

RECEIVED: August 18, 2022

REVISED: October 19, 2022

ACCEPTED: February 22, 2023

PUBLISHED: March 8, 2023

Moduli stabilization in type IIB orientifolds at $h^{2,1} = 50$

Konstantinos Tsagkaris and Erik Plauschinn

*Institute for Theoretical Physics, Utrecht University,
Princetonplein 5, Utrecht 3584CC, The Netherlands*

E-mail: konst.uni@gmail.com, e.plauschinn@uu.nl

ABSTRACT: We study moduli stabilization in Calabi-Yau orientifold compactifications of type IIB string theory with O3- and O7-planes. We consider a Calabi-Yau three-fold with Hodge number $h^{2,1} = 50$ and stabilize all axio-dilaton and complex-structure moduli by three-form fluxes. This is a challenging task, especially for large moduli-space dimensions. To address this question we develop an algorithm to generate 10^5 flux vacua with small flux number N_{flux} . Based on recent results by Crinò et al. we estimate the bound imposed by the tadpole-cancellation condition as $N_{\text{flux}} \leq \mathcal{O}(10^3)$, however, the smallest flux number we obtain in our search is of order $N_{\text{flux}} = \mathcal{O}(10^4)$. This implies, in particular, that for all solutions to the F-term equations in our data set the tadpole conjecture is satisfied.

KEYWORDS: Flux Compactifications, Superstring Vacua

ARXIV EPRINT: [2207.13721](https://arxiv.org/abs/2207.13721)

Contents

1	Introduction	1
2	Moduli stabilization	2
3	Tadpole contribution of fluxes	5
4	An example with $h_{-}^{2,1} = 50$	7
5	Results and discussion	10
A	Some details on the model	14

1 Introduction

String theory is a consistent theory of quantum gravity including gauge interactions. The theory is defined in ten space-time dimensions — and the connection to physics in four dimensions is typically made by compactifying on Calabi-Yau three-folds. The resulting effective theory is determined largely by the geometry of the compact space, in particular, deformations that preserve the Calabi-Yau condition correspond to massless scalar fields. However, such fields are in conflict with experimental observations. One way to resolve this conflict is to include fluxes which can give masses to the moduli. Well-understood settings for this procedure are orientifold compactifications of type IIB string theory with O3- and O7-planes, where three-form fluxes generate a potential for the axio-dilaton and complex-structure moduli [1]. Stabilizing moduli in this way is the first step in the KKLT [2] and Large Volume Scenarios [3].

One may expect that a generic choice of fluxes will stabilize all axio-dilaton and complex-structure moduli. This expectation has been challenged recently. Fluxes are restricted by the geometry and topology of the compact space, more concretely, the flux number N_{flux} is bounded through the tadpole-cancellation condition. In [4] it has been argued that stabilizing moduli near a conifold locus requires large fluxes which are incompatible with the tadpole-cancellation condition and in [5–7] it has been discussed that for control of perturbative corrections in the Large Volume Scenario large flux numbers (likely exceeding the tadpole bound) are needed. In [8] we showed that the tadpole-cancellation condition can force moduli to be stabilized in a perturbatively poorly-controlled regime and in [9] it is argued that stabilizing all moduli in M-theory can be in tension with the tadpole condition. Taking these arguments one step further, the authors of [10] made the *tadpole conjecture* which implies that stabilizing a large number of moduli by fluxes is not possible.

The purpose of the present work is to investigate the tadpole conjecture for a concrete setting. We consider a Calabi-Yau three-fold with 50 complex-structure moduli for which the tadpole conjecture is applicable. All axio-dilaton and complex-structure moduli are stabilized by fluxes in the large-complex-structure limit. This is a challenging task — and we have developed an algorithm that allows us to construct a large number of flux vacua with a small flux number N_{flux} . Based on results of [11] we estimate a bound on N_{flux} from the tadpole-cancellation condition as $N_{\text{flux}} \leq \mathcal{O}(10^3)$ and compare with our solutions. We find that the vacua in our data set satisfy $N_{\text{flux}} \geq \mathcal{O}(10^4)$ and therefore exceed this bound. In particular, for the vacua we obtain the tadpole conjecture is satisfied.

This paper is organized as follows: in section 2 we briefly review moduli stabilization for type IIB orientifolds. In section 3 we discuss the tadpole conjecture in some detail, in section 4 we introduce a concrete setting for studying moduli stabilization, and in section 5 we present and discuss our results.

2 Moduli stabilization

In this section we introduce the setting for our subsequent discussion. We focus on the material necessary for moduli stabilization in the large-complex-structure regime and refer for instance to [12] for a more detailed introduction to this topic.

Moduli. We consider orientifold compactifications of type IIB string theory on Calabi-Yau three-folds \mathcal{X} with O3- and O7-planes. The orientifold projection splits the cohomologies of \mathcal{X} into even and odd eigenspaces $H_{\pm}^{p,q}(\mathcal{X})$, whose dimensions will be denoted by $h_{\pm}^{p,q}$. The effective four-dimensional theory obtained after compactification contains massless scalar fields, in particular, the axio-dilaton τ , $h_{-}^{2,1}$ complex-structure moduli z^i , $h_{+}^{1,1}$ Kähler moduli T_a , and $h_{-}^{1,1}$ moduli $G_{\hat{a}}$. We parametrize the first two as

$$\tau = c + is, \quad z^i = u^i + iv^i, \quad i = 1, \dots, h_{-}^{2,1}, \quad (2.1)$$

and the physical region of the dilaton is characterized by $s > 0$. The Kähler potential for these fields is given by

$$\mathcal{K} = -\log[-i(\tau - \bar{\tau})] - \log\left[+i \int_{\mathcal{X}} \Omega \wedge \bar{\Omega}\right] - 2\log \mathcal{V}, \quad (2.2)$$

where Ω denotes the holomorphic three-form of \mathcal{X} which depends on the complex-structure moduli z^i and \mathcal{V} denotes the volume of \mathcal{X} which depends on the Kähler moduli T_a and on the moduli $G_{\hat{a}}$.

