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Post-Minkowskian radial action from soft limits and velocity cuts

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Abstract: We consider gravitational massive scalar-scalar scattering from unitarity and demonstrate how intermediate soft graviton behavior and the concept of extracting classical physics from localization of integrands on velocity cuts devise an efficient extraction scheme for computing the classical post-Minkowskian radial action perturbatively. We demonstrate the computational efficiency by deriving the scattering amplitudes in the probe regime to the fifth post-Minkowskian order in arbitrary dimensions.

Keywords: Scattering Amplitudes, Classical Theories of Gravity, Effective Field Theories

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JHEP03(2022)071

Contents

JHEP03(2022)071

1 Introduction

In an age where gravitational waves in the Universe can be witnessed from black hole and neutron star mergers [1, 2], an exciting particle physics theme is computation of relativistic classical interactions from gravitational quantum scattering [3, 4]. Here modern amplitudes techniques are handy [5–8], and adaptable for the provision of precision predictions in gravity at relativistic velocities [9–26] (for some applications with spin see for instance [27–30]).

At heart of exploring classical general relativity from quantum scattering amplitudes [31, 32] is the correspondence principle of quantum mechanics [33]. It stipulates the emergence of classical observables when quantum numbers are large, and naturally, an ideal application is gravitational quantum scattering amplitudes for superheavy black hole point particles [34–37], in context of Weinberg's [38] widely celebrated idea of general relativity as a low-energy effective field theory [39–43].

Current applications revolve around obtaining low-energy quantum S-matrix elements in an asymptotic Minkowskian flat background arranged in powers of Newton's constant $\mathcal{O}(G_N)$, $\mathcal{O}(G_N^2), \ldots$. Thus deriving $L+1$ post-Minkowskian order classical physics require L-loop scattering amplitudes [6, 7, 44, 45].

In such computations (and contrasting most other precision physics loop amplitude computations), only long-distance (non-analytic) components with varying orders of Planck's constant $\hbar$ have to be computed [6, 7, 39, 40]. To compute a classical Hamiltonian from a scattering amplitude, we can use either Born subtractions in the context of the Lippmann-Schwinger equation [12, 14, 15], or equivalently an effective field theory matching procedure [8, 9, 13, 18, 22]. At low orders in perturbation, this computational scheme is very efficient, but each new loop order poses a challenge in part because of the new integrals involved, but also since the number of amplitude pieces that have to be identified, discarded, subtracted, and kept growing factorially with the perturbative order considered.

The purpose of this paper is to devise a refined computational technology that allows focusing on exactly those amplitude integrand components that integrate to the classical radial action. This work generalises to all loop orders the understanding of how classical physics appear in such computations at one- and two-loop orders [23, 24]. The method uses the *velocity cuts* formalism introduced in refs. [23, 24]. It was shown that reorganizing some combination of propagators in the integrand (see for example eq. (3.4) and (3.6) in ref. [24])

$$\left(\frac{1}{(p_a\cdot\ell_a+i\varepsilon)(p_a\cdot\ell_b-i\varepsilon)}-\frac{1}{(p_a\cdot\ell_b+i\varepsilon)(p_a\cdot\ell_a-i\varepsilon)}\right)\times \left(\frac{1}{(p_b\cdot\ell_a-i\varepsilon)(p_b\cdot\ell_c+i\varepsilon)}-\frac{1}{(p_b\cdot\ell_c-i\varepsilon)(p_b\cdot\ell_a+i\varepsilon)}\right), \tag{1.1}$$

in terms of delta functions

$$\left(\frac{\delta(p_a\cdot\ell_a)}{p_a\cdot\ell_b+i\varepsilon}-\frac{\delta(p_a\cdot\ell_b)}{p_b\cdot\ell_a+i\varepsilon}\right)\times\left(\frac{\delta(p_b\cdot\ell_c)}{p_b\cdot\ell_a+i\varepsilon}-\frac{\delta(p_b\cdot\ell_a)}{p_b\cdot\ell_c+i\varepsilon}\right), \tag{1.2}$$

before integration, lead to considerable simplifications of the classical part of one- and two-loop amplitude computations. We use a multi-soft graviton expansion to organize the $\hbar\to 0$ expansion of the integrand. This allows us to collect terms in the integrand according to unitarity relations in the exponential representation of the S-matrix of [26]. This organization of the multi-loop integrand combined with the velocity cut formalism provides a direct identification of the classical radial action N_L at each loop order.

As we will see this saves computational resources, and we will demonstrate efficiency by direct calculation of the probe limit of the radial action until the fifth post-Minkowskian order. We will verify calculations by deriving the scattering angle in the probe limit in arbitrary dimensions, and compare to known results derived from the Schwarzschild-Tangherlini metric.

JHEP03(2022)071

The paper is organized as follows. First, we review needed background knowledge for computations and the gravitational coupling of scalar massive fields in the context of the Einstein-Hilbert Lagrangian. This is followed by the section 3 where we give compact expressions for the scalar-graviton tree-level amplitudes, that will be used in the loop amplitude computations. In section 4 we give a new organization of the tree-level scalar-multi-graviton amplitude, and in section 4.4 we show how it is possible to manifest the multi-soft behaviour in a way that allows us to devise a direct extraction scheme for classical physics from loop-integrands using velocity cuts. In section 5 we evaluate the multi-loop two-body scattering amplitudes in the probe limit up to the fifth post-Minkowskian order. Section 6 contains our conclusion. Details of the numerators factors and tree-level amplitudes are given in appendix B.

2 Classical physics from quantum amplitudes

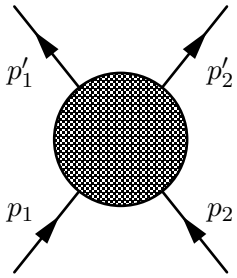
We will focus on the minimal gravitational coupling of scalar fields $(\phi_1, \phi_2) \to (\phi_1, \phi_2)$ with masses m_1 and m_2 from the effective field theory Einstein-Hilbert Lagrangian,

$$\mathcal{L} = \int d^4x \sqrt{-\det(g)} \left[\frac{R}{16\pi G_N} + \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi_1 \partial_\nu \phi_1 + \partial_\mu \phi_2 \partial_\nu \phi_2) - m_1^2 \phi_1^2 - m_2^2 \phi_2^2 \right] + \mathcal{L}_{\text{EFT}} \,, \tag{2.1}$$

here $\mathcal{L}_{\text{EFT}}$ denote effective field theory operators necessary to define a well-behaved low-energy quantum gravity theory at any perturbative order, G_N is Newton's constant, R is the Ricci scalar, and the expansion of the metric is defined from $g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + \sqrt{32\pi G_N} h_{\mu\nu}(x)$ where $\eta_{\mu\nu}$ is the mostly minus Minkowskian metric.

