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Fully nonlinear transformations of the Weyl-Bondi-Metzner-Sachs asymptotic symmetry group

Éanna É. Flanagan[a,b] **and David A. Nichols**[c]

[a]*Department of Physics, Cornell University, Ithaca, New York 14853, U.S.A.*
[b]*Cornell Center for Astrophysics and Planetary Science (CCAPS), Cornell University, Ithaca, New York 14853, U.S.A.*
[c]*Department of Physics, University of Virginia, P.O. Box 400714, Charlottesville, Virginia 22904-7414, U.S.A.*

E-mail: eef3@cornell.edu, david.nichols@virginia.edu

Abstract: The asymptotic symmetry group of general relativity in asymptotically flat spacetimes can be extended from the Bondi-Metzner-Sachs (BMS) group to the generalized BMS (GMBS) group suggested by Campiglia and Laddha, which includes arbitrary diffeomorphisms of the celestial two-sphere. It can be further extended to the Weyl BMS (BMSW) group suggested by Freidel, Oliveri, Pranzetti and Speziale, which includes general conformal transformations. We compute the action of fully nonlinear BMSW transformations on the leading order Bondi-gauge metric functions: specifically, the induced metric, Bondi mass aspect, angular momentum aspect, and shear. These results generalize previous linearized results in the BMSW context by Freidel et al., and also nonlinear results in the BMS context by Chen, Wang, Wang and Yau. The transformation laws will be useful for exploring implications of the BMSW group.

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1 Introduction

The systematic study of spacetimes that become asymptotically flat far from an isolated source was initiated in the 1960s. Bondi, van der Burg, and Metzner [1] as well as Sachs [2] used coordinates (now called Bondi coordinates) that are adapted to the outgoing null rays of an isolated gravitating and radiating system. Their analysis allowed them to compute the asymptotic Einstein equations in Bondi coordinates and the symmetry transformations that preserve the metric at infinity and the Bondi coordinate conditions. The group of such transformations, the Bondi-Metzner-Sachs (BMS) group, is larger than the Poincaré group of transformations that preserve Minkowski spacetime. Lorentz transformations are contained in the BMS group, and the group has a semi-direct product structure similar to that of the Poincaré group. However, the infinite-dimensional commutative group of supertranslations replaces the four-dimensional group of spacetime translations in the Poincaré group, though the spacetime translations remain a subgroup of the supertranslations.

More recently, the analysis of asymptotically flat spacetimes has been revisited, and larger groups of symmetry transformations were found which preserve different geometric quantities at future null infinity (this is described in more detail in section 2) [3, 4]. These new groups include the extended BMS group [5, 6], the generalized BMS group [7], and the Weyl-BMS (BMSW) group [8–12]. While the action of BMS transformations on the Bondi metric functions has been computed to nonlinear order [13–16], the actions of the extended BMS, generalized BMS and BMSW transformations have been computed to linear order only [8, 17]. In this paper, we derive the nonlinear transformations of the leading-order metric functions in vacuum for the generalized BMS and BMSW groups. These transformations reduce to the known nonlinear BMS and linearized BMSW results in the appropriate limits.

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Fully nonlinear transformation laws can be useful for exploring the viability of alternative definitions of charges associated with symmetries. An example is the recent investigation of ref. [18] of continuity of various charges as the cross-section of future null infinity is varied. They are also useful for understanding the space of vacua which is relevant to the quantum theory [8, 17].

The remainder of this paper contains a review of the different asymptotic symmetry groups in section 2. The main results are derived in section 3 and compared with existing results in section 4. Some applications of these results are given in section 5.

2 Review of asymptotic symmetry groups in asymptotically flat spacetimes

In this section, we review the different asymptotic symmetry groups in asymptotically flat spacetime to establish our notation and conventions.

We use retarded Bondi coordinates $(u, r, \theta^1, \theta^2) = (u, r, \theta^A)$ near future null infinity, following refs. [1, 5, 6, 8, 9, 14, 17, 19–23]. We will use throughout the notations and conventions of Flanagan and Nichols [14] (henceforth FN). The key metric functions in Bondi gauge are the induced metric $h_{AB}(\theta^A)$, the Bondi mass aspect $m(u, \theta^A)$, the angular momentum aspect $N_A(u, \theta^A)$, and the shear $C_{AB}(u, \theta^A)$. The metric expansion obtained from the asymptotic conditions, the gauge conditions and the vacuum Einstein equations is

$$ds^2 = -\left[\frac{1}{2}\mathcal{R} - \frac{2m}{r} + O\left(\frac{1}{r^2}\right)\right] du^2 - 2\left[1 - \frac{C_{AB}C^{AB}}{16r^2} + O\left(\frac{1}{r^3}\right)\right] dudr$$
$$+ r^2\left[h_{AB} + \frac{1}{r}C_{AB} + \frac{C_{CD}C^{CD}}{4r^2}h_{AB} + \left(\frac{1}{r^3}\right)\right](d\theta^A - \mathcal{U}^A du)(d\theta^B - \mathcal{U}^B du). \quad (2.1)$$

Here

$$\mathcal{U}^A = -\frac{D_B C^{AB}}{2r^2} + \frac{1}{r^3}\left[-\frac{2}{3}N^A + \frac{1}{16}D^A(C_{BC}C^{BC}) + \frac{1}{2}C^{AB}D^C C_{BC}\right] + O\left(\frac{1}{r^4}\right), \quad (2.2)$$

$\mathcal{R}$ is the two dimensional Ricci scalar of $h_{AB}(\theta^A)$, and A, B are angular indices which run over the values $1, 2$ and are raised and lowered with h^{AB} and h_{AB}, respectively. Also we have generalized the treatment of FN, following Compère, Fiorucci and Ruzziconi [17], to allow the induced metric h_{AB} to differ from the canonical round metric, so that $\mathcal{R}$ is allowed to be an arbitrary function of θ^A instead of being constrained to $\mathcal{R} = 2$. This generalization requires replacing the leading term in the expansion (FN,2.3b) with $\mathcal{R}/2$, adding the term $D^2\mathcal{R}/8$ to the right hand side of the evolution equation (FN,2.11a) for the Bondi mass aspect, and adding the term $C_{AB}D^B\mathcal{R}/4$ to the evolution equation (FN,2.11b) for the angular momentum aspect [17]. In this paper we specialize to spacetimes which are vacuum near $\mathscr{I}^+$, and so the subleading shear tensor $\mathcal{D}_{AB}$ of (FN,2.3c) vanishes, by (FN,2.10).

