

Spectral curves and \mathcal{W} -representations of matrix models

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ABSTRACT: We explain how the spectral curve can be extracted from the \mathcal{W} -representation of a matrix model. It emerges from the part of the \mathcal{W} -operator, which is linear in time-variables. A possibility of extracting the spectral curve in this way is important because there are models where matrix integrals are not yet available, and still they possess all their important features. We apply this reasoning to the family of WLZZ models and discuss additional peculiarities which appear for the non-negative value of the family parameter n , when the model depends on additional couplings (dual times). In this case, the relation between topological and $1/N$ expansions is broken. On the other hand, all the WLZZ partition functions are τ -functions of the Toda lattice hierarchy, and these models also celebrate the superintegrability properties.

KEYWORDS: Matrix Models, $1/N$ Expansion, Integrable Hierarchies

ARXIV EPRINT: [2210.09993](https://arxiv.org/abs/2210.09993)

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1 Introduction

Many properties of matrix models [1–11] are defined by their spectral curves, which define the distribution of eigenvalues in the large N limit, and is a generating function of all the genus zero contributions to the single-trace correlators. This does not seem to be much, still, if the whole set of Virasoro-like constraints is available, the spectral curve (with some simple additional data) is sufficient to reproduce all correlators at all genera, the relevant procedure is known as the AMM-EO topological recursion [12–18]. In this sense, the knowledge of the spectral curve is nearly equivalent to that of the entire matrix model.

On the other hand, nowadays matrix model partition functions are defined not only by an explicit matrix (or eigenvalue) integral, but also by action of an operator $\hat{\mathcal{W}}$ on a trivial state in the space of matrix model couplings p_k :

$$Z\{p\} = e^{\hat{\mathcal{W}}\{p\}} \cdot 1 \tag{1.1}$$

Such a realization is called W -representation [19–25] (see also similar realizations in [12–14, 26–31]). Sometimes, it is better to present it in the form

$$Z\{p\} = e^{\hat{\mathcal{W}}\{p\}} \cdot e^{\sum_k g_k p_k / k} \tag{1.2}$$

where g_k are parameters. Such second form can be definitely reduced to (1.1) using the Campbell-Hausdorff formula, but the resulting $\hat{\mathcal{W}}$ -operator is too complicated. Hence, the form (1.2) is more preferable in such a case.

It is a natural question to ask what is the spectral curve, and the topological recursion in terms of this W -representation. Once understood, this would provide spectral curves for models that are not defined through any integrals.

Our claim in this paper is that the spectral curve is associated with a peculiar part $\hat{\mathcal{W}}^{\text{spec}}$ of the $\hat{\mathcal{W}}$ -operator. We demonstrate this in detail for the Gaussian model, and then discuss implication for the other cases, mostly for the WLZZ models [32]. In particular, we give a general recipe for constructing $\hat{\mathcal{W}}^{\text{spec}}$ in these cases. More formally, our claims are:

- $\hat{\mathcal{W}}^{\text{spec}}$ is made from all the terms of $\hat{\mathcal{W}}$, which are linear in p (but these terms can be non-linear in p -derivatives, like $\frac{\partial^2}{\partial p^2}$ in the Gaussian model)
- The action of $e^{\hat{\mathcal{W}}^{\text{spec}}}$ produces an exponential of an expression, which is *linear in p* (and no longer contains p -derivatives)

$$e^{\hat{\mathcal{W}}^{\text{spec}}} \cdot 1 = \exp(\mathcal{P}) \tag{1.3}$$

Thus we see a *mysterious role of the exponential function*.

- Making the substitution $p_k \rightarrow z^k p_k$ allows one to generate the function (resolvent) $y(z)$ such that

$$\mathcal{P}(z) = \oint V(xz) y(x) dx \tag{1.4}$$

where $V(z) = \sum_k p_k z^k / k$ is the matrix model potential. The resolvent $y(z)$ satisfies the spectral curve equation.

- For WLZZ models [32] with negative grading $m < 0$, the resolvent

$$y_m(z) = \sum_k \frac{|m|k}{z^{|m|k+1}} \frac{\partial \log \mathcal{Z}}{\partial p_{|m|k}} \tag{1.5}$$

satisfies the spectral curve that is a simple generalization of the semicircle distribution equation at $m = -2$:

$$y_m^{|m|} - zy_m + 1 = 0 \tag{1.6}$$

This spectral curve equation describes the large N limit, and corresponds to the leading behaviour of the topological expansion. The complete expansion is constructed from the full set of W -constraints

$$\left(n \frac{\partial}{\partial p_n} + \widetilde{W}_{n-m}^{(-,m)} \right) Z = 0 \tag{1.7}$$

where the operators \hat{W}_{-m} generating the W -representation are connected with $\widetilde{W}_{n-m}^{(-,m)}$ by the relation

$$\hat{W}_{-m} = \sum_{k=1}^{\infty} p_k \widetilde{W}_{n-m}^{(-,m)} \tag{1.8}$$

- In the WLZZ model at $m = 0$, the \mathcal{W}_0 annihilates 1, and one should act on a non-trivial state $\exp(\beta p_1)$. The spectral curve is given by the Lambert curve. One can naturally extend this \mathcal{W}_0 -operator to a series of operators associated with the generalized cut-and-join operators $W_{[s]}$, which gives rise to higher Lambert curves (the \mathcal{W}_0 case corresponds to $s = 2$)

$$ye^{-z^{s-1}y^{s-1}} = \frac{\beta}{z^2} \tag{1.9}$$

- In the WLZZ model at $m > 0$, the \mathcal{W}_m also annihilates 1, but one should act on a non-trivial state $\exp(\sum_k g_k p_k)$. This makes the situation more involved and intriguing, because one acquires new parameters (dual time-variables) g_k . In this case, the spectral curve is given by the equation

$$y = \sum_{k=2} \frac{g_k}{z^{k+1}} \left(1 + z^{2m-1} y^{m-1} \right)^{k-1} \tag{1.10}$$

and is associated with the *small(!)* N expansion, so that the interpretation in terms of topological expansion is no longer straightforward.

- Superintegrability relations trivialize in the large N limit: the averages become *linear* in each sector of a given grading

$$\langle S_R \rangle_{\infty} = N^{|R|} S_R \{ \delta_{k,n} \} \tag{1.11}$$

and multiple correlators of characters factorize

$$\langle \prod_i S_{R_i} \rangle_{\infty} = \prod_i \langle S_{R_i} \rangle_{\infty} \tag{1.12}$$

where S_R are the Schur functions [33]. The Schur functions can be treated as symmetric polynomials of variables x_i , or graded polynomials of power sums $p_k = \sum_i x_i^k$. In the later case, we use the notation $S_R\{p_k\}$. We also use the shortened notation $S_R\{N\} = S_R\{p_k = N\}$ and $S_R\{\delta_{k,m}\} = S_R\{p_k = \delta_{k,m}\}$.

In the paper, we are mostly considering the WLZZ partition functions [32], which are introduced by W -representations:

$$Z_m = e^{\hat{W}_m/m} \cdot \begin{cases} 1 & \text{for } m < 0 \\ \exp\left(\sum_k \frac{g_k p_k}{k}\right) & \text{for } m > 0 \end{cases} \quad (1.13)$$

and the Hurwitz partition function [27, 34, 35]

$$Z_0 = e^{\hat{W}_0} \cdot e^{\beta p_1} \quad (1.14)$$

These definitions are inverse to the ordinary definition, when one searches for a \mathcal{W} -representation for a given model with nice properties. There is no *a priori* reason to expect that this new formulation is *obliged* to provide interesting results. However, we demonstrate that it does. The results look relatively simple for the negative branch of the WLZZ models. In this case, the WLZZ models are described by the two-matrix models of the Hurwitz type [36–39] counting the isomorphism classes of Belyi pairs, arising in the study of equilateral triangulations and Grothendieck’s *dessins d’enfant* [40–43].

Things get more intriguing for $m = 0$, related both with the Hurwitz numbers and, by the ELSV formula, with the Kontsevich model deformation involving the Hodge integrals [29, 30], and even more intriguing for positive m , when the WLZZ models are described by two-matrix models in an external field [38, 39].

The WLZZ family includes two one-matrix model examples: for $m = \pm 2$. Another example well-studied earlier is $m = 0$. We underline the subsections devoted to these three particular cases.

Let us stress that our definition of the spectral curve follows the recursion procedure worked out in [28]: one starts from the loop equation for the resolvent and then considers the leading order that excludes terms with derivatives of the resolvent. In this leading order, the loop equation becomes an algebraic equation, which is the spectral curve. The recursion is further constructed as an expansion of the resolvent over this spectral curve. In this paper, we directly applied this procedure in the cases of $m = \pm 2$ for the corresponding one-matrix models, and managed to confirm our scheme for constructing the spectral curve (in particular, the expansion parameters of large and small N for $m = 2$ and $m = -2$, correspondingly).

Finally, let us note that manifest expressions for the results of action of the W -operators like (3.31) in this paper are conjectural and checked by computer evaluations of many terms of expansion of the exponential and at many values of m (both negative and positive). One can prove such formulas in simple cases like (2.11), (3.26) reducing them to an equation like (3.4), however, in other, more complicated cases this way of proof becomes already too involved.

2 Basic example: Hermitian Gaussian model

2.1 Description of the model

In this section, we explain how one can construct the spectral curve from the W -representation in the simplest case of the Hermitian Gaussian matrix model. The partition function of this model satisfies an infinite set of Virasoro constraints (generated by the Borel subalgebra of the Virasoro algebra):

$$L_n Z_{-2} = (n+2) \frac{\partial Z_{-2}}{\partial p_{n+2}}, \quad n \geq -1$$

$$L_n = \sum_k (k+n)p_k \frac{\partial}{\partial p_{k+n}} + \sum_{a=1}^{n-1} a(n-a) \frac{\partial^2}{\partial p_a \partial p_{n-a}} + 2Nn \frac{\partial}{\partial p_n} + N^2 \delta_{n,0} + Np_1 \delta_{n+1,0}$$
(2.1)

and has a W -representation of the form

$$Z_{-2} = e^{\frac{1}{2} \hat{W}_{-2}} \cdot 1$$
(2.2)

$$\hat{W}_{-2} = \sum_k p_k \hat{L}_{k-2} = \sum_k (k+l-2)p_k p_l \frac{\partial}{\partial p_{k+l-2}} + \sum_{k,l} kl p_{k+l+2} \frac{\partial^2}{\partial p_k \partial p_l} + 2N \sum_k k p_{k+2} \frac{\partial}{\partial p_k} + N^2 p_2 + Np_1^2$$

Because the r.h.s. of the Virasoro constraints is of different grading, this partition function is an unambiguous solution to the non-trivial equation

$$(\hat{l}_0 - \hat{W}_{-2}) Z_{-2} = 0, \quad \hat{l}_0 := \sum_k k p_k \frac{\partial}{\partial p_k}$$
(2.3)

This solution is actually equal to

$$Z_{-2} = \sum_R \frac{S_R\{N\} S_R\{\delta_{k,2}\}}{d_R} S_R\{p_k\}$$
(2.4)

which is a sum over the Schur functions with factorized coefficients proportional to the same Schur functions evaluated at special loci. Such specific form of a sum is nicknamed a *superintegrability* property [44–46].

2.2 Spectral curve

Let us now make the following trick:

- 1) introduce the variables $t_k := \frac{p_k}{k\hbar}$,
- 2) introduce t_0 such that $\frac{\partial Z_{-2}}{\partial t_0} := N Z_{-2}$.

Then, one can rewrite the Virasoro generators (2.1) in the form

$$L_n = \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \hbar^2 \sum_{k \geq 0} \frac{\partial^2}{\partial t_k \partial t_{n-k}} \quad (2.5)$$

Let us now define the resolvent

$$\begin{aligned} \rho_{-2}(z|t_k) &:= \hat{\nabla}_z \mathcal{F}_{-2} = \sum_{k \geq 0} \frac{1}{z^{k+1}} \frac{\partial \mathcal{F}_{-2}}{\partial t_k} \\ \mathcal{F}_{-2} &:= \hbar^2 \log Z_{-2} \end{aligned} \quad (2.6)$$

Then, the generating function of all the Virasoro constraints by converting L_n 's with z^{-n-2} can be rewritten in the form:

$$\rho_{-2}(z|t_k)^2 - z \rho_{-2}(z|t_k) + N + \hbar^2 \hat{\nabla}_z \rho_{-2}(z|t_k) + \left[V'(z) \rho_{-2}(z|t_k) \right]_- \quad (2.7)$$

where $V(z) := \sum_{k=1}^{\infty} \frac{p_k}{k} z^k = \sum_{k=1}^{\infty} t_k z^k$ is the potential of the matrix model, and $\left[\dots \right]_-$ denotes projection to the negative powers of z .

