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N^3LO quadratic-in-spin interactions for generic compact binaries

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Abstract: We derive the third subleading (N^3LO) corrections of the quadratic-in-spin sectors via the EFT of spinning objects in post-Newtonian (PN) gravity. These corrections consist of contributions from 4 sectors for generic compact binaries, that enter at the fifth PN order. One of these contributions is due to a new tidal interaction, that is unique to the sectors with spin, and complements the first tidal interaction that also enters at this PN order in the simple point-mass sector. The evaluation of Feynman graphs is carried out in a generic dimension via advanced multi-loop methods, and gives rise to dimensional-regularization poles in conjunction with logarithms. At these higher-spin sectors the reduction of generalized Lagrangians entails redefinitions of the position beyond linear order. We provide here the most general Lagrangians and Hamiltonians. We then specify the latter to simplified configurations, and derive the consequent gauge-invariant relations among the binding energy, angular momentum, and frequency. We end with a derivation of all the scattering angles that correspond to an extension of our Hamiltonians to the scattering problem in the simplified aligned-spins configuration, as a guide to scattering-amplitudes studies.

Keywords: Effective Field Theories, Renormalization and Regularization, Black Holes, Scattering Amplitudes

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Contents

1 Introduction

The inspirals and mergers of compact binaries have been successfully measured through their emitted gravitational waves (GWs) as of 7 years now, with the first black-hole (BH) binary merger detected [1] by the advanced LIGO [2] in collaboration with Advanced VIRGO [3]. By now there is already an operational array of second-generation ground-based GW detectors on the northern hemisphere, including also KAGRA in Japan, and several elaborate plans for diverse ground- and space-based GW experiments that will extend the measured bandwidth of frequencies to no less than 16 orders of magnitude.

Since these compact-binary sources of GWs spend most of their lifetime with their components orbiting in non-relativistic (NR) velocities, they have been analytically studied in the post-Newtonian (PN) approximation of General Relativity (GR) [4]. With the input from PN theory, supplemented with numerical simulations that cover the short-lived phase of the binary in strong gravity, the effective one-body (EOB) approach [5] enables the generation of complete theoretical waveforms to be tested against our real-world measurements. The wealth of data that is being recorded [6–8], contains coveted information on gravity and QCD theory due to the diverse sources involved, including rotating BHs in a surprising span of masses [9–11], and neutron stars (NSs), whose interiors probe QCD in extreme conditions that only nature can generate [12, 13].

These impressive advances in the experimental front of GW science have catalysed a resurgence of theoretical studies aimed at improving the precision of predicted gravitational waveforms. Modern HEP approaches have joined traditional GR efforts to better describe such GW sources, yet they both essentially share a similar perspective of capturing these two-body systems via effective descriptions that consider either the single compact object, or the whole binary, as a point particle [4, 14]. The UV physics of their internal structure is then captured by some unknown characteristic coefficients, which remain to be specified. For the conservative point-mass and spin-orbit sectors of the two-body interactions, such UV coefficients show up only as of the high fifth PN (5PN) accuracy [14, 15], which has been reached in the point-mass sector via heroic undertakings only recently in both traditional GR [16–18], and EFT methods [19]. Subsequently, the spin-orbit sector at the 4.5PN accuracy has been approached following a similar approach to [16, 17] via traditional GR methods, without the need to account for any UV coefficients [20, 21]. At the same time, this third subleading spin-orbit sector was approached, completed, and verified in [22–24], and following those was also derived in [25], via the EFT of spinning gravitating objects [26], and the public `EFTofPNG` code [27].

However more generally, namely beyond the spin-orbit sector which requires only the minimal coupling to gravity, much lower PN orders than the present precision frontier necessitate to take into account finite-size effects related with such unknown UV coefficients, due to the rotation of the individual compact objects. The leading finite-size effects, starting from the 2PN order [28], are due to spin-induced multipolar deformations, which have been uniquely formulated within the EFT of spinning gravitating objects to all orders in spin [26, 29]. Indeed the EFT presented in [26], has granted unique access to higher-spin sectors, and thus the completion of the current state of the art at the 4PN order in sectors with spins [29–33]. At the present 5PN order however, another type of finite-size effects comes into play due to the tidal response to the gravitational field. The most simple unknown UV coefficients of this type arise in the point-mass sector, and are referred to as "Love numbers" in traditional GR. These have been studied for decades, see e.g. [34–37], with a recent booming activity, that notably considers also the real case of rotating compact objects, e.g. [38–43]. Interestingly almost all studies to date indicate that such simple "Love numbers" vanish for BHs in 4-dimensional GR. On the other hand, the UV coefficients of tidal effects can be significantly large for NSs, and thus these effects are very important for GW astronomy [44].

In a recent letter by the present authors, which reported on the completion of all quadratic-in-spin interactions at the 5PN order [45], a new finite-size feature was found, related with these tidal effects at the point-mass sector. This new feature, unique to spinning objects, comes with a new unknown UV coefficient, which thus provides a new unique probe for gravity and QCD. In the present paper we provide in full detail the derivation of all these quadratic-in-spin sectors at the third subleading order (N^3LO), for generic compact binaries, and in the most general settings. These results constitute the PN state of the art, and we provide the consequent useful GW quantities and observables. The derivations are carried out via the EFT of spinning gravitating objects [26], and see [27, 46], which provides a unique independent conceptual framework to tackle general sectors in PN theory, such as those in the present challenging overlap of high-loop and higher-spin orders. The overall derivation builds on the public code `EFTofPNG` introduced in [27], and on extensions of this framework in [15, 22, 23, 29–33, 47–49].

In section 2 we present the formulation of the EFT required for the present higher-spin sectors, which importantly entails non-minimal coupling of spin to gravity. Moreover, in section 2.1 we present the extension of the effective theory of a spinning particle to couplings that are quadratic in the curvature, which is required at the present 5PN order, where we point out the relation to the point-mass sector. In section 3 we present in full detail the evaluation of the diagrammatic expansion, which consists of 4 independent sectors of quadratic-in-spin corrections at the 5PN order. The Feynman graphs are evaluated via advanced multi-loop methods with a computational load of graphs, and a generic dimensional expansion, due to the appearance of dimensional-regularization (DimReg) poles across all loop orders, which we handled with independent development of the `EFTofPNG` code, similar to [23]. In section 3.1 we provide the full generalized Lagrangians for all of the 4 independent contributions included in the present quadratic-in-spin sectors.

We proceed in section 4 to reduce these effective actions via a formal procedure that involves redefinitions of the position and rotational variables through lower-order sectors, which requires in higher-spin sectors such as the present ones, an intricate implementation of the redefinitions beyond linear order. We provide the full redefinitions in section 4.1, and present in section 4.2 the consequent reduced actions of all 4 subsectors for the first time. In section 5 we present the full general Hamiltonians of the relevant 4 subsectors for the first time, with the consequent useful simplified Hamiltonians in section 5.1. Finally, in section 5.2 we proceed to discuss the consequent gauge-invariant binding-energy relations to the angular momentum and orbital frequency, noting in particular the binding energy associated to the new tidal effect with spins, in comparison to that from the point-mass sector, which also contributes at the present 5PN order.

We end in section 5.3 with a derivation of all the extrapolated scattering angles that correspond to an extension of our Hamiltonians to the scattering problem in the simplified aligned-spins configuration, as a guide to scattering-amplitudes studies.

2 EFT of spinning gravitating objects

We start by reviewing the EFT of spinning gravitating objects [14, 26, 46]. At the orbital scale the compact-binary inspiral is described as a two-particle system with an effective action that consists of the purely gravitational action with the relevant weak field modes, augmented by a point-particle action, $S_{\rm pp}$, for each of the compact objects in the binary [14]:

$$S_{\rm eff} = S_{\rm gr}[g_{\mu\nu}] + \sum_{a=1}^{2} S_{\rm pp}(\lambda_a). \tag{2.1}$$

$S_{\rm gr}$, the action of the field in some theory of gravity, must be supplemented by $S_{\rm pp}$, a new infinite tower of interactions between the weak field modes and the new worldline degrees of freedom at this scale. These interactions are localized on the worldline of each object in the binary, parametrized by λ_a for the a-th object.

For the gravitational action we consider here the theory of GR, which we supplement with the fully-harmonic gauge-fixing term:

$$S_{\rm gr}[g_{\mu\nu}] = S_{\rm EH} + S_{\rm GF} = -\frac{1}{16\pi G_d}\int d^Dx\sqrt{g}\,R + \frac{1}{32\pi G_d}\int d^Dx\sqrt{g}\,g_{\mu\nu}\Gamma^\mu\Gamma^\nu, \tag{2.2}$$

with $\Gamma^\mu \equiv \Gamma^\mu_{\rho\sigma}g^{\rho\sigma}$. As of third subleading orders in PN theory, such as in this work, the dimension must be kept generic throughout the whole evaluation of two-body interactions, including a modified minimal subtraction ($\overline{\rm MS}$) (see e.g. [50]), for the d-dimensional coupling constant of the NR theory:

$$G_d \equiv G_N\left(\sqrt{4\pi e^\gamma}\,R_0\right)^{d-3}, \tag{2.3}$$

with $d \equiv D-1$ the number of spatial dimensions, $G_N \equiv G$ Newton's gravitational constant in three-dimensional space, γ Euler's constant, and R_0 some renormalization scale.

For the NR gravitational field it is beneficial to use a $d+1$ Kaluza-Klein reduction over the time dimension [51, 52]:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \equiv e^{2\phi}\left(dt - A_i dx^i\right)^2 - e^{-\frac{2}{d-2}\phi}\gamma_{ij}dx^i dx^j, \tag{2.4}$$

which defines the set of fields: ϕ, A_i, and $\gamma_{ij} \equiv \delta_{ij} + \sigma_{ij}$. This decomposition has been useful also in sectors with spin since its application in [53], and as applied later all across the public `EFTofPNG` code [27]. The propagators in a NR approximation are instantaneous, leaving the momenta integrals as purely spatial, namely Euclidean in d dimensions. The relativistic corrections in this approximation are then obtained as quadratic insertions on the propagators with two time derivatives. All the Feynman rules which are required for the present sectors in d dimensions, involving only the gravitational field, including the propagators, their relativistic perturbative corrections, and the self-interaction vertices, are similar to those provided in [23].

For rotating objects one then needs to specify the effective action of a spinning particle as the point-particle action of each object. This effective action starts with the leading couplings of mass and spin, whose associated position and rotational degrees of freedom (DOFs), respectively, are minimally coupled to gravity [26, 46, 54]:

$$S_{\rm pp}(\lambda) = \int d\lambda\left[-m\sqrt{u^2} - \frac{1}{2}\hat{S}_{\mu\nu}\hat{\Omega}^{\mu\nu} - \frac{\hat{S}^{\mu\nu}p_\nu}{p^2}\frac{Dp_\mu}{D\lambda}\right]. \tag{2.5}$$

Here we introduced the mass m, the 4-velocity u^μ, the linear momentum p_μ, as well as generic angular velocity and spin variables, $\hat{\Omega}^{\mu\nu}$ and $\hat{S}_{\mu\nu}$, respectively. The latter generic rotational variables, together with the extra term in eq. (2.5) with the covariant derivative, enable to switch the rotational gauge along the worldline [26], which was not accounted for in past formulations of spin in relativity [55–58].

Next, we must consider the part of the point-particle action that accounts for finite-size effects, of which we have in the present sectors two types: spin-induced multipolar, and tidal deformations, that are both uniquely due to the presence of spin. Tidal effects enter (only) as of the present high 5PN order, and thus we need to consider them properly for the present sectors, which we do in section 2.1 below. For the spin-induced multipolar deformations, we consider the leading non-minimal couplings to gravity to all orders in spin, that are linear in the curvature, which we introduced in [26, 29]:

$$L_{\mathrm{NMC(RS^\infty)}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{ES^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \cdots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} \bullet S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n-1}} S^{\mu_{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{BS^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \cdots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} \bullet S^{\mu_1} S^{\mu_2} \cdots S^{\mu_{2n}} S^{\mu_{2n+1}}, \tag{2.6}$$

where $\bullet$ stands for tensor contraction of the spin-induced multipole tensors with the definite-parity electric and magnetic components of the curvature:

$$E_{\mu\nu} \equiv R_{\mu\alpha\nu\beta} u^\alpha u^\beta, \tag{2.7}$$

$$B_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}{}_{\delta\nu} u^\gamma u^\delta, \tag{2.8}$$

and their covariant derivatives, D_μ. This infinite series of operators in eq. (2.6) is preceded by a corresponding series of Wilson coefficients, that would be referred to as "multipolar deformation parameters" in traditional GR. From this series only the leading term is needed in the present sectors, with the electric component and no further covariant derivatives, and whose coefficient corresponds to the quadrupolar deformation constant, as in [28].

From the effective action of a spinning particle in eqs. (2.5), (2.6), all the necessary worldline couplings for the present sectors can be extracted using the `EFTofPNG` code, including both mass and spin, but excluding new couplings that are beyond linear in the curvature, which are discussed in the next section 2.1. We remind that in our spin couplings we implement the generalized canonical gauge that we introduced in [26], and so all the indices in the Feynman rules are Euclidean. All of these rules are provided in both human- and machine-readable formats in the supplementary material to this publication.

2.1 New physics

At such high perturbative orders the point-particle effective action needs to be further extended to include non-minimal coupling to gravity that is quadratic in the curvature. In the EFT formulation of the point-mass sector, defined as that whose worldline DOFs are only the position coordinates, two leading tidal operators were presented already in [14]:

$$L^{\mathrm{LO}}_{\mathrm{NMC(R^2S^0)}} = C_{\mathrm{E}^2} \frac{E_{\alpha\beta} E^{\alpha\beta}}{\sqrt{u^2}^{\,3}} + C_{\mathrm{B}^2} \frac{B_{\alpha\beta} B^{\alpha\beta}}{\sqrt{u^2}^{\,3}}, \tag{2.9}$$

where the coefficients in this expression contain all dependence in the mass and coupling constant, and numerical symmetry factors, in contrast to the new coefficients we introduced in eq. (2.6). Initial power-counting indicates that these operators may enter at the 5PN order, notably being the leading finite-size effects to affect the point-mass sector, where this high PN accuracy has been obtained only recently in this simplest of all sectors [16–19].

An extension of the effective action of a spinning particle to include such general operators was outlined in [26], and approached in [15, 49]. There it was shown that quadratic-in-curvature operators, which are linear in the spin may enter only as of the 6.5PN order, notably making also the spin-orbit sector affected solely by minimal coupling to gravity to a very high PN order. Following [15] and then [45], we consider up to quadratic order in the spins and at the present 5PN accuracy, the following extension to the non-minimal coupling of spin to gravity:

$$L^{\rm LO}_{\rm NMC(R^2S^2)} = \frac{1}{2} C_{\rm E^2} G^4 m^5 \frac{E_{\alpha\beta}E^{\alpha\beta}}{\sqrt{u^2}^3} + \frac{1}{2} C_{\rm B^2} G^4 m^5 \frac{B_{\alpha\beta}B^{\alpha\beta}}{\sqrt{u^2}^3} \tag{2.10a}$$
$$+ \frac{1}{2} C_{\rm E^2S^2} G^2 m \frac{E_{\mu\alpha}E_{\nu}{}^{\alpha}}{\sqrt{u^2}^3} S^\mu S^\nu + \frac{1}{2} C_{\rm B^2S^2} G^2 m \frac{B_{\mu\alpha}B_{\nu}{}^{\alpha}}{\sqrt{u^2}^3} S^\mu S^\nu. \tag{2.10b}$$

As detailed in [26] we construct the effective action of a spinning particle by considering the contraction of SO(3) tensors in the local frame, and specifically for rotating objects — in the body-fixed frame. Accordingly, the SO(3) building blocks are symmetrized and their traces are removed.

Note that in eq. (2.10) we already defined the dimensionless normalized Wilson coefficients of the new operators, similar to eq. (2.6). Note that these are the leading operators to carry an additional scaling in the gravitational coupling-constant G, in contrast to all operators encountered up to this PN order, including the operator that is quadratic in curvature and quartic in spin at the 5PN order [49]. The signs of new operators are fixed according to the leading non-minimal coupling, and the number of derivatives. The Wilson coefficients in eq. (2.10a) correspond to the long-studied "Love numbers" in traditional GR. Trace terms of the new spin operators in eq. (2.10b) are absorbed in the usual "Love numbers" from the point-mass sector, e.g. $C_{\rm E^2} \supset \frac{C_{\rm E^2S^2}}{3}\frac{S^2}{G^2m^4}$ (recall that $[S^2/(G^2m^4)] \leq 1$ for Kerr black holes). This is similar to that the basic constant scalar spin length, S^2, which is the trace of the square of the spin tensor, is absorbed in the mass, and other Wilson coefficients, and is therefore omitted from our traceless tensors [26]. For this reason such trace terms cannot be matched independently of the mass. Yet, as we shall clearly see the new operators in eq. (2.10b) represent a new type of tidal effects that are only relevant to spinning objects, and are thus preceded by new Wilson coefficients [45]. Their intrinsic G scaling gives further indication that the operators in eq. (2.10b) belong to a different type of effects than spin-induced multipolar deformations as in eq. (2.6), and that the new Wilson coefficients may correspond to a generic concept of "Love numbers" in gravity [59].

Proceeding to derive the consequent Feynman rules, one finds that the operators in eq. (2.10), which contain the magnetic curvature component, actually enter only as of the 6PN order, and thus we are left with the contributions:

$$L^{\rm 5PN}_{\rm NMC(R^2S^2)} = \frac{1}{2} C_{\rm E^2} G^4 m^5 \frac{E_{ij}E_{ij}}{\sqrt{u^2}^3} - \frac{1}{2} C_{\rm E^2S^2} G^2 m \left[\frac{E_{ik}E_{jk}}{\sqrt{u^2}^3} S^i S^j - \frac{S^2}{3}\frac{E_{ij}E_{ij}}{\sqrt{u^2}^3}\right]. \tag{2.11}$$

Order in G	1	2	3	4	Total
$\mathrm{S_1S_2}$	10	94	294	73	471
$\mathrm{S_1^2}$	0	50	181	52	283
$\mathrm{C_ES_1^2}$	9	107	213	38	367
$\mathrm{C_{E^2}S_1^2}$	0	1	0	0	1
Total in $\mathrm{S^2}$	19	252	688	163	1122

Table 1. The number of graphs across topology orders in G in the $\mathrm{N^3LO}$ quadratic-in-spin sectors.

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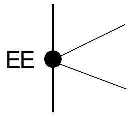
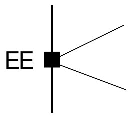
From eq. (2.11) we can now derive the Feynman rules for the point-mass tidal two-graviton coupling as (using JaxoDraw [60–62]):



$$= \frac{C_{\mathrm{E}^2}\, G^4 m^5}{2} \int dt\, \phi_{,ij}\phi_{,ij}\,, \tag{2.12}$$

and for the quadratic-in-spin new two-graviton coupling as:



$$= -\frac{C_{\mathrm{E}^2\mathrm{S}^2}\, G^2 m}{2} \int dt \left[S_i S_j \phi_{,ik}\phi_{,jk} - \frac{S^2}{3}\phi_{,ij}\phi_{,ij} \right]. \tag{2.13}$$

Let us highlight here again that these types of tidal operators for the point-mass and for the quadratic-in-spin sectors, will enter the potentials at orders G^6 and G^4, respectively, though they start to contribute through the graph topology of two-graviton exchange at order G^2. This is in contrast with all other (non-tidal) PN contributions up to the present 5PN order, where the order in G has always been identical to the order of the graph topology.

3 Diagrammatic expansion

The Wick contractions and corresponding Feynman graphs in the diagrammatic expansion for the present sectors were then generated using the `EFTofPNG` code. The distribution of graphs among the 4 topology orders in G, and among the 4 quadratic-in-spin sectors at the $\mathrm{N^3LO}$, is displayed in table 1. There are 1122 graphs altogether, with 1 graph originating from the new coupling beyond linear in curvature from section 2.1, as shown in figure 1(b). The topologies at order G^4, and their corresponding integral structure, were studied in [15, 22], where 163 graphs at this order in the present sectors were evaluated. The analysis of all graphs in the present sectors builds on the $\mathrm{N^2LO}$ quadratic-in-spin sectors completed via our EFT in [30, 32].

We provide all the graphs and their values enumerated in 3 separate lists, which correspond to the interactions of spin_1 with spin_2, of each spin non-linearly with itself, and of each spin-induced quadrupole. These lists are included in the supplementary material to this publication in both human visual and machine-readable formats. The graphs are indexed as (n_1, n_2, n_3), with n_1 for order in G, n_2 for topology type (for the topologies at each order in G, see [22, 46]), and n_3 as a graph counter within each topology. Only unique graphs with spin on worldline "1" are presented in each list, while a copy of each list from the exchange of worldline labels, $1 \leftrightarrow 2$, is suppressed.

Dimensional regularization (DimReg) is used to evaluate all integrals, and as in all third and higher subleading sectors [22], the dimension needs to be kept generic across the whole derivation. An expansion in the dimensional parameter, $\epsilon_d \equiv d - 3$, is imperative due to the appearance of DimReg poles (in conjunction with logarithms) in the values of individual graphs across all loop orders, and in their summation. This dimensional expansion is the most time-consuming and demanding task in the evaluation of the present sectors. Similar to the spin-orbit sector in [23], there is also a concentration of graphs in the highest-loop of each order in G (see [22] for the definition of loop order in the worldline picture). At order G^2 where one-loop is the highest loop order, 192 out of 252 graphs are in the one-loop topology, which in the present sectors already contain DimReg poles and logarithms, and at order G^3, 384 of the 688 graphs are of two-loop order.

In the present sectors similar to the spin-orbit sector, the topologies at order G^3, whose evaluation is the most demanding, are of the nested type, with 214 graphs, and of the rank-two topology (see [22] for the definition of such rank), which requires the use of integration-by-parts (IBP) methods, with 87 graphs overall. We also recall that at the N^{3}LO the graphs at order G^1 can carry up to 3 relativistic corrections (with 2 time derivatives each) on their propagators, and that these insertions are also allowed on two-loop graphs at order G^3. Therefore we implemented upgrades of the `EFTofPNG` code, which included improvements of the IBP and projection methods, see e.g. [63, 64] and [65, 66], respectively, for reviews of such modern multi-loop integration methods. Independent development, implementation and evaluation, were carried out in parallel to verify the validity of all results.

We note that similar to the spin-orbit sector [23], factorizable topologies gave rise either to zeros due to contact-interaction terms [22, 30], or to factors of the transcendental Riemann zeta value $\zeta(2) \propto \pi^2$, which also appeared in the single rank-two topology up to order G^3.

Finally, let us examine the contributions from the quadratic-in-curvature operators in eq. (2.11) to our present sectors. It is clear that there are no explicit contributions to the present sectors from contractions/graphs that contain the point-mass tidal coupling in eq. (2.12). Yet let us evaluate the related leading contribution as a point of reference. It gives rise to a graph of two-graviton exchange, shown in figure 1(a), which equals:

$$\text{figure } 1(a) = 3C_{1(\mathrm{E}^2)} \frac{G^6 m_1^5 m_2^2}{r^6}. \tag{3.1}$$

The new quadratic-in-spin coupling in eq. (2.13) however, gives rise to a graph of two-graviton exchange, shown in figure 1(b), that contributes to the present sectors:

$$\text{figure } 1(b) = \frac{C_{1(\mathrm{E}^2\mathrm{S}^2)}}{2} \frac{G^4\, m_1 m_2^2}{r^6} \left[S_1^2 - 3(\vec{S}_1 \cdot \vec{n})^2 \right]. \tag{3.2}$$

Thus this contribution introduces a new Wilson coefficient, possibly of a "Love number" type, appearing first in the present sectors.

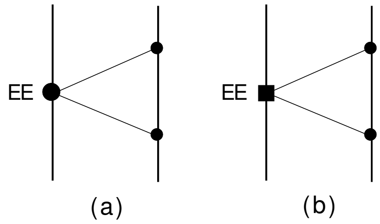


Figure 1. The leading Feynman graphs of two-graviton exchange due to quadratic-in-curvature operators entering first at the 5PN order. (a) The tidal interaction in the point-mass sector. (b) The new interaction in the present quadratic-in-spin sectors, due to the new quadratic-in-curvature coupling to spin.

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3.1 Effective actions

From the sum of all graphs an effective action is obtained, which contains higher-order time derivatives up to the sixth order in total, beyond the standard velocity and spin variables. In the present quadratic-in-spin sectors this effective action consists of 4 independent parts for the potential, $-V_{\text{S}^2}^{\text{N}^3\text{LO}} \subset L$:

$$V_{\text{S}^2}^{\text{N}^3\text{LO}} = V_{\text{S}_1\text{S}_2}^{\text{N}^3\text{LO}} + V_{\text{S}_1^2}^{\text{N}^3\text{LO}} + C_{1\text{ES}^2} V_{C_{1\text{ES}^2}}^{\text{N}^3\text{LO}} + C_{1\text{E}^2\text{S}^2} V_{C_{1\text{E}^2\text{S}^2}}^{\text{N}^3\text{LO}} + (1 \leftrightarrow 2), \tag{3.3}$$

where

$$V_{\text{S}_1\text{S}_2}^{\text{N}^3\text{LO}} = \sum_{i=0}^{6} \overset{(i)}{V}_{3,\text{S}_1\text{S}_2}, \tag{3.4}$$

$$V_{\text{S}_1^2}^{\text{N}^3\text{LO}} = \sum_{i=0}^{4} \overset{(i)}{V}_{3,\text{S}_1^2}, \tag{3.5}$$

$$V_{C_{1\text{ES}^2}}^{\text{N}^3\text{LO}} = \sum_{i=0}^{5} \overset{(i)}{V}_{3,\text{E}}, \tag{3.6}$$

$$V_{C_{1\text{E}^2\text{S}^2}}^{\text{N}^3\text{LO}} = -\frac{G^4 m_1 m_2{}^2}{2r^6}\left[S_1^2 - 3(\vec{S}_1\cdot\vec{n})^2\right], \tag{3.7}$$

where the superscript (i) stands for the total number of higher-order time derivatives in each of these pieces, and the subscript contains the subleading/maximal-loop counter n (see [48] where the notation of sectors, (n,l), was introduced), and a notation for the respective specific sector within the present quadratic-in-spin sectors. Notice that the part of the potential that is due to the non-linear interaction of the spin with itself contains only up to 4 higher-order time derivatives in total, since it enters only as of the NLO at order G^2. Since this unreduced action of all the present sectors is very lengthy, we provided it in a separate tex file, on top of two machine-readable files, which cover separately the part bilinear in the spins, and the part that is square in each spin, in the supplementary material to this publication. Notice that all in all, at this stage there are DimReg poles with logarithms at orders G^2 and G^3, and transcendental $\zeta(2)$ factors at orders G^3 and G^4.

l \ n	(N^0)LO	$\mathrm{N}^{(1)}$LO	N^2LO	N^3LO
S^0			+	+
S^1	+	++	++	
S^2		++	++	++

Table 2. The notation (n, l) of sectors was introduced in [48]. The 11 sectors that contribute to the present $(3, 2)$ one through redefinition of variables. "+" marks sectors that require only position shifts to be fixed, whereas "++" marks sectors that require redefinition of both position and rotational variables to be fixed.

4 Reduction of effective actions

We press on to reduce the action obtained directly from the EFT computation, which contains many higher-order time derivatives beyond velocity and spin, to our final action. The formal reduction procedure we carry out in the present sectors builds on that which was presented in detail for the N^3LO spin-orbit sector [23]. It proceeds with gradual subleading redefinitions of the position and rotational variables, where the formulation to handle rotational variables was first introduced in [47]. Table 2 shows all the sectors that are involved in the reduction process for the present sectors. Note that redefinitions are applied on all these sectors, some of which do not require themselves any redefinitions in order to be fixed, as they yield contributions to the present sector.

First, as pointed out in [26] at higher-spin sectors, such as the present ones, position shifts need to be applied beyond linear order already at the NLO due to the leading position shift that is required already at the LO spin-orbit sector. Since position shifts scale like the sector they originated from in their spin and PN orders [47], we only need to apply here such shifts originating from the spin-orbit sector, to quadratic order, while all other position shifts can be applied at linear order. Second, for redefinitions of the rotational variables, that are clearly required only in sectors with spins, their power counting is $S^{-1}v^{-1}$ relative to their originating sector [47]. This power counting, together with the extension of the formal procedure of redefinition for rotational variables beyond linear order, that was carried out in [23], show that in the present sectors such redefinitions are only needed to be applied at linear order.

4.1 Redefinition of degrees of freedom

As noted we follow here closely the reduction procedure which was presented in detail for the N^3LO spin-orbit sector in [23]. As table 2 shows, we proceed through the application of redefinitions fixed in 8 relevant sectors including the present ones, according to their increasing PN order, see [48] for our general PN-counting formula for any sector, with or without spins. Yet, the redefinitions in 5 of these sectors, which are below quadratic in the spin, are not modified with respect to [23], and thus we need to consider here in addition only the 3 sectors that are quadratic in the spin, as summarized in tables 3–5.

