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A note on the asymptotic symmetries of electromagnetism

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Abstract: We extend the asymptotic symmetries of electromagnetism in order to consistently include angle-dependent $u(1)$ gauge transformations ϵ that involve terms growing at spatial infinity linearly and logarithmically in r, $\epsilon \sim a(\theta, \varphi) r + b(\theta, \varphi) \ln r + c(\theta, \varphi)$. The charges of the logarithmic $u(1)$ transformations are found to be conjugate to those of the $\mathcal{O}(1)$ transformations (abelian algebra with invertible central term) while those of the $\mathcal{O}(r)$ transformations are conjugate to those of the subleading $\mathcal{O}(r^{-1})$ transformations. Because of this structure, one can decouple the angle-dependent $u(1)$ asymptotic symmetry from the Poincaré algebra, just as in the case of gravity: the generators of these internal transformations are Lorentz scalars in the redefined algebra. This implies in particular that one can give a definition of the angular momentum which is free from $u(1)$ gauge ambiguities. The change of generators that brings the asymptotic symmetry algebra to a direct sum form involves non linear redefinitions of the charges. Our analysis is Hamiltonian throughout and carried at spatial infinity.

Keywords: Gauge Symmetry, Global Symmetries, Space-Time Symmetries

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Contents

1 Introduction

In a recent paper [1], we extended the Hamiltonian formulation of Einstein theory in the asymptotically flat context by allowing more flexible boundary conditions that involved logarithmic terms. This generalization led to an enlargement of the asymptotic symmetry, from the original BMS_4 algebra [2–4] to the log-BMS_4 algebra, which contains, besides the familiar angle-dependent supertranslations, angle-dependent logarithmic supertranslations [5–11]. This extension was carried out at spatial infinity following the approach developed in [12, 13].

The presence in the asymptotic symmetry algebra of logarithmic supertranslations had a dramatic impact, in that it enabled one to completely disentangle the Poincaré subalgebra from the supertranslations (ordinary and logarithmic), realizing at spatial infinity a mechanism similar to the one described at null infinity in [14–17] (see also [18–20] in that context).

The purpose of this note is to carry out the analogous construction in the electromagnetic case, where angle-dependent $u(1)$ gauge transformations play the role of the supertranslations [21–25]. We consistently enlarge the boundary conditions of [26] in such a way that the asymptotic symmetries contain angle-dependent gauge transformations that grow at spatial infinity not only logarithmically in r, but also linearly in r,

$$\epsilon \sim a(\theta,\varphi)r + b(\theta,\varphi)\ln r + c(\theta,\varphi) + o(1).$$

The term linear in r is included because it is associated with the subleading soft theorems [27–29].

In the standard approach where ϵ is restricted to take the form $\epsilon \sim c(\theta,\varphi) + o(1)$, the angle-dependent $u(1)$ asymptotic transformations of order one transform in a non-trivial representation of the Poincaré algebra [26, 30]. As in the case of gravity, we show that the enlargement of the symmetry enables one to disentangle the internal asymptotic angle-dependent $u(1)$ symmetries from the Poincaré algebra: the asymptotic symmetry algebra is the direct sum of the two. In retrospect, this result is perhaps not too surprising as it is in the line of the Coleman-Mandula theorem [31] (even though the hypotheses of this theorem are not all fulfilled). The fact that the internal $u(1)$ improper gauge symmetries commute with the Lorentz transformations leads to an angular momentum that is free from ambiguities under asymptotic angle-dependent $u(1)$ transformations.

The change of generators that brings the asymptotic symmetry algebra to a direct sum form involves redefinitions of the Poincaré generators by the addition of field-dependent gauge transformations corresponding to a specific non linear redefinitions of the charges.

Our paper is organized as follows. In section 2, we give the form of the new, more flexible, boundary conditions and verify the finiteness of the action. We then describe the asymptotic gauge symmetries in section 3. Poincaré invariance is established in section 4 and further discussed in appendix A. The algebra of the asymptotic symmetries and the redefinitions that bring it to a direct sum form are successively analysed in sections 5 and 6. In the concluding section 7, we outline some potential future developments.

2 Action and asymptotic conditions

2.1 Extended Hamiltonian action

In the formulation where the conjugate momentum to A_0 is kept, the phase space of the Maxwell theory is spanned by the components (A_i, A_0) of the vector potential and their conjugate momenta (π^i, π^0). These are subject to the "primary constraint"

$$\pi^0 \approx 0\,, \tag{2.1}$$

and Gauss's law, which arises as a "secondary constraint",

$$\mathcal{G} = -\partial_i \pi^i \approx 0\,. \tag{2.2}$$

It is customary to eliminate π^0 to obtain a reduced theory where the canonical variables are (A_i, π^i), the temporal component A_0 appearing then as a Lagrange multiplier for the ("secondary") constraint $\mathcal{G} \approx 0$. We shall refrain from doing so here because A_0 does carry degrees of freedom at infinity when one imposes boundary conditions that are invariant under an angle-dependent $u(1)$ symmetry [12, 26]. It is in that case useful to keep its conjugate momentum in the Hamiltonian description.

The extended Hamiltonian action of Maxwell theory is given by

$$I_H[A_i, \pi^i, A_0, \pi^0; \psi, \lambda] = \int dt \int d^3x \Big[\pi^i \dot{A}_i + \pi^0 \dot{A}_0 - (\mathcal{H} + \tilde{\epsilon}\mathcal{G} + \tilde{\mu}\pi^0) - \psi\mathcal{G} - \lambda\pi^0\Big] + \mathfrak{B}, \quad (2.3)$$

$$\mathcal{H} = \frac{1}{2\sqrt{g}}\pi^i\pi_i + \frac{\sqrt{g}}{4}F^{ij}F_{ij} + A_0\mathcal{G} - \partial^i\pi^0 A_i\,, \quad (2.4)$$

where ψ and λ are the respective Lagrange multipliers for the constraints (2.2) and (2.1) and where the Hamiltonian contains constraint terms besides the usual energy density $\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$, which we have included following [12, 26] for later convenience. We have set

$$\tilde{\epsilon} = \frac{\ln r}{r}\tilde{\epsilon}^{(1)}_{\log} + \frac{1}{r}\tilde{\epsilon}^{(1)} + o\left(r^{-1}\right)\,, \quad (2.5)$$

$$\tilde{\mu} = \frac{\ln r}{r^2}\tilde{\mu}^{(1)}_{\log} + \frac{1}{r^2}\tilde{\mu}^{(1)} + o\left(r^{-2}\right)\,, \quad (2.6)$$

with

$$\tilde{\mu}^{(1)}_{\log} - \tilde{\mu}^{(1)} = \overline{D}_A\overline{A}^A + 3\overline{A}_r\,, \quad (2.7)$$

$$\tilde{\epsilon}^{(1)}_{\log} - \tilde{\epsilon}^{(1)} = \overline{\Psi}\,. \quad (2.8)$$

Here, the functions of the angles $\overline{A}^A$, $\overline{A}_r$ and $\overline{\Psi}$ are the $\mathcal{O}(1)$ coefficients appearing in the asymptotic expansion of the vector potential A_μ, see (2.15)–(2.18) below. The functions $\tilde{\mu}^{(1)}_{\log}$, $\tilde{\mu}^{(1)}$, $\tilde{\epsilon}^{(1)}_{\log}$ and $\tilde{\epsilon}^{(1)}$ are not completely determined by the equations (2.7) and (2.8) but only the combinations $\tilde{\mu}^{(1)}_{\log} - \tilde{\mu}^{(1)}$ and $\tilde{\epsilon}^{(1)}_{\log} - \tilde{\epsilon}^{(1)}$ are physically relevant (see below). With the inclusion of the constraint terms, the Hamiltonian density $\mathcal{H} + \tilde{\epsilon}\mathcal{G} + \tilde{\mu}\pi^0$ coincides with the density of the Poincaré generator of time translations discussed below. Finally, the term $\mathfrak{B}$ is the integral over time of a surface term which we will write explicitly once we have given the boundary conditions.

The constraint function $\mathcal{G}$ generates the gauge transformation

$$\delta A_i = \partial_i\epsilon\,, \quad (2.9)$$

while the other constraint function π^0 generates the gauge transformations

$$\delta A_0 = \mu\,. \quad (2.10)$$

These are proper in the sense of [32] if ϵ and μ decrease sufficiently fast at infinity, i.e.

$$\epsilon = o\left(r^{-1}\right)\,, \qquad \mu = o\left(r^{-2}\right)\,, \quad (2.11)$$

as we shall explicitly show in section 3 below. We will assume that these fall-off conditions are fulfilled until we discuss improper gauge transformations, which can only be meaningfully done after the boundary conditions have been made precise, a task which we have not achieved yet.

Since the Lagrange multipliers parametrize the gauge transformation performed in the course of the evolution (on top of the evolution generated by the Hamiltonian), we take for them the same asymptotic decay

$$\psi = o\left(r^{-1}\right) , \qquad \lambda = o\left(r^{-2}\right) . \tag{2.12}$$

Again, more general asymptotic behaviours can be considered once one has full control of the improper gauge symmetries.

The action is invariant under the gauge transformations (2.9) and (2.10) provided one transforms at the same time the Lagrange multipliers as

$$\delta\psi = \dot{\epsilon} - \mu - \delta\tilde{\epsilon} , \qquad \delta\lambda = \dot{\mu} - \triangle\epsilon - \delta\tilde{\mu} \tag{2.13}$$

($\delta\tilde{\epsilon}$ and $\delta\tilde{\mu}$ preserve the asymptotic decays of ψ and λ because $\delta\tilde{\mu}^{(1)}_{\log} = \delta\tilde{\mu}^{(1)} = \delta\tilde{\epsilon}^{(1)}_{\log} = \delta\tilde{\epsilon}^{(1)} = 0$). The action (2.3) is called the "extended action" because the gauge parameters ϵ, μ of the constraints are taken to be independent [33, 34]. This is the form that exhibits most explicitly the symmetry. The formulation that emerges from the Maxwell Lagrangian is characterized by $\psi = -\tilde{\epsilon}$ (a condition that is permissible once one allows for more general asymptotic behaviours of the Lagrange multipliers), which relates ϵ and μ through $\mu = \dot{\epsilon}$ so that $\delta A_\mu = \partial_\mu \epsilon$. The two formulations are physically equivalent because $\dot{\epsilon}$ and ϵ are independent at any given time [33, 34].

