

Analytic thin wall false vacuum decay rate

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ABSTRACT: We derive a closed-form false vacuum decay rate at one loop in the thin wall limit, where the true and false vacua are nearly degenerate. We obtain the bounce configuration in D dimensions, together with the Euclidean action with a higher order correction, counter-terms and renormalization group running. We extract the functional determinant via the Gel'fand-Yaglom theorem for low and generic orbital multipoles. The negative and zero eigenvalues appear for low multipoles and the translational zeroes are removed. We compute the fluctuations for generic multipoles, multiply and regulate the orbital modes. We find an explicit finite renormalized decay rate in $D = 3, 4$ and give a closed-form expression for the finite functional determinant in any dimension.

KEYWORDS: Phase Transitions in the Early Universe, Cosmology of Theories BSM, Solitons Monopoles and Instantons, Nonperturbative Effects

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1 Introduction

Local ground states in physical systems, described by a field theory, may not be stable. A deeper minimum may exist, or appear with varying temperature, and quantum/thermal tunneling can trigger a first-order phase transition. Such a transition takes the system from the metastable false vacuum (FV) to a deeper minimum, called the true vacuum (TV). It is initiated by a sudden appearance of a bubble of TV upon the homogenous configuration of FV and the probability of such events is given by the FV decay rate.

The theory of bubble nucleation was pioneered by the works of Langer [1], and applied to field theory by Kobzarev et al. [2]. These initial results were rigorously derived and extended in seminal papers by Coleman on the bounce [3] (see also [4, 5]), the first quantum corrections with Callan [6] and gravitational effects with de Luccia [7]. Derivation of the rate was revisited in [8] and recently in [9, 10], elucidating and confirming previous results.

Developing a theoretical understanding of bubble nucleation and subsequent expansion is important for several reasons. The stability of the potential in the Standard Model (SM)

depends largely on the masses of the SM Higgs and the top. The bound on the Higgs mass was historically considered in [11, 12], based on absolute stability, and in [13, 14] from metastability/longevity. There has been tremendous progress on quantifying the SM rate, for example in recent works [15, 16] and references therein. In most theories beyond the SM, the presence of new particles may affect the stability of the Higgs vacuum or even alter the landscape of the vacua altogether. From these, only the physical vacua need to be selected, which requires them to be either globally stable or at least sufficiently long-lived when metastable. Thus, a precise knowledge of the decay rate puts specific constraints on the parameter space of such models.

Furthermore, the shape of the potential may change significantly at high temperatures in the early universe, leading to a first-order phase transition [17–19]. At a sufficiently high temperature, the time dimension compactifies and gets replaced by temperature [20, 21]. The FV decay rate at finite T then depends on the three-dimensional bounce and the fluctuations come from the $D = 3$ determinant as well [22], see also the recent works [23–27] regarding theoretical progress on the uncertainties of thermal rates.

Colliding bubbles in the early universe create gravitational waves [28–37]. These may be strong enough to be detected by the present observatories aLIGO [38] and aVIRGO [39], although most TeV scale phase transitions typically predict signals in reach of upcoming detectors, such as LISA [40–42], DECIGO [43] and BBO [44, 45]. Moreover, the expanding bubble walls may carry CP-violating interactions and be responsible for the creation of matter over anti-matter, for example in electroweak baryogenesis [46] or in extended scalar sectors. Other examples in which phase transitions are relevant include the creation of primordial B -fields [47–51], relation to [52, 53] and creation of dark matter [54], neutrino physics [56, 57] and primordial black holes [58], among others.

The usual approach to the calculation of the bubble nucleation rate is based on the semi-classical saddle point approximation in Euclidean spacetime. This requires finding the bounce solution, which is an unstable [59] instanton configuration that extremizes the action and interpolates between the two minima. An important simplification is that the bounce is proven to be $O(D)$ symmetric [60] for single field theories in flat spacetime with mild restrictions on the potential. This proof was extended to multi-fields in [61].

Currently, various ways of finding the bounce configuration are available. The first solution was found already in the original work [3] and was derived in the so-called thin wall (TW) limit, where the two minima are nearly degenerate. In such a setting, the size of the instanton becomes large, and therefore the transition region (or the wall of the bounce) from FV to TV becomes narrow, hence thin wall. This limit may be useful as an order of magnitude estimate, e.g. in thermal field theory when the nucleation temperature is not far away from the critical one. However, its applicability is limited and deteriorates [62, 63] when minima become more separated.

It is obviously desirable to have analytic control over the bounce part of the rate and some examples of exact solutions exist, for instance the Fubini-Lipatov instanton [64, 65] and its generalization [66], pure quartic [67], logarithmic [68, 69] and quartic-quartic potentials [70, 71]. Also notable is the triangular potential with two piecewise linear segments that was solved exactly in $D = 4$ [72], see also [73]. The validity of such kink

solutions was studied in [74] for a single field and in [75] for multi-field theories. The linear-quadratic solution was found in [70], while [76] performed the analytic continuation to Minkowski space.

Based upon the piecewise linear solution [72], a general construction was developed in [77]. An arbitrary number of linear segments were joined into a polygonal shape, which was solved semi-analytically in $D = 2, 3, 4, 6, 8$ dimensions. The piecewise linear was perturbatively expanded to second-order corrections in the potential and most significantly to any number of fields. This approach was implemented in the `FindBounce` package [78], which allows for a fast and arbitrarily precise evaluation of the bounce. In recent years, new approaches to the bounce calculation were proposed, based on the tunneling potential [79, 80], machine learning techniques [81] and real time formalism [9, 82–84]. Nowadays, several publicly available tools exist that tackle the issue, even with multiple fields [85–88].

Once the bounce is found, one can include one quantum corrections [6], which is a rather more involved calculation. It includes fluctuations from all the fields that couple to the bounce and thus enter into the second variation of the action, which ought to be evaluated in the presence of the bounce. This is similar to the path integral in quantum mechanics [89, 90], where one computes the (semi)classical particle trajectory and then performs the path integral around it. To compute the path integral in field theory, we have to find the eigenvalues of the fluctuation operator, which is the second functional derivative of the action, and then integrate over the resulting Gaussian coefficients. We are thus left with a functional determinant [6], a product of all the eigenvalues of the operator. Because the bounce is $O(D)$ symmetric, the fluctuations also come from a radially symmetric operator, which can therefore be decomposed into orbital multipoles. The product over all the multipoles is divergent and has to be properly regulated, e.g. with dimensional regularization, where poles of Γ functions and the renormalization scale enter. When the rate is written in terms of renormalized couplings, these artifacts go away, and one is left with a finite result. This much was stated already in the original work, however, an explicit solution is missing.

Following the original derivation of the rate [6], a couple of works found solutions in the TW, in particular [91] in $D = 4$ and [92] in $D = 3$. In more recent years, another calculation [93] was done using Green’s functions, claiming a numerical discrepancy with [91]. These works used explicit eigenfunctions with fixed (Dirichlet) boundary conditions. Instead of having to find all of these, one can use the Gel’fand-Yaglom [94] theorem¹ and consider solving a semi-open boundary condition, which automatically gives the product of eigenvalues without having to compute them.² This approach was adopted when deriving the renormalized rate using the WKB [100] and ζ function [101] formalism, using a finite orbital cutoff [100]. It is well suited for numerically evaluating the rate [102] and was used to obtain an analytic result for the complete parameter space of the quartic-quartic potential [71] in $D = 4$. Progress has also been made regarding the precise treatment of gauge bosons, leading to gauge invariant rates [103, 104] in single and multi-field theories [105].

¹The paper [94] is a survey article, where the first instance of the actual theorem is attributed to Cameron & Martin [95] and the derivation due to Montroll [96], see also [5].

²For pedagogical examples involving functional determinants and physical applications, see [90, 97–99].

The aim of the present paper is to work out the TW decay rate analytically, including the bounce and quantum fluctuations via the Gel'fand-Yaglom theorem, while keeping the number of dimensions arbitrary. The paper is organized in the following way. We start by giving the expressions for the decay rate in $D = 3, 4$ in section 2. This is followed by the general setup for the calculation of the rate at one loop in section 3. The entire section 4 is devoted to the Euclidean action, where we define the thin wall expansion and give the bounce field configuration in section 4.1, followed by the counter-terms and the running of parameters in section 4.2. Section 5 focuses on the quantum fluctuations. After defining the problem and setting up the expectations for the behavior of the fluctuations, we first derive the results for low multipoles in section 5.1. These contain the single negative unstable eigenvalue and the D translational zero modes, which are removed perturbatively in section 5.2. We then move on to high orbital modes in section 5.3 and find an expression, which is valid in the UV. In section 6 we regularize the functional determinant and show that the final decay rate is free of divergences and independent of the renormalization scale. We conclude and give an outlook in section 7. Technicalities are delegated to the appendices. The Euclidean action, valid to higher orders can be found in the appendix A, details on the functional determinants are given in B and the UV integrals that enter in the renormalization procedure are collected in C. In appendix D we compute the renormalized functional determinant for the single real quartic using the ζ formalism and show that it matches the one using Feynman diagrams. The final appendix E contains a general result for the functional determinant in any dimension.

2 Summary of the results

We consider the following potential for a real scalar field ϕ ,

$$V = \frac{\lambda}{8} (\phi^2 - v^2)^2 + \lambda \Delta v^3 (\phi - v), \tag{2.1}$$

which is a slightly rewritten form of what was considered in [3, 6]. This notation introduces a dimensionless coupling Δ , which governs the tunneling and defines the TW limit when $\Delta \rightarrow 0$. The linear term in ϕ is proportional to Δ and tilts the mexican hat potential, which breaks the degeneracy of the two minima, as seen in figure 1.

We assume that $0 < \lambda \ll 1$ and $0 < \Delta \ll 1$ are small enough for the theory to remain perturbative and for the TW expansion to be applicable. We chose $\Delta > 0$ to keep the FV on the right (see figure 1), while an upper bound from having the second minimum is given by $\Delta < 1/(3\sqrt{3}) \simeq 0.19$. In this regime we compute the FV decay rate analytically and find

$$\frac{\Gamma}{\mathcal{V}} \simeq \left(\left(\frac{S}{2\pi} \right) \frac{12}{e^{D-1}} \lambda v^2 \right)^{D/2} \exp \left[-S - \frac{1}{\Delta^{D-1}} \begin{cases} \frac{20+9\ln 3}{54}, & D = 3, \\ \frac{45-4\pi\sqrt{3}}{192}, & D = 4, \end{cases} \right] \tag{2.2}$$

with the Euclidean bounce action given by

$$S = \frac{1}{\Delta^{D-1}} \begin{cases} \frac{2^5 \pi v}{3^4 \sqrt{\lambda}} \left(1 - \left(\frac{9\pi^2}{4} - 1 \right) \Delta^2 \right), & D = 3, \\ \frac{\pi^2}{3\lambda} \left(1 - \left(2\pi^2 + \frac{9}{2} \right) \Delta^2 \right), & D = 4. \end{cases} \tag{2.3}$$

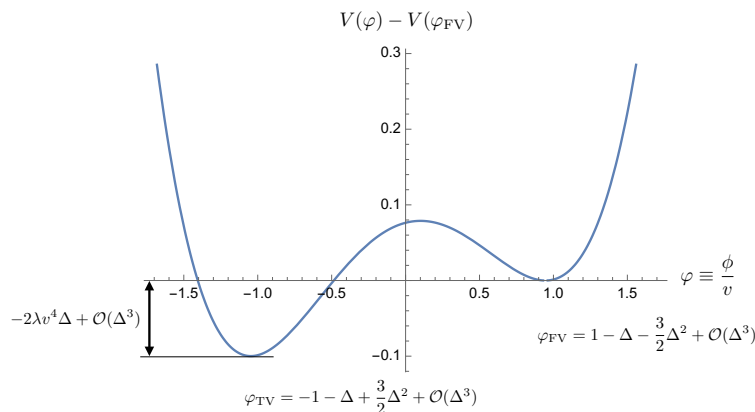


Figure 1. The potential (2.1) as a function of $\varphi \equiv \phi/v$ with V_{FV} subtracted, such that the difference vanishes at the FV. We set $\lambda v^4 = 1$, $\Delta = 0.05$ in the plot.

Here the Δ^3 corrections vanish, hence S is valid up to $\mathcal{O}(\Delta^4)$. The rate in (2.2) consists of the prefactor and the exponent. Both λ and v are dimensional parameters in general D ,³ but λv^2 always has mass dimension 2, or equivalently $(\text{length})^{-2}$, and Δ is dimensionless. The argument of the exponent in (2.2) is dimensionless and has two parts. The first comes from the renormalized bounce action in (2.3), given in terms of renormalized couplings, defined at the scale $\mu_0 = \sqrt{\lambda}v$. The second one comes from the functional determinant (6.9) in $D = 3$, and (6.20) in $D = 4$. We omit the $\mathcal{O}(\Delta^2/\Delta^{D-1})$ and higher corrections in such a term. As expected, the rate goes to zero either when $\lambda \rightarrow 0$, in which case the potential vanishes, or when $\Delta \rightarrow 0$, in which case the vacua are degenerate. Depending on the relative size of the couplings, the Δ^2 correction of the action in (2.3) and the one loop contribution from the functional determinant in (2.2) may be comparable and they should both be included.

3 Setup

The problem of computing the FV decay rate was lucidly set up in [3, 6]. The rate at one loop is given by (3.11) of [6], which we write for generic D dimensions in Euclidean space as

$$\frac{\Gamma}{\mathcal{V}} = \left(\frac{S_R}{2\pi\hbar} \right)^{\frac{D}{2}} \left| \frac{\det' \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right|^{-\frac{1}{2}} e^{-\frac{S_R}{\hbar} - S_{\text{ct}}} (1 + \mathcal{O}(\hbar)) , \quad (3.1)$$

where $\mathcal{O} = -\partial_\mu \partial^\mu + V^{(2)}$, S_R is the renormalized bounce action and S_{ct} is its one loop counterterm. The second derivative of the potential with respect to ϕ is denoted by $V^{(2)}$ and is calculated on the bounce field configuration, while $V_{\text{FV}}^{(2)}$ is evaluated at the FV. The \det' signifies the omission of zero eigenvalues from the determinant. There are D such eigenvalues associated with the translational symmetry in D dimensions. The removal of each zero mode brings about the insertion of $\sqrt{S/(2\pi\hbar)}$, hence we get the first factor in the equation above. In order to keep the tree level action finite, we subtract the constant FV

³In particular, at tree level $[\lambda] = 4 - D$, $[\phi] = [v] = D/2 - 1$, and $[\Delta] = 0$.

part of the potential, as shown in figure 1. Likewise, the FV of the action counter-term must be subtracted to keep S_{ct} finite at one loop.

The rate is thus a properly renormalized expression, free of UV divergences and independent of the renormalization scale μ at a given loop level. As we will see, the $\ln \mu$ and $1/\epsilon$ terms⁴ coming from S_R and S_{ct} in the exponent, will be compensated by the terms coming from the regularized functional determinant.

We kept factors of \hbar explicit in (3.1), but from here on we set $\hbar = 1$. One can recover the \hbar counting from the dimensionless action by considering the kinetic term to be $(\partial_\mu \phi)^2 = v^2 (\partial_\mu \varphi)^2$, while the potential term has an overall λv^4 factor when expressed in terms of φ . Rescaling $v \rightarrow \hbar^{-1/2} v$ and $\lambda \rightarrow \hbar \lambda$ gives $S \rightarrow S/\hbar$.

4 The bounce

Let us begin with the bounce field configuration, which enjoys a spherical $O(D)$ symmetry, as shown in [3, 60]. The bounce extremizes the Euclidean action

$$S = \int_D \left(\frac{1}{2} \dot{\phi}^2 + V - V_{\text{FV}} \right), \quad \int_D = \Omega \int_0^\infty d\rho \rho^{D-1}, \quad \Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \quad (4.1)$$

where $\rho^2 = t^2 + x_i^2$ is the Euclidean radius, Ω is the D -dimensional surface element and the dot denotes a derivative over ρ . We need to subtract the FV constant V_{FV} , i.e. the value of the potential at the FV, to make the action finite. The extremization of S corresponds to solving the Euler-Lagrange equations of motion with appropriate boundary conditions:

$$\begin{aligned} \ddot{\phi} + \frac{D-1}{\rho} \dot{\phi} &= \frac{dV}{d\phi}, & \dot{\phi}(0) &= \dot{\phi}(\infty) = 0, \\ \phi(0) &= \phi_{\text{in}}, & \phi(\infty) &= \phi_{\text{FV}}. \end{aligned} \quad (4.2)$$

These conditions ensure that the solution remains finite [3], while ϕ_{in} is in principle arbitrary. As we shall see, the TW expansion automatically sets ϕ_{in} to the TV value ϕ_{TV} .

4.1 Thin wall expansion of the action

Throughout this work, we use Δ as an expansion parameter for the bounce action and the fluctuations. The entire perturbation series is set in powers of Δ : the expansion for the fields, the Euclidean action and the functional determinant. It is convenient to work with the dimensionless field φ and the dimensionless Euclidean coordinate z , defined as

$$\frac{\phi}{v} \equiv \varphi = \sum \varphi_n \Delta^n, \quad z = \sqrt{\lambda v} \rho - r, \quad (4.3)$$

where $n \geq 0$ counts the powers of Δ , starting from the leading $n = 0$ term. The size of the bounce instanton (or the bubble wall) is set by the dimensionless Euclidean radius r , which becomes infinite in the TW, when $\Delta \rightarrow 0$. This justifies the expansion

$$r = \frac{1}{\Delta} \sum r_n \Delta^n. \quad (4.4)$$

⁴The $\ln \mu$ is only present in even D dimensions. Using dimensional regularization, we define $\epsilon = 4 - D$ in four dimensions, and analogously in generic D .

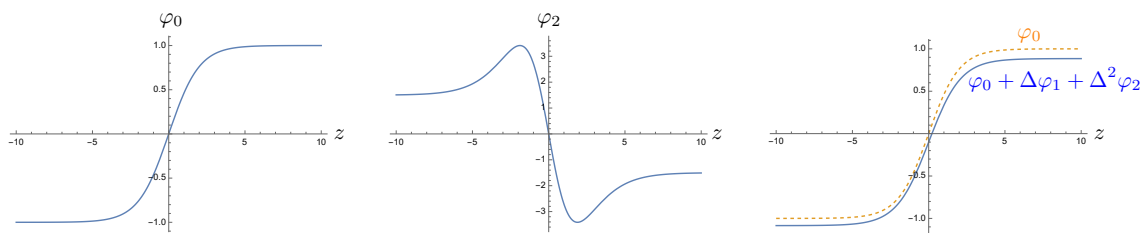


Figure 2. Bounce field configuration in $D = 4$ with $\Delta = 0.1$. Left: leading order φ_0 , center: φ_2 from (4.7) and on the right: $\varphi = \varphi_0 + \Delta\varphi_1 + \Delta^2\varphi_2$.

The bounce equation in (4.2) is rewritten using dimensionless variables, expanded in powers of Δ and solved in appendix A at each order, such that

$$\varphi_0 = \text{th} \frac{z}{2}, \quad (4.5)$$

$$\varphi_1 = -1, \quad (4.6)$$

$$\varphi_2 = \frac{3}{4(D-1)\text{ch}^2(z/2)} \left((2-D-2(4+\text{ch}z)\ln(1+e^z))\text{sh}z - z(D-e^z(4+\text{sh}z)) + 3(\text{Li}_2(-e^z) - \text{Li}_2(-e^{-z})) \right). \quad (4.7)$$

At the leading order the bounce solution φ_0 interpolates between the two symmetric minima. At next-to-leading order the bounce gets shifted by a constant φ_1 , while at $n = 2$ the φ_2 describes a non-trivial deformation of the bubble profile, as illustrated on figure 2. Examining the asymptotics at $\rho = 0, \infty$ or $z = \pm\infty$, we have

$$\varphi_0(z \rightarrow \pm\infty) = \pm 1, \quad \varphi_1(z \rightarrow \pm\infty) = -1, \quad \varphi_2(z \rightarrow \pm\infty) = \mp \frac{3}{2}, \quad (4.8)$$

as in figure 1. This implies that the extremal ends of the bounce correspond to the two vacua $\phi(\rho = 0) = \phi_{\text{in}} = \phi_{\text{TV}}$, $\phi(\rho \rightarrow \infty) = \phi_{\text{FV}}$, with increasing level of precision at higher orders of Δ . While this construction does not allow for an arbitrary ϕ_{in} outside of the TV, one does get a more accurate estimate of the action at higher powers of Δ , as reported in appendix A, where we go up to φ_3 .

Solving the bounce equation at the leading order gave us the shape of the bounce in φ_0 , but not its position. In order to complete the extremization, we need to fix the bubble radius r . To fix $r \sim r_0/\Delta$ we have to go to $n = 1$. This can be done by either extremizing the action explicitly as in [3] or implicitly by solving (4.2) at $n = 1$. These two procedures are equivalent and interchangeable at any order. As explained in appendix A, the radii up to second order, are given by

$$r_0 = \frac{D-1}{3}, \quad r_1 = 0, \quad r_2 = \frac{6\pi^2 - 40 + D(26 - 4D - 3\pi^2)}{3(D-1)}. \quad (4.9)$$

With the shape and position of the bounce fixed, the leading order action is given by

$$S_0 = \frac{\Omega v^{4-D}}{\lambda^{D/2-1} \Delta^{D-1}} \left(\frac{D-1}{3} \right)^{D-1} \frac{2}{3D}. \quad (4.10)$$

While v and λ are dimensional, the combination $v^{4-D}/\lambda^{D/2-1}$ that enters in S_0 is dimensionless in any D . The action diverges as $1/\Delta^{D-1}$ and thereby suppresses the rate in the TW limit. In $D = 4$ we have $S_0 = \pi^2/(3\lambda\Delta^3)$, so the decay rate goes as $\Gamma \propto \exp(-\pi^2/(3\lambda\Delta^3))$. This is reminiscent of the $\exp(-1/g^2)$ behaviour of gauge instantons [106], with g the gauge coupling, since quartics like λ act as g^2 . It is also closely related to the unstable quartic of the SM, where $\Gamma \propto \exp(-8\pi^2/(3\lambda))$. The TW result in (4.10) multiplies it by $1/\Delta^3$, which further suppresses the rate.⁵

Proceeding to higher orders, as worked out in appendix A, we get the following correction

$$S = S_0 \left(1 + \Delta^2 \left(\frac{1 + D(25 - 8D - 3\pi^2)}{2(D - 1)} \right) \right), \tag{4.11}$$

which is valid up to $\mathcal{O}(\Delta^4)$, since the Δ^3 term vanishes. This result is of interest for two reasons: it provides a more precise estimate of the TW action in any D , and serves as an upper bound for Δ , below which the TW approximation is still valid.

4.2 Renormalized bounce action

Because we are calculating a one loop quantity, we need to renormalize the bounce action, which requires adding the action counter-terms and running the relevant couplings. We follow the usual procedure with dimensional regularization, where one encounters a pole near integer even dimensions. Here we discuss the $D = 4$ case, where the deviation from the integer D is given by the usual $\varepsilon = 4 - D$.

Four dimensions.

Counterterms. We begin with the textbook derivation of counter-terms that renormalize the potential without the Δ term. We then turn on the linear Δ term and show that it does not affect the UV structure of the theory at one loop. In other words, the Δ counter-term is zero and this parameter does not run.

Let us define the counter-term potential V_{ct} and a shorthand $V^{(n)}$ for the n -th derivative, evaluated at $\phi = \langle\phi\rangle$,

$$V_{\text{ct}} = \frac{\delta m^2}{2} \phi^2 + \frac{\delta\lambda}{4} \phi^4, \quad V^{(n)} \equiv \frac{d^n V}{d\phi^n}(\langle\phi\rangle). \tag{4.12}$$

Removing the infinity from the four-point function fixes the quartic counter-term

$$\text{Diagram 1} + \text{Diagram 2} = 0, \quad \Rightarrow \quad \delta\lambda = \frac{1}{(4\pi)^2 2\varepsilon} V^{(4)2}. \tag{4.13}$$

The $1/\varepsilon$ pole in the 3-point function vanishes under the condition

$$\text{Diagram 3} + \text{Diagram 4} = 0, \quad \text{iff} \quad V^{(3)} = \langle\phi\rangle V^{(4)}. \tag{4.14}$$

⁵As discussed above, the dimensions of λ and v are such that their ratio $v^{4-D}/\lambda^{D/2-1}$ in (4.10) is dimensionless. However, in $D = 4$ thermal field theory fields have dimension 1, while $[\lambda] = 0$, whereby the S_3 action from (4.10) has dimension 1. In such an instance, the 3D action gets divided by T to keep the exponent S_3/T in the rate dimensionless.

This relation is automatically satisfied when the cubic term, i.e. $V^{(3)}$, comes solely from expanding the quartic after spontaneous breaking. Moving on to the tadpole, we remove the divergence with the quadratic counter-term

$$\begin{array}{c} \circ \\ \vdots \end{array} + \begin{array}{c} \otimes \\ \vdots \end{array} = 0, \quad \delta_{m^2} = \frac{1}{(4\pi)^2 \varepsilon} V^{(4)} \left(V^{(2)} - \frac{1}{2} V^{(4)} \langle \phi \rangle^2 \right). \quad (4.15)$$

Using this counter-term and the relation in (4.14), the two-point infinities cancel automatically

$$\begin{array}{c} \circ \\ \vdots \end{array} + \begin{array}{c} \circ \\ \vdots \end{array} + \begin{array}{c} \otimes \\ \vdots \end{array} = 0. \quad (4.16)$$

This procedure was set up for an arbitrary potential, now we can apply it to our V , given in (2.1). When the two minima are degenerate, the potential reduces to the simple quartic with $\langle \phi \rangle = v$, where all the consistency relations, derived above, are valid.

Now turn on the linear Δ term. The fourth and third derivatives remain the same, such that the δ_λ in (4.13) and the running of λ is unaffected. On the other hand, the position of the minima shifts away from v , so $\langle \phi \rangle$ is Δ -dependent. However, the finiteness of tadpoles and 2-point functions was derived for arbitrary $\langle \phi \rangle$ and remains valid for any Δ . Thus, even if we were to add a counter-term for Δ , we would have to set it to zero and Δ does not run.

With the δ_λ and δ_{m^2} at hand, we can calculate the counter-term for the Euclidean action. Plugging the V from (2.1) into (4.13) and (4.15), we have

$$\delta_\lambda = \frac{9\lambda^2}{(4\pi)^2 2\varepsilon}, \quad \delta_{m^2} = -\frac{3\lambda^2 v^2}{(4\pi)^2 2\varepsilon}. \quad (4.17)$$

The $\langle \phi \rangle$ terms in (4.15) cancel away and δ_{m^2} does not depend on Δ to all orders in n . With (4.17) we can compute the one-loop counter-term action:

$$S_{\text{ct}} = \int_D (V_{\text{ct}} - V_{\text{ctFV}}) = \frac{3\lambda^2}{8(4\pi)^2 \varepsilon} \int_D \left(3(\phi^4 - \phi_{\text{FV}}^4) - 2v^2(\phi^2 - \phi_{\text{FV}}^2) \right) \simeq -\frac{3}{16\varepsilon \Delta^3}. \quad (4.18)$$

This integration was performed in the TW expansion up to $\mathcal{O}(\Delta^2)$, which requires including φ_0 and φ_1 , see (A.61) and (A.64).

