z + \log \Gamma(z) = (z+1/2) \log z - z + \frac{1}{2} \log(2\pi) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)z^{2m-1}}
 \tag{2.11}$$

so that (... denotes here inessential terms)

$$\log \Gamma(z+1) - \log \Gamma(1-z) = 2z \left(\log z - 1 + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)z^{2m}} \right) + \dots
 \tag{2.12}$$

one obtains ($\mathcal{F}^{\text{class}}$ is the classical quadratic part of the prepotential)

$$\begin{aligned}
 - \frac{\partial (\mathcal{F}^{\text{class}} + \mathcal{F}^{\text{pert}})}{\partial a} &= 8a \log \frac{\epsilon_1}{\Lambda} + 2\epsilon_1 \log \Gamma \left(1 + \frac{2a}{\epsilon_1} \right) - 2\epsilon_1 \log \Gamma \left(1 - \frac{2a}{\epsilon_1} \right) \\
 &= 8a \left(\log \frac{2a}{\Lambda} - 1 \right) + 8a \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} \left(\frac{\epsilon_1}{2a} \right)^{2m} + \dots
 \end{aligned}
 \tag{2.13}$$

so that

$$\begin{aligned}
 \mathcal{F}(a|\epsilon_1) &= \mathcal{F}^{\text{class}}(a|\epsilon_1) + \mathcal{F}^{\text{pert}}(a|\epsilon_1) + \mathcal{F}^{\text{inst}}(a|\epsilon_1) \\
 &= (\text{quadratic part in } a) - 4a^2 \log \frac{a}{\Lambda} - \frac{\epsilon_1^2}{6} \log a \\
 &\quad - \left(\frac{\Lambda^4}{2a^2} + \frac{5\Lambda^8}{64a^6} + \dots \right) - \epsilon_1^2 \left(\frac{\Lambda^4}{8a^4} + \frac{21\Lambda^8}{128a^8} + \dots \right) + O(\epsilon_1^4)
 \end{aligned}
 \tag{2.14}$$

Note that only even powers of ϵ_1 appear in this formula.

Our goal in this paper is to provide an alternative description of this $\mathcal{F}(a|\epsilon_1)$ in terms of SW-like relation (1.9), where $\Pi(C)$ are the exact BS periods (monodromies of the exact wave function) of the 0 + 1 dimensional sine-Gordon model.

3 Exact eigenvalues from quantized BS periods (monodromies)

Spectrum of the Shrödinger operator $-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$ is defined in the quasiclassical approximation by the BS quantization rule

$$\oint p_E(x) dx = \oint \sqrt{2m(E - V(x))} dx = 2\pi\hbar \left(n + \frac{1}{2} \right)
 \tag{3.1}$$

In fact, WKB theory allows one to calculate arbitrary corrections to the quasiclassical approximation, up to any desired power in \hbar . Remarkably, exact E is defined by the same quantization rule,

$$\oint P_E(x) dx = 2\pi\hbar \left(n + \frac{1}{2} \right), \quad (3.2)$$

only $p_E(x)$ should be substituted by $P_E(x)$, where

$$\Psi_E(x) = \exp \left(\frac{i}{\hbar} \int^x P_E dx \right) \quad (3.3)$$

is the exact solution of the stationary Shrödinger equation,

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Psi_E(x) = E \Psi_E(x) \quad (3.4)$$

In what follows we omit index E from $p_E(x)$ and $P_E(x)$ to avoid further overloading formulas.

For thorough discussion of the quantization rule (3.2) see [52–56], it can be justified by analysis of the Stokes phenomenon and by study of the Airy function asymptotic of $\Psi_E(x)$ in the vicinity of the turning points. An advantage of this formula is that the \hbar series for $P(x)$ is constructed by a simple iteration: substituting $P(x) = \sum_{k=0}^{\infty} \hbar^k P_k(x)$, into $i\hbar P' = P^2 - p^2$, one gets

$$\begin{aligned} P_0(x) &= p(x) = \sqrt{2m(E - V)}, \\ P_1(x) &= -i \frac{V'}{4(E - V)} = \frac{i}{4} \left[\log(E - V) \right]' = \frac{i}{2} [\log P_0]', \\ P_2(x) &= \frac{1}{32} \frac{5V'^2 + 4V''(E - V)}{\sqrt{2m(E - V)}^{5/2}}, \\ P_3(x) &= \frac{i}{64} \frac{4V'''(E - V)^2 + 18V'V''(E - V) + 15V'^2}{2m(E - V)^4} = i \left[\frac{P_2}{2P_0} \right]', \\ &\dots \end{aligned} \quad (3.5)$$

In what follows we put $m = 1$.

