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Analytic results for massless three-loop form factors

R.N. Lee,[a] **A.V. Smirnov**[b] **and V.A. Smirnov**[c]

[a] *Budker Institute of Nuclear Physics and Novosibirsk State University, 630090, Novosibirsk, Russia*

[b] *Scientific Research Computing Center, Moscow State University, 119992 Moscow, Russia*

[c] *Skobeltsyn Institute of Nuclear Physics of Moscow State University, 119992 Moscow, Russia*

E-mail: R.N.Lee@inp.nsk.su, asmirnov80@gmail.com, smirnov@theory.sinp.msu.ru

Abstract: We evaluate, exactly in d, the master integrals contributing to massless three-loop QCD form factors. The calculation is based on a combination of a method recently suggested by one of the authors (R.L.) with other techniques: sector decomposition implemented in `FIESTA`, the method of Mellin-Barnes representation, and the PSLQ algorithm. Using our results for the master integrals we obtain analytical expressions for two missing constants in the ϵ-expansion of the two most complicated master integrals and present the form factors in a completely analytic form.

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Contents

1 Introduction

Recently the evaluation of the QCD form factors at the three-loop level has attracted much attention. The form factors constitute important building blocks for a number of physical applications. Among them are the two-jet cross section in e^+e^- collisions, the Higgs-boson production in the gluon fusion and the lepton pair production in proton collisions via the Drell-Yan mechanism. The three-loop corrections to the form factors of the photon-quark and the effective gluon-Higgs boson vertex appear after integrating out the heavy top-quark loops. Let Γ_q^μ and $\Gamma_g^{\mu\nu}$ be the corresponding vertex functions. Then the form factors are defined by

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2}\mathrm{Tr}\left(\not{q}_2\,\Gamma_q^\mu \not{q}_1\,\gamma_\mu\right)\,, \tag{1.1}$$

$$F_g(q^2) = \frac{(q_1\cdot q_2\; g_{\mu\nu} - q_{1,\mu}\,q_{2,\nu} - q_{1,\nu}\,q_{2,\mu})}{2(1-\epsilon)}\Gamma_g^{\mu\nu}\,, \tag{1.2}$$

where q_1 and $-q_2$ are the momenta of the incoming and outgoing particles (quarks, for the case of F_q, and gluons, for the case of F_g), and $q = q_1 + q_2$ is the momentum transfer. Here and below, if not stated otherwise, we put $d = 4 - 2\epsilon$. Within perturbative expansion, the form factors take the form

$$F_x = 1 + \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \left(\frac{\mu^2}{Q^2}\right)^{n\epsilon} F_x^{(n)}\,, \tag{1.3}$$

where $Q^2 = -q^2$, and x is either q(quark) or g(gluon). One deals with the three-loop order and splits $F_q^{(3)}$ into the singlet, fermionic and remaining gluonic part

$$F_q^{(3)} = F_q^{(3),g} + F_q^{(3),n_f} + \sum_{q'} Q_{q'} F_q^{(3),sing}\,, \tag{1.4}$$

where n_f stands for the number of active quarks, and Q_q is the charge of the quark q. The pole parts of $F_q^{(3),g}$ and $F_g^{(3)}$ in ϵ were presented in eqs. (3.7) of ref. [1] and eqs. (9) of ref. [2], respectively. The finite parts $F_q^{(3),g}$, $F_q^{(3),sing}$ and $F_g^{(3)}$ were presented in ref. [3].

The integration-by-part reduction reduces the problem to the calculation of a small number of master integrals. All the master integrals apart from three most complicated master integrals contributing to the three-loop massless form factors have been evaluated in [4, 5]. In fact, the word *evaluated* means here the evaluation up to the order of ϵ which appears in the finite part of the form factors. Mathematically, this means the evaluation up to transcendentality weight six. About one year ago, one of the three most complicated master integrals (called $A_{9,1}$ in [3–6]) and the pole parts of $A_{9,4}$ and $A_{9,2}$ (shown in figures 1 and 2 in the next section) were evaluated analytically, while the ϵ^0 parts of $A_{9,4}$ and $A_{9,2}$ were evaluated numerically — see [3, 6]. Therefore, only the two (apparently, most complicated) pieces of the whole family of three-loop massless form factor master integrals are missing at the moment. Mathematically and aesthetically, it is desirable to obtain completely analytic results, and this is the problem we are going to solve in the present paper.