2.2 Asymptotic conditions

The gauge transformations (2.9), (2.10) and (2.11) are abelian so that their finite form coincides with their infinitesimal ones. The boundary conditions on the canonical variables are taken to differ from those of [12, 26] by gauge transformation terms $\Delta A_i = \partial_i \Theta$, $\Delta A_0 = \Xi$ with finite gauge parameter (Θ, Ξ) that contain contributions that are of order $\mathcal{O}(r)$ and $\mathcal{O}(\ln r)$ with respect to the leading orders present in [12, 26], i.e.

$$\Theta = r\Phi_{\text{lin}} + \ln r\,\Phi_{\log} + \Phi + \frac{\ln r}{r}\Phi^{(1)}_{\log} + o\left(\frac{\ln r}{r}\right) ,$$
$$\Xi = \Psi_{\text{lin}} + \frac{\ln r}{r}\Psi_{\log} + \frac{1}{r}\overline{\Psi} + \frac{\ln r}{r^2}\Psi^{(1)}_{\log} + o\left(\frac{\ln r}{r^2}\right) .$$

The terms Φ_{lin}, $\Phi_{\log}$, $\Phi^{(1)}_{\log}$, Ψ_{lin}, $\Psi^{(1)}_{\log}$ and $\Psi_{\log}$ are absent in [12, 26].

These gauge transformations only affect the components of the vector potential since the momenta are gauge invariant. In polar coordinates,

$$ds^2 = -dt^2 + dr^2 + g_{AB}dx^A dx^B , \qquad g_{AB} = r^2\overline{g}_{AB} , \tag{2.14}$$

where $\overline{g}_{AB}$ $(A,B=1,2)$ is the round metric of the unit sphere, the asymptotic conditions read explicitly

$$A_r = \Phi_{\rm lin} + \frac{1}{r}\overline{A}_r + \frac{\ln r}{r^2}A_r^{\log(2)} + \frac{1}{r^2}A_r^{(2)} + o\left(r^{-2}\right) , \tag{2.15}$$

$$A_A = r\partial_A \Phi_{\rm lin} + \ln r\, \partial_A \Phi_{\log} + \overline{A}_A + \frac{\ln r}{r}A_A^{\log(2)} + \frac{1}{r}A_A^{(2)} + o\left(r^{-2}\right) , \tag{2.16}$$

$$A_r^{\log(2)} = -\Phi_{\log}^{(1)}, \quad A_A^{\log(2)} = \partial_A \left(\Phi_{\log}^{(1)}\right) \quad \Leftrightarrow \quad \partial_A A_r^{\log(2)} + A_A^{\log(2)} = 0\,, \tag{2.17}$$

$$A_0 = \Psi_{\rm lin} + \frac{\ln r}{r}\Psi_{\log} + \frac{1}{r}\overline{\Psi} + \frac{\ln r}{r^2}\Psi_{\log}^{(1)} + \frac{1}{r^2}\Psi^{(1)} + o\left(r^{-2}\right) , \tag{2.18}$$

for the vector potential and

$$\pi^r = \overline{\pi}^r + \frac{1}{r}\pi^r_{(2)} + \frac{1}{r^2}\pi^r_{(3)} + \mathcal{O}\left(r^{-3}\right) , \tag{2.19}$$

$$\pi^A = \frac{1}{r}\overline{\pi}^A + \frac{1}{r^2}\pi^A_{(2)} + \frac{1}{r^3}\pi^A_{(3)} + \mathcal{O}\left(r^{-4}\right) , \tag{2.20}$$

$$\pi^0 = \frac{1}{r^2}\pi^{(2)}_{\Psi} + o\left(r^{-2}\right) , \tag{2.21}$$

for the momenta, which carry density weight one. Here, the coefficients in the expansion of r and $\ln r$ depend only on the angles and are subject to the following parity conditions,

$$\overline{A}_r = \text{odd}\,, \quad \overline{\pi}^r = \text{even}\,, \quad \overline{A}_A = (\overline{A})_{\rm even} + \partial_A \Phi \quad (\Phi = \text{even})\,, \quad \overline{\pi}^A = \text{odd}\,, \tag{2.22}$$

$$\Phi_{\rm lin} = \text{odd}\,, \quad \Phi_{\log} = \text{odd}\,, \quad \Psi_{\rm lin} = \text{even}\,, \quad \Psi_{\log} = \text{even}\,. \tag{2.23}$$

The parity conditions (2.22) reduce to those of [12, 26]. We will mention as we proceed where the parity conditions (2.23) on the new terms are needed. The parity conditions (2.23) are simply enforced by allowing only Θ and Ξ in $\Delta A_i = \partial_i \Theta$, $\Delta A_0 = \Xi$ that fulfill these parity conditions. Note that if Φ had an odd part, it could be absorbed in $(\overline{A})_{\rm even}$ so that we can assume that Φ is even.

We have kept the terms of order $\mathcal{O}(r^{-2})$ in A_r and $\mathcal{O}(r^{-1})$ in A_A because they now contribute to the charges. Similarly, $\pi^r_{(2)}$ becomes physically relevant. Note that the intermediate terms $A_r^{\log(2)}$ and $A_A^{\log(2)}$, absent in [12, 26], have their entire origin in the gauge transformation with gauge parameter $\epsilon = \frac{\ln r}{r}\Phi^{(1)}_{\log}$, hence (2.17).

Once the boundary conditions have been specified, one can write down explicitly the boundary terms in the action. The constructive procedure that we have followed for doing so is explained in appendix A. It leads to

$$\mathfrak{B} = \int dt \oint_{S_2^\infty} d^2x \mathcal{S} - \int dt \oint_{S_2^\infty} d^2x \mathcal{B}\,, \tag{2.24}$$

where $\oint d^2x \mathcal{S}$ is a surface integral which is linear in the time derivatives of the canonical variables,

$$\mathcal{S} = \sqrt{\overline{g}}\Big(- \overline{A}_r \dot{\overline{\Psi}} + \Psi_{\log}\dot{\Phi} - A_r^{(2)}\dot{\Psi}_{\rm lin} + \overline{\Psi}^{(1)}\dot{\Phi}_{\rm lin}\Big)\,, \qquad \overline{\Psi}^{(1)} \equiv \Psi^{(1)}_{\log} - \Psi^{(1)}\,, \tag{2.25}$$

while $\mathcal{B}$ does not depend on the time derivatives of the canonical variables and is given by

$$\mathcal{B} = \overline{\Pi}^r \Psi_{\rm lin} + \sqrt{\overline{g}}\,\partial_A \overline{A}_r \overline{D}^A \Phi_{\rm lin} - 2\sqrt{\overline{g}}\,\overline{A}_r \Phi_{\rm lin}\,, \tag{2.26}$$

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with

$$\overline{\Pi}^r = \overline{\pi}^r + \sqrt{\bar{g}}\,\Psi_{\log}\,. \tag{2.27}$$

It should be noted that the zero mode of Φ, which drops from the potential A_i, is present in the action through the boundary kinetic term. Its conjugate is non trivial even in the absence of charged fields, and given by the zero mode of $\Psi_{\log}$.

The boundary term $\oint d^2x \mathcal{S}$ completes the boundary term introduced in [26] in a way that integrability of the boost charges, which was the very reason for introducing it there, is maintained with our more general boundary conditions. Because $\mathcal{S}$ involves the time derivatives of the fields, it contributes to the symplectic form, which reads explicitly

$$\begin{aligned}\Omega = &\int d^3x (d_V \pi^i d_V A_i + d_V \pi^0 d_V A_0) \\ &- \oint d^2x \sqrt{\bar{g}} \left[d_V \overline{A}_r d_V \overline{\Psi} - d_V \Psi_{\log} d_V \Phi + d_V A_r^{(2)} d_V \Psi_{\text{lin}} - d_V \overline{\Psi}^{(1)} d_V \Phi_{\text{lin}} \right] .\end{aligned} \tag{2.28}$$

This symplectic form is non-degenerate in the sense that if X is a phase space vector field such that $\iota_X \Omega = 0$, then $X = 0$.

The symplectic form pairs $\overline{A}_r$ with $\overline{\Psi}$, as in [26], and also introduces a surface conjugate $\Psi_{\log}$ to Φ, which had none with the earlier boundary conditions since there was no logarithmic term $\ln r/r$ in A_0. The new variables Ψ_{lin} and Φ_{lin}, which would be absent if we had limited the extension of the gauge transformations of [26] to gauge transformations blowing only logarithmically at infinity (with no linear term), are naturally paired with the subleading terms in the expansion of the components of the potential. This canonical structure will be reflected in the Poisson brackets of the improper gauge charges. In fact, the brackets of these charges conversely restrict the symplectic structure to the above form, which provides an independent argument (besides integrability of the boost charges) for extending the $\mathcal{S}$ of [26] as in (2.25).

As a side final comment to this subsection, we note that the pair (A_0, π^0) was denoted (Ψ, π_Ψ) in [12, 26], and that the Lagrange multiplier ψ was denoted A_0. This was due to the peculiar constructive way they were arrived at by studying the dynamics at the boundary. We revert here to more familiar notations.

2.3 Asymptotic form of the constraints

We shall also require that the constraints decay one order faster than the one prescribed in [26]. This yields (2.21) for π^0 as well as $\mathcal{G} = \mathcal{O}(r^{-3})$, which implies, since

$$\mathcal{G} = \frac{1}{r}\partial_A \overline{\pi}^A + \frac{1}{r^2}\left(\partial_A \pi^A_{(2)} - \pi^r_{(2)}\right) + \frac{1}{r^3}\left(\partial_A \pi^A_{(3)} - 2\pi^r_{(3)}\right) + \mathcal{O}\left(r^{-4}\right), \tag{2.29}$$

the conditions

$$\partial_A \overline{\pi}^A = 0 \quad \text{and} \quad \partial_A \pi^A_{(2)} - \pi^r_{(2)} = 0\,. \tag{2.30}$$

These conditions are part of the asymptotic conditions on the canonical variables.