To extract classical physics from quantum amplitudes, we consider scalar four-point L-loop scattering processes. We derive such amplitudes from $(L+1)$-graviton generalised unitarity cuts. Computations are organized in a perturbative expansion in Newton's constant G_N.

$$\mathcal{M}(p_1, p_2, p_1', p_2') = \sum_{L=0}^{\infty} \mathcal{M}_L(p_1, p_2, p_1', p_2') = \tag{2.2}$$



where $\mathcal{M}_L$ denotes the L-loop scalar four-point scattering process of order G_N^{L+1}. We will suppress the Newton constant and only reinstate it in section 5 where we compute the probe radial action. We have defined p_1 and p_2 and p_1' and p_2' as on-shell incoming and outgoing momenta respectively, and $p_1^2 \equiv {p_1'}^2 \equiv m_1^2$, $p_2^2 \equiv {p_2'}^2 \equiv m_2^2$. The center-of-mass energy is $E_{CM}^2 = s \equiv (p_1 + p_2)^2 \equiv (p_1' + p_2')^2$, and we introduce the relativistic factor $\sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}$. The transfer momentum is $q = p_1 - p_1' = p_2' - p_2$. We remark that $q \cdot p_1 = -q \cdot p_1' = \frac{q^2}{2}$.

JHEP03(2022)071

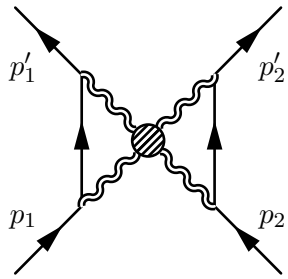


Figure 1. The bowtie diagram that arises at two loops.

We will consider L-loop integrands from generalised $(L+1)$ graviton unitarity cuts[1] (We refer to [23] for the $\hbar$ factors in the loop amplitudes.)

$$i\mathcal{M}_L^{\text{cut}}(\sigma,q^2)=\hbar^{3L+1}\int(2\pi)^D\delta(q+\ell_2+\cdots+\ell_{L+2})\prod_{i=2}^{L+2}\frac{i}{\ell_i^2}\prod_{i=2}^{L+2}\frac{d^D\ell_i}{(2\hbar\pi)^D}$$
$$\frac{1}{(L+1)!}\sum_{h_i=\pm 2}M_{\text{Left}}^{\text{tree}}(p_1,\ell_2^{h_2},\ldots,\ell_{L+2}^{h_{L+2}},-p_1')M_{\text{Right}}^{\text{tree}}(p_2,-\ell_2^{h_2},\ldots,-\ell_{L+2}^{h_{L+2}},-p_2')^\dagger, \tag{2.3}$$

where $M_{\text{Left}}^{\text{tree}}(p_1,\ell_2,\ldots,\ell_{L+2},-p_1')$ and $M_{\text{Right}}^{\text{tree}}(p_2,-\ell_2,\ldots,-\ell_{L+2},-p_2')$ are tree-level amplitudes of multi-graviton emission from a massive scalar. We take the convention that all graviton lines are incoming in the left tree-level factor and out-going in the right tree-level factor, with the momentum conservation

$$q=p_1-p_1'=-\sum_{i=2}^{L+2}\ell_i. \tag{2.4}$$

The multi-graviton cut is not enough for reconstructing the full classical L-loop amplitude. We need to add terms that are not contained in the cut in eq. (2.3). A first type of contributions are multi-graviton cuts factorising the amplitude into a product of two scalar-graviton amplitudes times graviton amplitudes

$$M_{\text{Left}}^{\text{tree}}(p_1,\ell_2,\ldots,\ell_n,-p_1')\times M^{\text{grav.}}(-\ell_2,\ldots,-\ell_n,\ell_{n+1},\ldots,\ell_m)$$
$$\times M_{\text{Right}}^{\text{tree}}(p_2,-\ell_{n+1},\ldots,\ell_m,-p_2')^\dagger, \tag{2.5}$$

where $M^{\text{grav.}}$ is a pure gravity amplitude. Such a contribution arises at two-loop order from the bow-tie graph in figure 1 for which $M^{\text{grav.}}$ is a four-graviton tree amplitude. Since the construction of the integrands presented in this paper is an organisation the scalar-graviton tree-level amplitudes, there is no obstacle from including such contributions.

A second type of contributions are the self-energy ones (see section 4 of [24] and [19, 46]). After cutting all the graviton lines these amplitudes factorise two tree-level scalar-graviton amplitudes times possibly multi-graviton amplitudes. Again the manipulations of the scalar-graviton tree amplitude presented in this paper can be applied to this case, but their analysis is beyond the scope of the present paper.

[1] From now we will remove the polarisation label, and indicate it only when evaluating the cut in section 5.

JHEP03(2022)071

3 Compact massive-scalar-graviton tree amplitudes

In this section, we give the tree-level multi-graviton emission from a scalar line. We use the scattering equations formalism [47–50] which derives amplitudes for a large class of field theories through moduli integrations over string-theory-like integrands in very compact ways. Systematic algebraic construction of numerators for gluon in the scattering equation framework was pioneered by Fu, Du, Huang, and Feng in [51] and in refs. [52, 53] developed into diagrammatic methods.

Provided the scattering equation prescription for the colour-ordered multi-gluon amplitude in [48, 54] and following the construction in [53] for tree-level amplitudes, the Yang-Mills multi-gluon amplitude (with Yang-Mills coupling at unity) has the form

$$A_{n-2}(1,\{2,\ldots,n-1\},n) = \int \frac{\prod_{i=1}^{n} dz_i}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C}))} \prod_{i=1}^{n} \delta'\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{k_i\cdot k_j}{z_{ij}}\right) \frac{1}{z_{12}\cdots z_{n-1\,n}} \times \sum_{\beta\in\mathfrak{S}_{n-2}} \frac{N_{n-2}(1,\beta(2,\ldots,n-1),n)}{z_{1\beta(2)}z_{\beta(2)\beta(3)}\cdots z_{\beta(n-1)\,n}}\,, \tag{3.1}$$

where we have made use of the notation $z_{ij} = z_i - z_j$, and $N_{n-2}(1,\beta(2,\ldots,n-1),n)$ are numerator expressions containing ith gluon polarisation vectors ζ_i and momenta k_i. In contrast to the numerators developed in [51] we will as suggested in [53] average over reference orderings since it provides a computational advantage. Integration over the scattering equations can be done numerically but for arriving at analytic expressions one can explore the one-to-one link between integration measures and integrands in scattering equations and traditional string theory [55] and use it to formulate scattering equation integration rules [56–59].