Consider now the solution at the leading (spherical) order at small \hbar and at zero p_k at all k , $y := \rho_{-2}(z|0) \Big|_{\hbar \rightarrow 0}$. Then, one obtains the equation

$$y^2 - zy + N = 0 \quad (2.8)$$

which is exactly the spectral curve. Actually, in this particular case this is the Riemann sphere in hyperelliptic representation.

Parameter \hbar can be used to define topological expansion [28, 47–49] and AMM-EO topological recursion [12–18]. In this case, one can identify $\hbar = 1/N$ by rescaling time-variables $t_k \rightarrow N t_k$ (i.e. by making the substitution $\log Z_{-2} = N^2 \mathcal{F}_{-2}$): this allows one to identify topological and $1/N$ expansions. As we shall see, this identification appears consistent for all WLZZ models with $m < 0$, but breaks down for $m \geq 0$.

2.3 Spectral curve from the \mathcal{W} -representation

Now let us note that the leading order in the loop equations is completely due to the second derivatives terms in \hat{L}_n and, hence, in $\hat{\mathcal{W}}_{-2}$. Hence, one could naturally expect that, in order to generate the leading term \mathcal{F}_0 , one has to truncate the \mathcal{W} -representation to the second derivative terms only:

$$\hat{\mathcal{W}}_{-2}^{\text{spec}} := \sum_{a,b \geq 0} (a+b+2) t_{a+b+2} \frac{\partial^2}{\partial t_a \partial t_b} = \sum_{a,b \geq 1} a b p_{a+b+2} \frac{\partial^2}{\partial p_a \partial p_b} + 2N \sum_{a \geq 1} a p_{a+2} \frac{\partial}{\partial p_a} + N^2 p_2 \quad (2.9)$$

Now we define

$$e^{\frac{z}{2} \hat{\mathcal{W}}_{-2}^{\text{spec}}} \cdot 1 = e^{\mathcal{P}_{-2}(z)} \quad (2.10)$$

and realize that

$$\mathcal{P}_{-2}(z) = N \sum_{k=1}^{\infty} \frac{p_{2k}}{2k} \frac{2 \cdot \Gamma(2k)}{\Gamma(k+2)\Gamma(k)} (N z^2)^k \quad (2.11)$$

Note that the exponential \mathcal{P} is linear in p_k . This *exponentiation* phenomenon in (2.10) takes place for all other models in this text. It is not a priori that obvious, and it is explained by the structure of the Campbell-Hausdorff formula, see section 6.

On the other hand, expanding (2.8) at large z , one obtains

$$y(z) = \frac{z - \sqrt{z^2 - 4N}}{2} = y(z) = \frac{N}{z} + \sum_{k=1}^{\infty} \frac{2 \cdot \Gamma(2k)}{\Gamma(k+2)\Gamma(k)} \cdot z^{-2k-1} N^{k+1} \quad (2.12)$$

Thus, one obtains

$$\mathcal{P}_{-2}(z) = \oint V(xz) y(x) dx \quad (2.13)$$

As we already pointed out, one expects that $\mathcal{P}_{-2}(z)$ is a leading contribution to the partition function at small g . Indeed, this formula can be compared with [50, eq. (48)] to see this is really the case. Here we derived this contribution completely in terms of the W -representation.

Note that the N -dependence for $\hat{\mathcal{W}}^{\text{spec}}$ can be fully eliminated by the change of variables $p_k \rightarrow p_k/N$ and $z^2 \rightarrow z^2/N$. Remarkably, this is a kind of opposite to the change $p_k \rightarrow Np_k$, relevant for the definition of the $1/N$ expansion at the end of the section 2.2.

Note that the true partition function corresponds to the particular value $z^2 = 1$ in (2.10) with the full-fledged operator $\hat{\mathcal{W}}_{-2}$. Spectral curve, however, appears when we truncate the operator to $\hat{\mathcal{W}}_{-2}^{\text{spec}}$ and *release* z . As we explain in sections 4–5, this procedure can be formulated in another, more universal way by making a substitution $p_k \rightarrow z^k p_k$ instead of releasing z .

3 An infinite set of WLZZ models. Negative branch

3.1 Description of the models

The authors of [32] proposed an infinite set of models parameterized by integers. The models parameterized by negative integers generalize the Hermitian Gaussian model and are described by the following procedure: one constructs the \mathcal{W} -representations of these models using the operators built by a recursive procedure

$$\hat{\mathcal{W}}_{-m-1} = \frac{1}{m!} [\hat{\mathcal{W}}_{-1}, \hat{\mathcal{W}}_{-m}], \quad m \geq 2 \quad (3.1)$$

where

$$\hat{\mathcal{W}}_{-1} := \sum_k (k+l-1) p_k p_l \frac{\partial}{\partial p_{k+l-1}} + \sum_{k,l} k l p_{k+l+1} \frac{\partial^2}{\partial p_k \partial p_l} + 2N \sum_k k p_{k+1} \frac{\partial}{\partial p_k} + N^2 p_1 \quad (3.2)$$

Every such operator gives rise to a partition function

$$Z_{-m} = e^{\frac{1}{m} \hat{\mathcal{W}}_{-m}} \cdot 1 \quad (3.3)$$

which is an unambiguous solution to the equation

$$\left(\hat{l}_0 - m \hat{\mathcal{W}}_{-m} \right) Z_{-n} = 0, \quad \hat{l}_0 := \sum_k k p_k \frac{\partial}{\partial p_k} \quad (3.4)$$

Among other things, this means that the matrix models that describes Z_{-m} could not be of the X^m -potential type: there should be only one possible integration contour. This is, indeed, the case, see (3.7).

The partition function, (3.3) is equal (for $m \geq 2$) to

$$Z_{-m} = \sum_R \frac{S_R\{N\}S_R\{\delta_{k,m}\}}{d_R} S_R\{p_k\} \quad (3.5)$$

and is a KP τ -function of the hypergeometric type [36, 51–54].

Representation (3.5) for the partition function implies the superintegrability relation for the correlator [45],

$$\langle S_R\{P_k\} \rangle = \frac{S_R\{N\}S_R\{\delta_{k,m}\}}{S_R\{\delta_{k,1}\}} \quad (3.6)$$

where $P_k = \text{tr } M^k$ are traces of matrices at the matrix WLZZ model.

This partition function (3.5) can be realized by the two-matrix integral [36, 37]

$$Z_{-m} = \iint_{N \times N} dX dY \exp \left(-\text{Tr } XY + \sum_k \frac{p_k}{k} \text{Tr } X^k + \frac{1}{m} \text{Tr } Y^m \right) \quad (3.7)$$

Here the integral is understood as integration of a power series in p_k , and X are Hermitian $N \times N$ matrices, while Y are anti-Hermitian $N \times N$ ones. Such partition function satisfies the so called \widetilde{W} -algebra constraints [55]:

$$\widetilde{W}_n^{(-,m)} Z_{-m} = (n+m) \frac{\partial Z_{-m}}{\partial p_{n+m}}, \quad n \geq -m+1 \quad (3.8)$$

There are spin m \widetilde{W} -algebras of two types which we denote $\widetilde{W}^{(\pm,m)}$, their generators being defined [55, 56] (see also [36, section 7]) by any of the following three relations:

$$\left(\frac{\partial}{\partial \Lambda} \right)^{m+1} f(p_k) = \sum_{n \geq 1} \Lambda^{\pm n-1} \widetilde{W}_{n \pm m}^{(\pm, m+1)}(p_k) f(p_k) \Big|_{p_k = \text{Tr } \Lambda^{\pm k}} \quad (3.9)$$

or¹

$$\widetilde{W}_{n \pm m}^{(\pm, m+1)}(t) e^{\sum_{k \geq 0} t_k \text{Tr } \Lambda^{\mp k}} = \text{Tr} \left\{ \left(\frac{\partial}{\partial \Lambda} \right)^m \Lambda^{\mp n} \right\} e^{\sum_{k \geq 0} t_k \text{Tr } \Lambda^{\mp k}} \quad (3.10)$$

or

$$\widetilde{W}_n^{(\pm, m+1)}(p) = \sum_{k \geq 1} p_k \widetilde{W}_{n+k}^{(\pm, m)}(p) + \sum_{k=1}^{n \mp (m-1/2) - 1/2} k \frac{\partial}{\partial p_k} \widetilde{W}_{n-k}^{(\pm, m)}(p) + N \widetilde{W}_n^{(\pm, m-1)} \quad (3.11)$$

The last recurrence relation should be supplemented by “the initial condition”

$$\begin{aligned} \widetilde{W}_n^{(\pm, 1)} &= n \frac{\partial}{\partial p_n}, \quad n \geq 1 \\ \widetilde{W}_0^{(\pm, 1)} &= N \end{aligned} \quad (3.12)$$

and one generally requires that

$$\widetilde{W}_n^{(\pm, m)} = 0, \quad n \leq -m \quad (3.13)$$

¹In eq. (3.10), we again introduced $t_k := p_k/k$ in order to switch on the variable t_0 , which makes the formulas simpler.

3.2 Model with $m \geq 3$

3.2.1 \mathcal{W} -representation of the $m = 3$ model

Let us start from the simplest example of $m = 3$. Then, one can obtain from (3.1) and (2.2)

$$\begin{aligned}
 \hat{\mathcal{W}}_{-3} = \frac{1}{2}[\hat{\mathcal{W}}_{-1}, \hat{\mathcal{W}}_{-2}] &= \sum_{k,l,m} mp_k p_l p_{m-k-l+3} \frac{\partial}{\partial p_m} + \frac{3}{2} \sum_{k,l,m} k m p_l p_{k+m-l+3} \frac{\partial^2}{\partial p_m \partial p_k} \\
 &+ \sum_{k,l,m} k l m p_{k+l+m+3} \frac{\partial^3}{\partial p_k \partial p_l \partial p_m} + \frac{1}{2} \sum_k k(k+1)(k+2) p_{k+3} \frac{\partial}{\partial p_k} \\
 &+ 3N \sum_{k,l} k l p_{k+l+3} \frac{\partial^2}{\partial p_k \partial p_l} + 3N \sum_{k,l} l p_k p_{l-k+3} \frac{\partial}{\partial p_l} + N(p_1^3 + p_3) \\
 &+ 2N^2 \sum_k k p_{k+3} \frac{\partial}{\partial p_k} + 3N^2 p_1 p_2 + N^3 p_3
 \end{aligned} \tag{3.14}$$

In accordance with the general rule of constructing \mathcal{W} -representations [24], this operator can be rewritten as

$$\hat{\mathcal{W}}_{-3} = \sum_k p_k \hat{W}_{k-3}^{(3)} \tag{3.15}$$

with a simple and immediate operator

$$\begin{aligned}
 \hat{W}_n^{(3)} &= \sum_{m,l} (n+m+l) p_l p_m \frac{\partial}{\partial p_{m+l+n}} + \frac{3}{2} \sum_{m,l} l m p_{l+m-n} \frac{\partial^2}{\partial p_l \partial p_m} \\
 &+ \sum_{m,l} (n-m-l) m l \frac{\partial^3}{\partial p_m \partial p_l \partial p_{n-m-l}} + \frac{n(n+1)(n+2)}{2} \frac{\partial}{\partial p_n} \\
 &+ 3N \sum_k (k+n) p_k \frac{\partial}{\partial p_{k+n}} + 3N \sum_k k(n-k) \frac{\partial^2}{\partial p_k \partial p_{n-k}} + 3N^2 n \frac{\partial}{\partial p_n} \\
 &+ N(N^2 + 1) \delta_{n,0} + 3N^2 p_1 \delta_{n,-1} + N p_1^2 \delta_{n,-2}
 \end{aligned} \tag{3.16}$$

One may think that again, similarly to the $m = -2$ case, the infinite set of constraints satisfied by the partition function is

$$\hat{W}_n^{(3)} Z_{-3} \stackrel{?}{=} (n+3) \frac{\partial Z_{-3}}{\partial p_{n+3}}, \quad n \geq -2 \tag{3.17}$$

However, this is not the case: for instance,

$$\begin{aligned}
 \hat{W}_{-2}^{(3)} Z_{-3} - \frac{\partial Z_{-3}}{\partial p_1} &= -2N^2 p_2 + \dots \\
 \hat{W}_{-1}^{(3)} Z_{-3} - 2 \frac{\partial Z_{-3}}{\partial p_2} &= 2N^2 p_1 + \dots
 \end{aligned} \tag{3.18}$$

and only in the sum (3.15) the first terms cancel with each other. It is related to the fact is that the set of equations (3.17) does not admit any non-trivial solutions.