We will consider three different asymptotic groups obtained from three different phase space definitions (see refs. [4, 12] for reviews). In the coordinate system $y^i = (u, \theta^A)$ on $\mathscr{I}^+$, the diffeomorphisms $\psi : \mathscr{I}^+ \to \mathscr{I}^+$ have the following form for all three groups:

$$\bar{u} = e^{\alpha(\theta^A)}\left[u + \beta(\theta^A)\right], \quad (2.3a)$$
$$\bar{\theta}^A = \chi^A(\theta^B), \quad (2.3b)$$

where $\chi : S^2 \to S^2$ is a diffeomorphism of the two-sphere S^2, and for a point $\mathcal{P}$ on $\mathscr{I}^+$ we have defined $y^i = y^i(\mathcal{P})$ and $\bar{y}^i = y^i(\psi(\mathcal{P}))$. The groups are:

- The Weyl BMS (BMSW) group suggested by Freidel, Oliveri, Pranzetti and Speziale [8–12]. For this group the two-sphere diffeomorphism χ and the functions β and α can be freely chosen.

- The generalized BMS (GBMS) group suggested by Campiglia and Laddha [7] and further studied in refs. [17, 24–29]. For this group the function α is determined as a function of χ as follows. Let ϵ_{AB} be one of the two volume forms on the two-sphere that are determined up to sign by the metric h_{AB}. Define the function ω_χ by

$$\chi_* \epsilon_{AB} = \frac{1}{\omega_\chi} \epsilon_{AB}, \tag{2.4}$$

  where χ_* is the pullback operator. Then we have

$$e^{2\alpha} = \frac{1}{|\omega_\chi|}, \tag{2.5}$$

  which will have the consequence that GBMS transformations preserve the volume form ϵ_{AB} up to a sign (see section 3.4 below).

- The BMS group [1, 2, 30–32], the subgroup of GBMS for which the diffeomorphisms χ are restricted to be global conformal isometries of the two-sphere. As a consequence the metric h_{AB} is preserved under BMS transformations.

3 Derivation of transformation laws

In this section, we derive the transformation properties of the metric functions under nonlinear BMSW transformations, for solutions which are vacuum near $\mathscr{I}^+$.

3.1 Supertranslations

Consider first supertranslations, and specifically the finite supertranslation $\psi : \mathscr{I}^+ \to \mathscr{I}^+$. Denoting the coordinates (u, θ^A) by y^i and defining $\bar{y}^i = y^i \circ \psi = (\bar{u}, \bar{\theta}^A)$ the mapping ψ on $\mathscr{I}^+$ is given by

$$\bar{u} = u + \beta(\theta^A), \qquad \bar{\theta}^A = \theta^A. \tag{3.1}$$

Under this mapping the metric functions transform as

$$\bar{h}_{AB}(\theta^A) = h_{AB}(\theta^A), \tag{3.2a}$$

$$\bar{C}_{AB}(u,\theta^A) = C_{AB}(u+\beta,\theta^A) - 2D_AD_B\beta + h_{AB}D^2\beta, \tag{3.2b}$$

$$\begin{aligned}\bar{m}(u,\theta^A) &= m(u+\beta,\theta^A) + \frac{1}{2}D_AN^{AB}(u+\beta,\theta^A)D_B\beta + \frac{1}{4}N^{AB}(u+\beta,\theta^A)D_AD_B\beta \\ &\quad + \frac{1}{4}\dot{N}^{AB}(u+\beta,\theta^A)D_A\beta D_B\beta + \frac{1}{4}D^A\beta D_A\mathcal{R},\end{aligned} \tag{3.2c}$$

$$\begin{aligned}\bar{N}_A(u,\theta^A) &= N_A(u+\beta,\theta^A) + 3m(u+\beta,\theta^A)D_A\beta + \frac{3}{4}D_A\beta D^B\beta D_B\mathcal{R} - \frac{3}{8}D_A\mathcal{R}(D\beta)^2 \\ &\quad + \frac{3}{4}D_B\beta\left[D_AD_CC^{BC}(u+\beta,\theta^A) - D^BD^CC_{CA}(u+\beta,\theta^A)\right] \\ &\quad + \frac{3}{4}D_B\beta\, C_{AC}(u+\beta,\theta^A)\, N^{BC}(u+\beta,\theta^A) + \frac{3}{2}D_A\beta\, D_B\beta\, D_CN^{BC}(u+\beta,\theta^A) \\ &\quad - \frac{3}{4}(D\beta)^2\, D^CN_{AC}(u+\beta,\theta^A) + \frac{1}{2}D_A\beta\, D_B\beta\, D_C\beta\dot{N}^{BC}(u+\beta,\theta^A) \\ &\quad - \frac{1}{4}(D\beta)^2\, D^C\beta\dot{N}_{CA}(u+\beta,\theta^A).\end{aligned} \tag{3.2d}$$

Here dots denote derivatives with respect to u, D_A is the covariant derivative associated with h_{AB}, and $\mathcal{R}$ is the Ricci scalar of h_{AB}. Also $N_{AB} = \dot{C}_{AB}$ is the Bondi news tensor, and $(D\beta)^2 = D^A\beta D_A\beta$. The transformation laws (3.2) apply to all three groups, BMSW, GBMS and BMS.

To derive the transformation laws (3.2), we extend the diffeomorphism (3.1) to a diffeomorphism $\eta : M \to M$ which coincides with ψ on $\mathscr{I}^+$, by assuming a general power series expansion in $1/r$ and using the notation $x^\alpha = (u, r, \theta^A)$ and $\bar{x}^\alpha = x^\alpha \circ \eta$:

$$\bar{u} = u + \beta(\theta^A) + \frac{1}{r}\beta^{(1)}(u,\theta^A) + \frac{1}{r^2}\beta^{(2)}(u,\theta^A) + O\left(\frac{1}{r^3}\right), \tag{3.3a}$$

$$\bar{r} = r + R^{(1)}(u,\theta^A) + \frac{1}{r}R^{(2)}(u,\theta^A) + O\left(\frac{1}{r^2}\right), \tag{3.3b}$$

$$\bar{\theta}^A = \theta^A + \frac{1}{r}\chi^{(1)A}(u,\theta^A) + \frac{1}{r^2}\hat{\chi}^{(2)A}(u,\theta^A) + \frac{1}{r^3}\hat{\chi}^{(3)A}(u,\theta^A) + O\left(\frac{1}{r^4}\right). \tag{3.3c}$$