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from \ to	(0P)N	LO S^1
LO S^1	$(\Delta\vec{x})^2$	$\Delta\vec{x}$
NLO S^2	$\Delta\vec{x}$, $\Delta\vec{S}$	

Table 3. Contributions to the NLO S^2 sectors from position shifts and spin redefinitions in lower-order sectors.

The algorithm we use for the reduction procedure is similar to what we described in [23], with the tables here understood in a similar manner to what was explained therein. However, as can be inferred from table 2, and tables 3–5, the reduction procedure in the present sectors is more laborious and computationally demanding, mainly due to the application of position shifts to quadratic order. Nevertheless, our streamlined code following the algorithm we described in [23] still completes the necessary reduction for the present sectors efficiently and rapidly.

Let us then go over all the relevant sectors in order, also reviewing those which were covered in [23]. We remind that all of our unreduced actions are computed directly with the `EFTofPNG` code. First, we note that for the LO spin-orbit, 2PN, NLO spin-orbit, and 3PN sectors our unreduced actions and redefinitions are identical to those presented in [23]. We remind that in the latter sector, namely at 3PN, there are DimReg poles and logarithms showing up for the first time, and so we also add in advance a total time derivative (TTD) to the unreduced action in order to land on a final action without these unphysical features. Moreover, since redefinitions in the 3PN sector also contain such poles, they should be applied on LO potentials, expanded to linear order in their DimReg zeros and containing further logarithms. For the present sectors it means considering beyond the Newtonian potential, also the following piece of the LO quadratic-in-spin potentials:

$$
\begin{aligned}
L^{\rm LO}_{\rm S^2}|_{\mathcal{O}(\epsilon^1)} = \epsilon \Bigg[& -\frac{G}{r^3}\left(\left[2-\frac{3}{2}\log\left(\frac{r}{R_0}\right)\right]\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}-\left[1-\frac{1}{2}\log\left(\frac{r}{R_0}\right)\right]\vec{S}_1\cdot\vec{S}_2\right)\\
& +\frac{GC_{1(\rm ES^2)}m_2}{4m_1r^3}\left(\left[5-2\log\left(\frac{r}{R_0}\right)\right]S_1^2-\left[11-6\log\left(\frac{r}{R_0}\right)\right](\vec{S}_1\cdot\vec{n})^2\right)\Bigg]\\
& +(1\leftrightarrow 2)\,.
\end{aligned} \tag{4.1}
$$

At this point we proceed to the NLO quadratic-in-spin sectors captured in table 3. The unreduced potentials are identical to those presented in our [26]. The new redefinitions applied in these sectors for the position and spin variables can be written as:

$$
\begin{aligned}
(\Delta\vec{x}_1)^{\rm NLO}_{\rm S^2} = & \frac{2G}{m_1r^2}\left[\vec{S}_1\cdot\vec{S}_2\vec{n}-\vec{S}_1\cdot\vec{n}\vec{S}_2\right]-\frac{Gm_2}{8{m_1}^2r^2}\left[S_1^2\vec{n}-\vec{S}_1\cdot\vec{n}\vec{S}_1\right]+\frac{GC_{1(\rm ES^2)}m_2}{{m_1}^2r^2}\vec{S}_1\cdot\vec{n}\vec{S}_1\\
& -\frac{1}{4{m_1}^2}\left[\dot{\vec{S}}_1\cdot\vec{v}_1\vec{S}_1-\vec{S}_1\cdot\vec{v}_1\dot{\vec{S}}_1\right]-\frac{3}{8{m_1}^2}\left[S_1^2\vec{a}_1-\vec{S}_1\cdot\vec{a}_1\vec{S}_1\right],
\end{aligned} \tag{4.2}
$$

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to / from	(0P)N	1PN	LO S^1	LO S^2	NLO S^1
LO S^1		$(\Delta\vec{x})^2$			$\Delta\vec{x}$
2PN				$\Delta\vec{x}$	
NLO S^1			$\Delta\vec{x}$	$\Delta\vec{S}$	
NLO S^2		$\Delta\vec{x}$	$\Delta\vec{S}$		
N^2LO S^2	$\Delta\vec{x}$, $\Delta\vec{S}$				

Table 4. Contributions to the N^2LO S^2 sector from position shifts and spin redefinitions in lower-order sectors.

and

$$\begin{aligned}(\omega_1^{ij})^{\rm NLO}_{\rm S^2} = & -\frac{G}{2r^2}\Big[3S_2^{ki}n^k\vec{v}_2\cdot\vec{n}n^j - 3S_2^{ki}v_2^kn^j - S_2^{kj}n^k(4v_1^i - v_2^i) + S_2^{ij}\vec{v}_2\cdot\vec{n}\Big] \\ & -\frac{Gm_2}{4m_1r^2}\Big[S_1^{ki}v_1^kn^j + S_1^{kj}n^kv_1^i\Big] - \frac{GC_{1(\rm ES^2)}m_2}{2m_1r^2}\Big[3S_1^{ki}n^k\vec{v}_2\cdot\vec{n}n^j - 2S_1^{ki}v_1^kn^j \\ & +3S_1^{ki}v_2^kn^j + S_1^{kj}n^k(2v_1^i - 3v_2^i) - S_1^{ij}(2\vec{v}_1\cdot\vec{n} - 3\vec{v}_2\cdot\vec{n})\Big] \\ & -\frac{1}{4m_1}\Big[S_1^{kj}v_1^ka_1^i + S_1^{kj}a_1^kv_1^i\Big] - \frac{1}{8m_1}v_1^i\dot{S}_1^{kj}v_1^k \\ & -\frac{G}{4r}\Big[\dot{S}_2^{ki}n^kn^j + \dot{S}_2^{ij}\Big] - (i\leftrightarrow j). \end{aligned} \tag{4.3}$$

Note that these redefinitions are different than those we used in [26]. Next, the N^2LO spin-orbit sector is unaffected by the redefinitions in the latter sectors, that are quadratic in the spin, and thus its unreduced action and redefinitions are again identical to those presented in [23].

We proceed then to the N^2LO quadratic-in-spin sectors captured in table 4. The unreduced potential can be written as the sum:

$$V^{\rm N^2LO}_{\rm S^2} = V^{\rm N^2LO}_{\rm S_1S_2} + V^{\rm N^2LO}_{\rm S_1^2} + C_{1\rm ES^2}V^{\rm N^2LO}_{C_{1\rm ES^2}} + (1\leftrightarrow 2), \tag{4.4}$$

where

$$V^{\rm N^2LO}_{\rm S_1S_2} = \sum_{i=0}^{4}\overset{(i)}{V}_{2,\rm S_1S_2}, \tag{4.5}$$

$$V^{\rm N^2LO}_{\rm S_1^2} = \sum_{i=0}^{2}\overset{(i)}{V}_{2,\rm S_1^2}, \tag{4.6}$$

$$V^{\rm N^2LO}_{C_{1\rm ES^2}} = \sum_{i=0}^{3}\overset{(i)}{V}_{2,\rm E}\ . \tag{4.7}$$

The new redefinitions that are fixed in these sectors are then written for the position shift as:

$$(\Delta\vec{x}_1)^{\rm N^2LO}_{\rm S^2} = (\Delta\vec{x}_1)^{\rm N^2LO}_{\rm S_1S_2} + (\Delta\vec{x}_1)^{\rm N^2LO}_{\rm S_1^2} + C_{\rm 1ES^2}\,(\Delta\vec{x}_1)^{\rm N^2LO}_{C_{\rm 1ES^2}} \\ + (\Delta\vec{x}_1)^{\rm N^2LO}_{\rm S_2^2} + C_{\rm 2ES^2}\,(\Delta\vec{x}_1)^{\rm N^2LO}_{C_{\rm 2ES^2}}\,, \tag{4.8}$$

where

$$(\Delta\vec{x}_1)^{\rm N^2LO}_{\rm S_1S_2} = \sum_{i=0}^{2} \overset{(i)}{\Delta\vec{x}}_{1(2,{\rm S_1S_2})}, \tag{4.9}$$

$$(\Delta\vec{x}_1)^{\rm N^2LO}_{\rm S_1^2} = \sum_{i=0}^{1} \overset{(i)}{\Delta\vec{x}}_{1(2,{\rm S_1^2})}, \tag{4.10}$$

$$(\Delta\vec{x}_1)^{\rm N^2LO}_{C_{\rm 1ES^2}} = \sum_{i=0}^{1} \overset{(i)}{\Delta\vec{x}}_{1(2,{\rm ES_1^2})}, \tag{4.11}$$

$$(\Delta\vec{x}_1)^{\rm N^2LO}_{\rm S_2^2} = \sum_{i=0}^{1} \overset{(i)}{\Delta\vec{x}}_{1(2,{\rm S_2^2})}, \tag{4.12}$$

$$(\Delta\vec{x}_1)^{\rm N^2LO}_{C_{\rm 2ES^2}} = \sum_{i=0}^{2} \overset{(i)}{\Delta\vec{x}}_{1(2,{\rm ES_2^2})}\,. \tag{4.13}$$

The redefinition of spin can be written as:

$$\left(\omega_1^{ij}\right)^{\rm N^2LO}_{\rm S^2} = \left(\omega_1^{ij}\right)^{\rm N^2LO}_{\rm S_1S_2} + \left(\omega_1^{ij}\right)^{\rm N^2LO}_{\rm S_1^2} + C_{\rm 1ES^2}\left(\omega_1^{ij}\right)^{\rm N^2LO}_{C_{\rm 1ES^2}} - (i \leftrightarrow j), \tag{4.14}$$

where

$$\left(\omega_1^{ij}\right)^{\rm N^2LO}_{\rm S_1S_2} = \sum_{i=0}^{3} \overset{(i)}{\omega}{}^{ij}_{1\,(2,{\rm S_1S_2})}, \tag{4.15}$$

$$\left(\omega_1^{ij}\right)^{\rm N^2LO}_{\rm S_1^2} = \sum_{i=0}^{1} \overset{(i)}{\omega}{}^{ij}_{1\,(2,{\rm S_1^2})}, \tag{4.16}$$

$$\left(\omega_1^{ij}\right)^{\rm N^2LO}_{C_{\rm 1ES^2}} = \sum_{i=0}^{2} \overset{(i)}{\omega}{}^{ij}_{1\,(2,{\rm E})}\,. \tag{4.17}$$

Since the expressions for the unreduced action and for the redefinitions are lengthy, they are presented in a separate tex file, on top of machine-readable files for the spin$_1$-spin$_2$ and spin-squared parts, in the supplementary material to this publication.

Finally, we arrive at the present N^{3}LO quadratic-in-spin sectors as captured in table 5. Similar to the N^{3}LO sectors of 3PN and spin-orbit, we also add here a TTD to the

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from \ to	(0P)N	1PN	LO S^1	2PN	LO S^2	NLO S^1	NLO S^2	N^2LO S^1
LO S^1				$(\Delta\vec{x})^2$				$\Delta\vec{x}$
2PN							$\Delta\vec{x}$	
NLO S^1	$(\Delta\vec{x})^2$					$\Delta\vec{x}$	$\Delta\vec{S}$	
3PN					$\Delta\vec{x}$			
NLO S^2				$\Delta\vec{x}$		$\Delta\vec{S}$		
N^2LO S^1			$\Delta\vec{x}$		$\Delta\vec{S}$			
N^2LO S^2		$\Delta\vec{x}$	$\Delta\vec{S}$					
N^3LO S^2	$\Delta\vec{x}$, $\Delta\vec{S}$							

Table 5. Contributions to the N^3LO S^2 sector from position shifts and spin redefinitions in lower-order sectors.

unreduced potential:

$$\begin{aligned}
\Delta V = \frac{d}{dt}\Bigg[\Bigg(&-\frac{19G^3{m_1}^2}{5r^4}\Big[5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2\vec{v}_1\cdot\vec{n}-\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\Big]\\
&+\frac{G^3{m_2}^2}{5r^4}\Big[5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\vec{v}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2\vec{v}_1\cdot\vec{n}-\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1-\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\Big]\\
&+\frac{G^3C_{1(\mathrm{ES}^2)}m_1m_2}{70r^4}\Big[S_1^2(73\vec{v}_1\cdot\vec{n}-38\vec{v}_2\cdot\vec{n})+146\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1-76\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\\
&-5(73\vec{v}_1\cdot\vec{n}-38\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^3C_{1(\mathrm{ES}^2)}{m_2}^3}{2m_1r^4}\Big[S_1^2(2\vec{v}_1\cdot\vec{n}-3\vec{v}_2\cdot\vec{n})\\
&+4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1-6\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2-5(2\vec{v}_1\cdot\vec{n}-3\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{G^3m_1m_2}{70r^4}\Big[S_1^2(43\vec{v}_1\cdot\vec{n}+67\vec{v}_2\cdot\vec{n})+86\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1+134\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\\
&-5(43\vec{v}_1\cdot\vec{n}+67\vec{v}_2\cdot\vec{n})(\vec{S}_1\cdot\vec{n})^2\Big]\Bigg)\left(\frac{1}{\epsilon}-3\log\frac{r}{R_0}\right)\Bigg]+(1\leftrightarrow 2), \qquad (4.18)
\end{aligned}$$

in order to land on a reduced potential free of poles and logarithms, after the redefinition of variables that we will fix here, will be carried out. The new position shifts fixed in the present sectors can be written as:

$$\begin{aligned}
(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{\mathrm{S^2}} = {}&(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}}+(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{\mathrm{S_1^2}}+C_{1\mathrm{ES}^2}\,(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}}\\
&+(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{\mathrm{S_2^2}}+C_{2\mathrm{ES}^2}\,(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{C_{2\mathrm{ES}^2}}\,, \qquad (4.19)
\end{aligned}$$

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where

$$(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} = \sum_{i=0}^{4} \overset{(i)}{\Delta\vec{x}}_{1(3,\mathrm{S_1S_2})}, \tag{4.20}$$

$$(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} = \sum_{i=0}^{3} \overset{(i)}{\Delta\vec{x}}_{1(3,\mathrm{S_1^2})}, \tag{4.21}$$

$$(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}} = \sum_{i=0}^{3} \overset{(i)}{\Delta\vec{x}}_{1(3,\mathrm{ES_1^2})}, \tag{4.22}$$

$$(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{\mathrm{S_2^2}} = \sum_{i=0}^{3} \overset{(i)}{\Delta\vec{x}}_{1(3,\mathrm{S_2^2})}, \tag{4.23}$$

$$(\Delta\vec{x}_1)^{\mathrm{N^3LO}}_{C_{2\mathrm{ES}^2}} = \sum_{i=0}^{4} \overset{(i)}{\Delta\vec{x}}_{1(3,\mathrm{ES_2^2})}, \tag{4.24}$$

The new redefinitions of spin that are fixed in the present sectors can be written as:

$$\left(\omega_1^{ij}\right)^{\mathrm{N^3LO}}_{\mathrm{S^2}} = \left(\omega_1^{ij}\right)^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} + \left(\omega_1^{ij}\right)^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} + C_{1\mathrm{ES}^2}\left(\omega_1^{ij}\right)^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}} - (i \leftrightarrow j), \tag{4.25}$$

where

$$\left(\omega_1^{ij}\right)^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} = \sum_{i=0}^{5} \overset{(i)}{\omega}{}_{1\ (3,\mathrm{S_1S_2})}^{ij}, \tag{4.26}$$

$$\left(\omega_1^{ij}\right)^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} = \sum_{i=0}^{3} \overset{(i)}{\omega}{}_{1\ (3,\mathrm{S_1^2})}^{ij}, \tag{4.27}$$

$$\left(\omega_1^{ij}\right)^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}} = \sum_{i=0}^{3} \overset{(i)}{\omega}{}_{1\ (3,\mathrm{E})}^{ij}\ . \tag{4.28}$$

These redefinitions for the position and spin for the present sectors are extremely lengthy, and thus we provide them in separate tex and machine-readable formats in the supplementary material to this publication.

4.2 Reduced actions

The action that we obtain after the reduction procedure detailed above is standard in the velocity, but with the additional spin variable in the potential, that contains the following 4 sectors:

$$\hat{V}^{\mathrm{N^3LO}}_{\mathrm{S^2}} = \hat{V}^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} + \hat{V}^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} + C_{1\mathrm{ES}^2}\hat{V}^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}} + C_{1\mathrm{E}^2\mathrm{S}^2}V^{\mathrm{N^3LO}}_{C_{1\mathrm{E}^2\mathrm{S}^2}} + (1 \leftrightarrow 2), \tag{4.29}$$

where we provide the explicit expressions of all these sectors in appendix A below. This action is also provided in machine-readable format in the supplementary material to this publication.

Similar to the spin-orbit sector, we find the transcendental Riemann zeta value $\zeta(2) \propto \pi^2$ at orders G^3 and G^4. Similar to the treatment in the spin-orbit sector [23], we obtained

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an action without poles and logarithms, thanks to the addition of TTDs we made at the level of unreduced actions. Let us reiterate that this removal of the poles at the level of the action is not imperative, and one could alternatively carry such poles through to the computation of observables, where these unphysical features would vanish anyway. The equations of motion (EOMs) for both the position and spin could be obtained directly via variations of the unreduced actions, or more readily from the reduced ones that we provided here, as discussed for both cases in our [26], thanks to the generalized canonical gauge that we formulated therein.

5 Hamiltonians and observables

In our formulation [26] the full general Hamiltonian is directly obtained from the reduced action through a straightforward Legendre transform with respect to the position variables. In this Legendre transform all the reduced actions in the sectors that are noted in table 2 must be included. Our general Hamiltonian for the present sectors, that contains 4 parts, can then be written as:

$$H^{\mathrm{N^3LO}}_{\mathrm{S^2}} = H^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} + H^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} + C_{1\mathrm{ES}^2} H^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}} + C_{1\mathrm{E}^2\mathrm{S}^2} H^{\mathrm{N^3LO}}_{C_{1\mathrm{E}^2\mathrm{S}^2}} + (1 \leftrightarrow 2), \tag{5.1}$$

where we provide the explicit expressions of all these sectors in appendix B below. These Hamiltonians are also provided in machine-readable format in the supplementary material to this publication. Our general Hamiltonians here have been fully verified in [59], via the completion of the Poincaré algebra, which constitutes the most stringent check of validity.

5.1 Specialized Hamiltonians

In this section we derive specialized Hamiltonians by gradually constraining the kinematics, and using simplified notation to obtain distilled expressions, see also e.g. [23, 47]. First we remind various conventional mass quantities of the binary: total mass $m \equiv m_1 + m_2$, mass ratio $q \equiv m_1/m_2$, reduced mass $\mu \equiv m_1 m_2/m$, and symmetric mass-ratio $\nu \equiv m_1 m_2/m^2 = \mu/m = q/(1+q)^2$. All variables are then rescaled to be dimensionless, with Gm and μ for the length and mass units, respectively. The dimensionless variables are then all denoted by a tilde.

The first simplification is to move to the center-of-mass (COM) frame where $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$. With this we define the orbital angular momentum as $\vec{L} \equiv r\vec{n} \times \vec{p}$. The Hamiltonian of the present sectors in the COM frame is then written as:

$$\tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S^2}} = \tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} + \tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} + C_{1\mathrm{ES}^2} \tilde{H}^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}} + C_{1\mathrm{E}^2\mathrm{S}^2} \tilde{H}^{\mathrm{N^3LO}}_{C_{1\mathrm{E}^2\mathrm{S}^2}} + (1 \leftrightarrow 2), \tag{5.2}$$

where we provide the explicit expressions of all these sectors in appendix C below. As can be easily seen, the COM specification is actually a major simplification to the general Hamiltonians.

The second common simplification is of the spins being aligned with the orbital angular momentum, that is for both spins we have $\vec{S}_a \cdot \vec{n} = \vec{S}_a \cdot \vec{p} = 0$. In the spin-orbit sector, where the coupling of the single spin and angular momentum is uniquely simple, this constraint does not affect the COM Hamiltonian. However, in all other sectors of higher spin — this is a dramatic simplification, which yields a significant loss of physical information on the system. Specifically in our present sectors the additional aligned-spins constraints yield:

$$\begin{aligned}
\tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} = & \frac{\nu\tilde{S}_1\tilde{S}_2}{2\tilde{r}^6}\Bigg[\left(\left(\frac{101641}{600}-\frac{81\pi^2}{8}\right)\nu+\frac{4537}{100}+\left(-\frac{209\nu^2}{8}+\left(\frac{141\pi^2}{64}-\frac{256207}{1200}\right)\nu\right.\right.\\
&\left.-\frac{6337}{100}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{483\nu^2}{16}+\frac{339\nu}{16}-\frac{57}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(-\frac{25\nu^3}{32}-\frac{39\nu^2}{16}+\frac{69\nu}{16}-\frac{17}{16}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{109\nu^2}{8}+\left(\frac{306553}{1200}-\frac{141\pi^2}{16}\right)\nu+\frac{1112}{25}+\left(\frac{215\nu^2}{32}-\frac{209\nu}{32}-\frac{27}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}\right.\\
&\left.+\left(-\frac{237\nu^3}{64}-\frac{81\nu^2}{8}+\frac{369\nu}{32}-\frac{15}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{575\nu^2}{32}-\frac{231\nu}{8}+\frac{3}{8}+\left(-\frac{111\nu^3}{16}-\frac{237\nu^2}{16}-3\nu+\frac{21}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^6\tilde{r}^3\left(-\frac{187\nu^3}{64}+\frac{153\nu^2}{8}-\frac{327\nu}{32}+\frac{19}{16}\right)\Bigg],
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} = & \frac{\nu^2\tilde{S}_1^2}{\tilde{r}^6}\Bigg[\left(\frac{45\pi^2}{64}-\frac{6679}{3920}\right)\nu+\frac{3987}{2450}+\left(\left(-\frac{96683}{3920}-\frac{45\pi^2}{64}\right)\nu\right.\\
&\left.-\frac{840921}{19600}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{19\nu^2}{8}+\frac{191\nu}{32}-\frac{75}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(-\frac{119\nu^2}{64}+\frac{91\nu}{32}-\frac{45}{64}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\left(\left(\frac{45\pi^2}{16}-\frac{143221}{7840}\right)\nu-\frac{54379}{4900}+\left(\frac{283\nu^2}{32}+\frac{2075\nu}{16}-\frac{11}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\right.\\
&\left.+\left(-\frac{897\nu^2}{128}+\frac{195\nu}{32}-\frac{99}{128}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{177\nu^2}{32}+\frac{43\nu}{32}+\frac{13}{2}+\left(-\frac{165\nu^2}{16}+\frac{57\nu}{32}+\frac{9}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^6\tilde{r}^3\left(-\frac{31\nu^2}{128}-\frac{47\nu}{32}+\frac{81}{128}\right)\Bigg]\\
&+\frac{\nu\tilde{S}_1^2}{q\tilde{r}^6}\Bigg[\left(\frac{45\pi^2}{64}-\frac{6679}{3920}\right)\nu^2-\frac{11\nu}{4}+9\\
&+\left(\left(-\frac{37019}{1960}-\frac{45\pi^2}{64}\right)\nu^2+\left(\frac{63\pi^2}{2048}-36\right)\nu+\frac{441}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\\
&+\left(\frac{19\nu^3}{8}-\frac{13\nu^2}{4}-\frac{1147\nu}{32}+\frac{31}{4}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(-\frac{13\nu^3}{8}+\frac{127\nu^2}{16}-\frac{19\nu}{8}+\frac{45}{64}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\left(\left(\frac{45\pi^2}{16}-\frac{132651}{7840}\right)\nu^2+\left(-\frac{1835}{32}-\frac{63\pi^2}{512}\right)\nu-1\right.
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{283\nu^3}{32}+\frac{323\nu^2}{4}-\frac{1177\nu}{16}+\frac{3}{4}\right)\frac{\tilde{L}^2}{\tilde{r}} \\
&+\left(-\frac{189\nu^3}{32}+\frac{537\nu^2}{32}-\frac{849\nu}{64}+\frac{99}{128}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\Bigg) \\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{177\nu^3}{32}+\frac{371\nu^2}{8}+\frac{177\nu}{32}-\frac{11}{2}+\left(-\frac{267\nu^3}{32}-\frac{3\nu^2}{2}-\frac{261\nu}{32}-\frac{9}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right) \\
&+\tilde{p}_r^6\tilde{r}^3\left(-\frac{25\nu^3}{32}-\frac{137\nu^2}{64}+\frac{175\nu}{64}-\frac{81}{128}\right)\Bigg], \qquad (5.4)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}} = &\,\frac{\nu^2\tilde{S}_1^2}{\tilde{r}^6}\Bigg[\frac{177\nu}{7}-\frac{14894}{1225}+\left(\frac{74413}{2450}-\frac{561\nu}{56}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{21\nu^2}{16}-\frac{223\nu}{16}-\frac{79}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2} \\
&+\left(\frac{15\nu^2}{16}+\frac{\nu}{4}-\frac{9}{16}\right)\frac{\tilde{L}^6}{\tilde{r}^3} \\
&+\tilde{p}_r^2\tilde{r}\left(\frac{1445\nu}{14}-\frac{41779}{4900}+\left(-\frac{37\nu^2}{8}+\frac{57\nu}{2}+\frac{65}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}+\frac{27\nu^2}{8}\frac{\tilde{L}^4}{\tilde{r}^2}\right) \\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{77\nu^2}{16}-\frac{269\nu}{16}+\frac{243}{16}+\left(3\nu^2-\frac{3\nu}{4}+\frac{27}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right) \\
&+\tilde{p}_r^6\tilde{r}^3\left(-6\nu^2-\frac{\nu}{2}+\frac{9}{8}\right)\Bigg] \\
&+\frac{\nu\tilde{S}_1^2}{q\tilde{r}^6}\Bigg[\frac{177\nu^2}{7}+\left(\frac{1861}{24}-\frac{21\pi^2}{4}\right)\nu+\frac{73}{4}+\Bigg(-\frac{1237\nu^2}{112} \\
&+\left(\frac{2703\pi^2}{2048}-\frac{979}{16}\right)\nu-\frac{311}{8}\Bigg)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{21\nu^3}{16}-\frac{35\nu^2}{4}+\frac{57\nu}{16}+\frac{13}{4}\right)\frac{\tilde{L}^4}{\tilde{r}^2} \\
&+\left(\frac{25\nu^3}{32}-\frac{77\nu^2}{32}+\frac{3\nu}{16}+\frac{11}{32}\right)\frac{\tilde{L}^6}{\tilde{r}^3} \\
&+\tilde{p}_r^2\tilde{r}\Bigg(\frac{3771\nu^2}{28}+\left(\frac{861}{8}-\frac{2703\pi^2}{512}\right)\nu+\frac{123}{8} \\
&+\left(-\frac{37\nu^3}{8}+114\nu^2+\frac{1557\nu}{16}-\frac{39}{2}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{21\nu^3}{8}-\frac{279\nu^2}{32}+\frac{63\nu}{16}-\frac{21}{32}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\Bigg) \\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{77\nu^3}{16}-\frac{193\nu^2}{4}+\frac{173\nu}{8}-\frac{67}{4}+\left(\frac{3\nu^3}{2}-\frac{63\nu^2}{8}+\frac{117\nu}{16}-\frac{75}{32}\right)\frac{\tilde{L}^2}{\tilde{r}}\right) \\
&+\tilde{p}_r^6\tilde{r}^3\left(-8\nu^3+5\nu^2+\frac{57\nu}{16}-\frac{43}{32}\right)\Bigg], \qquad (5.5)
\end{aligned}
$$

$$
\tilde{H}^{\mathrm{N^3LO}}_{C_{1\mathrm{E}^2\mathrm{S}^2}} = -\frac{\nu^3\tilde{S}_1^2}{2\tilde{r}^6}\left(1+\frac{1}{q}\right). \qquad (5.6)
$$

We note that in [21] an aligned-spins EOB Hamiltonian was presented for the simple bilinear in spin, spin$_1$-spin$_2$ sector, via some ansatz, assumptions from the EOB approach, and available results from self-force theory, see e.g. recent review in [67]. Both that EOB

Hamiltonian, and our Hamiltonian in eq. (5.3), are specified to the COM frame and subject to the aligned-spins constraints. Yet, the simplified EOB Hamiltonian in [21] is based on some EOB ansatz, and is specified to the so-called "quasi-isotropic" gauge, where dependence in factors of L^2 is hidden.