2.4 Finiteness of the kinetic term

It is clear that the only potentially divergent term in the action is the bulk kinetic term $\int dtd^3x\, \pi^i \dot{A}_i$. We check in this subsection that it is finite. One finds by direct substitution

$$\int dt\, dr\, d^2x\, \pi^i \dot{A}_i = \int dt\, dr\, d^2x \Big[\overline{\pi}^r \dot{\Phi}_{\text{lin}} + \overline{\pi}^A \partial_A \dot{\Phi}_{\text{lin}} + \frac{\ln r}{r} \overline{\pi}^A \partial_A \dot{\Phi}_{\text{log}} \tag{2.31}$$

$$+ \frac{1}{r} \left(\overline{\pi}^A \dot{\overline{A}}_A + \overline{\pi}^r \dot{\overline{A}}_r + \pi^r_{(2)} \dot{\Phi}_{\text{lin}} + \pi^A_{(2)} \partial_A \dot{\Phi}_{\text{lin}} \right) + \mathcal{O}\left(r^{-2}\right) \Big] . \tag{2.32}$$

i) For the linear divergence, we note immediately that the first term is zero because Φ_{lin} is *odd*, while the second vanishes by virtue of the condition $\partial_A \overline{\pi}^A = 0$.

ii) The term proportional to $r^{-1} \ln r$ vanishes by virtue of $\partial_A \overline{\pi}^A = 0$.

iii) The terms proportional to the logarithmic divergence reduce to (after considering parity conditions)

$$\oint d^2x \left[-\partial_A \overline{\pi}^A \dot{\Phi} - \left(\partial_A \pi^A_{(2)} - \pi^r_{(2)} \right) \dot{\Phi}_{\text{lin}} \right] , \tag{2.33}$$

which vanish by virtue of the conditions in (2.30).

The bulk kinetic term and hence the action are finite. The parity conditions on Φ_{lin} and on the asymptotic fields appearing already in [12, 26], as well as the fast asymptotic decay of the constraint functions, are key for this result.

2.5 Equations of motion

The equations of motion that follow from the action can be analyzed as follows.

- Variation with respect to the Lagrange multipliers yields the constraints $\mathcal{G} \approx 0$ and $\pi^0 \approx 0$.

- Variation with respect to the field A_0 yields, besides the bulk equation of motion $\dot{\pi}^0 + \mathcal{G} = 0$ which is a consequence of the constraints, the following conditions on the asymptotic fields,

$$\dot{\overline{A}}_r = \dot{\Phi}_{\text{lin}} = 0 , \tag{2.34}$$

$$\dot{A}^{(2)}_r = \frac{1}{\sqrt{\overline{g}}} \left(\overline{\pi}^r + \sqrt{\overline{g}}\, \Psi_{\text{log}} \right) , \tag{2.35}$$

$$\dot{\Phi} = \Psi_{\text{lin}} . \tag{2.36}$$

- Variation with respect to π^0 yields the bulk equation of motion

$$\dot{A}_0 - \lambda - \partial^i A_i - \tilde{\mu} = 0 \tag{2.37}$$

(without surface term contribution because π^0 decreases sufficiently fast at infinity). This equation implies

$$\dot{\Psi}_{\text{lin}} = 0\,, \tag{2.38}$$

$$\dot{\Psi}_{\text{log}} = 0\,, \tag{2.39}$$

$$\dot{\overline{\Psi}} = \overline{\triangle}\,\Phi_{\text{lin}} + 2\Phi_{\text{lin}}\,, \tag{2.40}$$

$$\dot{\overline{\Psi}}^{(1)} = \overline{\triangle}\,\Phi_{\text{log}} + 2\overline{A}_r\,, \tag{2.41}$$

to the leading orders to which $\lambda = o\left(r^{-2}\right)$ does not contribute. The next orders can then be used to express the subleading terms in $\dot{A}_0$ to the Lagrange multiplier λ.

- Variation of the action with respect to the conjugate momenta π^i yields the equation

$$\dot{A}_i - \partial_i\left(A_0 + \psi + \tilde{\epsilon}\right) - \frac{\pi_i}{\sqrt{g}} = 0\,, \tag{2.42}$$

or equivalently

$$\pi_i = \sqrt{g}\left(\partial_t A_i - \partial_i\left(A_0 + \psi + \tilde{\epsilon}\right)\right)\,. \tag{2.43}$$

The asymptotic form of the field equations yields the following conditions

$$\dot{\Phi}_{\text{lin}} = 0\,, \tag{2.44}$$

$$\dot{\overline{A}}_r = 0\,, \tag{2.45}$$

$$\dot{A}_r^{\log(2)} + \Psi_{\log} = 0\,, \tag{2.46}$$

$$\overline{\pi}^r = \sqrt{\overline{g}}\left(\dot{A}_r^{(2)} - \Psi_{\log}\right)\,, \tag{2.47}$$

$$\pi^r_{(2)} = \sqrt{\overline{g}}\left(\dot{A}_r^{(3)} + 2\Psi^{(1)} - \Psi^{(1)}_{\log} + 2\psi^{(2)} - \psi^{(2)}_{\log} + 2\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(2)}_{\log}\right)\,, \tag{2.48}$$

which are consistent with the equations (2.34) and (2.35) obtained previously.

- Variation of the action with respect to the gauge field A_i yields for the bulk term (up to a total time derivative):

$$\delta I_{\text{bulk}} = \int dt\Big[-\int d^3x\Big(\dot{\pi}^i + \sqrt{g}\nabla_j F^{ij} - \partial^i\pi^0\Big)\delta A_i - \oint d^2x\sqrt{\overline{g}}\,\overline{\triangle}\left(\overline{A}_r - \Phi_{\log}\right)\delta\Phi_{\text{lin}}\Big]\,, \tag{2.49}$$

and for the boundary term

$$\delta I_{\text{boundary}} = -\int dt\oint d^2x\sqrt{\overline{g}}\Big\{\Big[\dot{\overline{\Psi}} - \left(\overline{\triangle}\,\Phi_{\text{lin}} + 2\Phi_{\text{lin}}\right)\Big]\delta\overline{A}_r + \dot{\Psi}_{\log}\delta\Phi \tag{2.50}$$

$$+ \dot{\Psi}_{\text{lin}}\delta A_r^{(2)} + \Big[\dot{\overline{\Psi}}^{(1)} - \left(\overline{\triangle}\,\overline{A}_r + 2\overline{A}_r\right)\Big]\delta\Phi_{\text{lin}}\Big\}\,. \tag{2.51}$$

This implies the equations

$$\dot{\pi}^i = -\sqrt{g}\nabla_j F^{ij} + \partial^i\pi^0\,, \tag{2.52}$$

$$\dot{\overline{\Psi}}^{(1)} = -\overline{\triangle}\left(\overline{A}_r - \Phi_{\log}\right) + \overline{\triangle}\,\overline{A}_r + 2\overline{A}_r = \overline{\triangle}\,\Phi_{\log} + 2\overline{A}_r\,, \tag{2.53}$$

$$\dot{\overline{\Psi}} = \overline{\triangle}\,\Phi_{\text{lin}} + 2\Phi_{\text{lin}}\,, \tag{2.54}$$

$$\dot{\Psi}_{\log} = \dot{\Psi}_{\text{lin}} = 0\,, \tag{2.55}$$

which are consistent with the dynamical equations for the asymptotic components of A_0 derived above.

The time derivatives of the first three leading orders of the conjugate momentum are then given by

$$\dot{\overline{\pi}}^r = 0\,, \tag{2.56}$$

$$\dot{\pi}^r_{(2)} = \sqrt{\overline{g}}\,\overline{\triangle}\left(\overline{A}_r - \Phi_{\log}\right), \tag{2.57}$$

$$\dot{\pi}^r_{(3)} = \sqrt{\overline{g}}\left[\overline{\triangle}\, A_r^{(2)} + \overline{D}_A\left(A^{(2)A} - A^{\log(2)A}\right)\right]. \tag{2.58}$$

Note that while the leading "monopole" term $\overline{\pi}^r$ is conserved, the subsequent terms are not. This is of course well known, but will not prevent us from defining a conserved quantity involving the subleading term $\pi^r_{(2)}$ and generating gauge transformations that blow up at infinity, as achieved in [35] along different lines.

This concludes the discussion of the equations of motion.

3 Improper gauge symmetries

If one allows the parameters ϵ, μ in (2.9) and (2.10) to decrease slowlier at infinity, in a way compatible with the asymptotic form of the gauge potential,

$$\epsilon = r\epsilon_{\text{lin}} + \ln r\,\epsilon_{\log} + \overline{\epsilon} + \frac{\ln r}{r}\epsilon^{(1)}_{\log} + \frac{1}{r}\epsilon^{(1)} + o\left(r^{-1}\right), \tag{3.1}$$

$$\mu = \mu_{\text{lin}} + \frac{\ln r}{r}\mu_{\log} + \frac{1}{r}\overline{\mu} + \frac{\ln r}{r^2}\mu^{(1)}_{\log} + \frac{1}{r^2}\mu^{(1)} + o\left(r^{-2}\right), \tag{3.2}$$

with

$$\epsilon_{\text{lin}} = \text{odd}\,, \quad \epsilon_{\log} = \text{odd}\,, \quad \overline{\epsilon} = \text{even}\,, \quad \mu_{\text{lin}} = \text{even}\,, \quad \mu_{\log} = \text{even}\,, \tag{3.3}$$

one finds that the transformations are still symmetries of the action but now of the "improper gauge type" [32]. The bulk part of the corresponding generator is again given by the combination $\int d^3x\,(\mu\pi^0 + \epsilon\mathcal{G})$ of the constraints, but there are in addition non-vanishing surface terms at infinity.

The complete generator Q_X of the transformation generated by the phase space vector field X is obtained through the general formula

$$\iota_X\Omega = -d_V Q_X\,, \tag{3.4}$$

which receives in our case non trivial contributions both from the bulk part of the symplectic form and from its surface part $\mathcal{S}$ [26]. (If Ω reduced to its canonical bulk part, the rule (3.4) would reproduce the integrability condition of [36].)