Following this analytic procedure, we arrive at the following expression for the amplitude

$$A_{n-2}(1,\beta(2,\ldots,n-1),n) = \sum_{\gamma\in\mathfrak{S}_{n-2}} m^{\mathrm{tree}}(1,\beta(1,\ldots,n),n|1,\gamma(2,\ldots,n-1),n) N_{n-2}(1,\gamma(2,\ldots,n-1),n)\,, \tag{3.2}$$

where $m^{\mathrm{tree}}(1,\beta(2,\ldots,n-2),n,n-1|1,\gamma(2,\ldots,n-2),n-1,n) = \mathcal{S}^{-1}(\beta|\gamma)|_{p_1}$ can be interpreted as the inverse momentum kernel as demonstrated in [48]. Plugging this in the momentum kernel expression for gravity amplitudes [60–63] immediately yields

$$M^{\mathrm{tree}}_{n-2}(1,2,\ldots,n-1,n) = (-1)^n \sum_{\beta,\gamma\in\mathfrak{S}_{n-3}} A_{n-2}(1,\beta(2,\ldots,n-2),n-1,n)\mathcal{S}(\beta|\gamma)|_{p_1} A_{n-2}(p_1,n-1,\gamma(2,\ldots,n-2),n)\,, \tag{3.3}$$

leading to the following compact prescription for computing tree-level multi-graviton amplitudes

$$M^{\mathrm{tree}}_{n-2}(1,2,\ldots,n) = i \sum_{\beta\in\mathfrak{S}_{n-2}} N_{n-2}(1,\beta(2,\cdots,n-1),n) A_{n-2}(1,\beta(2,\ldots,n-1),n)\,, \tag{3.4}$$

JHEP03(2022)071

where $A_{n-2}(1,\gamma(2,\ldots,n-1),n)$ are the colour-ordered multi-gluon tree-level amplitudes. As explained in [48, 53] we can derive numerators with gluons states replaced by massive scalar states $k_1 \to p$, $k_n \to -p'$ by dimensional reduction with $p^2 = (p')^2 = m^2$. This has the effect of replacing the multi-gluon numerator factors $N_{n-2}(1,\beta(2,\ldots,n-1),n)$ by $N_{n-2}(p,\beta(2,\ldots,n-1),-p')$ so that the amplitude for multi-gluon emission from a massive scalar reads

$$A_{n-2}(p,\beta(2,\ldots,n-1),-p') = \sum_{\gamma\in\mathfrak{S}_{n-2}} m^{\text{tree}}(p,\beta(2,\ldots,n-1),-p'|p,\gamma(2,\ldots,n-1),-p')N_{n-2}(p,\gamma(2,\ldots,n-1),-p')\,. \tag{3.5}$$

Thus the amplitude for emission of gravitons from a massive scalar can be written

$$M^{\text{tree}}_{n-2}(p,2,\ldots,n-1,-p') = i\sum_{\beta\in\mathfrak{S}_n} N_{n-2}(p,\beta(2,\ldots,n-1),-p')A_{n-2}(p,\beta(2,\ldots,n-1),-p')\,. \tag{3.6}$$

The numerator factors for the scalar multi-gluon amplitudes we will use here are constructed using the method of ref. [53]. The expressions for the numerator factors and Yang-Mills tree-level amplitudes are collected in appendix B. A consequence of the colour-kinematics representation provided is that the numerator factors for gravity amplitudes are perfect squares of linear combinations of Yang-Mills numerators. We summarise the required amplitudes below.

- At three-point order we have

$$M^{\text{tree}}_1(p,\ell_2,-p') = i\,N_1(p,\ell_2,-p')A_1(p,\ell_2,-p') = i\,N_1(p,\ell_2,-p')^2, \tag{3.7}$$

  where the numerator is given in appendix B.1.

- At four-points order we have

$$\begin{aligned} M^{\text{tree}}_2(p,\ell_2,\ell_3,-p') &= i\,N_2(p,2,3,-p')A_2(p,2,3,-p')+\text{perm.}\{2,3\} \\ &= \frac{iN_2(p,2,3,-p')^2}{(\ell_2+p)^2-m^2+i\varepsilon}+\frac{iN_2(p,3,2,-p')^2}{(\ell_3+p)^2-m^2+i\varepsilon}+\frac{i(N_2^{[2,3]})^2}{(\ell_2+\ell_3)^2+i\varepsilon}\,, \end{aligned} \tag{3.8}$$

  with $N_2^{[2,3]} \equiv N_2(p,2,3,-p') - N_2(p,3,2,-p')$. The numerator is given in appendix B.2.

JHEP03(2022)071

- At five-point order we have

$$
\begin{aligned}
M_3^{\text{tree}}(p,\ell_2,\ell_3,\ell_4,-p') &= i\,N_3(p,2,3,4,-p')A_3(p,2,3,4,-p') + \text{perm.}\{2,3,4\} \\
&= \frac{i\,({N_3}^{2,3,4})^2}{((p+\ell_2)^2-m^2+i\varepsilon)((p+\ell_2+\ell_3)^2-m^2+i\varepsilon)} \\
&\quad + \frac{i({N_3}^{[2,3],4})^2}{2((p+\ell_2+\ell_3)^2-m^2+i\varepsilon)((\ell_2+\ell_3)^2+i\varepsilon)} \\
&\quad + \frac{i({N_3}^{2,[3,4]})^2}{2((p+\ell_2)^2-m^2+i\varepsilon)((\ell_3+\ell_4)^2+i\varepsilon)} \\
&\quad + \frac{i({N_3}^{[2,[3,4]]})^2}{4((\ell_2+\ell_4)^2+i\varepsilon)((\ell_2+\ell_3+\ell_4)^2+i\varepsilon)} \\
&\quad + \frac{i({N_3}^{[[2,3],4]})^2}{4((\ell_2+\ell_3)^2+i\varepsilon)((\ell_2+\ell_3+\ell_4)^2+i\varepsilon)} + \text{perm.}\{2,3,4\}\,,
\end{aligned}
\tag{3.9}
$$

  where the numerator is given in appendix B.3.

- For six-point we have

$$
\begin{aligned}
M_4^{\text{tree}}(p,\ell_2,\ell_3,\ell_4,\ell_5,-p') &= i\,N_4(p,2,3,4,5,-p')A_4(p,2,3,4,5,-p') \\
&\quad + \text{perm.}\{2,3,4,5\},
\end{aligned}
\tag{3.10}
$$

  which gives

$$
\begin{aligned}
&M_4^{\text{tree}}(p,\ell_2,\ell_3,\ell_4,\ell_5,-p') = \\
&\quad \frac{i\,({N_4}^{2,3,4,5})^2}{s_{2p}s_{23p}s_{234p}} + \frac{i({N_4}^{2,3,[4,5]})^2}{2s_{45}s_{2p}s_{23p}} + \frac{i({N_4}^{2,[3,4],5})^2}{2s_{34}s_{2p}s_{234p}} + \frac{i({N_4}^{[2,3],4,5})^2}{2s_{23}s_{23p}s_{234p}} \\
&\quad + \frac{i({N_4}^{[2,3],[4,5]})^2}{4s_{23}s_{45}s_{23p}} + \frac{i({N_4}^{[[2,3],4],5})^2}{4s_{23}s_{234}s_{234p}} + \frac{i({N_4}^{[2,[3,4]],5})^2}{4s_{34}s_{234}s_{234p}} \\
&\quad + \frac{i({N_4}^{2,[[3,4],5]})^2}{4s_{34}s_{345}s_{2p}} + \frac{i({N_4}^{2,[3,[4,5]]})^2}{4s_{45}s_{345}s_{2p}} \\
&\quad + \frac{i({N_4}^{[[2,3],[4,5]]})^2}{8s_{23}s_{45}s_{2345}} + \frac{i({N_4}^{[[[2,3],4],5]})^2}{8s_{23}s_{234}s_{2345}} + \frac{i({N_4}^{[[2,[3,4]],5]})^2}{8s_{34}s_{234}s_{2345}} \\
&\quad + \frac{i({N_4}^{[2,[[3,4],5]]})^2}{8s_{34}s_{345}s_{2345}} + \frac{i({N_4}^{[2,[3,[4,5]]]})^2}{8s_{45}s_{345}s_{2345}} + \text{perm.}\{2,3,4,5\},
\end{aligned}
\tag{3.11}
$$

  where we have used the short-hand notations $s_{i_1,\ldots,i_r,p} = (p+\sum_{j=1}^r \ell_{i_r})^2 - m^2 + i\varepsilon$, $s_{i_1,\ldots,i_r} = (\sum_{j=1}^r \ell_{i_j})^2 + i\varepsilon$. The Yang-Mills amplitude is given in appendix B.4.