The energy levels are defined by the *exact* Bohr-Sommerfeld rule

$$\oint P(x) dx = 2\pi\hbar(n + 1/2) \quad (3.6)$$

i.e.

$$\begin{aligned} \Pi &= \frac{1}{\sqrt{2}} \oint P(x) dx \\ &= \oint \sqrt{E - V} dx - \frac{\hbar^2}{64} \oint \frac{V'^2 dx}{(E - V)^{5/2}} - \frac{\hbar^4}{8192} \oint \left(\frac{49V'^4}{(E - V)^{11/2}} - \frac{16V'V'''}{(E - V)^{7/2}} \right) dx + \dots \end{aligned} \quad (3.7)$$

can be considered as a quantum deformation of the quasiclassical periods $\Pi^{(0)} = \oint \sqrt{E - V} dx$. To simplify formulas, hereafter we divide periods by $\sqrt{2}$.

In this formula integration by parts is allowed and, therefore, it looks simpler than (3.5). Only \hbar^{2k} corrections survive (\hbar and \hbar^3 are indeed absent). This fact will be important to match the absence of odd powers of ϵ_1 in (2.14).

The \hbar^2 -term can be alternatively represented as

$$\begin{aligned} \Pi^{(2)} &= -\frac{\hbar^2}{64} \oint \frac{V'^2 dx}{(E-V)^{5/2}} = +\frac{\hbar^2}{96} \oint \frac{V'' dx}{(E-V)^{3/2}} \\ &= -\frac{\hbar^2}{96} \oint \frac{\gamma \cos \phi d\phi}{(E-\gamma \cos \phi)^{3/2}} = -\frac{\gamma}{24} \partial_{E\gamma}^2 \oint \sqrt{E-\gamma \cos \phi} d\phi = -\frac{\hbar^2 \gamma}{24} \partial_{E\gamma}^2 \Pi^{(0)} \end{aligned} \quad (3.8)$$

where in the second line we substituted the concrete potential of the sine-Gordon model, $V(\phi) = \gamma \cos \phi$.

Similarly, integrating by parts one can rewrite the \hbar^4 -term as

$$\begin{aligned} \Pi^{(4)} &= -\frac{\hbar^4}{3 \cdot 2048} \oint \left(\frac{7V''^2}{(E-V)^{7/2}} + \frac{2V''''}{(E-V)^{5/2}} \right) \\ &\stackrel{V=\gamma \cos \phi}{=} \frac{9\hbar^4}{128} \gamma \left(-\frac{2}{5} E \partial_E + \gamma \partial_\gamma \right) \partial_E^2 \partial_\gamma \oint \sqrt{E-\gamma \cos \phi} d\phi \end{aligned} \quad (3.9)$$

4 Quantum corrections to BS periods in the sine-Gordon case

4.1 Picard-Fuchs equation [42–44, 57]

The simplest way to evaluate the periods is to make use of the Picard-Fuchs equation [42–44, 57]

$$\left(\gamma(\partial_E^2 + \partial_\gamma^2) + 2E\partial_{E\gamma}^2 \right) \Pi^{(0)} = 0 \quad (4.1)$$

This equation follows from the simple fact:

$$\begin{aligned} \left(\gamma(\partial_E^2 + \partial_\gamma^2) + 2E\partial_{E\gamma}^2 \right) \sqrt{E-\gamma \cos \phi} d\phi &= \frac{2E \cos \phi - \gamma(1 + \cos^2 \phi)}{4(E-\gamma \cos \phi)^{3/2}} d\phi \\ &= d \left(\frac{\sin \phi}{2\sqrt{E-\gamma \cos \phi}} \right) \end{aligned} \quad (4.2)$$

We need to construct the two solutions of this equation with asymptotics $\sqrt{2E} + O(\gamma)$ and $\sqrt{2E} \log(E/\gamma) + O(\gamma)$. Since $E^{\frac{1}{2}+\epsilon} = \sqrt{E} \left(1 + \epsilon \log E + O(\epsilon^2) \right)$ both periods can be obtained simultaneously, by substituting into (4.1) the formal series

$$\Pi_\epsilon^{(0)} = \Pi^{(0)} + \epsilon \Pi^{(0)'} + O(\epsilon^2) = \sqrt{2E} E^{\frac{1}{2}+\epsilon} \left(1 + \sum_{n>0} s_n \left(\frac{\gamma}{E} \right)^{2n} \right) \quad (4.3)$$