Recently, in ref. [7] a method of multiloop calculations based on the use of dimensional recurrence relations (DRR) [8] and analytic properties of Feynman integrals as functions of the parameter of dimensional regularization, d, has been suggested. In the present paper we apply this method to evaluate, exactly in d, the master integrals contributing to massless three-loop QCD form factors. Using the derived expressions, we obtain analytic results for the missing two constants and thereby arrive at analytic expressions for the form factors.

The key point of the approach of ref. [7] is the analysis of the analytic properties of a given integral in a basic stripe of the complex plane d. The proper choice of the master integral, the basic stripe, and the summation factor can essentially simplify the analysis reducing the number of (or totally fixing) the constants parametrizing the homogeneous solution of DRR. The freedom of this choice, being an advantage, is also the only heuristic part of the method. For the case of massive tadpoles, this choice is relatively simple due to the possibility to get rid of the infrared and ultraviolet singularities by performing an analysis in an infrared-safe region $d \in (d_0, d_0 + 2)$ and raising, if necessary, the powers of the massive denominators (see, e.g., example 2 in ref. [7]). For the case of massless on-shell vertex integrals this recipe does not necessarily work because raising the powers of the massless denominators also makes worse the infrared and collinear behavior of the integral. Thus, in this case, one should rely on an analysis of the corresponding parametric representation. A manual analysis of the parametric representation for the purpose of revealing the position and the order of the poles can still be a very complicated problem for the cases considered in this paper. Fortunately, the current version of the code `FIESTA` based on sector decompositions provides the possibility to solve this problem automatically. So, in order to apply the method of ref. [7] to the calculation of a given master integral, we apply a complete set of various techniques:

(i) a reduction to master integrals by two alternative ways: by a code based on [9] and the code called `FIRE` [10] to obtain DRR,

(ii) a sector decomposition [11–17] implemented in the code `FIESTA` [17, 18] to determine the position and the order of the poles in the basic stripe,

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(iii) the method of Mellin-Barnes representation [19**?** –24] to fix the remaining constants parametrizing the homogeneous solution (if any),

(iv) PSLQ [25] to guess the analytical expression for both the constants parametrizing the homogeneous solution and for the ϵ-expansion of the master integral around $d = 4$.

As a result, we obtain representations for all master integrals in arbitrary d. The representations have the form of convergent series which allow, in particular, a fast high-precision calculation of the ϵ-expansion around $d = 4$.

The paper is organized as follows. In the next section we present an example of the calculation for the integral $A_{7,2}$ and give exact results for this integral and for the lower master integral $A_{6,3}$. We also present analytical expressions for the ϵ-expansion of the integrals $A_{9,2}$ and $A_{9,4}$.

In the conclusion, starting from results for the form factors of ref. [3] and substituting the two constants by our analytic values, we present completely analytic expressions for the finite parts of the form factors.

2 Master integrals for massless three-loop form factors

Master integrals naturally form a partially ordered set. One master integral is said to be *lower* than the other master integral if the Feynman graph for the former can be obtained by contracting some internal lines from the Feynman graph of the latter. This ordering enables us to introduce the notion of *complexity level* of a given master integral which is the maximal number of nested lower master integrals. Owing to this definition, the master integrals with zero complexity level have no lower master integrals. The DRR for such integral is obviously homogeneous and its explicit solution is expressed in terms of Γ-functions. Moreover, it turns out that for three-loop on-shell massless vertex master integrals any integral expressed in terms of Γ-functions has zero complexity level. We expect this situation to be general.