Prepotential. For the third cohomology of the Calabi-Yau three-fold \mathcal{X} we can choose an integral symplectic basis $\{\alpha_I, \beta^I\} \in H_{-}^3(\mathcal{X}, \mathbb{Z})$. The holomorphic three-form is expanded in this basis in the following way

$$\Omega = X^I \alpha_I - \mathcal{F}_I \beta^I, \quad I = 0, \dots, h_{-}^{2,1}, \quad (2.3)$$

where the periods \mathcal{F}_I can be expressed using a prepotential \mathcal{F} as $\mathcal{F}_I = \partial_I \mathcal{F}$ with $\partial_I = \partial/\partial X^I$. The complex-structure moduli z^i are written in terms of the projective coordinates

X^I as $z^i = X^i/X^0$. In this paper we are interested in the large-complex-structure regime, for which the prepotential splits into a perturbative and a non-perturbative part

$$\mathcal{F} = \mathcal{F}_{\text{pert}} + \mathcal{F}_{\text{inst}}. \tag{2.4}$$

The perturbative part takes the following form [13] (we follow the discussion and conventions of [14])

$$\mathcal{F}_{\text{pert}} = -\frac{1}{3!} \frac{\tilde{\kappa}_{ijk} X^i X^j X^k}{X^0} + \frac{1}{2!} a_{ij} X^i X^j + b_i X^i X^0 + \frac{1}{2!} c (X^0)^2, \tag{2.5}$$

where $\tilde{\kappa}_{ijk}$ are the triple intersection numbers of the mirror-dual three-fold $\tilde{\mathcal{X}}$ and the constants a_{ij} , b_i , and c are given by

$$a_{ij} = \frac{1}{2} \begin{cases} \tilde{\kappa}_{iij} & i \geq j, \\ \tilde{\kappa}_{ijj} & i < j, \end{cases} \quad b_i = \frac{1}{24} \int_{\tilde{\mathcal{X}}} c_2(\tilde{\mathcal{X}}) \wedge \tilde{\beta}_i, \quad c = \frac{\zeta(3)\chi(\tilde{\mathcal{X}})}{(2\pi i)^3}. \tag{2.6}$$

Here, $c_2(\tilde{\mathcal{X}})$ denotes the second Chern class of $\tilde{\mathcal{X}}$, $\{\tilde{\beta}_i\}$ is a basis of $H^2(\tilde{\mathcal{X}}, \mathbb{Z})$ mirror-dual to the three-forms $\beta_i \in H^3(\mathcal{X}, \mathbb{Z})$, and $\chi(\tilde{\mathcal{X}})$ is the Euler number of $\tilde{\mathcal{X}}$. The non-perturbative part of the prepotential originates from world-sheet instanton corrections in the mirror-dual theory. It takes the form

$$\mathcal{F}_{\text{inst}} = -\frac{1}{(2\pi i)^3} (X^0)^2 \sum_{\vec{q}} N_{\vec{q}} \text{Li}_3 \left(e^{2\pi i q_i X^i / X^0} \right), \tag{2.7}$$

where Li_s denotes the polylogarithm, the vector \vec{q} represents effective curve classes in $H_2(\tilde{\mathcal{X}}, \mathbb{Z})$, and $N_{\vec{q}}$ are the genus-zero Gopakumar-Vafa invariants of $\tilde{\mathcal{X}}$.

Fluxes. In order to stabilize the axio-dilaton and complex-structure moduli we consider non-vanishing Neveu-Schwarz-Neveu-Schwarz (NS-NS) and Ramond-Ramond (R-R) three-form fluxes H_3 and F_3 . These fluxes are integer quantized and can be expanded as

$$H_3 = h^I \alpha_I - h_I \beta^I, \quad F_3 = f^I \alpha_I - f_I \beta^I, \tag{2.8}$$

where $h^I, h_I, f^I, f_I \in \mathbb{Z}$. They generate a scalar potential for the four-dimensional theory that can be computed from the superpotential

$$W = \int_{\mathcal{X}} \Omega \wedge G_3, \quad G_3 = F_3 - \tau H_3. \tag{2.9}$$

Since the Kähler potential (2.2) we are considering is of no-scale type and because the superpotential (2.9) does not depend on the moduli T_a or $G_{\hat{a}}$, the standard F-term scalar potential can be brought into the form

$$V = e^{\mathcal{K}} F_{\alpha} G^{\alpha\bar{\beta}} \bar{F}_{\bar{\beta}}. \tag{2.10}$$

The Kähler potential \mathcal{K} was shown in (2.2), the F-terms are given by the Kähler-covariant derivative of W as $F_{\alpha} = \partial_{\alpha} W + (\partial_{\alpha} \mathcal{K}) W$ with $\alpha = (\tau, z^i)$, and $G^{\alpha\bar{\beta}}$ denotes the inverse of the Kähler metric $G_{\alpha\bar{\beta}} = \partial_{\alpha} \partial_{\bar{\beta}} \mathcal{K}$.

Tadpole-cancellation condition. The fixed loci of the orientifold projection give rise to orientifold planes. These objects are charged under the R-R gauge potentials and therefore contribute to the corresponding Bianchi identities as sources. To solve these identities one typically has to introduce D-branes, which for our setting are D3- and D7-branes. Integrating the Bianchi identities leads to the tadpole-cancellation conditions, and relevant for our discussion is the D3-brane tadpole given by (for details on the derivation see for instance [15])

$$0 = N_{\text{flux}} + 2N_{\text{D3}} + Q_{\text{D3}}, \tag{2.11}$$

where we defined

$$N_{\text{flux}} = \int_{\mathcal{X}} F_3 \wedge H_3, \tag{2.12}$$

$$Q_{\text{D3}} = -\frac{N_{\text{O3}}}{2} - \sum_{\text{D7}_i} \left[\int_{\Gamma_{\text{D7}_i}} \text{tr} \left[F_{\text{D7}_i}^2 \right] + N_{\text{D7}_i} \frac{\chi(\Gamma_{\text{D7}_i})}{12} \right] - \sum_{\text{O7}_j} \frac{\chi(\Gamma_{\text{O7}_j})}{6}. \tag{2.13}$$

The flux number N_{flux} is non-negative in our conventions, N_{D7_i} denotes the number of D7-branes in a stack labelled by i wrapping a four-cycle Γ_{D7_i} in \mathcal{X} , and N_{D3} is the total number of D3-branes. Both of these numbers are counted without the orientifold images. Furthermore, F_{D7_i} is the open-string gauge flux for a stack i , N_{O3} is the total number of O3-planes and $\chi(\Gamma)$ denotes the Euler number of the cycle Γ .

Minima. We are interested in global minima of the scalar potential (2.10). These are given by vanishing F-terms $F_\alpha = 0$ and to implement these conditions we expand the three-form flux G_3 appearing in (2.9) in the integral symplectic basis $\{\alpha_I, \beta^I\}$ as

$$G_3 = m^I \alpha_I - e_I \beta^I. \tag{2.14}$$

Next, we denote the derivative of the periods by $\mathcal{F}_{IJ} = \partial_I \mathcal{F}_J$ and define the complex matrix

$$\mathcal{N}_{IJ} = \overline{\mathcal{F}}_{IJ} + 2i \frac{(\text{Im } \mathcal{F})_{IM} X^M (\text{Im } \mathcal{F})_{JN} X^N}{X^R (\text{Im } \mathcal{F})_{RS} X^S}, \tag{2.15}$$

with $I, J, \dots = 0, \dots, h_-^{2,1}$. The requirement of vanishing F-terms, that is the minimum conditions, can then be expressed as the following complex-valued matrix equation

$$e_I - \overline{\mathcal{N}}_{IJ} m^J = 0. \tag{2.16}$$

Stretched Kähler cone. The prepotential (2.4) is valid in the large-complex-structure regime of the complex-structure moduli space. Away from that limit the instanton sum appearing in (2.7) may not converge and the large-complex-structure expansion may break down. The validity of the expansion can be characterized using a *stretched Kähler cone* [16] (see also [17]). More concretely,

- the geometry of the complex-structure moduli space of \mathcal{X} in the large-complex-structure regime is mirror-dual to the geometry of the Kähler moduli space of $\tilde{\mathcal{X}}$ in the large-volume regime. For the latter one finds a cone structure and hence, by mirror symmetry, also the complex-structure moduli are restricted to lie in a cone.