With similar expressions for the seven-point amplitude, where the Yang-Mills amplitude is given in the appendix B.5.

4 Unitarity and multi-graviton emission

We consider the amplitude in (2.3) with $L+1$ graviton generalised unitarity cuts. We will make use of the notation

$$
M_{\text{Left}}^{\text{tree}}(p_1,\ell_2,\ldots,\hat{\ell}_i,\ldots,\ell_{L+1},\ell_{L+2},-p_1'),
\tag{4.1}
$$

JHEP03(2022)071

where we indicate the leg on which we have used momentum conservation by a hat

$$\ell_i = -q - \sum_{\substack{2\le j\le L+2\\ j\neq i}} \ell_j. \tag{4.2}$$

The tree-level amplitudes in (4.1) have two kinds of massive propagators. The propagators containing a 'hatted' leg read

$$\begin{aligned}&\frac{1}{(p_1-\ell_{i_2}-\cdots-\ell_{i_j}-q)^2-m^2+i\varepsilon}\\ &\quad=\frac{-1}{2p_1\cdot(\ell_{i_2}+\cdots+\ell_{i_j})-(\ell_{i_2}+\cdots+\ell_{i_j}+q)^2-i\varepsilon},\end{aligned} \tag{4.3}$$

while the propagators which do not involve the 'hatted' leg are

$$\frac{1}{(p_1+\ell_{i_2}+\cdots+\ell_{i_j})^2-m^2+i\varepsilon}=\frac{1}{2p_1\cdot(\ell_{i_2}+\cdots+\ell_{i_j})+(\ell_{i_2}+\cdots+\ell_{i_j})^2+i\varepsilon}\,. \tag{4.4}$$

In the above equations we need to take $1<j\le L+2$.

Using the identity

$$\lim_{\varepsilon\to 0^+}\left(\frac{1}{\eta-i\varepsilon}-\frac{1}{\eta+i\varepsilon}\right)=\lim_{\varepsilon\to 0^+}\frac{2i\varepsilon}{\eta^2+\varepsilon^2}=\hat{\delta}(\eta), \tag{4.5}$$

we rewrite propagators with a 'hatted' leg

$$\begin{aligned}\frac{1}{(p_1-\ell_{i_2}-\cdots-\ell_{i_j}-q)^2-m^2+i\varepsilon}&=\hat{\delta}((p_1-\ell_{i_2}-\cdots-\ell_{i_j}-q)^2-m^2)\\ &\quad+\frac{1}{(p_1-\ell_{i_2}-\cdots-\ell_{i_j}-q)^2-m^2-i\varepsilon}\,,\end{aligned} \tag{4.6}$$

where we have introduced the notation $\hat{\delta}(x)\equiv -2\pi i\delta(x)$.

We denote by $M^{\text{tree}(+)}_{L+1}(p_1,\ell_2\,,\ldots,\ell_{L+2},-p_1')$ the tree-level amplitudes where all the propagators of the type $2p_1\cdot(\sum_r\ell_r)-(q+\sum_r\ell_r)^2-i\varepsilon$ are flipped to $2p_1\cdot(\sum_r\ell_r)-(q+\sum_r\ell_r)^2+i\varepsilon$. We denote $M^{\text{tree}(-)}_{L+1}(p_1,\ell_2,\ldots,\ell_{L+2},-p_1')$ the tree-level amplitudes where all the propagators of the type $2p_1\cdot(\sum_r\ell_r)+(\sum_r\ell_r)^2+i\varepsilon$ are flipped to $2p_1\cdot(\sum_r\ell_r)+(\sum_r\ell_r)^2-i\varepsilon$.

In the next sections, we focus first on two- and three-graviton emission tree-level amplitudes and the relations between the amplitudes M^{tree}_{L+1}, $M^{\text{tree}(+)}_{L+1}$ and $M^{\text{tree}(-)}_{L+1}$ with $L=1$ and $L=2$ followed by a generalisation to generic multi-graviton emission.

4.1 The four-point case

We write the four-point Feynman tree-level amplitude in the form

$$M^{\text{tree}}_2(p_1,\hat{\ell}_2,\ell_3,-p_1')=\frac{n_{p_1+\hat{\ell}_2}}{(p_1+\hat{\ell}_2)^2-m_1^2+i\varepsilon}+\frac{n_{p_1+\ell_3}}{(p_1+\ell_3)^2-m_1^2+i\varepsilon}+\frac{n_q}{q^2}, \tag{4.7}$$

with generic (off-shell) numerator factors $n_{p_1+\ell_2}$, $n_{p_1+\ell_3}$ and n_q and where momentum conservation is imposed on leg $\ell_2=-\ell_3-q$. Applying the relation (4.6) on the first propagator only, the amplitude reads

$$M^{\text{tree}}_2(p_1,\hat{\ell}_2,\ell_3,-p_1')=\hat{\delta}((p_1+\hat{\ell}_2)^2-m_1^2)n_{p_1+\hat{\ell}_2}+M^{\text{tree}(+)}_2(p_1,\hat{\ell}_2,\ell_3,-p_1'), \tag{4.8}$$

JHEP03(2022)071

with

$$M_2^{\text{tree}(+)}(p_1,\hat{\ell}_2,\ell_3,-p_1') \equiv -\frac{n_{p_1-q-\ell_3}}{2p_1\cdot(q+\ell_3)+i\varepsilon}+\frac{n_{p_1+\ell_3}}{2p_1\cdot\ell_3+i\varepsilon}+\frac{n_q}{q^2}. \tag{4.9}$$

Using the factorisation theorem on the pole $(p_1+\hat{\ell}_2)^2-m_1^2=0$

$$M_2^{\text{tree}}(p_1,\hat{\ell}_2,\ell_3,-p_1') \sim M_1^{\text{tree}}(p_1,\hat{\ell}_2,-p_1-\hat{\ell}_2)\frac{i}{(p_1+\hat{\ell}_2)^2-m_1^2}M_1^{\text{tree}}(p_1+\hat{\ell}_2,\ell_3,-p_1'), \tag{4.10}$$

implies that the support of the delta-function the numerator factor $n_{p_1+\hat{\ell}_2}$ factorises into the product of two tree-level amplitudes. The tree amplitude takes the form (with the cut conditions $\hat{\ell}_2^2=\ell_3^2=0$)

$$\begin{aligned} M_2^{\text{tree}}(p_1,\hat{\ell}_2,\ell_3,-p_1') = {} & M_2^{\text{tree}(+)}(p_1,\hat{\ell}_2,\ell_3,-p_1') \\ & +\hat{\delta}((p_1+\hat{\ell}_2)^2-m_1^2)M_1^{\text{tree}}(p_1,\hat{\ell}_2,-p_1-\hat{\ell}_2)M_1^{\text{tree}}(p_1+\hat{\ell}_2,\ell_3,-p_1'). \end{aligned} \tag{4.11}$$