An additional common assumption in the inspiral phase, where the orbit rapidly gets circularized, is that of circular orbits, namely of satisfying the necessary condition: $p_r \equiv \vec{p}\cdot\vec{n} = 0 \Rightarrow p^2 = p_r^2 + L^2/r^2 \to L^2/r^2$. After applying this circular-orbit condition our Hamiltonians become:

$$\begin{aligned}\tilde{H}^{\rm N^3LO}_{\rm S_1S_2} &= \frac{\nu\tilde{S}_1\tilde{S}_2}{2\tilde{r}^6}\Bigg[\left(\frac{101641}{600}-\frac{81\pi^2}{8}\right)\nu+\frac{4537}{100}\\ &+\left(-\frac{209\nu^2}{8}+\left(\frac{141\pi^2}{64}-\frac{256207}{1200}\right)\nu-\frac{6337}{100}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{483\nu^2}{16}+\frac{339\nu}{16}-\frac{57}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\ &+\left(-\frac{25\nu^3}{32}-\frac{39\nu^2}{16}+\frac{69\nu}{16}-\frac{17}{16}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\Bigg], \end{aligned}\tag{5.7}$$

$$\begin{aligned}\tilde{H}^{\rm N^3LO}_{\rm S_1^2} &= \frac{\nu^2\tilde{S}_1^2}{\tilde{r}^6}\Bigg[\left(\frac{45\pi^2}{64}-\frac{6679}{3920}\right)\nu+\frac{3987}{2450}+\left(\left(-\frac{96683}{3920}-\frac{45\pi^2}{64}\right)\nu-\frac{840921}{19600}\right)\frac{\tilde{L}^2}{\tilde{r}}\\ &+\left(\frac{19\nu^2}{8}+\frac{191\nu}{32}-\frac{75}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(-\frac{119\nu^2}{64}+\frac{91\nu}{32}-\frac{45}{64}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\ &+\frac{\nu\tilde{S}_1^2}{q\tilde{r}^6}\Bigg[\left(\frac{45\pi^2}{64}-\frac{6679}{3920}\right)\nu^2-\frac{11\nu}{4}+9\\ &+\left(\left(-\frac{37019}{1960}-\frac{45\pi^2}{64}\right)\nu^2+\left(\frac{63\pi^2}{2048}-36\right)\nu+\frac{441}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\\ &+\left(\frac{19\nu^3}{8}-\frac{13\nu^2}{4}-\frac{1147\nu}{32}+\frac{31}{4}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(-\frac{13\nu^3}{8}+\frac{127\nu^2}{16}-\frac{19\nu}{8}+\frac{45}{64}\right)\frac{\tilde{L}^6}{\tilde{r}^3}, \end{aligned}\tag{5.8}$$

$$\begin{aligned}\tilde{H}^{\rm N^3LO}_{C_{\rm 1ES^2}} &= \frac{\nu^2\tilde{S}_1^2}{\tilde{r}^6}\Bigg[\frac{177\nu}{7}-\frac{14894}{1225}+\left(\frac{74413}{2450}-\frac{561\nu}{56}\right)\frac{\tilde{L}^2}{\tilde{r}}\\ &+\left(-\frac{21\nu^2}{16}-\frac{223\nu}{16}-\frac{79}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(\frac{15\nu^2}{16}+\frac{\nu}{4}-\frac{9}{16}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\ &+\frac{\nu\tilde{S}_1^2}{q\tilde{r}^6}\Bigg[\frac{177\nu^2}{7}+\left(\frac{1861}{24}-\frac{21\pi^2}{4}\right)\nu+\frac{73}{4}\\ &+\left(-\frac{1237\nu^2}{112}+\left(\frac{2703\pi^2}{2048}-\frac{979}{16}\right)\nu-\frac{311}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\\ &+\left(-\frac{21\nu^3}{16}-\frac{35\nu^2}{4}+\frac{57\nu}{16}+\frac{13}{4}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(\frac{25\nu^3}{32}-\frac{77\nu^2}{32}+\frac{3\nu}{16}+\frac{11}{32}\right)\frac{\tilde{L}^6}{\tilde{r}^3}, \end{aligned}\tag{5.9}$$

$$\tilde{H}^{\rm N^3LO}_{C_{\rm 1E^2S^2}} = -\frac{\nu^3\tilde{S}_1^2}{2\tilde{r}^6}\left(1+\frac{1}{q}\right). \tag{5.10}$$

5.2 GW observables

Despite the various uses of general Lagrangians and Hamiltonians, both formally and phenomenologically, it is essential to obtain physical observables that can be tested through real-world measurements of GW experiments, such as LIGO, Virgo or KAGRA. This is especially critical in high-loop and higher-spin sectors such as those we currently tackled, where considering gauge-invariant quantities is crucial in order to remove possible ambiguities, that multiply when going to higher orders. Yet, it is important to bear in mind that the determination of such observables is always possible only under some specific kinematic constraints, whereas general Hamiltonians provide the full generic information on the system, despite their gauge-dependent "overload". Thus applying kinematic constraints, such as those presented in the previous section, enable to define from the related simplified Hamiltonians, the binding energy $e \equiv \tilde{H}$ in terms of gauge-invariant quantities. Such gauge-invariant relations are regularly used as critical tools in the theoretical construction of gravitational waveforms for the LIGO, Virgo, and KAGRA experiments.

Specifically, in order to eliminate the coordinate dependence from the above Hamiltonians, we can use the additional condition for circular orbits $\dot{p}_r = -\partial\tilde{H}(\tilde{r},\tilde{L})/\partial\tilde{r} = 0$, to obtain $\tilde{r}(\tilde{L})$. Using this condition one can get the binding energy as a function of the gauge-invariant angular momentum, $e(\tilde{L})$. The binding energy can also be expressed in terms of the gauge-invariant PN parameter $x \equiv \tilde{\omega}^{2/3}$, which stands for the orbital frequency that is directly measured through the frequency of emitted GWs, and is inversely related to the orbital separation. From Hamilton's equation for the orbital phase, $d\phi/d\tilde{t} \equiv \tilde{\omega} = \partial\tilde{H}(\tilde{r},\tilde{L})/\partial\tilde{L} = 0$, one can then express the angular momentum as a function of x, which is given in our sectors by:

$$\begin{aligned}\frac{1}{\tilde{L}^2} \supset x^6\nu\Bigg[&\frac{1}{2}\left(-\frac{403\nu^3}{96}+\frac{49\nu^2}{2}+\left(\frac{41\pi^2}{4}-\frac{14489}{48}\right)\nu+\frac{81}{4}\right)\tilde{S}_1\tilde{S}_2\\&+\nu\left(-4\nu C_{1(\mathrm{E^2S^2})}+\left(-\frac{14\nu^2}{3}-\frac{17\nu}{2}+\frac{13}{2}\right)C_{1(\mathrm{ES^2})}-\frac{3607\nu^2}{192}-\frac{7099\nu}{80}-\frac{2279}{64}\right)\tilde{S}_1{}^2\\&+q^{-1}\left(-4\nu^2C_{1(\mathrm{E^2S^2})}+\left(-\frac{14\nu^3}{3}-\frac{253\nu^2}{14}+\left(\frac{1375\pi^2}{256}-\frac{509}{24}\right)\nu\right)C_{1(\mathrm{ES^2})}\right.\\&\left.-\frac{267\nu^3}{16}-\frac{65251\nu^2}{1120}+\left(\frac{63\pi^2}{256}-\frac{2865}{32}\right)\nu+\frac{1863}{64}\right)\tilde{S}_1{}^2\Bigg]+[1\leftrightarrow 2].\end{aligned}\tag{5.11}$$

From the previous relations, one can also obtain the binding energy as a function of the frequency, $e(x)$. The binding energy of the present sectors as a function of the frequency x, and as a function of the angular momentum $\tilde{L}$, were already presented in our [45]. In particular we find binding-energy relations due to the new effect discussed in sections 2.1 and 3 above:

$$(e)^{\mathrm{N^3LO}}_{\mathrm{C_{1(E^2S^2)}S_1^2}}(\tilde{L}) = -\frac{1}{2}C_{1(\mathrm{E^2S^2})}\tilde{S}_1^2\frac{\nu}{\tilde{L}^{12}}\left[\nu^2\left(1+q^{-1}\right)\right], \tag{5.12}$$

$$(e)^{\mathrm{N^3LO}}_{\mathrm{C_{1(E^2S^2)}S_1^2}}(x) = \frac{3}{2}C_{1(\mathrm{E^2S^2})}\tilde{S}_1^2x^6\nu^2\left[\nu(1+q^{-1})\right]. \tag{5.13}$$

The effect, quadratic in the curvature and in the spins, and at the 5PN order, thus represents a real new physical effect which is *unique to spinning objects*. For reference, we find that the binding energy associated with the point-mass tidal effect in figure 1(a) equals:

$$(e)^{5\mathrm{PN}}_{\mathrm{C}_{1(\mathrm{E}^2)}}(x) = 9C_{1(\mathrm{E}^2)}x^6\nu^2\left[-\nu + q(1-\nu)\right]. \tag{5.14}$$

The new binding energy with spins in eq. (5.13) is always positive, and thus it reduces the binding of the compact binary, similar to the point-mass tidal effect in eq. (5.14). Most importantly, we found here a new independent Wilson coefficient that goes beyond what was presented in [14, 26], which is *unique to spinning objects*, and should provide new information on the UV physics of gravity and QCD. Such a coefficient can be significantly large for NSs, and thus tidal effects are important for GW astronomy [44].

5.3 Scattering observables

First, let us clarify on the weak-field, or so-called post-Minkowskian (PM) approximation, which treats the scattering problem in a perturbative expansion in G. It can only be useful via its gauge-dependent COM Hamiltonians, or even extrapolated gauge-invariant binding-energy relations, if it can provide the N^nLO PM results, namely of a similar (or higher) loop order, n, to the target PN results of real binary inspirals. In particular, no PM results for scattering, equivalent to our present ones at the N^3LO, exist to date, and therefore no comparison whatsoever (even partial) is possible with the above critical quantities (gauge-invariant or not) for GW science.

Moreover, a comparison of scattering results in the overlap with the PN approximation, is only possible provided that some limited link exists between some kinematic configuration of the scattering problem to binary inspirals, which are the events targeted in GW measurements. Such a link of the scattering angle, which is already restricted only to aligned spins, to PN inspirals, becomes challenging as of the N^2LO, once radiation-reaction effects come in, and infeasible beyond the N^3LO, where physical logarithms show up systematically. On the other hand, a unique novel approach was recently devised, which incorporates scattering-amplitudes methods directly in the binary-inspiral problem [68]. This approach fully captures radiation-reaction effects at any loop order, and is thus free of all the aforementioned limitations and obstacles of all other scattering-amplitudes approaches to the classical gravitational scattering problem.

With all that in mind, we can still assume in the present sectors that our binding energy which the Hamiltonian stands for, can be extended to a kinetic energy of scattering, and compute the associated scattering angle in the restricted aligned-spins configuration. As noted, such a link is possible only as long as the Hamiltonians do not contain logarithms, which can still be achieved — as we have shown in the previous sections — up to the N^3LO of PN binary inspirals. We then start our computation with our PN aligned-spin Hamiltonians in eqs. (5.3), (5.4), (5.5), (5.6), which depend on r, p_r, L, and $S_{1,2}$. Note that we compute the scattering angles with our Hamiltonians not being specified to the "quasi-isotropic" gauge, by using the integration considerations in [69]. Thus derivations of scattering angles do not require Hamiltonians to be in the "quasi-isotropic" gauge, as was specified in e.g. [21].

Our computation first yields the scattering angle θ as a function of E, L, $S_{1,2}$. Since we start with a PN Hamiltonian, we arrive at a PN expanded scattering angle θ, and unlike the Hamiltonian, each PN order contains terms at all orders in G. To switch to PM results in the overlapping region of the PN and PM approximations, one then needs to expand the scattering angle to a desired order in G, and switch to some standard scattering variables. Importantly also, due to different natural choices of gauge of the rotational DOFs, it is necessary to transform the "canonical" L in the Hamiltonian to the "covariant" one $L_{\rm cov}$, corresponding to the respective canonical and covariant gauges:

$$L = L_{\rm cov} + \Delta L, \quad \Delta L = \left(\sqrt{m_1^2 + p_\infty^2} - m_1\right)\frac{S_1}{m_1} + (1 \leftrightarrow 2), \tag{5.15}$$

with

$$p_\infty = \frac{m_1 m_2}{E}\sqrt{\gamma^2 - 1}, \quad E = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma}, \quad \gamma = \frac{1}{\sqrt{1 - v_\infty^2}}. \tag{5.16}$$

The relation between $L_{\rm cov}$ and the impact parameter b is $L_{\rm cov} = p_\infty b$. The above relations enable us to transform the scattering angle θ from a function of $\{E, L\}$ to a function of $\{v_\infty, b\}$.

With the following notations:

$$\tilde{b} = \frac{v_\infty^2}{Gm} b, \qquad \tilde{v} = \frac{v_\infty}{c}, \qquad \tilde{a}_i = \frac{S_i}{b m_i c}, \qquad \Gamma = \frac{E}{mc^2} = \sqrt{1 + 2\nu(\gamma - 1)}, \tag{5.17}$$

our consequent scattering angles in the sectors with spin are given by:

$$\theta = \left(\theta_{\rm SO} + \theta_{\rm S_1S_2} + \theta_{\rm S_1^2} + C_{\rm 1ES^2}\theta_{C_{\rm 1ES^2}} + C_{\rm 1E^2S^2}\theta_{C_{\rm 1E^2S^2}}\right) + (1 \leftrightarrow 2), \tag{5.18}$$

with the following pieces:

$$\begin{aligned}
\frac{\theta_{\rm SO}}{\Gamma} = \tilde{v}\tilde{a}_1 \Bigg[& -\frac{4}{\tilde{b}} + \frac{\pi}{\tilde{b}^2}\left(-4 + \nu + \left(\frac{3\nu}{2} - 6\right)\tilde{v}^2 + \frac{\nu}{q}\left(1 + \frac{3}{2}\tilde{v}^2\right)\right) \\
& + \frac{1}{\tilde{b}^3}\Bigg(-12 + 4\nu + (50\nu - 120)\,\tilde{v}^2 + \left(\frac{117\nu}{2} - 60\right)\tilde{v}^4 + \frac{177\nu}{4}\tilde{v}^6 \\
& + \frac{\nu}{q}\left(4 + 40\tilde{v}^2 + 20\tilde{v}^4\right)\Bigg) + \frac{\pi}{\tilde{b}^4}\Bigg(\left(-\frac{3}{2}\nu^2 + 42\nu - 84\right)\tilde{v}^2 \\
& + \left(-\frac{45}{4}\nu^2 + \frac{363\nu}{2} - 210\right)\tilde{v}^4 + \left(\left(-\frac{257}{48} - \frac{251\pi^2}{128}\right)\nu^2 + \left(\frac{3233}{12} - \frac{241\pi^3}{128}\right)\nu \right. \\
& \left. - \frac{105}{2}\right)\tilde{v}^6 + \frac{\nu}{q}\Bigg(\left(\frac{63}{2} - \frac{3\nu}{2}\right)\tilde{v}^2 + \left(\frac{315}{4} - \frac{45\nu}{4}\right)\tilde{v}^4 \\
& + \left(\left(-\frac{257}{48} - \frac{251\pi^2}{128}\right)\nu + \frac{315}{16}\right)\tilde{v}^6\Bigg)\Bigg)\Bigg],
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
\frac{\theta_{\rm S_1S_2}}{\Gamma} = \tilde{a}_1\tilde{a}_2 \Bigg[& \frac{1}{\tilde{b}}\left(2 + 2\tilde{v}^2\right) + \frac{\pi}{\tilde{b}^2}\left(\frac{3}{2} + \frac{45}{4}\tilde{v}^2 + \frac{9}{4}\tilde{v}^4\right) + \frac{1}{\tilde{b}^3}\Bigg(4 + (4\nu + 140)\,\tilde{v}^2 \\
& + (220 - 33\nu)\,\tilde{v}^4 + \left(20 - \frac{1093\nu}{10}\right)\tilde{v}^6\Bigg) + \frac{\pi}{\tilde{b}^4}\Bigg(\left(\frac{15\nu}{4} + \frac{315}{4}\right)\tilde{v}^2 \\
& + \left(\frac{3675}{8} - \frac{495\nu}{8}\right)\tilde{v}^4 + \left(\left(\frac{1845\pi^2}{512} - \frac{7995}{16}\right)\nu + \frac{9975}{32}\right)\tilde{v}^6\Bigg)\Bigg],
\end{aligned} \tag{5.20}$$

$$\begin{aligned}\frac{\theta_{\mathrm{S}_1^2}}{\Gamma} &= \tilde{a}_1^2\left[\frac{\pi}{\tilde{b}^2}\left(\left(6-\frac{15\nu}{4}\right)\tilde{v}^2+\left(\frac{15\nu}{32}+\frac{3}{32}\right)\tilde{v}^4+\frac{\nu}{q}\left(-\frac{15}{4}\tilde{v}^2+\frac{15}{32}\tilde{v}^4\right)\right)\right.\\
&+\frac{1}{\tilde{b}^3}\left((96-72\nu)\tilde{v}^2+\left(\frac{1136}{7}-\frac{956\nu}{7}\right)\tilde{v}^4+\left(\frac{16}{7}-\frac{993\nu}{35}\right)\tilde{v}^6\right.\\
&\left.+\frac{\nu}{q}\left(-64\tilde{v}^2-\frac{688}{7}\tilde{v}^4+\frac{144}{35}\tilde{v}^6\right)\right)+\frac{\pi}{\tilde{b}^4}\left(\left(60-\frac{195\nu}{4}\right)\tilde{v}^2\right.\\
&+\left(\frac{165\nu^2}{8}-\frac{45555\nu}{112}+\frac{47295}{112}\right)\tilde{v}^4+\left(\frac{34521\nu^2}{448}+\left(-\frac{105303}{224}-\frac{945\pi^2}{16384}\right)\nu\right.\\
&\left.\left.\left.+\frac{8655}{32}\right)\tilde{v}^6+\frac{\nu}{q}\left(-\frac{165}{4}\tilde{v}^2+\left(\frac{165\nu}{8}-\frac{31545}{112}\right)\tilde{v}^4+\left(\frac{34521\nu}{448}-165\right)\tilde{v}^6\right)\right)\right],
\end{aligned}\tag{5.21}$$

$$\begin{aligned}\frac{\theta_{C_{1\mathrm{ES}^2}}}{\Gamma} &= \tilde{a}_1^2\left[\frac{1}{\tilde{b}}\left(2+2\tilde{v}^2\right)+\frac{\pi}{\tilde{b}^2}\left(\frac{3}{2}+\left(\frac{15}{2}-\frac{3\nu}{2}\right)\tilde{v}^2+\left(\frac{87}{32}-\frac{57\nu}{32}\right)\tilde{v}^4\right.\right.\\
&\left.+\frac{\nu}{q}\left(-\frac{3}{2}\tilde{v}^2-\frac{57}{32}\tilde{v}^4\right)\right)+\frac{1}{\tilde{b}^3}\left(4+(84-20\nu)\tilde{v}^2+\left(\frac{964}{7}-\frac{563\nu}{7}\right)\tilde{v}^4\right.\\
&\left.+\left(\frac{180}{7}-\frac{6421\nu}{70}\right)\tilde{v}^6+\frac{\nu}{q}\left(-16\tilde{v}^2-\frac{432}{7}\tilde{v}^4-\frac{704}{35}\tilde{v}^6\right)\right)+\frac{\pi}{\tilde{b}^4}\left(\left(45-\frac{45\nu}{4}\right)\tilde{v}^2\right.\\
&+\left(\frac{15\nu^2}{4}-\frac{14925\nu}{112}+\frac{26205}{112}\right)\tilde{v}^4+\left(\frac{5925\nu^2}{448}+\left(\frac{58095\pi^2}{16384}-\frac{89955}{224}\right)\nu\right.\\
&\left.\left.\left.+\frac{6045}{32}\right)\tilde{v}^6+\frac{\nu}{q}\left(-\frac{15}{2}\tilde{v}^2+\left(\frac{15\nu}{4}-\frac{9405}{112}\right)\tilde{v}^4+\left(\frac{5925\nu}{448}-\frac{3495}{32}\right)\tilde{v}^6\right)\right)\right],
\end{aligned}\tag{5.22}$$

$$\frac{\theta_{C_{1\mathrm{E}^2\mathrm{S}^2}}}{\Gamma} = -\tilde{a}_1^2\frac{\pi}{\tilde{b}^4}\frac{15}{16}\tilde{v}^6\left(\nu^2-\nu+\frac{\nu^2}{q}\right).\tag{5.23}$$

The above results are organized according to their PM order first, $1/\tilde{b}\sim G$, and the nth subleading contribution then corresponds to $\tilde{v}^{2n}$ terms in even-in-spin sectors, and $\tilde{v}^{2n+1}$ in odd-in-spin sectors. The scattering angles due to the present N^{3}LO quadratic-in-spin sectors are thus:

$$\frac{\theta^{\mathrm{N^3LO}}_{\mathrm{S}_1\mathrm{S}_2}}{\Gamma} = \tilde{a}_1\tilde{a}_2\tilde{v}^6\left[\frac{1}{\tilde{b}^3}\left(20-\frac{1093\nu}{10}\right)+\frac{\pi}{\tilde{b}^4}\left(\left(\frac{1845\pi^2}{512}-\frac{7995}{16}\right)\nu+\frac{9975}{32}\right)\right],\tag{5.24}$$

$$\begin{aligned}\frac{\theta^{\mathrm{N^3LO}}_{\mathrm{S}_1^2}}{\Gamma} &= \tilde{a}_1^2\tilde{v}^6\left[\frac{1}{\tilde{b}^3}\left(\frac{16}{7}-\frac{993\nu}{35}+\frac{144}{35}\frac{\nu}{q}\right)+\frac{\pi}{\tilde{b}^4}\left(\frac{34521\nu^2}{448}+\left(-\frac{105303}{224}-\frac{945\pi^2}{16384}\right)\nu\right.\right.\\
&\left.\left.+\frac{8655}{32}+\left(\frac{34521\nu}{448}-165\right)\frac{\nu}{q}\right)\right],
\end{aligned}\tag{5.25}$$

$$\begin{aligned}\frac{\theta^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}}}{\Gamma} &= \tilde{a}_1^2\tilde{v}^6\left[\frac{1}{\tilde{b}^3}\left(\frac{180}{7}-\frac{6421\nu}{70}+\frac{704}{35}\frac{\nu}{q}\right)+\frac{\pi}{\tilde{b}^4}\left(\frac{5925\nu^2}{448}+\left(\frac{58095\pi^2}{16384}-\frac{89955}{224}\right)\nu\right.\right.\\
&\left.\left.+\frac{6045}{32}+\left(\frac{5925\nu}{448}-\frac{3495}{32}\right)\frac{\nu}{q}\right)\right],
\end{aligned}\tag{5.26}$$

$$\frac{\theta^{\mathrm{N^3LO}}_{C_{1\mathrm{E}^2\mathrm{S}^2}}}{\Gamma} = \frac{\theta_{C_{1\mathrm{E}^2\mathrm{S}^2}}}{\Gamma}.\tag{5.27}$$

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Our results in eqs. (5.19), and (5.20), (5.24), for the spin-orbit, and spin$_1$-spin$_2$ sectors, respectively, agree with [21], where they were obtained via traditional GR methods, and these agreements were already indicated in [23, 45]. Our results in eqs. (5.21), (5.25), and (5.22), (5.26), agree with those obtained at the NLO PM approximation for sectors quadratic in the spin of generic objects in [70] via EFT and scattering-amplitudes methods, and in [71] via EFT methods. Finally, our results in eqs. (5.21), (5.25), and (5.22), (5.26), also agree with those obtained at the N^{2}LO PM approximation for sectors quadratic in the spin of generic objects in [72] via EFT methods.

Such agreements with [70, 72] are not surprising, as the latter approaches have built closely on our theory presented in detail in [26], and on our comprehensive results along the years, see e.g. [46], and thus [70, 72] are clearly dependent on our framework. Moreover, all the approaches noted above still provide only very partial results, that are below the higher orders of either loop or spin of our present new results, and are only in the restricted aligned-spins configuration, which is significantly less informative at higher-spin sectors. In contrast, our results in this work cover all quadratic-in-spin sectors, are at the higher N^{3}LO (or three-loop) level, and are not limited to the aligned-spins simplification, but rather hold for any generic spin orientations, and moreover — in general reference frames.

6 Conclusions

In this paper we presented for the first time the complete N^{3}LO quadratic-in-spin interactions that push the state of the art of conservative dynamics of binary inspirals at the 5PN order. Our results cover all the relevant sectors, and are obtained for generic compact binaries in generic kinematic settings, via the EFT of spinning gravitating objects introduced in [26]. Our independent framework, including the public `EFTofPNG` code [27], enables to study PN theory in any generic sector, which is critical to advance the present high precision frontier in both high-loop and higher-spin orders, such as in the present sectors. Such generic theoretical input that our framework provides, improves the ability to mine unknown physics from real-world GW data. Moreover, the unique treatment of the present sectors, also at the 5PN order, via the EFT framework, provides even more invaluable new information on the UV physics of strong gravity and extreme QCD.

The N^{3}LO quadratic-in-spin corrections consist of 4 independent sectors, one of which is new, first appearing in the present 5PN order, similar to the related tidal effect in the point-mass sector, which also enters first at this order. The 3 other quadratic-in-spin sectors here scale as the single N^{3}LO spin-orbit sector. The overall computational load of the Feynman evaluation is similar to the N^{3}LO spin-orbit sector [23], with an imperative use of advanced multi-loop methods, and a generic overall dimensional treatment due to the appearance of DimReg poles across all loop orders. Yet, the procedure of redefinitions of the position and rotational variables is somewhat more intricate and demanding in higher-spin sectors such as the present ones, compared to the point-mass or spin-orbit sectors, e.g. at the N^{3}LO [23]. Nevertheless, the streamlined automated algorithm for this procedure, which we set up in [23], still performed in an efficient and rapid manner.

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We presented here the reduced Lagrangians of all relevant sectors for the first time, which can be used to directly derive the EOMs for both position and spin [26]. We also presented here the most general Hamiltonians of these sectors for the first time, and their consequent useful simplifications. Our general Hamiltonians here have been fully verified in [59] via the completion of the Poincaré algebra, which constitutes the most stringent check of validity. Finally, in these high-order sectors it is especially crucial to derive unambiguous observables, which correspond to real-world GW measurements, and we presented here the gauge-invariant relation of the angular momentum to the GW frequency. We also showed the binding energy of the new tidal effect with spins in comparison with the tidal effect from the point-mass sector, which also enters at the present 5PN order.

We ended with a derivation of all the extrapolated scattering angles that hold for aligned spins, which correspond to an extension of our Hamiltonians to the scattering problem. Complete agreement was found with the limited available results at lower orders obtained via traditional GR, and via EFT and scattering-amplitudes methods. Such agreements with [70, 72] are not surprising, as the latter approaches have built closely on our theory presented in detail in [26], and on our comprehensive results along the years, see e.g. [46], and thus [70, 72] are clearly dependent on our framework. Moreover, all the approaches noted above still provide only very partial results, that are below the higher orders of either loop or spin of our present new results, and are only in the restricted aligned-spins configuration, which is significantly less informative at higher-spin sectors. In contrast, our results in this work cover all quadratic-in-spin sectors, are at the higher $\mathrm{N}^3\mathrm{LO}$ (or three-loop) level, and are not limited to the aligned-spins simplification, but rather hold for any generic spin orientations, and moreover — in general reference frames.

We found a new independent Wilson coefficient in the present sectors, that is unique to spinning objects, yet seems to be related to the long-studied "Love numbers". Determining the value of such characteristic UV coefficients constitutes one of the most challenging tasks in an EFT framework. Fortunately the EFT framework is also well-suited to carry out such so-called matching of its unknown coefficients, either via analytical studies, or by comparing to numerical or real-world data. Indeed in recent years with the incredible success of GW measurements came a notable thrust in the theoretical efforts to study such "Love numbers", that have been explored for both BHs and NSs, notably extending studies to the case of rotating objects, see e.g. [38–43].

Interestingly, almost all studies to date indicate that such "Love numbers" vanish for BHs in 4-dimensional GR. On the other hand, the Wilson coefficients of tidal effects can be significantly large for NSs, and thus tidal effects are important for GW astronomy [44]. Despite the long history and enormous body of work that has accumulated of studying such "Love numbers", we believe that the intriguing studies of such coefficients are only in their infancy. There is a huge challenge in thoroughly studying them, using various approaches, for either BHs or NSs, and in general candidate-theories of gravity [73]. Such studies are immensely important to unveil the new physics that is encrypted in the continually increasing influx of GW data.