The point is that the representation of $\hat{\mathcal{W}}_{-3}$ as a sum (3.15) is not unique, and one can equivalently realize $\hat{\mathcal{W}}_{-3}$ instead of (3.15) as

$$\hat{\mathcal{W}}_{-3} = \sum_k p_k \widetilde{\mathcal{W}}_{k-3}^{(-,3)} \quad (3.19)$$

with

$$\begin{aligned} \widetilde{\mathcal{W}}_n^{(-,3)} = & \sum_{a+b+c=n} abc \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} + \sum_{k=1} p_k \left(\sum_{a=1}^{n+1} + \sum_{a=1}^{n+k-1} \right) a(k+n-a) \frac{\partial^2}{\partial p_a \partial p_{k+n-a}} \\ & + \sum_{a,b=1} (a+b+n) p_a p_b \frac{\partial}{\partial p_{a+b+n}} + \frac{n(n+1)(n+2)}{2} \frac{\partial}{\partial p_n} \\ & + 3N \sum_{a=1}^{n-1} a(n-a) \frac{\partial^2}{\partial p_a \partial p_{n-a}} + 3N \sum_{k=1} (n+k) p_k \frac{\partial}{\partial p_{n+k}} + 3N^2 n \frac{\partial}{\partial p_n} + N(n+1) p_1 \frac{\partial}{\partial p_{n+1}} \\ & + N(N^2+1) \delta_{n,0} + 2N^2 p_1 \delta_{n,-1} + N(p_1^2 + N p_2) \delta_{n,-2} - N \sum_k k p_{k+2} \frac{\partial}{\partial p_k} \delta_{n,-2} \end{aligned} \quad (3.20)$$

and the equation

$$\widetilde{\mathcal{W}}_n^{(-,3)} Z_{-3} = (n+3) \frac{\partial Z_{-3}}{\partial p_{n+3}}, \quad n \geq -2 \quad (3.21)$$

is satisfied, in accordance with (3.8).

3.2.2 Spectral curve

In order to construct the loop equation, one has to repeat the procedure of section 2.2. The only difference is that now one has to convert the constraints (3.21) at $m = 3$ with z^{-n-3} . Again,

$$\rho_{-3}(z|t_k) := \hat{\nabla}_z \mathcal{F}_{-3} = \sum_{k \geq 0} \frac{1}{z^{k+1}} \frac{\partial \mathcal{F}_{-3}}{\partial t_k} \quad (3.22)$$

and introducing $y := \rho_{-3}(z|0) \Big|_{g \rightarrow 0}$, one obtains the spectral curve equation

$$y^3 - zy + N = 0 \quad (3.23)$$

3.2.3 Spectral curve from the \mathcal{W} -representation

In order to relate solution of the spectral curve equation, (3.23) with the \mathcal{W} -representation, we again truncate the \mathcal{W} -representation to the third derivative terms only:

$$\begin{aligned} \hat{\mathcal{W}}_{-3}^{\text{spec}} & := \sum_{a,b,c \geq 0} (a+b+c+3) t_{a+b+c+3} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c} \\ & = N^3 p_3 + 3N^2 \sum_{a=1}^{\infty} a p_{a+3} \frac{\partial}{\partial p_a} + 3N \sum_{a,b=1}^{\infty} a b p_{a+b+3} \frac{\partial^2}{\partial p_a \partial p_b} + \sum_{a,b,c=1}^{\infty} a b c p_{a+b+c+3} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} \end{aligned} \quad (3.24)$$

We again define

$$e^{\frac{z}{3} \hat{\mathcal{W}}_{-3}^{\text{spec}}} \cdot 1 = e^{\mathcal{P}_{-3}(z)} \quad (3.25)$$

and realize that

$$\mathcal{P}_{-3}(z) = N \sum_{k=1}^{\infty} \frac{3\Gamma(3k)(z^3 N^2)^k}{\Gamma(2k+2)\Gamma(k)} \frac{p_{3k}}{3k} \tag{3.26}$$

On the other hand, expanding (3.23) at large z , one obtains

$$y(z) = \frac{N}{z} + \sum_{k=1}^{\infty} \frac{3 \cdot \Gamma(3k)}{\Gamma(2k+2)\Gamma(k)} \cdot z^{-3k-1} N^{2k+1} \tag{3.27}$$

Thus, one obtains

$$\mathcal{P}_{-3}(z) = \oint V(xz) y(x) dx \tag{3.28}$$

3.3 Model with generic m

Formulas are just the same for any $\hat{\mathcal{W}}_{-m}^{\text{spec}}$, for example

$$\begin{aligned} \hat{\mathcal{W}}_{-4}^{\text{spec}} := & N^4 p_4 + 4N^3 \sum_{a=1}^{\infty} a p_{a+4} \frac{\partial}{\partial p_a} + 6N^2 \sum_{a,b=1}^{\infty} a b p_{a+b+4} \frac{\partial^2}{\partial p_a \partial p_b} \\ & + 4N \sum_{a,b,c=1}^{\infty} a b c p_{a+b+c+4} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} + \sum_{a,b,c,d=1}^{\infty} a b c d p_{a+b+c+d+4} \frac{\partial^4}{\partial p_a \partial p_b \partial p_c \partial p_d} \end{aligned} \tag{3.29}$$

Defining again

$$e^{\frac{z^m}{m} \hat{\mathcal{W}}_{-m}^{\text{spec}}} \cdot 1 = e^{\mathcal{P}_{-m}(z)} \tag{3.30}$$

one obtains

$$\mathcal{P}_{-m}(z) = N \sum_{k=1}^{\infty} \frac{m\Gamma(mk)(z^m N^{m-1})^k}{\Gamma((m-1)k+2)\Gamma(k)} \frac{p_{mk}}{mk} \tag{3.31}$$

The spectral curve equation in this case is

$$y^m - zy + N = 0 \tag{3.32}$$

Expanding its solution at large z , one obtains

$$y(z) = \frac{N}{z} + \sum_{k=1}^{\infty} \frac{m \cdot \Gamma(mk)}{\Gamma((m-1)k+2)\Gamma(k)} \cdot z^{-mk-1} N^{(m-1)k+1} \tag{3.33}$$

Thus, one again obtains

$$\mathcal{P}_{-m}(z) = \oint V(xz) y(x) dx \tag{3.34}$$

One can always invert the procedure and start from the \mathcal{W} -representation, pick up the terms with maximal number of derivatives, and calculate the corresponding \mathcal{P} . After this, one calculates $y(x)$ (up to the first term, which is always N/z) and then find an equation that is satisfied by this y . This equation is exactly the spectral curve. When only p -linear terms are kept, i.e. when we deal with $\hat{\mathcal{W}}^{\text{spec}}$, the matrix size N can always be eliminated by the change $p_k \rightarrow p_k/N$, $z \rightarrow z/N^{\frac{m-1}{m}}$. This scheme works for all the models we considered so far, however, it has to be improved in some points as we shall see in the next two sections.

4 From Hurwitz model to Lambert curves

4.1 Hurwitz model and its spectral curve

There is also a model in between negative and positive branches of the WLZZ models.² This model is given by the \mathcal{W} -representation

$$\hat{\mathcal{W}}_0 = \sum_{a,b} abp_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} + \sum_{a,b} (a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + N \sum_a ap_a \frac{\partial}{\partial p_a} \quad (4.1)$$

and is nothing but the Hurwitz partition function [34, 35, 57]

$$Z_0 = e^{x\hat{\mathcal{W}}_0} \cdot e^{\beta p_1} = \sum_R \beta^{|R|} S_R\{\delta_{k,1}\} S_R\{p_k\} e^{xC_2(R)} \quad (4.2)$$

where $C_2(R)$ denotes the eigenvalue of second Casimir operator: $C_2(R) = \sum_{i,j \in R} (N+j-i)$. This partition function is also a KP τ -function of the hypergeometric type [27, 36, 51, 54].

Now the spherical limit again is governed by the part of this operator with maximal number of derivatives:

$$\hat{\mathcal{W}}_0^{\text{spec}} = \sum_{a,b} abp_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} + N \sum_a ap_a \frac{\partial}{\partial p_a} \quad (4.3)$$

One obtains

$$e^{x\hat{\mathcal{W}}_0^{\text{spec}}} \cdot e^{\beta p_1} = e^{\mathcal{P}_0(x)} \quad (4.4)$$

with

$$\mathcal{P}_0(x) = \frac{1}{x} \sum_{k \geq 1} \frac{(2k)^{k-1}}{k!} (\beta e^{xN} x)^k \frac{p_k}{k} \quad (4.5)$$

This is the first time, when we can observe that the coefficient in front of the \mathcal{W} -operator, x does not provide a good spectral parameter, since it no longer provides grading (because of the term e^{xN}). This is because the exponential of \mathcal{W} -operator acts not on the unity, but on the exponential of times. Hence, from now on, we use another procedure, which does not give anything new in the earlier considered cases, but will be of use in the forthcoming considerations. That is, we use x as a free parameter that can be chosen in a convenient way (it can be removed by rescalings of other parameters), and instead we *introduce the spectral parameter z by making a substitution $p_k \rightarrow z^k p_k$* .

Hence, our general prescription for making the spectral curve from the $\hat{\mathcal{W}}^{\text{spec}}$ -operator is:

²The operator (4.1) of this model generates both the operator

$$\hat{\mathcal{W}}_1 = \left[\hat{\mathcal{W}}_0, \left[\hat{\mathcal{W}}_0, \frac{\partial}{\partial p_1} \right] \right]$$

and the operator

$$\hat{\mathcal{W}}_{-1} = \left[\hat{\mathcal{W}}_0, \left[\hat{\mathcal{W}}_0, p_1 \right] \right]$$

generating the positive and negative branches accordingly, sections 5 and 3.

- (i) to use this operator instead of the full operator in the \mathcal{W} -representation;
- (ii) to demonstrate that this produces a linear exponential in p_k 's, and gives rise to $\mathcal{P}(z)$, where the z -dependence is introduced by making the substitution $p_k \rightarrow z^k p_k$;
- (iii) to use the formula

$$\mathcal{P}_0(z) = \oint V(wz) y(w) dw$$

in order to generate $y(z)$, which satisfies the spectral curve equation.

In particular, in the model under consideration, we obtain from (4.5)

$$y(z) = \frac{1}{2x} \sum_{k \geq 1} \frac{k^{k-1} (2x\beta e^{xN})^k}{k!} z^{-k-1} \tag{4.6}$$

This is a large z expansion of the spectral curve

$$2xye^{-2xyz} = \frac{\xi}{z^2}, \quad \xi = 4\beta x^2 e^{xN} \tag{4.7}$$

which is the Lambert curve, in accordance with what should be the spectral curve for the Hurwitz theory, [29, 30]. Since the parameter x provides just a trivial rescaling, we choose it, for the sake of simplicity, equal to 1/2. Hence, the spectral curve finally has the form³

$$ye^{-yz} = \frac{\beta e^{N/2}}{z^2} \tag{4.8}$$

Note that N is now eliminated by the change of variables $p_k \rightarrow p_k/N$, $z \rightarrow z/N$, $\beta \rightarrow \beta N$. However, because of additional exponential dependence of x and additional factor of β in (4.5), the full N -dependence gets very different from that in the $m < 0$ models. In particular, it no longer has any straightforward relation to topological expansion. Because of it, and since the N -dependence reduces just to simple rescalings, for the sake of simplicity, we just ignore a possibility of adding N -dependent terms in the next subsection.