Here the functions $\beta^{(1)}$, $\beta^{(2)}$, $R^{(1)}$, $R^{(2)}$, $\chi^{(1)}$, $\hat{\chi}^{(2)}$ and $\hat{\chi}^{(3)}$ are arbitrary. One subtlety is that although $\chi^{(1)A}$ transforms as a vector under diffeomorphisms of the two-sphere S^2, the higher order functions $\hat{\chi}^{(2)A}$ and $\hat{\chi}^{(3)A}$ in eq. (3.3c) do not. We remedy this by parameterizing the diffeomorphism $\varphi : S^2 \to S^2$ on the two sphere at fixed u and r in terms of three vector fields $\chi^{(1)A}$, $\chi^{(2)A}$ and $\chi^{(3)A}$:

$$\varphi = \varphi_{\vec{\chi}^{(3)}}(r^{-3}) \circ \varphi_{\vec{\chi}^{(2)}}(r^{-2}) \circ \varphi_{\vec{\chi}^{(1)}}(r^{-1})\left[1 + O\left(\frac{1}{r^4}\right)\right]. \tag{3.4}$$

Here for any vector field $\vec{\chi}$ on S^2 the map $\varphi_{\vec{\chi}}(\varepsilon)$ is the knight diffeomorphism that moves any point ε units along the integral curve of $\vec{\chi}$ that passes through that point [33, 34]. By

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comparing eqs. (3.3c) and (3.4) we find that[1]

$$\hat{\chi}^{(2)A} = \chi^{(2)A} + \frac{1}{2}\chi^{(1)B}\partial_B\chi^{(1)A}, \tag{3.5a}$$

$$\hat{\chi}^{(3)A} = \chi^{(3)A} + \chi^{(1)B}\partial_B\chi^{(2)A} + \frac{1}{6}\chi^{(1)B}\partial_B\left(\chi^{(1)C}\partial_C\chi^{(1)A}\right). \tag{3.5b}$$

We next take the pullback $\bar{g}_{ab} = \eta_* g_{ab}$ of the metric (2.1) using the expansions (3.3) and imposing that it has the same form as the metric (2.1), but with different metric functions which we denote with overbars. We adopt the shorthand notation that $O(\alpha\beta, n)$ means the $O(r^{-n})$ piece of the $(\alpha\beta)$ component of this metric comparison. The derivation takes place in a series of steps where each step is simplified using results from the previous steps. From $O(uu,0)$, $O(ur,1)$, $O(rA,0)$, $O(rr,2)$ and the trace part of $O(AB,-1)$, respectively, we find that

$$\dot{R}^{(1)} = 0, \tag{3.6a}$$

$$\dot{\beta}^{(1)} = 0, \tag{3.6b}$$

$$\chi^{(1)A} = -h^{AB}D_B\beta, \tag{3.6c}$$

$$\beta^{(1)} = -\frac{1}{2}(D\beta)^2, \tag{3.6d}$$

$$R^{(1)} = \frac{1}{2}D^2\beta. \tag{3.6e}$$

We then obtain from $O(AB,-2)$ and from the trace-free part of $O(AB,-1)$ the transformation laws (3.2a) and (3.2b) for the metric h_{AB} and shear tensor C_{AB}.

Next we find from $O(rA,1)$, $O(rr,3)$, $O(ru,2)$, and $O(rA,2)$ respectively that

$$\chi^{(2)A} = \frac{1}{2}C^{AB}(u+\beta,\theta^A)D_B\beta - \frac{1}{2}D^AD_B\beta D^B\beta + \frac{1}{2}D^A\beta D^2\beta, \tag{3.7a}$$

$$\beta^{(2)} = \frac{1}{4}C_{AB}(u+\beta,\theta^A)D^A\beta D^B\beta + \frac{1}{4}(D\beta)^2D^2\beta, \tag{3.7b}$$

$$\begin{aligned}R^{(2)} = &-\frac{1}{2}D_AC^{AB}(u+\beta,\theta^A)D_B\beta - \frac{1}{4}N^{AB}(u+\beta,\theta^A)D_A\beta D_B\beta + \frac{1}{4}\mathcal{R}(D\beta)^2\\ &-\frac{1}{4}C^{AB}(u+\beta,\theta^A)D_AD_B\beta + \frac{1}{4}D_AD_B\beta D^AD^B\beta - \frac{1}{8}(D^2\beta)^2,\end{aligned} \tag{3.7c}$$

$$\begin{aligned}\chi^{(3)A} = &-\frac{1}{16}D^A\beta C_{BC}(u+\beta,\theta^A)C^{BC}(u+\beta,\theta^A) + \frac{1}{3}N^{AB}(u+\beta,\theta^A)D_B\beta(D\beta)^2\\ &-\frac{1}{6}N^{BC}(u+\beta,\theta^A)D^A\beta D_B\beta D_C\beta + \frac{1}{4}C_{BC}(u+\beta,\theta^A)D^A\beta D^BD^C\beta\\ &-\frac{1}{2}C_{BC}(u+\beta,\theta^A)D^C\beta D^AD^B\beta - \frac{1}{4}D^BC_{BC}(u+\beta,\theta^A)D^A\beta D^C\beta\\ &+\frac{1}{4}D_BC^{AC}(u+\beta,\theta^A)D^B\beta D_C\beta + \frac{1}{2}D^A\beta D_BD^2\beta D^B\beta - \frac{1}{3}D^AD_BD_C\beta D^B\beta D^C\beta\\ &-\frac{5}{24}D^A\beta(D^2\beta)^2 + \frac{1}{12}D^A\beta D_BD_C\beta D^BD^C\beta + \frac{1}{6}D^AD^B\beta D_B\beta D^2\beta\\ &+\frac{1}{6}\mathcal{R}D^A\beta(D\beta)^2,\end{aligned} \tag{3.7d}$$

[1] This is most easily derived using the formula for the pullback of a knight diffeomorphism in terms of Lie derivatives: $\varphi_{\vec{\chi}}(\varepsilon)_* = \sum_n \varepsilon^n(\pounds_{\vec{\chi}})^n/n!$.

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where $D^2 = D_A D^A$ and $(D\beta)^2 = D^A\beta D_A\beta$. Finally from $O(uu,1)$ and $O(uA,1)$ we obtain the transformation laws (3.2c) and (3.2d) for the Bondi mass aspect m and angular momentum aspect N_A.