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A Reduced actions

The action that we obtain after the reduction procedure detailed in section 4 is standard in the velocity, with the additional spin variable, and contains the following 4 sectors:

$$\hat{V}^{\mathrm{N^3LO}}_{\mathrm{S^2}} = \hat{V}^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} + \hat{V}^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} + C_{\mathrm{1ES^2}}\hat{V}^{\mathrm{N^3LO}}_{C_{\mathrm{1ES^2}}} + C_{\mathrm{1E^2S^2}}V^{\mathrm{N^3LO}}_{C_{\mathrm{1E^2S^2}}} + (1 \leftrightarrow 2), \tag{A.1}$$

where

$$\begin{aligned}
\hat{V}^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} = \frac{G}{128r^3}\Big[& 3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(85v_1^2\vec{v}_1\cdot\vec{v}_2v_2^2 - 192v_1^2(\vec{v}_1\cdot\vec{v}_2)^2 + 44(\vec{v}_1\cdot\vec{v}_2)^3 + 8v_1^6 \\
& +124\vec{v}_1\cdot\vec{v}_2v_1^4 - 24v_2^2v_1^4 - 48v_1^2v_2^4 + 280\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 - 60\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}v_2^2 \\
& -640v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2 - 120v_1^2v_2^2(\vec{v}_1\cdot\vec{n})^2 - 320v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2 \\
& +480(\vec{v}_1\cdot\vec{n})^2(\vec{v}_1\cdot\vec{v}_2)^2 + 40\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2 + 480v_1^2v_2^2(\vec{v}_2\cdot\vec{n})^2 \\
& -280\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_1^4 + 120(\vec{v}_2\cdot\vec{n})^2v_1^4 + 560\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^4 + 840v_1^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2 \\
& +280\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^3 - 560v_1^2(\vec{v}_2\cdot\vec{n})^4 \\
& -770\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2 - 420(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^3\big) + \vec{S}_1\cdot\vec{S}_2\big(154v_1^2\vec{v}_1\cdot\vec{v}_2v_2^2 \\
& +240v_1^2(\vec{v}_1\cdot\vec{v}_2)^2 - 160(\vec{v}_1\cdot\vec{v}_2)^3 - 104v_1^6 + 48\vec{v}_1\cdot\vec{v}_2v_1^4 - 224v_2^2v_1^4 \\
& +48v_1^2v_2^4 - 216\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2 - 489\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}v_2^2 + 912\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2 \\
& +456v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2 + 1032v_1^2v_2^2(\vec{v}_1\cdot\vec{n})^2 - 480\vec{v}_1\cdot\vec{v}_2v_2^2(\vec{v}_1\cdot\vec{n})^2 \\
& -24v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2 - 768(\vec{v}_1\cdot\vec{n})^2(\vec{v}_1\cdot\vec{v}_2)^2 - 180\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2 \\
& -288(\vec{v}_2\cdot\vec{n})^2(\vec{v}_1\cdot\vec{v}_2)^2 - 864v_1^2v_2^2(\vec{v}_2\cdot\vec{n})^2 + 600\vec{v}_1\cdot\vec{v}_2v_2^2(\vec{v}_2\cdot\vec{n})^2 \\
& +288(\vec{v}_1\cdot\vec{n})^2v_1^4 + 876\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_1^4 - 120(\vec{v}_2\cdot\vec{n})^2v_1^4 + 528(\vec{v}_1\cdot\vec{n})^2v_2^4 \\
& -1320\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^4 - 2880v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3 - 720\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^4 \\
& +240\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^3 - 960v_2^2(\vec{v}_1\cdot\vec{n})^4 + 1080\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^3 \\
& -1680v_1^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2 + 240\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^3 + 1200v_1^2(\vec{v}_2\cdot\vec{n})^4 \\
& +3120\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2 - 1200\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^3 - 480\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^4 \\
& -720v_2^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2 + 2880\vec{v}_1\cdot\vec{n}v_2^2(\vec{v}_2\cdot\vec{n})^3 + 1680\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^5 \\
& +3360(\vec{v}_1\cdot\vec{n})^4(\vec{v}_2\cdot\vec{n})^2 - 3430(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^3 - 1680\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^5\big) \\
& -24\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(7\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_1\cdot\vec{v}_2 - 3\vec{v}_1\cdot\vec{n}v_1^2v_2^2 - 4v_1^2\vec{v}_2\cdot\vec{n}v_2^2 + 6\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2 \\
& -2\vec{v}_1\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2 + \vec{v}_1\cdot\vec{n}v_1^4 - 6\vec{v}_1\cdot\vec{n}v_2^4 - 35v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2 \\
& -20\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2 - 15v_2^2(\vec{v}_1\cdot\vec{n})^3 + 15\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^2 \\
& +15\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^2 + 15v_1^2(\vec{v}_2\cdot\vec{n})^3 - 10\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2
\end{aligned}$$

$$
\begin{aligned}
&+60\vec{v}_1\cdot\vec{n}v_2^2(\vec{v}_2\cdot\vec{n})^2+105(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^2-70(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^3\\
&-70\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^4)+3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1\big(96\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_1\cdot\vec{v}_2-48v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-176\vec{v}_1\cdot\vec{n}v_1^2v_2^2+191v_1^2\vec{v}_2\cdot\vec{n}v_2^2+144\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2-480\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2\\
&+64\vec{v}_1\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-20\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-96\vec{v}_1\cdot\vec{n}v_1^4+4\vec{v}_2\cdot\vec{n}v_1^4-176\vec{v}_1\cdot\vec{n}v_2^4\\
&+440\vec{v}_2\cdot\vec{n}v_2^4+960v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2-160\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+320v_2^2(\vec{v}_1\cdot\vec{n})^3\\
&-240\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^2-200\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^2-240v_1^2(\vec{v}_2\cdot\vec{n})^3\\
&-80\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+640\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^3+240\vec{v}_1\cdot\vec{n}v_2^2(\vec{v}_2\cdot\vec{n})^2\\
&-960v_2^2(\vec{v}_2\cdot\vec{n})^3-560\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^4-1120(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^2\\
&+910(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^3+560(\vec{v}_2\cdot\vec{n})^5\big)+8\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1\big(2v_1^2\vec{v}_1\cdot\vec{v}_2+28v_1^2v_2^2\\
&-8\vec{v}_1\cdot\vec{v}_2v_2^2-6(\vec{v}_1\cdot\vec{v}_2)^2+13v_1^4-6v_2^4-111\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}-72\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-9\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2-63v_2^2(\vec{v}_1\cdot\vec{n})^2+15v_1^2(\vec{v}_2\cdot\vec{n})^2-42\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2\\
&+108v_2^2(\vec{v}_2\cdot\vec{n})^2+285(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2+30\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3-150(\vec{v}_2\cdot\vec{n})^4\big)\\
&+3\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(176\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_1\cdot\vec{v}_2-25\vec{v}_1\cdot\vec{n}v_1^2v_2^2-68\vec{v}_1\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-68\vec{v}_1\cdot\vec{n}v_1^4\\
&+640v_1^2(\vec{v}_1\cdot\vec{n})^3-440v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2-480\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^3\\
&+160\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+40\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^2-560(\vec{v}_1\cdot\vec{n})^5\\
&+630(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^2\big)-2\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\big(96v_1^2\vec{v}_1\cdot\vec{v}_2+33v_1^2v_2^2-96(\vec{v}_1\cdot\vec{v}_2)^2\\
&+32v_1^4-468\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+132\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-264\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&+372v_1^2(\vec{v}_1\cdot\vec{n})^2-288\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2-84v_1^2(\vec{v}_2\cdot\vec{n})^2-144\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2\\
&+300v_2^2(\vec{v}_2\cdot\vec{n})^2-360(\vec{v}_1\cdot\vec{n})^4-120\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3+780(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&+360\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3-240(\vec{v}_2\cdot\vec{n})^4\big)+24\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2\big(12\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_1\cdot\vec{v}_2\\
&+2v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-3\vec{v}_1\cdot\vec{n}v_1^2v_2^2+22\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2-25\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2\\
&-10\vec{v}_1\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2+12\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-7\vec{v}_1\cdot\vec{n}v_1^4\\
&+2\vec{v}_2\cdot\vec{n}v_1^4-10v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2-20\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^3-5\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+10\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^2-30\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+20\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^3\\
&+35(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^2\big)-4\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2\big(50v_1^2\vec{v}_1\cdot\vec{v}_2+7v_1^2v_2^2-22(\vec{v}_1\cdot\vec{v}_2)^2\\
&+12\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}-90\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-42v_1^2(\vec{v}_1\cdot\vec{n})^2-60\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&-24v_2^2(\vec{v}_1\cdot\vec{n})^2-66v_1^2(\vec{v}_2\cdot\vec{n})^2-60\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3+345(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\big)\\
&-8\vec{S}_1\cdot\vec{v}_2\vec{S}_2\cdot\vec{v}_2\big(14v_1^2\vec{v}_1\cdot\vec{v}_2-25v_1^4-3\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+36\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-75\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2+66v_1^2(\vec{v}_1\cdot\vec{n})^2-36\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+66v_2^2(\vec{v}_1\cdot\vec{n})^2-60(\vec{v}_1\cdot\vec{n})^4+45\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3-90(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&+60\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3\big)\Big]\\
&+\frac{G^2m_1}{96r^4}\Big[\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(11154v_1^2\vec{v}_1\cdot\vec{v}_2+1076v_1^2v_2^2-4544(\vec{v}_1\cdot\vec{v}_2)^2\\
&-5859v_1^4-32220\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+26142\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+41211v_1^2(\vec{v}_1\cdot\vec{n})^2\\
&-37152\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2-5367v_1^2(\vec{v}_2\cdot\vec{n})^2-2208(\vec{v}_1\cdot\vec{n})^4-12882\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3\\
&+6738(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\big)-\vec{S}_1\cdot\vec{S}_2\big(4995v_1^2\vec{v}_1\cdot\vec{v}_2+2349v_1^2v_2^2-2702(\vec{v}_1\cdot\vec{v}_2)^2
\end{aligned}
$$

$$
\begin{aligned}
&-2595v_1^4-11223\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+12480\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+3048\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&+24009v_1^2(\vec{v}_1\cdot\vec{n})^2-35685\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+1898v_2^2(\vec{v}_1\cdot\vec{n})^2-11200v_1^2(\vec{v}_2\cdot\vec{n})^2\\
&-198\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-26268(\vec{v}_1\cdot\vec{n})^4+46683\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3\\
&-2712(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-6489\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3)+\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(2211\vec{v}_1\cdot\vec{n}v_1^2\\
&-294v_1^2\vec{v}_2\cdot\vec{n}-4698\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+320\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+250\vec{v}_1\cdot\vec{n}v_2^2\\
&-576\vec{v}_2\cdot\vec{n}v_2^2-37083(\vec{v}_1\cdot\vec{n})^3+44856\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+3477\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2)\\
&+\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1(1062\vec{v}_1\cdot\vec{n}v_1^2-4071v_1^2\vec{v}_2\cdot\vec{n}-8106\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+4718\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+670\vec{v}_1\cdot\vec{n}v_2^2+3624\vec{v}_2\cdot\vec{n}v_2^2-27012(\vec{v}_1\cdot\vec{n})^3+51435\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&-9762\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2-6489(\vec{v}_2\cdot\vec{n})^3)-\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1(2211v_1^2-1176\vec{v}_1\cdot\vec{v}_2\\
&-2387v_2^2+11076\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-24747(\vec{v}_1\cdot\vec{n})^2+9590(\vec{v}_2\cdot\vec{n})^2)\\
&-6\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2(1164\vec{v}_1\cdot\vec{n}v_1^2-928\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-7494(\vec{v}_1\cdot\vec{n})^3\\
&+6999\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2)+\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(3561v_1^2-3086\vec{v}_1\cdot\vec{v}_2+17354\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-31449(\vec{v}_1\cdot\vec{n})^2-198(\vec{v}_2\cdot\vec{n})^2)+6\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2(1528\vec{v}_1\cdot\vec{n}v_1^2-590v_1^2\vec{v}_2\cdot\vec{n}\\
&-1169\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-33\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-243(\vec{v}_1\cdot\vec{n})^3+942\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2)\\
&+2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2(174v_1^2-346\vec{v}_1\cdot\vec{v}_2+562\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-5247(\vec{v}_1\cdot\vec{n})^2)\\
&+6\vec{S}_1\cdot\vec{v}_2\vec{S}_2\cdot\vec{v}_2(118v_1^2+33\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+1145(\vec{v}_1\cdot\vec{n})^2)\Big]\\
&-\frac{G^2m_2}{96r^4}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(59v_1^2\vec{v}_1\cdot\vec{v}_2+226v_1^2v_2^2-12v_1^4+222\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\\
&-885\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2-451v_1^2(\vec{v}_2\cdot\vec{n})^2)-\vec{S}_1\cdot\vec{S}_2(927v_1^2\vec{v}_1\cdot\vec{v}_2+1078v_1^2v_2^2\\
&-48v_1^4-330\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}-6568\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+3474\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&-240v_1^2(\vec{v}_1\cdot\vec{n})^2-2082\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2-5018v_2^2(\vec{v}_1\cdot\vec{n})^2-668v_1^2(\vec{v}_2\cdot\vec{n})^2\\
&-3678\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-2250\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3+17382(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&-5412\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3)+2\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(36\vec{v}_1\cdot\vec{n}v_1^2+66v_1^2\vec{v}_2\cdot\vec{n}+408\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-868\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-1067\vec{v}_1\cdot\vec{n}v_2^2+384\vec{v}_2\cdot\vec{n}v_2^2-720\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&+2199\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2-696(\vec{v}_2\cdot\vec{n})^3)-2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1(120\vec{v}_1\cdot\vec{n}v_1^2+273v_1^2\vec{v}_2\cdot\vec{n}\\
&-3084\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+3784\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+2120\vec{v}_1\cdot\vec{n}v_2^2-1353\vec{v}_2\cdot\vec{n}v_2^2\\
&+1125\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2-8022\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2+2010(\vec{v}_2\cdot\vec{n})^3)-2\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1(24v_1^2\\
&+504\vec{v}_1\cdot\vec{v}_2-539v_2^2+48\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+491(\vec{v}_2\cdot\vec{n})^2)\\
&-6\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2(229\vec{v}_1\cdot\vec{n}v_1^2-885(\vec{v}_1\cdot\vec{n})^3)+3\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(713v_1^2\\
&+912\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-2750(\vec{v}_1\cdot\vec{n})^2-1226(\vec{v}_2\cdot\vec{n})^2)-6\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2(640\vec{v}_1\cdot\vec{n}v_1^2\\
&-746v_1^2\vec{v}_2\cdot\vec{n}-456\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+613\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2)\\
&-12\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2(373\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-320(\vec{v}_1\cdot\vec{n})^2)+6\vec{S}_1\cdot\vec{v}_2\vec{S}_2\cdot\vec{v}_2(613\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-456(\vec{v}_1\cdot\vec{n})^2)\Big]\\
&+\frac{G^3{m_2}^2}{300r^5}\Big[75\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(16v_1^2+31(\vec{v}_1\cdot\vec{n})^2)-\vec{S}_1\cdot\vec{S}_2(1536v_1^2-36485\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&+1395(\vec{v}_1\cdot\vec{n})^2)-5\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(99\vec{v}_1\cdot\vec{n}+3236\vec{v}_2\cdot\vec{n})-5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1(174\vec{v}_1\cdot\vec{n}
\end{aligned}
$$

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$$
\begin{aligned}
&+4061\vec{v}_2\cdot\vec{n})+843\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1\Big]-\frac{G^3{m_1}^2}{600r^5}\Big[50\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(352v_1^2\\
&+451\vec{v}_1\cdot\vec{v}_2-4453\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+4298(\vec{v}_1\cdot\vec{n})^2)-\vec{S}_1\cdot\vec{S}_2(20368v_1^2\\
&+9529\vec{v}_1\cdot\vec{v}_2-117090\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+21760(\vec{v}_1\cdot\vec{n})^2)-20\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(5758\vec{v}_1\cdot\vec{n}\\
&-1303\vec{v}_2\cdot\vec{n})-20\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1(3718\vec{v}_1\cdot\vec{n}-1693\vec{v}_2\cdot\vec{n})+24300\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\\
&-13200\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}+40984\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1-19099\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
&+3626\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2\Big]+\frac{G^3m_1m_2}{192r^5}\Big[\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}((33424-2115\pi^2)v_1^2\\
&-(47208-2115\pi^2)\vec{v}_1\cdot\vec{v}_2+(81252-14805\pi^2)\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-(84526-14805\pi^2)(\vec{v}_1\cdot\vec{n})^2)-\vec{S}_1\cdot\vec{S}_2((13736-423\pi^2)v_1^2\\
&-(24672-423\pi^2)\vec{v}_1\cdot\vec{v}_2+(46992-2115\pi^2)\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-(38338-2115\pi^2)(\vec{v}_1\cdot\vec{n})^2)+2\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1((6316-2115\pi^2)\vec{v}_1\cdot\vec{n}\\
&-(7149-2115\pi^2)\vec{v}_2\cdot\vec{n})-2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_1((5774+2115\pi^2)\vec{v}_1\cdot\vec{n}\\
&-(4299+2115\pi^2)\vec{v}_2\cdot\vec{n})+14238\vec{S}_2\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}+11142\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{v}_2\vec{v}_1\cdot\vec{n}\\
&+2(2596+423\pi^2)\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_1-3(3088+141\pi^2)\vec{S}_2\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
&-141(44+3\pi^2)\vec{S}_1\cdot\vec{v}_1\vec{S}_2\cdot\vec{v}_2\Big]\\
&-\frac{G^4{m_1}^3}{100r^6}\Big[5131\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}-2137\vec{S}_1\cdot\vec{S}_2\Big]\\
&-\frac{G^4{m_1}^2m_2}{1200r^6}\Big[9(86423-4050\pi^2)\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}-(286139-12150\pi^2)\vec{S}_1\cdot\vec{S}_2\Big], \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{V}^{\mathrm{N^3LO}}_{\mathrm{S}_1^2}=&-\frac{Gm_2}{128m_1r^3}\Big[S_1^2(1336v_1^2\vec{v}_1\cdot\vec{v}_2v_2^2-1044v_1^2(\vec{v}_1\cdot\vec{v}_2)^2+624(\vec{v}_1\cdot\vec{v}_2)^3\\
&-1056v_2^2(\vec{v}_1\cdot\vec{v}_2)^2-74v_1^6+488\vec{v}_1\cdot\vec{v}_2v_1^4-352v_2^2v_1^4-380v_1^2v_2^4+456\vec{v}_1\cdot\vec{v}_2v_2^4\\
&+4536\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-2280\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}v_2^2+2112\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2\\
&-876v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+594v_1^2v_2^2(\vec{v}_1\cdot\vec{n})^2-516\vec{v}_1\cdot\vec{v}_2v_2^2(\vec{v}_1\cdot\vec{n})^2\\
&-3996v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+984(\vec{v}_1\cdot\vec{n})^2(\vec{v}_1\cdot\vec{v}_2)^2-3312\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2\\
&+3072(\vec{v}_2\cdot\vec{n})^2(\vec{v}_1\cdot\vec{v}_2)^2+2256v_1^2v_2^2(\vec{v}_2\cdot\vec{n})^2-2544\vec{v}_1\cdot\vec{v}_2v_2^2(\vec{v}_2\cdot\vec{n})^2\\
&+75(\vec{v}_1\cdot\vec{n})^2v_1^4-1224\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_1^4+1116(\vec{v}_2\cdot\vec{n})^2v_1^4-258(\vec{v}_1\cdot\vec{n})^2v_2^4\\
&+168\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^4+1620v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3-1200\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^3\\
&+150v_2^2(\vec{v}_1\cdot\vec{n})^4-900\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^3-3750v_1^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&+4020\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^3-1920v_1^2(\vec{v}_2\cdot\vec{n})^4+2520\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&-3840\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^3+2160\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^4+1680v_2^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&-240\vec{v}_1\cdot\vec{n}v_2^2(\vec{v}_2\cdot\vec{n})^3-1050(\vec{v}_1\cdot\vec{n})^4(\vec{v}_2\cdot\vec{n})^2+2520(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^3\\
&-1680(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^4)+6\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(80\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_1\cdot\vec{v}_2-56v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-70\vec{v}_1\cdot\vec{n}v_1^2v_2^2+8v_1^2\vec{v}_2\cdot\vec{n}v_2^2+48\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2+64\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2\\
&-28\vec{v}_1\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-8\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-43\vec{v}_1\cdot\vec{n}v_1^4+18\vec{v}_2\cdot\vec{n}v_1^4-14\vec{v}_1\cdot\vec{n}v_2^4\\
&-28\vec{v}_2\cdot\vec{n}v_2^4-140v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+60\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2-30v_2^2(\vec{v}_1\cdot\vec{n})^3
\end{aligned}
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\begin{aligned}
&+120\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^2+280\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^2-30v_1^2(\vec{v}_2\cdot\vec{n})^3\\
&-40\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-80\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^3+40\vec{v}_1\cdot\vec{n}v_2^2(\vec{v}_2\cdot\vec{n})^2+40v_2^2(\vec{v}_2\cdot\vec{n})^3\\
&+210(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^2-420(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^3\big)-24\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(18\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_1\cdot\vec{v}_2\\
&-25\vec{v}_1\cdot\vec{n}v_1^2v_2^2+32\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2-15\vec{v}_1\cdot\vec{n}v_1^4-\vec{v}_2\cdot\vec{n}v_1^4\\
&-16\vec{v}_1\cdot\vec{n}v_2^4-5v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+70\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^2-80\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2\\
&+80\vec{v}_1\cdot\vec{n}v_2^2(\vec{v}_2\cdot\vec{n})^2-70\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^4\big)+4\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\big(64v_1^2\vec{v}_1\cdot\vec{v}_2-68v_1^2v_2^2\\
&+264\vec{v}_1\cdot\vec{v}_2v_2^2-156(\vec{v}_1\cdot\vec{v}_2)^2+15v_1^4-114v_2^4-132\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\\
&+840\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-624\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2-138v_1^2(\vec{v}_1\cdot\vec{n})^2-234\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+60v_2^2(\vec{v}_1\cdot\vec{n})^2+207v_1^2(\vec{v}_2\cdot\vec{n})^2-768\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+636v_2^2(\vec{v}_2\cdot\vec{n})^2\\
&+270\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3-570(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2+1080\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3-540(\vec{v}_2\cdot\vec{n})^4\big)\\
&-3\big(204v_1^2\vec{v}_1\cdot\vec{v}_2v_2^2-92v_1^2(\vec{v}_1\cdot\vec{v}_2)^2-128v_2^2(\vec{v}_1\cdot\vec{v}_2)^2-49v_1^6+124\vec{v}_1\cdot\vec{v}_2v_1^4\\
&-74v_2^2v_1^4-114v_1^2v_2^4+128\vec{v}_1\cdot\vec{v}_2v_2^4+120\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-60\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}v_2^2\\
&-10v_1^2v_2^2(\vec{v}_1\cdot\vec{n})^2-560v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+320(\vec{v}_2\cdot\vec{n})^2(\vec{v}_1\cdot\vec{v}_2)^2\\
&+640v_1^2v_2^2(\vec{v}_2\cdot\vec{n})^2-640\vec{v}_1\cdot\vec{v}_2v_2^2(\vec{v}_2\cdot\vec{n})^2-60\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_1^4+210(\vec{v}_2\cdot\vec{n})^2v_1^4\\
&+70v_1^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-560v_1^2(\vec{v}_2\cdot\vec{n})^4+560\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^4\big)(\vec{S}_1\cdot\vec{n})^2\\
&-2\big(274v_1^2\vec{v}_1\cdot\vec{v}_2-176v_1^2v_2^2+532\vec{v}_1\cdot\vec{v}_2v_2^2-404(\vec{v}_1\cdot\vec{v}_2)^2-37v_1^4\\
&-190v_2^4-558\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+1848\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-1116\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&-18v_1^2(\vec{v}_1\cdot\vec{n})^2-480\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+198v_2^2(\vec{v}_1\cdot\vec{n})^2+558v_1^2(\vec{v}_2\cdot\vec{n})^2\\
&-1584\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+1128v_2^2(\vec{v}_2\cdot\vec{n})^2+480\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3\\
&-1350(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2+1920\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3-960(\vec{v}_2\cdot\vec{n})^4\big)(\vec{S}_1\cdot\vec{v}_1)^2\\
&-4\big(5v_1^4-69v_1^2(\vec{v}_1\cdot\vec{n})^2-96v_2^2(\vec{v}_1\cdot\vec{n})^2+240(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&-\frac{G^2m_2}{96r^4}\Big[S_1^2\big(6v_1^2\vec{v}_1\cdot\vec{v}_2-230v_1^2v_2^2-1632\vec{v}_1\cdot\vec{v}_2v_2^2+1775(\vec{v}_1\cdot\vec{v}_2)^2-84v_1^4\\
&+156v_2^4-7524\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+1772\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-432\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&+4728v_1^2(\vec{v}_1\cdot\vec{n})^2-5376\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+2678v_2^2(\vec{v}_1\cdot\vec{n})^2+620v_1^2(\vec{v}_2\cdot\vec{n})^2\\
&+4752\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-36v_2^2(\vec{v}_2\cdot\vec{n})^2-4845(\vec{v}_1\cdot\vec{n})^4+15000\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3\\
&-9732(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-1800\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3\big)+2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(258\vec{v}_1\cdot\vec{n}v_1^2\\
&-888v_1^2\vec{v}_2\cdot\vec{n}+636\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+2306\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-1340\vec{v}_1\cdot\vec{n}v_2^2+84\vec{v}_2\cdot\vec{n}v_2^2\\
&+12345(\vec{v}_1\cdot\vec{n})^3-24318\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+8778\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2+900(\vec{v}_2\cdot\vec{n})^3\big)\\
&-4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(120\vec{v}_1\cdot\vec{n}v_1^2-419v_1^2\vec{v}_2\cdot\vec{n}-952\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+1692\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+114\vec{v}_1\cdot\vec{n}v_2^2-36\vec{v}_2\cdot\vec{n}v_2^2+5190(\vec{v}_1\cdot\vec{n})^3-8016\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&+1026\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)-2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\big(162v_1^2+1127\vec{v}_1\cdot\vec{v}_2-588v_2^2\\
&+4918\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-6858(\vec{v}_1\cdot\vec{n})^2+1800(\vec{v}_2\cdot\vec{n})^2\big)-\big(4812v_1^2\vec{v}_1\cdot\vec{v}_2-1010v_1^2v_2^2\\
&+576\vec{v}_1\cdot\vec{v}_2v_2^2-2848(\vec{v}_1\cdot\vec{v}_2)^2-1668v_1^4+144v_2^4-22788\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\\
&+28848\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-1944\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2+14301v_1^2(\vec{v}_1\cdot\vec{n})^2\\
&-15768\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+1596v_2^2(\vec{v}_1\cdot\vec{n})^2+5376v_1^2(\vec{v}_2\cdot\vec{n})^2-7848\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2
\end{aligned}
$$