4.2 Cut-and-join operators and higher Lambert curves

Let us note that the \mathcal{W} -operator $\hat{\mathcal{W}}_0$, (4.1) is nothing but the cut-and-join operator $\hat{W}_{[2]}$ [34, 35, 57]. Hence, let us consider the next non-trivial generalized cut-and-join operator $\hat{W}_{[3]}$ [34, 35]:

$$\begin{aligned} \hat{W}_{[3]} = & \sum_{a,b,c \geq 1} abc p_{a+b+c} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} + \frac{3}{2} \sum_{a+b=c+d} cd (1 - \delta_{ac} \delta_{bd}) p_a p_b \frac{\partial^2}{\partial p_c \partial p_d} \\ & + \sum_{a,b,c \geq 1} (a+b+c) (p_a p_b p_c + p_{a+b+c}) \frac{\partial}{\partial p_{a+b+c}} \end{aligned} \tag{4.9}$$

³The parameterization of the Lambert curve here is different from that in [29], since, in accordance with the definition of the resolvent [28], we choose to work at the vicinity of infinity (at large z) with the local parameter $\frac{1}{z}$ instead of x around $x = 0$ in [29, eq. (2.18)], and the leading term is $\frac{1}{z}$. Thus, this formula and [29, eq. (2.16)] are identified upon the redefinition $z \rightarrow 1/x$ and $y \rightarrow yx$.

As we explained in the previous subsection, we do not need to add any N -dependent terms. This operator generates the Hurwitz partition function [34, 35]

$$Z_{[3]} = e^{x\hat{\mathcal{W}}_{[3]}} \cdot e^{\beta p_1} = \sum_R \beta^{|R|} S_R\{\delta_{k,1}\} S_R\{p_k\} e^{xC_3(R)} \quad (4.10)$$

where $C_3(R)$ denotes the eigenvalue of second Casimir operator: $C_3(R) = \sum_{R_i} [(R_i - i + 1/2)^3 - (-i + 1/2)^3]$.

In order to construct $\hat{\mathcal{W}}_{[3]}^{\text{spec}}$, as before, we pick up only the terms with maximal number of derivatives. Then,

$$\hat{\mathcal{W}}_{[3]}^{\text{spec}} = \sum_{a,b,c \geq 1}^{\infty} abc p_{a+b+c} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} \quad (4.11)$$

and

$$e^{\frac{x}{3}\hat{\mathcal{W}}_{[3]}^{\text{spec}}} \cdot e^{\beta p_1} = \exp\left(\beta \sum_{n=0}^{\infty} \frac{(2n+1)^{n-1}}{n!} (x\beta^2)^n \frac{p_{2n+1}}{2n+1}\right) \quad (4.12)$$

This leads to the spectral curve (we choose $x = 1$)

$$ye^{-z^2 y^2} = \frac{\beta}{z^2} \quad (4.13)$$

which is the higher Lambert curve.

Similarly, the higher generalized cut-and-join operators $\hat{\mathcal{W}}_{[s]}$ [34, 35] generate higher Hurwitz partition function [34, 35] associated with ‘‘completed cycles’’ [58, 59] (or r -spin Hurwitz numbers [60–62])

$$Z_{[r]} = e^{x\hat{\mathcal{W}}_{[r]}} \cdot e^{\beta p_1} = \sum_R \beta^{|R|} S_R\{\delta_{k,1}\} S_R\{p_k\} e^{xC_r(R)} \quad (4.14)$$

where $C_r(R)$ denotes the eigenvalue of second Casimir operator: $C_r(R) = \sum_{R_i} [(R_i - i + 1/2)^r - (-i + 1/2)^r]$. They corresponds to

$$\hat{\mathcal{W}}_{[s]}^{\text{spec}} = \sum_{\{a_i\}} \left(\prod_{i=1}^s a_i\right) p_{\sum_i a_i} \frac{\partial^s}{\partial a_1 \dots \partial a_s} \quad (4.15)$$

and

$$e^{\frac{x}{k}\hat{\mathcal{W}}_{[s]}^{\text{spec}}} \cdot e^{\beta p_1} = \exp\left(\beta \sum_{n=0}^{\infty} \frac{((s-1)n+1)^{n-1}}{n!} (x\beta^{s-1})^n \frac{p_{(s-1)n+1}}{(s-1)n+1}\right) \quad (4.16)$$

which leads to the higher Lambert curve ($x = 1$)

$$ye^{-z^{s-1} y^{s-1}} = \frac{\beta}{z^2} \quad (4.17)$$

The higher Lambert curves as the spectral curves for the r -spin Hurwitz numbers were earlier discussed in [60–62].

Note that one could consider a generic generalized cut-and-join operator \hat{W}_Δ with Δ being an arbitrary partition [34, 35]. However, it acts trivially on $e^{\beta p_1}$ if the partition Δ has more than one part, or more than one line in terms of Young diagrams. Consider, for instance, $\Delta = [2, 1]$. Then,

$$\hat{W}_{[2,1]}^{\text{spec}} = \sum_{a,b=1} ab(a+b-2)p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} = 2p_3 \frac{\partial^2}{\partial p_1 \partial p_2} + \dots \quad (4.18)$$

Similarly,

$$\begin{aligned} \hat{W}_{[2,2]}^{\text{spec}} &= \sum_{a,b=1} abc(a+b-2)p_{a+b+c} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} = 2p_4 \frac{\partial^3}{\partial p_1^2 \partial p_2} + \dots \\ \hat{W}_{[3,1]}^{\text{spec}} &= \sum_{a,b=1} abc(a+b+c-3)p_{a+b+c} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} = 2p_4 \frac{\partial^3}{\partial p_1^2 \partial p_2} + \dots \\ \hat{W}_{[2,1,1]}^{\text{spec}} &= \sum_{a,b=1} ab(a+b-2)(a+b-3)p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} = 6p_4 \frac{\partial^2}{\partial p_1 \partial p_3} + 8p_4 \frac{\partial^2}{\partial p_2^2} + \dots \\ \hat{W}_{[1^k]}^{\text{spec}} &= \sum_{a,b=1} a(a-1)\dots(a-k+1)p_a \frac{\partial}{\partial p_a} = k!p_k \frac{\partial}{\partial p_k} + \dots \end{aligned} \quad (4.19)$$

and, in all these cases,

$$e^{\hat{W}_\Delta^{\text{spec}}} \cdot e^{\beta p_1} = e^{\beta p_1} \quad \Delta \neq [s] \quad (4.20)$$

4.3 Completed cycles or not?

One can also consider linear and even multi-linear combinations of W_Δ -operators. Adding lower order operators does not change the \hat{W}^{spec} -operator and, hence, does not change the answer for the spectral curve obtained by our procedure. However, among all these combinations, there are some distinguished ones, which provide integrable partition functions [34, 35, 51, 54]. They are exactly those associated with completed cycles (4.14). For instance, for $s = 1, 2, 3, 4$, these are the operators (one at each level)

$$\hat{W}_{[1]}, \quad \hat{W}_{[2]}, \quad \hat{W}_{[3]} + \frac{1}{2}\hat{W}_{[1]}^2, \quad \hat{W}_{[4]} + 2\hat{W}_{[1]}\hat{W}_{[2]} \quad (4.21)$$

and their arbitrary linear combinations. We expect that our procedure of getting spectral curves is most immediately applied exactly to such W -operators.

Alternatively, one can take *other* combinations and claim that, perhaps, integrability is not that necessary for superintegrability of the system, since the partition function ($\alpha_{\Delta,k}$ are arbitrary coefficients)

$$e^{\sum_{\Delta} \alpha_{\Delta,k} \hat{W}_\Delta^k} \cdot e^{\beta p_1} = \sum_R \beta^{|R|} S_R\{\delta_{k,1}\} S_R\{p_k\} e^{\sum_{\Delta} \alpha_{\Delta,k} \phi_R(\Delta)^k} \quad (4.22)$$

has a clear superintegrable structure despite not being integrable at generic $\alpha_{\Delta,k}$ [54]. In this expression, $\phi_R(\Delta)$ is a peculiarly normalized character of the symmetric group S_∞ [34, 35], and the formula is based on the defining property of the \hat{W}_Δ -operators [34, 35]

$$\hat{W}_\Delta S_R = \phi_R(\Delta) S_R \quad (4.23)$$

Thus, these models at $m = 0$ and at higher s allows one to study the questions of connections of integrability and superintegrability as well as of relation to the topological recursion and topological expansions, which were difficult to ask in simpler cases. Note that the topological recursion in the completed cycle case was studied earlier in [60–62].

4.4 Spectral densities with $\text{tr } A^k = \delta_{k,s}$

Note that the higher Lambert curves are surprisingly related to the negative branch of the WLZZ models: their superintegrability relation essentially involves $S_R\{\delta_{k,m}\}$, see (3.5), i.e., if the variables p_k are expressed through the matrices A , $p_k = \text{tr } A^k$, it is related to solutions to the equation

$$\text{tr } A^k = \delta_{k,s} \quad \text{for all } k \in \mathbb{Z}_+ \tag{4.24}$$

Equivalently, one may ask what are the variables a_i , or the eigenvalues of the matrix A such that

$$\sum_{i=1} a_i^k = \delta_{k,s} \quad \text{for all } k \in \mathbb{Z}_+ \tag{4.25}$$

This is a very natural question since the Schur function S_R is a symmetric function just of a_i .

In fact, this problem is difficult to solve, however, one may consider the matrix A of a large size N , and to study the density of a_i in the large N limit:

$$\rho(z)dz = \sum_{i=1}^N \delta(z - a_i) \tag{4.26}$$

so that

$$\int z^k \rho(z) dz = \text{tr } A^k \tag{4.27}$$

Hence, one has to solve the equation

$$\int z^k \rho(z) dz = \delta_{k,s} \tag{4.28}$$

A solution to this equation is related to a remarkable property at the large N limit [63]: the variables z lying on the higher Lambert curve

$$ze^{-z^s} = w = e^{i\phi} \tag{4.29}$$

satisfies the relation

$$\oint z^{-k} d\phi = \delta_{k,s} \tag{4.30}$$

i.e. only for $k = s$ the series z^{-k} does not have the term w^0 (this is non-trivial for all $k = ms$ with $m > 1$).

5 An infinite set of WLZZ models. Positive branch

5.1 Description of the models

The WLZZ proposal at positive integers is to use another pair of operators in order to generate \mathcal{W} -representations, and act with it on an exponential linear in variables p_k instead of unity. More precisely, the procedure is as follows. One starts with the two operators

$$\hat{\mathcal{W}}_1 = \sum_{k,l} (k+l+1)p_k p_l \frac{\partial}{\partial p_{k+l+1}} + \sum_{k,l} kl p_{k+l-1} \frac{\partial^2}{\partial p_k \partial p_l} + 2N \sum_k (k+1)p_k \frac{\partial}{\partial p_{k+1}} + N^2 \frac{\partial}{\partial p_1} \quad (5.1)$$

$$\hat{\mathcal{W}}_2 = \sum_{k,l} (k+l+2)p_k p_l \frac{\partial}{\partial p_{k+l+2}} + \sum_{k,l} kl p_{k+l-2} \frac{\partial^2}{\partial p_k \partial p_l} + 2N \sum_k (k+2)p_k \frac{\partial}{\partial p_{k+2}} + 2N^2 \frac{\partial}{\partial p_2} + N \frac{\partial^2}{\partial p_1^2} \quad (5.2)$$

which give rise to an infinite set of operators

$$\hat{\mathcal{W}}_{m+1} = \frac{1}{m} [\hat{\mathcal{W}}_m, \hat{\mathcal{W}}_1], \quad m \geq 2 \quad (5.3)$$

In fact, these operators can be manifestly described as invariant operators on matrices: with an $N \times N$ matrix Λ , one can define

$$\hat{\mathcal{W}}_m = \text{Tr} \left(\frac{\partial^m}{\partial \Lambda^m} \right), \quad m \geq 2 \quad (5.4)$$

When acting on invariant functions, i.e. functions of $p_k = \text{Tr} \Lambda^k$, these operators coincide [32] with (5.3).