which provides a recursion relation

$$s_{n+1} = \frac{\left(n + \frac{1}{4} - \frac{\epsilon}{2} \right) \left(n + \frac{1}{4} - \frac{\epsilon}{2} \right)}{(n+1)(n+1-\epsilon)} s_n = \left(\frac{n^2 - \frac{1}{16}}{(n+1)^2} - \frac{n + \frac{1}{16}}{(n+1)^3} \epsilon + O(\epsilon^2) \right) s_n \quad (4.4)$$

and

$$\begin{aligned} \Pi_\varepsilon^{(0)} = & \sqrt{2E} \left(1 - \frac{1}{16} \left(\frac{\gamma}{E} \right)^2 - \frac{15}{2^{10}} \left(\frac{\gamma}{E} \right)^4 + \dots \right) + \\ & + \varepsilon \left\{ \sqrt{2E} \log E \left(1 - \frac{1}{16} \left(\frac{\gamma}{E} \right)^2 - \frac{15}{2^{10}} \left(\frac{\gamma}{E} \right)^4 + \dots \right) \right. \\ & \left. - \sqrt{2E} \left(\frac{1}{16} \left(\frac{\gamma}{E} \right)^2 + \frac{13}{2^{11}} \left(\frac{\gamma}{E} \right)^4 + \dots \right) \right\} + O(\varepsilon^2) \quad (4.5) \end{aligned}$$

According to Seiberg-Witten theory, we identify $\Pi^{(0)} = a$, $\Pi^{(0)'} = 1/4\partial F_{\text{SW}}(a)/\partial a$. It follows that

$$\sqrt{2E} = a \left(1 + \frac{1}{4} \left(\frac{\gamma}{a^2} \right)^2 + \frac{3}{64} \left(\frac{\gamma}{a^2} \right)^4 + \dots \right) \quad (4.6)$$

and substituting this into $\Pi^{(0)'}$, we get:

$$-\frac{1}{4} \frac{\partial F_{\text{SW}}(a)}{\partial a} = \Pi^{(0)'} = 2a \log a + \frac{1}{4} a \left(\left(\frac{\gamma}{a^2} \right)^2 + \frac{15}{32} \left(\frac{\gamma}{a^2} \right)^4 + \dots \right) \quad (4.7)$$

i.e.

$$F_{\text{SW}}(a) = -4a^2(\log a + \text{const}) - \frac{\gamma^2}{2a^2} - \frac{5\gamma^4}{64a^6} + \dots \quad (4.8)$$

This is a well-known formula in Seiberg-Witten theory. Since $\gamma = \Lambda^2$, one sees that it is in accordance with the ε_1 -independent term in formula (2.14) obtained entirely within conformal field theory, from the 1-point toric conformal block.

4.2 Prepotential in the order \hbar^2

According to (3.8), acting by the operator $\left(1 - \frac{\hbar^2 \gamma}{24} \partial_{E\gamma}^2 \right)$ on $\Pi^{(0)}$, one obtains

$$\begin{aligned} \Pi_\varepsilon^{(0)} + \Pi_\varepsilon^{(2)} = & \left(1 - \frac{\hbar^2 \gamma}{24} \partial_{E\gamma}^2 \right) \left[\sqrt{2E} \left(1 - \frac{1}{16} \left(\frac{\gamma}{E} \right)^2 - \frac{15}{2^{10}} \left(\frac{\gamma}{E} \right)^4 + \dots \right) + \right. \\ & + \varepsilon \left\{ \sqrt{2E} \log \frac{E}{\gamma} \left(1 - \frac{1}{16} \left(\frac{\gamma}{E} \right)^2 - \frac{15}{2^{10}} \left(\frac{\gamma}{E} \right)^4 + \dots \right) \right. \\ & \left. \left. - \sqrt{2E} \left(\frac{1}{16} \left(\frac{\gamma}{E} \right)^2 + \frac{13}{2^{11}} \left(\frac{\gamma}{E} \right)^4 + \dots \right) \right\} + O(\varepsilon^2) \right] \\ = & \sqrt{2E} \left(1 - \frac{1}{16} \left(\frac{\gamma}{E} \right)^2 \left(1 + \frac{\hbar^2}{8E} \right) - \frac{15}{2^{10}} \left(\frac{\gamma}{E} \right)^4 \left(1 + \frac{7\hbar^2}{12E} \right) + \dots \right) + \\ & + \varepsilon \left\{ \sqrt{2E} \log \frac{E}{\gamma} \left(1 - \frac{1}{16} \left(\frac{\gamma}{E} \right)^2 \left(1 + \frac{\hbar^2}{8E} \right) - \frac{15}{2^{10}} \left(\frac{\gamma}{E} \right)^4 \left(1 + \frac{7\hbar^2}{12E} \right) + \dots \right) - \right. \\ & - \sqrt{2E} \left(\frac{1}{16} \left(\frac{\gamma}{E} \right)^2 \left(1 + \frac{\hbar^2}{8E} \right) + \frac{13}{2^{11}} \left(\frac{\gamma}{E} \right)^4 \left(1 + \frac{7\hbar^2}{12E} \right) + \dots \right) + \\ & + \frac{\hbar^2 \sqrt{2E}}{24E} \left(\frac{2}{16} \left(\frac{\gamma}{E} \right)^2 + \frac{15}{2^8} \left(\frac{\gamma}{E} \right)^4 + \dots \right) \\ & \left. + \frac{\hbar^2 \sqrt{2E}}{48E} \left(1 + \frac{3}{16} \left(\frac{\gamma}{E} \right)^2 + \frac{105}{2^{10}} \left(\frac{\gamma}{E} \right)^4 + \dots \right) \right\} + O(\varepsilon^2) \quad (4.9) \end{aligned}$$