Running. With the δ_λ computed in (4.13), we get the usual one loop β_λ function that defines the running of λ

$$\beta_\lambda = \frac{d\lambda}{d \ln \mu} = \frac{9\lambda^2}{(4\pi)^2}, \quad \lambda(\mu) \simeq \lambda_0 + \frac{9\lambda_0^2}{(4\pi)^2} \ln \frac{\mu}{\mu_0}. \quad (4.19)$$

Here $\mu_0^2 \sim V_{\text{FV}}^{(2)} \sim \lambda_0 v^2$ is the scale where the renormalized λ_0 is measured by an anxious observer in the FV. Plugging the running coupling $\lambda(\mu)$ into the action, and including the counter-terms in (4.18), we obtain

$$S_R + S_{\text{ct}} = S \left(1 - \frac{9\lambda_0}{(4\pi)^2} \left(\frac{1}{\varepsilon} + \ln \frac{\mu}{\mu_0} \right) \right), \quad (4.20)$$

with S the tree-level bounce action given in (4.11). This is the combination that enters at the exponent in (3.1), and as we shall see, these are precisely the terms that will cancel the divergence and the μ -dependence from the functional determinant. Moreover, while the semi-classical S in (4.10) goes as \hbar^{-1} , the running and counterterms are one loop suppressed, and thus one order higher in \hbar , i.e. \hbar^0 .

Other dimensions. In the $\overline{\text{MS}}$ scheme the poles of Γ functions vanish in odd dimensions and there are no associated infinities and no renormalization scale, such that $S_R + S_{\text{ct}} = S$. In higher even dimensions, diagrams with more than two propagators become divergent, which means one has to introduce further counterterms for ϕ^6 in $D = 6$ and so on. These all get fixed, starting from the last divergent diagram with the highest number of external legs, and then going down to the quartics and tadpoles. In the end, all of them depend only on λ and v , and no new parameters enter. In case there is a clear physical motivation to consider a particular dimension $D \geq 6$, one can follow through these steps. However, the main focus of this work is on the functional determinant, therefore a general treatment of counterterms for any D goes beyond our scope.

5 Fluctuations

With the Euclidean action and the associated bounce at hand, we are ready to move on to the main objective, the calculation of the fluctuations. We wish to evaluate the ratio of determinants in (3.1)

$$\left| \frac{\det' \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right|^{-\frac{1}{2}} = \left| \prod_{l=0}^{\infty} \frac{\det' \mathcal{O}_l}{\det \mathcal{O}_{\text{FV}}} \right|^{-\frac{1}{2}}. \tag{5.1}$$

Exploiting the $O(D)$ symmetry of the problem, we perform the angular separation of the radial variable ρ , and expand the fluctuations in hyper-spherical multipoles l :

$$\mathcal{O}_l = -\frac{d^2}{d\rho^2} - \frac{D-1}{\rho} \frac{d}{d\rho} + \frac{l(l+D-2)}{\rho^2} + V^{(2)}, \tag{5.2}$$

where $V^{(2)} = d^2V/d\phi^2$ is evaluated on the bounce. The operator \mathcal{O}_{FV} is the same as \mathcal{O}_l , with $V^{(2)}$ replaced by $V_{\text{FV}}^{(2)}$. The prime in \det' means that the zero eigenvalues are removed.

The functional determinant $\det \mathcal{O}_l$ is computed around the bounce configuration, i.e. $\phi = \bar{\phi} + \psi$, where $\bar{\phi}$ is the bounce solution computed in the previous section and ψ are the fluctuations around it. To compute the determinant, one needs to define the basis for ψ and find the eigenvalues of the fluctuation operator with fixed Dirichlet boundary conditions at $\rho = 0$ and ∞ , and then multiply them. However, instead of solving the entire eigensystem, we take advantage of the Gel'fand-Yaglom theorem [94], according to which

$$\frac{\det \mathcal{O}_l}{\det \mathcal{O}_{\text{FV}}} = \lim_{\rho \rightarrow \infty} \left(\frac{\psi_l(\rho)}{\psi_{\text{FV}}(\rho)} \right)^{d_l}, \tag{5.3}$$

where the degeneracy is given by [90, 101]

$$d_l = \frac{(2l+D-2)(l+D-3)!}{l!(D-2)!}. \tag{5.4}$$

Setting $l = 0$ we get $d_0 = 1$ regardless of D and thereby a single s -wave eigenmode, whose eigenvalue should be negative [107]. On the other hand, plugging $l = 1$ into (5.4) gives $d_1 = D$ and corresponds to translations of the nucleated bubble in all possible D directions. Since the bounce is $O(D)$ symmetric and can nucleate anywhere, we expect to get D eigenmodes with zero eigenvalues due to translational symmetry. We will take care of them in section 5.2.

The ψ_l and $\psi_{l\text{FV}}$ that enter in (5.3) are now solutions of

$$\mathcal{O}_l \psi_l = 0, \quad \mathcal{O}_{l\text{FV}} \psi_{l\text{FV}} = 0, \quad (5.5)$$

with the boundary conditions $\psi_l(\rho \sim 0) \sim \rho^l$, $\psi_{l\text{FV}}(\rho \sim 0) \sim \rho^l$ fixed only at the origin. The theorem allows to compute the ratio of the two determinants

$$R_l \equiv \frac{\psi_l}{\psi_{l\text{FV}}}, \quad \text{and} \quad \frac{\det \mathcal{O}_l}{\det \mathcal{O}_{l\text{FV}}} = R_l(\infty)^{d_l}, \quad (5.6)$$

without explicitly computing all the eigenvalues. We can trade the ψ_l in (5.5) for R_l (see appendix B), and solve

$$\ddot{\psi}_{l\text{FV}} + \frac{D-1}{\rho} \dot{\psi}_{l\text{FV}} = \left(\frac{l(l+D-2)}{\rho^2} + V_{\text{FV}}^{(2)} \right) \psi_{l\text{FV}}, \quad (5.7)$$

$$\ddot{R}_l + 2 \left(\frac{\dot{\psi}_{l\text{FV}}}{\psi_{l\text{FV}}} \right) \dot{R}_l = \left(V^{(2)} - V_{\text{FV}}^{(2)} \right) R_l, \quad (5.8)$$

with boundary conditions

$$\psi_{l\text{FV}}(\rho \sim 0) \sim \rho^l, \quad R_l(\rho = 0) = 1, \quad \dot{R}_l(\rho = 0) = 0. \quad (5.9)$$

The main advantage of analyzing R_l directly [108] is that it is bounded on the entire ρ interval, because $V^{(2)} - V_{\text{FV}}^{(2)} \xrightarrow{\rho \rightarrow \infty} 0$. This was not the case for ψ_l and $\psi_{l\text{FV}}$, which diverge exponentially when $\rho \rightarrow \infty$, even though their ratio stays finite. Once we solve (5.8) and find $R_l(\infty)$, we expect the following behavior

$$R_l(\infty) = \begin{cases} < 0, & l = 0, \\ 0, & l = 1, \\ 1, & l \gg 1. \end{cases} \quad (5.10)$$

The first two come from the instability of the bounce and translational invariance. We will calculate the spectrum in this low multipole/IR limit in the upcoming section 5.1 and then show how to remove the zeroes from the determinant in 5.2. The last limit in (5.10), when $l \gg 1$, becomes obvious upon examining (5.2). In this UV limit, the orbital term dominates over the rest, and the $V^{(2)}$ term can be neglected. Therefore, there is no significant difference between \mathcal{O}_l and $\mathcal{O}_{l\text{FV}}$, so that $\psi_l \rightarrow \psi_{l\text{FV}}$ or $R_l \rightarrow 1$. The behavior of R_l in the UV will be examined separately in section 5.3.

5.1 Low multipoles

To study the low multipoles, let us start with the FV part in (5.7). Introducing

$$\nu = l + \frac{D}{2} - 1, \quad (5.11)$$

rescaling $\psi_{l\text{FV}} \rightarrow \rho^{\frac{D-1}{2}} \psi_{l\text{FV}}$ and switching to dimensionless variables, (5.7) simplifies to

$$\frac{d\psi_{l\text{FV}}^2}{dz^2} = \left(\frac{\nu^2 - \frac{1}{4}}{(z+r)^2} + \tilde{V}_{\text{FV}}^{(2)} \right) \psi_{l\text{FV}}, \quad (5.12)$$

with the FV constant given by

$$\tilde{V}_{\text{FV}}^{(2)} = \frac{1}{2}(3\varphi_{\text{FV}}^2 - 1) = 1 - 3\Delta - 3\Delta^2. \quad (5.13)$$

We are only interested in terms up to Δ^2 and will consistently be dropping the $\Delta^{p>2}$ powers in this section, such that

$$\frac{d\psi_{l\text{FV}}^2}{dz^2} = \left(1 - 3\Delta - 3\Delta^2 + \Delta^2 \left(\frac{\nu^2 - \frac{1}{4}}{r_0^2} \right) \right) \psi_{l\text{FV}}. \quad (5.14)$$

It becomes clear why we need to go to $\mathcal{O}(\Delta^2)$: there is no l dependence at lower orders. Moreover, up to this order, there is no z dependence⁶ on the right hand side of (5.14), so the FV solution is simply

$$\psi_{l\text{FV}}(z) \simeq c_{\text{FV}} \exp \left[\left(1 - \frac{3}{2}\Delta + \left(\frac{\nu^2 - \frac{1}{4}}{2r_0^2} - \frac{21}{8} \right) \Delta^2 \right) z \right]. \quad (5.15)$$

We expanded the exponent in powers of Δ and dropped the term proportional to e^{-z} in order to satisfy the boundary condition (5.9) at $z \rightarrow -\infty$, corresponding to $\rho = 0$. The FV solution has the form $\psi_{l\text{FV}} = \psi_{l\text{FV}0} \psi_{l\text{FV}1}^{\Delta} \psi_{l\text{FV}2}^{\Delta^2} \dots$ that motivates taking a *multiplicative expansion* for R_l

$$R_l = \prod_{n \geq 0} R_{ln}^{\Delta^n}, \quad \ln R_l = \sum_{n \geq 0} \ln R_{ln} \Delta^n. \quad (5.16)$$

One further simplifying technicality in the calculation of R_l is to work with an exponentially compensating variable $x = e^z$, which turns (5.8) into

$$\frac{d^2 R_l}{dx^2} + \frac{1}{x} \left(2 \frac{d\psi_{l\text{FV}}/dz}{\psi_{l\text{FV}}} + 1 \right) \frac{dR_l}{dx} = \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right) R_l. \quad (5.17)$$

Here $\tilde{V}^{(2)} = \frac{1}{2}(3\varphi^2 - 1)$, with φ the bounce solution from section 4.1. We plug the bounce expansion from (5.16) into (5.17) and solve it up to order Δ^2 . Up to this order, the combination $\frac{d\psi_{l\text{FV}}/dz}{\psi_{l\text{FV}}}$ does not depend on x . Further details of the calculation of R_l are

⁶The z dependence enters from $\mathcal{O}(\Delta^3)$ onwards.

deferred to appendix B, here we only discuss the results. From the first equation in (5.16), we have

$$R_l = R_{l0} \left(1 + \Delta \ln R_{l1} + \Delta^2 \left(\frac{1}{2} \ln^2 R_{l1} + \ln R_{l2} \right) \right), \quad (5.18)$$

and the solutions of (5.17) up to $n = 2$ are found to be

$$R_{l0} = \frac{1}{(1+x)^2}, \quad (5.19)$$

$$\ln R_{l1} = 3(r + \ln x), \quad (5.20)$$

$$\ln R_{l2}(x \rightarrow \infty) = \frac{3(l-1)(l+D-1)}{4(D-1)^2} x^2. \quad (5.21)$$

In the last equation, we only show the high- x part of $\ln R_{l2}$, as prescribed by the Gel'fand-Yaglom theorem (5.3), which grows as x^2 . In this limit, the R_{l0} goes as x^{-2} , implying that $R_l \rightarrow 0$ for the first two orders and a non-vanishing result comes in at $\mathcal{O}(\Delta^2)$. Indeed, $\ln R_{l2} \propto x^2$, which compensates exactly the large- x behavior of R_{l0} . As anticipated, the first non-trivial l -dependent term appears at second order in Δ and is obtained by plugging the solutions into (5.18), such that

$$R_l(\infty) = \Delta^2 e^{D-1} \frac{3(l-1)(l+D-1)}{4(D-1)^2}. \quad (5.22)$$

Some comments on the low- l fluctuations are in order.

- The final expression in (5.22) conforms to expectations from (5.10): the ratio of multiplied eigenvalues is negative for $l = 0$ and vanishes for $l = 1$.
- The $R_l(\infty)$ does not go to 1 when $l \gg 1$, as argued in (5.10), because the entire setup applies only for low l . Looking back at our starting point in (5.14), the l dependence enters via the $\Delta^2 \nu^2$ term, which was counted as $\mathcal{O}(\Delta^2)$. Such power counting only makes sense as long as $\nu < 1/\Delta$. In the following section, we will have to adapt it in order to access the higher multipoles.
- The factor e^{D-1} in (5.22) comes from $R_{l1}^\Delta = e^{(3r_0/\Delta + 3 \ln x)\Delta}$, tracing back to the $3r$ term in (5.20). This in turn is needed to implement the boundary condition at the origin $\rho = 0$, corresponding to $z = -r$. As explained in appendix B, proper implementation of the boundary conditions is crucial in order to keep the Δ power counting valid.

5.2 Zero removal

The $\det' \mathcal{O}_l$ that enters (5.1) and the total rate in (3.1), contains the truncated product of eigenvalues with the removed $l = 1$ zeroes due to the translational invariance of the bounce. In the Gel'fand-Yaglom approach, we cannot simply handpick and remove any individual eigenvalue, because all the eigenvalues with the same l are multiplied together. Instead, we subtract the zeroes perturbatively [16, 103, 104, 109] by first off-setting the fluctuation

potential with a small dimensionful parameter μ_ε^2 and then finding the corresponding solution of

$$\left(\mathcal{O}_1 + \mu_\varepsilon^2\right) \psi_1^\varepsilon = 0. \tag{5.23}$$

By introducing such a dimensional regulator in the off-set, the ratio of determinants does not approach zero as in (5.22), but instead goes to

$$R_1^\varepsilon(\infty) = \frac{\psi_1^\varepsilon(\infty)}{\psi_1^{\text{FV}}(\infty)} \simeq \frac{(\mu_\varepsilon^2 + \gamma_1) \prod_{n=2}^\infty \gamma_{\mathbf{n}}}{\prod_{n=1}^\infty \gamma_{\mathbf{n}}^{\text{FV}}} = \mu_\varepsilon^2 R_1'(\infty). \tag{5.24}$$

Here, the \mathbf{n} refers to the collective index over all the eigenvalues and γ_1 corresponds to the D -fold degenerate $l = 1$ eigenvalue. It also becomes clear from (5.24) that the dimension of R_1' is reduced by 2 with respect to R_l . To calculate the reduced determinant, we thus have to compute

$$R_1'(\infty) = \lim_{\mu_\varepsilon^2 \rightarrow 0} \frac{1}{\mu_\varepsilon^2} R_1^\varepsilon(\infty). \tag{5.25}$$

In the TW power counting scheme, the l dependence in the determinant appears at the Δ^2 order. Therefore, it is enough to lift the fluctuation potential at the same order, i.e. $V^{(2)} \rightarrow V^{(2)} + \Delta^2 \mu_\varepsilon^2$ in the R_l equation (5.8). This does not affect the FV, which has no zeroes to begin with. It also does not affect the $l = 1$ fluctuations at the leading and next-to-leading order, i.e. R_{l0} and R_{l1} remain the same. However, it does impact the R_{l2} fluctuations at the second order, where the solution for $l = 1$ is found to be

$$\ln R_{12}^\varepsilon = \frac{1}{12} \frac{\mu_\varepsilon^2}{\lambda v^2} x^2. \tag{5.26}$$

The technical details of this derivation are left to the appendix B, see (B.62)–(B.68). Plugging the result back into the multiplicative expansion defined in (5.18) and using (5.25), we have

$$R_1'(\infty) = R_{l0}(\infty) e^{D-1} \lim_{\mu_\varepsilon^2 \rightarrow 0} \frac{1}{\Delta^2 \mu_\varepsilon^2} \Delta^2 \ln R_{21}^\varepsilon = \frac{e^{D-1}}{12} \frac{1}{\lambda v^2}. \tag{5.27}$$

This is exactly what we would expect on dimensional grounds because each removal of zero gives us an additional power of $1/(\lambda v^2)$ and $[\lambda v^2] = 2$. The first part of the determinant is thus

$$\left| \frac{\det' \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right|^{-\frac{1}{2}} = \left(|R_{l0}| R_1'^D \prod_{l=2}^\infty \frac{\det \mathcal{O}_l}{\det \mathcal{O}_{l\text{FV}}} \right)^{-\frac{1}{2}}, \tag{5.28}$$

which is how the dimensional part enters in (2.2).

It may be insightful to derive this result in an alternative way. We can look for the explicit $l = 1$ eigenvalue γ_1 with Dirichlet boundary conditions, and then divide R_l by it. To this end, consider the ansatz $\psi_l = Y_{lm} \dot{\Phi}$, inspired by the fact that $\partial_\mu \Phi$ satisfies the equations of motion with zero eigenvalues [110]. Plugging it into $\mathcal{O}_l \psi_l = \gamma_1 \psi_l$ with \mathcal{O}_l from (5.2), we have

$$-\ddot{\Phi} - \frac{D-1}{\rho} \dot{\Phi} + V^{(2)} \dot{\Phi} + \frac{l(l+D-2)}{\rho^2} \dot{\Phi} = \gamma_1 \dot{\Phi}. \tag{5.29}$$

Taking the derivative of the bounce equation (4.2) with respect to ρ and rearranging the terms, we get to the following identity

$$-\ddot{\Phi} - \frac{D-1}{\rho}\dot{\Phi} + V^{(2)}\dot{\Phi} = -\frac{D-1}{\rho^2}\Phi. \quad (5.30)$$

Plugging it into (5.29), using $\rho = (r+z)/(\sqrt{\lambda}v)$, $r = r_0/\Delta$, with $r_0 = (D-1)/3$, and expanding in Δ , we obtain the desired Dirichlet eigenvalue

$$\gamma_1 = 9\lambda v^2 \Delta^2 \frac{(l-1)(l+D-1)}{(D-1)^2}. \quad (5.31)$$

After dividing R_l in (5.22) by γ_1 , the reduced determinant comes about

$$R'_l(\infty) = \frac{R_l(\infty)}{\gamma_1} = \frac{e^{D-1}}{12} \frac{1}{\lambda v^2}, \quad (5.32)$$

which is in agreement with (5.27).

5.3 Generic multipoles

The functional determinant in (5.1) involves a product of orbital multipoles up to infinity. However, the ratio of eigenvalues R_l , given in section 5.1, only applies for low l . We need to modify the Δ power counting such that we can access the UV part of the spectrum with arbitrary high ν , given by the UV-valid ratio R_ν .

The orbital numbers ν enter in the R_l equation (5.8) via the FV solution ψ_{lFV} that comes from solving (5.12). The key change here is to treat the combination $\Delta\nu$, which enters into the FV equation (5.12) in the TW expansion, as an order one parameter. With such modification, the ν multipoles enter the iterative expansion already at the leading order (instead of Δ^2 as in section 5.1). The leading order FV solution is thus given by

$$\psi_{\nu FV} \simeq e^{k_\nu z}, \quad k_\nu^2 = 1 + \frac{\Delta^2 \nu^2}{r_0^2}, \quad (5.33)$$

where the multipoles are characterized by k_ν , which goes from 1 for low multipoles (in agreement with the low multipole solution) and up to ∞ . Next, we consider the UV-valid equation for R_ν at the leading order, written in terms of the FV fluctuations

$$\frac{d^2}{dx^2} R_\nu + \frac{2k_\nu + 1}{x} \frac{d}{dx} R_\nu - \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{FV}^{(2)} \right) R_\nu = 0, \quad (5.34)$$

which is solved by

$$R_{\nu 0} = \frac{1}{(1+x)^2} \left(1 + 4 \frac{k_\nu - 1}{2k_\nu + 1} x + \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)} x^2 \right). \quad (5.35)$$

For low multipoles we have $k_\nu \rightarrow 1$, which reduces the above to $R_{\nu 0} \simeq 1/(1+x)^2$ and matches with (5.19). On the other hand, at large x we have

$$R_{\nu 0}(\infty) = \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)}. \quad (5.36)$$

However, the leading term alone is not enough, one needs to proceed to all orders in n , to recuperate and sum all the $\mathcal{O}(\Delta^0)$ terms. A systematic procedure of isolating all such terms is explained in detail in appendix B. While getting the individual terms is rather complicated, the final result after all the orders are summed up is remarkably simple

$$U = 3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1} \right). \quad (5.37)$$

Together with $R_{\nu 0}$ in (5.36), one gets the complete leading-order result

$$R_\nu(\infty) = R_{\nu 0}(\infty) e^U, \quad (5.38)$$

$$\ln R_\nu(\infty) = \ln \frac{(k_\nu - 1)(2k_\nu - 1)}{(k_\nu + 1)(2k_\nu + 1)} + 3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1} \right). \quad (5.39)$$

Before proceeding, let us comment on the salient features of this elegant result.

- The final result in (5.38) is a *complete* expression for the product of eigenvalues at a given multipole, which is non-zero at the leading order in Δ . This is in contrast to the low- l result in (5.22), where the first non-vanishing term appeared at $\mathcal{O}(\Delta^2)$. If we expand (5.38) for low multipoles ($\nu \ll 1/\Delta$) up to $\mathcal{O}(\Delta^2)$, using the definition of k_ν in (5.33), we do not recover (5.22). This is not a surprise: the result (5.38) is only valid at the leading order, $\mathcal{O}(\Delta^0)$, does not include all the $\mathcal{O}(\Delta^2)$ corrections, so it cannot reproduce (5.22). That is why we needed a separate calculation for the low multipoles. Note when $k_\nu \rightarrow 1$ one gets $U(k_\nu \rightarrow 1) = 3r_0$, confirming the e^{D-1} factor which appears in (5.22), and traces back to (5.20). As we shall see in the following section, the large multipoles are the ones that dominate in the determinant, and at the leading order those are fully accounted for by (5.39).
- At large multipoles when $k_\nu \rightarrow \infty$, the determinant goes to unity, in agreement with the expectation from (5.10). Higher order corrections in U behave as $\sim 1/k_\nu$ and vanish for large k_ν , such that they do not affect the correct behavior of $R_{\nu 0}$. However, they do provide sub-leading powers of k_ν , which are important in the sum we have to perform to obtain the determinant, as we will see in the next section.

6 Renormalized determinant

In the previous section we computed the negative $l = 0$ eigenmodes and removed the zero modes with $l = 1$, now we wish to evaluate the infinite product in (5.28). Taking the log and using Gel'fand-Yaglom (5.3), one has

$$\ln \left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \sum_{\nu=D/2-1}^{\infty} d_\nu \ln R_\nu, \quad (6.1)$$

where in this section we denote the asymptotic value $R_\nu(\infty)$ with R_ν , to ease the notation. The sum in (6.1) formally starts from $l = 0$, corresponding to $\nu = D/2 - 1$, and goes to ∞ ,

but as we shall shortly see, it is dominated by large multipoles in the TW limit.⁷ The low multipoles are suppressed by powers of Δ and we can use the generic multipole fluctuations from (5.39) to obtain the dominant contribution to the sum.

Let us examine the structure of divergences in the UV limit when $\nu \rightarrow \infty$. The degeneracy factor in terms of ν is given by

$$d_\nu = \frac{2\nu \left(\nu + \frac{D}{2} - 2\right)!}{(D-2)! \left(\nu - \frac{D}{2} + 1\right)!} \simeq \frac{2}{(D-2)!} \nu^{D-2}, \quad (6.2)$$

where the last approximation is valid in general D for large ν , but is exact for any ν in $D = 2, 3, 4$. Then we expand (5.39) up to including $\mathcal{O}(\nu^{-3})$

$$\sum_{\nu \gg 1} d_\nu \ln R_{\nu \gg 1} \sim -\frac{3r_0(2-r_0)}{(D-2)!\Delta} \sum_{\nu \gg 1} \nu^{D-2} \left(\frac{1}{\nu} - \frac{1}{\nu^3} \left(\frac{r_0}{2\Delta} \right)^2 \right). \quad (6.3)$$

This sum clearly diverges and the number of divergent terms depends on the dimension D . In $D = 2, 3$, only the first term diverges: it produces a log divergence in $D = 2$ and a linear one in $D = 3$. In $D = 4$, the first term gives a quadratic divergence and the second a logarithmic one, and so on to higher dimensions.

This is hardly surprising — the functional determinant is a one-loop quantity, which may diverge and must be regularized. To obtain a consistent finite result, we have to use the same regularization scheme that was adopted to compute the renormalized bounce action in section 4.2, i.e. the $\overline{\text{MS}}$ scheme. That can be done either by computing the one-loop effective action using Feynman diagrams [108], or with the ζ -function regularization scheme developed in [101], which was shown to be equivalent to $\overline{\text{MS}}$.⁸ In the rest of this section we compute the renormalized determinant explicitly in three and four dimensions. We begin with some specifications useful in generic dimensions.

Finite sum. To perform the sum over the multipoles in (6.1) we use the Euler-Maclaurin (EuMac) approximation. For later convenience it is useful to define Σ_D as the log of the determinant for generic D , with individual terms in the sum σ_D given by

$$\Sigma_D = \sum_{\nu=\nu_0}^{\infty} \sigma_D = \sum_{\nu=\nu_0}^{\infty} d_\nu (\ln R_\nu - \ln R_\nu^a). \quad (6.4)$$

The lower boundary is given by $\nu_0 = D/2 - 1$ and the $\ln R_\nu^a$ term stands for the generic asymptotic subtraction, needed to regulate the UV part ($\nu \rightarrow \infty$) and make the sum finite. The EuMac formula approximates the sum as an integral, Σ_D^{\int} , plus corrections from the

⁷The sum starting at $l = 0$ is in contrast with the product left in (5.28) starting at $l = 2$. However it is convenient to define the starting point at $l = 0$ here, in order to deal properly with the regularization, as we will see shortly.

⁸As a further check that the two schemes are indeed equivalent we provide an explicit example in appendix D, where we use the bounce solution for a purely quartic potential [16], which has a simpler form compared to the TW bounce.

boundaries, Σ_D^{bnd} , which in our case consist of the summand σ_D and its derivatives $\sigma_D^{(j)}$ evaluated at ν_0 ,

$$\Sigma_D \simeq \Sigma_D^f + \Sigma_D^{\text{bnd}} + R_p, \tag{6.5}$$

$$\Sigma_D^f = \int_{\nu_0}^{\infty} d\nu \sigma_D, \quad \Sigma_D^{\text{bnd}} = \frac{1}{2} \sigma_D(\nu_0) - \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \frac{B_{2j}}{(2j)!} \sigma_D^{(2j-1)}(\nu_0). \tag{6.6}$$

Here, B_{2j} are the Bernoulli numbers and R_p is the remainder, dependent on p , which is the desired order of approximation. In general Σ_D^{bnd} also contains σ_D evaluated at the upper boundary. Here, such terms $\sigma_D^{(p \geq 0)}(\infty)$, vanish because the subtraction of the asymptotic part guarantees that the sum is finite.