Similarly, by flipping the other propagator we have

$$\begin{aligned} M_2^{\text{tree}}(p_1,\hat{\ell}_2,\ell_3,-p_1') = {} & M_2^{\text{tree}(-)}(p_1,\hat{\ell}_2,\ell_3,-p_1') \\ & +\hat{\delta}((p_1+\ell_3)^2-m_1^2)M_1^{\text{tree}}(p_1,\ell_3,-p_1-\ell_3)M_1(p_1+\ell_2,\hat{\ell}_2,-p_1'), \end{aligned} \tag{4.12}$$

with

$$M_2^{\text{tree}(-)}(p_1,\hat{\ell}_2,\ell_3,p_1') \equiv -\frac{n_{p_1-q-\ell_3}}{2p_1\cdot(q+\ell_3)-i\varepsilon}+\frac{n_{p_1+\ell_3}}{2p_1\cdot\ell_3-i\varepsilon}+\frac{n_q}{q^2}. \tag{4.13}$$

In the case the 'hatted' momentum enters the incoming momentum as in $M_2^{\text{tree}}(p_1+\hat{\ell}_4,\ell_2,\ell_3,-p_1')$ with $\ell_4=-q+\ell_2+\ell_3$ we have

$$\begin{aligned} M_2^{\text{tree}}(p_1+\hat{\ell}_4,\ell_2,\ell_3,-p_1') = {} & M_2^{\text{tree}(+)}(p_1+\hat{\ell}_4,\ell_2,\ell_3,-p_1') \\ & +\hat{\delta}((p_1+\hat{\ell}_4+\ell_2)^2-m_1^2)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4,\ell_2,-p_1-\hat{\ell}_4-\ell_2)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4+\ell_2,\ell_3,-p_1') \\ & +\hat{\delta}((p_1+\hat{\ell}_4+\ell_3)^2-m_1^2)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4,\ell_3,-p_1-\hat{\ell}_4-\ell_3)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4+\ell_3,\ell_2,-p_1'), \end{aligned} \tag{4.14}$$

where we have made use of the notation $\hat{\delta}(x)\equiv -2\pi i\delta(x)$, and we used that for the three-point functions

$$M_1^{\text{tree}}(p_1+\hat{\ell}_4+\ell_2,\ell_3,-p_1') = M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4+\ell_2,\ell_3,-p_1') = M_1^{\text{tree}(-)}(p_1+\hat{\ell}_4+\ell_2,\ell_3,-p_1'). \tag{4.15}$$

4.1.1 The multi-soft expansion of the four-point amplitude

We consider the following multi-soft scaling of graviton legs in the context of the two-scalar-$(L+1)$-graviton tree amplitudes described in the above sections. We take $\ell_i\to|\vec{q}|\tilde{\ell}_i$ with $|\vec{q}|\to 0$, so that

$$q=p-p'=-|\vec{q}|(\tilde{\ell}_2+\tilde{\ell}_3). \tag{4.16}$$

For a conventional Feynman gravity amplitude (see appendix A for a derivation) we have the universal results

$$\lim_{|\vec{q}|\to 0} M_2^{\text{tree}}(p,|\vec{q}|\tilde{\ell}_2,|\vec{q}|\tilde{\ell}_3,-p') \sim \frac{1}{|\vec{q}|}\,. \tag{4.17}$$

JHEP03(2022)071

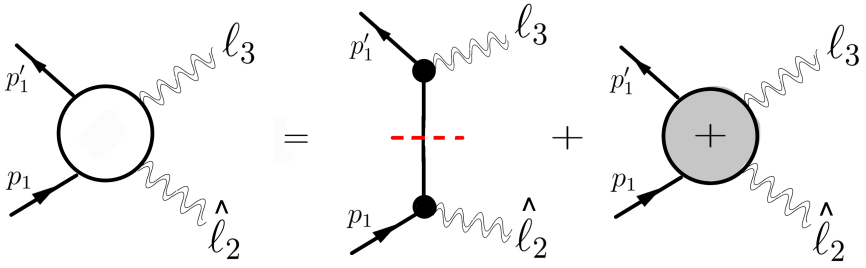


Figure 2. The four-point relation graphically outlined. The red line symbolises a delta-function constraint, the grey blob is $M_2^{\text{tree}(+)}$.

We now derive explicitly the soft scaling behaviour for the four-point amplitude. The four-point graviton amplitude reads (see section 3 for details)

$$M_2^{\text{tree}}(p,\ell_2,\ell_3,-p') = \frac{iN_2(p,2,3,-p')^2}{(p+\ell_2)^2-m^2+i\varepsilon} + \frac{iN_2(p,3,2,-p')^2}{(p+\ell_3)^2-m^2+i\varepsilon} + \frac{i\left(N_2^{[2,3]}\right)^2}{q^2}\,. \tag{4.18}$$

Performing the soft expansion of numerators (see appendix B.2 for their expressions) we get for the amplitude the soft expansion (with $\ell_i = |q|\,\tilde{\ell}_i$ and $|q| \to 0$)

$$\begin{aligned} M_2^{\text{tree}}(p,\ell_2,\hat{\ell}_3,-p') = &-\frac{2i(p\cdot\zeta_2)^2(p\cdot\zeta_3)^2}{|\vec{q}|}\left(\frac{1}{p\cdot\tilde{\ell}_2+i\varepsilon}-\frac{1}{p\cdot\tilde{\ell}_2-i\varepsilon}\right) \\ &+\frac{4i(p\cdot\tilde{F}_2\cdot\tilde{F}_3.p)^2}{(p\cdot\tilde{\ell}_2)^2}+\mathcal{O}(|\vec{q}|)\,, \end{aligned} \tag{4.19}$$

where we defined the field strength $\tilde{F}_i^{\mu\nu} = \tilde{\ell}_i^\mu\zeta_i^\nu - \tilde{\ell}_i^\nu\zeta_i^\mu$. This means that we have the following soft expansion of the tree amplitudes

$$M_2^{\text{tree}}(p,\ell_2,\hat{\ell}_3,-p') = -\frac{4\pi(p\cdot\zeta_2)^2(p\cdot\zeta_3)^2\delta(p.\tilde{\ell}_2)}{|\vec{q}|}+\frac{4i(p\cdot\tilde{F}_2\cdot\tilde{F}_3.p)^2}{(p\cdot\tilde{\ell}_2)^2}+\mathcal{O}(|\vec{q}|)\,, \tag{4.20}$$

so that $M_2^{\text{tree}} \sim \mathcal{O}(\frac{1}{|\vec{q}|})$ as given in (4.17). The same derivation for the $M_2^{\text{tree}(+)}(p,\ell_2,\hat{\ell}_3,-p')$ amplitude in (4.9) on the other hand is

$$\begin{aligned} M_2^{\text{tree}(+)}(p,\ell_2,\hat{\ell}_3,-p') = &-\frac{2i(p\cdot\zeta_2)^2(p\cdot\zeta_3)^2}{|\vec{q}|}\left(\frac{1}{p\cdot\tilde{\ell}_2+i\varepsilon}-\frac{1}{p\cdot\tilde{\ell}_2+i\varepsilon}\right) \\ &+\frac{4i(p\cdot\tilde{F}_2\cdot\tilde{F}_3.p)^2}{(p\cdot\tilde{\ell}_2)^2}+\mathcal{O}(|\vec{q}|)\,, \end{aligned} \tag{4.21}$$

showing that

$$M_2^{\text{tree}(+)}(p,\ell_2,\hat{\ell}_3,-p') = \frac{4i(p\cdot\tilde{F}_2\cdot\tilde{F}_3.p)^2}{(p\cdot\tilde{\ell}_2)^2}+\mathcal{O}(|\vec{q}|) = \mathcal{O}(|\vec{q}|^0)\,. \tag{4.22}$$