Applying this general rule to the transformations (2.9) and (2.10), which imply the following transformations of the asymptotic fields

$$\delta_\epsilon\Phi_{\text{lin}} = \epsilon_{\text{lin}}\,, \quad \delta_\epsilon\Phi_{\log} = \epsilon_{\log}\,, \quad \delta_\epsilon\Phi = \overline{\epsilon}\,, \quad \delta_\epsilon\overline{A}_r = \epsilon_{\log}\,, \quad \delta_\epsilon A_r^{(2)} = \overline{\epsilon}^{(1)}\,, \quad \overline{\epsilon}^{(1)} \equiv \epsilon^{(1)}_{\log} - \epsilon^{(1)}\,, \tag{3.5}$$

$$\delta_\mu\Psi_{\text{lin}} = \mu_{\text{lin}}\,, \quad \delta_\mu\Psi_{\log} = \mu_{\log}\,, \quad \delta_\mu\overline{\Psi} = \overline{\mu}\,, \quad \delta_\mu\overline{\Psi}^{(1)} = \overline{\mu}^{(1)}\,, \quad \overline{\mu}^{(1)} \equiv \mu^{(1)}_{\log} - \mu^{(1)}\,, \tag{3.6}$$

one finds that the canonical generator of the improper gauge symmetries reads

$$G_{\mu,\epsilon} = \int d^3x \left(\mu\pi^0 + \epsilon\mathcal{G}\right) + Q_{\epsilon_{\rm lin}} + Q_{\epsilon_{\rm log}} + Q_{\overline{\epsilon}} + Q_{\overline{\epsilon}^{(1)}} + Q_{\mu_{\rm lin}} + Q_{\mu_{\rm log}} + Q_{\overline{\mu}} + Q_{\overline{\mu}^{(1)}} \,, \tag{3.7}$$

with

$$Q_{\epsilon_{\rm lin}} = \oint d^2x \epsilon_{\rm lin} \left[\pi^r_{(2)} + \sqrt{\bar{g}}\left(\Psi^{(1)}_{\rm log} - \Psi^{(1)}\right)\right] \,, \tag{3.8}$$

$$Q_{\epsilon_{\rm log}} = \oint d^2x \sqrt{\bar{g}}\, \epsilon_{\rm log} \overline{\Psi} \,, \tag{3.9}$$

$$Q_{\overline{\epsilon}} = \oint d^2x \overline{\epsilon} \left(\overline{\pi}^r + \sqrt{\bar{g}}\, \Psi_{\rm log}\right) \,, \tag{3.10}$$

$$Q_{\overline{\epsilon}^{(1)}} = \oint d^2x \sqrt{\bar{g}}\, \overline{\epsilon}^{(1)} \Psi_{\rm lin} \,, \tag{3.11}$$

and

$$Q_{\mu_{\rm lin}} = -\oint d^2x \sqrt{\bar{g}}\, \mu_{\rm lin} A_r^{(2)} \,, \tag{3.12}$$

$$Q_{\mu_{\rm log}} = -\oint d^2x \sqrt{\bar{g}}\, \mu_{\rm log}\, \Phi \,, \tag{3.13}$$

$$Q_{\overline{\mu}} = -\oint d^2x \sqrt{\bar{g}}\, \overline{\mu}\, \overline{A}_r \,, \tag{3.14}$$

$$Q_{\overline{\mu}^{(1)}} = -\oint d^2x \sqrt{\bar{g}}\, \overline{\mu}^{(1)} \Phi_{\rm lin} \,. \tag{3.15}$$

In the charge $Q_{\overline{\epsilon}^{(1)}}$, only the even part of $\overline{\epsilon}^{(1)}$ contributes since the field $\Psi_{\rm lin}$ is even. The odd part of $\overline{\epsilon}^{(1)}$ drops out and defines a proper gauge symmetry. To emphasize this point, one sometimes writes $Q_{\overline{\epsilon}^{(1)}_{\rm even}}$. Therefore, among the charges generating the improper gauge symmetries parametrized by ϵ, there are two charges characterized by parameters that are even functions on the sphere ($Q_{\overline{\epsilon}}$ and $Q_{\overline{\epsilon}^{(1)}_{\rm even}}$), and two charges characterized by odd parameters ($Q_{\epsilon_{\rm lin}}$ and $Q_{\epsilon_{\rm log}}$).

Similarly, among the charges generating the improper gauge symmetries parametrized by μ, there are two charges characterized by parameters that are odd functions on the sphere ($Q_{\overline{\mu}_{\rm odd}}$ and $Q_{\overline{\mu}^{(1)}_{\rm odd}}$), and two charges characterized by even parameters ($Q_{\mu_{\rm lin}}$ and $Q_{\mu_{\rm log}}$). This is because $\overline{A}_r$ and $\Phi_{\rm lin}$ are odd funtions on the sphere, so that $\mu_{\rm even}$ and $\overline{\mu}^{(1)}_{\rm even}$ drop out from the charges and define proper gauge symmetries.

Furthermore, if $\epsilon^{(1)}_{\rm log}$ and $\epsilon^{(1)}$ are such that $\overline{\epsilon}^{(1)} = 0$, the corresponding gauge transformation is proper since it has zero charge. A similar property holds for $\mu^{(1)}_{\rm log}$ and $\mu^{(1)}$.

There is thus a total of eight non trivial improper gauge charges, each characterized by a definite parity under the sphere antipodal map. The brackets among these charges are easily computed. They are found to form a centrally extended Abelian algebra with

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the following non-zero central charges

$$\{G_{\epsilon_{\rm lin}}, G_{\overline{\mu}^{(1)}}\} = -\{G_{\overline{\mu}^{(1)}}, G_{\epsilon_{\rm lin}}\} = \oint d^2x \sqrt{\bar{g}}\, \epsilon_{\rm lin}\, \overline{\mu}^{(1)}\,, \tag{3.16}$$

$$\{G_{\epsilon_{\rm log}}, G_{\overline{\mu}}\} = -\{G_{\overline{\mu}}, G_{\epsilon_{\rm log}}\} = \oint d^2x \sqrt{\bar{g}}\, \epsilon_{\rm log}\, \overline{\mu}\,, \tag{3.17}$$

$$\{G_{\overline{\epsilon}}, G_{\mu_{\rm log}}\} = -\{G_{\mu_{\rm log}}, G_{\overline{\epsilon}}\} = \oint d^2x \sqrt{\bar{g}}\, \overline{\epsilon}\, \mu_{\rm log}\,, \tag{3.18}$$

$$\{G_{\overline{\epsilon}^{(1)}}, G_{\mu_{\rm lin}}\} = -\{G_{\mu_{\rm lin}}, G_{\overline{\epsilon}^{(1)}}\} = \oint d^2x \sqrt{\bar{g}}\, \overline{\epsilon}^{(1)}\, \mu_{\rm lin}\,. \tag{3.19}$$

In the paper [26] where the boundary conditions are more restrictive, only the improper gauge charges $G_{\overline{\epsilon}}$ and $G_{\overline{\mu}}$ are present. These commute, $\{G_{\overline{\epsilon}}, G_{\overline{\mu}}\} = 0$ so that the central terms are absent. The improper gauge charges associated with logarithmic gauge transformations ($G_{\epsilon_{\rm log}}$ and $G_{\mu_{\rm log}}$) are conjugate to these charges. The improper gauge transformations linear in r bring in two additional new charges ($G_{\epsilon_{\rm lin}}$ and $G_{\mu_{\rm lin}}$) as expected, but at the same time they turn on the subleading gauge transformations, which become non-trivial and also bring in two additional non trivial charges. These form canonically conjugate pairs. It is because of the central terms present in the extended improper gauge algebra that one can make the improper gauge generators commute with the Poincaré generators in the asymptotic symmetry algebra, as in the gravity case [1], and as we shall show explicitly below.

Among the improper gauge charges, $G_{\overline{\epsilon}}$ and $G_{\overline{\mu}}$ combine to form the angle-dependent $u(1)$ charge seen at null infinity [26]. The charge $G_{\epsilon_{\rm lin}}$ involves the $1/r^3$ component of the electric field with an odd gauge parameter and is related to the electric dipole moment for the $\ell = 1$ spherical harmonic. Together with its companion $G_{\mu_{\rm lin}}$, we expect that it should be connected to the charges that underlie the subleading soft photon theorems [27–29].

Note that the integrands of the charges $Q_{\epsilon_{\rm lin}}$ and $Q_{\epsilon_{\rm log}}$ are not conserved in time since it follows from the equations of motion that

$$\partial_t \left(\pi^r_{(2)} + \sqrt{\bar{g}}\, \overline{\Psi}^{(1)}\right) = \overline{\triangle}\, \overline{A}_r + 2\overline{A}_r\,, \qquad \partial_t \overline{\Psi} = \overline{\triangle}\, \Phi_{\rm lin} + 2\Phi_{\rm lin}\,. \tag{3.20}$$

This expresses the fact that $Q_{\epsilon_{\rm lin}}$ and $Q_{\epsilon_{\rm log}}$ do not commute with the generator H of time translations and hence are conserved only at the price of an explicit time dependence. Indeed the Poisson brackets with the Poincaré generators (prior to decoupling) imply, as we shall see,

$$\{G_{\epsilon_{\rm lin}}, H\} = G_{\overline{\mu}}\,, \qquad \overline{\mu} = -(\overline{\triangle}\, \epsilon_{\rm lin} + 2\epsilon_{\rm lin}) \tag{3.21}$$

and

$$\{G_{\epsilon_{\rm log}}, H\} = G_{\overline{\mu}^{(1)}}\,, \qquad \overline{\mu}^{(1)} = -(\overline{\triangle}\, \epsilon_{\rm log} + 2\epsilon_{\rm log})\,. \tag{3.22}$$

Similarly,

$$\{G_{\mu_{\rm lin}}, H\} = G_{\overline{\epsilon}}\,, \qquad \overline{\epsilon} = -\mu_{\rm lin} \tag{3.23}$$

and

$$\{G_{\mu_{\rm log}}, H\} = G_{\overline{\epsilon}^{(1)}}\,, \qquad \overline{\epsilon}^{(1)} = -\mu_{\rm log}\,, \tag{3.24}$$

in agreement with (2.35) and (2.36).

The equations of motion have been derived in subsection 2.5 assuming that the Lagrange multipliers ψ and λ decrease sufficiently fast at infinity so that they define proper gauge symmetries. The surface term in the action was adjusted under this assumption. In fact, if we had not known that $\psi = o(r^{-1})$ and $\lambda = o(r^{-2})$, we would have derived from the extremization of the action (2.3) *(with that boundary term)* that consistency of the equations of motion implies that ψ and λ should define proper gauge transformations ($\psi_{\text{lin}} = 0 = \psi_{\text{log}} = \overline{\psi} = \overline{\psi}^{(1)}$ and $\lambda_{\text{lin}} = 0 = \lambda_{\text{log}} = \overline{\lambda} = \overline{\lambda}^{(1)}$).

A more general behaviour of the Lagrange multipliers ψ and λ can be accomodated. One can allow them to define improper gauge symmetries provided one includes in the action the corresponding surface integral. This would be in particular needed if one wanted to impose the Lorenz gauge $\dot{A}_0 = \partial^i A_i$ which requires from (2.37) that λ should be equal to $-\tilde{\mu}$.

4 Poincaré invariance

4.1 Poincaré transformations

The analysis of Poincaré invariance proceeds along the lines of [26], adapted to the less stringent boundary conditions considered in this paper. We shall therefore give only the final results and check their consistency. A more constructive approach, which was actually the approach we followed to arrive at the results presented below, is given in appendix A.