Three dimensions. The renormalized functional determinant [100, 101] requires the subtraction of a single term

$$\ln \left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \sum_{\nu} d_{\nu} \left(\ln R_{\nu} - \frac{1}{2\nu} I_1 \right). \tag{6.7}$$

Here $d_{\nu} = 2\nu$. The first term in the parenthesis is the ratio of determinants that we computed in (5.39). The second term is the asymptotic subtraction $\ln R_{\nu}^a$, which makes the sum finite. Note we only have to remove terms proportional to $1/\nu$. The I_1 coefficient is given by [100, 101]

$$I_1 = \int_0^{\infty} d\rho \rho \left(V^{(2)} - V_{\text{FV}}^{(2)} \right) \simeq -3(2 - r_0) \left(\frac{r_0}{\Delta} \right). \tag{6.8}$$

It is convenient to introduce $y = \Delta\nu/r_0$, which counts as $\mathcal{O}(\Delta^0)$ and thus helps to clarify the overall Δ dependence. The variable k_{ν} , which enters in $\ln R_{\nu}$, see (5.39), is related to y as $k_{\nu} = \sqrt{1 + y^2}$. Using (6.8), $r_0 = 2/3$, and comparing (6.7) to (6.4), we get the asymptotic form $\ln R_{\nu}^a = -2/y$. The integral piece of the EuMac approximation, after performing the change of variable $\nu \rightarrow y$, is given by

$$\Sigma_3^f \simeq 2 \left(\frac{r_0}{\Delta} \right)^2 \int_{y_0}^{\infty} dy y \left(\ln R_{\nu} + \frac{2}{y} \right) = \frac{1}{\Delta^2} \frac{20 + 9 \ln 3}{27}, \tag{6.9}$$

and does not depend on the precise value of the lower boundary of y , which can be extended to zero. Here it becomes clearer why the low-multipole result (5.22) is not important when we sum (integrate) over multipoles. Were we to use (5.22), instead of R_{ν} from (5.38), in (6.9) to integrate over low multipoles, we would get a contribution suppressed by Δ^2 compared to (6.9). That is subdominant, and we are only interested in the leading order terms in this section. We learn that the integral in (6.9) is manifestly dominated by the high multipoles.

For the remaining terms in the EuMac, we need to evaluate $\sigma_3(\nu_0)$ by treating $\nu = \mathcal{O}(1)$ and expanding in small Δ . The leading term comes from the asymptotic part and goes as $\sigma_3 \simeq 8/(3\Delta) + \mathcal{O}(\Delta^0)$, and is therefore suppressed compared to the leading Δ^{-2} dependence in (6.9). When taking higher derivatives over ν , we get corrections that are further suppressed by powers of Δ ; the Bernoulli terms are thus irrelevant.

The final result for the determinant is $\ln \left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \Sigma_3^f$, with Σ_3^f given in (6.9). This enters directly in the final expression for the decay rate (2.2). Recall that we are working in the $\overline{\text{MS}}$ scheme, so there are no counter-terms and no running in $D = 3$.

Thermal field theory. At high temperatures, when the periodic boundary conditions go below the size of the instanton, the time coordinate in 4D quantum field theory gets compactified and we end up with a 3D theory. In such case [20–22] (see also [19]), the rate is given by

$$\frac{\Gamma}{\mathcal{V}} \simeq \left(\frac{\lambda_-}{2\pi} \right) \left(\frac{S_3}{2\pi T} \right)^{\frac{3}{2}} \sqrt{\frac{\det \mathcal{O}_{\text{FV}}}{\det' \mathcal{O}}} e^{-S_3/T}, \quad (6.10)$$

where $(-\lambda_-^2)$ is the single negative eigenvalue of the 3D fluctuation operator. The Euclidean action is exactly the one given as S_3 in (4.10), but with the usual 4D counting of dimensions. Likewise, the determinant in (6.10) has to be calculated in $D = 3$, precisely what we got in (6.9) above, and is given by

$$\ln \sqrt{\frac{\det \mathcal{O}_{\text{FV}}}{\det' \mathcal{O}}} \simeq -\frac{1}{2} \Sigma_3. \quad (6.11)$$

Four dimensions. In this case the renormalized determinant needs the following subtractions [100, 101],

$$\ln \left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \sum_{\nu} d_{\nu} \left(\ln R_{\nu} - \frac{1}{2\nu} I_1 + \frac{1}{8\nu^3} I_2 \right) - \frac{1}{8} \tilde{I}_2. \quad (6.12)$$

Here we have $d_{\nu} = \nu^2$. The I_1 coefficient in (6.8) is also valid for $D = 4$, while I_2 and \tilde{I}_2 are obtained by inserting two powers of $V^{(2)}$. In the TW limit one has

$$I_2 = \int_0^{\infty} d\rho \rho^3 \left(V^{(2)2} - V_{\text{FV}}^{(2)2} \right) \simeq -3(2 - r_0) \left(\frac{r_0}{\Delta} \right)^3, \quad (6.13)$$

$$\begin{aligned} \tilde{I}_2 &= \int_0^{\infty} d\rho \rho^3 \left(V^{(2)2} - V_{\text{FV}}^{(2)2} \right) \left(\frac{1}{\varepsilon} + \gamma_E + 1 + \ln \left(\frac{\mu\rho}{2} \right) \right) \\ &\simeq I_2 \left(\frac{1}{\varepsilon} + \gamma_E + 1 + \ln \left(\frac{\mu r_0}{2\sqrt{\lambda\nu\Delta}} \right) \right). \end{aligned} \quad (6.14)$$

We anticipate that the I_2 will be valid for the asymptotic subtractions in higher D . Note that the \tilde{I}_2 part of the renormalized determinant is outside the sum. It contains the $1/\varepsilon$ pole and depends on the renormalization scale μ . The role of $\ln R_{\nu}^a = I_1/(2\nu) - I_2/(8\nu^3)$, inside the sum, is to remove the divergent parts of $\ln R_{\nu}$: the I_1 term regulates the quadratic divergence, while I_2 removes the logarithmic one.

Using $r_0 = 1$ and the variable $y = \Delta\nu/r_0$, we have $\ln R_{\nu}^a = -3/(2y) + 3/(8y^3)$ that enters in (6.4). The integral part of the EuMac approximation is then

$$\Sigma_4^f \simeq \frac{1}{\Delta^3} \int_{y_0}^{\infty} dy y^2 \left(\ln R_{\nu} + \frac{3}{2y} - \frac{3}{8y^3} \right) = \frac{3}{8\Delta^3} \left(\frac{9 - 4\sqrt{3}\pi}{36} + \ln 2y_0 \right). \quad (6.15)$$

Here, the finite piece $\frac{3}{8\Delta^3} \left(\frac{9-4\sqrt{3}\pi}{36} \right)$ comes from intermediate multipoles with $\Delta\nu \sim 1$ and behaves as $1/\Delta^3$. As in the $D = 3$ case discussed above, the low-multipole contribution from (5.22) would be suppressed, so we do not need to account for it in the integral. On the other hand, we get a $\ln y_0$ term that seems to depend on the low multipoles. It comes solely from the term in the asymptotic subtraction $\ln R_\nu^a$ which goes as $1/y^3$ and integrates into $\ln y_0$, due to lower boundary $y_0 = \Delta\nu_0/r_0$ in (6.15). However, this is an artifact of the ζ -regularization scheme. As we shall see, y_0 is just a regulator that will disappear from the final result.

Let us move on to the EuMac corrections given by Σ_D^{bnd} in (6.5). Here the lower boundary is at $\nu_0 \sim 1$, so we can expand the summands σ_D from (6.4) in small Δ and keep only the leading part, which is of order Δ^{-3} . The only terms that contribute come from the $3/8y^3$ piece of $\ln R_\nu^a$, and are of the form $\sigma_4^{(j)}(\nu_0) = 3(-)^{j+1}j!/(8\Delta^3\nu_0^{j+1})$, with $j \geq 0$. Recall that the superscript (j) here denotes the j -th derivative with respect to ν . Since $\nu_0 \sim 1$, when we plug $\sigma_4^{(j)}(\nu_0)$ into the EuMac approximation in (6.5), the Bernoulli terms eventually grow large and the sum starts to diverge.

We can delay the onset of this divergence by raising the ν_0 in the Bernoulli terms. To accomplish this, we split the original sum in (6.4) as follows

$$\Sigma_D = \Sigma_D^{\text{low}} + \Sigma_D^{\text{high}} = \sum_{\nu=\nu_0}^{\nu_1} \sigma_D + \sum_{\nu=\nu_1+1}^{\infty} \sigma_D. \quad (6.16)$$

Here the splitting point is such that $\mathcal{O}(1) = \nu_0 \ll \nu_1 < 1/\Delta$. To evaluate Σ_D^{high} we use the EuMac formula and replace ν_0 in (6.6) with $\nu_1 + 1 \simeq \nu_1$. Following the reasoning above, we now find that the EuMac boundary corrections Σ^{bnd} containing $\sigma_4^{(j)}(\nu_1)$ are formally suppressed, because $\nu_1 \gg 1$, so we can neglect them. Then the high sum is well approximated by the integral (6.15), with y_0 replaced by $y_1 = \Delta\nu_1/r_0$, such that

$$\Sigma_4^{\text{high}} = \frac{3}{8\Delta^3} \left(\frac{9 - 4\sqrt{3}\pi}{36} + \ln 2y_1 \right). \quad (6.17)$$

The low sum Σ_4^{low} can be performed explicitly. In the range up to $\nu_1 < 1/\Delta$ the leading piece in the summand σ_4 comes from the $3/(8y^3)$ term in $\ln R_\nu^a$, that is $3/(8\Delta^3\nu)$ once multiplied by $d_\nu = \nu^2$. Compared to it, all the other terms in σ_4 are suppressed by powers of Δ , so we neglect them. Then Σ_4^{low} is given by the Harmonic number H_{ν_1} , which can be expanded for $\nu_1 \gg 1$

$$\Sigma_4^{\text{low}} = -\frac{3}{8\Delta^3} \sum_{\nu=1}^{\nu_1} \frac{1}{\nu} = -\frac{3}{8\Delta^3} H_{\nu_1} \simeq -\frac{3}{8\Delta^3} (\ln \nu_1 + \gamma_E). \quad (6.18)$$

The low-multipole sum of the fluctuations from (5.22) still do not play any role. They would give terms suppressed by Δ^2 compared to (6.18), which we neglect. Adding up (6.18) and (6.17), we get

$$\Sigma_4 = \frac{3}{8\Delta^3} \left(\frac{9 - 4\sqrt{3}\pi}{36} - \gamma_E + \ln 2\Delta \right). \quad (6.19)$$

It is reassuring that the dependence on the arbitrary point of separation ν_1 is gone.

Renormalized rate. To complete the calculation of the decay rate, we need to sum up the renormalized functional determinant, consisting of the finite sum Σ_2 above and the \tilde{I}_2 term in (6.14),

$$\ln \left(\frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \Sigma_4 - \frac{\tilde{I}_2}{8}, \tag{6.20}$$

with the renormalized Euclidean action $S_R + S_{\text{ct}}$ in (4.20), that gives

$$\ln \frac{\Gamma}{\mathcal{V}} \ni -S_R - S_{\text{ct}} - \frac{1}{2} \left(\Sigma_4 - \frac{\tilde{I}_2}{8} \right) = -S - \frac{1}{\Delta^3} \frac{45 - 4\pi\sqrt{3}}{192}. \tag{6.21}$$

The action S is given in (4.11) in terms of renormalized couplings. The finite expression (6.21) enters in the final result of (2.2). Both, the $1/\varepsilon$ pole and the $\ln \mu$ from the running of λ have canceled out. The remaining $\mu_0 \simeq \sqrt{\lambda}v$ in (4.20) cancels with $\sqrt{\lambda}v$ in the \ln part of (6.14), and the combination $-\gamma_E + \ln 2\Delta$ in (6.19) also cancels against the analogous terms in (6.14).

The procedure of obtaining a finite functional determinant can be generalized to any even or odd dimension in general. This is done in appendix E.

7 Conclusions and outlook

In this work we presented a dimensionally unified treatment of the false vacuum decay for a single real scalar in the benchmark TW limit. We started by reviewing the bounce solution, which is valid for any D , including the second order correction in the expansion parameter Δ , as well as the counter-terms and RGE running. Employing the Gel'fand-Yaglom theorem, we found the orbital multipoles of the functional determinant around the bounce in two regions. The first region was the low multipole one, where the negative eigenvalues and the translational zeroes appear. The zeroes are removed perturbatively and that is how we get the correct physical dimension of the rate. The second region is computed by modifying the Δ counting such that $\Delta l = \mathcal{O}(1)$ and is therefore valid for higher multipoles, going up to infinity.

The renormalization section shows how the infinities in the orbital summation get regulated in the TW limit. In particular, we review how the subtractions in inverse powers of multipoles, $1/\varepsilon$ poles and the μ dependence are derived and evaluated in the TW limit, which gives us the asymptotic part of the determinant that subtracts the leading divergencies. Finally, we show how the multipoles are added up using the Euler-Maclaurin approximation.

It turns out that the summations are somewhat different for even and odd dimensions. In odd dimensions, the dominant contribution to the determinant comes from higher multipoles and no renormalization scale dependence appears. In even dimensions, the single term from the asymptotic subtraction brings in a dependence on low multipoles, which has to be treated carefully. We resolve this issue by splitting the sum into two parts and deriving a consistent result for lower and larger multipoles, while showing that the exact point of separation does not enter in the final result.

A natural extension of this work would be to include additional degrees of freedom, such as a globally symmetric potential with additional scalars, would-be-Goldstones, gauge bosons and fermions [111], analytically in the TW limit. This would require a somewhat general approach to renormalization, similar to the simplification that was done for the running of couplings [112]. It may also be desirable to develop a framework for evaluating the total rate at one loop for general potentials. Such a formalism may be developed from a semi-analytical polygonal bounce [77] setup, where the shape of the bounce is given explicitly on each segment. The upshot of this construction is that it applies to a broad class of models and matches exactly to the known Euclidean action in the TW limit. Most significantly, the bounce part was already suited for multi-fields, which would make it a good starting point for developing the theory of one-loop fluctuations for generic BSM setups.

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A Calculation of the bounce and the Euclidean action

Here we provide a detailed derivation of the bounce field configuration and the bounce action. The starting point is eq. (4.2), which in terms of the dimensionless variables reads

$$\frac{d^2\varphi}{dz^2} + \frac{D-1}{z+r} \frac{d\varphi}{dz} - \frac{1}{2}\varphi(\varphi^2 - 1) - \Delta = 0. \tag{A.1}$$

This is subject to the boundary conditions

$$\frac{d\varphi}{dz}(z = \infty) = \frac{d\varphi}{dz}(z = -r \rightarrow -\infty) = 0, \tag{A.2}$$

$$\varphi(z = \infty) = \varphi_{\text{FV}} = 1 - \Delta - \frac{3}{2}\Delta^2 - 4\Delta^3 + \mathcal{O}(\Delta^4). \tag{A.3}$$

It is useful to recall that at the true vacuum

$$\varphi_{\text{TV}} = -1 - \Delta + \frac{3}{2}\Delta^2 - 4\Delta^3 + \mathcal{O}(\Delta^4). \tag{A.4}$$

Using the expansions

$$\varphi = \sum_{n \geq 0} \Delta^n \varphi_n, \quad r = \frac{1}{\Delta} \sum_{n \geq 0} \Delta^n r_n, \tag{A.5}$$

in (A.1) we get

$$\frac{d^2\varphi_0}{dz^2} - \frac{1}{2}\varphi_0(\varphi_0^2 - 1) \tag{A.6}$$

$$+ \Delta \left[\frac{d^2\varphi_1}{dz^2} + \frac{1}{2}\varphi_1(1 - 3\varphi_0^2) - 1 + (D-1)\frac{1}{r_0}\frac{d\varphi_0}{dz} \right] \tag{A.7}$$

$$+ \Delta^2 \left[\frac{d^2\varphi_2}{dz^2} + \frac{1}{2}\varphi_2(1 - 3\varphi_0^2) - \frac{3}{2}\varphi_0\varphi_1^2 + (D-1)\left(\frac{1}{r_0}\frac{d\varphi_1}{dz} - \frac{z+r_1}{r_0^2}\frac{d\varphi_0}{dz}\right) \right] \tag{A.8}$$

$$+ \Delta^3 \left[\frac{d^2\varphi_3}{dz^2} + \frac{1}{2}\varphi_3(1 - 3\varphi_0^2) - 3\varphi_0\varphi_1\varphi_2 - \frac{1}{2}\varphi_1^3 \right. \\ \left. + (D-1)\left(\frac{1}{r_0}\frac{d\varphi_2}{dz} - \frac{z+r_1}{r_0^2}\frac{d\varphi_1}{dz} + \frac{z^2+2zr_1+r_1^2-r_0r_2}{r_0^3}\frac{d\varphi_0}{dz}\right) \right] \tag{A.9}$$

$$+ \mathcal{O}(\Delta^4) = 0. \tag{A.10}$$

The strategy is to first solve the $n = 0$ part, (A.6), plug the φ_0 solution into the $n = 1$ equation, (A.7) solve for φ_1 , and so on iterating to higher orders. We will clarify how solving the equations for $n \geq 1$ fixes the r_n coefficients.

The leading order. By noting that

$$\frac{d^2\varphi_0}{dz^2} = \frac{d(d\varphi_0/dz)}{d\varphi_0} \frac{d\varphi_0}{dz}, \tag{A.11}$$

we can write (A.6) as

$$d(d\varphi_0/dz)\frac{d\varphi_0}{dz} = \frac{1}{2}\varphi_0(\varphi_0^2 - 1)d\varphi_0. \tag{A.12}$$

Integrating both sides, requiring the derivative to vanish at $z \rightarrow \infty$, gives

$$\frac{d\varphi_0}{dz} = -\frac{1}{2}(\varphi_0^2 - 1), \tag{A.13}$$

where we chose the minus sign in order to satisfy the boundary conditions.

Integrating once more with the boundary condition (A.3), that is $\varphi_0(z = \infty) = 1$, we get $\int d\varphi_0/(1 - \varphi_0^2) = \text{ath}\varphi_0 = z/2$. The bounce configuration at the leading order is thus

$$\varphi_0 = \text{th}\frac{z}{2}. \tag{A.14}$$

This is odd under $z \rightarrow -z$, and interpolates between the FV at $z = \infty$ and the TV at $z = -\infty$.

Plugging the leading order solution into the action (4.1), we have

$$S_0 = \frac{\Omega v^{4-D}}{\lambda^{D/2-1}\Delta^{D-1}} \int_{-r}^{\infty} dz (r_0 + \Delta z)^{D-1} \left(\frac{1}{2} \left(\frac{d\varphi_0}{dz} \right)^2 + \frac{1}{8} (\varphi_0^2 - 1)^2 + \Delta (\varphi_0 - 1) \right). \tag{A.15}$$

Here the index n in S_n refers to the order of the expansion in Δ^n . The first two terms of the integral are even under parity and vanish exponentially as $z \rightarrow \pm\infty$, because

$(d\varphi_0/dz)^2 = 1/4(\varphi_0^2 - 1)^2 \propto \exp(\mp 2z)$. For such terms we can extend the lower limit of integration from $-r \sim -r_0/\Delta \xrightarrow{\Delta \rightarrow 0} -\infty$, which amounts to neglecting terms⁹ of the form $e^{-1/\Delta}$. The integral then becomes, upon using (A.13),

$$\int_{-r}^{\infty} dz r_0^{D-1} \left(\frac{1}{2} \left(\frac{d\varphi_0}{dz} \right)^2 + \frac{1}{8} (\varphi_0^2 - 1)^2 \right) = r_0^{D-1} \int_{-\infty}^{\infty} \left(\frac{d\varphi_0}{dz} \right)^2 dz \quad (\text{A.16})$$

$$= r_0^{D-1} \int_{-1}^1 d\varphi_0 \frac{d\varphi_0}{dz} = \frac{2}{3} r_0^{D-1}, \quad (\text{A.17})$$

where we omitted the sub-leading Δz terms, which will be added once we get to the Δ^2 corrections below. The last term in (A.15) looks as if it should not belong to $n = 0$, because it is proportional to Δ . However, this is not the case. The combination $(\varphi_0 - 1)$ goes to a constant, -2 , as $z \rightarrow -r \rightarrow -\infty$, implying that the lower limit of integration cannot be extended to $-\infty$. We can either integrate the term directly, keeping $-r$ as the lower boundary, which brings in the $1/\Delta$ power to bring down the term to $n = 0$, or equivalently integrate by parts (using Δz as the integration variable):

$$\int_{-r}^{\infty} d\Delta z (r_0 + \Delta z)^{D-1} (\varphi_0 - 1) \quad (\text{A.18})$$

$$= \frac{1}{D} (r_0 + \Delta z)^D (\varphi_0 - 1) \Big|_{-r_0}^{\infty} - \frac{1}{D} \int_{-\infty}^{\infty} dz (r_0 + \Delta z)^D \frac{d\varphi_0}{dz} \quad (\text{A.19})$$

$$= -\frac{1}{D} r_0^D \int_{-\infty}^{\infty} dz \frac{d\varphi_0}{dz} = -\frac{1}{D} r_0^D \varphi_0 \Big|_{-1}^1 = -\frac{2}{D} r_0^D. \quad (\text{A.20})$$

In (A.19) the boundary term vanishes, while in the integral with $d\varphi_0/dz$ we can extend the lower limit to $-\infty$, committing only an exponentially small error. We keep only the r_0^D term, since the rest is sub-leading in Δ . We refer to this as a volume term, since it is proportional to r_0^D , while (A.17) is a surface term, proportional to r^{D-1} . Combining them we get

$$S_0 = \frac{\Omega v^{4-D}}{\lambda^{D/2-1} \Delta^{D-1}} \left(\frac{2}{3} r_0^{D-1} - \frac{2}{D} r_0^D \right), \quad (\text{A.21})$$

that can be extremized over r_0

$$\frac{dS_0}{dr_0} = 0 \quad \Rightarrow \quad r_0 = \frac{D-1}{3}. \quad (\text{A.22})$$

We end up with the Euclidean bounce action (4.10) at the leading order. A sanity check can be made by splitting the action into the kinetic \mathcal{T} term, which is 1/2 of the surface term in (A.17), and the potential piece \mathcal{V} , coming from the other half of the surface plus the volume term:

$$S_0 = \frac{\Omega v^{4-D}}{\lambda^{D/2-1} \Delta^{D-1}} \frac{r_0^{D-1}}{3} \left(1 + \frac{D-6r_0}{D} \right) = \mathcal{T} + \mathcal{V}. \quad (\text{A.23})$$

This verifies that the two terms of the action are indeed related by

$$(D-2)\mathcal{T} = -D\mathcal{V}, \quad (\text{A.24})$$

as stated by Derrick's [59] theorem, once the action is properly extremized by r_0 in (A.22).

⁹Terms with $e^{-1/\Delta}$ go to zero faster than any power of Δ when $\Delta \rightarrow 0$. That is why it is consistent in our approach, based on counting powers of Δ , to always drop such exponentially suppressed terms.

Higher order corrections. The equations for $n \geq 1$, as we see in (A.7), (A.8), (A.9), all have the form

$$\frac{d^2\varphi_n}{dz^2} + \frac{1}{2}\varphi_n(1 - 3\varphi_0^2) = f^{\text{nhom}}(\varphi_{j < n}(z), r_{j < n}), \quad (\text{A.25})$$

with the homogeneous part on the left hand side, the non-homogeneous part on the right hand side. With $\varphi_0 = \text{th}(z/2)$, the solution to the homogeneous part,

$$\varphi_n^{\text{hom}} = \frac{c_{1n}}{4\text{ch}^2(z/2)} + \frac{c_{2n}}{4\text{ch}^2(z/2)}[6z + 8\text{sh}(z) + \text{sh}(2z)], \quad (\text{A.26})$$

is the sum of an even function in z , multiplied by c_{1n} , and an odd function, multiplied by c_{2n} .

At $n = 1$ we have (A.7):

$$\frac{d^2\varphi_1}{dz^2} + \frac{1}{2}\varphi_1(1 - 3\varphi_0^2) = 1 - (D - 1)\frac{1}{r_0}\frac{d\varphi_0}{dz}. \quad (\text{A.27})$$

We can understand some features by inspection. As φ_0 is an odd function of z , φ_0^2 and $d\varphi_0/dz$ are even functions. Hence the equation admits a solution φ_1 which is even. Moreover, as $\varphi_0^2(z = \pm\infty) = 1$ and the derivatives of the field must vanish at $z = \pm\infty$, we have $\varphi_1(z = \pm\infty) = -1$. Therefore we know immediately that φ_1 is forced to interpolate between the FV and the TV at order Δ . The solution to (A.27) is

$$\varphi_1 = \frac{c_{11}}{4\text{ch}^2(z/2)} + \frac{c_{21}}{4\text{ch}^2(z/2)}[6z + 8\text{sh}(z) + \text{sh}(2z)] \quad (\text{A.28})$$

$$+ \frac{1}{3r_0(e^z + 1)^2} [\text{ch}(z)((D - 1)(3z + 1) - 9r_0(z + 1)) + (D - 1)(3z + 2)\text{sh}(z) - 4D - 3r_0(\text{ch}(2z) + \text{sh}(z)(3z + 2\text{ch}(z) + 4)) + 9r_0 + 4]. \quad (\text{A.29})$$

The boundary condition $\varphi_1(z = \infty) = -1$ requires $c_{21} = 0$. This sets $d\varphi_1/dz = 0$ at $z = \infty$. Requiring $d\varphi_1/dz = 0$ also at $z = -\infty$ fixes $r_0 = (D - 1)/3$. With these, the solution simplifies to

$$\varphi_1 = -1 + \frac{c_{11}}{4\text{ch}^2(z/2)}. \quad (\text{A.30})$$

This is an even function, as anticipated, and we see that the boundary conditions do not fix the constant c_{11} . We are free to set it to zero, so that

$$\varphi_1 = -1. \quad (\text{A.31})$$

Note that solving the bounce equation at this order gives us the same r_0 as we found in (A.22) by extremizing the action, and provides a further consistency check.

At $n = 2$, using φ_1 and r_0 from above into (A.8), we get

$$\frac{d^2\varphi_2}{dz^2} + \frac{1}{2}\varphi_2(1 - 3\varphi_0^2) = \frac{3}{2}\varphi_0 + \frac{9}{D - 1}\frac{d\varphi_0}{dz}(z + r_1). \quad (\text{A.32})$$

With the derivatives vanishing at $z = \pm\infty$, this equation implies $\varphi_2(z = \pm\infty) = \mp\frac{3}{2}$. Hence, φ_2 is also forced to interpolate between true and false vacua. Note that setting $r_1 = 0$ at

this stage would make it apparent that (A.32) admits a solution φ_2 odd under z . Let us keep r_1 in the game for a moment. The solution then is

$$\varphi_2 = \frac{c_{12}}{4\text{ch}^2(z/2)} + \frac{c_{22}}{4\text{ch}^2(z/2)}[6z + 8\text{sh}(z) + \text{sh}(2z)] + \varphi_2^{\text{nhom}}(z, r_1), \quad (\text{A.33})$$

with $\varphi_2^{\text{nhom}}(z, r_1)$ a function of z and r_1 computed from the non-homogeneous part on the right hand side of (A.32). The boundary condition $\varphi_2(z = \infty) = -\frac{3}{2}$ requires $c_{22} = 0$, which also makes $d\varphi_2/dz = 0$ at $z = \infty$. Asking that the derivative vanish also at $z = -\infty$ requires $r_1 = 0$. Thus, the boundary condition forces this parameter to vanish. We can fix the remaining parameter by demanding that the solution be an odd function of z ,

$$c_{12} = \frac{3(\pi^2 + 3D - 9)}{2(D - 1)}, \quad (\text{A.34})$$

so that

$$\varphi_2 = \frac{1}{4r_0\text{ch}^2(z/2)} \left((2 - D - 2(4 + \text{ch}z) \ln(1 + e^z)) \text{sh}z - z(D - e^z(4 + \text{sh}z)) + 3(\text{Li}_2(-e^z) - \text{Li}_2(-e^{-z})) \right), \quad (\text{A.35})$$

with the asymptotic behavior $\varphi_2(\pm\infty) = \mp 3/2$.