The two terms in the last line come from differentiation of $\log E$ and $\log \gamma$ respectively.

Expressing E through a by solving the equation $\Pi = a$ and substituting the result into Π' one obtains instead of (4.6) and (4.7)

$$\sqrt{2E} = a \left\{ 1 + \frac{1}{4} \left(\frac{\gamma}{a^2} \right)^2 \left(1 + \frac{\hbar^2}{4a^2} \right) + \frac{3}{64} \left(\frac{\gamma}{a^2} \right)^4 + \frac{19\hbar^2}{128a^2} \left(\frac{\gamma}{a^2} \right)^4 + O(\gamma^6, \hbar^4) \right\} \quad (4.10)$$

and

$$\begin{aligned} -\frac{1}{4} \frac{\partial F(a, \hbar)}{\partial a} &= \Pi^{(0)'} & (4.11) \\ &= 2a \log a + \frac{\hbar^2}{24a} + \frac{1}{4} a \left\{ \left(\frac{\gamma}{a^2} \right)^2 \left(1 + \frac{\hbar^2}{2a^2} \right) + \frac{15}{32} \left(\frac{\gamma}{a^2} \right)^4 + \frac{21\hbar^2}{16} \left(\frac{\gamma}{a^2} \right)^4 + \dots \right\} \end{aligned}$$

and finally, instead of (4.8),

$$F(a, \hbar) = -4a^2(\log a + \text{const}) - \frac{\hbar^2}{6} \log a - \frac{\gamma^2}{2a^2} - \frac{5\gamma^4}{64a^6} - \frac{\hbar^2\gamma^2}{8a^4} - \frac{21\hbar^2\gamma^4}{128a^8} + O(\gamma^6, \hbar^4) \quad (4.12)$$

This reproduces (2.14), provided one identifies $\gamma = \Lambda^2$ and $\hbar = \epsilon_1$.

4.3 Prepotential in the order \hbar^4

In this order we perform calculation only for the γ -independent terms in $\mathcal{F}(a|\epsilon_1)$. Such contributions come from the action on $\sqrt{2E} \log \frac{2E}{\gamma}$ in $\Pi^{(0)}$:

$$\begin{aligned} &\left\{ 1 - \frac{\hbar^2}{24} \gamma \frac{\partial^2}{\partial E \partial \gamma} + \frac{\hbar^4}{9 \cdot 128} \gamma \left(-\frac{2}{5} E \partial_E + \gamma \partial_\gamma \right) \frac{\partial^3}{\partial E^2 \partial \gamma} + \dots \right\} \sqrt{2E} \log \frac{2E}{\gamma} \\ &= \sqrt{2E} \left(\log \frac{2E}{\gamma} + \frac{\hbar^2}{48E} - \frac{\hbar^4}{32 \cdot 360E^2} + \dots \right) \\ &= 2\sqrt{2E} \left(\log \sqrt{\frac{2E}{\gamma}} + \frac{B_2}{2} \left(\frac{\hbar^2}{8E} \right) + \frac{B_4}{12} \left(\frac{\hbar^4}{8E} \right)^2 + O(\gamma^2, \hbar^6) \right) \end{aligned} \quad (4.13)$$

with $B_2 = 1/6$, $B_4 = -1/30$. This is again in agreement with (2.14) if $\frac{\hbar^2}{8E} = \frac{\epsilon_1^2}{(2a)^2} + O(\gamma)$. Since $2E = a^2 + O(\gamma)$ this implies that $\hbar = \epsilon_1$.