3.2 Conformal transformations of the Weyl BMS group

We now turn to conformal transformations $\psi : \mathscr{I}^+ \to \mathscr{I}^+$ of the form

$$\bar{u} = e^{\tau(\theta^A)} u, \qquad \bar{\theta}^A = \theta^A. \tag{3.8}$$

Under this mapping the metric functions transform as

$$\bar{h}_{AB}(\theta^A) = e^{-2\tau} h_{AB}(\theta^A), \tag{3.9a}$$

$$\bar{C}_{AB}(u,\theta^A) = e^{-\tau} C_{AB}(e^\tau u, \theta^A) - 2ue^{-\tau}\left(D_A D_B - \frac{1}{2} h_{AB} D^2\right) e^\tau, \tag{3.9b}$$

$$\begin{aligned}\bar{m}(u,\theta^A) = {} & e^{3\tau} m(e^\tau u,\theta^A) + \frac{1}{4} e^{2\tau} C^{AB}(e^\tau u,\theta^A) D_A D_B e^\tau + \frac{1}{4} u e^{3\tau} N^{AB}(e^\tau u,\theta^A) D_A D_B e^\tau \\ & + \frac{1}{2} u e^{4\tau} D_A N^{AB}(e^\tau u,\theta^A) D_B \tau + \frac{1}{4} e^{5\tau} u^2 \dot{N}^{AB}(e^\tau u,\theta^A) D_A\tau D_B\tau \\ & - \frac{1}{2} u e^{2\tau}\left[D_A D_B e^\tau D^A D^B e^\tau - \frac{1}{2}(D^2 e^\tau)^2\right] + \frac{1}{4} u e^{4\tau} D^A \tau D_A \mathcal{R},\end{aligned} \tag{3.9c}$$

$$\begin{aligned}\bar{N}_A(u,\theta^A) = {} & e^{2\tau} N_A(e^\tau u,\theta^A) + 3ue^{3\tau} m(e^\tau u,\theta^A) D_A\tau \\ & + \frac{3}{4} u^2 e^{4\tau}\left(D_A\tau D^B\tau D_B\mathcal{R} - \frac{1}{2} D_A\mathcal{R} D^B\tau D_B\tau\right) \\ & + \frac{1}{2} u^3 e^{5\tau} D_A\tau\, D_B\tau\, D_C\tau \dot{N}^{BC}(e^\tau u,\theta^A) - \frac{1}{4} u^3 e^{5\tau}(D\tau)^2\, D^C\tau \dot{N}_{CA}(e^\tau u,\theta^A) \\ & + \frac{3}{2} u^2 e^{4\tau} D_A\tau\, D_B\tau\, D_C N^{BC}(e^\tau u,\theta^A) - \frac{3}{4} u^2 e^{4\tau}(D\tau)^2\, D^C N_{AC}(e^\tau u,\theta^A) \\ & + \frac{3}{4} u e^{3\tau} D_B\tau\left[D_A D_C C^{BC}(e^\tau u,\theta^A) - D^B D^C C_{CA}(e^\tau u,\theta^A)\right] \\ & + \frac{3}{4} u e^{3\tau} D_B\tau\, C_{AC}(e^\tau u,\theta^A)\, N^{BC}(e^\tau u,\theta^A).\end{aligned} \tag{3.9d}$$

To derive the transformation laws (3.9), we again extend the diffeomorphism (3.8) to a diffeomorphism $\eta : M \to M$ which coincides with ψ on $\mathscr{I}^+$, by assuming a general power series expansion in $1/r$ and using the notation $x^\alpha = (u, r, \theta^A)$ and $\bar{x}^\alpha = x^\alpha \circ \eta$:

$$\bar{u} = e^{\tau(\theta^A)}\left[u + \frac{1}{r}\beta^{(1)}(u,\theta^A) + \frac{1}{r^2}\beta^{(2)}(u,\theta^A)\right] + O\left(\frac{1}{r^3}\right), \tag{3.10a}$$

$$\bar{r} = e^{-\tau(\theta^A)}\left[r + R^{(1)}(u,\theta^A) + \frac{1}{r} R^{(2)}(u,\theta^A)\right] + O\left(\frac{1}{r^2}\right), \tag{3.10b}$$

$$\bar{\theta}^A = \theta^A + \frac{1}{r}\chi^{(1)A}(u,\theta^A) + \frac{1}{r^2}\hat{\chi}^{(2)A}(u,\theta^A) + \frac{1}{r^3}\hat{\chi}^{(3)A}(u,\theta^A) + O\left(\frac{1}{r^4}\right). \tag{3.10c}$$

Here as before the functions $\beta^{(1)}$, $\beta^{(2)}$, $R^{(1)}$, $R^{(2)}$, $\chi^{(1)}$, $\hat{\chi}^{(2)}$ and $\hat{\chi}^{(3)}$ are arbitrary, and we use instead of $\hat{\chi}^{(2)A}$ and $\hat{\chi}^{(3)A}$ the covariant quantities $\chi^{(2)A}$ and $\chi^{(3)A}$ defined by eqs. (3.5).

We next take the pullback $\bar{g}_{ab} = \eta_* g_{ab}$ of the metric and follow the same steps as in section 3.1. From $O(rA,0)$, the trace part of $O(AB,-1)$, and $O(rr,2)$, respectively, we

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find that

$$\chi^{(1)A} = -ue^{2\tau}h^{AB}D_B\tau, \tag{3.11a}$$

$$R^{(1)} = \frac{1}{2}ue^{2\tau}h^{AB}(D_A\tau D_B\tau + D_AD_B\tau), \tag{3.11b}$$

$$\beta^{(1)} = -\frac{1}{2}u^2e^{2\tau}(D\tau)^2. \tag{3.11c}$$

We then obtain from $O(AB,-2)$ and from the trace-free part of $O(AB,-1)$ the transformation laws (3.9a) and (3.9b) for the metric h_{AB} and shear tensor C_{AB}.

Next we find from $O(rA,1)$, $O(rr,3)$, $O(ru,2)$ and $O(rA,2)$ respectively that

$$\begin{aligned}\chi^{(2)A} =& \frac{1}{2}ue^{3\tau}C^{AB}(e^\tau u,\theta^A)D_B\tau \\ &+\frac{1}{2}u^2e^{4\tau}\left(D^A\tau D^2\tau - D^AD_B\tau D^B\tau - D^A\tau D_B\tau D^B\tau\right),\end{aligned} \tag{3.12a}$$

$$\beta^{(2)} = \frac{1}{4}u^2e^{3\tau}C_{AB}(e^\tau u,\theta^A)D^A\tau D^B\tau + \frac{1}{4}u^3e^{3\tau}(D\tau)^2D^2e^\tau, \tag{3.12b}$$