The 10-dimensional Poincaré symmetry is generated by the vector fields conveniently parametrized in spherical coordinates as

$$\xi = br + T\,, \tag{4.1}$$

$$\xi^r = W\,, \tag{4.2}$$

$$\xi^A = Y^A + \frac{1}{r}\overline{D}^A W\,, \tag{4.3}$$

where

$$\overline{D}_A\overline{D}_B b + \overline{g}_{AB} b = 0\,, \quad \overline{D}_A\overline{D}_B W + \overline{g}_{AB} W = 0\,, \quad \overline{D}_A Y_B + \overline{D}_B Y_A = 0\,, \quad \partial_A T = 0\,. \tag{4.4}$$

The boost function $b(x^A)$ and the spatial translation function $W(x^A)$ each belong to the three-dimensional space of the spin-1 representation of the rotation group. The vector field Y^A generates infinitesimal rotations. It is a Killing vector on the sphere and depends also on three parameters. The last parameter of the Poincaré group is given by T, which is constant.

The Poincaré transformations of the fields are given by

$$\delta_{\xi,\xi^i} A_i = \frac{\xi\pi_i}{\sqrt{g}} + \partial_i\left(\xi A_0\right) + \mathcal{L}_\xi A_i + \delta_{\epsilon_{(T,W)}} A_i\,, \tag{4.5}$$

$$\delta_{\xi,\xi^i} \pi^i = \sqrt{g}\nabla_m\left(F^{mi}\xi\right) + \xi\partial^i\pi^0 + \mathcal{L}_\xi \pi^i\,, \tag{4.6}$$

$$\delta_{\xi,\xi^i} A_0 = \nabla_i\left(\xi A^i\right) + \mathcal{L}_\xi A_0 + \delta_{\mu_{(b,T,W)}} A_0\,, \tag{4.7}$$

$$\delta_{\xi,\xi^i} \pi^0 = \xi\partial_i\pi^i + \mathcal{L}_\xi \pi^0\,, \tag{4.8}$$

These transformations take the same form as in [26], except for the additional correction terms $\delta_{\epsilon_{(T,W)}} A_i$ and $\delta_{\mu_{(b,T,W)}} A_0$ that must be added in order to preserve integrability of the Poincaré charges (see below). These transformations are improper gauge symmetries with parameters

$$\epsilon_{(T,W)} = \frac{\ln r}{r}\epsilon^{(1)}_{\log(T,W)} + \frac{1}{r}\epsilon^{(1)}_{(T,W)} + o\left(r^{-1}\right) , \tag{4.9}$$

and

$$\mu_{(b,T,W)} = \frac{\ln r}{r^2}\mu^{(1)}_{\log(b,T,W)} + \frac{1}{r^2}\mu^{(1)}_{(b,T,W)} + o\left(r^{-2}\right) , \tag{4.10}$$

with

$$\bar{\epsilon}^{(1)}_{(T,W)} = T\overline{A}_0 + \overline{D}^A W \overline{A}_A - W\overline{A}_r , \tag{4.11}$$

and

$$\overline{\mu}^{(1)}_{(b,T,W)} = 4bA_r^{(2)} + T\left(\overline{D}_A\overline{A}^A + 3\overline{A}_r\right) - \left(W\overline{\pi}^r - \partial_A W\overline{\pi}^A\right) - \left[-\partial_A W\overline{D}^A\overline{\Psi} + W\left(\Psi_{\log} + \overline{\Psi}\right)\right] . \tag{4.12}$$

Asymptotically expanding (4.5)–(4.8) leads to the following transformation laws:

- For the leading orders:

$$\delta_{\xi,\xi^i}\Phi_{\text{lin}} = \mathcal{L}_Y\Phi_{\text{lin}} + b\Psi_{\text{lin}} , \tag{4.13}$$

$$\delta_{\xi,\xi^i}\Phi_{\log} = \mathcal{L}_Y\Phi_{\log} + b\Psi_{\log} , \tag{4.14}$$

$$\delta_{\xi,\xi^i}\Psi_{\text{lin}} = \mathcal{L}_Y\Psi_{\text{lin}} + \overline{D}_A\left(b\overline{D}^A\Phi_{\text{lin}}\right) + 3b\Phi_{\text{lin}} , \tag{4.15}$$

$$\delta_{\xi,\xi^i}\Psi_{\log} = \mathcal{L}_Y\Psi_{\log} + \overline{D}_A\left(b\overline{D}^A\Phi_{\log}\right) , \tag{4.16}$$

- For the subleading orders:

$$\delta_{\xi,\xi^i}\overline{A}_r = \mathcal{L}_Y\overline{A}_r + \frac{b}{\sqrt{\bar{g}}}\overline{\pi}_r + b\Psi_{\log} , \tag{4.17}$$

$$\delta_{\xi,\xi^i}\overline{A}_A = \mathcal{L}_Y\overline{A}_A + \frac{b}{\sqrt{\bar{g}}}\overline{\pi}_A + \partial_A\left(W\Phi_{\text{lin}} + \overline{D}^B W\partial_B\Phi_{\text{lin}} + b\overline{\Psi} + T\Psi_{\text{lin}}\right) , \tag{4.18}$$

$$\delta_{\xi,\xi^i}\overline{\pi}^r = \mathcal{L}_Y\overline{\pi}^r + \sqrt{\bar{g}}\,\overline{D}_A\left[b\overline{D}^A\left(\overline{A}_r - \Phi_{\log}\right)\right] , \tag{4.19}$$

$$\delta_{\xi,\xi^i}\overline{\pi}^A = \mathcal{L}_Y\overline{\pi}^A - \sqrt{\bar{g}}\,\bar{g}^{AB}\overline{D}^C\left(b\overline{F}_{BC}\right) , \tag{4.20}$$

$$\delta_{\xi,\xi^i}\overline{\Psi} = \mathcal{L}_Y\overline{\Psi} + \overline{D}^A W\partial_A\Psi_{\text{lin}} + \overline{D}_A(b\overline{A}^A) + 2b\overline{A}_r + T\left(\triangle\Phi_{\text{lin}} + 2\Phi_{\text{lin}}\right) , \tag{4.21}$$

which implies that

$$\delta_{\xi,\xi^i}\Phi = \mathcal{L}_Y\Phi + W\Phi_{\text{lin}} + \overline{D}^B W\partial_B\Phi_{\text{lin}} + b\overline{\Psi}^{\text{odd}} + T\Psi_{\text{lin}} . \tag{4.22}$$

- For the sub-subleading orders:

$$\delta_{\xi,\xi^i} A_r^{(2)} = \mathcal{L}_Y A_r^{(2)} + \overline{D}^A W \left(\partial_A \overline{A}_r - \overline{A}_A\right) - W\overline{A}_r + \frac{b}{\sqrt{\overline{g}}}\pi^r_{(2)} + b\left(\Psi^{\log(1)} - \Psi^{(1)}\right) + \frac{T}{\sqrt{\overline{g}}}\overline{\pi}^r + T\left(\Psi_{\log} - \overline{\Psi}\right) + \overline{\epsilon}^{(1)}_{(T,W)}, \quad (4.23)$$

$$\delta_{\xi,\xi^i} A_A^{(2)} = \mathcal{L}_Y A_A^{(2)} + \overline{D}_B \overline{D}_A W \overline{A}^B + \overline{D}^B W \overline{D}_B \overline{A}_A + \partial_A W \overline{A}_r + W\partial_A \Phi_{\log} + \frac{b}{\sqrt{\overline{g}}}\pi_{(2)A} + \frac{T}{\sqrt{\overline{g}}}\overline{\pi}_A, \quad (4.24)$$

$$\delta_{\xi,\xi^i} \pi^r_{(2)} = \mathcal{L}_Y \pi^r_{(2)} + \overline{D}_A\left(\overline{D}^A W \overline{\pi}^r\right) - \partial_A W \overline{\pi}^A + \sqrt{\overline{g}}\,\overline{D}_A[b(\overline{D}^A A_r^{(2)} + A^{(2)A} - A^{\log(2)A})] + \sqrt{\overline{g}}\,\overline{D}_A\left[T\overline{D}^A\left(\overline{A}_r - \Phi_{\log}\right)\right], \quad (4.25)$$

$$\delta_{\xi,\xi^i} \pi^A_{(2)} = \mathcal{L}_Y \pi^A_{(2)} + \overline{D}_B\left(\overline{D}^B W \overline{\pi}^A\right) + \overline{D}_B \overline{D}^A W \overline{\pi}^B + \overline{D}^A W \overline{\pi}^r - W\overline{\pi}^A - \sqrt{\overline{g}}\,\overline{g}^{AB}\overline{D}^C(bF^{(2)}_{BC}) + \sqrt{\overline{g}}\,\overline{g}^{AB} b\left(\partial_B A_r^{(2)} + A_B^{(2)}\right) - \sqrt{\overline{g}}\,\overline{g}^{AB}\overline{D}^C(T\overline{F}_{BC}) + \sqrt{\overline{g}}\,T\overline{D}^A \overline{A}_r, \quad (4.26)$$

$$\delta_{\xi,\xi^i} \Psi^{(1)}_{\log} = \mathcal{L}_Y \Psi^{(1)}_{\log} + \partial_A W \overline{D}^A \Psi_{\log} - W\Psi_{\log} + \overline{D}_A\left(bA^{\log(2)A}\right) + bA_r^{\log(2)} + T\overline{\triangle}\,\Phi_{\log} + \mu^{(1)}_{\log(b,T,W)}, \quad (4.27)$$

$$\delta_{\xi,\xi^i} \Psi^{(1)} = \mathcal{L}_Y \Psi^{(1)} + \partial_A W \overline{D}^A \overline{\Psi} + W\left(\Psi_{\log} - \overline{\Psi}\right) + \overline{D}_A\left(bA^{(2)A}\right) + b\left(A_r^{(2)} + A_r^{\log(2)}\right) + T\left(\overline{D}_A \overline{A}^A + \overline{A}_r\right) + \mu^{(1)}_{(b,T,W)}, \quad (4.28)$$

$$\delta_{\xi,\xi^i} \Phi^{(1)}_{\log} = \mathcal{L}_Y \Phi^{(1)}_{\log} - \partial_A W \overline{D}^A \Phi_{\log} + b\Psi^{(1)} + T\Psi_{\log} - \epsilon^{(1)}_{\log(T,W)}. \quad (4.29)$$

4.2 Poincaré generators

The justification of the above definitions of the Poincaré transformations of the fields and of the symplectic structure is that the latter is invariant under the former,

$$\mathcal{L}_{X_{\xi,\xi^i}}\Omega = 0 \quad (4.30)$$

with X_{ξ,ξ^i} the phase space vector field defined by (4.5)–(4.8). The verification of (4.30) is somewhat cumbersome and involves the following key ingredients:

- The divergent terms in $\mathcal{L}_{X_{\xi,\xi^i}}\Omega$ are zero thanks to the parity conditions and the equation (2.17). In particular, the parity conditions (2.23) on the new terms in the asymptotic expansion of the vector potential are needed.