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$$
\begin{aligned}
&-108v_2^2(\vec{v}_2\cdot\vec{n})^2+3360(\vec{v}_1\cdot\vec{n})^4-11328\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3+6240(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&+1152\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{n})^2+2\big(126v_1^2+15\vec{v}_1\cdot\vec{v}_2+73v_2^2+5466\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-4728(\vec{v}_1\cdot\vec{n})^2-166(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_1)^2+\big(599v_1^2+528\vec{v}_1\cdot\vec{v}_2-144v_2^2\\
&+4128\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-6728(\vec{v}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_2)^2\Big]-\frac{G^2{m_2}^2}{128m_1r^4}\Big[S_1^2\big(1784v_1^2\vec{v}_1\cdot\vec{v}_2\\
&-1396v_1^2v_2^2+3180\vec{v}_1\cdot\vec{v}_2v_2^2-1524(\vec{v}_1\cdot\vec{v}_2)^2-409v_1^4\\
&-1632v_2^4-1368\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+1944\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-2568\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&+720v_1^2(\vec{v}_1\cdot\vec{n})^2-3104\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+3436v_2^2(\vec{v}_1\cdot\vec{n})^2+1412v_1^2(\vec{v}_2\cdot\vec{n})^2\\
&-2340\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+3264v_2^2(\vec{v}_2\cdot\vec{n})^2-192(\vec{v}_1\cdot\vec{n})^4+2304\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3\\
&-3492(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-24\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3-432(\vec{v}_2\cdot\vec{n})^4\big)\\
&-8\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(171\vec{v}_1\cdot\vec{n}v_1^2-399v_1^2\vec{v}_2\cdot\vec{n}-295\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+735\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+109\vec{v}_1\cdot\vec{n}v_2^2-517\vec{v}_2\cdot\vec{n}v_2^2-48(\vec{v}_1\cdot\vec{n})^3+156\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+186\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\\
&+573(\vec{v}_2\cdot\vec{n})^3\big)+8\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(189\vec{v}_1\cdot\vec{n}v_1^2-405v_1^2\vec{v}_2\cdot\vec{n}-75\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+521\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-33\vec{v}_1\cdot\vec{n}v_2^2-396\vec{v}_2\cdot\vec{n}v_2^2-240\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&+918\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2+648(\vec{v}_2\cdot\vec{n})^3\big)-4\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\big(436v_1^2-318\vec{v}_1\cdot\vec{v}_2\\
&+299v_2^2-282\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-386(\vec{v}_1\cdot\vec{n})^2-717(\vec{v}_2\cdot\vec{n})^2\big)-\big(2920v_1^2\vec{v}_1\cdot\vec{v}_2\\
&-1492v_1^2v_2^2+5752\vec{v}_1\cdot\vec{v}_2v_2^2-3224(\vec{v}_1\cdot\vec{v}_2)^2-801v_1^4-3152v_2^4\\
&+1296\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}-5904\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+3600\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2+192v_1^2(\vec{v}_1\cdot\vec{n})^2\\
&+960v_2^2(\vec{v}_1\cdot\vec{n})^2-1356v_1^2(\vec{v}_2\cdot\vec{n})^2+576\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2\\
&+2352v_2^2(\vec{v}_2\cdot\vec{n})^2-1008(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-288\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3\\
&+1392(\vec{v}_2\cdot\vec{n})^4\big)(\vec{S}_1\cdot\vec{n})^2+4\big(102v_1^2-6\vec{v}_1\cdot\vec{v}_2+71v_2^2-248\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-38(\vec{v}_1\cdot\vec{n})^2-27(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_1)^2+4\big(341v_1^2-512\vec{v}_1\cdot\vec{v}_2+420v_2^2\\
&+242\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-586(\vec{v}_1\cdot\vec{n})^2-1076(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&-\frac{G^3{m_2}^3}{16m_1r^5}\Big[S_1^2\big(16v_1^2-127\vec{v}_1\cdot\vec{v}_2+320v_2^2-257\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-(\vec{v}_1\cdot\vec{n})^2\\
&-146(\vec{v}_2\cdot\vec{n})^2\big)-2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(39\vec{v}_1\cdot\vec{n}+322\vec{v}_2\cdot\vec{n}\big)+2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(121\vec{v}_1\cdot\vec{n}\\
&+621\vec{v}_2\cdot\vec{n}\big)+144\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-\big(5v_1^2-357\vec{v}_1\cdot\vec{v}_2+686v_2^2-285\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-80(\vec{v}_1\cdot\vec{n})^2+395(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-12(\vec{S}_1\cdot\vec{v}_1)^2-335(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&-\frac{G^3m_1m_2}{117600r^5}\Big[S_1^2\big(1389116v_1^2+2980008\vec{v}_1\cdot\vec{v}_2-489024v_2^2-3207240\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-4741105(\vec{v}_1\cdot\vec{n})^2+3087720(\vec{v}_2\cdot\vec{n})^2\big)-10\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(364471\vec{v}_1\cdot\vec{n}\\
&-346476\vec{v}_2\cdot\vec{n}\big)+480\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(11287\vec{v}_1\cdot\vec{n}+7888\vec{v}_2\cdot\vec{n}\big)\\
&-3287544\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-25\big(61561v_1^2+143376\vec{v}_1\cdot\vec{v}_2-49152v_2^2\\
&+17136\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-406672(\vec{v}_1\cdot\vec{n})^2+326424(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2\\
&-48788(\vec{S}_1\cdot\vec{v}_1)^2-245568(\vec{S}_1\cdot\vec{v}_2)^2\Big]-\frac{G^3{m_2}^2}{6144r^5}\Big[S_1^2\big((39936-189\pi^2)v_1^2
\end{aligned}
$$

$$
\begin{aligned}
&+(242432-3942\pi^2)\vec{v}_1\cdot\vec{v}_2+(22272+4131\pi^2)v_2^2\\
&-(497536-19710\pi^2)\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-(206016-945\pi^2)(\vec{v}_1\cdot\vec{n})^2\\
&+(441600-20655\pi^2)(\vec{v}_2\cdot\vec{n})^2)-4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1((86016-945\pi^2)\vec{v}_1\cdot\vec{n}\\
&+(36992-9855\pi^2)\vec{v}_2\cdot\vec{n})+4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2((74176+9855\pi^2)\vec{v}_1\cdot\vec{n}\\
&+(225792-20655\pi^2)\vec{v}_2\cdot\vec{n})-4(8672+1971\pi^2)\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\\
&-((65856-945\pi^2)v_1^2+(276736-19710\pi^2)\vec{v}_1\cdot\vec{v}_2+(28416+20655\pi^2)v_2^2\\
&-(765952-137970\pi^2)\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-(677760-6615\pi^2)(\vec{v}_1\cdot\vec{n})^2\\
&+(1389312-144585\pi^2)(\vec{v}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2-54(64+7\pi^2)(\vec{S}_1\cdot\vec{v}_1)^2\\
&-6(42752-1377\pi^2)(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&-\frac{G^4{m_1}^2m_2}{2450r^6}\Big[5813S_1^2-3684(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{29G^4{m_2}^4}{16m_1r^6}\Big[S_1^2-(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{3G^4{m_2}^3}{4r^6}\Big[29S_1^2+11(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G^4m_1{m_2}^2}{78400r^6}\Big[(2549096-55125\pi^2)S_1^2\\
&-(958088-165375\pi^2)(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
\hat{V}^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}}=&\frac{Gm_2}{32m_1r^3}\Big[S_1^2(5v_1^2\vec{v}_1\cdot\vec{v}_2v_2^2+26v_1^2(\vec{v}_1\cdot\vec{v}_2)^2-22(\vec{v}_1\cdot\vec{v}_2)^3-14v_2^2(\vec{v}_1\cdot\vec{v}_2)^2-25v_1^6\\
&+27\vec{v}_1\cdot\vec{v}_2v_1^4-6v_2^2v_1^4-10v_1^2v_2^4+31\vec{v}_1\cdot\vec{v}_2v_2^4-11v_2^6-204\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+57\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}v_2^2-60\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2+12v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+21v_1^2v_2^2(\vec{v}_1\cdot\vec{n})^2+39\vec{v}_1\cdot\vec{v}_2v_2^2(\vec{v}_1\cdot\vec{n})^2+27v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2\\
&-120(\vec{v}_1\cdot\vec{n})^2(\vec{v}_1\cdot\vec{v}_2)^2+150\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-27v_1^2v_2^2(\vec{v}_2\cdot\vec{n})^2\\
&+42(\vec{v}_1\cdot\vec{n})^2v_1^4+87\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_1^4-9(\vec{v}_2\cdot\vec{n})^2v_1^4+15(\vec{v}_1\cdot\vec{n})^2v_2^4\\
&-21\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^4-180v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3+240\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^3-30v_2^2(\vec{v}_1\cdot\vec{n})^4\\
&-105\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^3-45v_1^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2+75\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^3\\
&-135\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2+135v_2^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2+210(\vec{v}_1\cdot\vec{n})^4(\vec{v}_2\cdot\vec{n})^2\\
&-175(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^3)+6\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1(10\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_1\cdot\vec{v}_2+6v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-2\vec{v}_1\cdot\vec{n}v_1^2v_2^2-v_1^2\vec{v}_2\cdot\vec{n}v_2^2-6\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2-4\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2+10\vec{v}_1\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2\\
&-10\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-11\vec{v}_1\cdot\vec{n}v_1^4-\vec{v}_1\cdot\vec{n}v_2^4+7\vec{v}_2\cdot\vec{n}v_2^4+35v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&-50\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+5v_2^2(\vec{v}_1\cdot\vec{n})^3+15\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^2-15\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^2\\
&-5v_1^2(\vec{v}_2\cdot\vec{n})^3+50\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-30\vec{v}_1\cdot\vec{n}v_2^2(\vec{v}_2\cdot\vec{n})^2-35(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^2\\
&+35(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^3)-6\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2(4\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_1\cdot\vec{v}_2+10v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-5\vec{v}_1\cdot\vec{n}v_1^2v_2^2-6v_1^2\vec{v}_2\cdot\vec{n}v_2^2+4\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2+10\vec{v}_1\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2-7\vec{v}_1\cdot\vec{n}v_1^4\\
&-6\vec{v}_2\cdot\vec{n}v_1^4-7\vec{v}_1\cdot\vec{n}v_2^4+30v_1^2\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2-50\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+5v_2^2(\vec{v}_1\cdot\vec{n})^3+30\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^2+15\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^2-35(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^2)\\
&-2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2(6v_1^2\vec{v}_1\cdot\vec{v}_2-v_1^2v_2^2-4\vec{v}_1\cdot\vec{v}_2v_2^2-10(\vec{v}_1\cdot\vec{v}_2)^2\\
&+7v_2^4-18\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+60\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-36\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2+21v_1^2(\vec{v}_1\cdot\vec{n})^2\\
&-30\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+9v_2^2(\vec{v}_1\cdot\vec{n})^2-9v_1^2(\vec{v}_2\cdot\vec{n})^2-30\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3
\end{aligned}
$$

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$$
\begin{aligned}
&+45(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\big)-3\big(19v_1^2\vec{v}_1\cdot\vec{v}_2v_2^2-6v_1^2(\vec{v}_1\cdot\vec{v}_2)^2-2(\vec{v}_1\cdot\vec{v}_2)^3\\
&-6v_2^2(\vec{v}_1\cdot\vec{v}_2)^2-11v_1^6+17\vec{v}_1\cdot\vec{v}_2v_1^4-8v_2^2v_1^4-8v_1^2v_2^4+17\vec{v}_1\cdot\vec{v}_2v_2^4\\
&-11v_2^6-20\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+25\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}v_2^2-20\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2v_2^2\\
&+15v_1^2v_2^2(\vec{v}_1\cdot\vec{n})^2-25\vec{v}_1\cdot\vec{v}_2v_2^2(\vec{v}_1\cdot\vec{n})^2-25v_1^2\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2\\
&-50\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{v}_2)^2+15v_1^2v_2^2(\vec{v}_2\cdot\vec{n})^2+35\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_1^4+15(\vec{v}_2\cdot\vec{n})^2v_1^4\\
&+15(\vec{v}_1\cdot\vec{n})^2v_2^4+35\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^4-35\vec{v}_2\cdot\vec{n}v_2^2(\vec{v}_1\cdot\vec{n})^3-105v_1^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&-35\vec{v}_1\cdot\vec{n}v_1^2(\vec{v}_2\cdot\vec{n})^3+175\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-105v_2^2(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2\\
&+105(\vec{v}_1\cdot\vec{n})^3(\vec{v}_2\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{n})^2-2\big(5v_1^2\vec{v}_1\cdot\vec{v}_2-2v_1^2v_2^2-\vec{v}_1\cdot\vec{v}_2v_2^2+10(\vec{v}_1\cdot\vec{v}_2)^2\\
&-7v_1^4-4v_2^4+33\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}-60\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+27\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&-12v_1^2(\vec{v}_1\cdot\vec{n})^2+15\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+15\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-9v_2^2(\vec{v}_2\cdot\vec{n})^2\\
&+15\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3-45(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2+15\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{v}_1)^2\\
&+2\big(5v_1^2\vec{v}_1\cdot\vec{v}_2-3v_1^2v_2^2-3v_1^4+9\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+9v_1^2(\vec{v}_1\cdot\vec{n})^2-15\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+9v_2^2(\vec{v}_1\cdot\vec{n})^2-15\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&+\frac{G^2m_2}{16r^4}\Big[S_1^2\big(35v_1^2\vec{v}_1\cdot\vec{v}_2-64v_1^2v_2^2+152\vec{v}_1\cdot\vec{v}_2v_2^2-92(\vec{v}_1\cdot\vec{v}_2)^2+15v_1^4-47v_2^4\\
&+308\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}-192\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-255v_1^2(\vec{v}_1\cdot\vec{n})^2+95\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+77v_2^2(\vec{v}_1\cdot\vec{n})^2-44v_1^2(\vec{v}_2\cdot\vec{n})^2+4\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-2v_2^2(\vec{v}_2\cdot\vec{n})^2+92(\vec{v}_1\cdot\vec{n})^4\\
&-84\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3+78(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-56\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3\big)\\
&-2\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(879\vec{v}_1\cdot\vec{n}v_1^2-418v_1^2\vec{v}_2\cdot\vec{n}-851\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+388\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&+68\vec{v}_1\cdot\vec{n}v_2^2-50\vec{v}_2\cdot\vec{n}v_2^2-164(\vec{v}_1\cdot\vec{n})^3-33\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+108\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\\
&-28(\vec{v}_2\cdot\vec{n})^3\big)+4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(206\vec{v}_1\cdot\vec{n}v_1^2+13v_1^2\vec{v}_2\cdot\vec{n}-180\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-46\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+17\vec{v}_1\cdot\vec{n}v_2^2-6\vec{v}_2\cdot\vec{n}v_2^2-6(\vec{v}_1\cdot\vec{n})^3-63\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&+60\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2\big)-\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\big(221v_1^2-224\vec{v}_1\cdot\vec{v}_2+40v_2^2-56\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&+13(\vec{v}_1\cdot\vec{n})^2+20(\vec{v}_2\cdot\vec{n})^2\big)+\big(540v_1^2\vec{v}_1\cdot\vec{v}_2+111v_1^2v_2^2-400\vec{v}_1\cdot\vec{v}_2v_2^2\\
&-44(\vec{v}_1\cdot\vec{v}_2)^2-336v_1^4+137v_2^4-2454\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+2112\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-192\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2+2646v_1^2(\vec{v}_1\cdot\vec{n})^2-2322\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+51v_2^2(\vec{v}_1\cdot\vec{n})^2\\
&-66v_1^2(\vec{v}_2\cdot\vec{n})^2+288\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+30v_2^2(\vec{v}_2\cdot\vec{n})^2-592(\vec{v}_1\cdot\vec{n})^4\\
&+6\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3+546(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-320\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{n})^2+\big(231v_1^2\\
&-252\vec{v}_1\cdot\vec{v}_2+39v_2^2-38\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-11(\vec{v}_1\cdot\vec{n})^2+26(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_1)^2-2\big(5v_1^2\\
&-12\vec{v}_1\cdot\vec{v}_2-2v_2^2+24\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-13(\vec{v}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&+\frac{G^2{m_2}^2}{16m_1r^4}\Big[S_1^2\big(124v_1^2\vec{v}_1\cdot\vec{v}_2-174v_1^2v_2^2+407\vec{v}_1\cdot\vec{v}_2v_2^2-370(\vec{v}_1\cdot\vec{v}_2)^2-8v_1^4\\
&+17v_2^4+208\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}+2680\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-2156\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2\\
&-160v_1^2(\vec{v}_1\cdot\vec{n})^2-256\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2+710v_2^2(\vec{v}_1\cdot\vec{n})^2+457v_1^2(\vec{v}_2\cdot\vec{n})^2\\
&-2517\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2+989v_2^2(\vec{v}_2\cdot\vec{n})^2+96(\vec{v}_1\cdot\vec{n})^4-128\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3\\
&-3768(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2+5940\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3-1966(\vec{v}_2\cdot\vec{n})^4\big)
\end{aligned}
$$

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\begin{aligned}
&-\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(148\vec{v}_1\cdot\vec{n}v_1^2-112v_1^2\vec{v}_2\cdot\vec{n}-36\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+1362\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2\\
&-293\vec{v}_1\cdot\vec{n}v_2^2-1122\vec{v}_2\cdot\vec{n}v_2^2+96(\vec{v}_1\cdot\vec{n})^3-624\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2\\
&-357\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2+1002(\vec{v}_2\cdot\vec{n})^3\big)-\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(156\vec{v}_1\cdot\vec{n}v_1^2+578v_1^2\vec{v}_2\cdot\vec{n}\\
&+1672\vec{v}_1\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-3692\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2-613\vec{v}_1\cdot\vec{n}v_2^2+1982\vec{v}_2\cdot\vec{n}v_2^2-128(\vec{v}_1\cdot\vec{n})^3\\
&-4086\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^2+6933\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^2-3130(\vec{v}_2\cdot\vec{n})^3\big)-2\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2\big(32v_1^2\\
&-241\vec{v}_1\cdot\vec{v}_2+49v_2^2+355\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}+22(\vec{v}_1\cdot\vec{n})^2-391(\vec{v}_2\cdot\vec{n})^2\big)-\big(364v_1^2\vec{v}_1\cdot\vec{v}_2\\
&-156v_1^2v_2^2-721\vec{v}_1\cdot\vec{v}_2v_2^2+104(\vec{v}_1\cdot\vec{v}_2)^2-74v_1^4+468v_2^4-516\vec{v}_1\cdot\vec{n}v_1^2\vec{v}_2\cdot\vec{n}\\
&-1656\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\vec{v}_1\cdot\vec{v}_2+2391\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}v_2^2-48v_1^2(\vec{v}_1\cdot\vec{n})^2-384\vec{v}_1\cdot\vec{v}_2(\vec{v}_1\cdot\vec{n})^2\\
&+249v_2^2(\vec{v}_1\cdot\vec{n})^2+126v_1^2(\vec{v}_2\cdot\vec{n})^2+2703\vec{v}_1\cdot\vec{v}_2(\vec{v}_2\cdot\vec{n})^2-3042v_2^2(\vec{v}_2\cdot\vec{n})^2\\
&+512\vec{v}_2\cdot\vec{n}(\vec{v}_1\cdot\vec{n})^3+735(\vec{v}_1\cdot\vec{n})^2(\vec{v}_2\cdot\vec{n})^2-1623\vec{v}_1\cdot\vec{n}(\vec{v}_2\cdot\vec{n})^3\\
&+1072(\vec{v}_2\cdot\vec{n})^4\big)(\vec{S}_1\cdot\vec{n})^2+\big(12v_1^2+40\vec{v}_1\cdot\vec{v}_2-89v_2^2-600\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&+204(\vec{v}_1\cdot\vec{n})^2+221(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_1)^2+\big(211v_1^2-752\vec{v}_1\cdot\vec{v}_2+257v_2^2\\
&+2240\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-850(\vec{v}_1\cdot\vec{n})^2-1267(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&-\frac{G^3m_1m_2}{58800r^5}\Big[S_1^2(468263v_1^2-731426\vec{v}_1\cdot\vec{v}_2+323538v_2^2+2235430\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-744265(\vec{v}_1\cdot\vec{n})^2-726240(\vec{v}_2\cdot\vec{n})^2)-20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(100574\vec{v}_1\cdot\vec{n}\\
&-128539\vec{v}_2\cdot\vec{n}\big)+20\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(133789\vec{v}_1\cdot\vec{n}-83004\vec{v}_2\cdot\vec{n}\big)\\
&-341452\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-25\big(59347v_1^2-88090\vec{v}_1\cdot\vec{v}_2+40398v_2^2\\
&+427882\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-149947(\vec{v}_1\cdot\vec{n})^2-140784(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2\\
&+260326(\vec{S}_1\cdot\vec{v}_1)^2+103176(\vec{S}_1\cdot\vec{v}_2)^2\Big]-\frac{G^3{m_2}^3}{48m_1r^5}\Big[S_1^2(522v_1^2-110\vec{v}_1\cdot\vec{v}_2\\
&-673v_2^2+178\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-1476(\vec{v}_1\cdot\vec{n})^2+2147(\vec{v}_2\cdot\vec{n})^2)\\
&-4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1\big(1419\vec{v}_1\cdot\vec{n}-1622\vec{v}_2\cdot\vec{n}\big)+4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2\big(1265\vec{v}_1\cdot\vec{n}-1246\vec{v}_2\cdot\vec{n}\big)\\
&-880\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-\big(1218v_1^2-922\vec{v}_1\cdot\vec{v}_2-185v_2^2+12850\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-7464(\vec{v}_1\cdot\vec{n})^2-2887(\vec{v}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2+636(\vec{S}_1\cdot\vec{v}_1)^2+490(\vec{S}_1\cdot\vec{v}_2)^2\Big]\\
&-\frac{G^3{m_2}^2}{6144r^5}\Big[S_1^2((183680-8109\pi^2)v_1^2-(384256-16218\pi^2)\vec{v}_1\cdot\vec{v}_2\\
&+(242816-8109\pi^2)v_2^2+(972032-81090\pi^2)\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}\\
&-(324736-40545\pi^2)(\vec{v}_1\cdot\vec{n})^2-(684928-40545\pi^2)(\vec{v}_2\cdot\vec{n})^2)\\
&-4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_1((509152-40545\pi^2)\vec{v}_1\cdot\vec{n}-(672736-40545\pi^2)\vec{v}_2\cdot\vec{n})\\
&+4\vec{S}_1\cdot\vec{n}\vec{S}_1\cdot\vec{v}_2((447712-40545\pi^2)\vec{v}_1\cdot\vec{n}-(541024-40545\pi^2)\vec{v}_2\cdot\vec{n})\\
&-4(146240-8109\pi^2)\vec{S}_1\cdot\vec{v}_1\vec{S}_1\cdot\vec{v}_2-((670720-40545\pi^2)v_1^2\\
&-(1422848-81090\pi^2)\vec{v}_1\cdot\vec{v}_2+(869632-40545\pi^2)v_2^2\\
&+(7089152-567630\pi^2)\vec{v}_1\cdot\vec{n}\vec{v}_2\cdot\vec{n}-(2963200-283815\pi^2)(\vec{v}_1\cdot\vec{n})^2\\
&-(3748864-283815\pi^2)(\vec{v}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2+2(155840-8109\pi^2)(\vec{S}_1\cdot\vec{v}_1)^2\\
&+2(153536-8109\pi^2)(\vec{S}_1\cdot\vec{v}_2)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{19G^4{m_1}^2m_2}{4900r^6}\Big[1571S_1^2-4433(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^4{m_2}^4}{4m_1r^6}\Big[73S_1^2-119(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{G^4m_1{m_2}^2}{29400r^6}\Big[(4275319-154350\pi^2)S_1^2-3(3162179-154350\pi^2)(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{G^4{m_2}^3}{24r^6}\Big[(3175-126\pi^2)S_1^2-9(785-42\pi^2)(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{A.4}
$$

and

$$
V^{\mathrm{N^3LO}}_{C_{\mathrm{1E^2S^2}}}=-\frac{G^4m_1{m_2}^2}{2r^6}\Big[S_1^2-3(\vec{S}_1\cdot\vec{n})^2\Big].
\tag{A.5}
$$

This action is also provided in machine-readable format in the supplementary material to this publication.

B General Hamiltonians

Our general Hamiltonian for the present sectors, comprised of 4 parts, can be expressed as:

$$
H^{\mathrm{N^3LO}}_{\mathrm{S^2}}=H^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}}+H^{\mathrm{N^3LO}}_{\mathrm{S_1^2}}+C_{\mathrm{1ES^2}}H^{\mathrm{N^3LO}}_{C_{\mathrm{1ES^2}}}+C_{\mathrm{1E^2S^2}}H^{\mathrm{N^3LO}}_{C_{\mathrm{1E^2S^2}}}+(1\leftrightarrow 2),
\tag{B.1}
$$

where

$$
\begin{aligned}
H^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}}=&\frac{G}{16{m_1}^6r^3}\Big[15\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}p_1^6-15\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}p_1^4-15\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}p_1^4\\
&+17\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2p_1^4-\vec{S}_1\cdot\vec{S}_2\big(17p_1^6-15(\vec{p}_1\cdot\vec{n})^2p_1^4\big)\Big]\\
&-\frac{G}{32{m_1}^5m_2r^3}\Big[3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(41\vec{p}_1\cdot\vec{p}_2p_1^4+40p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\\
&-80\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_1^4-140\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^4\big)-6\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(16\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2\\
&+9\vec{p}_2\cdot\vec{n}p_1^4-55p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2+20\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^3\big)+3\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(3\vec{p}_1\cdot\vec{n}p_1^4\\
&+80p_1^2(\vec{p}_1\cdot\vec{n})^3\big)-3\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(22\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2\\
&+\vec{p}_2\cdot\vec{n}p_1^4-70p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2-140\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^4\big)\\
&+2\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(10p_1^2\vec{p}_1\cdot\vec{p}_2-147\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}+60\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)\\
&+2\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(5p_1^4-12p_1^2(\vec{p}_1\cdot\vec{n})^2-150(\vec{p}_1\cdot\vec{n})^4\big)-36\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}p_1^4\\
&+30\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2p_1^4-3\vec{S}_1\cdot\vec{S}_2\big(22\vec{p}_1\cdot\vec{p}_2p_1^4-30p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\\
&-105\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_1^4+190p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3-100\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^4\big)\Big]\\
&+\frac{3G}{16m_1{m_2}^5r^3}\Big[14\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^4-\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(23\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2-6\vec{p}_1\cdot\vec{n}p_2^4\\
&+25\vec{p}_1\cdot\vec{n}p_2^2(\vec{p}_2\cdot\vec{n})^2\big)-11\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2p_2^2(\vec{p}_2\cdot\vec{n})^2+11\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2\\
&+\vec{S}_1\cdot\vec{S}_2\big(23\vec{p}_1\cdot\vec{p}_2p_2^2(\vec{p}_2\cdot\vec{n})^2-20\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^4+25\vec{p}_1\cdot\vec{n}p_2^2(\vec{p}_2\cdot\vec{n})^3\big)\Big]\\
&+\frac{G}{16{m_1}^4{m_2}^2r^3}\Big[3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(2p_1^2(\vec{p}_1\cdot\vec{p}_2)^2+3p_2^2p_1^4-95\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&-5p_1^2p_2^2(\vec{p}_1\cdot\vec{n})^2+60(\vec{p}_1\cdot\vec{n})^2(\vec{p}_1\cdot\vec{p}_2)^2-25(\vec{p}_2\cdot\vec{n})^2p_1^4+35p_1^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)\\
&-3\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(12p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+17\vec{p}_1\cdot\vec{n}p_1^2p_2^2-14\vec{p}_1\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2
\end{aligned}
$$

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$$
\begin{aligned}
&+10\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-15p_2^2(\vec{p}_1\cdot\vec{n})^3-85\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^2\\
&+105(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^2\big)-3\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(12\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2\\
&-2\vec{p}_2\cdot\vec{n}p_1^4-105p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2-20\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^3+70\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^4\big)\\
&+3\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(8p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-5\vec{p}_1\cdot\vec{n}p_1^2p_2^2\\
&+2\vec{p}_1\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-40\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+15p_2^2(\vec{p}_1\cdot\vec{n})^3\\
&+60\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^2-105(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^2\big)+\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(10p_1^2p_2^2\\
&-6(\vec{p}_1\cdot\vec{p}_2)^2-96\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-63p_2^2(\vec{p}_1\cdot\vec{n})^2-63p_1^2(\vec{p}_2\cdot\vec{n})^2\\
&+405(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)-\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(10p_1^2\vec{p}_1\cdot\vec{p}_2+147\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&-108\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\big)-24\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\big(\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2-5p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)\\
&+3\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\big(7p_1^2\vec{p}_1\cdot\vec{p}_2-32\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}+6\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\big)\\
&-\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\big(5p_1^4+78p_1^2(\vec{p}_1\cdot\vec{n})^2-60(\vec{p}_1\cdot\vec{n})^4\big)-\vec{S}_1\cdot\vec{S}_2\big(4p_1^2(\vec{p}_1\cdot\vec{p}_2)^2\\
&+8p_2^2p_1^4-387\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-132p_1^2p_2^2(\vec{p}_1\cdot\vec{n})^2+114(\vec{p}_1\cdot\vec{n})^2(\vec{p}_1\cdot\vec{p}_2)^2\\
&-69(\vec{p}_2\cdot\vec{n})^2p_1^4+120\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^3+105p_2^2(\vec{p}_1\cdot\vec{n})^4\\
&+735p_1^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2-525(\vec{p}_1\cdot\vec{n})^4(\vec{p}_2\cdot\vec{n})^2\big)\Big]\\
&-\frac{G}{128{m_1}^3{m_2}^3r^3}\Big[3\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(11p_1^2\vec{p}_1\cdot\vec{p}_2p_2^2-44(\vec{p}_1\cdot\vec{p}_2)^3-140\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}p_2^2\\
&-160\vec{p}_1\cdot\vec{p}_2p_2^2(\vec{p}_1\cdot\vec{n})^2+320p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\\
&-40\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-280\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^3+770\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\\
&+420(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^3\big)-24\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(10p_1^2\vec{p}_2\cdot\vec{n}p_2^2+12\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2\\
&-12\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-35\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^2-5p_1^2(\vec{p}_2\cdot\vec{n})^3+25\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\\
&+70(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^3\big)-3\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(96p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+55\vec{p}_1\cdot\vec{n}p_1^2p_2^2\\
&-20\vec{p}_1\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-400\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-240p_2^2(\vec{p}_1\cdot\vec{n})^3\\
&-320\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^2+910(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^2\big)+3\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(p_1^2\vec{p}_2\cdot\vec{n}p_2^2\\
&+16\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2+68\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2+360\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^2\\
&-480\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-630(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^3\big)\\
&+24\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(4\vec{p}_1\cdot\vec{p}_2p_2^2-25\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2+28\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\\
&+15\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)+2\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(29p_1^2p_2^2-96(\vec{p}_1\cdot\vec{p}_2)^2+132\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&-324p_2^2(\vec{p}_1\cdot\vec{n})^2+780(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)-24\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\big(3\vec{p}_1\cdot\vec{n}p_1^2p_2^2\\
&+2\vec{p}_1\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-20\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-10p_2^2(\vec{p}_1\cdot\vec{n})^3+35(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^2\big)\\
&+4\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\big(17p_1^2p_2^2-22(\vec{p}_1\cdot\vec{p}_2)^2-90\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-66p_2^2(\vec{p}_1\cdot\vec{n})^2\\
&+12p_1^2(\vec{p}_2\cdot\vec{n})^2+345(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)+8\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\big(14p_1^2\vec{p}_1\cdot\vec{p}_2\\
&+9\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-78\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-30\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)-\vec{S}_1\cdot\vec{S}_2\big(226p_1^2\vec{p}_1\cdot\vec{p}_2p_2^2\\
&-160(\vec{p}_1\cdot\vec{p}_2)^3-921\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}p_2^2-1464\vec{p}_1\cdot\vec{p}_2p_2^2(\vec{p}_1\cdot\vec{n})^2\\
&+672p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-180\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2+1200\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^3\\
&+1080\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^3+3120\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2-3430(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^3\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3G}{16{m_1}^2{m_2}^4r^3}\Big[2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(p_1^2p_2^4+10p_1^2p_2^2(\vec{p}_2\cdot\vec{n})^2\\
&+5(\vec{p}_1\cdot\vec{n})^2p_2^4-35p_1^2(\vec{p}_2\cdot\vec{n})^4)-2\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}(7\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2+6\vec{p}_1\cdot\vec{n}p_2^4\\
&+10\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^3+5\vec{p}_1\cdot\vec{n}p_2^2(\vec{p}_2\cdot\vec{n})^2-35\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^4)\\
&+\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}(5p_1^2\vec{p}_2\cdot\vec{n}p_2^2+12\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2+6\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2\\
&+25\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^2+25p_1^2(\vec{p}_2\cdot\vec{n})^3\\
&+20\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-35(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^3)-2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2(\vec{p}_1\cdot\vec{n}p_2^4\\
&+10\vec{p}_1\cdot\vec{n}p_2^2(\vec{p}_2\cdot\vec{n})^2-35\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^4)+2\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(p_2^4\\
&+15p_2^2(\vec{p}_2\cdot\vec{n})^2-25(\vec{p}_2\cdot\vec{n})^4)-10\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2+2\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3)\\
&+2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2(3\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2+2\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2+5\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3)\\
&-2\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2(2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+3p_2^2(\vec{p}_1\cdot\vec{n})^2+5(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2)\\
&-\vec{S}_1\cdot\vec{S}_2(2p_1^2p_2^4-2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2+6(\vec{p}_2\cdot\vec{n})^2(\vec{p}_1\cdot\vec{p}_2)^2+25p_1^2p_2^2(\vec{p}_2\cdot\vec{n})^2\\
&-6(\vec{p}_1\cdot\vec{n})^2p_2^4-45p_1^2(\vec{p}_2\cdot\vec{n})^4+15p_2^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+35(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^4)\Big]\\
&-\frac{21G}{16{m_2}^6r^3}\Big[\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^4-\vec{S}_1\cdot\vec{S}_2(\vec{p}_2\cdot\vec{n})^2p_2^4\Big]\\
&+\frac{G^2m_2}{8{m_1}^4r^4}\Big[75\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}p_1^4-74\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}p_1^2-74\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}p_1^2\\
&+56\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2p_1^2-\vec{S}_1\cdot\vec{S}_2(57p_1^4-74p_1^2(\vec{p}_1\cdot\vec{n})^2)\Big]\\
&-\frac{G^2}{32{m_1}^3r^4}\Big[\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(1924p_1^2\vec{p}_1\cdot\vec{p}_2+1318p_1^4-2358\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&-10305p_1^2(\vec{p}_1\cdot\vec{n})^2-4086\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+796(\vec{p}_1\cdot\vec{n})^4)\\
&-\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}(1218\vec{p}_1\cdot\vec{n}p_1^2+558p_1^2\vec{p}_2\cdot\vec{n}+1554\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-10501(\vec{p}_1\cdot\vec{n})^3\\
&-2964\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2)-3\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}(42\vec{p}_1\cdot\vec{n}p_1^2-1327(\vec{p}_1\cdot\vec{n})^3)\\
&-\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2(1394\vec{p}_1\cdot\vec{n}p_1^2+754p_1^2\vec{p}_2\cdot\vec{n}+452\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-10149(\vec{p}_1\cdot\vec{n})^3\\
&-7050\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2)+\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(559p_1^2-220\vec{p}_1\cdot\vec{p}_2-966\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&-6341(\vec{p}_1\cdot\vec{n})^2)+\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(755p_1^2-5088(\vec{p}_1\cdot\vec{n})^2)\\
&-470\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}p_1^2+322\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2p_1^2-\vec{S}_1\cdot\vec{S}_2(961p_1^2\vec{p}_1\cdot\vec{p}_2\\
&+499p_1^4-2274\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-7395p_1^2(\vec{p}_1\cdot\vec{n})^2-5540\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\\
&+10657(\vec{p}_1\cdot\vec{n})^4+6945\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3)\Big]+\frac{G^2}{96{m_1}^2m_2r^4}\Big[6\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(631p_1^2\vec{p}_1\cdot\vec{p}_2\\
&-4p_1^2p_2^2-2646\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-3105\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-227p_2^2(\vec{p}_1\cdot\vec{n})^2\\
&-1092p_1^2(\vec{p}_2\cdot\vec{n})^2-275\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3)+2\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}(549p_1^2\vec{p}_2\cdot\vec{n}\\
&+2748\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-5333\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-2539\vec{p}_1\cdot\vec{n}p_2^2+9396\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\\
&+3741\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2)-\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}(16710\vec{p}_1\cdot\vec{n}p_1^2-8797p_1^2\vec{p}_2\cdot\vec{n}\\
&-7418\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-39246(\vec{p}_1\cdot\vec{n})^3-20199\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2)\\
&-6\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2(917p_1^2\vec{p}_2\cdot\vec{n}+114\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-4\vec{p}_1\cdot\vec{n}p_2^2-8592\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\\
&-1092\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2)+2\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2(204\vec{p}_1\cdot\vec{p}_2-455p_2^2-7416\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}
\end{aligned}
$$