Our primary goal is the calculation of the most complicated integrals, $A_{9,2}$ and $A_{9,4}$ which are the last integrals in figures 1 and 2. However, in order to be able to apply the method of ref. [7] we have to know all lower master integrals which are shown in the same figures. Four rows of diagrams in each figure correspond to complexity levels 0, 1, 2 and 3. Therefore, we start our calculation from the complexity level 1, then pass to the complexity level 2 and, finally, calculate the two master integrals of complexity level 3. Let us demonstrate an intermediate step of this procedure using the example of the integral $A_{7,2}$. We directly follow the path of ref. [7]:

1. There are four lower master integrals, A_4, $A_{5,1}$, $A_{5,2}$, and $A_{6,3}$. Three of them are expressed in terms of Γ-functions, while the last one, $A_{6,3}$, can be obtained using the

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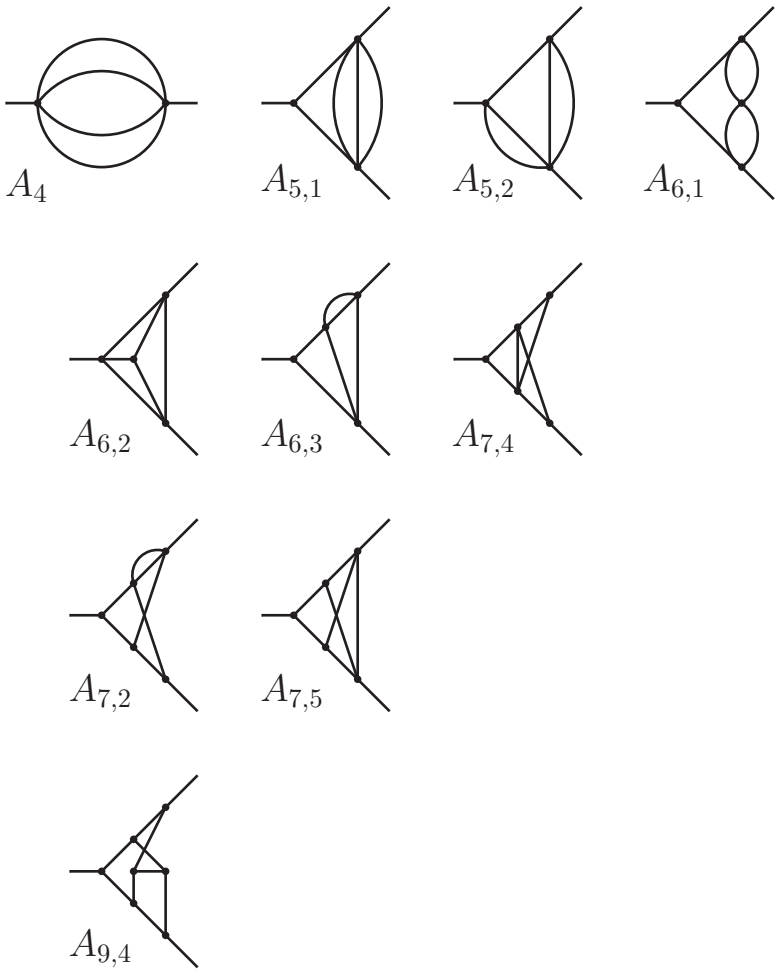

Figure 1. Master integrals for $A_{9,4}$.

same method with the final result conveniently represented as

$$\begin{aligned} A_{6,3}(d) &= A^{1,1}_{6,3}(d)\sum_{k=0}^{\infty} A^{1,2}_{6,3}(d+2k) + A^{2}_{6,3}(d)\,, \\ A^{1,1}_{6,3}(d) &= -\sin(\pi d) A^{2}_{6,3}(d) \\ &= \frac{\pi^4 2^{11-3d}\csc\left(\frac{3\pi d}{2}\right)\csc\left(\frac{\pi d}{2}\right)}{(3d-10)\Gamma\left(d-\frac{5}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}\,, \\ A^{1,2}_{6,3}(d) &= \frac{(7d-18)\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^3}{3\pi^2(d-3)\Gamma\left(\frac{3d}{2}-3\right)}\,. \end{aligned} \tag{2.1}$$

2. Here and in what follows, we omit, for brevity, a power-like dependence of the master integrals on $q^2 + i0$ which can easily be restored by power counting.