This completes our simple test of the claim (1.10).

5 Conclusion

In this letter we explicitly demonstrated that the deformation from Seiberg-Witten prepotential to the Nekrasov function continues to be described by the integrable system approach suggested in [38], at least, for a 1-parametric deformation to arbitrary ϵ_1 , with $\epsilon_2 = 0$. The deformed prepotential $\mathcal{F}(a|\epsilon_1)$ is given by exactly the same SW rule (1.9), only the SW differential pdx is deformed into the exact "quantum" differential Pdx . The main open question is what should be done with the same description when both ϵ_1 and ϵ_2 are non-zero. This generalization should be similar to the next, "elliptic" deformations of quantum groups, which are themselves deformations of the ordinary universal enveloping

algebras. As usual [58, 59], one could expect that such deformations will be also related to double loop algebras and to p -adic analysis.

There is also a number of technical questions at the level of $\mathcal{F}(a|\epsilon_1)$.

First of all, even in the case of $SU(2)$ we presented only the lowest terms of ϵ_1 expansion, one can look at generic terms and find a general proof of the statement.

Second, it can be easily generalized to other $SU(2)$ examples, that is, from the sine-Gordon to Calogero-Ruijenaars and magnetic systems, which are integrable system counterparts of various gauge theories under the GKMMM-DW correspondence of [38] and [41].

Third, one should sum up the \hbar series for Pdx , at least, conceptually, e.g. interpret deformed differential Pdx as a SW differential on a deformed (quantized) spectral curve.

Forth, generalization to higher rank groups requires a more detailed analysis. A piece of such analysis is presented in [1] in terms of advanced integrability theory *a la* [60–63], but it is desirable to convert it into a much simpler form, close to the one in the present paper.² The simplest option is to construct an \hbar -deformed SW differential and define $\mathcal{F}(a|\epsilon_1)$ from the system (1.9) with $2 \times \text{rank}$ different periods. For example, in the case of $SU(N)$ gauge theory, associated *a la* [38] with the N -body affine Toda model, the SW prepotential $\mathcal{F}(\vec{a}) = \mathcal{F}(a_1, \dots, a_{N-1})$ is defined through (1.9) by the $2N - 2$ periods of the SW differential pdx , where [39, 40]

$$p^N - \sum_{k=0}^{N-2} E_k p^k = 2\Lambda^2 \cos x \tag{5.1}$$

while $\mathcal{F}(\vec{a}|\epsilon_1)$ can be similarly defined through the same (1.9) by the $2N - 2$ periods of the deformed differential Pdx , which appears in solution $\Psi = \exp\left(\frac{i}{\hbar} \int^x Pdx\right)$ to the Baxter equation

$$\left\{ (-i\hbar\partial_x)^N - \sum_{k=0}^{N-2} E_k (-i\hbar\partial_x)^k - 2\Lambda^2 \cos x \right\} \Psi(x) = 0 \tag{5.2}$$

with $\hbar = \epsilon_1$. We remind [60–63] that the Fourier transform of this equation arises after separation of variables from the Shrödinger equation for N -body Toda theory. In the SW case, consistency of (1.9) for $N > 2$ is guaranteed by the holomorphicity of the differentials $\frac{\partial(pdx)}{\partial E_k}$ and the symmetry $T_{jk} = T_{kj}$ of the period matrix. In the deformed case, the idea can be that integration contours encircle all the singularities of ydx . The deformed SW differential Pdx will be obtained from the original one pdx by an action of operators like (3.8) and (3.9), whose explicit form remains to be found.

Fifth, since the AGT relation expresses the Nekrasov functions in terms of conformal blocks in $2d$ Liouville-Toda theories, the whole construction provides a new relation between open and periodic Toda systems, in particular, between Liouville and sine-Gordon models. It is well known that the free field formulation of Liouville theory *a la* [64, 65] requires its

² Another problem with the presentation of [1] is that only the second of the two equations in (1.9) is actually considered there (in somewhat different terms). This is enough to compare with the Nekrasov functions which already have the proper a as their argument, but not enough for the SW construction, where a still needs to be defined.

lifting to the sine-Gordon model, it would be nice to find an explicit connection between that construction and the one in the present paper, see also [8] for a related set of questions. Note that the BS description of the prepotential $\mathcal{F}(a|\epsilon_1)$ unavoidably contains its perturbative part and thus is sensitive to the choice of conformal model, not only to the chiral algebra.

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