In fact, these operators can be constructed from the generators of the \widetilde{W} -algebra of section 3.1. Indeed, from relation (3.9) it follows that

$$\hat{\mathcal{W}}_m = \text{Tr} \left(\frac{\partial^m}{\partial \Lambda^m} \right) = \sum_{n=1} p_n \widetilde{W}_{n+m}^{(+,m)} + N \widetilde{W}_m^{(+,m)} \quad (5.5)$$

This implies that the partition function of the corresponding matrix model satisfies the $\widetilde{W}^{(+)}$ -constraints (see a particular case in the next subsection). Note that, in variance with the negative branch of WLZZ models, where each degree of N was associated with the t_0 -derivative, for the positive branch of WLZZ models, N is just p_0 , and this formula can be rewritten in the form

$$\hat{\mathcal{W}}_m = \sum_{n=0} p_n \widetilde{W}_{n+m}^{(+,m)} \quad (5.6)$$

In particular,

$$\widetilde{W}_n^{(+,2)} = \sum_{k=1} (k+n)p_k \frac{\partial}{\partial p_{k+n}} + \sum_{k=1}^{n-1} k(n-k) \frac{\partial^2}{\partial p_{n-k} \partial p_k} + Nn \frac{\partial}{\partial p_n} + N^2 \delta_{n,0} + Np_1 \delta_{n+1,0} \quad (5.7)$$

The operators $\hat{\mathcal{W}}_m$ (5.3), (5.4) generate the partition function

$$Z_m = e^{\frac{\hat{\mathcal{W}}_m}{m}} \cdot e^{\sum_k g_k p_k / k} \quad (5.8)$$

where g_k are just arbitrary parameters. These partition functions have the superintegrable representations

$$\begin{aligned}
 Z_1 &= \sum_{R,Q} \left(\frac{S_R\{N\}S_Q\{\delta_{k,1}\}}{S_Q\{N\}S_R\{\delta_{k,1}\}} \right)^2 S_{R/Q}\{\delta_{k,1}\}S_R\{g_k\}S_Q\{p_k\} \\
 Z_m &= \sum_{R,Q} \frac{S_R\{N\}S_Q\{\delta_{k,1}\}}{S_Q\{N\}S_R\{\delta_{k,1}\}} S_{R/Q}\{\delta_{k,m}\}S_R\{g_k\}S_Q\{p_k\} \quad \text{at } m > 1
 \end{aligned} \tag{5.9}$$

where $S_{R/Q}$ is the skew Schur function [33]. This is a KP τ -function in variables p_k . It does not come as a surprise, since $e^{\sum_k g_k p_k/k}$ is a KP τ -function, and \hat{W}_n is an element of w_∞ -algebra [36, 64]. However, it turns out that this partition function is also a τ -function w.r.t. the second set of variables, g_k , which is far less evident. Moreover, even a more strong property is correct: Z_n is a τ -function of the Toda lattice hierarchy with N being the Toda zero time. It follows from the fact that it is a KP τ -function to the both sets of time variables⁴ kp_k and kg_k , and it satisfies to the lowest Toda-chain hierarchy.⁵

Representation (5.9) for the partition function implies that the superintegrability relation for the correlator

$$\langle S_Q\{P_k\} \rangle = \sum_R \frac{S_R\{N\}S_Q\{\delta_{k,1}\}}{S_Q\{N\}S_R\{\delta_{k,1}\}} S_{R/Q}\{\delta_{k,m}\}S_R\{g_k\} \tag{5.10}$$

where, as previously, $P_k = \text{tr } M^k$ are traces of matrices at the matrix WLZZ model.

Note that all the underlined terms in (5.1) and (5.2) break homogeneity in N under the substitution $p_k \rightarrow p_k/N$, this is already a signal that they all should be eliminated from $\hat{\mathcal{W}}^{\text{spec}}$, see below.

5.2 Model with $m = 2$

Like in the case of negative m , for one particular value $m = 2$ there is a known one-matrix model realization [32, 65]. In this case, this is the Hermitian matrix model in the external field Λ [56]:

$$Z_2 = \int dM \exp \left(-\frac{1}{2} \text{Tr } M^2 + \sum_k \frac{g_k}{k} \cdot \text{Tr}(M + \Lambda)^k \right) \tag{5.11}$$

and $p_k = \text{Tr } \Lambda^k$.

Shifting the variable of integration $M \rightarrow M - \Lambda$, one can rewrite this partition function in the form

$$Z_2 = e^{-\frac{1}{2}p_2} \int dM \exp \left(-\frac{1}{2} \text{Tr } M^2 + \text{Tr } M\Lambda + \sum_k \frac{g_k}{k} \cdot \text{Tr } M^k \right) \tag{5.12}$$

⁴Note that the traditional choice t_k of time variables of the KP hierarchy as compared with power sums p_k of variables in symmetric functions is $t_k = kp_k$.

⁵Indeed, one can check that the equation

$$Z_m(N) \cdot \frac{\partial^2 Z_m(N)}{\partial p_1 \partial g_1} - \frac{\partial Z_m(N)}{\partial p_1} \frac{\partial Z_m(N)}{\partial g_1} = Z_m(N+1)Z_m(N-1)$$

is satisfied. We checked it up to grading 10 with the computer. This equation along with the KP hierarchies w.r.t. to the two sets of time variables guarantees that $Z_m(N)$ is a τ -function of the Toda lattice hierarchy.

When only g_1 and g_2 are non-vanishing, we have the Gaussian integral, while when $g_3 \neq 0$, we get a more complicated integral, which requires a more advanced approach.

This kind of models was studied in [56], and this partition function describes a generalized Kontsevich model in the character phase. It satisfies the Ward identities [56, section 2.4.2]

$$\left(g_1 \delta_{n,1} + \delta_{n,2} - n \frac{\partial}{\partial p_n} + \sum_{k>1} g_k \widetilde{W}_{k+n-2}^{(+,k-1)} \right) e^{\frac{1}{2} p_2} Z_2 = 0, \quad n \geq 1 \quad (5.13)$$

5.3 Spectral curve, $m = 2$

Let us convert the Ward identities (5.13) with z^{-s-3} and, for the sake of simplicity, preserve only g_1, g_2 and g_3 . Then, one gets the spectral curve

$$\frac{g_1 z^2 + g_3 N z^2 + g_2 z + g_3}{z^6} + \frac{g_3 N z^2 + (g_2 - 1)z + 2g_3}{z^3} \left(y - \frac{N}{z} \right) - \frac{g_3 N \alpha}{z^3} + g_3 \left(y - \frac{N}{z} \right)^2 = 0 \quad (5.14)$$

where we used (3.12) and (5.7). Here α is an arbitrary constant, i.e. the spectral curve and the Ward identities have ambiguous solutions (parameterized by one constant).

Indeed, the set of Ward identities (5.13) can be rewritten as

$$W_n Z = \left((g_1 + g_3 N) \delta_{n,1} + g_2 \delta_{n,2} + g_3 \delta_{n,3} + (g_2 - 1) n \frac{\partial}{\partial p_n} + 2g_3 (n-1) \frac{\partial}{\partial p_{n-1}} + g_3 N (n+1) \frac{\partial}{\partial p_{n+1}} + g_3 \widetilde{W}_n^{(+,2)} \right) Z_2 = 0, \quad n \geq 1 \quad (5.15)$$

where $\widetilde{W}_n^{(+,2)}$ is given in (5.7).

Now, following the general procedure, one converts this infinite set of constraints with powers of z , rescales $p_k \rightarrow p_k/\hbar$, introduces $\mathcal{F} := \hbar^2 \log Z$, and rewrites the sum as an equation for the resolvent

$$\rho(z|p_k) := \hat{\nabla}_z \mathcal{F} = \sum_{k \geq 0} \frac{k}{z^{k+1}} \frac{\partial \mathcal{F}}{\partial p_k} \quad (5.16)$$

similarly to (2.7). It remains to put all p_k zero and take the leading behaviour at small \hbar in order to obtain for $y := \rho_{-2}(z|0) \Big|_{\hbar \rightarrow 0}$ equation (5.14).

5.4 Spectral curve vs. $\hat{\mathcal{W}}^{\text{spec}}, m = 2$

Now again let us compare this result with an alternative procedure, which we advocate in this paper. Namely, keep the highest derivatives terms in (5.2):

$$\hat{\mathcal{W}}_2^{\text{spec}} = \sum_{k,l} k l p_{k+l-2} \frac{\partial^2}{\partial p_k \partial p_l} + 2N \sum_k (k+2) p_k \frac{\partial}{\partial p_{k+2}} + \underline{2N^2 \frac{\partial}{\partial p_2} + N \frac{\partial^2}{\partial p_1^2}} \quad (5.17)$$

As we shall see, the underlined terms should better be omitted, i.e. the true definition of $\hat{\mathcal{W}}^{\text{spec}}$ should be reduced to terms, which are linear in p . Then,

$$e^x \hat{\mathcal{W}}_2^{\text{spec}} \cdot e^{g_1 p_1} = e^{xN} g_1^2 \cdot e^{g_1 p_1} \quad (5.18)$$

and

$$e^{x\hat{\mathcal{W}}_2^{\text{spec}}} \cdot e^{g_1 p_1 + \frac{g_2 p_2}{2}} = \frac{e^{\frac{x N g_1^2}{1-2g_2 x}}}{(1-2g_2 x)^{N^2/2}} \cdot e^{\frac{1}{1-2g_2 x} \left(g_1 p_1 + \frac{g_2 p_2}{2} \right)} \quad (5.19)$$

In fact, in this case, one can just evaluate the Gaussian integral (5.12) to obtain the r.h.s. of this formula at $x = \frac{1}{2}$. Indeed, the action of $\hat{\mathcal{W}}_2^{\text{spec}}$ generates the full answer, since the first term in $\hat{\mathcal{W}}_2$ (5.2) does not contribute when all p_k 's but p_1 and p_2 are vanishing in the exponential (5.8).

The ugly prefactor at the r.h.s., which is independent of times, is generated by the underlined terms in (5.17). Omitting them from $\hat{\mathcal{W}}_2^{\text{spec}}$, we get just

$$e^{x\hat{\mathcal{W}}_2^{\text{spec}}} \cdot e^{g_1 p_1 + \frac{g_2 p_2}{2}} = e^{\frac{1}{1-2g_2 x} \left(g_1 p_1 + \frac{g_2 p_2}{2} \right)} \quad (5.20)$$

Note that, at the moment, our general principle is to leave in $\hat{\mathcal{W}}^{\text{spec}}$ only the terms with maximum number of derivatives. However, this principle in all cases considered earlier was equivalent to leaving only terms linear in time variables p_k . In the case of $\hat{\mathcal{W}}_2^{\text{spec}}$ we observe, for the first time, the difference between these two principles, and it becomes clear that we need to follow the second one.

Now note that from (5.14) it follows that, in the case of only g_1 and g_2 non-zero,

$$y = \frac{N}{z} + \frac{g_1 z + g_2}{(1-g_2)z^3} \quad (5.21)$$

Inserting this y into $\oint V(xz) y(x) dx$, one gets in the exponential

$$\mathcal{P}_2 = \frac{1}{1-g_2} \left(z g_1 p_1 + \frac{z^2 g_2 p_2}{2} \right) \quad (5.22)$$

instead of

$$\mathcal{P}'_2 = \frac{1}{1-2g_2 x} \left(z g_1 p_1 + \frac{z^2 g_2 p_2}{2} \right) \quad (5.23)$$

in (5.20) after making the substitution $p_k \rightarrow z^k p_k$. One could make these two expressions consistent choosing $x = \frac{1}{2}$.