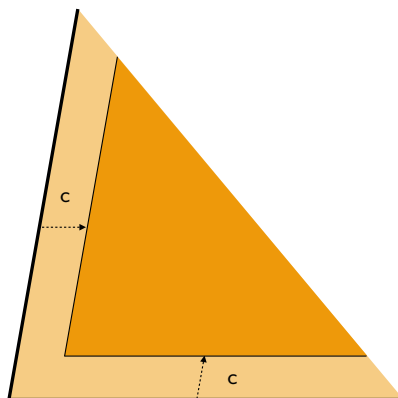


Figure 1. (Stretched) Kähler cones $\tilde{\mathcal{K}}_{\tilde{\mathcal{X}}}[0]$ and $\tilde{\mathcal{K}}_{\tilde{\mathcal{X}}}[c]$ for a two-dimensional setting. The darker shaded region is the stretched Kähler cone with parameter c .

- For the mirror-dual three-fold one can define a stretched Kähler cone in the following way [16]

$$\tilde{\mathcal{K}}_{\tilde{\mathcal{X}}}[c] = \left\{ \tilde{J} \in H^{1,1}(\tilde{\mathcal{X}}, \mathbb{R}) : \text{vol}(W) \geq c \quad \forall W \in \mathcal{W} \right\}, \quad (2.17)$$

where \tilde{J} denotes the Kähler form on $\tilde{\mathcal{X}}$, \mathcal{W} are all subvarieties (curves, divisors, $\tilde{\mathcal{X}}$) of the dual three-fold $\tilde{\mathcal{X}}$, and c is a constant in appropriate units (see figure 1 for an illustration). The ordinary Kähler cone is given by $\tilde{\mathcal{K}}_{\tilde{\mathcal{X}}}[0]$.

- Coming back to the prepotential (2.4), the sum of world-sheet instanton corrections (2.7) converges when the volumes of curves are sufficiently large. Hence, in order to trust the large-complex-structure expansion, the complex-structure moduli should be stabilized inside a stretched Kähler cone with parameter $c > 0$. We come back to this point below.

3 Tadpole contribution of fluxes

The NS-NS and R-R three-form fluxes H_3 and F_3 generate a potential in the effective four-dimensional theory that can stabilize the axio-dilaton and complex-structure moduli. However, H_3 and F_3 cannot be chosen arbitrarily but are restricted by the tadpole cancellation condition (2.11). In this section we discuss the interplay between these two questions and motivate the setting for our subsequent analysis.

The tadpole conjecture. Let us denote the number of axio-dilaton and complex-structure moduli which are stabilized through the flux superpotential (2.9) by n_{stab} . In [10] the authors conjectured that for $n_{\text{stab}} \gg 1$ the flux number N_{flux} shown in (2.12) grows at least linearly with n_{stab} . This conjecture has been called the *tadpole conjecture* and several versions thereof exist. In this paper we are interested in stabilizing all axio-dilaton and complex-structure moduli in type IIB orientifold compactifications by fluxes. In this case the conjecture takes the form

$$N_{\text{flux}} > 2\alpha(h_-^{2,1} + 1) \quad \text{for } h_-^{2,1} \gg 1, \quad (3.1)$$

and in the refined version the constant α is conjectured to be $\alpha = 1/3$. We briefly summarize the current status of this conjecture:

- The tadpole conjecture has been verified for F-theory compactifications on $K3 \times K3$ in [10, 18], for type IIB compactifications in the large complex-structure regime in [19], for F-theory compactifications on Calabi-Yau four-folds with a weak Fano base in [20], and for F-theory compactifications in asymptotic regimes in [21].
- A counter-example for the tadpole conjecture was proposed in [22], but was opposed in [23] and [24]. In particular, currently no concretely worked-out model that violates the tadpole conjecture is known to exist.
- The tadpole conjecture applies to smooth compactifications but could be violated for spaces that contain singularities. This was already noted in [10] and has been emphasized for instance in [6].

Contribution of orientifold planes and D7-branes. The flux number N_{flux} is bounded from below by the requirement to be positive and from above by the charge Q_{D3} in the tadpole cancellation condition (2.11). The contribution of orientifold planes appearing in Q_{D3} depends on the chosen orientifold projection and it is difficult to give a precise estimate on how it varies with $h_-^{2,1}$. For some classes of models complete classifications of the possible orientifold projections have been provided — see for instance [25] for a classification of orientifolds of $\mathbb{T}^6/\mathbb{Z}_M$ or $\mathbb{T}^6/\mathbb{Z}_M \times \mathbb{Z}_N$ and [26] for a classification for del-Pezzo surfaces. Furthermore, databases for orientifold compactifications have been constructed recently: in [27] orientifolds of CICYs have been classified, in [28] the authors extend their database on triangulations for the Kreuzer-Skarke list [29, 30] to include orientifold projections, and in [11] the authors systematically analyze the Kreuzer-Skarke database and classify orientifold projections. These analyses lead to the following bound on the D3-brane charge (2.13) in type IIB orientifold compactifications (see for instance equation (3.31) in [11])

$$Q_{D3} \geq -(2 + h^{1,1} + h^{2,1}). \tag{3.2}$$

Note that this bound is obtained for configurations in which D7-branes are placed on top of O7-planes and hence charges are cancelled locally. However, when considering D7-branes away from the orientifold locus (i.e. non-local cancellation of charges) smaller values for Q_{D3} can be found [31]. From the database of [11] we can infer the following bounds on Q_{D3} for local and non-local cancellation of D7-brane charges

$$Q_{D3} \geq \begin{cases} -48 & \text{local,} \\ -342 & \text{non-local,} \end{cases} \quad \text{for} \quad h^{1,1} = 5, \quad h_-^{2,1} = 50, \tag{3.3}$$

$$Q_{D3} \geq \begin{cases} -64 & \text{local,} \\ -722 & \text{non-local,} \end{cases} \quad \text{for} \quad h^{1,1} = 5, \quad 40 \leq h_-^{2,1} \leq 60. \tag{3.4}$$

Since the dependence of the bounds on Q_{D3} may fluctuate with $h_-^{2,1}$, in (3.4) we have broadened the search range to allow for variations in $h_-^{2,1}$. The values shown in (3.4) correspond to a model with $h^{2,1} = h_-^{2,1} = 57$.