Similar arguments imply that $M_2^{\text{tree}(-)} \sim \mathcal{O}(|\vec{q}|^0)$. Therefore the amplitude $M_2^{\pm}$ have the multi-soft behaviour

$$\lim_{|\vec{q}|\to 0} M_2^{\text{tree}(\pm)}(p,|\vec{q}|\tilde{\ell}_2,|\vec{q}|\hat{\tilde{\ell}}_3,-p') \sim |\vec{q}|^0\,. \tag{4.23}$$

This behaviour generalises to the general case $M_2^{\text{tree}(\pm)}(p,|\vec{q}|\tilde{\ell}_2,\ldots,|\vec{q}|\hat{\tilde{\ell}}_{L+2},-p')$.

JHEP03(2022)071

4.2 The five-point case

For making the multi-soft behaviour at five points, we start by rewriting the tree amplitude in the following form

$$M_3^{\text{tree}}(p_1,\ell_2,\ell_3,\hat{\ell}_4,-p_1') = \Bigg(\frac{n_{p_1+\ell_2,p_1+\ell_2+\ell_3}}{((p_1+\ell_2)^2-m_1^2+i\varepsilon)((p_1+\ell_2+\ell_3)^2-m_1^2+i\varepsilon)} + \frac{n_{p_1+\ell_2,\ell_3+\hat{\ell}_4}}{((p_1+\ell_2)^2-m_1^2+i\varepsilon)(\ell_3+\hat{\ell}_4)^2} + \frac{n_{p_1+\ell_2+\ell_3,\ell_2+\ell_3}}{((p_1+\ell_2+\ell_3)^2-m_1^2+i\varepsilon)(\ell_2+\ell_3)^2} + \frac{n_{p_1+p_2,\ell_3+\hat{\ell}_4}}{q^2(\ell_3+\hat{\ell}_4)^2}\Bigg) + \text{perm.}\{\ell_2,\ell_3,\hat{\ell}_4\}\,, \tag{4.24}$$

where we impose momentum conservation on $\hat{\ell}_4 = -q-\ell_2-\ell_3$ and the cut condition $\ell_2^2=\ell_3^2=\hat{\ell}_4^2=0$. Flipping the propagators as in the four-point case, we get

$$\begin{aligned} M_3^{\text{tree}}(p_1,\ell_2,\ell_3,\hat{\ell}_4,-p_1') &= n_{p_1+\hat{\ell}_4,p_1+\ell_2+\hat{\ell}_4}\hat{\delta}((p_1+\hat{\ell}_4)^2-m_1^2)\hat{\delta}((p_1+\ell_2+\hat{\ell}_4)^2-m_1^2) \\ &+ n_{p_1+\hat{\ell}_4,p_1+\ell_3+\hat{\ell}_4}\hat{\delta}((p_1+\hat{\ell}_4)^2-m_1^2)\hat{\delta}((p_1+\ell_3+\hat{\ell}_4)^2-m_1^2) \\ &+ \left(\frac{n_{p_1+\ell_2,p_1+\ell_2+\hat{\ell}_4}}{(p_1+\ell_2)^2-m_1^2+i\varepsilon} + \frac{n_{p_1+\hat{\ell}_4,p_1+\ell_2+\hat{\ell}_4}}{(p_1+\hat{\ell}_4)^2-m_1^2+i\varepsilon} + \frac{n_{p_1+\ell_2+\hat{\ell}_4,\ell_1+\hat{\ell}_4}}{(\ell_2+\hat{\ell}_4)^2}\right)\hat{\delta}((p_1+\ell_2+\hat{\ell}_4)^2-m_1^2) \\ &\left(\frac{n_{p_1+\ell_3,p_1+\ell_3+\hat{\ell}_4}}{(p_1+\ell_3)^2-m_1^2+i\varepsilon} + \frac{n_{p_1+\hat{\ell}_4,p_1+\ell_3+\hat{\ell}_4}}{(p_1+\hat{\ell}_4)^2-m_1^2+i\varepsilon} + \frac{n_{p_1+\ell_3+\hat{\ell}_4,\ell_3+\hat{\ell}_4}}{(\ell_3+\hat{\ell}_4)^2}\right)\hat{\delta}((p_1+\ell_3+\hat{\ell}_4)^2-m_1^2) \\ &\left(\frac{n_{p_1+\hat{\ell}_4,p_1+\ell_2+\hat{\ell}_4}}{(p_1+\ell_2+\hat{\ell}_4)^2-m_1^2-i\varepsilon} + \frac{n_{p_1+\hat{\ell}_4,p_1+\ell_3+\hat{\ell}_4}}{(p_1+\ell_3+\hat{\ell}_4)^2-m_1^2-i\varepsilon} + \frac{n_{p_1+\hat{\ell}_4,\ell_2+\ell_3}}{(\ell_2+\ell_3)^2}\right)\hat{\delta}((p_1+\hat{\ell}_4)^2-m_1^2) \\ &+ M_3^{\text{tree}(+)}(p_1,\ell_2,\ell_3,\hat{\ell}_4,-p_1'). \end{aligned} \tag{4.25}$$

With $\hat{\delta}(x) \equiv -2\pi i\delta(x)$. From the factorisation property of the tree-level amplitude we know that the coefficient of the delta-function are products of tree-level amplitudes

$$\begin{aligned} &n_{p_1+\hat{\ell}_4,p_1+\ell_2+\hat{\ell}_4}\Big|_{(p_1+\hat{\ell}_4)^2-m_1^2=(p_1+\ell_2+\hat{\ell}_4)^2-m_1^2=0} = \\ &\quad M_1^{\text{tree}}(p_1,\hat{\ell}_4,-p_1-\hat{\ell}_4)M_1^{\text{tree}}(p_1+\hat{\ell}_4,\ell_2,-p_1-\hat{\ell}_4-\ell_2)M_1^{\text{tree}}(p_1+\hat{\ell}_4+\ell_2,\ell_3,-p_1'), \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} &n_{p_1+\hat{\ell}_4,p_1+\ell_3+\hat{\ell}_4}\Big|_{(p_1+\hat{\ell}_4)^2-m_1^2=(p_1+\ell_3+\hat{\ell}_4)^2-m_1^2=0} = \\ &\quad M_1^{\text{tree}}(p_1,\hat{\ell}_4,-p_1-\hat{\ell}_4)M_1^{\text{tree}}(p_1+\hat{\ell}_4,\ell_3,-p_1-\hat{\ell}_4-\ell_3)M_1^{\text{tree}}(p_1+\hat{\ell}_4+\ell_3,\ell_2,-p_1'), \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} &\left(\frac{n_{p_1+\ell_2,p_1+\ell_2+\hat{\ell}_4}}{(p_1+\ell_2)^2-m_1^2+i\varepsilon} + \frac{n_{p_1+\hat{\ell}_4,p_1+\ell_2+\hat{\ell}_4}}{(p_1+\hat{\ell}_4)^2-m_1^2+i\varepsilon} + \frac{n_{p_1+\ell_2+\hat{\ell}_4,\ell_2+\hat{\ell}_4}}{(\ell_2+\hat{\ell}_4)^2}\right)\Big|_{(p_1+\ell_2+\hat{\ell}_4)^2-m_1^2=0} \\ &\quad = M_2^{\text{tree}}(p_1,\ell_2,\hat{\ell}_4,-p_1-\ell_2-\hat{\ell}_4)M_1^{\text{tree}}(p_1+\hat{\ell}_4+\ell_2,\ell_3,-p_1'), \end{aligned} \tag{4.28}$$