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$$+2739(\vec{p}_2\cdot\vec{n})^2\big)+3\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(1187p_1^2-4010\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-4437(\vec{p}_1\cdot\vec{n})^2\big)$$
$$+6\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\big(185\vec{p}_1\cdot\vec{n}p_1^2-1191(\vec{p}_1\cdot\vec{n})^3\big)$$
$$-12\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\big(47p_1^2-456\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-345(\vec{p}_1\cdot\vec{n})^2\big)$$
$$-5705\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2(\vec{p}_1\cdot\vec{n})^2-\vec{S}_1\cdot\vec{S}_2\big(2667p_1^2\vec{p}_1\cdot\vec{p}_2$$
$$-910p_1^2p_2^2-19110\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-3248\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-6957\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2$$
$$-6129p_2^2(\vec{p}_1\cdot\vec{n})^2+2245p_1^2(\vec{p}_2\cdot\vec{n})^2+54132\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3$$
$$+27681(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)\Big]+\frac{G^2}{96m_1{m_2}^2r^4}\Big[\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(2399p_1^2p_2^2$$
$$+1846(\vec{p}_1\cdot\vec{p}_2)^2-2424\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+1014p_2^2(\vec{p}_1\cdot\vec{n})^2$$
$$-9765p_1^2(\vec{p}_2\cdot\vec{n})^2-3162(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)-\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(8416\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2$$
$$+8705\vec{p}_1\cdot\vec{n}p_2^2-12882\vec{p}_2\cdot\vec{n}p_2^2-22305\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2+120(\vec{p}_2\cdot\vec{n})^3\big)$$
$$+\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(9508p_1^2\vec{p}_2\cdot\vec{n}+18122\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-12030\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2$$
$$+2091\vec{p}_1\cdot\vec{n}p_2^2-31668\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2-33147\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big)$$
$$+3\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\big(666\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-827\vec{p}_1\cdot\vec{n}p_2^2-3693\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big)$$
$$+\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(2333p_2^2-2409(\vec{p}_2\cdot\vec{n})^2\big)$$
$$-2\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\big(1321\vec{p}_1\cdot\vec{p}_2-346\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\big)$$
$$-36\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\big(59\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-385\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)$$
$$+2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\big(458\vec{p}_1\cdot\vec{p}_2-1628\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-6597(\vec{p}_2\cdot\vec{n})^2\big)$$
$$+2\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\big(54p_1^2+6597\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-1633(\vec{p}_1\cdot\vec{n})^2\big)-\vec{S}_1\cdot\vec{S}_2\big(2211p_1^2p_2^2$$
$$-794(\vec{p}_1\cdot\vec{p}_2)^2+390\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+14973\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2-4089p_2^2(\vec{p}_1\cdot\vec{n})^2$$
$$-257p_1^2(\vec{p}_2\cdot\vec{n})^2-12030\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-10461(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2$$
$$-33267\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)\Big]+\frac{G^2}{32{m_2}^3r^4}\Big[1918\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2$$
$$-\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big(3756\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+576\vec{p}_1\cdot\vec{n}p_2^2+1630\vec{p}_2\cdot\vec{n}p_2^2-588\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2$$
$$-4337(\vec{p}_2\cdot\vec{n})^3\big)-2088\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2(\vec{p}_2\cdot\vec{n})^2+2088\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}$$
$$-\vec{S}_1\cdot\vec{S}_2\big(1342\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2-3756\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-1630p_2^2(\vec{p}_2\cdot\vec{n})^2$$
$$+588\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3+4337(\vec{p}_2\cdot\vec{n})^4\big)\Big]+\frac{7G^2m_1}{4{m_2}^4r^4}\Big[\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2$$
$$-\vec{S}_1\cdot\vec{S}_2p_2^2(\vec{p}_2\cdot\vec{n})^2\Big]$$
$$-\frac{1393G^3{m_1}^2}{20{m_2}^2r^5}\Big[\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2(\vec{p}_2\cdot\vec{n})^2\Big]$$
$$+\frac{G^3{m_2}^2}{100{m_1}^2r^5}\Big[25\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big(193p_1^2+25(\vec{p}_1\cdot\vec{n})^2\big)-4415\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}$$
$$-4325\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2\big(6337p_1^2-3820(\vec{p}_1\cdot\vec{n})^2\big)+5831\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\Big]$$
$$+\frac{G^3m_2}{960m_1r^5}\Big[5\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}\big((71716-2115\pi^2)p_1^2-(122788-14805\pi^2)(\vec{p}_1\cdot\vec{n})^2\big)$$
$$-2\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\big((9970+10575\pi^2)\vec{p}_1\cdot\vec{n}+23128\vec{p}_2\cdot\vec{n}\big)+83284\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_1\cdot\vec{n}$$

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$$
\begin{aligned}
&-140640\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2((290140-2115\pi^2)p_1^2+37028\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&-(423720-31725\pi^2)(\vec{p}_1\cdot\vec{n})^2)+10(18220+423\pi^2)\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\Big]\\
&-\frac{G^3m_1}{2400m_2r^5}\Big[200\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(1987\vec{p}_1\cdot\vec{p}_2-3652\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n})\\
&-174160\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}\vec{p}_2\cdot\vec{n}+5\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}(27808\vec{p}_1\cdot\vec{n}\\
&+(80650+10575\pi^2)\vec{p}_2\cdot\vec{n})+120000\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2\vec{p}_2\cdot\vec{n}\\
&-254400\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2(395416\vec{p}_1\cdot\vec{p}_2-605760\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&+(403250+52875\pi^2)(\vec{p}_2\cdot\vec{n})^2)+9104\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2+199904\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\Big]\\
&+\frac{G^3}{4800r^5}\Big[25\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}(17744p_1^2-(78366-2115\pi^2)\vec{p}_1\cdot\vec{p}_2\\
&+(123654-14805\pi^2)\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-58280(\vec{p}_1\cdot\vec{n})^2)+10\vec{p}_1\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}(50548\vec{p}_1\cdot\vec{n}\\
&-(930-10575\pi^2)\vec{p}_2\cdot\vec{n})+10\vec{p}_2\cdot\vec{S}_1\vec{S}_2\cdot\vec{n}((75405+10575\pi^2)\vec{p}_1\cdot\vec{n}\\
&-46622\vec{p}_2\cdot\vec{n})+150\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_2(484\vec{p}_1\cdot\vec{n}+1445\vec{p}_2\cdot\vec{n})\\
&+653100\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_2\vec{p}_1\cdot\vec{n}-\vec{S}_1\cdot\vec{S}_2(442456p_1^2-(1557600-10575\pi^2)\vec{p}_1\cdot\vec{p}_2\\
&+(2795100-52875\pi^2)\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-638160(\vec{p}_1\cdot\vec{n})^2-466220(\vec{p}_2\cdot\vec{n})^2)\\
&+96928\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2-75(5536+141\pi^2)\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{S}_2\\
&-75(7804+141\pi^2)\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_2\Big]\\
&-\frac{G^4{m_1}^3}{100r^6}\Big[7531\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}-4537\vec{S}_1\cdot\vec{S}_2\Big]\\
&-\frac{G^4{m_1}^2m_2}{600r^6}\Big[3(143047-6075\pi^2)\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n}-(183307-6075\pi^2)\vec{S}_1\cdot\vec{S}_2\Big], \qquad \text{(B.2)}
\end{aligned}
$$

$$
\begin{aligned}
H_{S_1^2}^{\mathrm{N^3LO}}=&\frac{3G}{8{m_1}^2{m_2}^4r^3}\Big[4\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(2\vec{p}_2\cdot\vec{n}p_2^4-5p_2^2(\vec{p}_2\cdot\vec{n})^3)+\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1(\vec{p}_1\cdot\vec{n}p_2^4\\
&+10\vec{p}_1\cdot\vec{n}p_2^2(\vec{p}_2\cdot\vec{n})^2-35\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^4)+\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1(5p_2^4-38p_2^2(\vec{p}_2\cdot\vec{n})^2\\
&+45(\vec{p}_2\cdot\vec{n})^4)-S_1^2(5\vec{p}_1\cdot\vec{p}_2p_2^4-38\vec{p}_1\cdot\vec{p}_2p_2^2(\vec{p}_2\cdot\vec{n})^2+8\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^4\\
&+45\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^4-20\vec{p}_1\cdot\vec{n}p_2^2(\vec{p}_2\cdot\vec{n})^3)-(\vec{p}_1\cdot\vec{p}_2p_2^4\\
&+10\vec{p}_1\cdot\vec{p}_2p_2^2(\vec{p}_2\cdot\vec{n})^2-35\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^4)(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{3Gm_2}{128{m_1}^7r^3}\Big[38\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}p_1^4+3S_1^2(10p_1^6-19(\vec{p}_1\cdot\vec{n})^2p_1^4)-33p_1^6(\vec{S}_1\cdot\vec{n})^2\\
&-2(15p_1^4-26p_1^2(\vec{p}_1\cdot\vec{n})^2)(\vec{p}_1\cdot\vec{S}_1)^2\Big]-\frac{G}{32{m_1}^6r^3}\Big[12\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2\\
&+3\vec{p}_2\cdot\vec{n}p_1^4-10p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2)-108\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}p_1^4+12\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1(p_1^4\\
&+5p_1^2(\vec{p}_1\cdot\vec{n})^2)+S_1^2(59\vec{p}_1\cdot\vec{p}_2p_1^4-264p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-189\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_1^4\\
&+360p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3)+60\vec{p}_1\cdot\vec{p}_2p_1^4(\vec{S}_1\cdot\vec{n})^2-(71p_1^2\vec{p}_1\cdot\vec{p}_2-153\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&-240\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+240\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3)(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&+\frac{G}{32{m_1}^4{m_2}^2r^3}\Big[3\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(p_1^2\vec{p}_2\cdot\vec{n}p_2^2-12\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2
\end{aligned}
$$

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$$+4\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-70\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^2-5p_1^2(\vec{p}_2\cdot\vec{n})^3+20\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2$$
$$+210(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^3\big)-12\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(4\vec{p}_1\cdot\vec{n}p_1^2p_2^2-25\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^2\big)$$
$$+3\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(17p_1^2p_2^2+52(\vec{p}_1\cdot\vec{p}_2)^2-280\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-42p_2^2(\vec{p}_1\cdot\vec{n})^2$$
$$-49p_1^2(\vec{p}_2\cdot\vec{n})^2+190(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)-S_1^2\big(295p_1^2\vec{p}_1\cdot\vec{p}_2p_2^2$$
$$+156(\vec{p}_1\cdot\vec{p}_2)^3-513\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}p_2^2-144\vec{p}_1\cdot\vec{p}_2p_2^2(\vec{p}_1\cdot\vec{n})^2$$
$$-939p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-828\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-240\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^3$$
$$+945\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^3+630\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+630(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^3\big)$$
$$+6\big(11p_1^2\vec{p}_1\cdot\vec{p}_2p_2^2-5\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}p_2^2-50p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2$$
$$+4\big(61\vec{p}_1\cdot\vec{p}_2p_2^2-129\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2-198\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2$$
$$+240\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)(\vec{p}_1\cdot\vec{S}_1)^2\Big]+\frac{G}{64{m_1}^5m_2r^3}\Big[6\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(12p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2$$
$$+15\vec{p}_1\cdot\vec{n}p_1^2p_2^2+14\vec{p}_1\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-30\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+15p_2^2(\vec{p}_1\cdot\vec{n})^3$$
$$-20\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^2-105(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^2\big)-12\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(14\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2$$
$$+\vec{p}_2\cdot\vec{n}p_1^4+5p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)-4\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(8p_1^2\vec{p}_1\cdot\vec{p}_2-42\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}$$
$$-117\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+135\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)+S_1^2\big(426p_1^2(\vec{p}_1\cdot\vec{p}_2)^2$$
$$+104p_2^2p_1^4-2076\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-333p_1^2p_2^2(\vec{p}_1\cdot\vec{n})^2-492(\vec{p}_1\cdot\vec{n})^2(\vec{p}_1\cdot\vec{p}_2)^2$$
$$-318(\vec{p}_2\cdot\vec{n})^2p_1^4+600\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^3-75p_2^2(\vec{p}_1\cdot\vec{n})^4$$
$$+1635p_1^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+525(\vec{p}_1\cdot\vec{n})^4(\vec{p}_2\cdot\vec{n})^2\big)+3\big(18p_1^2(\vec{p}_1\cdot\vec{p}_2)^2+15p_2^2p_1^4$$
$$+60\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-5p_1^2p_2^2(\vec{p}_1\cdot\vec{n})^2-55(\vec{p}_2\cdot\vec{n})^2p_1^4$$
$$+35p_1^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-2\big(52p_1^2p_2^2$$
$$+202(\vec{p}_1\cdot\vec{p}_2)^2-924\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-99p_2^2(\vec{p}_1\cdot\vec{n})^2-159p_1^2(\vec{p}_2\cdot\vec{n})^2$$
$$+675(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2+2\big(5p_1^4+27p_1^2(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]$$
$$-\frac{G}{64{m_1}^3{m_2}^3r^3}\Big[6\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(48\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2+13\vec{p}_1\cdot\vec{n}p_2^4-40\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^3$$
$$-100\vec{p}_1\cdot\vec{n}p_2^2(\vec{p}_2\cdot\vec{n})^2\big)+960\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2$$
$$+48\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(9\vec{p}_1\cdot\vec{p}_2p_2^2-24\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2-32\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2$$
$$+45\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)-S_1^2\big(432p_2^2(\vec{p}_1\cdot\vec{p}_2)^2+118p_1^2p_2^4-864\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2$$
$$-1536(\vec{p}_2\cdot\vec{n})^2(\vec{p}_1\cdot\vec{p}_2)^2-888p_1^2p_2^2(\vec{p}_2\cdot\vec{n})^2+93(\vec{p}_1\cdot\vec{n})^2p_2^4+960p_1^2(\vec{p}_2\cdot\vec{n})^4$$
$$+1920\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^3-1080p_2^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+840(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^4\big)$$
$$+15\big(p_1^2p_2^4-32(\vec{p}_2\cdot\vec{n})^2(\vec{p}_1\cdot\vec{p}_2)^2-32p_1^2p_2^2(\vec{p}_2\cdot\vec{n})^2+56p_1^2(\vec{p}_2\cdot\vec{n})^4\big)(\vec{S}_1\cdot\vec{n})^2$$
$$+2\big(59p_2^4-444p_2^2(\vec{p}_2\cdot\vec{n})^2+480(\vec{p}_2\cdot\vec{n})^4\big)(\vec{p}_1\cdot\vec{S}_1)^2$$
$$-480(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2(\vec{p}_2\cdot\vec{S}_1)^2\Big]$$
$$+\frac{G^2{m_2}^2}{16{m_1}^5r^4}\Big[3\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(79\vec{p}_1\cdot\vec{n}p_1^2-16(\vec{p}_1\cdot\vec{n})^3\big)+4S_1^2\big(31p_1^4-59p_1^2(\vec{p}_1\cdot\vec{n})^2$$
$$+6(\vec{p}_1\cdot\vec{n})^4\big)-\big(167p_1^4-24p_1^2(\vec{p}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2$$
$$-\big(121p_1^2-163(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2\Big]+\frac{G^2m_2}{32{m_1}^4r^4}\Big[\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(79\vec{p}_1\cdot\vec{n}p_1^2$$

$$-574p_1^2\vec{p}_2\cdot\vec{n}+452\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-7288(\vec{p}_1\cdot\vec{n})^3+258\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2)$$
$$+731\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}p_1^2+2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(217p_1^2-851(\vec{p}_1\cdot\vec{n})^2\big)$$
$$+S_1^2\big(26p_1^2\vec{p}_1\cdot\vec{p}_2+119p_1^4+148\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-1260p_1^2(\vec{p}_1\cdot\vec{n})^2$$
$$+1012\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+1144(\vec{p}_1\cdot\vec{n})^4-480\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)-\big(552p_1^2\vec{p}_1\cdot\vec{p}_2$$
$$+631p_1^4-129\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-3888p_1^2(\vec{p}_1\cdot\vec{n})^2-1120(\vec{p}_1\cdot\vec{n})^4\big)(\vec{S}_1\cdot\vec{n})^2-\big(199p_1^2$$
$$+460\vec{p}_1\cdot\vec{p}_2-578\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-2900(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2\Big]$$
$$+\frac{G^2}{32{m_1}^3r^4}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(303p_1^2\vec{p}_2\cdot\vec{n}+362\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+420\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2$$
$$-310\vec{p}_1\cdot\vec{n}p_2^2+5820\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2+1392\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big)$$
$$+2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(750\vec{p}_1\cdot\vec{n}p_1^2+23p_1^2\vec{p}_2\cdot\vec{n}-1231\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+3208(\vec{p}_1\cdot\vec{n})^3$$
$$-54\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)+2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(29p_1^2+302\vec{p}_1\cdot\vec{p}_2+262\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}$$
$$-2738(\vec{p}_1\cdot\vec{n})^2\big)+S_1^2\big(792p_1^2\vec{p}_1\cdot\vec{p}_2-48p_1^2p_2^2-525(\vec{p}_1\cdot\vec{p}_2)^2+30\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}$$
$$+66\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+1684\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-932p_2^2(\vec{p}_1\cdot\vec{n})^2$$
$$+1028p_1^2(\vec{p}_2\cdot\vec{n})^2-2792\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3+588(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)+\big(56p_1^2\vec{p}_1\cdot\vec{p}_2$$
$$+634p_1^2p_2^2+824(\vec{p}_1\cdot\vec{p}_2)^2-5208\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-1008\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2$$
$$-4752\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+264p_2^2(\vec{p}_1\cdot\vec{n})^2-2037p_1^2(\vec{p}_2\cdot\vec{n})^2-3776\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3$$
$$-540(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-\big(714\vec{p}_1\cdot\vec{p}_2-295p_2^2+1140\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}$$
$$+1347(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2-2\big(187p_1^2-752(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]$$
$$-\frac{G^2}{96{m_1}^2m_2r^4}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(2426\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-872\vec{p}_1\cdot\vec{n}p_2^2+777\vec{p}_2\cdot\vec{n}p_2^2$$
$$+12\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2+1107(\vec{p}_2\cdot\vec{n})^3\big)+2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(1366p_1^2\vec{p}_2\cdot\vec{n}$$
$$+5504\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-597\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-1707\vec{p}_1\cdot\vec{n}p_2^2+11928\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2$$
$$+4914\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big)-\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(5062\vec{p}_1\cdot\vec{p}_2-861p_2^2+4028\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}$$
$$+3213(\vec{p}_2\cdot\vec{n})^2\big)+S_1^2\big(550p_1^2p_2^2-213\vec{p}_1\cdot\vec{p}_2p_2^2$$
$$+4331(\vec{p}_1\cdot\vec{p}_2)^2-5908\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-1866\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2+3914p_2^2(\vec{p}_1\cdot\vec{n})^2$$
$$-5200p_1^2(\vec{p}_2\cdot\vec{n})^2+5049\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-804(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2$$
$$-2790\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)-2\big(629p_1^2p_2^2-441\vec{p}_1\cdot\vec{p}_2p_2^2+568(\vec{p}_1\cdot\vec{p}_2)^2$$
$$+10608\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+792\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2+654p_2^2(\vec{p}_1\cdot\vec{n})^2-1182p_1^2(\vec{p}_2\cdot\vec{n})^2$$
$$+4176\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2+2544(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+36\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{n})^2$$
$$-2\big(413p_2^2-2888(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2+\big(1043p_1^2-816\vec{p}_1\cdot\vec{p}_2+3126\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}$$
$$-9560(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]-\frac{G^2}{8m_1{m_2}^2r^4}\Big[24\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(4\vec{p}_2\cdot\vec{n}p_2^2+19(\vec{p}_2\cdot\vec{n})^3\big)$$
$$-2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(314\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+73\vec{p}_1\cdot\vec{n}p_2^2-34\vec{p}_2\cdot\vec{n}p_2^2-183\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2$$
$$+15(\vec{p}_2\cdot\vec{n})^3\big)+6\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(26p_2^2-123(\vec{p}_2\cdot\vec{n})^2\big)-S_1^2\big(180\vec{p}_1\cdot\vec{p}_2p_2^2+26p_2^4$$
$$+56\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2-818\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-8p_2^2(\vec{p}_2\cdot\vec{n})^2+504\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3$$
$$-105(\vec{p}_2\cdot\vec{n})^4\big)+\big(54\vec{p}_1\cdot\vec{p}_2p_2^2+22p_2^4+84\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2+42\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2$$