Contribution of fluxes. We note that a priori the bound shown in (3.2) is compatible with the tadpole conjecture. However, the flux number N_{flux} may not saturate (3.1) but exceed it. In fact, for a toroidal orientifold compactification it was observed in [8] that the flux number will generically diverge when moduli are stabilized near the boundary of moduli space while it will take its smallest values in the interior. In [19] this argument has been made for more general settings.

Motivated by the above works, we are interested in what values the flux number N_{flux} typically takes when many moduli are stabilized. To study this question we consider the large-complex-structure regime for which the prepotential can easily be determined via mirror symmetry. The validity of the large-complex-structure approximation can be parametrized by the stretched Kähler-cone parameter c and we expect that for decreasing c the minimal N_{flux} will decrease — while at the same time the large-complex-structure expansion becomes less reliable. Similarly, the weak-string-coupling approximation is controlled by the dilaton (encoded in) s . In view of the tadpole conjecture we are therefore interested in the question:

for a given bound on the Kähler-cone parameter c and on the dilaton s , what is the smallest value of N_{flux} such that all axio-dilaton and complex-structure moduli are stabilized by fluxes?

4 An example with $h_{-}^{2,1} = 50$

In this section we introduce a concrete setting for stabilizing all axio-dilaton and complex-structure moduli through NS-NS and R-R three-form fluxes. We chose an example with $h_{-}^{2,1} = 50$ for which the tadpole conjecture is applicable.

The compactification space. As discussed in section 2, we employ mirror symmetry to construct the prepotential for the complex-structure moduli space in the large-complex-structure regime. We use `CYTools` [32] to determine the data for the mirror-dual setting as follows:

- We have randomly chosen a polytope of the Kreuzer-Skarke database [33]. Its normal form is shown in appendix A. With the help of `CYTools` we then perform a Delaunay triangulation using the command `triangulate()` and corresponding data is displayed as well in appendix A. The Calabi-Yau three-fold $\tilde{\mathcal{X}}$ is then obtained by `get_cy()`.
- The Hodge numbers for the mirror three-fold are determined using the functions `h11` and `h12` as

$$\tilde{h}^{1,1} = 50, \quad \tilde{h}^{2,1} = 5. \tag{4.1}$$

The perturbative part of the prepotential is specified by the triple-intersection numbers of $\tilde{\mathcal{X}}$ and the second Chern class, which are found using `intersection_numbers(in_basis=true)` as well as using the command `second_chern_class(in_basis=true)`. The Euler number of the mirror-dual three-fold is computed from (4.1) as

$$\chi(\tilde{\mathcal{X}}) = 90. \tag{4.2}$$

- The Kähler-cone conditions are encoded in a matrix \mathcal{M} , which can be computed using `CYTools` through `toric_kahler_cone().hyperplanes()`. A stretched Kähler cone with parameter c is specified by the conditions

$$\mathcal{M}\vec{v} \geq \vec{c}, \tag{4.3}$$

where \vec{v} is a vector with components $v^i = \text{Im} z^i$ and \vec{c} is a vector where each component takes the value c .

- Finally, we assume that an orientifold projection can be chosen such that $h_-^{2,1} = h^{2,1}$, so that from (4.1) we have

$$h_-^{2,1} = 50. \tag{4.4}$$

The tadpole cancellation condition (2.11) relates the bounds on Q_{D3} shown in (3.3) and (3.4) for non-local cancellation of D7-branes charges to the following bound on the flux number

$$N_{\text{flux}} \leq \mathcal{O}(10^3). \tag{4.5}$$

Estimating the validity of the large-complex-structure approximation. The prepotential (2.4) can be separated into a perturbative and a non-perturbative contribution. The perturbative part (2.5) is specified by the data discussed above, however, for the non-perturbative part (2.7) the Gopakumar-Vafa invariants $N_{\vec{q}}$ are needed. For our setting we currently do not have access to that data, although this functionality is expected to be implemented in `CYTools` in the future. We therefore want to determine a criterion that characterizes when non-perturbative contributions to the prepotential can be neglected.

Gopakumar-Vafa invariants relevant for one particular point in complex-structure moduli space were kindly provided to us by J. Moritz (see [34]). In the following we will denote this point by $\vec{z}_* = i\vec{v}_*$ and the vector \vec{v}_* is shown in appendix A. This point is located at the tip of a stretched Kähler cone with parameter $c = 1$ and there are in total 230 effective curves with wrapping numbers \vec{q} that satisfy $\vec{q} \cdot \vec{v}_* \leq 7$. Next, when scaling the vector \vec{v}_* with a parameter $\alpha > 0$ we obtain a family with

$$z^i(\alpha) = i\alpha z_*^i, \quad c = \alpha. \tag{4.6}$$

For this family we compute $\mathcal{F}_I = \partial_I \mathcal{F}$ for the perturbative and non-perturbative part of the prepotential and define the ratios

$$r_I(\alpha) = \left. \frac{\mathcal{F}_{\text{inst}I}}{\mathcal{F}_{\text{pert}I}} \right|_{z^i(\alpha)}. \tag{4.7}$$

When $r_I = 1$ for some I the non-perturbative contribution is comparable to the perturbative part and instanton corrections cannot be neglected. We then determine numerically the largest α for which any ratio r_I becomes one. We obtain

$$\max r_I(\alpha) = 1 \quad \Rightarrow \quad \alpha \simeq 0.04. \tag{4.8}$$

We conclude from this analysis that along the ray $\vec{z}(\alpha) = \alpha \vec{z}_*$ the large-complex-structure approximation can at most be trusted for $\alpha \geq 0.04$ — although it is very likely that the actual lower bound on α is larger. In the absence of complete information of the Gopakumar-Vafa invariants for our model, we estimate that the large-complex-structure approximation requires stretched Kähler-cone parameters at least of the form

$$c \geq 10^{-2}. \tag{4.9}$$

Constructing solutions. In the following we want to solve the minimum conditions (2.16) computed from the perturbative prepotential $\mathcal{F}_{\text{pert}}$ while ignoring the non-perturbative part. These conditions form a system of 102 real coupled polynomial equations which (in practice) cannot be solved analytically. Even numerically it is extremely difficult to obtain solutions because search algorithms typically require a starting point near a minimum. Finding a suitable starting point by random sampling is highly unlikely, especially for high-dimensional moduli spaces.