- The surface term in the symplectic form has been adjusted so that the remaining terms in $\mathcal{L}_{X_{\xi,\xi^i}}\Omega$, which are finite surface terms (with no bulk contribution), exactly cancel, taking into account the contributions coming from the correcting gauge transformation terms $\delta_{\epsilon_{(T,W)}} A_i$ and $\delta_{\mu_{(b,T,W)}} A_0$. As in [26], it is actually the requirement that $\mathcal{L}_{X_{\xi,\xi^i}}\Omega$ should vanish for boosts ($\xi = br$, $\xi^i = 0$) that fixes the form of the surface terms to be added to the standard canonical bulk part of Ω.

Since $\mathcal{L}_{X_{\xi,\xi^i}}\Omega = d_V \iota_{X_{\xi,\xi^i}}\Omega$, one can now compute the Poincaré generators through the formula

$$\iota_{X_{\xi,\xi^i}}\Omega = -d_V P_{\xi,\xi^i} \,. \tag{4.31}$$

One gets

$$P_{\xi,\xi^i} = \int d^3x \left(\xi\mathcal{H} + \xi^i\mathcal{H}_i + \epsilon_{(T,W)}\mathcal{G} + \mu_{(b,T,W)}\pi^0\right) + \mathcal{B}_{\xi,\xi^i} \,, \tag{4.32}$$

where

$$\mathcal{H} = \frac{\sqrt{g}}{4}F^{ij}F_{ij} + \frac{1}{2\sqrt{g}}\pi^i\pi_i - \partial^i\pi^0 A_i - \partial_i\pi^i A_0 \,, \tag{4.33}$$

(as above) and

$$\mathcal{H}_i = F_{ij}\pi^j - \partial_j\pi^j A_i + \pi^0\partial_i A_0 \,, \tag{4.34}$$

$$\begin{aligned}\mathcal{B}_{\xi,\xi^i} = \oint d^2x \Big\{ & b\Big(\overline{\Pi}^r\overline{\Psi} + \Pi^r_{(2)}\Psi_{\text{lin}} + \sqrt{\overline{g}}\partial_A\overline{A}_r\overline{A}^A + \sqrt{\overline{g}}\,\partial_A A_r^{(2)}\overline{D}^A\Phi_{\text{lin}} - \sqrt{\overline{g}}\,A_r^{(2)}\Phi_{\text{lin}}\Big) \\ & + Y^A\left(\overline{\Pi}^r\overline{A}_A + \Pi^r_{(2)}\partial_A\Phi_{\text{lin}} + \sqrt{\overline{g}}\,\overline{\Psi}\partial_A\overline{A}_r + \sqrt{\overline{g}}\,\Psi_{\text{lin}}\partial_A A_r^{(2)}\right) \\ & + T\Big(\overline{\Pi}^r\Psi_{\text{lin}} + \sqrt{\overline{g}}\,\partial_A\overline{A}_r\overline{D}^A\Phi_{\text{lin}} - 2\sqrt{\overline{g}}\,\overline{A}_r\Phi_{\text{lin}}\Big) \\ & + W\Big[\Big(\overline{\triangle}\,\overline{\Pi}^r + 3\overline{\Pi}^r\Big)\Phi_{\text{lin}} + \overline{D}^A\overline{\Pi}^r\partial_A\Phi_{\text{lin}} + \sqrt{\overline{g}}\,\overline{A}_r\overline{\triangle}\,\Psi_{\text{lin}} + \sqrt{\overline{g}}\,\partial_A\overline{A}_r\overline{D}^A\Psi_{\text{lin}}\Big]\Big\} \,,\end{aligned} \tag{4.35}$$

with

$$\overline{\Pi}^r = \overline{\pi}^r + \sqrt{\overline{g}}\,\Psi_{\log} \,, \tag{4.36}$$

$$\Pi^r_{(2)} = \pi^r_{(2)} + \sqrt{\overline{g}}\,\left(\Psi^{(1)}_{\log} - \Psi^{(1)}\right) \,. \tag{4.37}$$

5 Asymptotic symmetry algebra

A direct computation shows that the algebra of the asymptotic symmetries as defined above is the semi-direct sum of the Poincaré algebra with the above Abelian set of improper gauge symmetries, endowed with non-trivial central charges. Indeed, the Poisson brackets of the generators are given by

$$\{P_{\xi_1,\xi_1^i}, P_{\xi_2,\xi_2^i}\} = P_{\hat{\xi},\hat{\xi}^i} \,, \tag{5.1}$$

$$\{G_{\mu,\epsilon}, P_{\xi,\xi^i}\} = G_{\hat{\mu},\hat{\epsilon}} \,, \tag{5.2}$$

$$\{G_{\mu_1,\epsilon_1}, G_{\mu_2,\epsilon_2}\} = C_{\{\mu_1,\epsilon_1;\mu_2,\epsilon_2\}} \,, \tag{5.3}$$

where

$$\hat{\xi} = \xi_1^i\partial_i\xi_2 - \xi_2^i\partial_i\xi_1 \,, \tag{5.4}$$

$$\hat{\xi}^i = \xi_1^j\partial_j\xi_2^i - \xi_2^j\partial_j\xi_1^i + g^{ij}\left(\xi_1\partial_j\xi_2 - \xi_2\partial_j\xi_1\right) \,, \tag{5.5}$$

and

$$\hat{\mu}_{\rm lin} = -Y^A\partial_A\mu_{\rm lin} - 3b\epsilon_{\rm lin} - \overline{D}_A\left(b\overline{D}^A\epsilon_{\rm lin}\right)\,, \tag{5.6}$$

$$\hat{\epsilon}_{\rm lin} = -Y^A\partial_A\epsilon_{\rm lin} - b\mu_{\rm lin}\,, \tag{5.7}$$

$$\hat{\mu}_{\rm log} = -Y^A\partial_A\mu_{\rm log} - \overline{D}_A\left(b\overline{D}^A\epsilon_{\rm log}\right)\,, \tag{5.8}$$

$$\hat{\epsilon}_{\rm log} = -Y^A\partial_A\epsilon_{\rm log} - b\mu_{\rm log}\,, \tag{5.9}$$

$$\hat{\overline{\mu}} = -Y^A\partial_A\overline{\mu} - \overline{D}_A\left(b\overline{D}^A\overline{\epsilon}\right) - T\left(\triangle\,\epsilon_{\rm lin} + 2\epsilon_{\rm lin}\right) - \partial_A W\overline{D}^A\mu_{\rm lin}\,, \tag{5.10}$$

$$\hat{\overline{\epsilon}} = -Y^A\partial_A\overline{\epsilon} - b\overline{\mu} - T\mu_{\rm lin} - W\epsilon_{\rm lin} - \partial_A W\overline{D}^A\epsilon_{\rm lin}\,, \tag{5.11}$$

$$\hat{\overline{\mu}}^{(1)} = -Y^A\partial_A\overline{\mu}^{(1)} + 3b\epsilon^{(1)} + \overline{D}_A\left(b\overline{D}^A\epsilon^{(1)}\right) - T\left(\triangle\,\epsilon_{\rm log} + 2\epsilon_{\rm log}\right) + 3W\mu_{\rm log} - \partial_A W\overline{D}^A\mu_{\rm log}\,, \tag{5.12}$$

$$\hat{\overline{\epsilon}}^{(1)} = -Y^A\partial_A\overline{\epsilon}^{(1)} - b\overline{\mu}^{(1)} - T\mu_{\rm log} - \partial_A\left(\overline{D}^A W\epsilon_{\rm log}\right)\,. \tag{5.13}$$

in addition to (3.16)–(3.19).

6 Algebraic decoupling of the $u(1)$ charges

The presence of invertible central terms in the algebra of the improper gauge symmetries can be used to redefine the Poincaré generators in such a way that the new form of the algebra has a direct sum structure: the improper gauge charges commute with the new Poincaré generators. The redefinition is non-linear and is obtained by just applying the general formulas derived in [1].

It is in our case explicitly achieved by adding to the Poincaré transformations the following field-dependent improper gauge transformations

$$\mu^{(b,Y)}_{\rm lin} = -\mathcal{L}_Y\Psi_{\rm lin} - 3b\Phi_{\rm lin} - \overline{D}_A\left(b\overline{D}^A\Phi_{\rm lin}\right)\,, \tag{6.1}$$

$$\epsilon^{(b,Y)}_{\rm lin} = -\mathcal{L}_Y\Phi_{\rm lin} - b\Psi_{\rm lin}\,, \tag{6.2}$$

$$\mu^{(b,Y)}_{\rm log} = -\frac{1}{\sqrt{g}}\mathcal{L}_Y\overline{\Pi}^r - \overline{D}_A\left(b\overline{D}^A\overline{A}_r\right)\,, \tag{6.3}$$

$$\epsilon^{(b,Y)}_{\rm log} = -\mathcal{L}_Y\overline{A}_r - \frac{b}{\sqrt{g}}\overline{\Pi}^r\,, \tag{6.4}$$

$$\overline{\mu}^{(b,Y,T,W)} = -\mathcal{L}_Y\overline{\Psi} - \overline{D}_A(b\overline{A}^A) - T\left(\triangle\,\Phi_{\rm lin} + 2\Phi_{\rm lin}\right) - \partial_A W\overline{D}^A\Psi_{\rm lin}\,, \tag{6.5}$$

$$\overline{\epsilon}^{(b,Y,T,W)} = -\mathcal{L}_Y\Phi - b\overline{\Psi} - T\Psi_{\rm lin} - W\Phi_{\rm lin} - \partial_A W\overline{D}^A\Phi_{\rm lin}\,, \tag{6.6}$$

$$\begin{aligned}\overline{\mu}^{(1)}_{(b,Y,T,W)} = &-\frac{1}{\sqrt{g}}\mathcal{L}_Y\Pi^r_{(2)} - 3bA^{(2)}_r - \overline{D}_A\left(b\overline{D}^A A^{(2)}_r\right)\\ &- T\left(\triangle\,\overline{A}_r + 2\overline{A}_r\right) + \frac{1}{\sqrt{g}}\left(-3W\overline{\Pi}^r + \partial_A W\overline{D}^A\overline{\Pi}^r\right)\,,\end{aligned} \tag{6.7}$$

$$\overline{\epsilon}^{(1)}_{(b,Y,T,W)} = -\mathcal{L}_Y A^{(2)}_r - \frac{b}{\sqrt{g}}\Pi^r_{(2)} - \frac{T}{\sqrt{g}}\overline{\Pi}^r + 2W\overline{A}_r - \partial_A W\overline{D}^A\overline{A}_r\,. \tag{6.8}$$