   Using the `FIESTA` program we determine the position and the order of the poles in the basic stripe which we choose as $S = \{d| \quad \mathrm{Re}\, d \in (4, 6]\}$. The syntax for

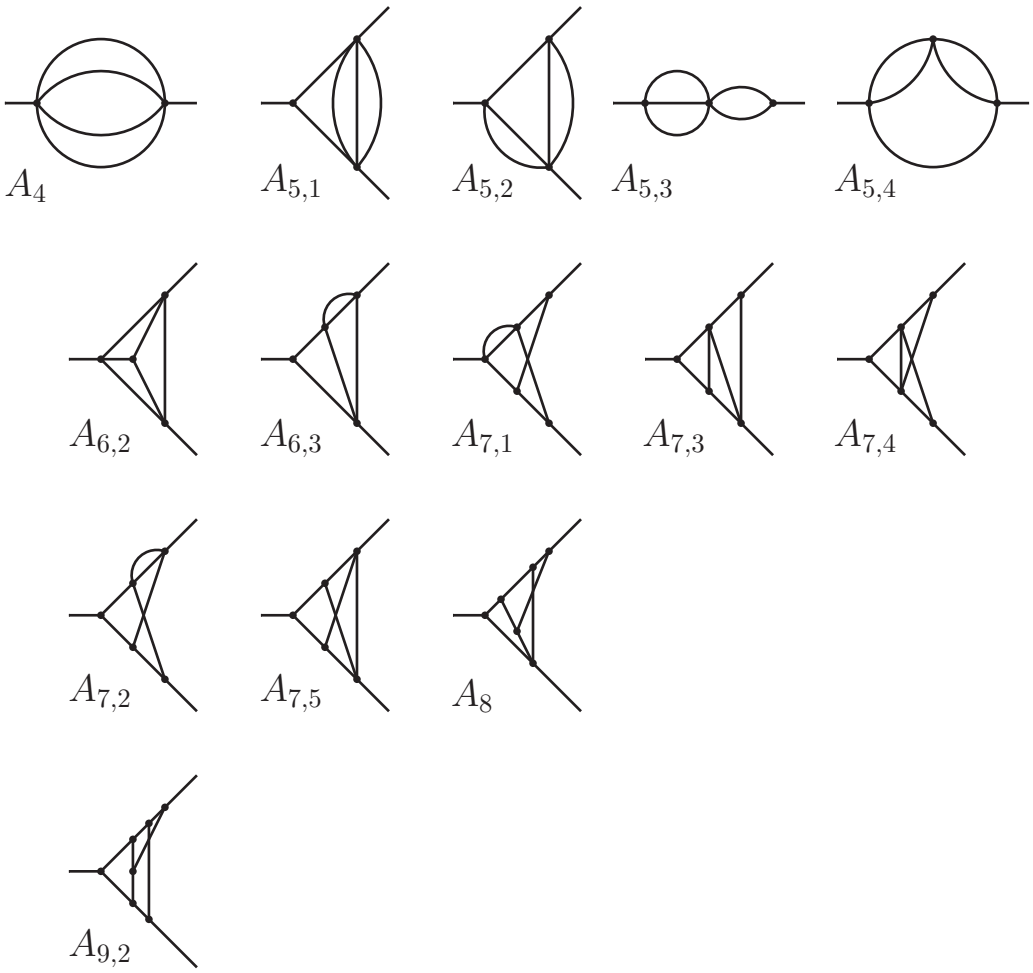

Figure 2. Master integrals for $A_{9,2}$.

this analysis is `SDAnalyze[U,F,h,degrees,order,dmin,dmax]`, where `U` and `F` are the basic functions in the parametric integral corresponding to the given Feynman integral, `h` is the number of loops, `degrees` are the indices, `order` is the required order in ϵ and `dmin` and `dmax` are values of the real part of d that determine the basic stripe. The output lists the values of d where the given Feynman integral can have poles. This feature appeared in the second version of FIESTA, but was not documented in [18] because testing was still in progress. So, after applying this procedure to $A_{7,2}$ we see[1] that the integral has simple poles at $d = 14/3, 5, 16/3, 6$.

[1]In fact, the overall factor $\Gamma(a - hd/2)$ where a is the sum of the indices is not taken into account by FIESTA but this can easily be done because the corresponding poles are explicit. Let us emphasize that FIESTA can report also on some fictitious poles. This can happen when contributions of individual sectors do have some additional poles which cancel in the sum. However, FIESTA itself can be used further to check whether the poles are indeed present or not.