At $n = 3$, using $\varphi_1 = -1$, $r_1 = 0$ into (A.9), we have

$$\frac{d^2\varphi_3}{dz^2} + \frac{1}{2}\varphi_3(1 - 3\varphi_0^2) = -3\varphi_0\varphi_2 - \frac{1}{2} - (D - 1) \left(\frac{1}{r_0} \frac{d\varphi_2}{dz} + \frac{z^2 - r_0r_2}{r_0^3} \frac{d\varphi_0}{dz} \right). \quad (\text{A.36})$$

As both φ_0 and φ_2 are odd functions of z , this equation admits an even solution φ_3 . Again, by inspection we see that the asymptotics are such that φ_3 interpolates between true and false vacua. The solution, after plugging in $\varphi_0, \varphi_2, r_0$, is

$$\varphi_3 = \frac{c_{13}}{4\text{ch}^2(z/2)} + \frac{c_{23}}{4\text{ch}^2(z/2)}[6z + 8\text{sh}(z) + \text{sh}(2z)] + \varphi_3^{\text{nhom}}(z, r_2). \quad (\text{A.37})$$

The boundary condition $\varphi_3(z = \infty) = -4$ fixes

$$c_{23} = -\frac{3\pi^2(D - 2)}{(D - 1)^2}, \quad (\text{A.38})$$

which results in $d\varphi_3/dz = 0$ at $z = \infty$. Asking that the derivative vanish at $z = -\infty$ fixes

$$r_2 = \frac{6\pi^2 - 40 + D(26 - 4D - 3\pi^2)}{3(D - 1)}. \quad (\text{A.39})$$

The parameter c_{13} , analogously to c_{11} at $n = 1$, remains unfixed, and we can choose to set it to zero. Finally we have

$$\begin{aligned}
 \varphi_3 = & -\frac{1}{16\text{ch}^2(z/2)(D-1)^2} \left[2\text{ch}(z) \left(16D^2 + D(-72z + 24\pi^2 - 77) \right. \right. \\
 & + 144(D-1)\log(e^z + 1) + 72z(z+1) - 48\pi^2 + 97) \\
 & + 216(3D-7)\text{Li}_3(-e^z) + 72\text{Li}_2(-e^z)(-3(D-3)z + 8(D-2)\text{sh}(z) \\
 & + (D-2)\text{sh}(2z)) + 27(D-3)z^2 + 9(4\pi^2(D-2) - 15(D-1))z \\
 & + 270D\log(e^z + 1) + 12\text{sh}(z)(3z(-5D + 4z + 7) \\
 & + 24(D-3)z\log(e^z + 1) + 4\pi^2(D-2)) \\
 & + 3\text{sh}(2z)(3z(-D + 2z + 1) + 12(D-3)z\log(e^z + 1) + 2\pi^2(D-2)) \\
 & + 3\text{ch}(2z)(3z(-D + 2z + 1) + 6(D-1)\log(e^z + 1) + 2\pi^2(D-2)) \\
 & \left. \left. + 16(D-4)(2D-5) - 18\pi^2(D-2) + 36z^3 - 270\log(e^z + 1) \right] \right]. \quad (\text{A.40})
 \end{aligned}$$

We did not manage to compute analytically the bounce solution beyond $n = 3$.

Comments on the bounce solution. We have seen that with this iterative setup, the bounce interpolates between the FV and the TV at any order n . This means that, with the Δ expansion initiated here, the field will never go beyond the TW in the sense that $\phi_{\text{in}} = \phi_{\text{TV}}$ for any power¹⁰ of Δ . By going to higher orders we are simply describing the TW instanton between the TV and the FV more accurately.

Note that a shift of the radius in the leading order bounce solution produces

$$\varphi_0(z + c_n^{\text{shift}}\Delta^n) = \varphi_0(z) + \frac{c_n^{\text{shift}}}{2\text{ch}^2(z/2)}\Delta^n + \mathcal{O}(\Delta^{2n}). \quad (\text{A.41})$$

Comparing to (A.26) and to the expansion of r (A.5), we see that a nonzero coefficient c_{1n} in (A.26) results in a shift $r_{n+1} \rightarrow r_{n+1} + c_{1n}/2$. At $n = 1$ we chose to set $c_{11} = 0$. An alternative choice would be to keep the c_{11} term in φ_1 and set $r_2 = 0$; in this case the derivative boundary condition for the $n = 3$ equation would fix c_{11} to twice the value of (A.39). There is no consistent solution with both $c_{11} = 0$ and $r_2 = 0$, but this becomes clear only when one gets to the $n = 3$ order. In conclusion, we learn that the Δ expansion of r is useful because by introducing more redundant parameters it allows for the bounce solution φ to take a simpler form.

The bounce action at $n = 2$. As we shall see, both r_2 and φ_2 enter in the kinetic and potential part of the action at Δ^2 , but eventually cancel out. So let us examine the several second-order sources of the Euclidean action. We found that the leading contribution to the action in (A.23) consists of the kinetic \mathcal{T} and potential \mathcal{V} terms at zero order

$$\mathcal{T}_0 = \frac{\Omega v^{4-D}}{\lambda^{D/2-1}\Delta^{D-1}} \frac{r_0^{D-1}}{3}, \quad \mathcal{V}_0 = \mathcal{T}_0 \left(1 - \frac{6r_0}{D} \right). \quad (\text{A.42})$$

¹⁰In order to have $\phi_{\text{in}} \neq \phi_{\text{TV}}$, one should reconsider the bounce equation and include the linear Δ term at the leading order and expand the friction term only.

These were obtained by extremizing the action over r and plugging in the leading order $r \sim r_0/\Delta$. Further expanding the bounce radius $r \sim (r_0 + \Delta^2 r_2)/\Delta$ to include the second order correction, we get

$$\mathcal{T}_2 \ni \frac{\Omega v^{4-D}}{\lambda^{D/2-1} \Delta^{D-1}} \frac{r_0^{D-1}}{3} \left(1 + \Delta^2 \frac{r_2}{r_0}\right)^{D-1} \simeq \mathcal{T}_0 \left(1 + 3\Delta^2 r_2\right), \quad (\text{A.43})$$

$$\mathcal{V}_2 \ni \mathcal{V}_0 - \mathcal{T}_0 3\Delta^2 r_2, \quad (\text{A.44})$$

where \mathcal{T}_2 and \mathcal{V}_2 stand for the kinetic and potential parts of the action up to second order. As expected, the action does not get corrected by r_2 , however individual kinetic and potential contributions are modified. Keeping all the terms lead to a non-trivial validation of extremization via Derrick's theorem in (A.24), which has to hold at all orders of Δ once all the terms in the action are obtained.

Further corrections come from the inclusion of friction when expanding the Δz polynomial in (A.15). We will explicitly attribute the various contributions to the kinetic or potential part, and check the validity of Derrick's theorem at the end of the calculation. We start with the corrections that concern only the effects of r_2 and the damping of φ_0 . The Δ corrections vanish since the integrand is odd in z , while the Δ^2 corrections give us the following higher-order terms in the action (in the following few lines we omit the overall constant $\Omega v^{4-D} \lambda^{1-D/2} \Delta^{1-D}$ in front of the action S_2 , not to overburden with notation), proportional to

$$S_2 \ni 2 r_0^{D-3} \Delta^2 \frac{(D-1)(D-2)}{4} \int_{-\infty}^{\infty} dz z^2 \left(\frac{d\varphi_0}{dz}\right)^2 \quad (\text{A.45})$$

$$- \frac{1}{D} r_0^{D-2} \Delta^2 \frac{D(D-1)}{2} \int_{-\infty}^{\infty} dz z^2 \frac{d\varphi_0}{dz} \quad (\text{A.46})$$

$$= \frac{3}{2} \Delta^2 r_0^{D-1} \left(3 \frac{D-2}{D-1} \int_{-\infty}^{\infty} dz z^2 \left(\frac{d\varphi_0}{dz}\right)^2 - \int_{-\infty}^{\infty} dz z^2 \frac{d\varphi_0}{dz}\right)$$

$$= -\Delta^2 \left(\frac{\pi^2 + 6(D-2)}{D-1}\right) r_0^{D-1}. \quad (\text{A.47})$$

We emphasize a factor of 2 in the first line of (A.45) as we have two equal contributions; one belonging to the kinetic part and one to the potential part. The second line belongs to the potential part only. The addition of $\varphi_1 = -1$ brings in the following terms, proportional to

$$S_2 \ni \int_{-r}^{\infty} dz (r_0 + \Delta z)^{D-1} \left(\frac{3}{4} \Delta^2 (\varphi_0^2 - 1) - \frac{1}{2} \Delta \varphi_0 (\varphi_0^2 - 1) - \frac{1}{2} \Delta^3 (\varphi_0 - 1)\right) \quad (\text{A.48})$$

$$= \int_{-r}^{\infty} dz (r_0 + \Delta z)^{D-1} \left(-\frac{3}{2} \Delta^2 \frac{d\varphi_0}{dz} + \Delta \varphi_0 \frac{d\varphi_0}{dz} - \frac{1}{2} \Delta^3 (\varphi_0 - 1)\right) \quad (\text{A.49})$$

$$= -\frac{3}{2} \Delta^2 r_0^{D-1} \int_{-1}^1 d\varphi_0 + r_0^{D-2} (D-1) \Delta^2 \int_{-\infty}^{\infty} dz z \varphi_0 \frac{d\varphi_0}{dz} + \frac{D-1}{3D} \Delta^2 r_0^{D-1} \quad (\text{A.50})$$

$$= \Delta^2 r_0^{D-1} \left(3 + \frac{r_0}{D}\right), \quad (\text{A.51})$$

where all the terms in (A.48) belong to the potential part. The final remaining pieces, depending on φ_2 , are found to be

$$S_2 \ni \Delta^2 \int_{-r}^{\infty} dz (r_0 + \Delta z)^{D-1} \left(\frac{d\varphi_0}{dz} \frac{d\varphi_2}{dz} - \varphi_0 \frac{d\varphi_0}{dz} \varphi_2 + 3\Delta \frac{d\varphi_0}{dz} \varphi_2\right) = 0. \quad (\text{A.52})$$

This can be understood from the following identity

$$\int_{-\infty}^{\infty} dz \frac{d}{dz} \left(\frac{d\varphi_0}{dz} \varphi_2 \right) = \frac{d\varphi_0}{dz} \varphi_2 \Big|_{-\infty}^{\infty} = 0 = \int_{-\infty}^{\infty} dz \left(\frac{d\varphi_0}{dz} \frac{d\varphi_2}{dz} + \frac{d^2\varphi_0}{dz^2} \varphi_2 \right) \quad (\text{A.53})$$

$$= \int_{-\infty}^{\infty} dz \left(\frac{d\varphi_0}{dz} \frac{d\varphi_2}{dz} + \frac{1}{2} \varphi_0 (\varphi_0^2 - 1) \varphi_2 \right) = \int_{-\infty}^{\infty} dz \left(\frac{d\varphi_0}{dz} \frac{d\varphi_2}{dz} - \varphi_0 \frac{d\varphi_0}{dz} \varphi_2 \right), \quad (\text{A.54})$$

which shows that the first two terms in (A.52) sum up to zero. The last term in (A.52) disappears, because φ_2 is odd and $d\varphi_0/dz$ is even and vanishes at the boundaries, allowing the limits of integration to be extended to $z \rightarrow \pm\infty$.

The final result is thus the Euclidean action, which can be written as

$$S_2 = S_0 \left(1 + \Delta^2 \left(\frac{1 + D(25 - 8D - 3\pi^2)}{2(D - 1)} \right) \right) + \mathcal{O}(\Delta^4). \quad (\text{A.55})$$

This result can be used to get an improved precision on the bounce part of the action in (4.10) for any D . The Δ^3 corrections vanish altogether, so this result is precise up to $\mathcal{O}(\Delta^4)$. Moreover, it serves an estimate on the upper bound on Δ , when the higher-order corrections become relevant and thus the entire expansion becomes unreliable. One should keep in mind that we have constructed such a TW setup, where higher orders of Δ simply describe the TV and FV more precisely and the bounce interpolates between those two. To go beyond this ansatz, one would have to construct a bounce solution with $\varphi_0 \neq \varphi_{\text{TV}}$ already at leading order, which might be feasible but is certainly beyond the scope of our paper.

It is useful to collect all of the corrections above and carefully attribute them to kinetic and potential parts: we get the integrated kinetic and potential parts of the action

$$\mathcal{T}_2 = \mathcal{T}_0 + \Delta^2 \mathcal{T}_0 \left(\frac{1 + D(25 - 8D - 3\pi^2)}{2(D - 1)} \right), \quad (\text{A.56})$$

$$\mathcal{V}_2 = \mathcal{V}_0 - \Delta^2 \mathcal{T}_0 \left(\frac{D - 2}{D} \right) \left(\frac{1 + D(25 - 8D - 3\pi^2)}{2(D - 1)} \right), \quad (\text{A.57})$$

which demonstrates that Derrick's theorem in (A.24) holds at second-order in Δ and therefore all of the terms at this order have been properly included.

Counter-term integration. When calculating the counterterm action in (4.18), we encountered the following two integrals

$$\int_D (\phi^2 - \phi_{\text{FV}}^2), \quad \text{and} \quad \int_D (\phi^4 - \phi_{\text{FV}}^4). \quad (\text{A.58})$$

These are evaluated in the TW limit using the same approach as above. To get the leading order term proportional to $1/\Delta^{D-1}$ correctly, we must include the bounce up to $n = 1$, that

is $\varphi = \varphi_0 + \Delta\varphi_1 = \varphi_0 - \Delta$. The first integral in (A.58) is

$$\frac{\Omega v^{2-D}}{\lambda^{D/2}\Delta^{D-1}} \int_{-r}^{\infty} dz (r_0 + \Delta z)^{D-1} \left((\varphi_0 - \Delta)^2 - (1 - \Delta)^2 \right) \quad (\text{A.59})$$

$$= \frac{\Omega v^{2-D}}{\lambda^{D/2}\Delta^{D-1}} \left(\int_{-\infty}^{\infty} dz r_0^{D-1} (\varphi_0^2 - 1) - 2 \int_{-r}^{\infty} dz (r_0 + \Delta z)^{D-1} \Delta (\varphi_0 - 1) \right) \quad (\text{A.60})$$

$$= \frac{\Omega v^{2-D}}{\lambda^{D/2}\Delta^{D-1}} \left(-4r_0^{D-1} + \frac{4}{D}r_0^D \right) = -4 \frac{\Omega v^{2-D}}{\lambda^{D/2}\Delta^{D-1}} r_0^{D-1} \left(1 - \frac{r_0}{D} \right), \quad (\text{A.61})$$

where we used the same integration by parts as in (A.19). Proceeding along the same lines, the second term in (A.58) integrates into

$$\frac{\Omega v^{4-D}}{\lambda^{D/2}\Delta^{D-1}} \int_{-r}^{\infty} dz (r_0 + \Delta z)^{D-1} \left((\varphi_0 - \Delta)^4 - (1 - \Delta)^4 \right) \quad (\text{A.62})$$

$$= \frac{\Omega v^{4-D}}{\lambda^{D/2}\Delta^{D-1}} \left(\int_{-\infty}^{\infty} dz r_0^{D-1} (\varphi_0^4 - 1) - 4 \int_{-r}^{\infty} dz (r_0 + \Delta z)^{D-1} \Delta (\varphi_0^3 - 1) \right) \quad (\text{A.63})$$

$$= \frac{\Omega v^{4-D}}{\lambda^{D/2}\Delta^{D-1}} \left(-\frac{16}{3}r_0^{D-1} + \frac{8}{D}r_0^D \right) = -\frac{16}{3} \frac{\Omega v^{4-D}}{\lambda^{D/2}\Delta^{D-1}} r_0^{D-1} \left(1 - \frac{3r_0}{2D} \right). \quad (\text{A.64})$$

B Calculation of the fluctuations

In this appendix, we provide the details of the calculation of the fluctuations discussed in 5. Starting from (5.1), We use the Gel'fand-Yaglom theorem, which states that

$$\frac{\det \mathcal{O}_l}{\det \mathcal{O}_{\text{IFV}}} = \lim_{\rho \rightarrow \infty} \left(\frac{\psi_l(\rho)}{\psi_{\text{IFV}}(\rho)} \right)^{d_l}. \quad (\text{B.1})$$

Recall that the radial operator

$$\mathcal{O}_l = -\frac{d^2}{d\rho^2} - \frac{D-1}{\rho} \frac{d}{d\rho} + \frac{l(l+D-2)}{\rho^2} + V^{(2)}, \quad (\text{B.2})$$

has degeneracy

$$d_l = \frac{(2l+D-2)(l+D-3)!}{l!(D-2)!}, \quad (\text{B.3})$$

and the functions ψ_l are solutions of

$$\mathcal{O}_l \psi_l = 0, \quad \mathcal{O}_{\text{IFV}} \psi_{\text{IFV}} = 0, \quad (\text{B.4})$$

with only one boundary condition fixed at $\rho \rightarrow 0$:

$$\psi_l(\rho \rightarrow 0) \sim \rho^l \quad \psi_{\text{IFV}}(\rho \rightarrow 0) \sim \rho^l. \quad (\text{B.5})$$

It is convenient to introduce

$$\nu = l + \frac{D}{2} - 1, \quad (\text{B.6})$$

and

$$\psi_\nu(\rho) = \rho^{\frac{D-1}{2}} \psi_l(\rho). \quad (\text{B.7})$$

Then from $\mathcal{O}_l \psi_l = 0$ we obtain $\mathcal{O}_\nu \psi_\nu = 0$, with

$$\mathcal{O}_\nu = -\frac{d^2}{d\rho^2} + \frac{\nu^2 - \frac{1}{4}}{\rho^2} + V^{(2)}. \quad (\text{B.8})$$

Thanks to the rescaling in (B.7), \mathcal{O}_ν has no friction term. Defining

$$R_\nu(\rho) \equiv \frac{\psi_\nu(\rho)}{\psi_{\nu\text{FV}}(\rho)} = \frac{\psi_l(\rho)}{\psi_{l\text{FV}}(\rho)}, \quad (\text{B.9})$$

we can rewrite the ratio of the determinants in equation (B.1) as

$$\frac{\det \mathcal{O}_l}{\det \mathcal{O}_{l\text{FV}}} = \frac{\det \mathcal{O}_\nu}{\det \mathcal{O}_{\nu\text{FV}}} = \left(\lim_{\rho \rightarrow \infty} R_\nu(\rho) \right)^{d_\nu}. \quad (\text{B.10})$$

Here the degeneracy factor

$$d_\nu = \frac{2\nu \left(\nu + \frac{D}{2} - 2 \right)!}{(D-2)! \left(\nu - \frac{D}{2} + 1 \right)!}, \quad (\text{B.11})$$

is obtained from (B.3) using (B.6). Taking the logarithm of equation (5.1) and using the Gel'fand-Yaglom theorem (B.10), we can finally write¹¹

$$\ln \left| \frac{\det \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right|^{-\frac{1}{2}} = -\frac{1}{2} \sum_{\nu=\frac{D}{2}-1}^{\infty} d_\nu \ln \left(\lim_{x \rightarrow \infty} R_\nu(x) \right). \quad (\text{B.12})$$

The parameter ν is an integer in even dimensions $D \geq 2$, a half-integer in odd dimensions $D \geq 3$. Our goal is to compute the quantity in (B.12), which implies eventually performing an infinite sum in ν . In this appendix we focus on computing the function R_ν . To begin with, we need an equation for it.

We can trade $\mathcal{O}_\nu \psi_\nu = 0$ for an equation with R_ν , which we derive as follows. From (B.9) we have

$$\frac{1}{R_\nu} \frac{d}{d\rho} R_\nu = \frac{1}{\psi_\nu} \frac{d}{d\rho} \psi_\nu - \frac{1}{\psi_{\nu\text{FV}}} \frac{d}{d\rho} \psi_{\nu\text{FV}}, \quad (\text{B.13})$$

$$\begin{aligned} \frac{1}{R_\nu} \frac{d^2}{d\rho^2} R_\nu - \left(\frac{1}{R_\nu} \frac{d}{d\rho} R_\nu \right)^2 &= \frac{1}{\psi_\nu} \frac{d^2}{d\rho^2} \psi_\nu - \left(\frac{1}{\psi_\nu} \frac{d}{d\rho} \psi_\nu \right)^2 \\ &\quad - \frac{1}{\psi_{\nu\text{FV}}} \frac{d^2}{d\rho^2} \psi_{\nu\text{FV}} + \left(\frac{1}{\psi_{\nu\text{FV}}} \frac{d}{d\rho} \psi_{\nu\text{FV}} \right)^2, \end{aligned} \quad (\text{B.14})$$

with (B.14) obtained by taking the derivative of (B.13). From (B.13) we have

$$\frac{1}{\psi_\nu} \frac{d}{d\rho} \psi_\nu = \frac{1}{R_\nu} \frac{d}{d\rho} R_\nu + \frac{1}{\psi_{\nu\text{FV}}} \frac{d}{d\rho} \psi_{\nu\text{FV}}, \quad (\text{B.15})$$

which plugged into (B.14) gives

$$\frac{1}{R_\nu} \frac{d^2}{d\rho^2} R_\nu + 2 \frac{1}{R_\nu} \frac{d}{d\rho} R_\nu \frac{1}{\psi_{\nu\text{FV}}} \frac{d}{d\rho} \psi_{\nu\text{FV}} = \frac{1}{\psi_\nu} \frac{d^2}{d\rho^2} \psi_\nu - \frac{1}{\psi_{\nu\text{FV}}} \frac{d^2}{d\rho^2} \psi_{\nu\text{FV}}. \quad (\text{B.16})$$

¹¹We drop the prime from the determinant at the numerator here, we will comment on the removal of the zero modes later on in the appendix.

The terms on the right-hand side, using (B.8), are

$$\frac{1}{\psi_\nu} \frac{d^2}{d\rho^2} \psi_\nu = \frac{\nu^2 - \frac{1}{4}}{\rho^2} + V^{(2)}. \quad (\text{B.17})$$

Thus, we arrive at the following system of equations:

$$\left(-\frac{d^2}{d\rho^2} + \frac{\nu^2 - \frac{1}{4}}{\rho^2} + V_{\text{FV}}^{(2)} \right) \psi_{\nu\text{FV}} = 0, \quad (\text{B.18})$$

$$\frac{1}{R_\nu} \frac{d^2 R_\nu}{d\rho^2} + 2 \frac{1}{R_\nu} \frac{dR_\nu}{d\rho} \left(\frac{d\psi_{\nu\text{FV}}/d\rho}{\psi_{\nu\text{FV}}} \right) - \left(V^{(2)} - V_{\text{FV}}^{(2)} \right) = 0, \quad (\text{B.19})$$

with boundary conditions

$$\psi_{\nu\text{FV}}(\rho \rightarrow 0) \sim \rho^{\nu + \frac{1}{2}}, \quad (\text{B.20})$$

$$R_\nu(\rho = 0) = 1, \quad \frac{dR_\nu}{d\rho}(\rho = 0) = 0. \quad (\text{B.21})$$

(B.21) is obtained from the definition (B.9) and the conditions (B.5). Note that in the second term of (B.19) the behavior

$$\frac{d\psi_{\nu\text{FV}}/d\rho}{\psi_{\nu\text{FV}}} \xrightarrow{\rho \rightarrow 0} \frac{1}{\rho}, \quad (\text{B.22})$$

is compensated by the derivative condition in (B.21).

Equation (B.18) can be solved exactly:

$$\psi_{\nu\text{FV}}(\rho) = c_{\text{FV}} \sqrt{\rho} I_\nu \left(\rho \sqrt{V_{\text{FV}}^{(2)}} \right), \quad (\text{B.23})$$

with I_ν a Bessel function. We have solved a second-order differential equation with one boundary condition on the derivative at the origin, so we are left with one integration constant, c_{FV} . There is no need to fix this constant, as it cancels out in the ratio $\frac{d\psi_{\nu\text{FV}}/d\rho}{\psi_{\nu\text{FV}}}$ that enters the R_ν equation, (B.19).

Equation (B.19) cannot be solved analytically. We are going to develop a method to solve it perturbatively, using expansions in the small parameter Δ . As we will see shortly, the parameter ν will enter the equations in the combination $\Delta\nu$. The final answer for the determinant, see (B.12), involves values of ν up to infinity. This implies that for $\nu > 1/\Delta$ the perturbation series in Δ will be ill defined. To fix it we will consider the combination $\Delta\nu$ as a leading order term in the expansion. First, however, we will go through the simpler procedure of solving the R_ν equation for $\nu \ll 1/\Delta$. The result, (B.61), is more illuminating written in terms of l rather than ν , as the negative eigenvalue for $l = 0$ and the zero eigenvalues for $l = 1$ are immediately apparent. For this reason, in the next section, we will relabel our functions with the subscript l instead of ν , to indicate that we are working with small values of l .

We will use the dimensionless variables

$$z = \sqrt{\lambda} v \rho - r, \quad \varphi = \phi/v, \quad \tilde{V}^{(2)} = \frac{1}{\lambda v^2} V^{(2)}. \quad (\text{B.24})$$

Perturbative solutions for low multipoles. Despite having the exact solution (B.23) for $\psi_{l\text{FV}}$ at hand, it is more useful to have an approximate, simpler solution to (B.18) with an explicit Δ dependence. We computed it in section 5.1 and found

$$\psi_{l\text{FV}}(z) = c_{\text{FV}} \text{Exp} \left[\left(1 - \frac{3}{2}\Delta + \frac{1}{2} \left(\frac{\nu^2 - \frac{1}{4}}{r_0^2} - \frac{21}{4} \right) \Delta^2 + \mathcal{O}(\Delta^3) \right) z \right]. \quad (\text{B.25})$$

From this, we get the ratio

$$g_l \equiv \frac{d\psi_{l\text{FV}}/dz}{\psi_{l\text{FV}}} = 1 - \frac{3}{2}\Delta + \frac{1}{2} \left(\frac{\nu^2 - \frac{1}{4}}{r_0^2} - \frac{21}{4} \right) \Delta^2 + \mathcal{O}(\Delta^3) \quad (\text{B.26})$$

$$= g_{l0} + \Delta g_{l1} + \Delta^2 g_{l2} + \mathcal{O}(\Delta^3). \quad (\text{B.27})$$

Introducing the variable

$$x = e^z, \quad (\text{B.28})$$

the R_ν equation (B.19) turns into

$$\frac{1}{R_l} \frac{d^2}{dx^2} R_l + \frac{2g_l + 1}{x} \frac{1}{R_l} \frac{d}{dx} R_l - \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right) = 0, \quad (\text{B.29})$$

with g_l given in (B.26). We relabeled $R_\nu \rightarrow R_l$ to indicate the low l regime. The boundary conditions from (B.21) become

$$R_l(x = e^{-r}) = 1, \quad e^{-r} \frac{d}{dx} R_l(x = e^{-r}) = 0. \quad (\text{B.30})$$

Note that we are retaining the e^{-r} in front of the derivative condition. The reason is that in our expansions we keep track of powers of Δ but always drop exponentials of the form $e^{-r} \sim e^{-1/\Delta}$. Hence, we keep e^{-r} explicit in (B.30) to indicate that even with $\frac{d}{dx} R_l(x = e^{-r})$ equal to a constant or proportional to powers of r , the boundary condition is satisfied.