JHEP03(2022)071

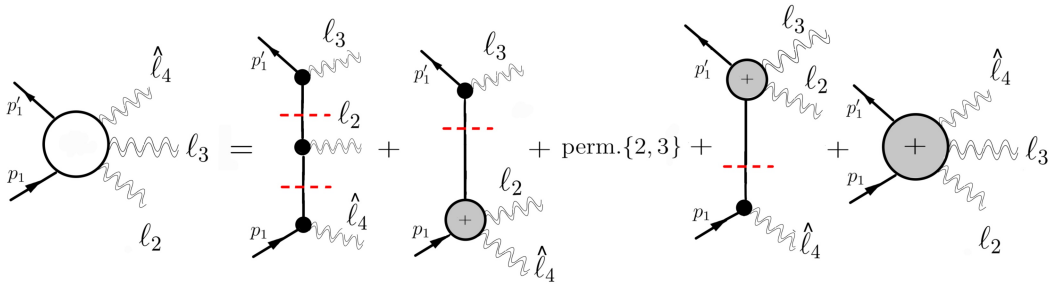


Figure 3. The five-point relation graphically outline depicted. The red line symbolises a delta-function constraint.

and

$$\left(\frac{n_{p_1+\ell_3,p_1+\ell_3+\hat{\ell}_4}}{(p_1+\ell_3)^2-m_1^2+i\varepsilon}+\frac{n_{p_1+\hat{\ell}_4,p_1+\ell_3+\hat{\ell}_4}}{(p_1+\hat{\ell}_4)^2-m_1^2+i\varepsilon}+\frac{n_{p_1+\ell_3+\hat{\ell}_4,\ell_3+\hat{\ell}_4}}{(\ell_3+\hat{\ell}_4)^2}\right)\Big|_{(p_1+\ell_3+\hat{\ell}_4)^2-m_1^2=0}$$
$$=M_2^{\text{tree}}(p_1,\ell_3,\hat{\ell}_4,-p_1-\ell_3-\hat{\ell}_4)M_1^{\text{tree}}(p_1+\hat{\ell}_4+\ell_3,\ell_2,-p_1'),\tag{4.29}$$

and

$$\left(\frac{n_{p_1+\hat{\ell}_4,p_1+\ell_2+\hat{\ell}_4}}{(p_1+\ell_2+\hat{\ell}_4)^2-m_1^2+i\varepsilon}+\frac{n_{p_1+\hat{\ell}_4,p_1+\ell_3+\hat{\ell}_4}}{(p_1+\ell_3+\hat{\ell}_4)^2-m_1^2+i\varepsilon}+\frac{n_{p_1+\hat{\ell}_4,\ell_2+\ell_3}}{(\ell_2+\ell_3)^2}\right)\Big|_{(p_1+\hat{\ell}_4)^2-m_1^2=0}$$
$$=M_1^{\text{tree}}(p_1,\hat{\ell}_4,-p_1-\hat{\ell}_4)M_2^{\text{tree}}(p_1+\hat{\ell}_4,\ell_2,\ell_3,-p_1').\tag{4.30}$$

The factorisation relations involve $M_2^{\text{tree}}(p_1+\hat{\ell}_4,\ell_2,\ell_3,-p_1')$ which is rewritten using (4.14). This leads to the following expression suitable for the multi-soft expansion

$$\begin{aligned}
M_3^{\text{tree}}(p_1,\ell_2,\ell_3,\hat{\ell}_4,-p_1') &= \hat{\delta}((p_1+\hat{\ell}_4)^2-m_1^2)\hat{\delta}((p_1+\ell_2+\hat{\ell}_4)^2-m_1^2)\\
&\times M_1^{\text{tree}(+)}(p_1,\hat{\ell}_4,-p_1-\hat{\ell}_4)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4,\ell_2,-p_1-\hat{\ell}_4-\ell_2)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4+\ell_2,\ell_3,-p_1')\\
&+\hat{\delta}((p_1+\hat{\ell}_4)^2-m_1^2)\hat{\delta}((p_1+\ell_3+\hat{\ell}_4)^2-m_1^2)\\
&\times M_1^{\text{tree}(+)}(p_1,\hat{\ell}_4,-p_1-\hat{\ell}_4)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4,\ell_3,-p_1-\hat{\ell}_4-\ell_3)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4+\ell_3,\ell_2,-p_1')\\
&+\hat{\delta}((p_1+\ell_2+\hat{\ell}_4)^2-m_1^2)M_2^{\text{tree}(+)}(p_1,\ell_2,\hat{\ell}_4,-p_1-\ell_2-\hat{\ell}_4)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4+\ell_2,\ell_3,-p_1')\\
&+\hat{\delta}((p_1+\ell_3+\hat{\ell}_4)^2-m_1^2)M_2^{\text{tree}(+)}(p_1,\ell_3,\hat{\ell}_4,-p_1-\ell_3-\hat{\ell}_4)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_4+\ell_3,\ell_2,-p_1')\\
&+\hat{\delta}((p_1+\hat{\ell}_4)^2-m_1^2)M_1^{\text{tree}(+)}(p_1,\hat{\ell}_4,-p_1-\hat{\ell}_4)M_2^{\text{tree}(+)}(p_1+\hat{\ell}_4,\ell_2,\ell_3,-p_1')\\
&+M_3^{\text{tree}(+)}(p_1,\ell_2,\ell_3,\hat{\ell}_4,-p_1'),
\end{aligned}\tag{4.31}$$

with a similar expression involving $M_3^{\text{tree}(-)}$ after flipping the $+i\varepsilon$ poles and with $\hat{\delta}(x)\equiv-2\pi i\delta(x)$.