$$\begin{aligned}R^{(2)} =& -\frac{1}{4}ue^{2\tau}C^{AB}(e^\tau u,\theta^A)D_AD_Be^\tau - \frac{1}{4}u^2e^{4\tau}N^{AB}(e^\tau u,\theta^A)D_A\tau D_B\tau \\ &-\frac{1}{2}e^{3\tau}uD_AC^{AB}(e^\tau u,\theta^A)D_B\tau + \frac{1}{4}e^{2\tau}u^2\left[D_AD_Be^\tau D^AD^Be^\tau - \frac{1}{2}(D^2e^\tau)^2\right] \\ &+\frac{1}{4}e^{4\tau}u^2\mathcal{R}(D\tau)^2,\end{aligned} \tag{3.12c}$$

$$\begin{aligned}\chi^{(3)A} =& -\frac{1}{16}ue^{4\tau}D^A\tau C_{BC}(e^\tau u,\theta^A)C^{BC}(e^\tau u,\theta^A) + \frac{1}{4}u^2e^{5\tau}D_BC^{AC}(e^\tau u,\theta^A)D^B\tau D_C\tau \\ &-\frac{1}{4}u^2e^{5\tau}D^BC_{BC}(e^\tau u,\theta^A)D^A\tau D^C\tau + \frac{1}{4}u^2e^{4\tau}C_{BC}(e^\tau u,\theta^A)D^A\tau D^BD^Ce^\tau \\ &-\frac{1}{2}u^2e^{4\tau}C_{BC}(e^\tau u,\theta^A)D^C\tau D^AD^Be^\tau + u^2e^{5\tau}C^{AB}(e^\tau u,\theta^A)D_B\tau(D\tau)^2 \\ &+\frac{1}{3}u^3e^{6\tau}N^{AB}(e^\tau u,\theta^A)D_B\tau(D\tau)^2 - \frac{1}{6}u^3e^{6\tau}N^{BC}(e^\tau u,\theta^A)D^A\tau D_B\tau D_C\tau \\ &+\frac{1}{2}u^3e^{5\tau}D^A\tau D^B\tau D_BD^2e^\tau - \frac{1}{3}u^3e^{5\tau}D^B\tau D^C\tau D^AD_BD_Ce^\tau \\ &-\frac{5}{24}u^3e^{6\tau}D^A\tau(D^2\tau)^2 + \frac{1}{12}u^3e^{6\tau}D^A\tau D_BD_C\tau D^BD^C\tau \\ &+\frac{1}{6}u^3e^{6\tau}D_B\tau D^AD^B\tau D^2\tau - \frac{5}{6}u^3e^{6\tau}D^AD^B\tau D_B\tau(D\tau)^2 + \frac{3}{4}u^3e^{6\tau}D^A\tau D^2\tau(D\tau)^2 \\ &-\frac{1}{2}u^3e^{6\tau}D^A\tau D_BD_C\tau D^B\tau D^C\tau - \frac{23}{24}u^3e^{6\tau}D^A\tau(D\tau)^4 \\ &+\frac{1}{6}u^3e^{6\tau}D^A\tau(D\tau)^2\mathcal{R}.\end{aligned} \tag{3.12d}$$

Here the angular indices A, B are raised and lowered with h_{AB} and not $\bar{h}_{AB}$ given by eq. (3.9a). Finally from $O(uu,1)$ and $(uA,1)$ we obtain the transformation laws (3.9c) and (3.9d) for the Bondi mass aspect m and angular momentum aspect N_A.

3.3 Two-sphere diffeomorphisms of the Weyl BMS group

The third category of transformations in the BMSW group are diffeomorphisms of the two sphere:

$$\bar{u} = u, \qquad \bar{\theta}^A = \chi^A(\theta^B). \tag{3.13}$$

This extends to the exact four-dimensional diffeomorphism

$$\bar{u} = u, \qquad \bar{\theta}^A = \chi^A(\theta^B), \qquad \bar{r} = r, \tag{3.14}$$

under which the metric functions transform by the pullback of the two-sphere diffeomorphism χ:

$$\bar{h}_{AB} = \chi_* h_{AB}, \tag{3.15a}$$
$$\bar{C}_{AB} = \chi_* C_{AB}, \tag{3.15b}$$
$$\bar{m} = \chi_* m, \tag{3.15c}$$
$$\bar{N}_A = \chi_* N_A. \tag{3.15d}$$

3.4 Two-sphere diffeomorphisms of the generalized BMS group

We now turn to the generalized BMS group instead of the Weyl BMS group, and focus again on two-sphere diffeomorphisms. The diffeomorphism ψ on $\mathscr{I}^+$ takes the form

$$\bar{u} = e^{\alpha(\theta^A)} u, \qquad \bar{\theta}^A = \chi^A(\theta^B). \tag{3.16}$$

Here the function $\alpha(\theta^A)$ is determined as a function of χ by the requirement (2.5).

To compute the transformation of the metric functions under the mapping (3.16) we combine the results of sections 3.2 and 3.3. We decompose ψ as $\psi = \psi_2 \circ \psi_1$, where ψ_1 is the two-sphere diffeomorphism (3.13), and ψ_2 is the conformal transformation (3.8) with τ chosen to be

$$\tau = \alpha \circ \chi^{-1}. \tag{3.17}$$

The corresponding spacetime diffeomorphisms are related by $\eta = \eta_2 \circ \eta_1$, and the pullback of the metric is then given by

$$\eta_* g_{ab} = \eta_{1\,*} \circ \eta_{2\,*} g_{ab}. \tag{3.18}$$

It follows that we can compute transformed metric functions as follows. Start with the metric functions h_{AB}, m, N_A, C_{AB}, and act with the pullback $\eta_{2\,*}$ which yields the transformed metric functions $\bar{h}_{AB}, \bar{m}, \bar{N}_A, \bar{C}_{AB}$ given by eqs. (3.9) using the parameter (3.17). Next, act with the pullback $\eta_{1\,*}$ using the prescription (3.15) which yields the final metric functions

$$\hat{h}_{AB} = \chi_* \bar{h}_{AB}, \quad \hat{m} = \chi_* \bar{m}, \quad \hat{N}_A = \chi_* \bar{N}_A, \quad \hat{C}_{AB} = \chi_* \bar{C}_{AB}. \tag{3.19}$$

In particular from eq. (3.9a) the final induced metric is given by $\hat{h}_{AB} = \chi_*(e^{-2\tau} h_{AB})$, and so the final volume form is

$$\hat{\epsilon}_{AB} = \chi_*(e^{-2\tau}\epsilon_{AB}) = e^{-2\alpha}\chi_*\epsilon_{AB} = \frac{e^{-2\alpha}}{\omega_\chi}\epsilon_{AB} = \pm\epsilon_{AB}, \tag{3.20}$$

where we have used eqs. (3.17), (2.4) and (2.5).