We take a multiplicative expansion for R_l ,

$$R_l = R_{l0} R_{l1}^\Delta R_{l2}^{\Delta^2} R_{l3}^{\Delta^3} \dots \quad (\text{B.31})$$

$$= R_{l0} \left(1 + \Delta \ln R_{l1} + \Delta^2 \left(\frac{1}{2} \ln^2 R_{l1} + \ln R_{l2} \right) + \mathcal{O}(\Delta^3) \right), \quad (\text{B.32})$$

and use the following expression

$$\begin{aligned} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right) &= \sum_{n=0}^{\infty} \Delta^n \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_n \\ &= \frac{3}{2}(\varphi_0^2 - 1) + 3\Delta(1 + \varphi_0\varphi_1) + \frac{3}{2}\Delta^2(2 + \varphi_1^2 + 2\varphi_0\varphi_2) + \mathcal{O}(\Delta^3). \end{aligned} \quad (\text{B.33})$$

At the zeroth order, $R_l = R_{l0}$, from (B.29) we have

$$\frac{1}{R_{l0}} \frac{d^2}{dx^2} R_{l0} + \frac{3}{x} \frac{1}{R_{l0}} \frac{d}{dx} R_{l0} - \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_0 = 0, \quad (\text{B.34})$$

with

$$\left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_0 = \frac{3}{2}(\varphi_0^2 - 1) = -\frac{6x}{(1+x)^2}. \quad (\text{B.35})$$

The solution is

$$R_{l0} = \frac{1}{(1+x)^2}. \quad (\text{B.36})$$

For orders Δ and higher, it is convenient to introduce

$$f_l \equiv \frac{1}{R_l} \frac{d}{dx} R_l, \quad (\text{B.37})$$

so that the second-order differential equation (B.29) turns into a first-order one,

$$\frac{d}{dx} f_l + f_l^2 + \frac{2g_l + 1}{x} f_l - \frac{1}{x^2} (\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)}) = 0, \quad (\text{B.38})$$

subject to the boundary condition

$$e^{-r} f_l(e^{-r}) = 0, \quad (\text{B.39})$$

where again we mean that even with $f_l(e^{-r})$ equal to a constant or proportional to powers of r , the boundary condition is satisfied (see the comment below (B.30)). We expand f_l in Δ ,

$$f_l = f_{l0} + \Delta f_{l1} + \Delta^2 f_{l2} + \mathcal{O}(\Delta^3), \quad (\text{B.40})$$

and plug it into (B.38), then we proceed to solve the equation order by order in Δ . At the lowest order, from (B.36) we have

$$f_{l0} = -\frac{2}{1+x}. \quad (\text{B.41})$$

At the order Δ^n , with $n \geq 1$ we have the equations

$$\frac{d}{dx} f_{ln} + \left(2f_{l0} + \frac{3}{x}\right) f_{ln} = P_{f_{ln}}, \quad (\text{B.42})$$

with the functions

$$P_{f_{ln}} = \frac{1}{x^2} (\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)})_n - \frac{2}{x} \sum_{j=1}^n g_{lj} f_{l(n-j)} - \sum_{j=1}^{n-1} f_{lj} f_{l(n-j)}, \quad (\text{B.43})$$

growing at every iteration. The last sum in (B.43) contributes only for $n \geq 2$. The solution to (B.42) is given by

$$f_{ln}(x) = \frac{(1+x)^4}{x^3} \int_{e^{-r}}^x dt \frac{t^3}{(1+t)^4} P_{f_{ln}}(t). \quad (\text{B.44})$$

Here the lower limit of integration is set formally to e^{-r} to guarantee the boundary condition (B.39). In practice, however, one can set such a lower limit to 0, and check at every order that this corresponds to dropping only terms exponentially suppressed in r , that means terms of the form $e^{-1/\Delta}$.

At the first order in Δ we have

$$\left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)}\right)_1 = 3(\varphi_0 \varphi_1 + 1) = \frac{6}{1+x}, \quad (\text{B.45})$$

$$P_{f_{l1}} = \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)}\right)_1 + \frac{3}{x} f_{l0} = \frac{6}{x^2} \frac{1-x}{1+x}, \quad (\text{B.46})$$

so that

$$f_{l1} = \frac{(1+x)^4}{x^3} \int_{e^{-r}}^x dt \frac{t^3}{(1+t)^4} P_{f_{l1}}(t) = \frac{(1+x)^4}{x^3} \left(\frac{3x^2}{(1+x)^4} - \frac{3e^{-2r}}{(1+e^{-r})^4} \right). \quad (\text{B.47})$$

We can drop the last term in the squared parentheses as it is exponentially suppressed in r . Then¹²

$$f_{l1} = \frac{3}{x}. \quad (\text{B.48})$$

At the second order we have

$$\begin{aligned} (\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)})_2 &= \frac{3}{2}(2 + \varphi_1^2 + 2\varphi_0\varphi_2) \\ &= \frac{3}{2} \left\{ 3 - \frac{1}{r_0 x (1+x)^3} (1-x) \left[x(-1 + \pi^2 x + x^2 - 3r_0(x^2 - 1)) \right. \right. \\ &\quad \left. \left. + x(-3 - 6r_0 + 8x + x^2) \ln(x) + 3x \ln^2(x) \right) \right. \\ &\quad \left. - (-1 - 8x + 8x^3 + x^4) \ln(1+x) + 12x^2 \text{Li}_2(-x) \right\}, \quad (\text{B.49}) \end{aligned}$$

and

$$P_{f_{l2}} = \frac{1}{x^2} (\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)})_2 + \frac{3}{x} f_{l1} - \frac{1}{x} \left(\frac{\nu^2 - \frac{1}{4}}{r_0^2} - \frac{21}{4} \right) f_{l0} - f_{l1}^2. \quad (\text{B.50})$$

Here note that the terms $\frac{3}{x} f_{l1} - f_{l1}^2$ cancel, given (B.48). We can compute the integral in (B.44) analytically with $P_{f_{l2}}$. Setting the lower limit of integration e^{-r} to zero, which is again equivalent to neglecting exponentially suppressed terms, we find

$$\begin{aligned} f_{l2} &= -\frac{1}{24r_0^2 x^3 (x+1)^2} \left(144r_0 x^3 \text{Li}_2(-x) + 9r_0^2 x^6 + 6r_0 x^6 + 12r_0 x^6 \ln(x+1) \right. \\ &\quad \left. + 54r_0^2 x^5 + 24r_0 x^5 + 36r_0 x^5 \ln(x+1) + 9r_0^2 x^4 + 60r_0 x^4 \right. \\ &\quad \left. - 36r_0 x^4 \ln(x+1) - 36r_0^2 x^3 + 12\pi^2 r_0 x^3 + 60r_0 x^3 + 36r_0 x^3 \ln^2(x) \right. \\ &\quad \left. + 30r_0 x^2 + 36r_0 x^2 \ln(x+1) - 12r_0 x^3 (6r_0 + x^3 + 3x^2 - 3x + 5) \ln(x) \right. \\ &\quad \left. + 12r_0 x - 36r_0 x \ln(x+1) - 12r_0 \ln(x+1) \right. \\ &\quad \left. - 4\nu^2 x^6 + x^6 - 24\nu^2 x^5 + 6x^5 - 36\nu^2 x^4 + 9x^4 - 16\nu^2 x^3 + 4x^3 \right). \quad (\text{B.51}) \end{aligned}$$

This function has the following asymptotic behavior:

$$f_{l2} \xrightarrow{x \rightarrow 0} \frac{8\nu^2 - 18r_0 \ln^2(x) + 6r_0 (6r_0 + 5) \ln(x) + 18r_0^2 - (55 + 6\pi^2) r_0 - 2}{12r_0^2}, \quad (\text{B.52})$$

$$f_{l2} \xrightarrow{x \rightarrow \infty} -\frac{x(-4\nu^2 + 9r_0^2 + 6r_0 + 1)}{24r_0^2}. \quad (\text{B.53})$$

Note that f_{l2} diverges linearly at large x .

Next, we compute $\ln R_{ln}$, which from the definition (B.37) is simply given by

$$\ln R_{ln}(x) = \int_{e^{-r}}^x dt f_{ln}(t). \quad (\text{B.54})$$

¹²The careful, and possibly skeptical reader might be bothered that (B.48) does not satisfy the boundary condition (B.39). However, one can check that using (B.47), which does satisfy (B.39), instead of (B.48) to continue the calculation at $n \geq 2$, produces the same results after longer and more tedious algebra. The lesson is that we can drop terms exponentially suppressed in r as soon as we encounter them.

Here, the lower limit of integration is to enforce the boundary condition $R_l(x = e^{-r}) = 1$. At $n = 1$, using (B.48), we have

$$\ln R_{l1}(x) = 3(\ln(x) + r), \quad (\text{B.55})$$

$$R_{l1}(x) = e^{\frac{3r_0}{\Delta} x^3}, \quad R_{l1}^\Delta(x) = e^{3r_0 x^3 \Delta}, \quad (\text{B.56})$$

where we have used $r = r_0/\Delta$. At $n = 2$ we have

$$\ln R_{l2}(x) = \int_0^x dt f_{l2}(t). \quad (\text{B.57})$$

Here we can set the lower limit of integration to zero. Doing so, given the asymptotic behavior (B.52), we are throwing away terms suppressed by e^{-r} . The integral (B.57) can be computed analytically plugging in (B.51). However, we only need its asymptotic behavior at large x , which is easily obtained by integrating (B.53):

$$\ln R_{l2}(x \rightarrow \infty) = \int^x dt f_{l2}(t \rightarrow \infty) = -\frac{x^2(-4\nu^2 + 9r_0^2 + 6r_0 + 1)}{48r_0^2}. \quad (\text{B.58})$$

We can now put together our low l result for R_l up to Δ^2 . Starting from (B.31) and (B.32), and using (B.56), we can factor out e^{3r_0} and write

$$R_l(x) = e^{3r_0} R_{l0}(x) \left(1 + 3\Delta \ln(x) + \Delta^2 \left(\frac{1}{2} 9 \ln^2(x) + \ln R_{l2}(x) \right) + \mathcal{O}(\Delta^3) \right). \quad (\text{B.59})$$

We are interested in the $x \rightarrow \infty$ limit. From (B.36) we have $R_{l0}(x \rightarrow \infty) = x^{-2}$, and using (B.58) we obtain

$$R_l(x \rightarrow \infty) = -\Delta^2 e^{3r_0} \frac{-4\nu^2 + 9r_0^2 + 6r_0 + 1}{48r_0^2} + \mathcal{O}(\Delta^3). \quad (\text{B.60})$$

Substituting $\nu = l + D/2 - 1$ and $r_0 = (D - 1)/3$ we can rewrite it as

$$\lim_{\rho \rightarrow \infty} R_l(\rho) = R_l(x \rightarrow \infty) = \Delta^2 \frac{3}{4} e^{D-1} \frac{(l-1)(l+D-1)}{(D-1)^2} + \mathcal{O}(\Delta^3), \quad (\text{B.61})$$

which corresponds to (5.22). Some comments are in order.

- The leading term in this low l result is of order Δ^2 , with the $\mathcal{O}(\Delta^0)$ and $\mathcal{O}(\Delta)$ terms vanishing. It has the structure we expected: it is negative for $l = 0$, it vanishes for $l = 1$. The latter signals the zero eigenvalues related to the translational invariance of the bounce. We will soon go through a procedure to remove them.
- The e^{3r_0} is an overall factor in (B.59) and as such it ends up contributing to the first term, that is zeroth order in the Δ expansion. Note we got e^{3r_0} at $n = 1$ [see (B.55) and (B.56)], rather than $n = 0$, as a consequence of the boundary condition at $x = e^{-r}$ (corresponding to $\rho = 0$). In other words, we set up our equations based on the Δ^n power counting, but then found a solution at $n = 1$ that fed back into $n = 0$. Let us refer to it as an order-breaking term. While this feature might be puzzling at first

sight, there is no reason to panic. The calculation is sensible and the procedure well defined: at each order one computes f_{ln} from (B.44), and then checks from (B.54) if there are contributions to $\ln R_{ln}$ from the boundary at $x = e^{-r}$, with $r = r_0/\Delta$, that feed back into lower orders. In the low l calculation this happens only at $n = 1$. In the next section, we will study the problem systematically for generic values of l (or, better, of ν) and show that order-breaking terms will appear for every $n \geq 1$, but that indeed at low l only the one from $n = 1$ survives.

- There is at least another way of obtaining the e^{3r_0} factor. Instead of using (B.31), one can approach the problem with an additive expansion, $R_l = R_{l0} + \Delta R_{l1} + \Delta^2 R_{l2} + \dots$, and solve the second order differential equation for R_l order by order. The calculation is more involved, compared to the one we have presented here, and one finds at each order boundary terms of the form $\Delta^n r^n$, which therefore feed back into $n = 0$. Summing all such terms, up to $n = \infty$, one recovers exactly the exponential e^{3r_0} .

Zero removal at low multipoles. With the explicit solution valid up to second-order corrections in Δ^2 , one can proceed to perturbatively remove the zero eigenvalue, as discussed in section 5.2. The orbital mode dependence comes in at $\mathcal{O}(\Delta^2)$. To remove the $l = 1$ modes it is then sufficient to off-set the potential at this order with an infinitesimal dimensional parameter

$$V^{(2)} \rightarrow V^{(2)} + \Delta^2 \mu_\varepsilon^2, \quad \text{or} \quad \tilde{V}^{(2)} \rightarrow \tilde{V}^{(2)} + \frac{\Delta^2 \mu_\varepsilon^2}{\lambda v^2}. \quad (\text{B.62})$$

This offset does not affect the zero and first-order solutions, once $\mu_\varepsilon \rightarrow 0$, thus R_0 and R_1 remain the same. On the other hand, the second-order solution with the above off-set is easily obtained starting from (B.44),

$$f_{l2}^\varepsilon(x) = \frac{(1+x)^4}{x^3} \int_{e^{-r}}^x dt \frac{t^3}{(1+t)^4} \left(P_{f_{l2}}(t) + \frac{\mu_\varepsilon^2}{\lambda v^2} \frac{1}{t^2} \right) \quad (\text{B.63})$$

$$= f_{l2} + \frac{\mu_\varepsilon^2}{\lambda v^2} \frac{(1+x)^4}{x^3} \int_0^\infty dt \frac{t}{(1+t)^4}. \quad (\text{B.64})$$

$$= f_{l2} + \frac{\mu_\varepsilon^2}{\lambda v^2} \frac{(1+x)^4}{6x^3}. \quad (\text{B.65})$$

From here, we plug the $f_{l2}^\varepsilon(x)$ into (B.54), such that

$$\ln R_{21}^\varepsilon = \int^x dt f_{l2}^\varepsilon(t) \quad (\text{B.66})$$

$$= \int^x dt \left(f_{l2} + \frac{\mu_\varepsilon^2}{\lambda v^2} \frac{(1+t)^4}{6t^3} \right) \quad (\text{B.67})$$

$$= \ln R_{21} + \frac{\mu_\varepsilon^2}{\lambda v^2} \frac{x^2}{12} = \frac{\mu_\varepsilon^2}{\lambda v^2} \frac{x^2}{12}, \quad (\text{B.68})$$

which is precisely what was given in (5.26).

Perturbative solutions for generic multipoles. In the previous section we computed $\lim_{x \rightarrow \infty} R_\nu(x)$, but our result was only valid for small values of ν . In this section we are going to revisit our calculation, generalizing it to generic values of ν . The starting point is to consider the combination $\Delta\nu$ as an order one parameter instead of a Δ suppressed one. Then the $\psi_{\nu\text{FV}}$ equation, see (5.14), has the leading order solution

$$\psi_{\nu\text{FV}}^{\text{lead}} = c_{\text{FV}} e^{k_\nu z}, \tag{B.69}$$

with

$$k_\nu = \sqrt{1 + \frac{\Delta^2 \nu^2}{r_0^2}}, \tag{B.70}$$

which counts as an $\mathcal{O}(\Delta^0)$ parameter. We define

$$\frac{1}{\psi_{\nu\text{FV}}} \frac{d}{dz} \psi_{\nu\text{FV}} \equiv g_\nu = \sum_{n=0}^{\infty} \Delta^n g_{\nu n}. \tag{B.71}$$

Given (B.69), we immediately have that $g_{\nu 0} = k_\nu$. Note, as a rather trivial check, that using (B.70), expanding in Δ and keeping only the zeroth order, that is $k_\nu = 1$, we match the low l result: $g_{\nu 0}(k_\nu = 1) = g_{l0} = 1$. With (B.71) the differential FV equation (5.12) turns into

$$\frac{d}{dz} g_\nu + g_\nu^2 = \frac{\nu^2 - \frac{1}{4}}{(z+r)^2} + \tilde{V}_{\text{FV}}^{(2)}. \tag{B.72}$$

With the counting stated above the right hand side has the expansion

$$\begin{aligned} \frac{\nu^2 - \frac{1}{4}}{(z+r)^2} + \tilde{V}_{\text{FV}}^{(2)} &= k_\nu^2 + \left(-3 - 2(k_\nu^2 - 1) \frac{z}{r_0}\right) \Delta + \left(-3 - \frac{1}{4r_0^2} + 3(k_\nu^2 - 1) \frac{z^2}{r_0^2}\right) \Delta^2 \\ &+ \left(-\frac{15}{2} + \frac{z}{2r_0^3} - 4(k_\nu^2 - 1) \frac{z^3}{r_0^3}\right) \Delta^3 + \mathcal{O}(\Delta^4). \end{aligned} \tag{B.73}$$

Using this and pugging (B.71) into (B.72), we get the $n = 1$ equation

$$\frac{d}{dz} g_{\nu 1} + 2k_\nu g_{\nu 1} = -3 - 2(k_\nu^2 - 1) \frac{z}{r_0}, \tag{B.74}$$

which has the solution

$$g_{\nu 1} = \frac{-2(k_\nu^2 - 1)}{2k_\nu} \frac{z}{r_0} + \frac{k_\nu^2 - 1 - 3k_\nu r_0}{2k_\nu^2 r_0}. \tag{B.75}$$

Note the first term is linear in z , while the second is z independent. With this we can solve (B.72) at $n = 2$, where we find

$$g_{\nu 2} = \frac{1}{2k_\nu} \left(3(k_\nu^2 - 1) - 4(k_\nu^2 - 1)^2 \frac{1}{(2k_\nu)^2} \right) \left(\frac{z}{r_0} \right)^2 + \text{terms with lower powers of } z. \tag{B.76}$$

Here we don't bother writing explicitly the terms with lower powers of z because they are not needed for the rest of the calculation, as we are going to explain. Let us keep the higher orders $g_{\nu n}$ on hold for a moment, and turn to R_ν .

The R_ν equation to solve for generic multipoles,

$$\frac{1}{R_\nu} \frac{d^2}{dx^2} R_\nu + \frac{2g_\nu + 1}{x} \frac{1}{R_\nu} \frac{d}{dx} R_\nu - \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right) = 0, \quad (\text{B.77})$$

reads the same as (B.29), but g_l from (B.26) is replaced with g_ν from (B.71). At order zero we have

$$\frac{1}{R_{\nu 0}} \frac{d^2}{dx^2} R_{\nu 0} + \frac{2k_\nu + 1}{x} \frac{1}{R_{\nu 0}} \frac{d}{dx} R_{\nu 0} - \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_0 = 0, \quad (\text{B.78})$$

with the last term given by (B.35). The solution is

$$R_{\nu 0}(x) = \frac{1 - 4x + x^2 + 2k_\nu^2(1+x)^2 + 3k_\nu(1-x^2)}{(1+k_\nu)(1+2k_\nu)(1+x)^2}. \quad (\text{B.79})$$

This matches the boundary condition $R_{\nu 0}(x \rightarrow 0) = 1$, reduces to (B.36) for $k_\nu = 1$, and has the asymptotic behavior

$$R_{\nu 0}(\infty) = \frac{(1-k_\nu)(1-2k_\nu)}{(1+k_\nu)(1+2k_\nu)}. \quad (\text{B.80})$$

Then we introduce

$$f_\nu \equiv \frac{1}{R_\nu} \frac{d}{dx} R_\nu, \quad (\text{B.81})$$

in terms of which (B.77) becomes

$$\frac{d}{dx} f_\nu + f_\nu^2 + \frac{2g_\nu + 1}{x} f_\nu - \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right) = 0, \quad (\text{B.82})$$

with the boundary condition

$$e^{-r} f_\nu(e^{-r}) = 0. \quad (\text{B.83})$$

We expand

$$f_\nu = \sum_{n=0}^{\infty} \Delta^n f_{\nu n}, \quad (\text{B.84})$$

and g_ν with (B.71), then solve (B.82) order by order in Δ . The $f_{\nu 0}$ can be immediately obtained from (B.79),

$$f_{\nu 0}(x) = \frac{-6(1+k_\nu-x+k_\nu x)}{(1+x)(1-4x+x^2+2k_\nu^2(1+x)^2+3k_\nu(1-x^2))}. \quad (\text{B.85})$$

For order $n \geq 1$, from (B.82) we get

$$\frac{d}{dx} f_{\nu n} + \left(2f_{\nu 0} + \frac{2k_\nu + 1}{x} \right) f_{\nu n} = P_{f_{\nu n}}, \quad (\text{B.86})$$

with

$$P_{f_{\nu n}} = \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_n - \frac{2}{x} \sum_{j=1}^n g_{\nu j} f_{\nu(n-j)} - \sum_{j=1}^{n-1} f_{\nu j} f_{\nu(n-j)}. \quad (\text{B.87})$$

Here the last sum contributes only for $n \geq 2$. The solution to (B.86) is

$$f_{\nu n}(x) = \frac{x^{-(2k_\nu+1)}(1+x)^4}{(1-4x+x^2+2k_\nu^2(1+x)^2+3k_\nu(1-x^2))^2} \tag{B.88}$$

$$\int_{e^{-r}}^x dt \frac{t^{2k_\nu+1}}{(1+t)^4} \left(1-4t+t^2+2k_\nu^2(1+t)^2+3k_\nu(1-t^2)\right)^2 P_{f_{\nu n}}(t),$$

and from the definition (B.81) we have

$$\ln R_{\nu n}(x) = \int_{e^{-r}}^x dt f_{\nu n}(t). \tag{B.89}$$

Analogously to the low l case, in (B.88) we can always set the lower limit of integration to zero, dropping only terms exponentially suppressed in r in doing so. On the contrary, in (B.89) it is crucial to maintain e^{-r} as the lower limit of integration, which implements the boundary condition $R_\nu(\rho=0) = R_\nu(x=e^{-r}) = 1$. Note that, as $R_{\nu 0}(x=e^{-r} \rightarrow 0) = 1$, we must require $\ln R_{\nu n}(x=e^{-r}) = 0$ for $n \geq 1$.

Formally we have the recipe to compute $\ln R_{\nu n}$ at any n and get the result

$$R_\nu = R_{\nu 0} R_{\nu 1}^\Delta R_{\nu 2}^{\Delta^2} R_{\nu 3}^{\Delta^3} \dots \tag{B.90}$$

$$= R_{\nu 0} \left(1 + \Delta \ln R_{\nu 1} + \Delta^2 \left(\frac{1}{2} \ln^2 R_{\nu 1} + \ln R_{\nu 2}\right) + \mathcal{O}(\Delta^3)\right), \tag{B.91}$$

up to the desired order in Δ . In practice, we can solve the integral in (B.88) analytically only for $n = 1$, the result containing hypergeometric functions, but already fail when trying to integrate it to compute the full form of $\ln R_{\nu 1}$. We have already found that $R_{\nu 0}(x \rightarrow \infty)$ in (B.80) does not vanish, in contrast with $R_{l 0}$ in the low- l regime, which did vanish at large x . Thus, it seems like we already have the leading order term in the expansion (B.90). But is that the full contribution at the leading order? In the low- l calculation we found an order-breaking term at $n = 1$ that fed back into $n = 0$. It came from the lower boundary of integration of $\ln R_{l 1}$. We should then check if we also get order-breaking terms in the current calculation for generic multipoles. To do so, we must study the lower boundary of the integral (B.89), for which it is sufficient to know the $x \rightarrow 0$ limit of $f_{\nu n}$. That is obtained from (B.88):

$$f_{\nu n}(x \rightarrow 0) = x^{-(2k_\nu+1)} \int_0^x dt t^{2k_\nu+1} P_{f_{\nu n}}(t \rightarrow 0). \tag{B.92}$$

At $n = 1$, from (B.87) we have

$$P_{f_{\nu 1}} = \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)}\right)_1 - \frac{2}{x} g_{\nu 1} f_{\nu 0}, \tag{B.93}$$

with [see (B.45) and (B.75)]

$$\left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)}\right)_1 = \frac{6}{1+x} \qquad g_{\nu 1}(x) = -2 \frac{k_\nu^2 - 1 \ln(x)}{2k_\nu r_0}, \tag{B.94}$$

and $f_{\nu 0}$ given in (B.85). Here we have kept only the highest power of $z = \ln(x)$ in $g_{\nu 1}$. In the limit $x \rightarrow 0$:

$$\frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_1 \xrightarrow{x \rightarrow 0} \frac{6}{x^2}, \quad (\text{B.95})$$

$$\frac{2}{x} g_{\nu 1} f_{\nu 0} \xrightarrow{x \rightarrow 0} 2 \frac{\ln(x)}{x} \left(-2 \frac{k_\nu^2 - 1}{2k_\nu} \frac{1}{r_0} \right) \left(\frac{-6}{1 + 2k_\nu} \right). \quad (\text{B.96})$$

From the first term, (B.95), using (B.92), we find

$$P_{f_{\nu 1}}(x \rightarrow 0) = \frac{6}{x^2}, \quad f_{\nu 1}(x \rightarrow 0) = \frac{3}{k_\nu x}, \quad (\text{B.97})$$

which plugged into (B.89) gives

$$\Delta \ln R_{\nu 1} \ni \Delta \int_{e^{-r}} dx f_{\nu 1}(x \rightarrow 0) = 3r_0 \frac{1}{k_\nu} \equiv U_1. \quad (\text{B.98})$$

This is an order-breaking term which feeds back into the $n = 0$ result. When $k_\nu \rightarrow 1$ it matches the factor in (B.56) we obtained in the low- l calculation. Note that U_1 originates from the second derivative of the potential, (B.95), calculated at $n = 1$, which involves the bounce up to first order in Δ , that is φ_1 . It is easy to check that the other term, (B.96), gives $f_{\nu 1}(x \rightarrow 0) \propto \ln(x)$, does not produce a boundary term in $\ln R_{\nu 1}$, and so does not feed back into $n = 0$. Recall that in (B.96) we have dropped a term coming from the x -independent piece in $g_{\nu 1}$, see (B.75). Clearly, that cannot produce a boundary term in $\ln R_{\nu 1}$ either.