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3. The dimensional recurrence reads[2]

$$A_{7,2}(d+2) = c_{7,2}(d)A_{7,2}(d) \qquad (2.2)$$
$$+c_{6,3}(d)A_{6,3}(d) + c_{5,2}(d)A_{5,2}(d) + c_{5,1}(d)A_{5,1}(d) + c_4(d)A_4(d)$$

where c_n are some rational functions of d presented in the appendix.

4. Using the explicit form of the coefficient $c_{7,2}(d)$, we choose the summing factor as

$$\Sigma(d) = \frac{(d-3)\cos\left(\frac{\pi d}{2}\right)\cos\left(\frac{\pi}{6}-\frac{\pi d}{2}\right)\cos\left(\frac{\pi d}{2}+\frac{\pi}{6}\right)\Gamma\left(\frac{5d}{2}-9\right)}{\Gamma\left(\frac{d}{2}-2\right)^2}. \qquad (2.3)$$

Passing to the function $\tilde{A}_{7,2}(d) = \Sigma(d)A_{7,2}(d)$, we obtain the following equation

$$\tilde{A}_{7,2}(d+2) = \tilde{A}_{7,2}(d) + \tilde{A}_{6,3}(d) + \tilde{A}_{5,2}(d) + \tilde{A}_{5,1}(d) + \tilde{A}_4(d)\,, \qquad (2.4)$$

where $\tilde{A}_n(d) = \Sigma(d+2)c_n(d)A_n(d)$. The general solution can easily be constructed using the explicit form of the integrals A_4, $A_{5,1}$, $A_{5,2}$, and $A_{6,3}$:

$$\tilde{A}_{7,2}(d) = \omega(z) + \sum_{l=0}^{\infty}\left[\tilde{A}_{5,2}(d-2-2l) + \tilde{A}_{5,1}(d-2-2l) + \tilde{A}^2_{6,3}(d-2-2l)\right]$$
$$-\sum_{l=0}^{\infty}\tilde{A}^{1,1}_{6,3}(d+2l)\sum_{k=0}^{\infty}A^{1,2}_{6,3}(d+2l+2k) - \sum_{l=0}^{\infty}\tilde{A}_4(d+2l), \qquad (2.5)$$

where $z = \exp[i\pi d]$.

5. The function $\Sigma(d)$ has simple zeros at $d = 14/3, 5, 16/3$, therefore, $\tilde{A}_{7,2}(d)$ is regular everywhere in S except the point $d = 6$, where it has a simple pole. Besides, from the explicit form of the summing factor and from the parametric representation of $A_{7,2}(d)$ it is immediately clear that $\tilde{A}_{7,2}(d)$ grows slower than any positive (negative) power of $|z|$ when $\mathrm{Im}\, d \to -\infty$ ($\mathrm{Im}\, d \to +\infty$). This fixes $\omega(z)$ up to a function

$$a_1 + a_2 \cot\left(\frac{\pi}{2}(d-6)\right) \qquad (2.6)$$

6. In order to fix the two remaining constants, we use data obtained from the Mellin-Barnes representation of $A_{7,2}(d)$ which can easily be obtained from the general Mellin-Barnes representation for the non-planar on-shell vertex diagram (see, e.g., Chap. 4 of [23, 24]):

$$A_{7,2}(d) = \frac{1}{(2\pi)^2}\int\int \frac{\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{d}{2}-1\right)^2\Gamma(d-3)\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)^2}{\Gamma(d-2)\Gamma\left(\frac{3d}{2}-5\right)\Gamma(2d-7)\Gamma(d-z_1-4)}$$
$$\times\frac{\Gamma\left(\frac{d}{2}-z_1-2\right)}{\Gamma\left(\frac{3d}{2}-z_1-5\right)}\Gamma\left(\frac{3d}{2}-z_2-6\right)\Gamma(z_1+z_2+1)\Gamma(d-z_1-z_2-5)$$
$$\times\Gamma\left(\frac{3d}{2}-z_1-z_2-6\right)\Gamma\left(-\frac{3d}{2}+z_1+z_2+7\right)\mathrm{d}z_1\,\mathrm{d}z_2\,. \qquad (2.7)$$

[2] We use the integration measure $\mathrm{d}^d k/(i\pi^{d/2})$ per loop.