The parameters of these transformations depend on the charges themselves and preserve integrability. The charges associated with these gauge transformations are given by

$$\begin{aligned} Q^{\text{extra}}_{\mu,\epsilon} = -\oint d^2x\Big\{ & b\Big(\overline{\Pi}^r\overline{\Psi} + \Pi^r_{(2)}\Psi_{\text{lin}} + \sqrt{\bar g}\partial_A\overline{A}_r\overline{A}^A + \sqrt{\bar g}\,\partial_A A^{(2)}_r\overline{D}^A\Phi_{\text{lin}} - \sqrt{\bar g}\,A^{(2)}_r\Phi_{\text{lin}}\Big) \\ & + Y^A\left(\overline{\Pi}^r\overline{A}_A + \Pi^r_{(2)}\partial_A\Phi_{\text{lin}} + \sqrt{\bar g}\,\overline{\Psi}\partial_A\overline{A}_r + \sqrt{\bar g}\,\Psi_{\text{lin}}\partial_A A^{(2)}_r\right) \\ & + T\Big(\overline{\Pi}^r\Psi_{\text{lin}} + \sqrt{\bar g}\,\partial_A\overline{A}_r\overline{D}^A\Phi_{\text{lin}} - 2\sqrt{\bar g}\,\overline{A}_r\Phi_{\text{lin}}\Big) \\ & + W\Big[\Big(\overline{\triangle}\,\overline{\Pi}^r + 3\overline{\Pi}^r\Big)\Phi_{\text{lin}} + \overline{D}^A\overline{\Pi}^r\partial_A\Phi_{\text{lin}} + \sqrt{\bar g}\,\overline{A}_r\overline{\triangle}\,\Psi_{\text{lin}} + \sqrt{\bar g}\,\partial_A\overline{A}_r\overline{D}^A\Psi_{\text{lin}}\Big]\Big\}\,. \end{aligned} \tag{6.9}$$

These boundary terms are supplemented by weakly vanishing bulk terms with gauge parameters $\epsilon^{\text{extra}}_{(b,Y,T,W)}$ and $\mu^{\text{extra}}_{(b,Y,T,W)}$, the fall-off of which is determined by the above parameters.

In fact, the above transformations were derived from these charges, which were constructed according to the general formulas of [1], so that integrabilty was not really an issue.

The new Poincaré generators read

$$\tilde P_{\xi,\xi^i} = P_{\xi,\xi^i} + G^{\text{extra}}_{\mu,\epsilon}\,, \tag{6.10}$$

Because

$$\mathcal{B}_{\xi,\xi^i} + Q^{\text{extra}}_{\mu,\epsilon} = 0\,, \tag{6.11}$$

the new Poincaré generators are pure bulk

$$\tilde P_{\xi,\xi^i} = \int d^3x\left[\xi\mathcal{H} + \xi^i\mathcal{H}_i + (\epsilon_{(T,W)} + \epsilon^{\text{extra}}_{(b,Y,T,W)})\mathcal{G} + (\mu_{(b,T,W)} + \mu^{\text{extra}}_{(b,Y,T,W)})\pi^0\right]\,, \tag{6.12}$$

without surface term. One can then check that their brackets with the $u(1)$ gauge charges are weakly zero:

$$\{G_{\mu,\epsilon}, \tilde P_{\xi,\xi^i}\} = 0\,. \tag{6.13}$$

This implies in particular that they are conserved with no explicit time dependence. One can also check that the new Poincaré generators satisfies the Poincaré algebra:

$$\{\tilde P_{\xi_1,\xi_1^i}, \tilde P_{\xi_2,\xi_2^i}\} = \tilde P_{\hat\xi,\hat\xi^i}\,. \tag{6.14}$$

The computation is direct. This result is actually guaranteed to hold by the general argument of [1].

7 Conclusions

In this paper, we have consistently extended the asymptotic symmetries of electromagnetism in four spacetime dimensions by allowing improper gauge transformations parametrized by coefficients that blow up at infinity like $\ln r$ and r. We have also shown that the structure of the algebra can be used to redefine the Poincaré generators in such a way that there is no $u(1)$ ambiguity in the definition of the Lorentz generators (angular momentum and boost generators (center of mass)). Even though parametrized by a function of the angles, the $u(1)$ charges transform in the trivial representation of the redefined Lorentz algebra. The necessary redefinitions are non-linear and can be performed in a direct manner because we have a well-defined Hamiltonian formulation (see [37] for a more complete discussion of this point in the context of Poisson manifolds). In particular, integrability of the redefined transformations is never an issue.

Similar results, which go beyond the analysis of [38] by adopting more flexible boundary conditions, can be established in higher dimensions as we shall show in future work.

One might wonder whether one could continue the construction and include gauge transformations that asymptotically blow up like r^2, or r^3 etc. While we have not made a systematic analysis of this question, preliminary investigations lead to a negative conclusion, because there are too many divergences in the formalism that cannot be cancelled by suitable choices of the parity conditions, which can only be of two types (even or odd). This agrees with the conclusion of [28] (note that the multipole charges of [35] generically diverge in a time-dependent context, since one might then have higher spherical harmonic terms in, say the $1/r$ order of the fields).

The terms linear in r in the gauge parameters are somewhat analogous to the superrotations or the Diff(S^2) transformations considered in the works [39–42]. Our study gives some hope that these can be consistently included at spatial infinity, but the non-linear complexity of Einstein theory calls for caution in drawing conclusions that might be premature. Further work is clearly needed to settle satisfactorily this issue.

Finally, it would be of great interest to investigate how our work is explicitly translated at null infinity. To achieve that task one should first go back to the standard 'non-extended' formalism where the Lagrange multiplier ψ is fixed in terms of the other fields since it is the formulation that is directly connected with the standard Lagrangian formulation. Work along these lines is in progress.

Note added. After this paper was completed, we became aware of the recent preprint [43] where the canonical formulation of gauge transformations that blow up at infinity is considered along different lines.

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A Poincaré invariance

In this appendix, we provide details on how the kinetic term (2.25) and the asymptotic form of the transformations of the fields under Poincaré transformation were arrived at.

A.1 Non-integrability of the boosts and time translations with standard symplectic form

We start with the familiar description of the canonical formalism of gravity in terms of the spatial components A_i of the vector potential and its conjugate momoentum π^i (electric field). The corresponding symplectic form is pure bulk and reads

$$\Omega = \int d^3x d_V \pi^i d_V A_i \,. \tag{A.1}$$

The change of the symplectic form Ω under the Poincaré diffeomorphism $\xi = br + T$ normal to the equal time hyperplanes is given by

$$\mathcal{L}_\xi \Omega = \oint d^2x \sqrt{\bar{g}}\, (br + T)\, \bar{g}^{AB} d_V F_{rA} d_V A_B \,. \tag{A.2}$$

Taking into account the fall-off of the mixed radial-angular component of the curvature

$$\begin{aligned} F_{rA} = &-\frac{1}{r}\left(\partial_A \overline{A}_r - \partial_A \Phi_{\log}\right) - \frac{\ln r}{r^2}\left(\partial_A A_r^{\log(2)} + A_A^{\log(2)}\right) \\ &- \frac{1}{r^2}\left(\partial_A A_r^{(2)} + A_A^{(2)} - A_A^{\log(2)}\right) + o\left(r^{-2}\right) , \end{aligned} \tag{A.3}$$

and of the angular component of the gauge field, we obtain the following expression,

$$\begin{aligned} \mathcal{L}_\xi \Omega = &-\oint d^2x \sqrt{\bar{g}}\, (br + T)\left(\partial_A d_V \overline{A}_r - \partial_A d_V \Phi_{\log}\right) \overline{D}^A d_V \Phi_{\text{lin}} \\ &- \ln r \oint d^2x \sqrt{\bar{g}}\, b \left(\partial_A d_V A_r^{\log(2)} + d_V A_A^{\log(2)}\right) \overline{D}^A d_V \Phi_{\text{lin}} \\ &- \ln r \oint d^2x \sqrt{\bar{g}}\, b \left(\partial_A d_V \overline{A}_r - \partial_A d_V \Phi_{\log}\right) D^A d_V \Phi_{\log} \\ &- \oint d^2x \sqrt{\bar{g}}\, b \left(\partial_A d_V \overline{A}_r - \partial_A d_V \Phi_{\log}\right) d_V \overline{A}^A \\ &- \oint d^2x \sqrt{\bar{g}}\, b \left(\partial_A d_V A_r^{(2)} + d_V A_A^{(2)} - d_V A_A^{\log(2)}\right) \overline{D}^A d_V \Phi_{\text{lin}} \,, \end{aligned} \tag{A.4}$$

We now make use of the fact that the components $A_i^{\log(2)}$ are pure gauge, then $\partial_A A_r^{\log(2)} + A_A^{\log(2)} = 0$, which takes care of the first logarithmic divergence. The remaining two divergent terms (in r and $\ln r$) vanish by assuming that $\Phi_{\log}$ is *odd*. After integration by parts and using that

$$\Phi_{\text{lin}} = \text{odd} \,, \tag{A.5}$$

$$\overline{A}_A = (\overline{A}_A)_{\text{even}} + \partial_A \Phi, \tag{A.6}$$

the finite terms can be re-written as

$$\begin{aligned} \mathcal{L}_\xi \Omega = &\oint d^2x \sqrt{\bar{g}}\, d_V \overline{A}_r \left[\partial_A \left(b d_V \overline{A}^A\right) + T \triangle d_V \Phi_{\text{lin}}\right] - \oint d^2x \sqrt{\bar{g}}\, \partial_A \left(b \overline{D}^A d_V \Phi_{\log}\right) d_V \Phi \\ &+ \oint d^2x \sqrt{\bar{g}}\, d_V A_r^{(2)} \partial_A \left(b \overline{D}^A d_V \Phi_{\text{lin}}\right) + \oint d^2x \sqrt{\bar{g}}\, \partial_A \left(b d_V A^{(2)A}\right) d_V \Phi_{\text{lin}} \\ &- \oint d^2x \sqrt{\bar{g}} \left[\partial_A \left(b d_V A^{\log(2)A}\right) + T \triangle d_V \Phi_{\log}\right] d_V \Phi_{\text{lin}} \,. \end{aligned} \tag{A.7}$$

These are not zero and hence, with the relaxed boundary conditions for A_i, Poincaré transformations are not canonical transformations for (A.1).