However, as stated at the end of section 3, we are interested in solutions with a small flux number N_{flux} for a given bound on the Kähler-cone parameter c such that all axio-dilaton and complex-structure moduli are stabilized. We developed an algorithm to construct such solutions with minimal N_{flux} is as follows:

- We first specify a stretched Kähler cone with parameter c . For each iteration of the algorithm c is chosen from a random uniform distribution in the range $c_{(0)} \in [0.1, 2]$.
- Next, we randomly choose a point $\{\tau_{(0)}, z_{(0)}^i\}$ in axio-dilaton and complex-structure moduli space. This point is taken from a uniform distribution in the ranges

$$s_{(0)} \in (0, 10], \quad c_{(0)} \in (-0.5, +0.5], \quad u_{(0)}^i \in (-0.5, +0.5]. \tag{4.10}$$

The point $v_{(0)}^i$ is obtained by determining the tip of the stretched Kähler cone $\tilde{\mathcal{K}}_{\tilde{\mathcal{X}}}[c_{(0)}]$. Note that in order to illustrate the dependence of N_{flux} on $s = \text{Im}\tau$ we sample $s_{(0)}$ also in the strong-coupling regime; we come back to this point below.

- The matrix \mathcal{N} appearing in the minimum conditions (2.16) scales approximately linearly with $v^i = \text{Im}z^i$. At large complex structure, the fluxes e_I therefore have to scale approximately as $v^i \times m^J$. Hence, to minimize the flux number we are interested in small values for m^J for which we choose randomly as

$$f_{(0)}^I, h_{(0)}^I \in \{-1, 0, +1\}. \tag{4.11}$$

- Inserting the values $\{\tau_{(0)}, z_{(0)}^i\}$ and the fluxes $\{f_{(0)}^I, h_{(0)}^I\}$ into the minimum condition (2.16), we can easily solve for the fluxes $f_{(0)I}$ and $h_{(0)I}$. These are in general not integer, but we round them to integer values denoted by $f_{(1)I}, h_{(1)I} \in \mathbb{Z}$.
- Next, we insert the integer-valued fluxes $\{f_{(0)}^I, h_{(0)}^I, f_{(1)I}, h_{(1)I}\}$ into the minimum condition. We solve (2.16) with Python's `scipy.optimize.root()` and specify as starting point for the search algorithm $\{\tau_{(0)}, z_{(0)}^i\}$. If for this choice of fluxes a solution $\{\tau_{(1)}, z_{(1)}^i\}$ is obtained, we determine the Hessian of the scalar potential at this point.

If the Hessian is of full rank, we add the solution to our data set. Note that these solutions can have a Kähler-cone parameter $c_{(1)}$ that is smaller than the initialization bound $c_{(0)} \geq 0.1$.

Using this strategy, we have generated $8.3 \cdot 10^4$ solutions in the large-complex-structure regime that stabilized all axio-dilaton and complex-structure moduli. Let us emphasize that our flux configurations are, on the one hand, non-generic among all possible flux configurations but, on the other hand, generic for solutions in the large-complex-structure regime. In particular, there can be (tuned) flux choices with a hierarchy different than the one described above which nevertheless stabilize moduli in the large-complex-structure regime with a small flux number.

Optimizing solutions. After having obtained solutions to the minimum conditions in the large-complex-structure regime, we use this data as starting points for minimizing the flux number N_{flux} . Our approach is as follows:

- We consider a flux configuration $\{h_{(1)I}, h_{(1)}^I, f_{(1)I}, f_{(1)}^I\}$ with flux number $N_{\text{flux}(1)}$ together with the corresponding minimum locus $\{\tau_{(1)}, z_{(1)}^i\}$.
- Next, we randomly change one of the flux quanta by one unit and denote the modified fluxes by $\{h_{(2)I}, h_{(2)}^I, f_{(2)I}, f_{(2)}^I\}$. If through that change the corresponding flux number $N_{\text{flux}(2)}$ is smaller than $N_{\text{flux}(1)}$, we numerically solve the minimum conditions for the new flux configuration with $\{\tau_{(1)}, z_{(1)}^i\}$ as starting point.
- If a solution $\{\tau_{(2)}, z_{(2)}^i\}$ is found that stabilizes all moduli (i.e. the corresponding Hessian of the scalar potential is of full rank) we repeat the algorithm with the new solution as starting point.

Through this mechanism we generated $1.7 \cdot 10^4$ additional solutions that we included in our data set.

5 Results and discussion

In this section we present the data obtained from the analysis outlined in section 4. This data, together with a corresponding `Mathematica` notebook explaining our conventions, can be found on the arXiv page¹ for this paper. In the following we show how the flux number N_{flux} depends on the Kähler-cone parameter c and on the string coupling encoded in s . We furthermore discuss the implications of our findings and give an outlook for future work.

Numerical results. In figures 2 and 3 we have shown how, in our data, the minimal flux number depends on the Kähler cone parameter c and on the string coupling s . We only included data points with $N_{\text{flux}} \leq 10^5$ in these plots, though we obtained many more with larger flux numbers. We make the following observations:

- The minimal flux number depends on the Kähler-cone parameter c . For smaller c we find a smaller N_{flux} (in agreement with [8, 19]), however, for smaller c the large-complex-structure approximation becomes less reliable. In figure 2 we have shown

¹[arXiv:2207.13721](https://arxiv.org/abs/2207.13721).

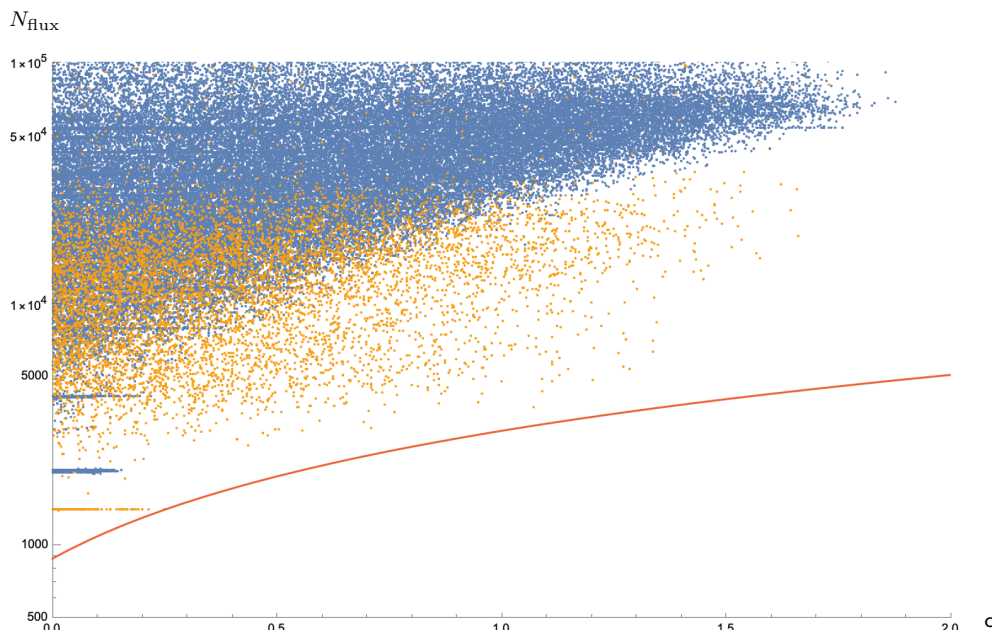


Figure 2. Dependence of N_{flux} on the Kähler-cone parameter c . Note that N_{flux} is shown on a logarithmic scale. The blue data points are solutions with $f^0 \neq 0$ or $h^0 \neq 0$, the orange data points are solutions with $f^0 = h^0 = 0$, and the red curve is the lower bound on N_{flux} shown in equation (5.1).