Now consider a more involved case of non-vanishing g_3 . To simplify the formulas, let $g_2 = g_1 = 0$. Then

$$e^{x\hat{\mathcal{W}}_2^{\text{spec}}} \cdot e^{g_3 p_3/3} = \exp \left(g_3 \sum_{k=1} \frac{(2k)!}{(k+1)!k!} (2g_3 x)^{k-1} \frac{p_{k+2}}{k+2} + N \sum_{k=1} \frac{(2k-1)!}{k!(k-1)!} (2g_3 x)^k \frac{p_k}{k} \right. \\ \left. + \sum_{m=1} (N g_3^2 x^3)^m \cdot \sum_{k=1} N \alpha_k^{(m)} \cdot (2g_3 x)^k \frac{p_k}{k} \right) \quad (5.24)$$

Numeric coefficients $\alpha_k^{(m)}$ are quite complicated, but all the underlined terms come with extra powers of $N g_3^2$. This provides a selection rule, which allows one to eliminate them in a regular way.

Now let us again choose $x = \frac{1}{2}$. Then, the spectral curve is associated with the main terms, which are not underlined is (in accordance with $\mathcal{P}_2 = \oint V(xz) y(x) dx$)

$$\begin{aligned}
 y_2(z) &= \sum_{k=1} \frac{2k!}{(k+1)!k!} \frac{g_3^k}{z^{k+3}} + N \sum_{k=1} \frac{(2k-1)!}{k!(k-1)!} \frac{g_3^k}{z^{k+1}} \\
 &= \frac{1}{2g_3 z^2} \left(1 - \sqrt{1 - \frac{4g_3}{z}} \right) - \frac{1}{z^3} + \frac{N}{2z} \left(\frac{1}{\sqrt{1 - \frac{4g_3}{z}}} + 1 \right) + O(N^2) \quad (5.25)
 \end{aligned}$$

In order to compare this curve with (5.14) note that, like in section 4.1, elimination of the underlined terms in (5.17) is done by the rescaling $p_k \rightarrow p_k/N$ and considering small N limit. Hence, the relation of topological and $1/N$ -expansion breaks down (similarly to what happened in section 4.1).

Consider now the leading order of (5.14) at *small* N :

$$\frac{g_3}{z^6} + \frac{2g_3 - z}{z^3} y^{(0)} + g_3 \left(y^{(0)} \right)^2 = 0 \quad (5.26)$$

Its solution is exactly the curve (5.25) at small N , the first two terms:

$$y^{(0)}(z) = \frac{1}{2g_3 z^2} \left(1 - \sqrt{1 - \frac{4g_3}{z}} \right) - \frac{1}{z^3} = y_2^{(0)}(z) \quad (5.27)$$

Now, one can consider the first small N correction to (5.14). It gives rise to a more complicated formula than just

$$y_2^{(1)}(z) = \frac{N}{2z} \left(\frac{1}{\sqrt{1 - \frac{4g_3}{z}}} + 1 \right) \quad (5.28)$$

in (5.25):

$$y^{(1)}(z) = \frac{N}{2z} \left(\frac{1 - 2\alpha g_3 + \frac{2g_3}{z}(\alpha - 1)}{\sqrt{1 - \frac{4g_3}{z}}} + 1 \right) \quad (5.29)$$

However, note that the rescaling $p_k \rightarrow p_k/N$ would imply also the rescaling $g_k \rightarrow Ng_k$ in order to preserve exponential intact. This means that one also has to consider a leading behaviour at small g_3 . The leading contribution at small g_3 in (5.29) (after the rescaling of $z \rightarrow g_3 z$) is exactly (5.28) upon the choice of $\alpha = 1$: $y^{(1)}(z) \rightarrow y_2^{(1)}(z)$, and finally we come to (5.25).

Note that one can introduce new variables $Y_2 = z^3 y_2^{(0)}$ and $x = Z/g_3$ such that the spectral curve (5.26) becomes

$$(ZY_2)^{1/2} - Y_2 - 1 = 0 \quad (5.30)$$

To summarize, we see that the spectral curve (5.14), which can be extracted from the matrix model realization (5.11), is consistent with our universal definition from the p -linear part \hat{W}_{spec} of the \hat{W} operator. However, in this case, one needs to deal with the small N limit, and the idea of large N expansion, which continued to be safe for the WLZZ models with $m < 0$ needs to be changed for its opposite at $m > 0$.

Thus, we finally can formulate the general prescription: in order to construct the operator $\hat{\mathcal{W}}^{\text{spec}}$, one has to leave in the original operator \hat{W} linear in p_k terms with maximum number of derivatives.

As for the large or small N limit and the topological expansion, as we demonstrated, it depends on the concrete model.

5.5 Spectral curve in the $m = 1$ model

Now consider models at other values of m . We consider only the leading small N order of the spectral curve.

We start with the very first example, $m = 1$. In this case, the \hat{W} -operator is given by formula (5.1), and, in accordance with our general rule,

$$\hat{\mathcal{W}}_1^{\text{spec}} = \sum_{k,l} klp_{k+l-1} \frac{\partial^2}{\partial p_k \partial p_l} + \underline{N^2 \frac{\partial}{\partial p_1}} \quad (5.31)$$

Since we deal with the small N limit, we drop out the underline term. Then,

$$e^{x\hat{\mathcal{W}}_1^{\text{spec}}} \cdot e^{g_1 p_1 + g_2 p_2 / 2} = \exp \left(\frac{g_1}{1-xg_1} p_1 + \frac{1}{x(1-xg_1)} \sum_{n \geq 1} \frac{1}{n} C_{n-1}^{3n} \left(\frac{xg_2}{(1-xg_1)^3} \right)^n \frac{p_{n+1}}{n+1} \right) \quad (5.32)$$

where C_k^n are the binomial coefficients.

In order to get the spectral curve, we choose $x = 1$. Then, the spectral curve is associated with the leading term at small N (in accordance with $\mathcal{P}_1 = \oint V(xz) y(x) dx$)

$$y_1^{(0)} = \frac{1}{(1-g_1)z^2} \left(g_1 + \underbrace{\sum_{n \geq 1} \frac{1}{n} C_{n-1}^{3n} \left(\frac{g_2}{z(1-g_1)^3} \right)^n}_{Y_1 - 1} \right) := \frac{1}{(1-g_1)z^2} (g_1 - 1 + Y_1) \quad (5.33)$$

Upon introducing also a new variable $Z = z(1-g_1)^3/g_2$, this sum satisfies the equation for the spectral curve

$$Z^{-1} Y_1^3 - Y_1 - 1 = 0 \quad (5.34)$$

Thus, one can see that the role of g_1 is basically to rescale g_2 for $g_2/(1-g_1)^3$, much similar to the rescaling with $1/(1-g_2)$ in the $m = 2$ case.

Hence, now we drop g_1 , and switch on the g_3 parameter instead:

$$e^{x\hat{\mathcal{W}}_1^{\text{spec}}} \cdot e^{g_2 p_2 / 2 + g_3 p_3 / 3} = \exp \left(\frac{1}{x} \sum_{n \geq 1, k \geq 0} \frac{1}{n} C_{n-1}^{3n+2k} C_k^n (xg_2)^{n-k} (xg_3)^k \frac{p_{n+k+1}}{n+k+1} \right) \quad (5.35)$$

The equation for the spectral curve for

$$y_1^{(0)} = \sum_{n \geq 1, k \geq 0} \frac{1}{n} C_{n-1}^{3n+2k} C_k^n g_2^{n-k} g_3^k z^{-n-k-2} = \frac{Y_1 - 1}{z^2} \quad (5.36)$$

is rather simple:

$$\frac{g_3}{z^2} Y_1^5 + \frac{g_2}{z} Y_1^3 - Y_1 + 1 = 0 \quad (5.37)$$

One can also easily restore the parameter g_1 in (5.35):

$$e^{x\hat{\mathcal{W}}_1^{\text{spec}}} \cdot e^{g_1 p_1 + g_2 p_2/2 + g_3 p_3/3} = \exp \left(\frac{g_1}{1-xg_1} p_1 + \frac{1}{x(1-xg_1)} \sum_{n \geq 1, k \geq 0} \frac{1}{n} C_{n-1}^{3n+2k} C_k^n \left(\frac{xg_2}{(1-xg_1)^3} \right)^{n-k} \left(\frac{xg_3}{(1-xg_1)^5} \right)^k \frac{p_{n+k+1}}{n+k+1} \right) \quad (5.38)$$

It again reduces to the rescalings $g_2 \rightarrow g_2/(1-xg_1)^3$, $g_3 \rightarrow g_3/(1-xg_1)^5$.

Now the general formula is clear: adding on more parameters g_k , $k = 2, \dots, K$ gives rise to the spectral curve

$$\sum_{k=2}^K \frac{g_k}{z^{k-1}} Y_1^{2k-1} - Y_1 + 1 = 0 \quad (5.39)$$

and, the parameters rescalings upon switching on g_1 are: $g_k \rightarrow g_k/(1-g_1)^{2k-1}$, $k = 1, \dots, K$.

5.6 Spectral curve in the $m = 3$ model

Our next example is $m = 3$, and the $\hat{\mathcal{W}}_3$ -operator is

$$\begin{aligned} \hat{\mathcal{W}}_3 = & \sum_{a,b,c \geq 1} abc p_{a+b+c-3} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} + \sum_{b,c \geq 1} \sum_{a=1}^{b+c+2} a(b+c-a+3) p_b p_c \frac{\partial^2}{\partial p_a \partial p_{b+c-a+3}} \\ & + \sum_{b,c \geq 1} \sum_{a=1}^{b+1} a(b+c-a+3) p_b p_c \frac{\partial^2}{\partial p_a \partial p_{b+c-a+3}} + \sum_{a,b,c \geq 1} (a+b+c+3) p_a p_b p_c \frac{\partial}{\partial p_{a+b+c+3}} \\ & + 3N \sum_{b \geq 1} \sum_{a=1}^{b+2} a(sb-a+3) p_b \frac{\partial^2}{\partial p_a \partial p_{b-a+3}} + 3N \sum_{a,b \geq 1} (a+b+3) p_a p_b \frac{\partial}{\partial p_{a+b+3}} \\ & + \sum_{a \geq 1} \frac{(a+1)(a+2)(a+3)}{2} p_a \frac{\partial}{\partial p_{a+3}} + 3N^2 \sum_{a \geq 1} (a+3) p_a \frac{\partial}{\partial p_{a+3}} \\ & + N \frac{\partial^3}{\partial p_1^3} + 6N^2 \frac{\partial^2}{\partial p_1 \partial p_2} + 3N(N^2+1) \frac{\partial}{\partial p_3} \end{aligned} \quad (5.40)$$

To get the operator $\hat{\mathcal{W}}_3^{\text{spec}}$, in accordance with the general principle, we leave only the first term in this expression:

$$\hat{\mathcal{W}}_3^{\text{spec}} = \sum_{a,b,c} abc p_{a+b+c-3} \frac{\partial^3}{\partial p_a \partial p_b \partial p_c} \quad (5.41)$$

In this case, the action of $e^{x\hat{\mathcal{W}}_3^{\text{spec}}}$ on $e^{g_1 p_1}$ is trivial:

$$e^{x\hat{\mathcal{W}}_3^{\text{spec}}} \cdot e^{g_1 p_1} = e^{g_1 p_1} \quad (5.42)$$

while the actions of $e^{x\hat{\mathcal{W}}_3^{\text{spec}}}$ on $e^{\frac{g_2 p_2}{2}}$ and $e^{\frac{g_3 p_3}{3}}$ are

$$e^{x\hat{\mathcal{W}}_3^{\text{spec}}} \cdot e^{\frac{g_2 p_2}{2}} = \exp\left(\sum_{n=0}^{\infty} \frac{g_2}{n+1} C_n^{2n} (xg_2^2)^n \frac{p_{n+2}}{n+2}\right) \quad (5.43)$$

$$e^{x\hat{\mathcal{W}}_3^{\text{spec}}} \cdot e^{\frac{g_3 p_3}{3}} = \exp\left(\sum_{n=0}^{\infty} \frac{2g_3}{3n+2} C_n^{4n+1} (xg_3^2)^n \frac{p_{3n+3}}{3n+3}\right) \quad (5.44)$$

Switching on the g_1 parameter in these cases, as previously, just rescales the parameters g_k . For instance, $g_2 \rightarrow g_2/\sqrt{1-4xg_1g_2}$. This also adds a contribution proportional to p_1 similar to $\frac{g_1}{(1-g_1)z^2}$ in (5.33).