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3.5 Conformal isometries of the BMS group

We now turn to the boosts and rotations of the BMS group. These are a special case of the GBMS two-sphere diffeomorphisms (3.16) where χ is restricted to be a global conformal isometry, so that

$$\chi_* h_{AB} = \frac{1}{\omega_\chi} h_{AB}. \tag{3.21}$$

Here ω_χ is defined by eq. (2.4), and we are excluding improper Lorentz transformations. The transformation laws are therefore given by combining eqs. (3.9), (3.17) and (3.19). It follows from the analysis that led to eq. (3.20) that the two-metric is preserved, $\hat{h}_{AB} = h_{AB}$. Moreover for rotations we have $\omega_\chi = 1$, and for boosts we have $\omega_\chi = (\cosh\gamma - \cos\Theta\sinh\gamma)^2$, where γ is the rapidity parameter and Θ is the angle between the boost direction and the direction determined by θ^A. From eqs. (2.4), (2.5) and (3.17) we obtain that

$$e^{2\tau} = \frac{1}{\omega_\chi \circ \chi^{-1}} = \omega_{\chi^{-1}} = (\cosh\gamma + \cos\Theta\,\sinh\gamma)^2. \tag{3.22}$$

It follows that e^τ is purely $l = 0$ and $l = 1$, and so it is annihilated by the differential operator $D_A D_B - h_{AB} D^2/2$. This implies that the following terms vanish in eqs. (3.9) for boosts and rotations: the second term in eq. (3.9b), and the second, third and sixth terms in eq. (3.9c). In addition the terms involving $\mathcal{R}$ in eqs. (3.2) and (3.9) will vanish when we specialize to round two-metrics with $\mathcal{R} = 2$, as is normally done for the BMS group.

4 Comparisons with previous results

Our results (3.2), (3.9), (3.15) and (3.19) agree (mostly) with a number of previous computations in special cases:

- The linearized BMSW computations of Freidel, Oliveri, Pranzetti and Speziale given in eqs. (4.42) of ref. [8], taking into account that the variables $(\bar{F}, T, W, \tau, \bar{P}_A)$ used there are given in terms of variables used here to linear order as $(\mathcal{R}/4, \beta, \tau, \beta + u\tau, N_A - D_A(C_{BC}C^{BC})/16 - C_{AB}D_C C^{BC}/4)$.

- The linearized GBMS computations of Compère, Fiorucci and Ruzziconi given in eqs. (2.20)–(2.24) of ref. [17], noting that their angular momentum aspect N_A is related to ours by their eq. (2.8).

- Our nonlinear computations in the BMS context given in appendix B of ref. [14] and appendix B of ref. [15]. These method used in those computations was to combine the Bondi-coordinate charge expression (FN,3.5) of ref. [14], which is known to be covariant, together with known nonlinear transformation of the symmetry generator vector fields on $\mathscr{I}^+$ to indirectly deduce the transformations of the metric functions. The results are limited to nonradiative regimes where $N_{AB} = 0$ since the charge expression (FN,3.5) was derived only in that context. Demonstrating consistency with the results of this paper requires using (i) the condition $N_{AB} = 0$; (ii) the simplifications discussed after eq. (3.22) above; (iii) the definition (3.6) of ref. [15] of the variable $\hat{N}_A$; and (iv) the

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evolution equation (FN,2.11a) for the Bondi mass aspect which shows that $\dot{m} = 0$ in vacuum when $N_{AB} = 0$.

- The nonlinear BMS transformation laws derived in appendices C.5 and C.6 of the book [13] by Chrusciel, Jezierski, and Kijowski, using the fact that their angular momentum aspect is related to ours by a factor of -3. For supertranslations the results agree, except for the transformation law for N_A, their eq. (C.124). The difference seems to arise from a discrepancy between our eq. (3.7b) for the function $\beta^{(2)}$ compared to their corresponding eq. (C.114). For boosts the results do not agree.[2]

- The nonlinear BMS transformation laws recently derived by Chen, Wang, Wang and Yau in ref. [16], which are consistent with our results when the simplifications discussed after eq. (3.22) above are used.

We also note that the action of finite transformations of the extended BMS group as well as Weyl transformations have been given by Barnich and Troessaert in ref. [35]. That paper uses the first-order Newman-Penrose formalism, which can be translated to the Bondi formalism used here using, for example, the results of ref. [36]. Also refs. [37, 38] have demonstrated that for non-radiative spacetimes, the mass and angular momentum aspects transform in the coadjoint representation of the extended BMS group, both at the linear and nonlinear levels.

5 Applications

5.1 Vacuum structure

One application of our nonlinear transformation results (3.2), (3.9), (3.15) and (3.19) is to obtain an explicit parameterization of "vacuum" states which in a local region of $\mathscr{I}^+$ are diffeomorphic to the data for Minkowski spacetime. This allows us to reproduce and generalize slightly the GBMS results of Compère, Fiorucci and Ruzziconi given in section 3 of ref. [17].

The result is as follows. First, following refs. [8, 9, 17] we define the Liouville or Geroch tensor $N^{\rm vac}_{AB}[h_{CD}]$, a function of the metric h_{CD}, as follows. We choose a conformal factor $e^{2\tau}$ to make $e^{2\tau}h_{AB}$ a unit round two-metric, by solving the equation

$$2D^2\tau + 2e^{2\tau} = \mathcal{R} \tag{5.1}$$

for τ. We then define the tensor

$$N^{\rm vac}_{AB} = 2e^{\tau}(D_A D_B - h_{AB}D^2/2)e^{-\tau}, \tag{5.2}$$

which has the property

$$D^A N^{\rm vac}_{AB} = -\frac{1}{2}D_B\mathcal{R}. \tag{5.3}$$

[2]While we have not been able to track down the source of the discrepancy in this case, we note that the boost results of Chrusciel et al. disagree with ref. [16] as well as with our results.

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General vacuum data can now be parameterized in terms of an arbitrary two-metric $h_{AB}(\theta^C)$ and a function $\beta(\theta^C)$ as follows:

$$N_{AB} = N^{\rm vac}_{AB}[h_{CD}], \tag{5.4a}$$
$$C_{AB} = (u+\beta)N^{\rm vac}_{AB}[h_{CD}] - 2(D_A D_B - h_{AB}D^2/2)\beta, \tag{5.4b}$$
$$m = -\frac{1}{8}C^{AB}N^{\rm vac}_{AB}[h_{CD}], \tag{5.4c}$$
$$N_A = 0. \tag{5.4d}$$

The result (5.4), when restricted to metrics h_{AB} whose volume form is fixed as appropriate for the GBMS group, agrees with eqs. (3.10) and (3.26) of [17][3]. The result (5.4) is also consistent with the results of section 4.7 of ref. [8] on the vacuum structure.