$N_{\text{flux}}(c)$ for which we determined the bound

$$N_{\text{flux}}(c) > 2100c + 880. \tag{5.1}$$

Note that for data points with $f^0 \neq 0$ or $h^0 \neq 0$ (shown in blue in figure 2) the bound on flux number scales as $N_{\text{flux}} \sim c^3$.

- The minimal flux number also depends on the value of the dilaton s . In figure 3 we have shown $N_{\text{flux}}(s)$, where we included points in the strong-coupling regime $s \simeq 1$. We furthermore determined the bound

$$N_{\text{flux}}(s) > 321s + \frac{282}{s}. \tag{5.2}$$

- In table 1 we summarized the minimal flux number for a given bound on Kähler-cone parameter c and the dilaton s . For values $s \lesssim 2$ and $c \lesssim 10^{-2}$ we do not believe that the corresponding vacua can be trusted and that corrections to the weak-coupling and large-complex-structure regime have to be taken into account. Such corrections are likely to modify these solutions, but it is beyond the scope of this paper to study this question here.
- The minimal flux numbers shown in table 1 are of the order $N_{\text{flux}} \geq \mathcal{O}(10^4)$ and therefore exceed the tadpole bound shown in (4.5) by about one order of magnitude. In particular, we did not find any flux choices that satisfy the tadpole cancellation condition and stabilize all axio-dilaton and complex-structure moduli.

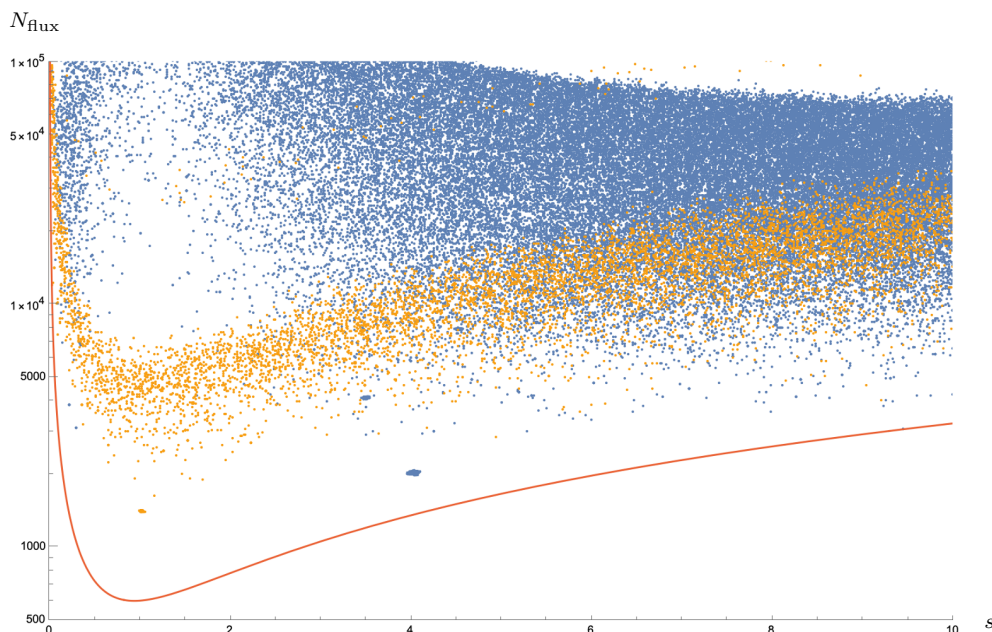


Figure 3. Dependence of N_{flux} on the dilaton s . Note that N_{flux} is shown on a logarithmic scale. The blue data points are solutions with $f^0 \neq 0$ or $h^0 \neq 0$, the orange data points are solutions with $f^0 = h^0 = 0$, and the red curve is the lower bound on N_{flux} shown in equation (5.2).

	$s \geq 1$	$s \geq 2$	$s \geq 5$	$s \geq 10$
$c \geq 10^{-3}$	1400	1991	3023	6157
$c \geq 10^{-2}$	1400	1991	3023	6157
$c \geq 10^{-1}$	1405	1992	3385	6157
$c \geq 0.5$	2993	3885	4886	13218
$c \geq 1$	3717	5379	9345	21384

Table 1. Smallest values for N_{flux} for given bounds on the Kähler-cone parameter c and the dilaton s .

- In figures 2 and 3 we used two different colors to distinguish between flux configurations with $f^0 = h^0 = 0$ and $f^0 \neq 0$ or $h^0 \neq 0$. We see that for $f^0 = h^0 = 0$ the minimal flux numbers is smaller as compared to flux choices with $f^0 \neq 0$ or $h^0 \neq 0$, which is in agreement with the discussion in [22].

Discussion. Let us now summarize our results and discuss them in a broader context:

- Stabilizing the axio-dilaton and complex-structure moduli by numerically minimizing a scalar potential becomes very difficult for large moduli-space dimensions. The main problem is to find a suitable starting point for the minimization procedure, which cannot be achieved by simple random searches. In this paper we presented an algorithm that allows us to find flux vacua for large $h_-^{2,1}$ that stabilize all moduli. To our knowledge, $h_-^{2,1} = 50$ is currently the largest moduli-space dimension for which all

axio-dilaton and complex-structure moduli are stabilized by fluxes (without imposing additional symmetries).

- The flux configurations we constructed have a specific structure and are therefore not generic within the flux space. However, our algorithm allows us to find a large number of flux vacua in the large-complex-structure limit that do not require a tuning of fluxes. In this sense these flux choices are generic (in the large-complex-structure regime).
- The bounds we obtain for the flux number N_{flux} are based on a data set with 10^5 flux vacua. By extending the search time one might be able to find vacua with lower N_{flux} .
- The lowest flux numbers we obtain in our search (cf. table 1) exceed the tadpole bound (4.5) by one order of magnitude. Our main conclusion therefore is that constructing consistent flux vacua in string theory that stabilize a large number of axio-dilaton and complex-structure moduli is generically difficult. In particular, we find that the tadpole conjecture [10] is satisfied for our data set.