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Using the codes of refs. [21, 22], at $d = 6 - 2\epsilon$ and $d = 5 - 2\epsilon$ we straightforwardly obtain

$$A_{7,2}(6-2\epsilon) = -\frac{41}{15552\epsilon} + O(\epsilon^0), \qquad A_{7,2}(5-2\epsilon) = -\frac{\pi^{5/2}}{24\epsilon} + O(\epsilon^0). \tag{2.8}$$

Using these two values and also taking into account the fact that the singularities of the inhomogeneous part should be cancelled, we obtain

$$\begin{aligned}\omega(z) = {} & \frac{\pi^3}{20\sqrt{5}}\tan\left(\frac{\pi}{10} - \frac{\pi d}{2}\right) - \frac{\pi^3}{36}\tan\left(\frac{\pi}{6} - \frac{\pi d}{2}\right) - \frac{\pi^3}{20\sqrt{5}}\tan\left(\frac{\pi d}{2} + \frac{\pi}{10}\right) \\ & + \frac{\pi^3}{36}\tan\left(\frac{\pi d}{2} + \frac{\pi}{6}\right) + \frac{\pi^3}{60}\cot^3\left(\frac{\pi d}{2}\right) + \frac{13\pi^3}{180}\cot\left(\frac{\pi d}{2}\right) \\ & + \frac{\pi^3}{20\sqrt{5}}\cot\left(\frac{\pi}{5} - \frac{\pi d}{2}\right) - \frac{\pi^3}{20\sqrt{5}}\cot\left(\frac{\pi d}{2} + \frac{\pi}{5}\right). \end{aligned} \tag{2.9}$$

Eqs. (2.5), (2.9), and (2.3) determine our final expression for $A_{7,2}(d)$.

Two remarks are in order. First, our choice of the summing factor, the basic stripe and the master integral itself (we could have considered instead, e.g., an integral with some denominators squared and/or with numerators) may be not the most optimal one. With some other choice, we might have been able to fix the homogeneous part of the solution entirely within the method. However, given the number of the integrals to be considered and the absence of the general recipe for this choice, it was much more convenient to use in such cases additional data from Mellin-Barnes representations. In fact, for other integrals the number of the constants to be fixed was not greater than two.

The second remark concerns the double sum in eq. (2.5). Making a shift $k \to k - l$, we obtain the following triangle sum with the factorized summand:

$$\sum_{l=0}^{\infty} \tilde{A}^{1,1}_{6,3}(d+2l) \sum_{k=l}^{\infty} A^{1,2}_{6,3}(d+2k)\,. \tag{2.10}$$

The factorized form of the summand essentially simplifies the numerical calculation of the sum, making it possible to organize the calculations without nested do-loops. Proceeding in the same way for the rest of the integrals, we finally obtain general expressions for $A_{9,4}$ and $A_{9,2}$. The resulting representations for arbitrary d are too lengthy to be presented here and can be obtained upon request from the authors. We present here only analytical results for the expansion of these two integrals around $d = 4$ which are most interesting for physical applications. The expansion for $A_{9,4}$ reads

$$\begin{aligned} A_{9,4}(4-2\epsilon) = {} & \mathrm{e}^{-3\gamma_E\epsilon}\left\{-\frac{1}{9\epsilon^6} - \frac{8}{9\epsilon^5} + \left[1 + \frac{43\pi^2}{108}\right]\frac{1}{\epsilon^4} + \left[\frac{109\zeta(3)}{9} + \frac{14}{9} + \frac{53\pi^2}{27}\right]\frac{1}{\epsilon^3}\right. \\ & + \left[\frac{608\zeta(3)}{9} - 17 - \frac{311\pi^2}{108} - \frac{481\pi^4}{12960}\right]\frac{1}{\epsilon^2} \\ & + \left[-\frac{949\zeta(3)}{9} - \frac{2975\pi^2\zeta(3)}{108} + \frac{3463\zeta(5)}{45} + 84 + \frac{11\pi^2}{18} + \frac{85\pi^4}{108}\right]\frac{1}{\epsilon} \\ & + \left[\frac{434\zeta(3)}{9} - \frac{299\pi^2\zeta(3)}{3} - \frac{3115\zeta(3)^2}{6} + \frac{7868\zeta(5)}{15}\right. \\ & \left.\left. -339 + \frac{77\pi^2}{4} - \frac{2539\pi^4}{2592} - \frac{247613\pi^6}{466560}\right] + O(\epsilon)\right\}, \end{aligned} \tag{2.11}$$