At $n = 2$ we have

$$P_{f_{\nu 2}} = \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_2 - \frac{2}{x} (g_{\nu 1} f_{\nu 1} + g_{\nu 2} f_{\nu 0}) - f_{\nu 1}^2. \quad (\text{B.99})$$

Using (B.49), (B.85), (B.75), (B.76), (B.97), for $x \rightarrow 0$ we get:

$$\frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_2 \xrightarrow{x \rightarrow 0} \frac{3 \left(12r_0(\ln(x) + 2) - 6 \ln^2(x) + 6 \ln(x) - 2\pi^2 - 15 \right)}{4r_0 x}, \quad (\text{B.100})$$

$$\frac{2}{x} g_{\nu 1} f_{\nu 1} \xrightarrow{x \rightarrow 0} 2 \cdot \frac{-2(k_\nu^2 - 1)}{2k_\nu} \frac{1}{r_0} \frac{3}{k_\nu} \frac{\ln(x)}{x^2}, \quad (\text{B.101})$$

$$\frac{2}{x} g_{\nu 2} f_{\nu 0} \xrightarrow{x \rightarrow 0} 2 \frac{k_\nu^2 - 1}{2k_\nu} \left(3 - 4(k_\nu^2 - 1) \frac{1}{(2k_\nu)^2} \right) \frac{1}{r_0^2} \frac{-6}{1 + 2k_\nu} \frac{\ln^2(x)}{x}, \quad (\text{B.102})$$

$$f_{\nu 1}^2 \xrightarrow{x \rightarrow 0} \frac{9}{k_\nu^2 x^2}. \quad (\text{B.103})$$

Here there are two contributions proportional to $1/x^2$: (B.103) and (B.101), which also contains $\ln(x)$ from $g_{\nu 1}$. From the latter we find

$$P_{f_{\nu 2}}(x \rightarrow 0) = -\frac{2}{x} g_{\nu 1} f_{\nu 1}(x \rightarrow 0) = \frac{6}{r_0} \frac{k_\nu^2 - 1}{k_\nu^2} \frac{\ln(x)}{x^2}, \quad (\text{B.104})$$

$$f_{\nu 2}(x \rightarrow 0) = \frac{6}{r_0} \frac{1}{2k_\nu} \frac{k_\nu^2 - 1}{k_\nu^2} \frac{\ln(x)}{x}, \quad (\text{B.105})$$

which gives the order-breaking contribution

$$\Delta^2 \ln R_{\nu 2} \ni \Delta^2 \int_{e^{-r}} dx f_{\nu 2}(x \rightarrow 0) = 3r_0 \left(-\frac{1}{2k_\nu} + \frac{1}{2k_\nu^3} \right) \equiv U_2, \quad (\text{B.106})$$

feeding back into $n = 0$. In the limit $k_\nu \rightarrow 1$ we get $U_2 \rightarrow 0$. This confirms the absence of such a contribution in the low-multipole limit, in agreement with our low- l result. From (B.103), one ends up with a boundary term $\Delta^2 \ln R_{\nu 2} \propto \Delta^2 r = \mathcal{O}(\Delta)$, which is also order-breaking, but feeds back into $n = 1$ rather than $n = 0$, so it is not a leading order contribution. From the other two terms, (B.100) and (B.102), both proportional to $1/x$, we get terms in $\Delta^2 \ln R_{\nu 2}$ which are genuinely of order Δ^2 .

At $n = 3$ we have

$$P_{f_{\nu 3}} = \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_3 - \frac{2}{x} (g_{\nu 1} f_{\nu 2} + g_{\nu 2} f_{\nu 1} + g_{\nu 3} f_{\nu 0}) - 2f_{\nu 1} f_{\nu 2}. \quad (\text{B.107})$$

Here, the term

$$\frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_3 = \frac{1}{x^2} \left(\frac{15}{2} + 3\varphi_1\varphi_2 + 3\varphi_0\varphi_3 \right) \xrightarrow{x \rightarrow 0} \frac{1}{x^2} (15 + \mathcal{O}(x)), \quad (\text{B.108})$$

gives a contribution $\Delta^3 \ln R_{\nu 3} \propto \mathcal{O}(\Delta^2)$, which feeds back into $n = 2$, but not into the leading order. The following pieces

$$P_{f_{\nu 3}}(x \rightarrow 0) = -\frac{2}{x} (g_{\nu 1} f_{\nu 2} + g_{\nu 2} f_{\nu 1})(x \rightarrow 0) = -\frac{9}{r_0^2} \frac{k_\nu^2 - 1}{k_\nu^4} \frac{\ln^2(x)}{x^2}, \quad (\text{B.109})$$

$$f_{\nu 3}(x \rightarrow 0) = -\frac{9}{r_0^2} \frac{1}{2k_\nu} \frac{k_\nu^2 - 1}{k_\nu^4} \frac{\ln^2(x)}{x}, \quad (\text{B.110})$$

give the contribution

$$\Delta^3 \ln R_{\nu 3} \ni \Delta^3 \int_{e^{-r}} dx f_{\nu 3}(x \rightarrow 0) = 3r_0 \left(-\frac{1}{2k_\nu^3} + \frac{1}{2k_\nu^5} \right) \equiv U_3, \quad (\text{B.111})$$

which feeds back into $n = 0$. Here we have used (B.97), (B.105), and we have kept only the terms with the leading power of $z = \ln(x)$ in $g_{\nu 1}$ and $g_{\nu 2}$ from (B.75) and (B.76); sub-leading powers of z do not contribute to (B.111). Likewise, one can check that the remaining terms in (B.107) don't produce a contribution that feeds back into $n = 0$.

At $n = 4$ we have

$$P_{f_{\nu 4}} = \frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_4 - \frac{2}{x} (g_{\nu 1} f_{\nu 3} + g_{\nu 2} f_{\nu 2} + g_{\nu 3} f_{\nu 1} + g_{\nu 4} f_{\nu 0}) - 2f_{\nu 1} f_{\nu 3} - f_{\nu 2}^2, \quad (\text{B.112})$$

with

$$\frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_4 = \frac{1}{x^2} \frac{3}{2} (16 + 2\varphi_0\varphi_4 + 2\varphi_1\varphi_3 + \varphi_2^2) \xrightarrow{x \rightarrow 0} \mathcal{O} \left(\frac{1}{x} \right). \quad (\text{B.113})$$

The order-breaking contribution feeding back into $n = 0$, comes from

$$P_{f_{\nu 4}}(x \rightarrow 0) = -\frac{2}{x} (g_{\nu 1} f_{\nu 3} + g_{\nu 2} f_{\nu 2} + g_{\nu 3} f_{\nu 1})(x \rightarrow 0) = -\frac{3}{r_0^3} \frac{5 - 6k_\nu^2 + k_\nu^4}{k_\nu^6} \frac{\ln^3(x)}{x^2}, \quad (\text{B.114})$$

$$f_{\nu 4}(x \rightarrow 0) = -\frac{3}{r_0^3} \frac{1}{2k_\nu} \frac{5 - 6k_\nu^2 + k_\nu^4}{k_\nu^6} \frac{\ln^3(x)}{x}, \quad (\text{B.115})$$

and is given by

$$\Delta^4 \ln R_{\nu 4} \ni \Delta^4 \int_{e^{-r}} dx f_{\nu 4}(x \rightarrow 0) = 3r_0 \left(\frac{1}{8k_\nu^3} - \frac{6}{8k_\nu^5} + \frac{5}{8k_\nu^7} \right) \equiv U_4. \quad (\text{B.116})$$

Here we have used $g_{\nu 3} \propto z^3 = \ln^3(x)$, see the derivation below. The terms in (B.112) not included in (B.114) do not feed back into $n = 0$.

The pattern should be clear at this point. We are after order-breaking contributions of the form $\Delta^n \ln R_{\nu n} = \mathcal{O}(\Delta^0)$. They come from the lower limit of the integral (B.89), when $f_{\nu n}$ has the form

$$f_{\nu n}(x \rightarrow 0) \propto \frac{\ln^{n-1}(x)}{x}. \quad (\text{B.117})$$

This, in turn, is obtained from

$$P_{f_{\nu n}}(x \rightarrow 0) \propto \frac{\ln^{n-1}(x)}{x^2}, \quad (\text{B.118})$$

as one can check by plugging it into (B.92),

$$x^{-(2k_\nu+1)} \int_0^x dt t^{2k_\nu+1} \frac{\ln^{n-1}(t)}{t^2} = \frac{1}{2k_\nu} \frac{\ln^{n-1}(x) + \dots}{x}, \quad (\text{B.119})$$

with the dots denoting terms with powers of $\ln(x)$ lower than $n - 1$ in the numerator, which we are dropping. Let us look into the general form of $P_{f_{\nu n}}$ in (B.87), which has the following behavior. For the odd orders

$$\frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_{2j+1} \xrightarrow{x \rightarrow 0} \frac{1}{x^2} [\text{constant} + \mathcal{O}(x)], \quad (\text{B.120})$$

while for the even orders

$$\frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_{2j} \xrightarrow{x \rightarrow 0} \frac{1}{x^2} [\mathcal{O}(x)]. \quad (\text{B.121})$$

Such a behavior is a direct consequence of the boundary condition $\varphi(x \rightarrow 0) = \varphi(\rho = 0) = \varphi_{\text{TV}}$ of the bounce solution. Then we see that only for $n = 1$ such a term is of the form (B.118), and from it we get first $f_{\nu 1}$, then U_1 . Proceeding to higher orders, $n \geq 2$, we can forget about the second-derivative-potential term in (B.87), as at most the odd orders (B.120) will give contributions feeding back into the previous order, not back into $n = 0$. We can also drop the last sum, $-\sum_{j=1}^{n-1} f_{\nu j} f_{\nu(n-j)}$, which never produces terms like (B.118). We are left with the sum $-\frac{2}{x} \sum_{j=1}^n g_{\nu j} f_{\nu(n-j)}$. We have seen that only the term with the highest power of z in $g_{\nu n}$, that is z^n , gives (B.118). Let us then compute the coefficients of such terms. To do so, we have to get back to (B.72). From (B.73) we see that we can write the expansion as

$$\frac{\nu^2 - \frac{1}{4}}{(z+r)^2} + \tilde{V}_{\text{FV}}^{(2)} = k_\nu^2 + (k_\nu^2 - 1) \sum_{n=1}^{\infty} (-)^n (n+1) \frac{\Delta^n z^n}{r_0^n} \quad (\text{B.122})$$

$$+ \sum_{n>q \geq 0} c_{n,q} \Delta^n z^q. \quad (\text{B.123})$$

Then we can write (B.72), using (B.71), for $n \geq 1$ as

$$\frac{d}{dz}g_{\nu n} + 2k_{\nu}g_{\nu n} = P_{g_n}, \tag{B.124}$$

with the function

$$P_{g_n} = (-)^n(n+1)(k_{\nu}^2 - 1)\frac{z^n}{r_0^n} - \sum_{j=1}^{n-1} g_{\nu j} g_{\nu(n-j)}, \tag{B.125}$$

built iteratively. Here the last sum is present only for $n \geq 2$. In P_{g_n} we have dropped the terms from (B.123), because they contain sub-leading powers of z , which we do not need. The solution to (B.124) is

$$g_{\nu n}(z) = e^{-2k_{\nu}z} \int_{-r}^z dt e^{2k_{\nu}t} P_{g_n}(t). \tag{B.126}$$

The lower limit of integration can be understood as follows. From (B.22), switching from ρ to z , we get the boundary condition

$$g_{\nu}(z \rightarrow -r) = \frac{1}{z+r} = \frac{\Delta}{r_0} \sum_{n=0}^{\infty} (-)^n \frac{\Delta^n z^n}{r_0^n}. \tag{B.127}$$

The sum with alternating signs on the right hand side, obtained after expanding in Δ , implies that $g_{\nu n}(z = -r = -r_0/\Delta) = 0$, as implemented in (B.126). We can infer from (B.125) that P_{g_n} is a polynomial in z of degree n . Hence, we can set $-r \rightarrow -\infty$ in the lower limit of integration of (B.126), dropping terms exponentially suppressed in r . At $n = 1$, from (B.125) we get

$$P_{g_1} = -2(k_{\nu}^2 - 1)\frac{z}{r_0}. \tag{B.128}$$

To get $g_{\nu 1}$ we must do the integral in (B.126). The following result is useful:

$$\frac{1}{r_0^n} e^{-2k_{\nu}z} \int_{-\infty}^z dt e^{2k_{\nu}t} t^n = \frac{1}{2k_{\nu}} \left(\frac{z}{r_0}\right)^n + \dots, \tag{B.129}$$

with the dots indicating terms with powers of z lower than n , which we will be dropping systematically in what follows. With this we reproduce the first term of $g_{\nu 1}$ in (B.75). Using it into (B.125), at $n = 2$ we have

$$P_{g_2} = \left(3(k_{\nu}^2 - 1) - 4(k_{\nu}^2 - 1)^2 \frac{1}{(2k_{\nu})^2}\right) \left(\frac{z}{r_0}\right)^2, \tag{B.130}$$

which upon integration with (B.129) gives $g_{\nu 2}$ in (B.76). We can systematize the procedure by writing

$$P_{g_n} = c_{g_n}(k_{\nu}) \left(\frac{z}{r_0}\right)^n, \tag{B.131}$$

and using (B.129) to retain only the highest power of z at any order n

$$g_{\nu n} = \frac{c_{g_n}(k_{\nu})}{2k_{\nu}} \left(\frac{z}{r_0}\right)^n. \tag{B.132}$$

The coefficients c_{g_n} , which are functions of k_ν , are built iteratively from (B.125):

$$c_{g_n}(k_\nu) = (-)^n(n+1)(k_\nu^2 - 1) - \frac{1}{(2k_\nu)^2} \sum_{j=1}^{n-1} c_{g_j}(k_\nu)c_{g_{n-j}}(k_\nu). \quad (\text{B.133})$$

Here the last sum is present only for $n \geq 2$. The explicit form of $g_{\nu 3}$, which we used to compute $f_{\nu 4}$ above, is

$$g_{\nu 3} = \frac{k_\nu^2 - 1}{2k_\nu} \left(-16 \frac{(k_\nu^2 - 1)^2}{(2k_\nu)^4} + 12 \frac{(k_\nu^2 - 1)}{(2k_\nu)^2} - 4 \right) \left(\frac{z}{r_0} \right)^3. \quad (\text{B.134})$$

With $g_{\nu n}$ at hand, we can return to the expression for $P_{f_{\nu n}}$ and focus on $-\frac{2}{x} \sum_{j=1}^n g_{\nu j} f_{\nu(n-j)}$. The last term in this sum, $\frac{2}{x} g_{\nu n} f_{\nu 0} \propto \frac{\ln^n(x)}{x}$, does not have the behavior of (B.118), and indeed does not feed back into $n = 0$, so we drop it. We are left with

$$P_{f_{\nu n}}(x \rightarrow 0) = -\frac{2}{x} \sum_{j=1}^{n-1} g_{\nu j} f_{\nu(n-j)} = \frac{3}{r_0^{n-1}} c_{f_{\nu n}}(k_\nu) \frac{\ln^{n-1}(x)}{x^2}, \quad n \geq 2, \quad (\text{B.135})$$

where we have introduced the coefficients $c_{f_{\nu n}}(k_\nu)$, which are functions only of k_ν . Integrating with (B.119) we get

$$f_{\nu n}(x \rightarrow 0) = \frac{3}{r_0^{n-1}} \frac{c_{f_{\nu n}}(k_\nu)}{2k_\nu} \frac{\ln^{n-1}(x)}{x}, \quad n \geq 2. \quad (\text{B.136})$$

Now we can proceed iteratively. Note (B.135) and (B.136) start at $n = 2$, so the iteration requires

$$c_{f_{\nu 1}} = 2, \quad f_{\nu 1} = \frac{3}{k_\nu x}, \quad (\text{B.137})$$

which we computed explicitly above, as the starting point. Plugging (B.132), with $z = \ln(x)$, and (B.136) into (B.135) we get the recursive relation

$$c_{f_{\nu n}}(k_\nu) = -\frac{1}{k_\nu^2} \left(c_{g_{n-1}}(k_\nu) + \frac{1}{2} \sum_{j=1}^{n-2} c_{g_j}(k_\nu) c_{f_{\nu(n-j)}}(k_\nu) \right), \quad n \geq 2, \quad (\text{B.138})$$

where we have used (B.137). Here the c_{g_n} coefficients are given by (B.133). We have thus managed to formally isolate all the contributions $\Delta^n \ln R_{\nu n} = \mathcal{O}(\Delta^0)$ which feed back into the leading order. They are given by

$$U_n \equiv \Delta^n \frac{3}{r_0^{n-1}} \frac{c_{f_{\nu n}}(k_\nu)}{2k_\nu} \int_{e^{-r_0/\Delta}} dx \frac{\ln^{n-1}(x)}{x} = 3r_0 \left(-\frac{(-)^n c_{f_{\nu n}}(k_\nu)}{n} \frac{1}{2k_\nu} \right). \quad (\text{B.139})$$

The U factor is obtained by summing¹³ over all of them:

$$U = \sum_{n=1}^{\infty} U_n = 3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1} \right). \quad (\text{B.140})$$

¹³What we have to perform is a complicated nested sum, given (B.138), which in turn contains c_{g_n} , given by (B.133). When we include enough orders in $c_{f_{\nu n}}$ for a fixed n , the sum turns into

$$\sum_{n=1}^{\infty} \left(-\frac{(-)^n c_{f_{\nu n}}(k_\nu)}{n} \frac{1}{2k_\nu} \right) = \frac{1}{k_\nu} \sum_{j=0}^{\infty} \frac{(2j)!}{2^{2j+1}(j+1)!j!} \frac{1}{k_\nu^{2j}} = k_\nu - \sqrt{k_\nu^2 - 1}.$$

With this, we have the full contribution at the leading order,

$$\begin{aligned} \lim_{x \rightarrow \infty} \ln R_\nu(x) &= \ln R_{\nu 0}(x \rightarrow \infty) + U + \mathcal{O}(\Delta) \\ &= \ln \frac{(1 - k_\nu)(1 - 2k_\nu)}{(1 + k_\nu)(1 + 2k_\nu)} + 3r_0 \left(k_\nu - \sqrt{k_\nu^2 - 1} \right) + \mathcal{O}(\Delta), \end{aligned} \quad (\text{B.141})$$

as in (5.39). After this long derivation, the result in (B.141) deserves some comments.

- Expression (B.141) holds in any number of dimensions D .
- Recall the definition of k_ν from (B.70) and remember we are treating k_ν as a leading order parameter in the expansion in Δ . One could be tempted to expand $R_\nu(\infty)$ for $\Delta\nu \ll 1$,

$$e^U \frac{(1 - k_\nu)(1 - 2k_\nu)}{(1 + k_\nu)(1 + 2k_\nu)} \xrightarrow{\Delta\nu \ll 1} \Delta^2 e^{3r_0} \frac{\nu^2}{12r_0^2}, \quad (\text{B.142})$$

and compare to the result (B.60). The two do not quite match, and indeed we should not expect so. The point is that the calculation we performed in this section is valid up to $\mathcal{O}(\Delta)$, as written explicitly in (B.141): we cannot expect to get the Δ^2 order correctly and match the result at low ν , (B.60), which was derived consistently up to that order in the earlier section. In other words, there are terms of order Δ^2 in $\mathcal{O}(\Delta)$ in (B.141) which we have not computed, and do not know how to compute for generic values of ν . Such terms have the form (B.123), and we have dropped them in our calculation. As we argued in section 6, when we perform the sum in ν our result (B.141) is sufficient to compute the log of the determinant at the leading order in Δ .

- When $\nu \ll 1/\Delta$, $k_\nu = 1 + \mathcal{O}(\Delta^2)$, and $U = U_1 = 3r_0$, with $U_n = 0$ for $n \geq 2$. This confirms what we anticipated at the end of the low l calculation: at low l (low ν) the only order-breaking contribution comes from $n = 1$ and results in the factor e^{3r_0} in R_l . For generic values of ν , we have found instead that there are order-breaking contributions for any $n \geq 1$ and we have managed to organize them systematically.
- The contribution to U from $n = 1$ is, in some sense, special. It is the only one that survives at low ν , and the only one that originates from the potential, that is from the term

$$\frac{1}{x^2} \left(\tilde{V}^{(2)} - \tilde{V}_{\text{FV}}^{(2)} \right)_1 = \frac{3}{x^2} (1 + \varphi_0 \varphi_1), \quad (\text{B.143})$$

in $P_{f_{\nu 1}}$, see (B.93). From this we got $f_{\nu 1}(x \rightarrow 0) = 3/(k_\nu x)$ and $U_1 = 3r_0/k_\nu$. Then, for $n \geq 2$ we just had to iterate (B.135) and (B.136), and we could do it to arbitrary order n . The knowledge of $f_{\nu 1}$ was the crucial starting point, then the iteration was automatic. This implies that our result (B.141) is constructed only based on knowing the bounce up to $n = 1$.

- What is often done in the literature, to compute analytically the fluctuations in the TW limit, is to set $z + r \simeq r$ in the denominator on the left hand side of (B.122), based on the observation that the bounce radius r is large. For us, on the contrary, it

is crucial to maintain the z dependence in that denominator, which then enters in the expansion on the right hand side of (B.122). It is from such a z dependence that we build the U factor, which is crucial to get the full correct result for the fluctuations at the leading order.

C UV integrals

In the process of renormalization, we encounter integrals of the generic form

$$I_m = \int_0^\infty d\rho \rho^{2m-1} \left(V^{(2)m} - V_{\text{FV}}^{(2)m} \right). \quad (\text{C.1})$$

This section of the appendix deals with the evaluation of these integrals, starting from the two explicit cases of $I_{1,2}$ and then with the generic expression, valid for any m . After the usual substitutions for ρ and the bounce, the calculation of I_1 goes as

$$I_1 = \int_{-r}^\infty dz (r+z) \left(\frac{3}{2} (\varphi^2 - 1) - 3\Delta \right). \quad (\text{C.2})$$

The second term in the bracket is of higher-order, so we drop it here to get the leading order

$$I_1^{n=0} = -6r \int_0^\infty \frac{dx}{x} \frac{x}{(1+x)^2} = -6r. \quad (\text{C.3})$$

Here we are allowed to extend the limits of integration from $x = e^{-r}$ to 0. Moving on to the higher-order and including φ_1 , we get

$$I_1^{n=1,2} = \Delta \int_{e^{-r}}^\infty \frac{dx}{x} (r + \ln x) \left(\frac{6}{1+x} \right), \quad (\text{C.4})$$

where the $\ln x$ term is Δ suppressed with respect to r and is thus counted as $n = 2$. These two higher orders are dominated by the lower boundary, which cannot be extended to $x = 0$. However, we may expand the integrand for small x and simplify the integration

$$I_1^{n=1} \simeq 6\Delta r \int_{e^{-r}} \frac{dx}{x} = 6\Delta r^2, \quad I_1^{n=2} = 6\Delta \int_{e^{-r}} \frac{dx}{x} \ln x = 6\Delta \int_{-r} dz z = -3\Delta r^2. \quad (\text{C.5})$$

Further including φ_2 does not bring any additional $1/\Delta$ terms to I_1 , they are Δ^2 suppressed. Combining all of the terms, we get

$$I_1 = -6r + 3\Delta r^2 \simeq -3(2 - r_0) \left(\frac{r_0}{\Delta} \right), \quad (\text{C.6})$$

which agrees with the expression in (6.8). Note that the $n = 0$ part of I_1 cancels the divergence of $\ln R_{\nu 0}$, while combining $n = 1, 2$ orders comes with an additional r_0 , just like the U -factor and it precisely cancels its dependence. In other words $n = 0$ part of I_1 constitutes the asymptotic $\ln R_{\nu 0}^a$ and the $n \geq 1$ go into the U^a part of the subtraction. Now that we have the complete expression for I_1 collected in (6.8), we can move on.

At the leading order, the I_2 UV integral (after all the substitutions) can be written as

$$I_2^{n=0} = -12r^3 \int_0^\infty \frac{dx}{x} \frac{x(1-x+x^2)}{(1+x)^4} = -6r^3. \quad (\text{C.7})$$

To get to the sub-leading orders, we add φ_1 into $V^{(2)2} - V_{\text{FV}}^{(2)2} = 12\Delta(1-x+4x^2)/(1+x)^3$, which brings in one power of Δ . We then expand the integrand at small x , because that is what dominates the integral, expand $\rho^3 = (r+z)^3$ and count each power of z as one order higher, to end up with

$$I_2^{n=1} = \int_{e^{-r}} \frac{dx}{x} r^3 12\Delta \left(\frac{1-x+4x^2}{(1+x)^3} \right) \simeq 12\Delta r^3 \int_{-r} dz = 12\Delta r^4, \quad (\text{C.8})$$

$$I_2^{n=2} = 12\Delta \int_{-r} dz 3r^2 z = -18\Delta r^4, \quad (\text{C.9})$$

$$I_2^{n=3} \simeq 12\Delta \int_{-r} dz 3r z^2 = 12\Delta r^4, \quad (\text{C.10})$$

$$I_2^{n=4} \simeq 12\Delta \int_{-r} dz z^3 = -3\Delta r^4. \quad (\text{C.11})$$

We dropped all the terms sub-leading in Δ and collected the leading Δ^0 powers. We now have five orders in $n = 0, \dots, 4$ and again the $n = 0$ matches the $\ln R_{\nu 0}^a$, while all the higher orders combine and give an extra factor of r_0 that matches to U^a ; altogether, we have

$$I_2^{\sum n} = -6r^3 + 3\Delta r^4 \simeq -3(2-r_0) \left(\frac{r_0}{\Delta} \right)^3, \quad (\text{C.12})$$

which is what is given in (6.13).

General order. We can carry out the above analysis for any order. First, expand the integrand up to Δ

$$V^{(2)m} - V_{\text{FV}}^{(2)m} \propto \frac{1}{2^m} (3\varphi_0^2 - 1)^m - 1 - 3\Delta m \left(\frac{1}{2^{m-1}} (3\varphi_0^2 - 1)^{m-1} \varphi_0 - 1 \right). \quad (\text{C.13})$$

In the first piece, coming from the $\mathcal{O}(\Delta^0)$ term in (C.13), we can extend the lower limit of integration to zero. Furthermore, we can expand the power of $\rho^{2m-1} \simeq r^{2m-1}$, the rest is subdominant in Δ

$$I_m^{n=0} = r^{2m-1} \int_0^\infty \frac{dx}{x} \left(\frac{1}{2^m} (3\varphi_0^2 - 1)^m - 1 \right), \quad \varphi_0 = \frac{x-1}{x+1}. \quad (\text{C.14})$$

This integral is given as a finite sum of incomplete Beta functions and Harmonic numbers

$$I_m^{n=0} = \frac{r^{2m-1}}{2^{m-1}} \sum_{i=0}^m \binom{m}{i} (-)^{i+1} 3^{m-i} \left(B(-1, 2(m-i), 0) + \log 2 + H_{2(m-i)-1} \right), \quad (\text{C.15})$$

here we list some of them from $m = 1 \dots 10$

$$\frac{I_m^{n=0}}{r^{2m-1}} = - \left\{ 6, 6, \frac{36}{5}, \frac{264}{35}, \frac{282}{35}, \frac{3222}{385}, \frac{43476}{5005}, \frac{44736}{5005}, \frac{780462}{85085}, \frac{1378158}{146965}, \dots \right\}. \quad (\text{C.16})$$

For the linear term in (C.13), we have to keep the lower limit of integration, but we can set $x = 0$, or equivalently $\varphi_0 = -1$, in the integrand, such that

$$I_m^{\geq 1} = 6\Delta m \int_{-r} dz (r+z)^{2m-1} = 6\Delta m \sum_{i=0}^{2m-1} \binom{2m-1}{i} r^{2m-1-i} \int_{-r} dz z^i \quad (\text{C.17})$$

$$= 6\Delta m r^{2m} \sum_{i=0}^{2m-1} \binom{2m-1}{i} \frac{(-1)^i}{i+1} = 6\Delta m r^{2m} \frac{1}{2m} \simeq 3r_0 \left(\frac{r_0}{\Delta}\right)^{2m-1}. \quad (\text{C.18})$$

D Quartic functional determinant

We discuss the calculation of the functional determinant of an (unstable) quartic potential, specified by $V = \lambda/4\phi^4$, where $\lambda < 0$. It can be computed either by use of Feynman diagrams or via the ζ /WKB formalism. The bounce is given by $\phi = \sqrt{8/(-\lambda)}R/(R^2 + \rho^2)$ and the ratio of multipoles R_l can be calculated using Gel'fand-Yaglom [15, 16, 71] with $R_l = l(l-1)/(l+2)/(l+3)$. To compute the determinant, we have to regulate the sum of $d_l \ln R_l$, such that we first subtract the divergent growth at large l and then renormalize it.