Outlook. We close this section with an outlook for future work:

- The results presented in this paper provide a starting point for more extensive numerical searches for flux vacua with small flux number. Such searches can both employ more computing time as well as extending the ranges from which random starting points are drawn (cf. equations (4.10) and (4.11)). Our algorithm may also be refined by taking into account further details of the minimization condition (2.16).
- In the future we hope to include the non-perturbative contribution to the prepotential (2.7) for the computation of the minimum condition (2.16). To determine the non-perturbative part of the prepotential the Gopakumar-Vafa invariants are needed which may be provided by `CYTools` in the future.
- It is important to determine how representative the vacua we found are with respect to more general flux configurations that stabilize moduli — not only in the large-complex-structure regime but also in different regions of moduli space. This would allow us to revisit arguments on the size of the flux landscape.
- Solving the tadpole cancellation condition with D7-branes away from the orientifold locus can, on the one hand, lower the charge Q_{D3} and allow for larger flux numbers [31]. On the other hand, for such D7-branes the NS-NS three-form flux H_3 can induce a Freed-Witten anomaly [35] (see [36] for a brief review). For such branes one therefore not only has to take into account the interplay between fluxes and D3-branes but also between fluxes and D7-branes. We are planning to address this question in the future.

Acknowledgments

We thank Ralph Blumenhagen, Thomas Grimm, Umut Gürsoy, Damian van de Heisteg, Sven Krippendorf, Jakob Moritz, Fabian Ruehle, Andreas Schachner, Lorenz Schlechter, and Alexandros Singh for very helpful discussions and communications. We furthermore

thank Jakob Moritz for providing us with Gopakumar-Vafa invariants for our model. The work of EP is supported by a Heisenberg grant of the *Deutsche Forschungsgemeinschaft* (DFG, German Research Foundation) with project-number 430285316.

A Some details on the model

In this appendix we collect some additional information about the model described in section 4.

- The polytope in normal form from which we construct the mirror Calabi-Yau threefold $\tilde{\mathcal{X}}$ is given by the 6×4 dimensional array

$$\begin{aligned} & [[1 & 0 & 0 & 0] \\ & [1 & 3 & 0 & 0] \\ & [1 & 0 & 3 & 0] \\ & [-2 & -3 & -3 & 0] \\ & [1 & 0 & 0 & 3] \\ & [-2 & 0 & 0 & -3]] \end{aligned}$$

- The triangulation is performed using `triangulate()` of `CYTools`. For cross-reference, we note that the points of this triangulation can be displayed using `points()` which gives in the following 55×4 dimensional array

$$\begin{aligned} & [[0 & 0 & 0 & 0] & [1 & 0 & 2 & 0] & [-1 & -1 & -1 & 0] \\ & [-2 & -3 & -3 & 0] & [1 & 0 & 2 & 1] & [-1 & -1 & 0 & -1] \\ & [-2 & 0 & 0 & -3] & [1 & 1 & 0 & 0] & [-1 & 0 & -1 & -1] \\ & [1 & 0 & 0 & 0] & [1 & 1 & 0 & 2] & [0 & -1 & -1 & 1] \\ & [1 & 0 & 0 & 3] & [1 & 2 & 0 & 0] & [0 & -1 & 0 & 0] \\ & [1 & 0 & 3 & 0] & [1 & 2 & 0 & 1] & [0 & -1 & 0 & 1] \\ & [1 & 3 & 0 & 0] & [-1 & -2 & -1 & 0] & [0 & 0 & -1 & 0] \\ & [-2 & -2 & -2 & -1] & [-1 & -1 & -2 & 0] & [0 & 0 & -1 & 1] \\ & [-2 & -1 & -1 & -2] & [-1 & 0 & 0 & -2] & [0 & 0 & 1 & -1] \\ & [-1 & -2 & -2 & 0] & [-1 & 0 & 0 & -1] & [0 & 0 & 1 & 0] \\ & [-1 & -2 & -2 & 1] & [0 & -1 & 1 & 0] & [0 & 1 & 0 & -1] \\ & [-1 & 0 & 1 & -2] & [0 & 0 & 0 & -1] & [0 & 1 & 0 & 0] \\ & [-1 & 1 & 0 & -2] & [0 & 0 & 0 & 1] & [0 & 1 & 1 & -1] \\ & [0 & -1 & -1 & 0] & [0 & 1 & -1 & 0] & [1 & 0 & 1 & 1] \\ & [0 & -1 & -1 & 2] & [1 & 0 & 0 & 1] & [1 & 1 & 0 & 1] \\ & [0 & 0 & 2 & -1] & [1 & 0 & 0 & 2] & [1 & 1 & 1 & 0] \\ & [0 & 2 & 0 & -1] & [1 & 1 & 2 & 0] & [1 & 1 & 1 & 1]] \\ & [1 & 0 & 1 & 0] & [1 & 2 & 1 & 0] \\ & [1 & 0 & 1 & 2] & [-1 & -1 & -1 & -1] \end{aligned}$$

- The vector \vec{v}_* introduced above equation (4.6) is specified by

$$\begin{bmatrix}
 141 & -14 & 13 & 81 & -9 & -9 & 84 & 31 & 94 & 109 & -23 & -25 & 49 \\
 85 & -24 & -26 & 1 & 45 & -7 & 14 & 46 & -7 & 16 & 71 & 71 & -8 \\
 2 & 15 & 32 & 15 & 32 & 55 & -15 & -15 & 38 & 53 & 19 & 19 & 65 \\
 31 & 48 & 31 & 48 & -14 & -15 & -28 & 21 & 21 & -10 & 12 & &
 \end{bmatrix}$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

- [1] S.B. Giddings, S. Kachru and J. Polchinski, *Hierarchies from fluxes in string compactifications*, *Phys. Rev. D* **66** (2002) 106006 [[hep-th/0105097](#)] [[INSPIRE](#)].
- [2] S. Kachru, R. Kallosh, A.D. Linde and S.P. Trivedi, *de Sitter vacua in string theory*, *Phys. Rev. D* **68** (2003) 046005 [[hep-th/0301240](#)] [[INSPIRE](#)].
- [3] V. Balasubramanian, P. Berglund, J.P. Conlon and F. Quevedo, *Systematics of moduli stabilisation in Calabi-Yau flux compactifications*, *JHEP* **03** (2005) 007 [[hep-th/0502058](#)] [[INSPIRE](#)].
- [4] I. Bena, E. Dudas, M. Graña and S. Lüst, *Uplifting Runaways*, *Fortsch. Phys.* **67** (2019) 1800100 [[arXiv:1809.06861](#)] [[INSPIRE](#)].
- [5] D. Junghans, *LVS de Sitter Vacua are probably in the Swampland*, [arXiv:2201.03572](#) [[INSPIRE](#)].
- [6] X. Gao, A. Hebecker, S. Schreyer and G. Venken, *The LVS parametric tadpole constraint*, *JHEP* **07** (2022) 056 [[arXiv:2202.04087](#)] [[INSPIRE](#)].
- [7] D. Junghans, *Topological constraints in the LARGE-volume scenario*, *JHEP* **08** (2022) 226 [[arXiv:2205.02856](#)] [[INSPIRE](#)].
- [8] P. Betzler and E. Plauschinn, *Type IIB flux vacua and tadpole cancellation*, *Fortsch. Phys.* **67** (2019) 1900065 [[arXiv:1905.08823](#)] [[INSPIRE](#)].
- [9] A.P. Braun and R. Valandro, *G_4 flux, algebraic cycles and complex structure moduli stabilization*, *JHEP* **01** (2021) 207 [[arXiv:2009.11873](#)] [[INSPIRE](#)].
- [10] I. Bena, J. Blåbäck, M. Graña and S. Lüst, *The tadpole problem*, *JHEP* **11** (2021) 223 [[arXiv:2010.10519](#)] [[INSPIRE](#)].
- [11] C. Crinò, F. Quevedo, A. Schachner and R. Valandro, *A database of Calabi-Yau orientifolds and the size of D3-tadpoles*, *JHEP* **08** (2022) 050 [[arXiv:2204.13115](#)] [[INSPIRE](#)].
- [12] R. Blumenhagen, B. Körs, D. Lüst and S. Stieberger, *Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes*, *Phys. Rept.* **445** (2007) 1 [[hep-th/0610327](#)] [[INSPIRE](#)].
- [13] S. Hosono, A. Klemm and S. Theisen, *Lectures on mirror symmetry*, in *Lecture Notes in Physics* **436**, Springer (1994), pp. 235–280 [[DOI:10.1007/3-540-58453-6_13](#)] [[hep-th/9403096](#)] [[INSPIRE](#)].