We obtain the form (5.4) of general vacuum data by applying a general BMSW transformation to Minkowski data. We parameterize the BMSW transformation by composing the two sphere diffeomorphism (3.13), the conformal transformation (3.8), and the supertranslation (3.1). We start with the Minkowski data

$$\mathring{h}_{AB}, \qquad \mathring{C}_{AB} = 0, \qquad \mathring{m} = 0, \qquad \mathring{N}_A = 0, \tag{5.5}$$

where $\mathring{h}_{AB}$ is a unit round two metric. Acting with the two-sphere diffeomorphism χ using eq. (3.15) gives the data

$$\bar{h}_{AB} = \chi_* \mathring{h}_{AB}, \qquad \bar{C}_{AB} = 0, \qquad \bar{m} = 0, \qquad \bar{N}_A = 0. \tag{5.6}$$

Next acting with the conformal transformation τ using eqs. (3.9) gives

$$\hat{h}_{AB} = e^{-2\tau}\chi_* \mathring{h}_{AB}, \tag{5.7a}$$
$$\hat{N}_{AB} = -2e^{-\tau}(\bar{D}_A\bar{D}_B - \bar{h}_{AB}\bar{D}^2/2)e^{\tau} = 2e^{\tau}(\hat{D}_A\hat{D}_B - \hat{h}_{AB}\hat{D}^2/2)e^{-\tau}, \tag{5.7b}$$
$$\hat{C}_{AB} = u\hat{N}_{AB}, \tag{5.7c}$$
$$\hat{m} = -\frac{1}{8}u\hat{N}_{AB}\hat{N}_{CD}\hat{h}^{AC}\hat{h}^{BD}, \tag{5.7d}$$
$$\hat{N}_A = 0. \tag{5.7e}$$

Here $\bar{D}_A$ and $\hat{D}_A$ are the derivative operators associated with $\bar{h}_{AB}$ and $\hat{h}_{AB}$ and we used $\bar{\mathcal{R}} = 2$. From eq. (5.7a) we can obtain generic two metrics $\hat{h}_{AB}$ by choosing χ and τ appropriately, and it will be convenient following [17] to use $\hat{h}_{AB}$ to parameterize the state rather than χ and τ. In the GBMS context of [17] the conformal transformation τ is constrained to be a function of χ, which constrains $\hat{h}_{AB}$ to have the same volume form as $\mathring{h}_{AB}$, as discussed in section 3.4 above.

[3]The comparison uses their relation (2.8) between the two different angular momentum aspects, and that their variables q_{AB}, Φ and C are given in terms of ours by $q_{AB} = h_{AB}$, $C = \beta$ and $\Phi = 2\tau - \ln\gamma_s$. Here γ_s is chosen so that the metric $\gamma_s^{-1}e^{2\tau}h_{AB}$ is flat. It is given explicitly by $\gamma_s = 2/(1+z\bar{z})^2$, where $(z,\bar{z})$ are the complex stereographic coordinates associated with the round metric $e^{2\tau}h_{AB}$.

The final step is to act with the supertranslation β using eqs. (3.2). This yields

$$h_{AB} = \hat{h}_{AB}, \tag{5.8a}$$
$$N_{AB} = \hat{N}_{AB} = 2e^{\tau}(D_A D_B - h_{AB} D^2/2)e^{-\tau}, \tag{5.8b}$$
$$C_{AB} = (u+\beta)N_{AB} - 2(D_A D_B - h_{AB} D^2/2)\beta, \tag{5.8c}$$
$$m = -\frac{1}{8} C_{AB} N_{CD} h^{AC} h^{BD}, \tag{5.8d}$$
$$N_A = 0. \tag{5.8e}$$

Here eqs. (5.8a) and (5.7a) yield the relation (5.1) that determines τ in terms of the final metric h_{AB}. We used the identities (5.3) and $\hat{N}_{AB}\hat{N}^B{}_C = \hat{N}_{BD}\hat{N}^{BD}\hat{h}_{AC}/2$, which is valid for any symmetric traceless tensor in two dimensions. The results (5.8) together with the definition (5.2) now yield the parameterization (5.4).

5.2 Other coordinates on phase space

The choice of Bondi coordinate metric functions is somewhat arbitrary, and is useful to consider other coordinates on the phase space that simplify the description. One organizing principle is to use variables which vanish in vacuum regions of $\mathscr{I}^+$ [17, 39]. Another is to use variables with simple transformation properties under conformal transformations [8–11, 40, 41]. These considerations lead to the following modified definitions of Bondi mass aspect, shear, and news tensor [8, 9, 17]:

$$\mathcal{M} = m + \frac{1}{8} C_{AB} N^{AB}, \tag{5.9a}$$
$$\mathcal{C}_{AB} = C_{AB} - u N^{\text{vac}}_{AB}[h_{CD}], \tag{5.9b}$$
$$\mathcal{N}_{AB} = \partial_u \mathcal{C}_{AB} = N_{AB} - N^{\text{vac}}_{AB}[h_{CD}], \tag{5.9c}$$

where N^{vac}_{AB} is defined in eq. (5.2). We do not need to redefine the angular momentum aspect N_A.[4] With these definitions $\mathcal{M}$, $\mathcal{N}_{AB}$ and N_A vanish in vacuum regions from eqs. (5.4)[5].

We can rewrite our transformation laws (3.2) and (3.9) in terms of the new variables (5.9), extending the linearized transformation laws of sections 4.5 of ref. [8] and 2.2 of ref. [9]. Under

[4]Note that our definition of N_A coincides with the covariant angular momentum aspect $\mathcal{P}_A$ of refs. [8, 9].