A.2 New symplectic form and invariance under boosts and time translations

In order to obtain an invariant symplectic form, we include A_0 and its conjugate π^0 with the asymptotic behaviour described in the text and adopt as new symplectic form

$$\Omega^{\text{new}} = \int d^3x (d_V\pi^i d_V A_i + d_V\pi^0 d_V A_0) \\ - \oint d^2x\sqrt{\overline{g}}\left[d_V\overline{A}_r d_V\overline{\Psi} - d_V\Psi_{\log}d_V\Phi + d_V A_r^{(2)} d_V\Psi_{\text{lin}} - \left(d_V\Psi_{\log}^{(1)} - d_V\Psi^{(1)}\right)d_V\Phi_{\text{lin}}\right], \tag{A.8}$$

(with $\pi^0 \approx 0$; Ω^{new} is just denoted Ω in the main text since it is the only one appearing there). The bulk term $d_V\pi^0 d_V A_0$ is the standard canonical one, the surface term $d_V\overline{A}_r d_V\overline{\Psi}$ has been introduced in [26] while the idea behind the introduction of the other surface terms is that if one takes $\delta_\xi\Psi_{\text{lin}} = \overline{D}_A\left(b\overline{D}^A\Phi_{\text{lin}}\right)$, $\delta_\xi\Psi_{\log} = \overline{D}_A\left(b\overline{D}^A\Phi_{\log}\right)$, $\delta_\xi\overline{\Psi} = \overline{D}_A(b\overline{A}^A) + T\overline{\triangle}\, d_V\Phi_{\text{lin}}$, $\delta_\xi\Psi_{\log}^{(1)} = \overline{D}_A\left(bA^{\log(2)A}\right) + T\overline{\triangle}\,\Phi_{\log}$ and $\delta_\xi\Psi^{(1)} = \overline{D}_A\left(bA^{(2)A}\right)$, one eliminates the non-integrable terms (A.7). However, this introduces other non-integrable terms (since there are additional variations coming from the new terms), so that the approach is not yet complete.

To get integrable boosts, we proceed as in [26] and adopt the transformation laws of the bulk fields shown there to work, namely,

$$\delta_\xi A_0 = \nabla_i\left(\xi A^i\right), \tag{A.9}$$

$$\delta_\xi\pi^0 = \xi\partial_i\pi^i, \tag{A.10}$$

$$\delta_\xi A_i = \frac{\xi\pi_i}{\sqrt{g}} + \partial_i(\xi\Psi), \tag{A.11}$$

$$\delta_\xi\pi^i = \sqrt{g}\nabla_m(F^{mi}\xi) + \xi\partial^i\pi^0. \tag{A.12}$$

This implies the following transformation laws for the asymptotic fields

$$\delta_\xi\Psi_{\text{lin}} = \overline{D}_A\left(b\overline{D}^A\Phi_{\text{lin}}\right) + 3b\Phi_{\text{lin}}, \tag{A.13}$$

$$\delta_\xi\Psi_{\log} = \overline{D}_A\left(b\overline{D}^A\Phi_{\log}\right), \tag{A.14}$$

$$\delta_\xi\overline{\Psi} = \overline{D}_A(b\overline{A}^A) + 2b\overline{A}_r + T\left(\overline{\triangle}\,\Phi_{\text{lin}} + 2\Phi_{\text{lin}}\right), \tag{A.15}$$

$$\delta_\xi\Psi_{\log}^{(1)} = \overline{D}_A\left(bA^{\log(2)A}\right) + bA_r^{\log(2)} + T\overline{\triangle}\,\Phi_{\log}, \tag{A.16}$$

$$\delta_\xi\Psi^{(1)} = \overline{D}_A\left(bA^{(2)A}\right) + b\left(A_r^{(2)} + A_r^{\log(2)}\right) + T\left(\overline{D}_A\overline{A}^A + \overline{A}_r\right). \tag{A.17}$$

$$\delta_\xi\Phi_{\text{lin}} = b\Psi_{\text{lin}}, \tag{A.18}$$

$$\delta_\xi\Phi = b\overline{\Psi}^{\text{odd}} + T\Psi_{\text{lin}}, \tag{A.19}$$

$$\delta_\xi\overline{A}_r = \frac{b}{\sqrt{\overline{g}}}\overline{\pi}_r + b\Psi_{\log}, \tag{A.20}$$

$$\delta_\xi A_r^{(2)} = \frac{b}{\sqrt{\overline{g}}}\pi^r_{(2)} + b\left(\Psi^{\log(1)} - \Psi^{(1)}\right) + \frac{T}{\sqrt{\overline{g}}}\overline{\pi}^r + T\left(\Psi_{\log} - \overline{\Psi}\right). \tag{A.21}$$

These contain additional terms besides the new ones written below (A.8).

The change in the bulk part of the symplectic form is found to be

$$\mathcal{L}_\xi\left(\Omega^{\text{new}}\right)_{\text{bulk}} = \int d^2S_i\xi d_V F^{ij} d_V A_j + r\oint d^2x b d_V\overline{\pi}^r d_V\Psi_{\text{lin}} + \ln r\oint d^2x b d_V \pi^r_{(2)} d_V\Psi_{\text{log}}$$
$$+\oint d^2x\left[b\left(d_V\overline{\pi}^r d_V\overline{\Psi} + d_V\pi^r_{(2)} d_V\Psi_{\text{lin}}\right) + T d_V\overline{\pi}^r d_V\Psi_{\text{lin}}\right]. \tag{A.22}$$

In order to eliminate the divergent terms we impose the following parity conditions

$$\Psi_{\text{lin}} = \text{even} \qquad \text{and} \qquad \Psi_{\text{log}} = \text{even}. \tag{A.23}$$

The variation of the boundary term of the symplectic form is given by

$$\mathcal{L}_\xi\left(\Omega^{\text{new}}\right)_{\text{boundary}} = \mathcal{L}_\xi\Omega^{\text{previous}} - \oint d^2x\left[b\left(d_V\overline{\pi}^r d_V\overline{\Psi} + d_V\pi^r_{(2)} d_V\Psi_{\text{lin}}\right) + T d_V\overline{\pi}^r d_V\Psi_{\text{lin}}\right]$$
$$-\oint d^2x\sqrt{\overline{g}}\left\{\left[4b d_V A_r^{(2)} + T d_V\left(\overline{D}_A\overline{A}^A + 3\overline{A}_r\right)\right] d_V\Phi_{\text{lin}} - T d_V\overline{\Psi} d_V\Psi_{\text{lin}}\right\}, \tag{A.24}$$

where $\mathcal{L}_\xi\Omega^{\text{previous}}$ is given in (A.7), and is identically cancelled by the first surface integral in (A.22).

Putting everything together, one finds that the variation of the new symplectic form reduces to the expression

$$\mathcal{L}_\xi\Omega^{\text{new}} = -\oint d^2x\sqrt{\overline{g}}\left\{\left[4b d_V A_r^{(2)} + T d_V\left(\overline{D}_A\overline{A}^A + 3\overline{A}_r\right)\right] d_V\Phi_{\text{lin}} - T d_V\overline{\Psi} d_V\Psi_{\text{lin}}\right\}. \tag{A.25}$$

These terms can be eliminated by performing the following field dependent subleading gauge transformations

$$\overline{\epsilon}^{(1)}_{(T)} = T\overline{\Psi}, \tag{A.26}$$
$$\overline{\mu}^{(1)}_{(b,T)} = 4bA_r^{(2)} + T\left(\overline{D}_A\overline{A}^A + 3\overline{A}_r\right), \tag{A.27}$$

where $\overline{\epsilon}^{(1)} = \epsilon^{(1)}_{\text{log}} - \epsilon^{(1)}$ and $\overline{\mu}^{(1)} = \mu^{(1)}_{\text{log}} - \mu^{(1)}$. Note that these correcting terms do not vanish even if the boosts are zero since they contain a contribution involving T.

A.3 Invariance of the new symplectic form under spatial translations

The change of the bulk term of the symplectic form under a spatial diffeomorphism is given by

$$\mathcal{L}_{\xi^i}\left(\Omega^{\text{new}}\right)_{\text{bulk}} = \int d^2S_I\xi^i\left(d_V\pi^j d_V A_j + d_V\pi^0 d_V A_0\right), \tag{A.28}$$

which reduces to the non-vanishing boundary term:

$$\mathcal{L}_{\xi^i}\left(\Omega^{\text{new}}\right)_{\text{bulk}} = \oint d^2x\left(W d_V\overline{\pi}^r - \partial_A W d_V\overline{\pi}^A\right) d_V\Phi_{\text{lin}}. \tag{A.29}$$

for our relaxed asymptotic conditions. On the other hand, the variation of the boundary term reads

$$\mathcal{L}_{\xi^i}\left(\Omega^{\text{new}}\right)_{\text{boundary}} = \oint d^2x\sqrt{\overline{g}}\left\{\left[-\partial_A W\overline{D}^A d_V\overline{\Psi} + W d_V\left(\Psi_{\text{log}} + \overline{\Psi}\right)\right] d_V\Phi_{\text{lin}}\right.$$
$$\left.+\left(\overline{D}^A W d_V\overline{A}_A - W d_V\overline{A}_r\right) d_V\Psi_{\text{lin}}\right\}. \tag{A.30}$$

The sum of these terms can be cancelled by corrective gauge transformations generated by the subleading parameters

$$\bar{\epsilon}^{(1)}_{(W)} = \overline{D}^A W \overline{A}_A - W \overline{A}_r \,, \tag{A.31}$$

$$\overline{\mu}^{(1)}_{(W)} = -\left(W\overline{\pi}^r - \partial_A W \overline{\pi}^A\right) - \left[-\partial_A W \overline{D}^A \overline{\Psi} + W\left(\Psi_{\log} + \overline{\Psi}\right)\right] , \tag{A.32}$$

which involve only spatial translations.

This completes the discussion of Poincaré invariance of the theory.

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