The spectral curves associated with (5.43) and (5.44) are accordingly

$$\begin{aligned} \frac{z^3}{g_2} y_3 &= 1 + z^5 y_3^2 \\ \frac{z^4}{g_3} y_3 &= (1 + z^5 y_3^2)^2 \end{aligned} \quad (5.45)$$

where we omitted the superscript 0 of $y^{(0)}$, for the sake of brevity.

Now the natural conjecture is that, for a non-zero parameter g_k , the curve looks like

$$\frac{z^{k+1}}{g_k} y_3 = (1 + z^5 y_3^2)^{k-1} \quad (5.46)$$

Indeed, let us consider the action of $e^{x\hat{\mathcal{W}}_3^{\text{spec}}}$ on $e^{\frac{g_4 p_4}{4}}$:

$$e^{x\hat{\mathcal{W}}_3^{\text{spec}}} \cdot e^{\frac{g_4 p_4}{4}} = \exp\left\{\sum_{k=1}^{\infty} \frac{3(6k-4)!}{(k-1)!(5k-2)!} (3x)^{k-1} g_4^{2k-1} \frac{p_{5k-1}}{5k-1}\right\} \quad (5.47)$$

In order to get the spectral curve, we choose $x = \frac{1}{3}$. Then, the spectral curve is associated with the leading term at small N is (in accordance with $\mathcal{P}_3 = \oint V(xz)y(x)dx$)

$$y_3 = \frac{3}{g_4} \sum_{k=1}^{\infty} \frac{(6k-4)!}{(k-1)!(5k-2)!} \left(\frac{g_4^2}{z^5}\right)^k \quad (5.48)$$

This sum satisfies the equation for the spectral curve (5.46).

At last, when the first K parameters g_k are non-zero, the spectral curve is

$$y_3 = \sum_{k=2}^K \frac{g_k}{z^{k+1}} (1 + z^5 y_3^2)^{k-1} \quad (5.49)$$

Note that, upon introducing new variables $Y_3 = g_4 y_3^{(0)}$, $Z = z^5/g_4^2$, sum (5.48) can be rewritten in the form

$$(ZY_3)^{1/3} - ZY_3^2 - 1 = 0 \quad (5.50)$$

5.7 Generic m

Generalization to all the WLZZ models with arbitrary $m > 0$ is now straightforward. For any $m > 0$, relevant in $\hat{\mathcal{W}}_m^{\text{spec}}$ is just the term:

$$\hat{\mathcal{W}}_m^{\text{spec}} = \sum_{\{a_i\}} \left(\prod_{i=1}^m a_i \right) p_{\sum_i a_i - m} \frac{\partial^m}{\partial a_1 \dots \partial a_m} \quad (5.51)$$

In the most interesting case, one gets

$$e^x \hat{\mathcal{W}}_m^{\text{spec}} \cdot e^{\frac{g_{m+1} p_{m+1}}{m+1}} = \exp \left\{ \sum_{k=0} \frac{1}{n_k} C_k^{n_k} m (mx)^k g_{m+1}^{(m-1)k+1} \cdot \frac{p_{n_k - k + 1}}{n_k - k + 1} \right\} \quad (5.52)$$

where $n_k = (m-1)(mk+1) + 1$.

As before, in order to get the spectral curve, we choose $x = \frac{1}{m}$. Then, the spectral curve is associated with the leading term at small N (in accordance with $\mathcal{P}_m = \oint V(xz) y(x) dx$)

$$y_m = g_{m+1}^{2-m} z^{(m+1)(m-3)} \sum_{k=0} \frac{m}{n_k} C_k^{n_k} \left(\frac{g_{m+1}^{m-1}}{z^{m^2-m-1}} \right)^{k+1} \quad (5.53)$$

Upon introducing new variables $Y_m = g_{m+1}^{m-2} z^{(m+1)(3-m)} y_m^{(0)}$, $Z = z^{m^2-m-1} / g_{m+1}^{m-1}$, this sum satisfies the equation for the spectral curve

$$(ZY_m)^{1/m} - Z^{m-2} Y_m^{m-1} - 1 = 0 \quad (5.54)$$

At the same time, the counterpart of (5.49) is

$$y = \sum_{k=2}^K \frac{g_k}{z^{k+1}} (1 + z^{2m-1} y^{m-1})^{k-1} \quad (5.55)$$

6 Exponentiation principle

We did not comment so far a miraculously looking property that action of the $\hat{\mathcal{W}}^{\text{spec}}$ operator on the exponential linear in p_k 's produces also an exponential linear in p_k 's. In fact, this property follows from the Campbell-Hausdorff formula (CHF) as we will discuss now.

Consider first the simplest case. Even getting the formula

$$\exp \left(\sum_{k=1}^{\infty} k p_{k+1} \frac{\partial}{\partial p_k} + N p_1 \right) \cdot 1 = \exp \left(N \sum_{k=1} \frac{p_k}{k} \right) \quad (6.1)$$

requires a few steps.

In fact, it results from a multiple application of the CHF, e.g.

$$\exp \left(y \frac{\partial}{\partial x} + N x \right) \cdot 1 = e^{-\frac{Ny}{2}} e^{y \frac{\partial}{\partial x}} e^{Nx} \cdot 1 = e^{-\frac{Ny}{2}} e^{N(x+y)} = e^{\frac{Ny}{2} + Nx} \quad (6.2)$$

where we used the CHF in the form

$$e^{A+B} = e^A e^B e^{-\frac{[A,B]}{2}} \quad (6.3)$$

which is valid when the commutator $[A, B]$ commutes with both A and B . Further,

$$\exp\left(2z\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} + Nx\right) \cdot 1 = e^{2z\frac{\partial}{\partial y}} e^{y\frac{\partial}{\partial x}} e^{-z\frac{\partial}{\partial x}} e^{-\frac{Ny}{2}} e^{\frac{Nz}{3}} e^{Nx} \cdot 1 = e^{2z\frac{\partial}{\partial y}} e^{-\frac{2Nz}{3} + \frac{Ny}{2} + Nx} = e^{\frac{Nz}{3} + \frac{Ny}{2} + Nx} \quad (6.4)$$

and so on. At this stage, we used the CHF in the form

$$e^{A+B+C} = e^A e^B e^{-\frac{[A,B]}{2}} e^{-\frac{[B,C]}{2}} e^{\frac{[[A,B],C]}{3}} e^C \quad (6.5)$$

where $[[A, B], C]$ commutes with all other quantities in this formula.

One can easily change the weights in (6.1):

$$\exp\left(\sum_{k=1}^{\infty} c_k p_{k+1} \frac{\partial}{\partial p_k} + N p_1\right) \cdot 1 = \exp\left(N \sum_{k=1}^{\infty} \frac{c_1 \cdots c_{k-1} p_k}{k!}\right) \quad (6.6)$$

However, if one attempts to substitute the exponential functions by anything else:

$$G\left(\sum_{k=1}^{\infty} k p_{k+1} \frac{\partial}{\partial p_k} + N p_1\right) \cdot 1 = H\left(N \sum_{k=1}^{\infty} \frac{p_k}{k}\right) \quad (6.7)$$

there will be no solutions different from $H(x) = G(x) = e^x$. In this sense, *the exponential function is distinguished*.

Our example demonstrates that, since $\hat{\mathcal{W}}^{\text{spec}}$ is linear in p_k 's though may involve higher derivatives, $\exp(\hat{\mathcal{W}}^{\text{spec}})$ upon acting on unity produces an exponential linear in p_k 's. In more involved examples of $\exp(\hat{\mathcal{W}}^{\text{spec}})$, the calculations are more tedious, however, they work same way. An even more complicated case is when $\exp(\hat{\mathcal{W}}^{\text{spec}})$ is acting on $\exp(\sum_k g_k p_k/k)$. However, the result is still an exponential linear in p_k 's. In order to prove this, one has to use the Dynkin form of the CHF [66, 67]:

$$\begin{aligned} \exp(\hat{A}) \cdot \exp(\hat{B}) = \exp\left(\sum_n \frac{(-1)^n}{n} \sum_{\{r_i+s_i>0\}} \frac{1}{\prod_{i=1}^n r_i! s_i! \cdot \sum_{i=1}^n (r_i + s_i)} \right. \\ \left. \times \underbrace{[\hat{A}, [\hat{A}, \dots [\hat{A}, \hat{B}, [\hat{B}, \dots [\hat{B}, \dots [\hat{A}, [\hat{A}, \dots [\hat{A}, [\hat{B}, [\hat{B}, \dots, \hat{B}]]]]]]]]]}_{r_1} \dots \underbrace{[\hat{A}, [\hat{A}, \dots [\hat{A}, [\hat{B}, [\hat{B}, \dots, \hat{B}]]]]]}_{s_1} \dots \underbrace{[\hat{A}, [\hat{A}, \dots [\hat{A}, [\hat{B}, [\hat{B}, \dots, \hat{B}]]]]]}_{r_n} \underbrace{[\hat{B}, [\hat{B}, \dots, \hat{B}]]]}_{s_n} \dots \right) \end{aligned} \quad (6.8)$$

where $[\hat{X}] := \hat{X}$.

It is clear from this formula that $\exp(\hat{\mathcal{W}}^{\text{spec}}) \cdot \exp(\sum_k g_k p_k/k)$ contains only commutators of operators of the form $\sum p_k \hat{D}_k$, where \hat{D}_k is a pure differential operator of a finite order, and commutators of these operators have also this form. Hence, we ultimately come to conclusion that $\exp(\hat{\mathcal{W}}^{\text{spec}}) \cdot \exp(\sum_k g_k p_k/k) = \exp(\sum p_k \hat{D}_k) \cdot 1$, and we return to our example above.

7 Comments

In this subsection, we mention some other promising directions for further development of the spectral curve theory for the WLZZ models. We do not elaborate on any of them, but hopefully they will attract attention in the future.

7.1 \mathcal{W} -representation vs. W -constraints

Actually, \mathcal{W} -representation (1.1) is naturally exponential, if $Z\{p\}$ satisfies the W -constraints [23–25].

For instance, as we discussed in sections 2–3 (eqs. (2.2), (3.19), etc.), the \mathcal{W} -representation in the negative branch WLZZ models is of the form

$$\hat{\mathcal{W}}_{-m} = \sum_{k=1}^{\infty} p_k \widetilde{W}_{k-m}^{(-,m)} \tag{7.1}$$

This allows one immediately to obtain the \mathcal{W} -representation: since the partition function Z_{-m} satisfies the set of $\widetilde{W}^{(-,m)}$ -algebra constraints (3.8),

$$\left(k \frac{\partial}{\partial p_k} - \widetilde{W}_{k-m}^{(-,m)} \right) Z = 0 \tag{7.2}$$

summing it up with p_k , we obtain

$$\left(\hat{l}_0 - \underbrace{\sum_k p_k \widetilde{W}_{k-m}^{(-,m)}\{p\}}_{\hat{\mathcal{W}}_{-m}\{p\}} \right) Z\{p\} = 0 \tag{7.3}$$

with the grading operator $\hat{l}_0 := \sum_k k p_k \frac{\partial}{\partial p_k}$, and with $\hat{\mathcal{W}}_{-m}$ having a given grading, m so that

$$[\hat{l}_0, \hat{\mathcal{W}}_{-m}] = m \hat{\mathcal{W}}_{-m} \tag{7.4}$$

Now, it is immediate to prove that

$$Z = e^{\frac{1}{m} \hat{\mathcal{W}}_{-m}} \cdot 1 \tag{7.5}$$

satisfies (7.3). Since the solution to (7.2) is unique, which can be checked following the line of [28], (or, equivalently, the solution to (7.3) is unique, which can be checked following the line of [22]), we obtain that (7.2) are equivalent to the W -representation.

This is a generalization of the elementary fact

$$\left(x \frac{d}{dx} - x^m \right) Z = 0 \implies Z \sim e^{x^m/m} \tag{7.6}$$

An interesting question is if we know a single $\hat{\mathcal{W}}$ in a more generic situation, can we find the entire set of constraints (7.2)? In particular, what is the set of constraints in the positive branch WLZZ models? This question stands from [30], where it was shown that a very simple $\hat{\mathcal{W}}$ is associated with somewhat non-trivial, “conjugate or deformed continuous” Virasoro constraints.