[5]Conversely, the condition that $\mathcal{M}, \mathcal{N}_{AB}$ and N_A all vanish is not sufficient to imply that the region is of the vacuum form (5.4). The additional condition needed is that the shear C_{AB} must have the form (5.4b), which is equivalent to the vanishing of the dual covariant mass (5.11) given that $\mathcal{N}_{AB} = 0$.

supertranslations we find from eqs. (3.2), (5.9) and (5.3) that

$$\begin{aligned}
\bar{h}_{AB}(\theta^C) &= h_{AB}(\theta^C), &(5.10a)\\
\bar{\mathcal{C}}_{AB}(u,\theta^A) &= \mathcal{C}_{AB}(u+\beta,\theta^A) - 2D_AD_B\beta + h_{AB}D^2\beta + \beta N^{\rm vac}_{AB}[h_{CD}], &(5.10b)\\
\bar{\mathcal{N}}_{AB}(u,\theta^A) &= \mathcal{N}_{AB}(u+\beta,\theta^A), &(5.10c)\\
\bar{\mathcal{M}}(u,\theta^A) &= \mathcal{M}(u+\beta,\theta^A) + \frac{1}{2}D_A\mathcal{N}^{AB}(u+\beta,\theta^A)D_B\beta \\
&\quad + \frac{1}{4}\dot{\mathcal{N}}^{AB}(u+\beta,\theta^A)D_A\beta D_B\beta, &(5.10d)\\
\bar{N}_A(u,\theta^A) &= N_A(u+\beta,\theta^A) + 3\mathcal{M}(u+\beta,\theta^A)D_A\beta + 3\mathscr{M}(u+\beta,\theta^A)\epsilon_A{}^BD_B\beta \\
&\quad + \frac{3}{2}D_A\beta\, D_B\beta\, D_C\mathcal{N}^{BC}(u+\beta,\theta^A) - \frac{3}{4}(D\beta)^2\, D^C\mathcal{N}_{AC}(u+\beta,\theta^A) \\
&\quad + \frac{1}{2}D_A\beta\, D_B\beta\, D_C\beta\dot{\mathcal{N}}^{BC}(u+\beta,\theta^A) \\
&\quad - \frac{1}{4}(D\beta)^2\, D^C\beta\dot{\mathcal{N}}_{CA}(u+\beta,\theta^A). &(5.10e)
\end{aligned}$$

Here $\mathscr{M}$ is the dual covariant mass defined by Freidel and Pranzetti [9] given by

$$\mathscr{M} = \frac{1}{8}C_{AB}\epsilon^{BC}N^A{}_C + \frac{1}{4}D_AD^B\epsilon^{AC}C_{BC}, \tag{5.11}$$

and the terms linear in β in eq. (5.10e) have been manipulated as described there. Note that the transformation laws (5.10) are much simpler than the original versions (3.2).

For conformal transformations we first compute the transformation law for $N^{\rm vac}_{AB}[h_{CD}]$. Denoting the solution of eq. (5.1) by $\tau[h_{CD}]$, we find that $\tau[e^{2\psi}h_{CD}] = \tau[h_{CD}] - \psi$, and it follows from the definition (5.2) that [cf. eq. (4.56) of ref. [8]]

$$N^{\rm vac}_{AB}[e^{2\psi}h_{CD}] = N^{\rm vac}_{AB}[h_{CD}] - 2e^{\psi}(D_AD_B - h_{AB}D^2/2)e^{-\psi}. \tag{5.12}$$

Now from eqs. (3.9), (5.9) and (5.3) we find that

$$\begin{aligned}
\bar{h}_{AB}(\theta^A) &= e^{-2\tau}h_{AB}(\theta^A), &(5.13a)\\
\bar{\mathcal{C}}_{AB}(u,\theta^A) &= e^{-\tau}\mathcal{C}_{AB}(e^\tau u,\theta^A), &(5.13b)\\
\bar{\mathcal{N}}_{AB}(u,\theta^A) &= \mathcal{N}_{AB}(e^\tau u,\theta^A), &(5.13c)\\
\bar{\mathcal{M}}(u,\theta^A) &= e^{3\tau}\mathcal{M}(e^\tau u,\theta^A) + \frac{1}{2}ue^{4\tau}D_A\mathcal{N}^{AB}(e^\tau u,\theta^A)D_B\tau \\
&\quad + \frac{1}{4}e^{5\tau}u^2\dot{\mathcal{N}}^{AB}(e^\tau u,\theta^A)D_A\tau D_B\tau, &(5.13d)\\
\bar{N}_A(u,\theta^A) &= e^{2\tau}N_A(e^\tau u,\theta^A) + 3ue^{3\tau}\mathcal{M}(e^\tau u,\theta^A)D_A\tau + 3ue^{3\tau}\mathscr{M}(e^\tau u,\theta^A)\epsilon_A{}^BD_B\tau \\
&\quad + \frac{3}{2}u^2e^{4\tau}D_A\tau\, D_B\tau\, D_C\mathcal{N}^{BC}(e^\tau u,\theta^A) - \frac{3}{4}u^2e^{4\tau}(D\tau)^2\, D^C\mathcal{N}_{AC}(e^\tau u,\theta^A) \\
&\quad + \frac{1}{2}u^3e^{5\tau}D_A\tau\, D_B\tau\, D_C\tau\dot{\mathcal{N}}^{BC}(e^\tau u,\theta^A) \\
&\quad - \frac{1}{4}u^3e^{5\tau}(D\tau)^2\, D^C\tau\dot{\mathcal{N}}_{CA}(e^\tau u,\theta^A). &(5.13e)
\end{aligned}$$

Again these transformation laws are much simpler than the original versions (3.9).

From eqs. (5.10) and (5.13) it follows that the transformation laws for the dual covariant mass $\mathscr{M}$ have the same forms (5.10d) and (5.13d) as for the mass $\mathcal{M}$, except that the modified news tensor $\mathcal{N}_{AB}$ is replaced by its dual $\tilde{\mathcal{N}}_{AB} = \epsilon_{AC}\mathcal{N}^C{}_B$. This was previously shown to linear order in ref. [9].

We also note that a different kind of simplification can achieved by defining the modified mass aspect

$$\mathbb{M} = m - \frac{1}{4}D^A D^B C_{AB}. \tag{5.14}$$

The transformation laws for this quantity under supertranslations and conformal transformations are

$$\bar{\mathbb{M}}(u,\theta^A) = \mathbb{M}(u+\beta,\theta^A) + \frac{1}{4}D^4\beta + \frac{1}{4}\mathcal{R}D^2\beta + \frac{1}{2}D^A\mathcal{R}D_A\beta, \tag{5.15}$$

and

$$\begin{aligned}\bar{\mathbb{M}}(u,\theta^A) = e^{3\tau}\mathbb{M}(e^\tau u,\theta^A) &+ \frac{1}{4}ue^{3\tau}\left[D^4e^\tau + \mathcal{R}D^2e^\tau + 2D^A\mathcal{R}D_Ae^\tau\right] \\ &+ \frac{1}{2}e^{2\tau}C^{AB}(e^\tau u,\theta^A)D_AD_Be^\tau - ue^{3\tau}(D^AD^Be^\tau)(D_AD_B - h_{AB}D^2/2)e^\tau. \end{aligned} \tag{5.16}$$

The terms on the right hand side in eq. (5.16) other than the first term vanish in the BMS context, when $\mathcal{R}=2$ and e^τ is purely $l=0$ and $l=1$. This mass definition is therefore natural to use in the BMS context, but somewhat less so in the more general BMSW context.

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