The Feynman diagrammatic approach [15, 16] specifies the subtraction of $\ln R_l$ in powers of $V^{(2)} - V_{\text{FV}}^{(2)}$ insertions. The renormalization in $D = 4$ is performed by calculating the Feynman diagrams with one and two insertions in momentum space, using the $\overline{\text{MS}}$ scheme. After performing the finite sum and adding back the renormalized determinant, one gets

$$-\frac{1}{2} \ln \left(\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = \frac{3}{\epsilon} - \frac{5}{4} \left(1 - 2 \ln \left(\frac{5}{6} \right) \right) + 3 \ln \left(\frac{\mu R}{2} \right) + 6\zeta'(-1), \quad (\text{D.1})$$

as stated e.g. in eq. (5.6) of [16] (here we translated to our convention $\epsilon = 4 - D$).

According to the ζ /WKB prescription, we use (6.12) with the UV integrals coming from integrating $V^{(2)} = 3\lambda\phi^2$ using the quartic bounce given above, such that

$$I_1 = -12, \quad I_2 = -48, \quad \tilde{I}_2 = 48 \left(\frac{1}{\epsilon} + 1 + \gamma_E + \ln \left(\frac{\mu R}{2} \right) \right). \quad (\text{D.2})$$

The finite sum is performed by starting from $l = 2$ (or $\nu = 3$)

$$S_{\text{fin}} = \sum_{\nu=3}^{\infty} \nu^2 \left(\ln R_l + \frac{6}{\nu} + \frac{6}{\nu^3} \right) = -\frac{37}{2} + 9 \ln 2 + 5 \ln 3 - 12\zeta'(-1). \quad (\text{D.3})$$

To complete the calculation, we have to add the $l = 0$ and $l = 1$ terms. This includes the $\ln R_l$ with zeroes removed, given by $R'_0 = -1/5$ and $R'_1 = 1/10$ (see e.g. (4.37) of [16]), as well as the asymptotic subtractions, such that

$$-\frac{1}{2} \ln \left(\frac{\det' \mathcal{O}}{\det \mathcal{O}_{\text{FV}}} \right) = -\frac{1}{2} \left(\ln |R'_0| + 4 \ln R'_1 + \sum_{\nu=1}^2 \left(6\nu + \frac{6}{\nu} \right) + S_{\text{fin}} - \frac{1}{8} \tilde{I}_2 \right). \quad (\text{D.4})$$

Although the intermediate steps in these two approaches differ significantly, the final result precisely agrees with (D.1). This further confirms that a single integration over ρ in (6.12) is equivalent to using Feynman diagrams in the $\overline{\text{MS}}$ scheme.

E Finite determinants in generic dimensions

The procedure that we discussed for the specific $D = 3, 4$ above, deriving the asymptotics, subtracting the infinities and applying the EuMac approximation, can be generalized for generic D . We first focus on the asymptotic behavior of the determinant that needs to be subtracted to regulate the sum, then on the sums in generic D . We treat odd and even dimensions in separate sub-sections, starting with the easier odd ones. Some parts of the summation are valid for any D , so we will keep the discussion general for as long as possible and reduce to a specific choice when this becomes unavoidable.

Asymptotics of determinants. We will derive the asymptotic behavior of the determinant for high multipoles in inverse powers of ν . To compute the finite sum in (6.4), we need to obtain $\ln R_\nu^a$ that regulates the infinities with a minimal number of ν^{-j} terms. The order depends on the dimension, because the degeneracy factor at high ν scales as $d_\nu \propto \nu^{D-2}$. We thus have to expand to high enough powers of ν^{-j} , to get to the lowest one, which is $d_\nu \ln R_\nu \sim \nu^{-1}$. This defines the notion of minimal subtraction.

To keep track of the calculation, we will separate the asymptotic into two pieces: the logarithmic one $\ln R_{\nu 0}^a$, and the U factor, such that $\ln R_\nu^a = \ln R_{\nu 0}^a + U^a$. In what follows we will be evaluating the integrals by performing a large k_ν expansion of $\ln R_\nu$, integrating the pieces of the sums and summing them back up. For this purpose, it is sensible to first expand in large k_ν , which is by definition larger than one, thus the sums with k_ν^{-j} converge, and then further expand in powers of ν^{-j} or better y^{-j} . The last step brings us to the well defined $1/\nu$ expansions, which we get during the process of renormalization.

Let us start with the logarithmic piece and rewrite the log of (5.36) as:

$$\ln R_{\nu 0} = -2 \sum_{j=1}^{\infty} \frac{1 + 2^{-2j+1}}{2j-1} k_\nu^{-2j+1}, \quad (\text{E.1})$$

where the powers of k_ν are going down by factors of 2. Further expanding the powers of k_ν in various roots of large $y \gg 1$, using

$$k_\nu^{-2j+1} = y^{-2j+1} (1 + y^{-2})^{-j+\frac{1}{2}} = y^{-2j+1} \sum_{p=0}^{\infty} \binom{-j+\frac{1}{2}}{p} y^{-2p}, \quad (\text{E.2})$$

we then express it as a double infinite sum

$$\ln R_{\nu 0} = -2 \sum_{j=1}^{\infty} \frac{1 + 2^{-2j+1}}{2j-1} \sum_{p=0}^{\infty} \binom{-j+\frac{1}{2}}{p} y^{-2(j+p)+1}. \quad (\text{E.3})$$

As discussed above, the powers in the asymptotic functions have to stop when we reach ν^{-1} . To derive the upper bound in the sum, note that the degeneracy factor goes as $d_\nu \propto y^{D-2}$ in the UV and we need to remove all the negative y powers, including the highest one with y^{-1} . Comparing the powers leads to the maximal one at $j = m$, for both odd and even dimensions, where $D = 2m + 1$ for the odd ones and $D = 2m$ for the even, thus

$$\ln R_{\nu 0}^a = -2 \sum_{j=1}^m \frac{1 + 2^{-2j+1}}{2j-1} \sum_{p=0}^{m-j} \binom{-j+\frac{1}{2}}{p} y^{-2(j+p)+1}. \quad (\text{E.4})$$

We proceed along similar lines to obtain the asymptotics of the U factor

$$U = -3r_0 \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} (-)^j k_{\nu}^{-2j+1} = -3r_0 \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} (-)^j y^{-2j+1} \sum_{p=0}^{\infty} \binom{-j + \frac{1}{2}}{p} y^{-2p}, \quad (\text{E.5})$$

and truncating the series at the appropriate maximal power

$$U^a = -3r_0 \sum_{j=1}^m \binom{\frac{1}{2}}{j} (-)^j y^{-2j+1} \sum_{p=0}^{m-j} \binom{-j + \frac{1}{2}}{p} y^{-2p}. \quad (\text{E.6})$$

With these minimal subtractions in powers of y , the sum in (6.4) becomes convergent and can be evaluated using the EuMac approximation in (6.5). Plugging the appropriate boundaries in (E.4) and (E.6), we get precisely the asymptotic functions $\ln R_{\nu}^a$, which were calculated in $D = 3, 4$ via the integrals $I_{1,2}$ in (6.8) and (6.13).

From the generic behavior of the asymptotics, we already see that the odd and even D will behave differently. Because the degeneracy factor grows as y^{D-2} and the UV expansion of $\ln R_{\nu_0} + U$ behave as y^{-2j+1} , we get y^{-1} terms only for even D . After the summation in ν , these become $\ln y$ divergent terms. Conversely, there are no log divergencies in odd dimensions, which makes them generally easier to handle.

Generic odd D . Let us generalize the summation procedure to generic D , beginning with an integral part of the U -factor in the EuMac approximation

$$\Sigma_D^{\int U} = \int_{\nu_0}^{\infty} d\nu d_{\nu} (U - U^a). \quad (\text{E.7})$$

We found out that in $D = 3$ the EuMac integral was insensitive to the lower bound/multipoles. We shall argue that this is true for any odd D . The dominant contribution to the integral comes from $\nu \sim 1/\Delta$ terms, where the degeneracy factor is approximated as

$$d_{\nu} = \frac{2^{4-D} \nu^{\frac{D-3}{2}}}{(D-2)!} \prod_{j=1}^{\frac{D-3}{2}} \left((2\nu)^2 - (2j-1)^2 \right) \simeq \frac{2}{(D-2)!} \nu^{D-2}. \quad (\text{E.8})$$

In the $\nu \sim 1/\Delta$ range, the $2j-1$ terms above are Δ^2 suppressed with respect to ν^2 . Using this approximation of d_{ν} , we get the following EuMac integral

$$\Sigma_D^{\int U} = \frac{2}{(D-2)!} \int_{\nu_0}^{\infty} d\nu \nu^{D-2} (U - U^a) \quad (\text{E.9})$$

$$= \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta} \right)^{D-1} \int_{\sqrt{1+y_0^2}}^{\infty} dk_{\nu} k_{\nu} (k_{\nu}^2 - 1)^{\frac{D-3}{2}} (U - U^a), \quad (\text{E.10})$$

where we will use the two expansions for U in (E.5) and the asymptotic U^a in (E.6).

After recasting the integrands U, U^a into sums over j , it becomes clear that we can divide (E.9) into a lower part with $j \in [1, m]$ that contains the subtraction of U^a and the

high region $j \in [m + 1, \infty]$, which remains unsubtracted. The high region is given by

$$\Sigma_D^{\int U \text{ uns}} = -\frac{6r_0}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \sum_{j=m+1}^{\infty} (-)^j \binom{\frac{1}{2}}{j} \int_{\sqrt{1+y_0^2}}^{\infty} dk_{\nu} k_{\nu}^{-2j+2} (k_{\nu}^2 - 1)^{\frac{D-3}{2}} \quad (\text{E.11})$$

$$= \frac{6r_0}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(D-2)! \left(m - \frac{D}{2}\right)!}{2^D \left(\frac{D}{2}\right)! m!}. \quad (\text{E.12})$$

This result is already valid for any D and does not depend on the lower boundary y_0 . The subtracted part contains a double sum coming from the lower part of the sum over j with $j \in [1, m]$. When we take the difference $U - U_a$, the lower part of the sum over p is removed, and we are left with the task of having to evaluate

$$\begin{aligned} \Sigma_D^{\int U \text{ sub}} &= -\frac{6r_0}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \sum_{j=1}^m \binom{\frac{1}{2}}{j} (-)^j \sum_{p=m-j+1}^{\infty} \binom{-j + \frac{1}{2}}{p} \\ &\quad \times \int_{\sqrt{1+y_0^2}}^{\infty} dk_{\nu} k_{\nu} (k_{\nu}^2 - 1)^{\frac{D}{2} - j - p - 1}. \end{aligned} \quad (\text{E.13})$$

Let us focus first on the integral and perform the sum over p

$$\sum_{p=m-j+1}^{\infty} \binom{-j + \frac{1}{2}}{p} \int_{\sqrt{1+y_0^2}}^{\infty} dk_{\nu} k_{\nu} (k_{\nu}^2 - 1)^{\frac{D}{2} - j - p - 1} \quad (\text{E.14})$$

$$= \sum_{p=m-j+1}^{\infty} \binom{-j + \frac{1}{2}}{p} \frac{y_0^{D-2(j+p)}}{D-2(j+p)} \quad (\text{E.15})$$

$$\begin{aligned} &= \binom{\frac{1}{2} - j}{-j + m + 1} \frac{-y_0^{D-2m-2}}{D-2m-2} \\ &\quad \times {}_3F_2 \left(1, m + \frac{1}{2}, m + 1 - \frac{D}{2}; m + 2 - \frac{D}{2}, m + 2 - j; -\frac{1}{y_0^2} \right). \end{aligned} \quad (\text{E.16})$$

The result is also valid for any D , however, the dependence on y_0 differs from odd to even. Therefore, we need to specify that we are working in odd dimensions, so we set $D = 2m + 1$ and expand for small y_0 to end up with a simple coefficient

$$(\text{E.16}) \xrightarrow[y_0 \rightarrow 0]{D=2m+1} \frac{(-)^m}{m \binom{m-j+\frac{1}{2}}{m}}. \quad (\text{E.17})$$

Now we come back to evaluate the remaining outer sum over j , which gives

$$\Sigma_{2m+1}^{\int U \text{ sub}} = -\frac{6r_0}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \sum_{j=1}^m \binom{\frac{1}{2}}{j} \frac{(-)^{j+m}}{m \binom{m-j+\frac{1}{2}}{m}} \quad (\text{E.18})$$

$$= -\frac{6r_0}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(-)^m}{m \binom{m+\frac{1}{2}}{\frac{1}{2}}} \left((m+1) \binom{\frac{1}{2}}{m+1} - \frac{1}{2} \right). \quad (\text{E.19})$$

Combining the subtracted sum with the unsubtracted part in (E.11), we get an elegant expression for the U part in odd dimensions

$$\Sigma_{2m+1}^{\int U} = \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(-)^m 3r_0}{2m \binom{m+\frac{1}{2}}{\frac{1}{2}}}. \quad (\text{E.20})$$

Now we understand why the integral is dominated by $\nu \sim 1/\Delta$: the lowest multipoles come with higher orders, expanded in y^2 and would be y_0^2 or Δ^2 suppressed. This justifies the approximation in (E.8). If we were to keep the additional constant terms next to ν^2 in (E.8), they would be Δ^2 suppressed once we reach the intermediate multipoles with $\nu \sim 1/\Delta$.

Let us show that this is already the final answer and discuss why the additional EuMac terms in (6.5) do not contribute. Inserting the lower boundary with $\nu_0, d_\nu \sim \mathcal{O}(1)$ and $y_0 \sim \mathcal{O}(\Delta)$, we get the following orders in Δ

$$\sigma_{2m+1}(\nu_0) \propto \mathcal{O}(1) \left(\sqrt{\Delta^2 + 1} - \Delta - \sum_{j=1}^m \binom{\frac{1}{2}}{j} \Delta^{-2j+1} \right). \quad (\text{E.21})$$

Remember that the leading order result in (E.7) goes as Δ^{-D+1} or equivalently Δ^{-2m} . On the other hand, the term with the most negative power of Δ in (E.21) is still Δ^{-2m+1} , which is suppressed compared to the leading one. The same argument goes through for the higher derivatives of $\sigma_{2m+1}(\nu_0)$: when we multiply the last term with the degeneracy factor, we will at most get a term $\propto \nu$. Taking higher derivatives of ν just makes it less important and therefore the Bernoulli terms are negligible.

Let us proceed with the evaluation of the $\ln R_{\nu_0}$ term using the same line of reasoning. Using the expansions in (E.1), (E.3) and (E.4), we again split the sum into two pieces,

$$\Sigma_{2m+1}^{\int \ln} = \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta} \right)^{D-1} \int_{y_0}^{\infty} dy y^{D-2} (\ln R_{\nu_0} - \ln R_{\nu_0}^a) \quad (\text{E.22})$$

$$= \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta} \right)^{D-1} \int_{y_0}^{\infty} dy y^{D-2} \left(\sum_{j=1}^m \dots + \sum_{j=m+1}^{\infty} \dots \right), \quad (\text{E.23})$$

where the first part in the parenthesis is the subtracted one and the second part is the unsubtracted one, independent of $\ln R_{\nu_0}^a$. The latter part is again simpler, valid for any D and independent of y_0

$$\begin{aligned} \Sigma_D^{\int \ln \text{ uns}} &= \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta} \right)^{D-1} \frac{1}{(D-1)(2m-1) \binom{m-\frac{D}{2}}{m-\frac{3}{2}}} \\ &\times \left(1 + \frac{D-1}{4^{m+1} (m+\frac{1}{2})} {}_2F_1 \left(1, m+1 - \frac{D}{2}; m + \frac{3}{2}; \frac{1}{4} \right) \right). \end{aligned} \quad (\text{E.24})$$

The subtracted piece is slightly more involved. Taking the difference of $\ln R_{\nu_0}$ and $\ln R_{\nu_0}^a$ in the subtracted region, where $j \in [1, m]$, gives us

$$\Sigma_D^{\int \ln \text{ sub}} = -\frac{4}{(D-2)!} \left(\frac{r_0}{\Delta} \right)^{D-1} \sum_{j=1}^m \frac{1+2^{-2j+1}}{2j-1} \sum_{p=m-j+1}^{\infty} \binom{-j+\frac{1}{2}}{p} \quad (\text{E.25})$$

$$\times \int_{\sqrt{1+y_0^2}}^{\infty} dk_\nu k_\nu (k_\nu^2 - 1)^{\frac{D}{2}-j-p-1}, \quad (\text{E.26})$$

which is very similar to the subtracted part of the $U - U^a$ integration in (E.13). Indeed, the integral and the sum over p are exactly the same as in (E.14) and once again we have

to have to separate the odd ones by setting $D = 2m + 1$ and expanding (E.14) for small y_0 to get (E.17). What remains is a slightly different sum over the j index

$$\Sigma_{2m+1}^{\int \ln \text{ sub}} = -\frac{4}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(-)^m}{m} \sum_{j=1}^m \frac{1+2^{-2j+1}}{(2j-1)\binom{m-j+\frac{1}{2}}{m}} \quad (\text{E.27})$$

$$= \frac{4}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(-)^{m+1}}{2m^2} \left(\frac{1+m {}_2F_1\left(1, m-\frac{1}{2}; \frac{3}{2}; \frac{1}{4}\right)}{\binom{m-\frac{1}{2}}{m}} - \frac{1+4^{-m} \frac{m}{2m+1} {}_2F_1\left(\frac{1}{2}, 1; m+\frac{3}{2}; \frac{1}{4}\right)}{\binom{-\frac{1}{2}}{m}} \right). \quad (\text{E.28})$$

To wrap up the $\ln R_\nu$ integration, we combine the subtracted part with the unsubtracted piece from (E.24), where we set $D = 2m + 1$, and end up with the following result

$$\Sigma_{2m+1}^{\int \ln} = \Sigma_{2m+1}^{\int \ln \text{ uns}} + \Sigma_{2m+1}^{\int \ln \text{ sub}} \quad (\text{E.29})$$

$$= \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(-)^{m+1}}{m^2 \binom{m-\frac{1}{2}}{m}} \left(1 + m {}_2F_1\left(1, \frac{1}{2} - m; \frac{3}{2}; \frac{1}{4}\right)\right). \quad (\text{E.30})$$

Before moving on to even dimensions, let us discuss the EuMac corrections for the log terms. The reasoning here goes along the same lines as for the U -factor above. The lowest power of Δ comes from taking the ultimate term with $p = m - j$ in the asymptotic $\ln R_{\nu_0}^a$, given in (E.4), such that

$$\ln R_{\nu_0}^a \ni \frac{-2}{y^{2m-1}} \sum_{j=1}^m \frac{1+2^{-2j+1}}{2j-1} \binom{-j+\frac{1}{2}}{-j+m} \quad (\text{E.31})$$

$$= -\frac{2}{y^{2m-1}} \frac{m}{m-\frac{1}{2}} \binom{\frac{1}{2}}{m} \left(1 + \left(m-\frac{1}{2}\right) {}_2F_1\left(1, 1-m; \frac{3}{2}; \frac{1}{4}\right)\right). \quad (\text{E.32})$$

Now we plug in the low values of ν_0 and perform the same Δ counting as in (E.21), to realize again that the term with the most negative power of Δ goes as Δ^{-2m+1} . This is suppressed compared to the leading one with Δ^{-2m} and the higher derivatives even more so, thus all the EuMac corrections are irrelevant here.

To summarize the situation for the odd D , we combine the integral of the log given by $\Sigma_{2m+1}^{\int \ln}$ in (E.30) with the integral of the U -factor given by $\Sigma_{2m+1}^{\int U}$ from (E.20). Perhaps surprisingly, the final expression comes out to be remarkably simple

$$\Sigma_{2m+1} = \Sigma_{2m+1}^{\int U} + \Sigma_{2m+1}^{\int \ln} \quad (\text{E.33})$$

$$= \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(-)^{m+1}}{m \binom{m-\frac{1}{2}}{m}} \left(\frac{1}{m} + {}_2F_1\left(1, \frac{1}{2} - m; \frac{3}{2}; \frac{1}{4}\right) - \frac{3}{2}r_0\right). \quad (\text{E.34})$$

Evaluating the hypergeometric function gives us the following numbers for a couple of low odd D s

$$\Sigma_D = \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \begin{cases} \frac{5}{6} + \frac{3}{8} \ln 3, & D = 3, \\ -\frac{37}{240} - \frac{9}{64} \ln 3, & D = 5, \\ -\frac{3}{224} + \frac{9}{128} \ln 3, & D = 7. \end{cases} \quad (\text{E.35})$$

Generic even D . The main difference for generic even dimensions is the appearance of a log term on the lower boundary and thus the importance of low multipoles in $\ln R_\nu^a$. This translates into the fact that the EuMac sum over the Bernoulli numbers diverges if we apply the EuMac approximation naïvely on the entire interval from $\nu_0 = \mathcal{O}(1)$ to ∞ , as we explained in section 6.

Let us discuss how these issues appear and how to resolve them in general $D = 2m$. In even dimensions, the asymptotic parts will always contain a term that goes as $1/\nu$. When we take higher derivatives in the last term of (6.5) to sum over the Bernoulli numbers, the $1/\nu$ will turn into a term that goes as $\nu^{-(p>1)}$. If we plug in the lower boundary with $\nu = D/2 - 1$, the sum over the Bernoulli numbers will eventually diverge.

A way out of this impasse is the same as in $D = 4$: split up the sum over ν into two parts, the first coming from $\nu_0 = \mathcal{O}(1)$ up to $\nu_1 \lesssim r_0/\Delta$ and the other from ν_1 to ∞ . As we shall see, the precise position of this separation is irrelevant, as long as $\nu_1 \gg 1$. Let us now consider the low part of the sum in more concrete detail. Here we need to extract the leading powers of Δ from the asymptotic terms $\ln R_{\nu_0}^a$ and U^a , which means we have to set $p = m - j$ and perform the sum over j , just as in (E.32). This gives us the most relevant low multipole terms, which can be written as

$$\sigma_{2m} \simeq \frac{d_\nu}{y^{2m-1}} C_m, \quad (\text{E.36})$$

where the ν -independent coefficient C_m is given by

$$C_m = \frac{2m}{m - \frac{1}{2}} \binom{\frac{1}{2}}{m} \left(1 + \left(m - \frac{1}{2} \right) {}_2F_1 \left(1, 1 - m; \frac{3}{2}; \frac{1}{4} \right) \right) + 3r_0 \binom{\frac{1}{2}}{m}. \quad (\text{E.37})$$

The sum over low multipoles gives

$$\Sigma_{2m}^{\text{low}U} = \sum_{\nu=m-1}^{\nu_1} \sigma_{2m} = \frac{2C_m}{(D-2)!} \left(\frac{r_0}{\Delta} \right)^{D-1} \prod_{j=1}^{2m-3} \frac{\nu - m + 1 + j}{\nu^{2m-2}}. \quad (\text{E.38})$$

We can further rework it into a product of pairs and ultimately into a sum

$$\prod_{j=1}^{2m-3} \frac{\nu - m + 1 + j}{\nu^{2m-2}} = \frac{1}{\nu^{2m-3}} \prod_{j=1}^{m-2} (\nu^2 - j^2) = \frac{1}{\nu} \sum_{j=0}^{m-2} \left\{ \begin{matrix} m-2 \\ j \end{matrix} \right\} (-)^j \nu^{-2j}, \quad (\text{E.39})$$

where the curly binomial stands for the multiply nested sum

$$\left\{ \begin{matrix} m-2 \\ j \end{matrix} \right\} \equiv \sum_{i_1=1}^{m-2} \sum_{i_2=i_1+1}^{m-2} \dots \sum_{i_j=i_{j-1}+1}^{m-2} (i_1 \dots i_j)^2. \quad (\text{E.40})$$

This is just a convenient way to write the numerical coefficients, which are independent of ν . The finite sums of powers of $1/\nu^{2p+1}$, on the other hand, are given by

$$\sum_{\nu=m-1}^{\nu_1} \frac{1}{\nu^n} = \begin{cases} \ln \nu_1 - \psi(m-1), & n = 1, \\ -\frac{1}{(n-1)!} \psi^{(n-1)}(m-1), & n \geq 1, \end{cases} \quad (\text{E.41})$$

where $\psi^{(n)}$ is the n -th derivative of the digamma function. We thus end up with

$$\Sigma_{2m}^{\text{low}} = \frac{2C_m}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \left(\ln \nu_1 - \psi(m-1) + \sum_{j=1}^{m-2} \binom{m-2}{j} \frac{(-)^{j+1}}{(2j)!} \psi^{(2j)}(m-1) \right). \quad (\text{E.42})$$

With the low- ν summation at hand, we can proceed to the second part of the sum over high multipoles $\nu \in [\nu_1, \infty]$ via the usual EuMac approximation in (6.5). Because we delayed the lower end of the sum to $\nu_1 \gg 1$, we do not have to worry about EuMac corrections in (6.5). All of the terms from the lower boundary at ν_1 are Δ suppressed and we can drop them safely.

Most of the work needed for the high-multipole sum has already been done in the previous section, so we can easily handle the EuMac integrals by specifying the even $D = 2m$ in the generic terms derived above. Let us begin with the simpler U -factor and augment the unsubtracted part in (E.11) to even dimensions, such that

$$\Sigma_{2m}^{\int U \text{ uns}} = \frac{6r_0}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{(2m-2)!}{4^m m!^2}. \quad (\text{E.43})$$

Moving on to the subtracted part, we take the expression from (E.16), specify $D = 2m$, shift the lower bound from $y_0 \rightarrow y_1$ and expand for smallish y_1

$$(\text{E.16}) \xrightarrow[y_1 \ll 1]{D=2m} \frac{j-m-1}{2m-1} \binom{\frac{1}{2}-j}{m+1-j} \left(H_{m-j} - H_{m-\frac{3}{2}} + 2 \ln y_1 \right), \quad (\text{E.44})$$

which leaves us with the following finite sum

$$\begin{aligned} \Sigma_{2m}^{\int U \text{ sub}} &= \frac{6r_0}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \sum_{j=1}^m \binom{\frac{1}{2}}{j} \frac{(-)^j (m+1-j)}{2m-1} \\ &\times \binom{\frac{1}{2}-j}{m+1-j} \left(H_{m-j} - H_{m-\frac{3}{2}} + 2 \ln y_1 \right). \end{aligned} \quad (\text{E.45})$$

While we do not get a closed-form expression when summing the H_{m-j} term, the two remaining terms, which are independent of j , nicely sum into

$$\Sigma_{2m}^{\int U \text{ sub}} \ni -\frac{6r_0}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \binom{\frac{1}{2}}{m} \left(\frac{1}{2} H_{m-\frac{3}{2}} - \ln y_1 \right). \quad (\text{E.46})$$

This again shows that when we combine with the low part of the sum in (E.42), the ν_1 parameter cancels out (i.e. the part that comes from the U -factor of C_m), so that the final result does not depend on the precise position where we split the sum.