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For $A_{9,2}$, we arrive at the following result:

$$
\begin{aligned}
A_{9,2}(4-2\epsilon) = \mathrm{e}^{-3\gamma_E\epsilon}\Bigg\{ & -\frac{2}{9\epsilon^6} - \frac{5}{6\epsilon^5} + \left[\frac{20}{9} + \frac{17\pi^2}{54}\right]\frac{1}{\epsilon^4} \\
& + \left[\frac{31\zeta(3)}{3} - \frac{50}{9} + \frac{181\pi^2}{216}\right]\frac{1}{\epsilon^3} \\
& + \left[\frac{347\zeta(3)}{18} + \frac{110}{9} - \frac{17\pi^2}{9} + \frac{119\pi^4}{432}\right]\frac{1}{\epsilon^2} \\
& + \left[-\frac{514\zeta(3)}{9} - \frac{341\pi^2\zeta(3)}{36} + \frac{2507\zeta(5)}{15} - \frac{170}{9} + \frac{19\pi^2}{6} + \frac{163\pi^4}{960}\right]\frac{1}{\epsilon} \\
& + \left[\frac{1516\zeta(3)}{9} - \frac{737\pi^2\zeta(3)}{24} - 29\zeta(3)^2 + \frac{2783\zeta(5)}{6}\right. \\
& \left. - \frac{130}{9} + \frac{\pi^2}{2} - \frac{943\pi^4}{1080} + \frac{195551\pi^6}{544320}\right] + O(\epsilon)\Bigg\} \,. \qquad (2.12)
\end{aligned}
$$

3 Conclusion

Eqs. (2.11) and (2.12) enable us to present completely analytic results for the three-loop corrections to the form factors defined by eqs. (1.1)–(1.4). Starting from eqs. (8)–(10) of ref. [3] and taking into account our analytic values of the ϵ^0 terms in (2.11) and (2.12) we obtain the following analytic expressions:

$$
\begin{aligned}
F_q^{(3),g+n_f}\Big|_{\rm fin} = {} & C_F^3\left[2669\zeta(3) + \frac{61\pi^2\zeta(3)}{6} - \frac{1826\zeta(3)^2}{3} + \frac{4238\zeta(5)}{5} - \frac{53675}{24} - \frac{13001\pi^2}{72}\right. \\
& \left. + \frac{12743\pi^4}{1440} - \frac{9095\pi^6}{54432}\right] + C_A C_F^2\left[-\frac{96715\zeta(3)}{18} + \frac{23\pi^2\zeta(3)}{27} + \frac{1616\zeta(3)^2}{3}\right. \\
& \left. - \frac{46594\zeta(5)}{45} + \frac{37684115}{5832} + \frac{664325\pi^2}{1944} - \frac{1265467\pi^4}{77760} - \frac{18619\pi^6}{272160}\right] \\
& + C_A^2 C_F\left[\frac{1341553\zeta(3)}{486} - \frac{355\pi^2\zeta(3)}{27} - \frac{1136\zeta(3)^2}{9} + \frac{2932\zeta(5)}{9} - \frac{52268375}{13122}\right. \\
& \left. - \frac{383660\pi^2}{2187} + \frac{152059\pi^4}{19440} - \frac{769\pi^6}{5103}\right] + C_F^2 n_f T\left[-\frac{2732173}{1458} - \frac{45235\pi^2}{486}\right. \\
& \left. + \frac{102010\zeta(3)}{81} + \frac{8149\pi^4}{3888} - \frac{343\pi^2\zeta(3)}{27} + \frac{556\zeta(5)}{45}\right] \\
& + C_A C_F n_f T\left[\frac{17120104}{6561} + \frac{442961\pi^2}{4374} - \frac{90148\zeta(3)}{81} - \frac{1093\pi^4}{486} + \frac{368\pi^2\zeta(3)}{27}\right. \\
& \left. - \frac{416\zeta(5)}{3}\right] + C_F n_f^2 T^2\left[-\frac{2710864}{6561} - \frac{124\pi^2}{9} + \frac{12784\zeta(3)}{243} - \frac{83\pi^4}{1215}\right] \,, \qquad (3.1)
\end{aligned}
$$