$$
\begin{aligned}
&+39p_2^2(\vec{p}_2\cdot\vec{n})^2-96\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3-81(\vec{p}_2\cdot\vec{n})^4)(\vec{S}_1\cdot\vec{n})^2+7\big(4\vec{p}_1\cdot\vec{p}_2+4p_2^2\\
&+56\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-19(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]-\frac{G^2}{8{m_2}^3r^4}\Big[36\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}p_2^2\\
&+S_1^2\big(13p_2^4-21p_2^2(\vec{p}_2\cdot\vec{n})^2\big)-3\big(4p_2^4+3p_2^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-12p_2^2(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^3{m_2}^3}{16{m_1}^3r^5}\Big[1458\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}+S_1^2\big(441p_1^2-457(\vec{p}_1\cdot\vec{n})^2\big)-16\big(55p_1^2\\
&+8(\vec{p}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-434(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^3{m_2}^2}{2048{m_1}^2r^5}\Big[12\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big((42112-105\pi^2)\vec{p}_1\cdot\vec{n}+960\vec{p}_2\cdot\vec{n}\big)\\
&-231040\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}+S_1^2\big((92032+63\pi^2)p_1^2-60032\vec{p}_1\cdot\vec{p}_2\\
&+260480\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-(17216+315\pi^2)(\vec{p}_1\cdot\vec{n})^2\big)+59904\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\\
&-\big((167872+315\pi^2)p_1^2-87424\vec{p}_1\cdot\vec{p}_2+128256\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&+(400512-2205\pi^2)(\vec{p}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-42(2816-3\pi^2)(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&-\frac{G^3m_1}{4900m_2r^5}\Big[402760\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}+187S_1^2\big(402p_2^2-85(\vec{p}_2\cdot\vec{n})^2\big)\\
&-25\big(1774p_2^2+16247(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-88632(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^3m_2}{3763200m_1r^5}\Big[10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(51835232\vec{p}_1\cdot\vec{n}-(6554240+2414475\pi^2)\vec{p}_2\cdot\vec{n}\big)\\
&-2450\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big((287584+9855\pi^2)\vec{p}_1\cdot\vec{n}+134688\vec{p}_2\cdot\vec{n}\big)+S_1^2\big(95727488p_1^2\\
&-(152723200-2414475\pi^2)\vec{p}_1\cdot\vec{p}_2-129595200p_2^2\\
&+(694467200-12072375\pi^2)\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}+144188960(\vec{p}_1\cdot\vec{n})^2\\
&-128184000(\vec{p}_2\cdot\vec{n})^2\big)+2450(53600+1971\pi^2)\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1-25\big(7852000p_1^2\\
&-(9715328-482895\pi^2)\vec{p}_1\cdot\vec{p}_2-9384480p_2^2+(6234368-3380265\pi^2)\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&+18470144(\vec{p}_1\cdot\vec{n})^2-9215136(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-180953984(\vec{p}_1\cdot\vec{S}_1)^2\\
&+122774400(\vec{p}_2\cdot\vec{S}_1)^2\Big]-\frac{G^3}{2508800r^5}\Big[85204480\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}\\
&+20\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(12050944\vec{p}_1\cdot\vec{n}+(29282400-1686825\pi^2)\vec{p}_2\cdot\vec{n}\big)\\
&+S_1^2\big(106223104\vec{p}_1\cdot\vec{p}_2+(197097600+1686825\pi^2)p_2^2-69048320\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&+(95491200-8434125\pi^2)(\vec{p}_2\cdot\vec{n})^2\big)-152297472\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\\
&-25\big(7097856\vec{p}_1\cdot\vec{p}_2+(9916032+337365\pi^2)p_2^2-412160\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&+(19769344-2361555\pi^2)(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-2450(98560-1377\pi^2)(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{9G^4{m_2}^4}{m_1r^6}\Big[S_1^2-(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^4{m_2}^3}{4r^6}\Big[97S_1^2-217(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^4{m_1}^2m_2}{2450r^6}\Big[26037S_1^2\\
&-28166(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^4m_1{m_2}^2}{78400r^6}\Big[(1895204+55125\pi^2)S_1^2\\
&-(3486212+165375\pi^2)(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
H^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}} = &-\frac{G}{4{m_1}^2{m_2}^4r^3}\Big[6\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}p_2^4+6\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}p_2^4-2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1p_2^4\\
&+S_1^2\big(\vec{p}_1\cdot\vec{p}_2p_2^4-3\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^4\big)+3\big(\vec{p}_1\cdot\vec{p}_2p_2^4-5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^4\big)(\vec{S}_1\cdot\vec{n})^2\Big]\\
&-\frac{7G}{32m_1{m_2}^5r^3}\Big[S_1^2p_2^6-3p_2^6(\vec{S}_1\cdot\vec{n})^2\Big]-\frac{G}{16{m_1}^6r^3}\Big[3\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(26\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2\\
&+9\vec{p}_2\cdot\vec{n}p_1^4-5p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)+24\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}p_1^4-3\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(3p_1^4\\
&-p_1^2(\vec{p}_1\cdot\vec{n})^2\big)+12S_1^2\big(3\vec{p}_1\cdot\vec{p}_2p_1^4-8p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_1^4\big)\\
&+12\big(\vec{p}_1\cdot\vec{p}_2p_1^4-5\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_1^4\big)(\vec{S}_1\cdot\vec{n})^2-\big(31p_1^2\vec{p}_1\cdot\vec{p}_2+3\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&-15\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-15\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&+\frac{Gm_2}{32{m_1}^7r^3}\Big[54\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}p_1^4+S_1^2\big(11p_1^6-54(\vec{p}_1\cdot\vec{n})^2p_1^4\big)+21p_1^6(\vec{S}_1\cdot\vec{n})^2\\
&-18p_1^4(\vec{p}_1\cdot\vec{S}_1)^2\Big]+\frac{G}{32{m_1}^5m_2r^3}\Big[6\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(30p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+6\vec{p}_1\cdot\vec{n}p_1^2p_2^2\\
&+10\vec{p}_1\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-50\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+5p_2^2(\vec{p}_1\cdot\vec{n})^3\\
&+5\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^2-35(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^2\big)+12\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(10\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_1\cdot\vec{p}_2\\
&+\vec{p}_2\cdot\vec{n}p_1^4-5p_1^2\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)-12\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(5p_1^2\vec{p}_1\cdot\vec{p}_2+\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&-5\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-5\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)+S_1^2\big(78p_1^2(\vec{p}_1\cdot\vec{p}_2)^2\\
&+16p_2^2p_1^4-228\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-33p_1^2p_2^2(\vec{p}_1\cdot\vec{n})^2-120(\vec{p}_1\cdot\vec{n})^2(\vec{p}_1\cdot\vec{p}_2)^2\\
&+9(\vec{p}_2\cdot\vec{n})^2p_1^4+240\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^3-30p_2^2(\vec{p}_1\cdot\vec{n})^4-135p_1^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\\
&+210(\vec{p}_1\cdot\vec{n})^4(\vec{p}_2\cdot\vec{n})^2\big)+3\big(2p_1^2(\vec{p}_1\cdot\vec{p}_2)^2+2p_2^2p_1^4-100\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&-5p_1^2p_2^2(\vec{p}_1\cdot\vec{n})^2-5(\vec{p}_2\cdot\vec{n})^2p_1^4+35p_1^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-2\big(8p_1^2p_2^2\\
&+10(\vec{p}_1\cdot\vec{p}_2)^2-60\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&+6p_1^2(\vec{p}_2\cdot\vec{n})^2-45(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2-2\big(p_1^4-3p_1^2(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{G}{32{m_1}^3{m_2}^3r^3}\Big[6\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(20\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2+7\vec{p}_1\cdot\vec{n}p_2^4-10\vec{p}_1\cdot\vec{n}p_2^2(\vec{p}_2\cdot\vec{n})^2\big)\\
&+12\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(p_1^2\vec{p}_2\cdot\vec{n}p_2^2+10\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2-5\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^2\big)\\
&-8\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(5\vec{p}_1\cdot\vec{p}_2p_2^2-3\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2\big)+S_1^2\big(38p_2^2(\vec{p}_1\cdot\vec{p}_2)^2\\
&+12p_1^2p_2^4-84\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2-9p_1^2p_2^2(\vec{p}_2\cdot\vec{n})^2-39(\vec{p}_1\cdot\vec{n})^2p_2^4\\
&+45p_2^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)+3\big(2p_2^2(\vec{p}_1\cdot\vec{p}_2)^2+2p_1^2p_2^4-100\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2\\
&-5p_1^2p_2^2(\vec{p}_2\cdot\vec{n})^2-5(\vec{p}_1\cdot\vec{n})^2p_2^4+35p_2^2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-6\big(2p_2^4\\
&-p_2^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2-2\big(p_1^2p_2^2-3p_2^2(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&-\frac{G}{32{m_1}^4{m_2}^2r^3}\Big[6\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(9p_1^2\vec{p}_2\cdot\vec{n}p_2^2+18\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2p_2^2\\
&+10\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-5\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^2+5p_1^2(\vec{p}_2\cdot\vec{n})^3\\
&-50\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-35(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^3\big)+30\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(2p_1^2\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&+\vec{p}_1\cdot\vec{n}p_1^2p_2^2+2\vec{p}_1\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-10\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+p_2^2(\vec{p}_1\cdot\vec{n})^3\\
&+3\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^2-7(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^2\big)-2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(9p_1^2p_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+10(\vec{p}_1\cdot\vec{p}_2)^2-60\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-3p_2^2(\vec{p}_1\cdot\vec{n})^2\\
&+9p_1^2(\vec{p}_2\cdot\vec{n})^2-45(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)+S_1^2\big(41p_1^2\vec{p}_1\cdot\vec{p}_2p_2^2\\
&+22(\vec{p}_1\cdot\vec{p}_2)^3-51\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}p_2^2-99\vec{p}_1\cdot\vec{p}_2p_2^2(\vec{p}_1\cdot\vec{n})^2-27p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\\
&-150\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2+45\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^3-75\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^3\\
&+135\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+175(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^3\big)+3\big(13p_1^2\vec{p}_1\cdot\vec{p}_2p_2^2\\
&-2(\vec{p}_1\cdot\vec{p}_2)^3-25\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}p_2^2-25\vec{p}_1\cdot\vec{p}_2p_2^2(\vec{p}_1\cdot\vec{n})^2-25p_1^2\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\\
&-50\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{p}_2)^2-35\vec{p}_2\cdot\vec{n}p_2^2(\vec{p}_1\cdot\vec{n})^3-35\vec{p}_1\cdot\vec{n}p_1^2(\vec{p}_2\cdot\vec{n})^3\\
&+175\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+105(\vec{p}_1\cdot\vec{n})^3(\vec{p}_2\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{n})^2\\
&-2\big(13\vec{p}_1\cdot\vec{p}_2p_2^2-15\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2\\
&-15\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-15\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)(\vec{p}_1\cdot\vec{S}_1)^2-2\big(5p_1^2\vec{p}_1\cdot\vec{p}_2+9\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&-15\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-15\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^2{m_2}^2}{8{m_1}^5r^4}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(97\vec{p}_1\cdot\vec{n}p_1^2-24(\vec{p}_1\cdot\vec{n})^3\big)+2S_1^2\big(13p_1^4-104p_1^2(\vec{p}_1\cdot\vec{n})^2\\
&+24(\vec{p}_1\cdot\vec{n})^4\big)+\big(67p_1^4-24p_1^2(\vec{p}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2\\
&-2\big(31p_1^2-33(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2\Big]-\frac{G^2m_2}{16{m_1}^4r^4}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(640\vec{p}_1\cdot\vec{n}p_1^2\\
&+138p_1^2\vec{p}_2\cdot\vec{n}+429\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-299(\vec{p}_1\cdot\vec{n})^3\big)\\
&+\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(189\vec{p}_1\cdot\vec{n}p_1^2-128(\vec{p}_1\cdot\vec{n})^3\big)-2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(2p_1^2+53(\vec{p}_1\cdot\vec{n})^2\big)\\
&+S_1^2\big(208p_1^2\vec{p}_1\cdot\vec{p}_2-109p_1^4+144\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}+293p_1^2(\vec{p}_1\cdot\vec{n})^2\\
&-488\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2+178(\vec{p}_1\cdot\vec{n})^4-496\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)+\big(272p_1^2\vec{p}_1\cdot\vec{p}_2\\
&+342p_1^4-1221\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-2136p_1^2(\vec{p}_1\cdot\vec{n})^2-384\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2\\
&+592(\vec{p}_1\cdot\vec{n})^4+512\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{n})^2-\big(135p_1^2\\
&+324\vec{p}_1\cdot\vec{p}_2-308\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-161(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^2}{16{m_1}^3r^4}\Big[\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(556p_1^2\vec{p}_2\cdot\vec{n}+742\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-498\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&+723\vec{p}_1\cdot\vec{n}p_2^2-930\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2+1287\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big)\\
&+2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(246\vec{p}_1\cdot\vec{n}p_1^2-221p_1^2\vec{p}_2\cdot\vec{n}-374\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-144(\vec{p}_1\cdot\vec{n})^3\\
&+2373\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\big)-3\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(43p_1^2-38\vec{p}_1\cdot\vec{p}_2+386\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&-57(\vec{p}_1\cdot\vec{n})^2\big)-S_1^2\big(357p_1^2\vec{p}_1\cdot\vec{p}_2-51p_1^2p_2^2-64(\vec{p}_1\cdot\vec{p}_2)^2-128\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}\\
&-2368\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-465\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-304p_2^2(\vec{p}_1\cdot\vec{n})^2\\
&-612p_1^2(\vec{p}_2\cdot\vec{n})^2-486\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3+4848(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)+\big(766p_1^2\vec{p}_1\cdot\vec{p}_2\\
&-15p_1^2p_2^2-302(\vec{p}_1\cdot\vec{p}_2)^2-1476\vec{p}_1\cdot\vec{n}p_1^2\vec{p}_2\cdot\vec{n}-252\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&-1548\vec{p}_1\cdot\vec{p}_2(\vec{p}_1\cdot\vec{n})^2-147p_2^2(\vec{p}_1\cdot\vec{n})^2-411p_1^2(\vec{p}_2\cdot\vec{n})^2+1056\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^3\\
&-1521(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-\big(22\vec{p}_1\cdot\vec{p}_2+245p_2^2-188\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&-81(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2+\big(199p_1^2-928(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]
\end{aligned}
$$

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$$
\begin{aligned}
&-\frac{G^2}{16{m_1}^2 m_2 r^4}\Big[2\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(58\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-128\vec{p}_1\cdot\vec{n}p_2^2\\
&-287\vec{p}_2\cdot\vec{n}p_2^2-102\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2+633(\vec{p}_2\cdot\vec{n})^3\big)+\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(20p_1^2\vec{p}_2\cdot\vec{n}\\
&+60\vec{p}_1\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-3200\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-527\vec{p}_1\cdot\vec{n}p_2^2-168\vec{p}_2\cdot\vec{n}(\vec{p}_1\cdot\vec{n})^2\\
&+7713\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\big)-2\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(2\vec{p}_1\cdot\vec{p}_2+23p_2^2-54\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&+465(\vec{p}_2\cdot\vec{n})^2\big)-S_1^2\big(79p_1^2p_2^2+203\vec{p}_1\cdot\vec{p}_2p_2^2\\
&+194(\vec{p}_1\cdot\vec{p}_2)^2-248\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2-2196\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2-134p_2^2(\vec{p}_1\cdot\vec{n})^2\\
&+4p_1^2(\vec{p}_2\cdot\vec{n})^2-2579\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-204(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2+6294\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)\\
&+\big(60p_1^2p_2^2-831\vec{p}_1\cdot\vec{p}_2p_2^2+242(\vec{p}_1\cdot\vec{p}_2)^2-132\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2\\
&+1599\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2+228p_2^2(\vec{p}_1\cdot\vec{n})^2-42p_1^2(\vec{p}_2\cdot\vec{n})^2+1929\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\\
&+288(\vec{p}_1\cdot\vec{n})^2(\vec{p}_2\cdot\vec{n})^2-2673\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\big)(\vec{S}_1\cdot\vec{n})^2+\big(59p_2^2\\
&+10(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_1\cdot\vec{S}_1)^2-2\big(p_1^2-339\vec{p}_1\cdot\vec{p}_2+1191\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&-5(\vec{p}_1\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]-\frac{G^2}{16m_1{m_2}^2r^4}\Big[8\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big(23\vec{p}_2\cdot\vec{n}p_2^2-7(\vec{p}_2\cdot\vec{n})^3\big)\\
&+2\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\big(92\vec{p}_2\cdot\vec{n}\vec{p}_1\cdot\vec{p}_2+116\vec{p}_1\cdot\vec{n}p_2^2+1179\vec{p}_2\cdot\vec{n}p_2^2-120\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^2\\
&-1565(\vec{p}_2\cdot\vec{n})^3\big)-4\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\big(13p_2^2-5(\vec{p}_2\cdot\vec{n})^2\big)+S_1^2\big(4\vec{p}_1\cdot\vec{p}_2p_2^2\\
&+147p_2^4-72\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2-4\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2-1595p_2^2(\vec{p}_2\cdot\vec{n})^2+56\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3\\
&+1966(\vec{p}_2\cdot\vec{n})^4\big)+2\big(96\vec{p}_1\cdot\vec{p}_2p_2^2+191p_2^4-348\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}p_2^2-144\vec{p}_1\cdot\vec{p}_2(\vec{p}_2\cdot\vec{n})^2\\
&-1356p_2^2(\vec{p}_2\cdot\vec{n})^2+160\vec{p}_1\cdot\vec{n}(\vec{p}_2\cdot\vec{n})^3+536(\vec{p}_2\cdot\vec{n})^4\big)(\vec{S}_1\cdot\vec{n})^2-\big(24\vec{p}_1\cdot\vec{p}_2\\
&+351p_2^2-48\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-1267(\vec{p}_2\cdot\vec{n})^2\big)(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^2}{16{m_2}^3r^4}\Big[24\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}p_2^2-S_1^2\big(27p_2^4-2p_2^2(\vec{p}_2\cdot\vec{n})^2\big)\\
&+5\big(17p_2^4-6p_2^2(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-4p_2^2(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^3{m_2}^3}{8{m_1}^3r^5}\Big[1382\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}-S_1^2\big(311p_1^2-434(\vec{p}_1\cdot\vec{n})^2\big)\\
&+\big(635p_1^2-1556(\vec{p}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-194(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^3{m_2}^2}{6144{m_1}^2r^5}\Big[20\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\big((67040-8109\pi^2)\vec{p}_1\cdot\vec{n}-61568\vec{p}_2\cdot\vec{n}\big)\\
&+3584\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_1\cdot\vec{n}-S_1^2\big((738944-8109\pi^2)p_1^2-592384\vec{p}_1\cdot\vec{p}_2\\
&+940544\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-(1429888-40545\pi^2)(\vec{p}_1\cdot\vec{n})^2\big)+65024\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\\
&+\big((1338112-40545\pi^2)p_1^2-908288\vec{p}_1\cdot\vec{p}_2+1640960\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&-(3014656-283815\pi^2)(\vec{p}_1\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-2(62144-8109\pi^2)(\vec{p}_1\cdot\vec{S}_1)^2\Big]\\
&+\frac{G^3m_1}{9800m_2r^5}\Big[453080\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}-S_1^2\big(83323p_2^2-150440(\vec{p}_2\cdot\vec{n})^2\big)\\
&+25\big(11437p_2^2-34048(\vec{p}_2\cdot\vec{n})^2\big)(\vec{S}_1\cdot\vec{n})^2-46596(\vec{p}_2\cdot\vec{S}_1)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G^3 m_2}{3763200 m_1 r^5}\Big[10\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1(14566912\vec{p}_1\cdot\vec{n}\\
&-(125871200-9933525\pi^2)\vec{p}_2\cdot\vec{n})-2450\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1((18016-40545\pi^2)\vec{p}_1\cdot\vec{n}\\
&+62080\vec{p}_2\cdot\vec{n})-S_1^2(77714432p_1^2-(972238400-9933525\pi^2)\vec{p}_1\cdot\vec{p}_2\\
&+215600000p_2^2+(2404763200-49667625\pi^2)\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-93496960(\vec{p}_1\cdot\vec{n})^2\\
&-399212800(\vec{p}_2\cdot\vec{n})^2)+2450(12032-8109\pi^2)\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1+25(7683712p_1^2\\
&-(65388736-1986705\pi^2)\vec{p}_1\cdot\vec{p}_2+10944640p_2^2\\
&+(178040128-13906935\pi^2)\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}-12089728(\vec{p}_1\cdot\vec{n})^2\\
&+1718528(\vec{p}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2-10075264(\vec{p}_1\cdot\vec{S}_1)^2+50489600(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&-\frac{G^3}{7526400 r^5}\Big[409968640\vec{S}_1\cdot\vec{n}\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{n}+20\vec{S}_1\cdot\vec{n}\vec{p}_2\cdot\vec{S}_1(23428352\vec{p}_1\cdot\vec{n}\\
&-(46561760-9933525\pi^2)\vec{p}_2\cdot\vec{n})-S_1^2(199932928\vec{p}_1\cdot\vec{p}_2\\
&-(1049148800-9933525\pi^2)p_2^2-386800640\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&+(2783670400-49667625\pi^2)(\vec{p}_2\cdot\vec{n})^2)-60640256\vec{p}_1\cdot\vec{S}_1\vec{p}_2\cdot\vec{S}_1\\
&+25(22151168\vec{p}_1\cdot\vec{p}_2-(70422016-1986705\pi^2)p_2^2-69859328\vec{p}_1\cdot\vec{n}\vec{p}_2\cdot\vec{n}\\
&+(185086720-13906935\pi^2)(\vec{p}_2\cdot\vec{n})^2)(\vec{S}_1\cdot\vec{n})^2\\
&-2450(3904+8109\pi^2)(\vec{p}_2\cdot\vec{S}_1)^2\Big]\\
&+\frac{19G^4{m_1}^2 m_2}{4900r^6}\Big[1571S_1^2-4433(\vec{S}_1\cdot\vec{n})^2\Big]+\frac{G^4{m_2}^4}{4m_1 r^6}\Big[73S_1^2-119(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{G^4 m_1 {m_2}^2}{29400r^6}\Big[(4275319-154350\pi^2)S_1^2-3(3162179-154350\pi^2)(\vec{S}_1\cdot\vec{n})^2\Big]\\
&+\frac{G^4{m_2}^3}{24r^6}\Big[(3175-126\pi^2)S_1^2-9(785-42\pi^2)(\vec{S}_1\cdot\vec{n})^2\Big],
\end{aligned}
\tag{B.4}
$$

and

$$
H^{\mathrm{N^3LO}}_{C_{1\mathrm{E^2S^2}}} = -\frac{G^4 m_1 {m_2}^2}{2r^6}\Big[S_1^2-3(\vec{S}_1\cdot\vec{n})^2\Big]. \tag{B.5}
$$

These Hamiltonians are also provided in machine-readable format in the supplementary material to this publication.

C COM Hamiltonians

The Hamiltonian of the present sectors in the COM frame is written as:

$$
\tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S^2}} = \tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} + \tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S_1^2}} + C_{1\mathrm{ES^2}}\tilde{H}^{\mathrm{N^3LO}}_{C_{1\mathrm{ES^2}}} + C_{1\mathrm{E^2S^2}}\tilde{H}^{\mathrm{N^3LO}}_{C_{1\mathrm{E^2S^2}}} + (1\leftrightarrow 2), \tag{C.1}
$$