7.2 Large N limit of superintegrability relation

In this subsection, we again return to the superintegrability relation (3.6) in the case of the negative branch of the WLZZ models.

For the sake of definiteness, we start with the Gaussian case $m = 2$. In this case, the resolvent (2.12) satisfies the spectral curve equation (2.8), and its imaginary part (jump

at the branch cut) $\rho(z) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0} (y(z - i\epsilon) - y(z + i\epsilon)) = \frac{\Im y(z)}{2\pi} \sim \sqrt{4N - z^2}$ is sometimes called spectral density since it provides the distribution of eigenvalues [1–5, 47–50] reasonable at large N , when multi-trace correlators factorize. This means that

$$\langle P_{k_1} P_{k_2} \rangle_\infty = \langle P_{k_1} \rangle_\infty \langle P_{k_2} \rangle_\infty \tag{7.7}$$

and

$$\langle P_k \rangle_\infty = \int z^k \rho(z) dz \tag{7.8}$$

It is instructive to see how the superintegrability relations (3.6) trivialize in this limit.

Since in Gaussian case $\langle P_{2k} \rangle_\infty \sim N^{k+1}$, dominating in the Schur average is the item with maximal number of P_2 :

$$\langle S_R \rangle_\infty = \langle S_R \{ \delta_{k,2} \} \cdot P_2^{|R|/2} \rangle_\infty = S_R \{ \delta_{k,2} \} \cdot \langle P_2 \rangle_\infty^{|R|/2} = N^{|R|} S_R \{ \delta_{k,2} \} \tag{7.9}$$

At the same time, this is exactly the large N limit of the r.h.s. of the superintegrability relation:

$$S_R \{ \delta_{k,2} \} \frac{S_R \{ N \}}{S_R \{ \delta_{k,1} \}} \xrightarrow{N \rightarrow \infty} N^{|R|} S_R \{ \delta_{k,2} \} \tag{7.10}$$

since dominating is the contribution from the maximal power of $p_k = N$, which is $p_1^{|R|}$.

The main point is that the superintegrability relation in the large N limit is just trivial: no requirements are imposed on actual values of $\langle P_{2k} \rangle_\infty$ for other $k \neq 1$. Completely the same consideration can be repeated for any negative branch WLZZ model.

To put it differently, the superintegrability relations in the large N limit become linear in the sector with definite grading $|R|$:

$$\langle S_R \rangle_\infty = N^{|R|} S_R \{ \delta_{k,m} \} \tag{7.11}$$

This means that they are not longer restricted to characters, one can take any linear combination of S_R with the same $|R|$:

$$\langle F \rangle_\infty = N^{|R|} F \{ \delta_{k,m} \} \tag{7.12}$$

for any $F = \sum_R$ with a given $|R|$ $f_R S_R$. In particular, one obtains a factorization: since

$$S_{R_1} S_{R_2} = \sum_{R_3: |R_3|=|R_1|+|R_2|} \mathcal{N}_{R_1 R_2}^{R_3} S_{R_3} \tag{7.13}$$

one gets

$$\begin{aligned} \langle S_{R_1} S_{R_2} \rangle_\infty &= \sum_{R_3} \mathcal{N}_{R_1 R_2}^{R_3} \langle S_{R_3} \rangle_\infty = N^{|R_1|+|R_2|} \sum_{R_3} \mathcal{N}_{R_1 R_2}^{R_3} S_{R_3} \{ \delta_{k,m} \} \\ &= N^{|R_1|+|R_2|} \cdot S_{R_1} \{ \delta_{k,m} \} S_{R_2} \{ \delta_{k,m} \} = \langle S_{R_1} \rangle_\infty \langle S_{R_2} \rangle_\infty \end{aligned} \tag{7.14}$$

7.3 Large N limit of double averages

Note that the factorization of correlators at large N should not be taken for granted. Consider, for instance, double averages in the negative branch WLZZ models that are factorized due to the superintegrability [64, 68]. These correlators are generated by the action of the W -operators $\hat{\mathcal{W}}_m$ on the Schur function S_R as functions of P_k ,

$$\langle S_Q\{\hat{\mathcal{W}}_k\} \cdot S_R\{P_k\} \rangle_{WLZZ-m} = \frac{S_{R/Q}\{\delta_{k,m}\} S_R\{N\}}{S_R\{\delta_{k,1}\}} \quad (7.15)$$

It is curious that, though these W -operators generate the positive branch of the WLZZ models, the correlators we are talking about are those in the negative branch models.

As we demonstrated in [64] for the Gaussian ($m = 2$) model, the averages (7.15) can be reduced to a correlator of the form

$$\langle S_Q\{\hat{\mathcal{W}}_k\} \cdot S_R\{P_k\} \rangle = \langle K_Q\{P_k\} \cdot S_R\{P_k\} \rangle \quad (7.16)$$

where the polynomials K_R form a complete basis, and celebrate the property

$$\langle K_R \cdot K_Q \rangle = \frac{S_R\{N\}}{S_R\{\delta_{k,1}\}} \delta_{RQ} \quad (7.17)$$

Examples of these polynomials can be found in [64, appendix].⁶

Now the point is that these double averages are not factorized. This is because the operators do not have a definite grading. Moreover, terms of different gradings come with N -dependent coefficients and are carefully matched to cancel the N -dependent contributions. In result, the average $\langle K_Q S_R \rangle$ does not grow as $N^{|Q|+|R|}$, it is rather $\sim N^{|R|}$. Moreover, $\langle K_Q P_R \rangle$ for individual time-variables P_R can grow even slower, e.g. $\langle K_{[1,1]} P_2 \rangle = 0$, while $\langle P_2 \rangle = N^2$.

This is consistent with the fact that the “eigenvalues” μ in

$$\langle K_Q S_R \rangle = \mu_{Q,R} \cdot \langle S_R \rangle, \quad \mu_{Q,R} = \frac{S_{R/Q}\{\delta_{k,2}\}}{S_R\{\delta_{k,2}\}} \quad (7.18)$$

do not depend on N (instead of growing like $N^{|Q|}$).

8 Conclusion

The main goal of this paper was to learn how the spectral curve for the resolvent $y(z)$ emerges from the \mathcal{W} -representation of the partition function. We demonstrated that, in

⁶Note that, throughout the paper [64], we discussed another basis of polynomials, K_Δ , the two related by the Fröbenius formula

$$K_R = \sum_{\Delta} \frac{\psi_R(\Delta)}{z_\Delta} K_\Delta$$

where $\psi_R(\Delta)$ is the symmetric group character, and z_Δ is the standard symmetric factor of the Young diagram (order of the automorphism) [69].

the standard examples of matrix models, it is described by a truncated version of \mathcal{W} -operator, $\hat{\mathcal{W}}^{\text{spec}}$. In order to construct the operator $\hat{\mathcal{W}}^{\text{spec}}$ from the full $\hat{\mathcal{W}}$, one has to leave in $\hat{\mathcal{W}}$ linear in p_k terms with maximum number of derivatives (taking into account that the coefficient N is also a derivative with respect to a variable t_0). Then,

$$e^{\hat{\mathcal{W}}^{\text{spec}}} \cdot 1 = \exp(\mathcal{P}) \tag{8.1}$$

and \mathcal{P} is *linear* in time variables. We explained in section 6 why this linearization happens exactly to the *exponential* functions on the both sides of (8.1). Now, the substitution $p_k \rightarrow z^k p_k$ makes \mathcal{P} depending on a spectral parameter z , and allows one to generate the function (resolvent) $y(z)$ such that

$$\mathcal{P}(z) = \oint V(xz) y(x) dx \tag{8.2}$$

where $V(z) = \sum_k p_k z^k / k$ is the matrix model potential. The resolvent $y(z)$ satisfies the spectral curve equation.

As a highly non-trivial check of this conjecture, we applied it to the intriguing family of WLZZ models [32], which so far were defined only through \mathcal{W} -representations. These models have a parameter m , which characterizes at once the grading of the operator and the maximal number of derivatives, the two *a priori* independent parameters. Somehow their identification seems to provide an especially interesting class of partition functions, which possess, apart from integrability, also a simple superintegrability property. We showed that, for negative m , the above prescription for $\hat{\mathcal{W}}^{\text{spec}}$ is just the correct one and leads to the family of spectral curves

$$y^{|m|} - zy + N = 0 \tag{8.3}$$

which generalize the one for the case of $m = -2$, the ordinary Hermitian model at the Gaussian point.

In order to check it, we needed to restore the W -constraints on the partition function in this case:

$$\left(n \frac{\partial}{\partial p_n} + \widetilde{W}_{n-m}^{(-,m)} \right) Z = 0 \tag{8.4}$$

where the operators $\hat{W}_n^{(m)}$ are obtained from the relation

$$\hat{W}_{-m} = \sum_{k=1}^{\infty} p_k \widetilde{W}_{k-m}^{(-,m)} \tag{8.5}$$

and then to construct the loop equations, the leading large N behaviour of them just giving rise to the spectral curve.

However, for positive m the situation is more complicated: in this case, there is no large N topological expansion, and the spectral curve limit rather corresponds to the small N limit. On the other hand, there is neither known a set of the W -constraints on the partition function. Hence, though, by our general procedure, we obtained the spectral curve

$$y = \sum_{k=2} \frac{g_k}{z^{k+1}} \left(1 + z^{2m-1} y^{m-1} \right)^{k-1} \tag{8.6}$$

we could check that it coincides with the correct one only in the case of $m = 2$ when there exists a realization of the partition function as the Gaussian matrix model in the external field [32, 56].

At the boundary between positive and negative m lies the case of $m = 0$, where it still makes sense to untie the number of derivatives s from the grading $m = 0$. This gives rise to a whole family of Lambert spectral curves

$$ye^{-z^{s-1}y^{s-1}} = \frac{\beta}{z^2} \tag{8.7}$$

of which $s = 1$ is the standard example of the Hurwitz model [29, 30]. In this case, the N dependence is already not quite simple, and the spectral curve is not described by a naive large N limit (one should rather substitute $\beta \rightarrow e^{-N/2}\beta$).

To conclude, our approach allows one to construct the spectral curves for the WLZZ models. The point is that the model defined via a \mathcal{W} -representation may be nice (in particular, superintegrable), and a nice expression is available for the would-be spectral curve even if a matrix model representation, or even a set of W -constraints on the partition function are unavailable. Moreover, the spectral curves are not obligatory related to the large N limit: the positive branch of WLZZ models is rather associated with the *small* N limit. What this means for the topological expansion and topological recursion still remains to be understood.

This study provides new insights into the notion of spectral curve, and thus of the AMM-EO topological recursion [12–18, 28]. It is an interesting question how the later one is constructed from \mathcal{W} -representations, and what are the restrictions (if any) on the possible choice of $\hat{\mathcal{W}}$ and the “vacuum” state. This is also related to the ambiguity problem of \mathcal{W} -representations [22, 28].

One of the straightforward generalizations of this investigation can be to confirm our general recipe for generating the spectral curve in the case of β -deformations, which are readily available for the WLZZ models [32].

To summarize, the WLZZ models provide us with entire three *families* of superintegrable theories: for $m < 0$; $m = 0$, $s \geq 2$ and $m > 0$, which generalize known and rather non-trivial examples at $m = \pm 2$ and $m = 0$, $s = 2$. This opens an opportunity of studying problems that could not be fully addressed before, like relation between super- and ordinary integrability (seemingly broken for $s > 2$), or relation between the spectral curves and the topological recursion and the large N expansion (broken at $m > 0$), or relation between the W -representations and the Ward identities. This makes further study of these models very promising and challenging. At the same time, it remains unclear what makes these models so successful, and which properties of the \mathcal{W} -operator are responsible for (super)integrability and even for the peculiar shape of the spectral curves. This adds to the older questions of ambiguity of W -representations and of possibility of selecting operators $\hat{\mathcal{W}}$ belonging to the w_∞ algebra. We hope that many of these questions will attract interest and will be addressed and answered in the near future.

Acknowledgments

We are grateful to V. Mishnyakov and A. Popolitov for useful discussions. We are also indebted to the referee of this paper for stimulating questions. This work was supported by the Russian Science Foundation (Grant No. 21-12-00400).

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