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$$
\begin{aligned}
F_g^{(3)}\Big|_{\rm fin} = {} & C_A^3\left[-\frac{68590\zeta(3)}{243}+\frac{77\pi^2\zeta(3)}{108}-\frac{1766\zeta(3)^2}{9}+\frac{20911\zeta(5)}{45}+\frac{14474131}{13122}\right.\\
& \left.+\frac{307057\pi^2}{8748}+\frac{8459\pi^4}{38880}-\frac{22523\pi^6}{58320}\right]\\
& +C_A^2 n_f T\left[-\frac{10021313}{6561}-\frac{37868\pi^2}{2187}-\frac{1508\zeta(3)}{27}+\frac{437\pi^4}{1080}-\frac{439\pi^2\zeta(3)}{27}+\frac{6476\zeta(5)}{45}\right]\\
& +C_F C_A n_f T\left[-\frac{155629}{243}-\frac{41\pi^2}{9}+\frac{23584\zeta(3)}{81}-\frac{8\pi^4}{45}+16\pi^2\zeta(3)+\frac{64\zeta(5)}{9}\right]\\
& +C_F^2 n_f T\left[\frac{608}{9}+\frac{592\zeta(3)}{3}-320\zeta(5)\right]+C_F n_f^2 T^2\left[\frac{42248}{81}-\frac{32\pi^2}{9}-\frac{2816\zeta(3)}{9}\right.\\
& \left.-\frac{112\pi^4}{135}\right]+C_A n_f^2 T^2\left[\frac{2958218}{6561}+\frac{152\pi^2}{81}+\frac{47296\zeta(3)}{243}+\frac{797\pi^4}{1215}\right]\,, \qquad (3.2)
\end{aligned}
$$

where $C_F = (N_c^2-1)/(2N_c)$, $C_A = N_c$, $T = 1/2$ and $d^{abc}d^{abc} = (N_c^2-1)(N_c^2-4)/N_c$.

We are confident that this technique can be applied to analytically evaluate master integrals appearing in various physical problems.

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A Coefficients in (2.2)

The coefficients of the dimensional recurrence relation for $A_{7,2}$ have the form

$$
\begin{aligned}
c_{7,2} = {} & -\frac{8(d-4)^2(d-3)}{5(d-2)(d-1)(5d-18)(5d-16)(5d-14)(5d-12)}\,,\\
c_{6,3} = {} & -\frac{(3d-10)\left(483d^4-5996d^3+27684d^2-56272d+42432\right)}{20(d-2)^2(d-1)(2d-5)(5d-18)(5d-16)(5d-14)(5d-12)}\,,\\
c_{5,2} = {} & -(d-3)[15(d-4)(d-2)^2(d-1)(2d-5)(3d-10)(3d-8)(5d-18)]^{-1}\\
& \times[(5d-16)(5d-14)(5d-12)]^{-1}\left[12447d^7-256626d^6+2261972d^5\right.\\
& \left.-11052152d^4+32339200d^3-56684032d^2+55123200d-22947840\right]\,,\\
c_{5,1} = {} & -\left[60(d-4)(d-2)^2(d-1)(2d-5)(3d-10)(5d-18)(5d-16)\right]^{-1}\\
& \times[(5d-14)(5d-12)]^{-1}\left[18909d^7-384006d^6+3329804d^5\right.\\
& \left.-15982952d^4+45870976d^3-78731008d^2+74846208d-30412800\right]\,,\\
c_4 = {} & -\left[90(d-3)(d-2)^2(d-1)(3d-10)(3d-8)(5d-16)\right]^{-1}\\
& \times[(5d-14)(5d-12)]^{-1}\left[38619d^6-651987d^5+4575500d^4\right.\\
& \left.-17083884d^3+35791888d^2-39892032d+18478080\right]\,. \qquad (A.1)
\end{aligned}
$$

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