- [14] M. Demirtas, M. Kim, L. McAllister, J. Moritz and A. Rios-Tascon, *Small cosmological constants in string theory*, *JHEP* **12** (2021) 136 [[arXiv:2107.09064](#)] [[INSPIRE](#)].
- [15] E. Plauschinn, *The Generalized Green-Schwarz Mechanism for Type IIB Orientifolds with D3- and D7-Branes*, *JHEP* **05** (2009) 062 [[arXiv:0811.2804](#)] [[INSPIRE](#)].
- [16] M. Demirtas, C. Long, L. McAllister and M. Stillman, *The Kreuzer-Skarke Axiverse*, *JHEP* **04** (2020) 138 [[arXiv:1808.01282](#)] [[INSPIRE](#)].
- [17] M. Cicoli, D. Ciupke, C. Mayrhofer and P. Shukla, *A Geometrical Upper Bound on the Inflaton Range*, *JHEP* **05** (2018) 001 [[arXiv:1801.05434](#)] [[INSPIRE](#)].
- [18] I. Bena, J. Blåbäck, M. Graña and S. Lüst, *Algorithmically Solving the Tadpole Problem*, *Adv. Appl. Clifford Algebras* **32** (2022) 7 [[arXiv:2103.03250](#)] [[INSPIRE](#)].
- [19] E. Plauschinn, *The tadpole conjecture at large complex-structure*, *JHEP* **02** (2022) 206 [[arXiv:2109.00029](#)] [[INSPIRE](#)].
- [20] I. Bena, C. Brodie and M. Graña, *D7 moduli stabilization: the tadpole menace*, *JHEP* **01** (2022) 138 [[arXiv:2112.00013](#)] [[INSPIRE](#)].
- [21] M. Graña et al., *The tadpole conjecture in asymptotic limits*, *JHEP* **08** (2022) 237 [[arXiv:2204.05331](#)] [[INSPIRE](#)].
- [22] F. Marchesano, D. Prieto and M. Wiesner, *F-theory flux vacua at large complex structure*, *JHEP* **08** (2021) 077 [[arXiv:2105.09326](#)] [[INSPIRE](#)].
- [23] S. Lüst, *Large complex structure flux vacua of IIB and the Tadpole Conjecture*, [arXiv:2109.05033](#) [[INSPIRE](#)].
- [24] T.W. Grimm, E. Plauschinn and D. van de Heisteeg, *Moduli stabilization in asymptotic flux compactifications*, *JHEP* **03** (2022) 117 [[arXiv:2110.05511](#)] [[INSPIRE](#)].
- [25] D. Lüst, S. Reffert, E. Scheidegger and S. Stieberger, *Resolved Toroidal Orbifolds and their Orientifolds*, *Adv. Theor. Math. Phys.* **12** (2008) 67 [[hep-th/0609014](#)] [[INSPIRE](#)].
- [26] R. Blumenhagen, V. Braun, T.W. Grimm and T. Weigand, *GUTs in Type IIB Orientifold Compactifications*, *Nucl. Phys. B* **815** (2009) 1 [[arXiv:0811.2936](#)] [[INSPIRE](#)].
- [27] F. Carta, J. Moritz and A. Westphal, *A landscape of orientifold vacua*, *JHEP* **05** (2020) 107 [[arXiv:2003.04902](#)] [[INSPIRE](#)].
- [28] R. Altman, J. Carifio, X. Gao and B.D. Nelson, *Orientifold Calabi-Yau threefolds with divisor involutions and string landscape*, *JHEP* **03** (2022) 087 [[arXiv:2111.03078](#)] [[INSPIRE](#)].
- [29] R. Altman, J. Gray, Y.-H. He, V. Jejjala and B.D. Nelson, *A Calabi-Yau Database: Threefolds Constructed from the Kreuzer-Skarke List*, *JHEP* **02** (2015) 158 [[arXiv:1411.1418](#)] [[INSPIRE](#)].
- [30] R. Altman, Y.-H. He, V. Jejjala and B.D. Nelson, *New large volume Calabi-Yau threefolds*, *Phys. Rev. D* **97** (2018) 046003 [[arXiv:1706.09070](#)] [[INSPIRE](#)].
- [31] A. Collinucci, F. Denef and M. Esole, *D-brane Deconstructions in IIB Orientifolds*, *JHEP* **02** (2009) 005 [[arXiv:0805.1573](#)] [[INSPIRE](#)].
- [32] M. Demirtas, A. Rios-Tascon and L. McAllister, *CYTools: A Software Package for Analyzing Calabi-Yau Manifolds*, [arXiv:2211.03823](#) [[INSPIRE](#)].

- [33] M. Kreuzer and H. Skarke, *Complete classification of reflexive polyhedra in four-dimensions*, *Adv. Theor. Math. Phys.* **4** (2000) 1209 [[hep-th/0002240](#)] [[INSPIRE](#)].
- [34] M. Demirtas, M. Kim, L. McAllister, J. Moritz and A. Rios-Tascon, *Computational Mirror Symmetry*, [arXiv:2303.00757](#) [MIT-CTP/5528] [[INSPIRE](#)].
- [35] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, *Asian J. Math.* **3** (1999) 819 [[hep-th/9907189](#)] [[INSPIRE](#)].
- [36] E. Plauschinn, *Non-geometric backgrounds in string theory*, *Phys. Rept.* **798** (2019) 1 [[arXiv:1811.11203](#)] [[INSPIRE](#)].