with

$$
\begin{aligned}
\tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S_1S_2}} = \frac{\nu\tilde{\vec{S}}_1\cdot\tilde{\vec{S}}_2}{\tilde{r}^6}\Bigg[\Bigg(\left(\frac{101641}{1200}-\frac{81\pi^2}{16}\right)\nu+\frac{4537}{200}+\Bigg(-3\nu^2+\left(\frac{423\pi^2}{128}-\frac{67947}{800}\right)\nu\\
-\frac{253}{100}\Bigg)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{\nu^2}{8}-\frac{9\nu}{4}-\frac{1}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(-\frac{5\nu^3}{32}+\frac{5\nu^2}{16}-\frac{3\nu}{32}\right)\frac{\tilde{L}^6}{\tilde{r}^3}
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{p}_r^2\tilde{r}\left(\frac{109\nu^2}{16}+\left(\frac{306553}{2400}-\frac{141\pi^2}{32}\right)\nu+\frac{556}{25}+\left(\frac{1373\nu^2}{64}-\frac{93\nu}{8}+\frac{1}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\right.\\
&\left.+\left(-\frac{33\nu^3}{128}+\frac{215\nu^2}{32}-\frac{237\nu}{64}+\frac{19}{32}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{575\nu^2}{64}-\frac{231\nu}{16}+\frac{3}{16}+\left(\frac{15\nu^3}{64}+\frac{353\nu^2}{16}-\frac{369\nu}{32}+\frac{19}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&\left.+\tilde{p}_r^6\tilde{r}^3\left(-\frac{187\nu^3}{128}+\frac{153\nu^2}{16}-\frac{327\nu}{64}+\frac{19}{32}\right)\right]\\
&+\frac{\nu\tilde{\vec{S}}_1\cdot\tilde{\vec{L}}\tilde{\vec{S}}_2\cdot\tilde{\vec{L}}}{\tilde{r}^7}\left[-\frac{161\nu^2}{16}+\left(-\frac{26183}{1200}-\frac{141\pi^2}{64}\right)\nu-\frac{5831}{200}\right.\\
&+\left(\frac{487\nu^2}{32}+\frac{411\nu}{32}-\frac{7}{2}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{15\nu^3}{64}-\frac{49\nu^2}{32}+\frac{9\nu}{4}-\frac{17}{32}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(-\frac{579\nu^2}{32}+\frac{535\nu}{64}-\frac{7}{2}+\left(-\frac{51\nu^3}{32}-\frac{377\nu^2}{32}+\frac{303\nu}{32}-\frac{17}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&\left.+\tilde{p}_r^4\tilde{r}^2\left(-\frac{237\nu^3}{64}-\frac{943\nu^2}{32}+\frac{321\nu}{32}-\frac{17}{32}\right)\right]\\
&+\frac{\nu\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\cdot\vec{n}}{\tilde{r}^6}\left[\left(\frac{243\pi^2}{16}-\frac{97861}{400}\right)\nu-\frac{7531}{200}+\left(-\frac{11\nu^2}{32}\right.\right.\\
&\left.+\left(\frac{362959}{2400}-\frac{987\pi^2}{128}\right)\nu-\frac{503}{100}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{801\nu^2}{64}-\frac{9\nu}{8}+\frac{19}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\left(\frac{75\nu^3}{128}-\frac{139\nu^2}{32}+\frac{87\nu}{64}-\frac{1}{16}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\left(-\frac{53\nu^2}{16}+\left(\frac{423\pi^2}{32}-\frac{158163}{800}\right)\nu-\frac{553}{25}+\left(-\frac{1583\nu^2}{32}+\frac{987\nu}{16}+1\right)\frac{\tilde{L}^2}{\tilde{r}}\right.\\
&\left.+\left(\frac{231\nu^3}{128}-\frac{565\nu^2}{32}+\frac{555\nu}{64}-\frac{23}{32}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{2943\nu^2}{64}+\frac{223\nu}{16}-\frac{3}{16}+\left(\frac{255\nu^3}{128}-\frac{725\nu^2}{32}+\frac{783\nu}{64}-\frac{5}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&\left.+\tilde{p}_r^6\tilde{r}^3\left(\frac{699\nu^3}{128}-\frac{217\nu^2}{16}+\frac{375\nu}{64}-\frac{19}{32}\right)\right]\\
&+\frac{\nu\tilde{p}_r\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\left[-\frac{1309\nu^2}{16}+\left(\frac{31469}{300}-\frac{141\pi^2}{8}\right)\nu+\frac{354}{25}\right.\\
&+\left(3\nu^3+\frac{1605\nu^2}{16}-\frac{181\nu}{16}-\frac{9}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{291\nu^3}{64}+\frac{17\nu^2}{2}-\frac{105\nu}{32}+\frac{1}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{27\nu^3}{2}+\frac{3409\nu^2}{16}-\frac{3631\nu}{32}-\frac{9}{4}+\left(-\frac{669\nu^3}{32}+\frac{59\nu^2}{4}-\frac{39\nu}{16}+\frac{1}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{2637\nu^3}{64}+25\nu^2-\frac{153\nu}{32}+\frac{1}{8}\right)\Bigg]\\
&+\frac{\nu\tilde{p}_r\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_2\times\tilde{\vec{L}}\cdot\vec{n}}{q\tilde{r}^6}\Bigg[-\frac{1221\nu^2}{16}-\frac{275\nu}{4}\\
&+\left(3\nu^3+\frac{677\nu^2}{16}+\frac{7\nu}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{27\nu^3}{8}+\frac{111\nu^2}{16}-\frac{21\nu}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{27\nu^3}{2}+\frac{695\nu^2}{4}+\frac{7\nu}{4}+\left(-\frac{129\nu^3}{8}+\frac{177\nu^2}{16}-\frac{21\nu}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{519\nu^3}{16}+\frac{69\nu^2}{4}-\frac{21\nu}{16}\right)\Bigg], \qquad\qquad (C.2)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{N^3LO}}_{\mathrm{S}_1^2}=\frac{\nu^2\tilde{S}_1^2}{\tilde{r}^6}&\Bigg[\left(\frac{45\pi^2}{64}-\frac{6679}{3920}\right)\nu+\frac{3987}{2450}+\Bigg(\left(\frac{29221}{3920}-\frac{135\pi^2}{64}\right)\nu\\
&+\frac{45257}{19600}\Bigg)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{9\nu^2}{16}+\frac{59\nu}{16}-\frac{5}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\Bigg(\left(\frac{45\pi^2}{16}-\frac{143221}{7840}\right)\nu-\frac{54379}{4900}+\left(\frac{53\nu^2}{32}+\frac{179\nu}{32}-\frac{5}{2}\right)\frac{\tilde{L}^2}{\tilde{r}}\\
&+\left(\frac{131\nu^2}{128}+\frac{43\nu}{8}-\frac{75}{128}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\Bigg)\\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{177\nu^2}{32}+\frac{43\nu}{32}+\frac{13}{2}+\left(\frac{185\nu^2}{64}-\frac{175\nu}{32}+\frac{3}{64}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^6\tilde{r}^3\left(-\frac{31\nu^2}{128}-\frac{47\nu}{32}+\frac{81}{128}\right)\Bigg]\\
&+\frac{\nu\tilde{S}_1^2}{q\tilde{r}^6}\Bigg[\left(\frac{45\pi^2}{64}-\frac{6679}{3920}\right)\nu^2-\frac{11\nu}{4}+9+\Bigg(\left(\frac{20751}{3920}-\frac{135\pi^2}{64}\right)\nu^2\\
&+\left(\frac{189\pi^2}{2048}-\frac{29}{2}\right)\nu+\frac{7}{16}\Bigg)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{9\nu^3}{16}+3\nu^2-\frac{55\nu}{16}+\frac{3}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\Bigg(\left(\frac{45\pi^2}{16}-\frac{132651}{7840}\right)\nu^2+\left(-\frac{1835}{32}-\frac{63\pi^2}{512}\right)\nu-1\\
&+\left(\frac{53\nu^3}{32}-\frac{35\nu^2}{32}+\frac{1077\nu}{32}+\frac{27}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{29\nu^3}{32}+\frac{67\nu^2}{16}-\frac{107\nu}{64}+\frac{75}{128}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\Bigg)\\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{177\nu^3}{32}+\frac{371\nu^2}{8}+\frac{177\nu}{32}-\frac{11}{2}+\left(\frac{79\nu^3}{32}-\frac{91\nu^2}{16}-\frac{103\nu}{16}-\frac{3}{64}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^6\tilde{r}^3\left(-\frac{25\nu^3}{32}-\frac{137\nu^2}{64}+\frac{175\nu}{64}-\frac{81}{128}\right)\Bigg]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\Bigg[\left(\frac{45\pi^2}{32}-\frac{7869}{245}\right)\nu-\frac{443089}{9800}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{29\nu^2}{16}+\frac{73\nu}{32}-\frac{145}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{29\nu^2}{16}+\frac{73\nu}{32}-\frac{145}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{115\nu^2}{16}+\frac{3971\nu}{32}+\frac{9}{8}+\left(-\frac{257\nu^2}{32}+\frac{23\nu}{32}-\frac{3}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{845\nu^2}{64}+\frac{29\nu}{4}+\frac{33}{64}\right)\Bigg]\\
&+\frac{\nu(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{q\tilde{r}^7}\Bigg[\left(\frac{45\pi^2}{32}-\frac{94789}{3920}\right)\nu^2+\left(-\frac{43}{2}-\frac{63\pi^2}{1024}\right)\nu+\frac{217}{8}\\
&+\left(\frac{29\nu^3}{16}-\frac{25\nu^2}{4}-\frac{1037\nu}{32}+\frac{121}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{13\nu^3}{8}+\frac{127\nu^2}{16}-\frac{19\nu}{8}+\frac{45}{64}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{115\nu^3}{16}+\frac{2619\nu^2}{32}-\frac{3431\nu}{32}-\frac{21}{8}+\left(-\frac{109\nu^3}{16}+\frac{403\nu^2}{32}-\frac{371\nu}{32}+\frac{3}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{173\nu^3}{16}+\frac{67\nu^2}{16}-\frac{55\nu}{32}-\frac{33}{64}\right)\Bigg]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\Bigg[\left(\frac{9627}{784}-\frac{135\pi^2}{64}\right)\nu-\frac{3058}{1225}+\Bigg(\left(\frac{315\pi^2}{64}-\frac{1197}{160}\right)\nu\\
&+\frac{26373}{1400}\Bigg)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{77\nu^2}{32}-\frac{433\nu}{16}+\frac{23}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}+\left(\frac{215\nu^2}{128}-\frac{7\nu}{4}+\frac{9}{128}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\Bigg(\left(\frac{770773}{7840}-\frac{135\pi^2}{16}\right)\nu+\frac{131497}{4900}+\left(-\frac{121\nu^2}{16}-\frac{1955\nu}{32}+\frac{115}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\\
&+\left(\frac{707\nu^2}{128}-\frac{323\nu}{32}+\frac{249}{128}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\Bigg)\\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{317\nu^2}{32}+\frac{601\nu}{32}-\frac{47}{8}+\left(\frac{823\nu^2}{128}-\frac{5\nu}{4}+\frac{159}{128}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^6\tilde{r}^3\left(\frac{31\nu^2}{128}+\frac{47\nu}{32}-\frac{81}{128}\right)\Bigg]\\
&+\frac{\nu(\tilde{\vec{S}}_1\cdot\vec{n})^2}{q\tilde{r}^6}\Bigg[\left(\frac{9627}{784}-\frac{135\pi^2}{64}\right)\nu^2-\frac{109\nu}{4}-9+\Bigg(\left(\frac{315\pi^2}{64}-\frac{1837}{560}\right)\nu^2\\
&+\left(\frac{2363}{32}-\frac{441\pi^2}{2048}\right)\nu-\frac{223}{8}\Bigg)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{77\nu^3}{32}-\frac{613\nu^2}{32}+\frac{277\nu}{16}-\frac{23}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\left(\frac{25\nu^3}{16}-\frac{23\nu^2}{4}+\frac{265\nu}{64}-\frac{9}{128}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\Bigg(\left(\frac{677323}{7840}-\frac{135\pi^2}{16}\right)\nu^2+\left(\frac{171}{32}+\frac{189\pi^2}{512}\right)\nu+1\\
&+\left(-\frac{121\nu^3}{16}-\frac{2301\nu^2}{32}+\frac{1407\nu}{32}-\frac{59}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{167\nu^3}{32}-\frac{607\nu^2}{32}+\frac{199\nu}{64}-\frac{249}{128}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{317\nu^3}{32}-24\nu^2-\frac{305\nu}{32}+\frac{11}{2}+\left(\frac{101\nu^3}{16}-\frac{139\nu^2}{64}+\frac{719\nu}{64}-\frac{159}{128}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^6\tilde{r}^3\left(\frac{25\nu^3}{32}+\frac{137\nu^2}{64}-\frac{175\nu}{64}+\frac{81}{128}\right)\Bigg]\\
&+\frac{\nu^2\tilde{p}_r\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\Bigg[\left(\frac{45\pi^2}{4}-\frac{91071}{1960}\right)\nu-\frac{812367}{9800}\\
&+\left(\frac{7\nu^2}{2}-\frac{373\nu}{16}-\frac{19}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{173\nu^2}{64}-\frac{29\nu}{8}+\frac{33}{64}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{43\nu^2}{4}+\frac{531\nu}{8}-\frac{297}{16}+\left(-\frac{151\nu^2}{16}+\frac{163\nu}{16}-\frac{45}{32}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{581\nu^2}{64}+\frac{131\nu}{16}-\frac{123}{64}\right)\Bigg]\\
&+\frac{\nu\tilde{p}_r\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{q\tilde{r}^6}\Bigg[\left(\frac{45\pi^2}{4}-\frac{193027}{3920}\right)\nu^2+\left(\frac{519}{16}-\frac{63\pi^2}{128}\right)\nu+\frac{295}{8}\\
&+\left(\frac{7\nu^3}{2}-\frac{3\nu^2}{2}-\frac{413\nu}{16}-\frac{5}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{79\nu^3}{32}+\frac{25\nu^2}{16}-\frac{79\nu}{32}-\frac{33}{64}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{43\nu^3}{4}+\frac{249\nu^2}{8}-97\nu+\frac{273}{16}+\left(-\frac{275\nu^3}{32}+\frac{655\nu^2}{32}+5\nu+\frac{45}{32}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(-8\nu^3+10\nu^2-\frac{241\nu}{32}+\frac{123}{64}\right)\Bigg],
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
\tilde{H}^{\mathrm{N^3LO}}_{C_{1\mathrm{ES}^2}}=&\frac{\nu^2\tilde{S}_1^2}{\tilde{r}^6}\Bigg[\frac{177\nu}{7}-\frac{14894}{1225}+\left(-\frac{1399\nu}{56}+\frac{244353}{4900}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{9\nu^2}{16}-\frac{115\nu}{16}+\frac{41}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(\frac{1445\nu}{14}-\frac{41779}{4900}+\left(-\frac{15\nu^2}{8}-\frac{35\nu}{2}+\frac{31}{2}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{9\nu^2}{16}-\frac{23\nu}{16}+\frac{9}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{77\nu^2}{16}-\frac{269\nu}{16}+\frac{243}{16}+\left(-\frac{45\nu^2}{16}-\frac{23\nu}{8}+\frac{9}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^6\tilde{r}^3\left(-6\nu^2-\frac{\nu}{2}+\frac{9}{8}\right)\Bigg]\\
&+\frac{\nu\tilde{S}_1^2}{q\tilde{r}^6}\Bigg[\frac{177\nu^2}{7}+\left(\frac{1861}{24}-\frac{21\pi^2}{4}\right)\nu+\frac{73}{4}+\Bigg(-\frac{1625\nu^2}{112}\\
&+\left(5+\frac{8109\pi^2}{2048}\right)\nu-\frac{505}{8}\Bigg)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{9\nu^3}{16}-\frac{41\nu^2}{8}+\frac{121\nu}{4}-\frac{9}{2}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\left(-\frac{5\nu^3}{32}-\frac{35\nu^2}{32}+\frac{17\nu}{16}-\frac{7}{32}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\Bigg(\frac{3771\nu^2}{28}+\left(\frac{861}{8}-\frac{2703\pi^2}{512}\right)\nu+\frac{123}{8}\\
&+\left(-\frac{15\nu^3}{8}+\frac{187\nu^2}{16}+\frac{1365\nu}{16}-19\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{15\nu^3}{16}-\frac{153\nu^2}{32}+\frac{53\nu}{8}-\frac{57}{32}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\tilde{p}_r^4\tilde{r}^2\left(-\frac{77\nu^3}{16}-\frac{193\nu^2}{4}+\frac{173\nu}{8}-\frac{67}{4}+\left(-3\nu^3-\frac{69\nu^2}{16}+\frac{161\nu}{16}-\frac{93}{32}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^6\tilde{r}^3\left(-8\nu^3+5\nu^2+\frac{57\nu}{16}-\frac{43}{32}\right)\Bigg]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{\tilde{r}^7}\Bigg[\frac{419\nu}{28}-\frac{95527}{4900}+\left(-\frac{3\nu^2}{4}-\frac{27\nu}{4}-\frac{15}{2}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{15\nu^2}{16}+\frac{\nu}{4}-\frac{9}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(-\frac{11\nu^2}{4}+46\nu+\frac{3}{4}+\left(\frac{63\nu^2}{16}+\frac{23\nu}{16}-\frac{9}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{93\nu^2}{16}+\frac{17\nu}{8}-\frac{9}{16}\right)\Bigg]\\
&+\frac{\nu(\tilde{\vec{S}}_1\cdot\tilde{\vec{L}})^2}{q\tilde{r}^7}\Bigg[\frac{97\nu^2}{28}+\left(-\frac{1059}{16}-\frac{2703\pi^2}{1024}\right)\nu+\frac{97}{4}\\
&+\left(-\frac{3\nu^3}{4}-\frac{29\nu^2}{8}-\frac{427\nu}{16}+\frac{31}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{15\nu^3}{16}-\frac{21\nu^2}{16}-\frac{7\nu}{8}+\frac{9}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\tilde{p}_r^2\tilde{r}\left(-\frac{11\nu^3}{4}+\frac{1637\nu^2}{16}+12\nu-\frac{1}{2}+\left(\frac{57\nu^3}{16}-\frac{63\nu^2}{16}-\frac{43\nu}{16}+\frac{9}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{9\nu^3}{2}-\frac{57\nu^2}{16}-\frac{11\nu}{4}+\frac{9}{16}\right)\Bigg]\\
&+\frac{\nu^2(\tilde{\vec{S}}_1\cdot\vec{n})^2}{\tilde{r}^6}\Bigg[\frac{15387}{1225}-\frac{286\nu}{7}+\left(\frac{3205\nu}{56}-\frac{24393}{350}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{9\nu^2}{16}-\frac{81\nu}{16}-\frac{169}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\left(\frac{15\nu^2}{16}+\frac{\nu}{4}-\frac{9}{16}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\left(-\frac{2823\nu}{14}-\frac{242933}{4900}+\left(\frac{21\nu^2}{8}+\frac{333\nu}{4}-\frac{39}{2}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{9\nu^2}{2}+\frac{23\nu}{8}-\frac{9}{4}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\right)\\
&+\tilde{p}_r^4\tilde{r}^2\left(\frac{217\nu^2}{16}+\frac{1129\nu}{16}-\frac{311}{16}+\left(\frac{69\nu^2}{8}+5\nu-\frac{45}{16}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)+\tilde{p}_r^6\tilde{r}^3\left(6\nu^2+\frac{\nu}{2}-\frac{9}{8}\right)\Bigg]\\
&+\frac{\nu(\tilde{\vec{S}}_1\cdot\vec{n})^2}{q\tilde{r}^6}\Bigg[-\frac{286\nu^2}{7}+\left(\frac{63\pi^2}{4}-\frac{1641}{8}\right)\nu-\frac{119}{4}+\left(\frac{629\nu^2}{16}\right.\\
&\left.+\left(-\frac{289}{16}-\frac{18921\pi^2}{2048}\right)\nu+\frac{829}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(\frac{9\nu^3}{16}-\frac{53\nu^2}{4}-\frac{1167\nu}{16}+\frac{129}{8}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\\
&+\left(\frac{45\nu^3}{32}+\frac{63\nu^2}{32}-\frac{65\nu}{16}+\frac{39}{32}\right)\frac{\tilde{L}^6}{\tilde{r}^3}\\
&+\tilde{p}_r^2\tilde{r}\left(-\frac{6161\nu^2}{28}+\left(\frac{8109\pi^2}{512}-\frac{1107}{8}\right)\nu+\frac{267}{8}\right.\\
&\left.+\left(\frac{21\nu^3}{8}+\frac{585\nu^2}{8}-\frac{1987\nu}{16}+\frac{119}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(6\nu^3+\frac{297\nu^2}{32}-\frac{275\nu}{16}+\frac{135}{32}\right)\frac{\tilde{L}^4}{\tilde{r}^2}\right)
\end{aligned}
$$

$$+\tilde{p}_r^4\tilde{r}^2\left(\frac{217\nu^3}{16}+\frac{115\nu^2}{2}-\frac{297\nu}{4}+\frac{193}{8}+\left(\frac{21\nu^3}{2}+12\nu^2-\frac{355\nu}{16}+\frac{153}{32}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)$$
$$+\tilde{p}_r^6\tilde{r}^3\left(12\nu^3-\frac{115\nu}{16}+\frac{57}{32}\right)\Bigg]$$
$$+\frac{\nu^2\tilde{p}_r\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{\tilde{r}^6}\Bigg[\frac{91\nu}{2}-\frac{428881}{4900}$$
$$+\left(-\frac{25\nu^2}{16}-\frac{493\nu}{16}-\frac{31}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{3\nu^2}{8}+\frac{19\nu}{16}-\frac{9}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}$$
$$+\tilde{p}_r^2\tilde{r}\left(-\frac{129\nu^2}{16}-\frac{105\nu}{16}-\frac{73}{4}+\left(-\frac{9\nu^2}{8}+\frac{23\nu}{16}-\frac{9}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)$$
$$+\tilde{p}_r^4\tilde{r}^2\left(\frac{3\nu^2}{16}-\frac{13\nu}{8}-\frac{9}{16}\right)\Bigg]$$
$$+\frac{\nu\tilde{p}_r\tilde{\vec{S}}_1\cdot\vec{n}\tilde{\vec{S}}_1\times\tilde{\vec{L}}\cdot\vec{n}}{q\tilde{r}^6}\Bigg[-\frac{743\nu^2}{28}+\left(-\frac{2249}{16}-\frac{2703\pi^2}{128}\right)\nu+\frac{497}{4}$$
$$+\left(-\frac{25\nu^3}{16}-29\nu^2-\frac{1043\nu}{16}+\frac{35}{4}\right)\frac{\tilde{L}^2}{\tilde{r}}+\left(-\frac{21\nu^3}{16}-\frac{5\nu}{16}+\frac{9}{16}\right)\frac{\tilde{L}^4}{\tilde{r}^2}$$
$$+\tilde{p}_r^2\tilde{r}\left(-\frac{129\nu^3}{16}+\frac{1411\nu^2}{16}-\frac{1331\nu}{16}+\frac{77}{4}+\left(-\frac{81\nu^3}{16}-3\nu^2+\frac{5\nu}{16}+\frac{9}{8}\right)\frac{\tilde{L}^2}{\tilde{r}}\right)$$
$$+\tilde{p}_r^4\tilde{r}^2\left(-\frac{15\nu^3}{2}-\frac{123\nu^2}{16}+\frac{5\nu}{2}+\frac{9}{16}\right)\Bigg], \tag{C.4}$$

$$\tilde{H}^{\mathrm{N^3LO}}_{C_{1\mathrm{E}^2\mathrm{S}^2}}=-\frac{\nu^3}{2\tilde{r}^6}\left(1+\frac{1}{q}\right)\left(\tilde{S}_1^2-3(\tilde{\vec{S}}_1\cdot\vec{n})^2\right). \tag{C.5}$$

References

[1] LIGO Scientific and Virgo collaborations, *Observation of Gravitational Waves from a Binary Black Hole Merger*, Phys. Rev. Lett. **116** (2016) 061102 [arXiv:1602.03837] [INSPIRE].

[2] LIGO Scientific collaboration, *Advanced LIGO*, Class. Quant. Grav. **32** (2015) 074001 [arXiv:1411.4547] [INSPIRE].

[3] VIRGO collaboration, *Advanced Virgo: a second-generation interferometric gravitational wave detector*, Class. Quant. Grav. **32** (2015) 024001 [arXiv:1408.3978] [INSPIRE].

[4] L. Blanchet, *Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries*, Living Rev. Rel. **17** (2014) 2 [arXiv:1310.1528] [INSPIRE].

JHEP03(2023)098

[5] A. Buonanno and T. Damour, *Effective one-body approach to general relativistic two-body dynamics*, *Phys. Rev. D* **59** (1999) 084006 [gr-qc/9811091] [INSPIRE].

[6] LIGO Scientific and Virgo collaborations, *GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs*, *Phys. Rev. X* **9** (2019) 031040 [arXiv:1811.12907] [INSPIRE].

[7] LIGO Scientific and Virgo collaborations, *GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run*, *Phys. Rev. X* **11** (2021) 021053 [arXiv:2010.14527] [INSPIRE].

[8] LIGO Scientific, Virgo and KAGRA collaborations, *GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo During the Second Part of the Third Observing Run*, arXiv:2111.03606 [INSPIRE].

[9] LIGO Scientific and Virgo collaborations, *Tests of general relativity with GW150914*, *Phys. Rev. Lett.* **116** (2016) 221101 [arXiv:1602.03841] [*Erratum ibid.* **121** (2018) 129902] [INSPIRE].

[10] LIGO Scientific and Virgo collaborations, *Properties of the Binary Black Hole Merger GW150914*, *Phys. Rev. Lett.* **116** (2016) 241102 [arXiv:1602.03840] [INSPIRE].

[11] LIGO Scientific and Virgo collaborations, *Improved analysis of GW150914 using a fully spin-precessing waveform Model*, *Phys. Rev. X* **6** (2016) 041014 [arXiv:1606.01210] [INSPIRE].

[12] LIGO Scientific and Virgo collaborations, *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*, *Phys. Rev. Lett.* **119** (2017) 161101 [arXiv:1710.05832] [INSPIRE].

[13] LIGO Scientific, KAGRA and Virgo collaborations, *Observation of Gravitational Waves from Two Neutron Star-Black Hole Coalescences*, *Astrophys. J. Lett.* **915** (2021) L5 [arXiv:2106.15163] [INSPIRE].

[14] W.D. Goldberger and I.Z. Rothstein, *An Effective field theory of gravity for extended objects*, *Phys. Rev. D* **73** (2006) 104029 [hep-th/0409156] [INSPIRE].

[15] M. Levi, A.J. Mcleod and M. Von Hippel, *N^3LO gravitational quadratic-in-spin interactions at G^4*, *JHEP* **07** (2021) 116 [arXiv:2003.07890] [INSPIRE].

[16] D. Bini, T. Damour and A. Geralico, *Novel approach to binary dynamics: application to the fifth post-Newtonian level*, *Phys. Rev. Lett.* **123** (2019) 231104 [arXiv:1909.02375] [INSPIRE].

[17] D. Bini, T. Damour and A. Geralico, *Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders*, *Phys. Rev. D* **102** (2020) 024062 [arXiv:2003.11891] [INSPIRE].

[18] D. Bini, T. Damour, A. Geralico, S. Laporta and P. Mastrolia, *Gravitational dynamics at $O(G^6)$: perturbative gravitational scattering meets experimental mathematics*, arXiv:2008.09389 [INSPIRE].

[19] J. Blümlein, A. Maier, P. Marquard and G. Schäfer, *The fifth-order post-Newtonian Hamiltonian dynamics of two-body systems from an effective field theory approach: potential contributions*, *Nucl. Phys. B* **965** (2021) 115352 [arXiv:2010.13672] [INSPIRE].

[20] A. Antonelli, C. Kavanagh, M. Khalil, J. Steinhoff and J. Vines, *Gravitational spin-orbit coupling through third-subleading post-Newtonian order: from first-order self-force to arbitrary mass ratios*, *Phys. Rev. Lett.* **125** (2020) 011103 [arXiv:2003.11391] [INSPIRE].

JHEP03(2023)098

[21] A. Antonelli, C. Kavanagh, M. Khalil, J. Steinhoff and J. Vines, *Gravitational spin-orbit and aligned spin$_1$-spin$_2$ couplings through third-subleading post-Newtonian orders*, *Phys. Rev. D* **102** (2020) 124024 [arXiv:2010.02018] [INSPIRE].

[22] M. Levi, A.J. Mcleod and M. Von Hippel, *N^3LO gravitational spin-orbit coupling at order G^4*, *JHEP* **07** (2021) 115 [arXiv:2003.02827] [INSPIRE].

[23] J.-W. Kim, M. Levi and Z. Yin, *N^3LO Spin-Orbit Interaction via the EFT of Spinning Gravitating Objects*, arXiv:2208.14949 [INSPIRE].

[24] M. Levi, R. Morales and Z. Yin, *From the EFT of Spinning Gravitating Objects to Poincaré and Gauge Invariance*, arXiv:2210.17538 [INSPIRE].

[25] M.K. Mandal, P. Mastrolia, R. Patil and J. Steinhoff, *Gravitational Spin-Orbit Hamiltonian at NNNLO in the post-Newtonian framework*, arXiv:2209.00611 [INSPIRE].

[26] M. Levi and J. Steinhoff, *Spinning gravitating objects in the effective field theory in the post-Newtonian scheme*, *JHEP* **09** (2015) 219 [arXiv:1501.04956] [INSPIRE].

[27] M. Levi and J. Steinhoff, *EFTofPNG: A package for high precision computation with the Effective Field Theory of Post-Newtonian Gravity*, *Class. Quant. Grav.* **34** (2017) 244001 [arXiv:1705.06309] [INSPIRE].

[28] B.M. Barker and R.F. O'Connell, *Gravitational Two-Body Problem with Arbitrary Masses, Spins, and Quadrupole Moments*, *Phys. Rev. D* **12** (1975) 329 [INSPIRE].

[29] M. Levi and J. Steinhoff, *Leading order finite size effects with spins for inspiralling compact binaries*, *JHEP* **06** (2015) 059 [arXiv:1410.2601] [INSPIRE].

[30] M. Levi, *Binary dynamics from spin1-spin2 coupling at fourth post-Newtonian order*, *Phys. Rev. D* **85** (2012) 064043 [arXiv:1107.4322] [INSPIRE].

[31] M. Levi and J. Steinhoff, *Next-to-next-to-leading order gravitational spin-orbit coupling via the effective field theory for spinning objects in the post-Newtonian scheme*, *JCAP* **01** (2016) 011 [arXiv:1506.05056] [INSPIRE].

[32] M. Levi and J. Steinhoff, *Next-to-next-to-leading order gravitational spin-squared potential via the effective field theory for spinning objects in the post-Newtonian scheme*, *JCAP* **01** (2016) 008 [arXiv:1506.05794] [INSPIRE].

[33] M. Levi and J. Steinhoff, *Complete conservative dynamics for inspiralling compact binaries with spins at the fourth post-Newtonian order*, *JCAP* **09** (2021) 029 [arXiv:1607.04252] [INSPIRE].

[34] T. Hinderer, *Tidal Love numbers of neutron stars*, *Astrophys. J.* **677** (2008) 1216 [arXiv:0711.2420] [INSPIRE].

[35] T. Damour and A. Nagar, *Relativistic tidal properties of neutron stars*, *Phys. Rev. D* **80** (2009) 084035 [arXiv:0906.0096] [INSPIRE].

[36] T. Binnington and E. Poisson, *Relativistic theory of tidal Love numbers*, *Phys. Rev. D* **80** (2009) 084018 [arXiv:0906.1366] [INSPIRE].

[37] B. Kol and M. Smolkin, *Black hole stereotyping: Induced gravito-static polarization*, *JHEP* **02** (2012) 010 [arXiv:1110.3764] [INSPIRE].

[38] A. Le Tiec and M. Casals, *Spinning Black Holes Fall in Love*, *Phys. Rev. Lett.* **126** (2021) 131102 [arXiv:2007.00214] [INSPIRE].

[39] A. Le Tiec, M. Casals and E. Franzin, *Tidal Love Numbers of Kerr Black Holes*, *Phys. Rev. D* **103** (2021) 084021 [arXiv:2010.15795] [INSPIRE].

[40] H.S. Chia, *Tidal deformation and dissipation of rotating black holes*, *Phys. Rev. D* **104** (2021) 024013 [arXiv:2010.07300] [INSPIRE].

[41] P. Charalambous, S. Dubovsky and M.M. Ivanov, *On the Vanishing of Love Numbers for Kerr Black Holes*, *JHEP* **05** (2021) 038 [arXiv:2102.08917] [INSPIRE].

[42] P. Charalambous, S. Dubovsky and M.M. Ivanov, *Hidden Symmetry of Vanishing Love Numbers*, *Phys. Rev. Lett.* **127** (2021) 101101 [arXiv:2103.01234] [INSPIRE].

[43] G. Castro, L. Gualtieri and P. Pani, *Hidden symmetry between rotational tidal Love numbers of spinning neutron stars*, *Phys. Rev. D* **104** (2021) 044052 [arXiv:2103.16595] [INSPIRE].

[44] L. Blanchet and G. Faye, *General relativistic dynamics of compact binaries at the third postNewtonian order*, *Phys. Rev. D* **63** (2001) 062005 [gr-qc/0007051] [INSPIRE].

[45] J.-W. Kim, M. Levi and Z. Yin, *Quadratic-in-spin interactions at fifth post-Newtonian order probe new physics*, *Phys. Lett. B* **834** (2022) 137410 [arXiv:2112.01509] [INSPIRE].

[46] M. Levi, *Effective Field Theories of Post-Newtonian Gravity: A comprehensive review*, *Rept. Prog. Phys.* **83** (2020) 075901 [arXiv:1807.01699] [INSPIRE].

[47] M. Levi and J. Steinhoff, *Equivalence of ADM Hamiltonian and Effective Field Theory approaches at next-to-next-to-leading order spin1-spin2 coupling of binary inspirals*, *JCAP* **12** (2014) 003 [arXiv:1408.5762] [INSPIRE].

[48] M. Levi, S. Mougiakakos and M. Vieira, *Gravitational cubic-in-spin interaction at the next-to-leading post-Newtonian order*, *JHEP* **01** (2021) 036 [arXiv:1912.06276] [INSPIRE].

[49] M. Levi and F. Teng, *NLO gravitational quartic-in-spin interaction*, *JHEP* **01** (2021) 066 [arXiv:2008.12280] [INSPIRE].

[50] M.E. Peskin and D.V. Schroeder, *An Introduction to quantum field theory*, Addison-Wesley, Reading, U.S.A. (1995).

[51] B. Kol and M. Smolkin, *Non-Relativistic Gravitation: From Newton to Einstein and Back*, *Class. Quant. Grav.* **25** (2008) 145011 [arXiv:0712.4116] [INSPIRE].

[52] B. Kol, M. Levi and M. Smolkin, *Comparing space+time decompositions in the post-Newtonian limit*, *Class. Quant. Grav.* **28** (2011) 145021 [arXiv:1011.6024] [INSPIRE].

[53] M. Levi, *Next to Leading Order gravitational Spin1-Spin2 coupling with Kaluza-Klein reduction*, *Phys. Rev. D* **82** (2010) 064029 [arXiv:0802.1508] [INSPIRE].

[54] M. Levi, *Next to Leading Order gravitational Spin-Orbit coupling in an Effective Field Theory approach*, *Phys. Rev. D* **82** (2010) 104004 [arXiv:1006.4139] [INSPIRE].

[55] A.J. Hanson and T. Regge, *The Relativistic Spherical Top*, *Annals Phys.* **87** (1974) 498 [INSPIRE].

[56] I. Bailey and W. Israel, *Lagrangian Dynamics of Spinning Particles and Polarized Media in General Relativity*, *Commun. Math. Phys.* **42** (1975) 65 [INSPIRE].

[57] K. Yee and M. Bander, *Equations of motion for spinning particles in external electromagnetic and gravitational fields*, *Phys. Rev. D* **48** (1993) 2797 [hep-th/9302117] [INSPIRE].

[58] R.A. Porto, *Post-Newtonian corrections to the motion of spinning bodies in NRGR*, *Phys. Rev. D* **73** (2006) 104031 [gr-qc/0511061] [INSPIRE].

[59] M. Levi and Z. Yin, *Completing the Fifth PN Precision Frontier via the EFT of Spinning Gravitating Objects*, arXiv:2211.14018 [INSPIRE].

[60] J.A.M. Vermaseren, *Axodraw*, Comput. Phys. Commun. **83** (1994) 45 [INSPIRE].

[61] D. Binosi and L. Theussl, *JaxoDraw: A Graphical user interface for drawing Feynman diagrams*, Comput. Phys. Commun. **161** (2004) 76 [hep-ph/0309015] [INSPIRE].

[62] D. Binosi, J. Collins, C. Kaufhold and L. Theussl, *JaxoDraw: A Graphical user interface for drawing Feynman diagrams. Version 2.0 release notes*, Comput. Phys. Commun. **180** (2009) 1709 [arXiv:0811.4113] [INSPIRE].

[63] V.A. Smirnov, *Feynman integral calculus*, Springer, Berlin, Germany (2006).

[64] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, Int. J. Mod. Phys. A **15** (2000) 5087 [hep-ph/0102033] [INSPIRE].

[65] R.H. Boels, Q. Jin and H. Luo, *Efficient integrand reduction for particles with spin*, arXiv:1802.06761 [INSPIRE].

[66] L. Chen, *A prescription for projectors to compute helicity amplitudes in D dimensions*, Eur. Phys. J. C **81** (2021) 417 [arXiv:1904.00705] [INSPIRE].

[67] L. Barack and A. Pound, *Self-force and radiation reaction in general relativity*, Rept. Prog. Phys. **82** (2019) 016904 [arXiv:1805.10385] [INSPIRE].

[68] A. Edison and M. Levi, *A tale of tails through generalized unitarity*, Phys. Lett. B **837** (2023) 137634 [arXiv:2202.04674] [INSPIRE].

[69] T. Damour and G. Schaefer, *Higher Order Relativistic Periastron Advances and Binary Pulsars*, Nuovo Cim. B **101** (1988) 127 [INSPIRE].

[70] D. Kosmopoulos and A. Luna, *Quadratic-in-spin Hamiltonian at $\mathcal{O}(G^2)$ from scattering amplitudes*, JHEP **07** (2021) 037 [arXiv:2102.10137] [INSPIRE].

[71] Z. Liu, R.A. Porto and Z. Yang, *Spin Effects in the Effective Field Theory Approach to Post-Minkowskian Conservative Dynamics*, JHEP **06** (2021) 012 [arXiv:2102.10059] [INSPIRE].

[72] G.U. Jakobsen and G. Mogull, *Conservative and Radiative Dynamics of Spinning Bodies at Third Post-Minkowskian Order Using Worldline Quantum Field Theory*, Phys. Rev. Lett. **128** (2022) 141102 [arXiv:2201.07778] [INSPIRE].

[73] M. Levi, *What is love? New physics at the state of the art with spins*, Kavli Institute for Theoretical Physics Audiovisual, https://doi.org/10.26081/K6N349 (2022).