The two final remaining pieces come from the integration of the log terms. The unsubtracted one is

$$\Sigma_{2m}^{\int \ln \text{ uns}} = -\frac{1}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \left(\frac{1}{\left(m-\frac{1}{2}\right)^2} + \frac{{}_2F_1\left(1, 1; m+\frac{3}{2}; \frac{1}{4}\right)}{4^{m+\frac{1}{2}} \left(m+\frac{3}{2}\right) \left(m+\frac{1}{2}\right)} \right). \quad (\text{E.47})$$

The subtracted part is very similar to the U -factor above, where the sum over p was given in (E.44). Here the coefficient enters into the sum defined by $\ln R_{\nu_0}^a$ in (E.4)

$$\begin{aligned} \Sigma_{2m}^{\int \ln \text{ sub}} &= \frac{4}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \sum_{j=1}^m \frac{1+2^{-2j+1}}{2j-1} \frac{m+1-j}{2m-1} \\ &\times \binom{\frac{1}{2}-j}{m+1-j} \left(H_{m-j} - H_{m-\frac{3}{2}} + 2 \ln y_1\right). \end{aligned} \quad (\text{E.48})$$

Just as in the case of the U -factor, we do not find a closed-form expression when summing over the H_{m-j} term, while the last two terms sum into

$$\begin{aligned} \Sigma_{2m}^{\int \ln \text{ sub}} \ni &-\frac{4}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \frac{m}{m-\frac{1}{2}} \binom{\frac{1}{2}}{m} \left(\frac{1}{2} H_{m-\frac{3}{2}} - \ln y_1\right) \\ &\times \left(1 + \binom{m-\frac{1}{2}}{2} {}_2F_1\left(1, 1-m; \frac{3}{2}; \frac{1}{4}\right)\right). \end{aligned} \quad (\text{E.49})$$

Let us compare this to the first part of the C_m coefficient in (E.37) that multiplies the $\ln \nu_1$ term in (E.42). Again, we see how the arbitrary splitting point defined by ν_1 cancels out and we are left with an unambiguous result for any even D .

To summarize this arduous section, the final result for the finite sum is obtained by collecting all of the five terms above into

$$\Sigma_{2m} = \Sigma_{2m}^{\text{low}} + \Sigma_{2m}^{\int} = \Sigma_{2m}^{\text{low}} + \Sigma_{2m}^{\int U \text{ uns}} + \Sigma_{2m}^{\int U \text{ sub}} + \Sigma_{2m}^{\int \ln \text{ uns}} + \Sigma_{2m}^{\int \ln \text{ sub}}. \quad (\text{E.50})$$

In this final result, the $\ln \nu_1$ terms partially cancel with $\ln y_1$ ones, leaving us with a residual dependence on $\ln \Delta/r_0$. Combining the closed-form expressions obtained above does not lead to a simple expression, as the one we found for odd D s. Still, plugging in the integers leads to rather compact expressions for the first couple of even D 's:

$$\Sigma_D^{\text{low}} = \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \begin{cases} \frac{5}{2} (\gamma_E + \ln \nu_1), & D = 2, \\ -\frac{3}{8} (\gamma_E + \ln \nu_1), & D = 4, \\ \frac{1}{10} (\gamma_E - \zeta(3) + \ln \nu_1), & D = 6, \end{cases} \quad (\text{E.51})$$

and

$$\Sigma_D^{\int} = \frac{2}{(D-2)!} \left(\frac{r_0}{\Delta}\right)^{D-1} \begin{cases} \frac{\pi}{2\sqrt{3}} - \frac{11}{4} - \frac{5}{2} \ln 2y_1, & D = 2, \\ -\frac{\pi}{8\sqrt{3}} + \frac{3}{32} + \frac{3}{8} \ln 2y_1, & D = 4, \\ \frac{3\sqrt{3}\pi}{160} - \frac{211}{4800} - \frac{1}{10} \ln 2y_1, & D = 6. \end{cases} \quad (\text{E.52})$$

Together with the result in generic odd D s, these finite sums provide us with the finite log of the determinant in any dimension. To complete the calculation of the decay rate one has to perform the renormalization procedure, as described in section 6 for even D s. This might be a useful exercise if there is enough physical motivation, but is beyond the scope of this paper. We point out that what enters in this calculation will be integrals with a certain number of insertions of $V^{(2)}$ and the accompanying powers of ρ to compensate for the dimensions. These types of integrals, as shown in (C.1), are considered and evaluated in closed form in the appendix C.

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References

- [1] J. Langer, *Theory of the condensation point*, *Annals Phys.* **41** (1967) 108 [[281](#) (2000) 941].
- [2] I.Y. Kobzarev, L.B. Okun and M.B. Voloshin, *Bubbles in Metastable Vacuum*, *Sov. J. Nucl. Phys.* **20** (1975) 644 [[INSPIRE](#)].
- [3] S.R. Coleman, *The Fate of the False Vacuum. 1. Semiclassical Theory*, *Phys. Rev. D* **15** (1977) 2929 [*Erratum ibid.* **16** (1977) 1248] [[INSPIRE](#)].
- [4] S.R. Coleman, *The Uses of Instantons*, *Subnucl. Ser.* **15** (1979) 805 [[INSPIRE](#)].
- [5] S. Coleman, *Aspects of Symmetry: Selected Erice Lectures*, Cambridge University Press, Cambridge, U.K. (1985), [[DOI](#)] [[INSPIRE](#)].
- [6] C.G. Callan Jr. and S.R. Coleman, *The Fate of the False Vacuum. 2. First Quantum Corrections*, *Phys. Rev. D* **16** (1977) 1762 [[INSPIRE](#)].
- [7] S.R. Coleman and F. De Luccia, *Gravitational Effects on and of Vacuum Decay*, *Phys. Rev. D* **21** (1980) 3305 [[INSPIRE](#)].
- [8] C.L. Hammer, J.E. Shrauner and B. DeFacio, *Alternate Derivation of Vacuum Tunneling*, *Phys. Rev. D* **19** (1979) 667 [[INSPIRE](#)].
- [9] A. Andreassen, D. Farhi, W. Frost and M.D. Schwartz, *Direct Approach to Quantum Tunneling*, *Phys. Rev. Lett.* **117** (2016) 231601 [[arXiv:1602.01102](#)] [[INSPIRE](#)].
- [10] A. Andreassen, D. Farhi, W. Frost and M.D. Schwartz, *Precision decay rate calculations in quantum field theory*, *Phys. Rev. D* **95** (2017) 085011 [[arXiv:1604.06090](#)] [[INSPIRE](#)].
- [11] S. Weinberg, *Mass of the Higgs Boson*, *Phys. Rev. Lett.* **36** (1976) 294 [[INSPIRE](#)].
- [12] A.D. Linde, *Dynamical Symmetry Restoration and Constraints on Masses and Coupling Constants in Gauge Theories*, *JETP Lett.* **23** (1976) 64 [*Pisma Zh. Eksp. Teor. Fiz.* **23** (1976) 73] [[INSPIRE](#)].
- [13] P.H. Frampton, *Consequences of Vacuum Instability in Quantum Field Theory*, *Phys. Rev. D* **15** (1977) 2922 [[INSPIRE](#)].
- [14] P.H. Frampton, *Vacuum Instability and Higgs Scalar Mass*, *Phys. Rev. Lett.* **37** (1976) 1378 [*Erratum ibid.* **37** (1976) 1716] [[INSPIRE](#)].
- [15] G. Isidori, G. Ridolfi and A. Strumia, *On the metastability of the standard model vacuum*, *Nucl. Phys. B* **609** (2001) 387 [[hep-ph/0104016](#)] [[INSPIRE](#)].
- [16] A. Andreassen, W. Frost and M.D. Schwartz, *Scale Invariant Instantons and the Complete Lifetime of the Standard Model*, *Phys. Rev. D* **97** (2018) 056006 [[arXiv:1707.08124](#)] [[INSPIRE](#)].
- [17] M. Quirós, *Finite temperature field theory and phase transitions*, [hep-ph/9901312](#) [[INSPIRE](#)].
- [18] J.I. Kapusta and C. Gale, *Finite-temperature field theory: Principles and applications*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (2011), [[DOI](#)] [[INSPIRE](#)].

- [19] M. Laine and A. Vuorinen, *Basics of Thermal Field Theory*, vol. 925, Springer (2016), [DOI] [arXiv:1701.01554] [INSPIRE].
- [20] A.D. Linde, *Fate of the False Vacuum at Finite Temperature: Theory and Applications*, *Phys. Lett. B* **100** (1981) 37 [INSPIRE].
- [21] A.D. Linde, *Decay of the False Vacuum at Finite Temperature*, *Nucl. Phys. B* **216** (1983) 421 [Erratum *ibid.* **223** (1983) 544] [INSPIRE].
- [22] I. Affleck, *Quantum Statistical Metastability*, *Phys. Rev. Lett.* **46** (1981) 388 [INSPIRE].
- [23] D. Croon, O. Gould, P. Schicho, T.V.I. Tenkanen and G. White, *Theoretical uncertainties for cosmological first-order phase transitions*, *JHEP* **04** (2021) 055 [arXiv:2009.10080] [INSPIRE].
- [24] A. Ekstedt, *Higher-order corrections to the bubble-nucleation rate at finite temperature*, *Eur. Phys. J. C* **82** (2022) 173 [arXiv:2104.11804] [INSPIRE].
- [25] O. Gould and J. Hirvonen, *Effective field theory approach to thermal bubble nucleation*, *Phys. Rev. D* **104** (2021) 096015 [arXiv:2108.04377] [INSPIRE].
- [26] J. Löfgren, M.J. Ramsey-Musolf, P. Schicho and T.V.I. Tenkanen, *Nucleation at finite temperature: a gauge-invariant, perturbative framework*, arXiv:2112.05472 [INSPIRE].
- [27] J. Hirvonen, J. Löfgren, M.J. Ramsey-Musolf, P. Schicho and T.V.I. Tenkanen, *Computing the gauge-invariant bubble nucleation rate in finite temperature effective field theory*, arXiv:2112.08912 [INSPIRE].
- [28] E. Witten, *Cosmic Separation of Phases*, *Phys. Rev. D* **30** (1984) 272 [INSPIRE].
- [29] C.J. Hogan, *Gravitational radiation from cosmological phase transitions*, *Mon. Not. Roy. Astron. Soc.* **218** (1986) 629 [INSPIRE].
- [30] A. Kosowsky, M.S. Turner and R. Watkins, *Gravitational waves from first order cosmological phase transitions*, *Phys. Rev. Lett.* **69** (1992) 2026 [INSPIRE].
- [31] A. Kosowsky, M.S. Turner and R. Watkins, *Gravitational radiation from colliding vacuum bubbles*, *Phys. Rev. D* **45** (1992) 4514 [INSPIRE].
- [32] C. Grojean and G. Servant, *Gravitational Waves from Phase Transitions at the Electroweak Scale and Beyond*, *Phys. Rev. D* **75** (2007) 043507 [hep-ph/0607107] [INSPIRE].
- [33] M. Hindmarsh, S.J. Huber, K. Rummukainen and D.J. Weir, *Gravitational waves from the sound of a first order phase transition*, *Phys. Rev. Lett.* **112** (2014) 041301 [arXiv:1304.2433] [INSPIRE].
- [34] M. Hindmarsh, S.J. Huber, K. Rummukainen and D.J. Weir, *Numerical simulations of acoustically generated gravitational waves at a first order phase transition*, *Phys. Rev. D* **92** (2015) 123009 [arXiv:1504.03291] [INSPIRE].
- [35] M. Hindmarsh, S.J. Huber, K. Rummukainen and D.J. Weir, *Shape of the acoustic gravitational wave power spectrum from a first order phase transition*, *Phys. Rev. D* **96** (2017) 103520 [Erratum *ibid.* **101** (2020) 089902] [arXiv:1704.05871] [INSPIRE].
- [36] D. Cutting, M. Hindmarsh and D.J. Weir, *Gravitational waves from vacuum first-order phase transitions: from the envelope to the lattice*, *Phys. Rev. D* **97** (2018) 123513 [arXiv:1802.05712] [INSPIRE].

- [37] D. Cutting, M. Hindmarsh and D.J. Weir, *Vorticity, kinetic energy, and suppressed gravitational wave production in strong first order phase transitions*, *Phys. Rev. Lett.* **125** (2020) 021302 [[arXiv:1906.00480](#)] [[INSPIRE](#)].
- [38] LIGO SCIENTIFIC collaboration, *Advanced LIGO*, *Class. Quant. Grav.* **32** (2015) 074001 [[arXiv:1411.4547](#)] [[INSPIRE](#)].
- [39] VIRGO collaboration, *Advanced Virgo: a second-generation interferometric gravitational wave detector*, *Class. Quant. Grav.* **32** (2015) 024001 [[arXiv:1408.3978](#)] [[INSPIRE](#)].
- [40] LISA collaboration, *Laser Interferometer Space Antenna*, [arXiv:1702.00786](#) [[INSPIRE](#)].
- [41] C. Caprini et al., *Science with the space-based interferometer eLISA. II: Gravitational waves from cosmological phase transitions*, *JCAP* **04** (2016) 001 [[arXiv:1512.06239](#)] [[INSPIRE](#)].
- [42] C. Caprini et al., *Detecting gravitational waves from cosmological phase transitions with LISA: an update*, *JCAP* **03** (2020) 024 [[arXiv:1910.13125](#)] [[INSPIRE](#)].
- [43] S. Kawamura et al., *The Japanese space gravitational wave antenna: DECIGO*, *Class. Quant. Grav.* **28** (2011) 094011 [[INSPIRE](#)].
- [44] J. Crowder and N.J. Cornish, *Beyond LISA: Exploring future gravitational wave missions*, *Phys. Rev. D* **72** (2005) 083005 [[gr-qc/0506015](#)] [[INSPIRE](#)].
- [45] V. Corbin and N.J. Cornish, *Detecting the cosmic gravitational wave background with the big bang observer*, *Class. Quant. Grav.* **23** (2006) 2435 [[gr-qc/0512039](#)] [[INSPIRE](#)].
- [46] J.M. Cline, *Baryogenesis*, [hep-ph/0609145](#) [[INSPIRE](#)].
- [47] T. Vachaspati, *Magnetic fields from cosmological phase transitions*, *Phys. Lett. B* **265** (1991) 258 [[INSPIRE](#)].
- [48] G. Sigl, A.V. Olinto and K. Jedamzik, *Primordial magnetic fields from cosmological first order phase transitions*, *Phys. Rev. D* **55** (1997) 4582 [[astro-ph/9610201](#)] [[INSPIRE](#)].
- [49] A. De Simone, G. Nardini, M. Quirós and A. Riotto, *Magnetic Fields at First Order Phase Transition: A Threat to Electroweak Baryogenesis*, *JCAP* **10** (2011) 030 [[arXiv:1107.4317](#)] [[INSPIRE](#)].
- [50] A.G. Tevzadze, L. Kisslinger, A. Brandenburg and T. Kahniashvili, *Magnetic Fields from QCD Phase Transitions*, *Astrophys. J.* **759** (2012) 54 [[arXiv:1207.0751](#)] [[INSPIRE](#)].
- [51] J. Ellis, M. Fairbairn, M. Lewicki, V. Vaskonen and A. Wickens, *Intergalactic Magnetic Fields from First-Order Phase Transitions*, *JCAP* **09** (2019) 019 [[arXiv:1907.04315](#)] [[INSPIRE](#)].
- [52] T.A. Chowdhury, M. Nemevšek, G. Senjanović and Y. Zhang, *Dark Matter as the Trigger of Strong Electroweak Phase Transition*, *JCAP* **02** (2012) 029 [[arXiv:1110.5334](#)] [[INSPIRE](#)].
- [53] S. Fabian, F. Goertz and Y. Jiang, *Dark matter and nature of electroweak phase transition with an inert doublet*, *JCAP* **09** (2021) 011 [[arXiv:2012.12847](#)] [[INSPIRE](#)].
- [54] M.J. Baker, J. Kopp and A.J. Long, *Filtered Dark Matter at a First Order Phase Transition*, *Phys. Rev. Lett.* **125** (2020) 151102 [[arXiv:1912.02830](#)] [[INSPIRE](#)].
- [55] A. Azatov, M. Vanvlasselaer and W. Yin, *Dark Matter production from relativistic bubble walls*, *JHEP* **03** (2021) 288 [[arXiv:2101.05721](#)] [[INSPIRE](#)].
- [56] V. Brdar, A.J. Helmboldt and J. Kubo, *Gravitational Waves from First-Order Phase Transitions: LIGO as a Window to Unexplored Seesaw Scales*, *JCAP* **02** (2019) 021 [[arXiv:1810.12306](#)] [[INSPIRE](#)].

- [57] V. Brdar, L. Graf, A.J. Helmboldt and X.-J. Xu, *Gravitational Waves as a Probe of Left-Right Symmetry Breaking*, *JCAP* **12** (2019) 027 [[arXiv:1909.02018](#)] [[INSPIRE](#)].
- [58] M.J. Baker, M. Breitbach, J. Kopp and L. Mittnacht, *Detailed Calculation of Primordial Black Hole Formation During First-Order Cosmological Phase Transitions*, [arXiv:2110.00005](#) [[INSPIRE](#)].
- [59] G.H. Derrick, *Comments on nonlinear wave equations as models for elementary particles*, *J. Math. Phys.* **5** (1964) 1252 [[INSPIRE](#)].
- [60] S.R. Coleman, V. Glaser and A. Martin, *Action Minima Among Solutions to a Class of Euclidean Scalar Field Equations*, *Commun. Math. Phys.* **58** (1978) 211 [[INSPIRE](#)].
- [61] K. Blum, M. Honda, R. Sato, M. Takimoto and K. Tobioka, *$O(N)$ Invariance of the Multi-Field Bounce*, *JHEP* **05** (2017) 109 [*Erratum ibid.* **06** (2017) 060] [[arXiv:1611.04570](#)] [[INSPIRE](#)].
- [62] F.C. Adams, *General solutions for tunneling of scalar fields with quartic potentials*, *Phys. Rev. D* **48** (1993) 2800 [[hep-ph/9302321](#)] [[INSPIRE](#)].
- [63] U. Sarid, *Tools for tunneling*, *Phys. Rev. D* **58** (1998) 085017 [[hep-ph/9804308](#)] [[INSPIRE](#)].
- [64] S. Fubini, *A New Approach to Conformal Invariant Field Theories*, *Nuovo Cim. A* **34** (1976) 521 [[INSPIRE](#)].
- [65] L.N. Lipatov, *Divergence of the Perturbation Theory Series and the Quasiclassical Theory*, *Sov. Phys. JETP* **45** (1977) 216 [[INSPIRE](#)].
- [66] F. Loran, *Fubini vacua as a classical de Sitter vacua*, *Mod. Phys. Lett. A* **22** (2007) 2217 [[hep-th/0612089](#)] [[INSPIRE](#)].
- [67] K.-M. Lee and E.J. Weinberg, *Tunneling Without Barriers*, *Nucl. Phys. B* **267** (1986) 181 [[INSPIRE](#)].
- [68] A. Ferraz de Camargo, R.C. Shellard and G.C. Marques, *Vacuum Decay in a Soluble Model*, *Phys. Rev. D* **29** (1984) 1147 [[INSPIRE](#)].
- [69] A. Aravind, B.S. DiNunno, D. Lorshbough and S. Paban, *Analyzing multifield tunneling with exact bounce solutions*, *Phys. Rev. D* **91** (2015) 025026 [[arXiv:1412.3160](#)] [[INSPIRE](#)].
- [70] K. Dutta, C. Hector, P.M. Vaudrevange and A. Westphal, *More Exact Tunneling Solutions in Scalar Field Theory*, *Phys. Lett. B* **708** (2012) 309 [[arXiv:1110.2380](#)] [[INSPIRE](#)].
- [71] V. Guada and M. Nemevšek, *Exact one-loop false vacuum decay rate*, *Phys. Rev. D* **102** (2020) 125017 [[arXiv:2009.01535](#)] [[INSPIRE](#)].
- [72] M.J. Duncan and L.G. Jensen, *Exact tunneling solutions in scalar field theory*, *Phys. Lett. B* **291** (1992) 109 [[INSPIRE](#)].
- [73] A. Amariti, *Analytic bounces in d dimensions*, [arXiv:2009.14102](#) [[INSPIRE](#)].
- [74] K. Dutta, C. Hector, T. Konstandin, P.M. Vaudrevange and A. Westphal, *Validity of the kink approximation to the tunneling action*, *Phys. Rev. D* **86** (2012) 123517 [[arXiv:1202.2721](#)] [[INSPIRE](#)].
- [75] A. Masoumi, K.D. Olum and J.M. Wachter, *Approximating tunneling rates in multi-dimensional field spaces*, *JCAP* **10** (2017) 022 [[arXiv:1702.00356](#)] [[INSPIRE](#)].
- [76] G. Pastras, *Exact Tunneling Solutions in Minkowski Spacetime and a Candidate for Dark Energy*, *JHEP* **08** (2013) 075 [[arXiv:1102.4567](#)] [[INSPIRE](#)].

- [77] V. Guada, A. Maiezza and M. Nemevšek, *Multifield Polygonal Bounces*, *Phys. Rev. D* **99** (2019) 056020 [[arXiv:1803.02227](#)] [[INSPIRE](#)].
- [78] V. Guada, M. Nemevšek and M. Pintar, *FindBounce: Package for multi-field bounce actions*, *Comput. Phys. Commun.* **256** (2020) 107480 [[arXiv:2002.00881](#)] [[INSPIRE](#)].
- [79] J.R. Espinosa, *A Fresh Look at the Calculation of Tunneling Actions*, *JCAP* **07** (2018) 036 [[arXiv:1805.03680](#)] [[INSPIRE](#)].
- [80] J.R. Espinosa and T. Konstandin, *A Fresh Look at the Calculation of Tunneling Actions in Multi-Field Potentials*, *JCAP* **01** (2019) 051 [[arXiv:1811.09185](#)] [[INSPIRE](#)].
- [81] R. Jinno, *Machine learning for bounce calculation*, [arXiv:1805.12153](#) [[INSPIRE](#)].
- [82] W.-Y. Ai, B. Garbrecht and C. Tamarit, *Functional methods for false vacuum decay in real time*, *JHEP* **12** (2019) 095 [[arXiv:1905.04236](#)] [[INSPIRE](#)].
- [83] M.P. Hertzberg and M. Yamada, *Vacuum Decay in Real Time and Imaginary Time Formalisms*, *Phys. Rev. D* **100** (2019) 016011 [[arXiv:1904.08565](#)] [[INSPIRE](#)].
- [84] J. Braden, M.C. Johnson, H.V. Peiris, A. Pontzen and S. Weinfurtnner, *New Semiclassical Picture of Vacuum Decay*, *Phys. Rev. Lett.* **123** (2019) 031601 [[arXiv:1806.06069](#)] [[INSPIRE](#)].
- [85] C.L. Wainwright, *CosmoTransitions: Computing Cosmological Phase Transition Temperatures and Bubble Profiles with Multiple Fields*, *Comput. Phys. Commun.* **183** (2012) 2006 [[arXiv:1109.4189](#)] [[INSPIRE](#)].
- [86] A. Masoumi, K.D. Olum and B. Shlaer, *Efficient numerical solution to vacuum decay with many fields*, *JCAP* **01** (2017) 051 [[arXiv:1610.06594](#)] [[INSPIRE](#)].
- [87] P. Athron, C. Balázs, M. Bardsley, A. Fowlie, D. Harries and G. White, *BubbleProfiler: finding the field profile and action for cosmological phase transitions*, *Comput. Phys. Commun.* **244** (2019) 448 [[arXiv:1901.03714](#)] [[INSPIRE](#)].
- [88] R. Sato, *SimpleBounce: a simple package for the false vacuum decay*, *Comput. Phys. Commun.* **258** (2021) 107566 [[arXiv:1908.10868](#)] [[INSPIRE](#)].
- [89] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, U.S.A. (1965). Emended by D F Styer, Dover, Mineola, New York, U.S.A. (2010).
- [90] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, World Scientific (2009) [[DOI](#)].
- [91] R.V. Konoplich, *Calculation of Quantum Corrections to Nontrivial Classical Solutions by Means of the Zeta Function*, *Theor. Math. Phys.* **73** (1987) 1286 [[INSPIRE](#)].
- [92] G. Münster and S. Rotsch, *Analytical calculation of the nucleation rate for first order phase transitions beyond the thin wall approximation*, *Eur. Phys. J. C* **12** (2000) 161 [[cond-mat/9908246](#)] [[INSPIRE](#)].
- [93] B. Garbrecht and P. Millington, *Green's function method for handling radiative effects on false vacuum decay*, *Phys. Rev. D* **91** (2015) 105021 [[arXiv:1501.07466](#)] [[INSPIRE](#)].
- [94] I.M. Gelfand and A.M. Yaglom, *Integration in functional spaces and its applications in quantum physics*, *J. Math. Phys.* **1** (1960) 48 [[INSPIRE](#)].
- [95] R.H. Cameron and W.T. Martin, *Evaluation of various Wiener integrals by use of certain Sturm-Liouville differential equations*, *Bull. Am. Math. Soc.* **51** (1945) 73.

- [96] E.W. Montroll, *Markoff chains, Wiener integrals, and quantum theory*, *Commun. Pure Appl. Math.* **5** (1952) 415.
- [97] K. Kirsten, *Spectral functions in mathematics and physics*, *AIP Conf. Proc.* **484** (1999) 106 [[hep-th/0005133](#)] [[INSPIRE](#)].
- [98] K. Kirsten, *Spectral functions in mathematics and physics*, Chapman & Hall/CRC Press, Boca Raton, FL, U.S.A. (2002).
- [99] G.V. Dunne, *Functional determinants in quantum field theory*, *J. Phys. A* **41** (2008) 304006 [[arXiv:0711.1178](#)] [[INSPIRE](#)].
- [100] G.V. Dunne and H. Min, *Beyond the thin-wall approximation: Precise numerical computation of prefactors in false vacuum decay*, *Phys. Rev. D* **72** (2005) 125004 [[hep-th/0511156](#)] [[INSPIRE](#)].
- [101] G.V. Dunne and K. Kirsten, *Functional determinants for radial operators*, *J. Phys. A* **39** (2006) 11915 [[hep-th/0607066](#)] [[INSPIRE](#)].
- [102] J. Hur and H. Min, *A Fast Way to Compute Functional Determinants of Radially Symmetric Partial Differential Operators in General Dimensions*, *Phys. Rev. D* **77** (2008) 125033 [[arXiv:0805.0079](#)] [[INSPIRE](#)].
- [103] M. Endo, T. Moroi, M.M. Nojiri and Y. Shoji, *On the Gauge Invariance of the Decay Rate of False Vacuum*, *Phys. Lett. B* **771** (2017) 281 [[arXiv:1703.09304](#)] [[INSPIRE](#)].
- [104] M. Endo, T. Moroi, M.M. Nojiri and Y. Shoji, *False Vacuum Decay in Gauge Theory*, *JHEP* **11** (2017) 074 [[arXiv:1704.03492](#)] [[INSPIRE](#)].
- [105] S. Chigusa, T. Moroi and Y. Shoji, *Precise Calculation of the Decay Rate of False Vacuum with Multi-Field Bounce*, *JHEP* **11** (2020) 006 [[arXiv:2007.14124](#)] [[INSPIRE](#)].
- [106] G. 't Hooft, *Computation of the Quantum Effects Due to a Four-Dimensional Pseudoparticle*, *Phys. Rev. D* **14** (1976) 3432 [*Erratum ibid.* **18** (1978) 2199] [[INSPIRE](#)].
- [107] S.R. Coleman, *Quantum Tunneling and Negative Eigenvalues*, *Nucl. Phys. B* **298** (1988) 178 [[INSPIRE](#)].
- [108] J. Baacke and G. Lavrelashvili, *One loop corrections to the metastable vacuum decay*, *Phys. Rev. D* **69** (2004) 025009 [[hep-th/0307202](#)] [[INSPIRE](#)].
- [109] A. Jevicki, *Treatment of Zero Frequency Modes in Perturbation Expansion About Classical Field Configurations*, *Nucl. Phys. B* **117** (1976) 365 [[INSPIRE](#)].
- [110] H.J. Lee, *Negative Modes in Vacuum Decay*, Ph.D. thesis, Columbia U., 2014. 10.7916/D84X55Z3 [[INSPIRE](#)].
- [111] M. Matteini, M. Nemevšek and L. Ubaldi, to appear.
- [112] D.R.T. Jones, *The Two loop β -function for a $G_1 \times G_2$ gauge theory*, *Phys. Rev. D* **25** (1982) 581 [[INSPIRE](#)].