4.3 General $(L+1)$ graviton case

The symbolic structure of the four-point amplitudes derived in the previous sections read

$$M_2^{\text{tree}}\sim(M_1^{\text{tree}(+)})^2\delta_i(\ldots)+M_2^{\text{tree}(+)},\tag{4.32}$$

JHEP03(2022)071

and the five-point amplitudes read

$$M_3^{\text{tree}} \sim (M_1^{\text{tree}(+)})^3 \prod_i^2 \delta_i(\ldots) + M_1^{\text{tree}(+)} M_2^{\text{tree}(+)} \delta_i(\ldots) + M_3^{\text{tree}(+)} \,. \tag{4.33}$$

Using the expression for the six- and seven-points amplitudes presented in section 3 we have derived similar expressions, which take the symbolic form

$$\begin{aligned} M_4^{\text{tree}} \sim{}& (M_1^{\text{tree}(+)})^4 \prod_i^3 \delta_i(\ldots) + (M_1^{\text{tree}(+)})^2 (M_2^{\text{tree}(+)}) \prod_i^2 \delta_i(\ldots) \\ &+ M_1^{\text{tree}(+)} M_3^{\text{tree}(+)} \delta(\ldots) + M_4^{\text{tree}(+)}, \end{aligned} \tag{4.34}$$

$$\begin{aligned} M_5^{\text{tree}} \sim{}& (M_1^{\text{tree}(+)})^5 \prod_i^4 \delta_i(\ldots) + (M_1^{\text{tree}(+)})^3 (M_2^{\text{tree}(+)}) \prod_i^3 \delta_i(\ldots) + \cdots \\ &+ M_1^{\text{tree}(+)} M_4^{\text{tree}(+)} \delta(\ldots) + M_5^{\text{tree}(+)}. \end{aligned} \tag{4.35}$$

It follows on general grounds from the above examples that we have the following structure for the general L point case organised by powers of unitarity cuts delta-functions

$$\begin{aligned} M_{L+1}^{\text{tree}} \sim{}& (M_1^{\text{tree}(+)})^{L+1} \prod_i^L \delta_i(\ldots) + (M_1^{\text{tree}(+)})^{L-1} (M_2^{\text{tree}(+)}) \prod_i^{L-1} \delta_i(\ldots) + \cdots \\ &+ M_1^{\text{tree}(+)} M_L^{\text{tree}(+)} \delta(\ldots) + M_{L+1}^{\text{tree}(+)} \,, \end{aligned} \tag{4.36}$$

with a similar expansion involving the $M_{L+1}^{\text{tree}(-)}$ amplitudes.

4.4 Multi-soft graviton of the tree-level amplitudes

We consider now the following multi-soft scaling of graviton legs in the context of the two-scalar-$(L+1)$-graviton tree amplitudes described in the above sections. We take $\ell_i \to |\vec{q}|\tilde{\ell}_i$ with $|\vec{q}| \to 0$, so that

$$q = p - p' = -|\vec{q}| \sum_{i=2}^{L+2} \tilde{\ell}_i. \tag{4.37}$$

We find the following results. For a conventional Feynman gravity amplitude (see appendix A for a derivation) we have the universal results

$$\lim_{|\vec{q}|\to 0} M_{L+1}^{\text{tree}} \left(p, |\vec{q}|\tilde{\ell}_2, \ldots, |\vec{q}|\tilde{\ell}_{L+2}, -p'\right) \sim |\vec{q}|^{-L} \,. \tag{4.38}$$

In contrast to this, the amplitude $M_{L+1}^{\pm}$ have the multi-soft behaviour

$$\lim_{|\vec{q}|\to 0} M_{L+1}^{\text{tree}(\pm)} \left(p, |\vec{q}|\tilde{\ell}_2, \ldots, |\vec{q}|\hat{\tilde{\ell}}_{L+2}, -p'\right) \sim |\vec{q}|^{0} \,. \tag{4.39}$$

This is trivial for the three-point amplitude $M_1^{\text{tree}(+)}(p, |\vec{q}|\hat{\tilde{\ell}}_2, -p')$ as such amplitudes are independent of the graviton momentum. The case of the four-point amplitude $L = 2$ has been shown in section 4.1.1. For higher-multiplicity amplitudes, this can be checked, considering the explicit expressions of the tree-level amplitudes provided up to seven points.

JHEP03(2022)071

4.4.1 The multi-soft expansion of the generic gravity amplitude

By combining (4.39) with the scaling of the delta-function

$$\delta\left(\left(p_1+\sum \ell_i\right)^2-m_1^2\right)=\delta\left(2|q|p_1\cdot\sum\tilde{\ell}_i+\mathcal{O}(|q|^2)\right)=\frac{1}{|q|}\delta\left(2p_1\cdot\sum\tilde{\ell}_i\right)+\mathcal{O}(|q|^0), \tag{4.40}$$

we deduce that the amplitude $\mathcal{M}_{L+1}$ has the multi-soft scaling

$$\lim_{|\vec{q}|\to 0} M_{L+1}^{\text{tree}}(p,|\vec{q}|\tilde{\ell}_2,\dots,|\vec{q}|\hat{\tilde{\ell}}_{L+2},-p')=\frac{(M_1^{\text{tree}})^{L+1}\delta(\cdots)^L}{|\vec{q}|^L}+\mathcal{O}\left(\frac{1}{|q|^{L-1}}\right). \tag{4.41}$$

4.5 The classical part from the multi-soft graviton expansion

We will now explain how the above multi-soft expansions of the tree-level amplitude in the multi-graviton cut imply a reorganisation of the computations of integrands, and allow us to easily identify classical components in the amplitude.

In the limit $\hbar\to 0$, $q\to 0$ with $\underline{q}=q/\hbar$ fixed, the multi-loop amplitude has the following $\hbar$-Laurent expansion (see section 3 of [23] for details)

$$\mathcal{M}_L(\sigma,|q|)=\frac{1}{\hbar^{L-1}|q|^{(4-D)L}}\sum_{r\geq -2}\mathcal{M}_L^{(r)}(\sigma,\epsilon)(\hbar|q|)^r. \tag{4.42}$$

We will now discuss how the organisation of the tree-level amplitudes across the cut in section 4 allows us to identify the classical part of the integrand.

When plugging the delta-function expansion from (4.36) in the product of tree-level amplitudes, the integrand of the cut integral in (2.3), becomes a sum of contributions organised as follows

$$\mathcal{M}_L^{\text{cut}}\sim\sum_{k=0}^{2L}\hbar^{3L+1}\int\frac{(d^D\ell)^L}{\hbar^{DL}}\frac{\left(\delta((p_1+\sum_i\ell_{\alpha_i})^2-m_1^2)\right)^k\times(\prod M^{\text{tree}(+)})\times(\prod M^{\text{tree}(-)\,\dagger})}{(\ell^2)^{L+1}}. \tag{4.43}$$

Now in the soft-expansion $|q|=\hbar\underline{q}$ with $\ell=\hbar|\underline{q}|\hat{\ell}$ for $\hbar\to 0$, the delta-function behaves at leading order as

$$\lim_{|q|\to 0}\delta((p+\ell)^2-m^2)\sim\delta(2\hbar|q|p\cdot\hat{\ell})\sim\frac{\delta(2p\cdot\hat{\ell})}{\hbar|\underline{q}|}. \tag{4.44}$$

The multi-soft expansion of section 4.4 stipulates that the amplitudes $M^{\pm}$ are of order $\mathcal{O}(|q|^0)$, so the generic integrals in the multi-graviton cut behaves as

$$\mathcal{M}_L^{\text{cut}}\sim\sum_{k=0}^{2L}\frac{\hbar^{L-1-k}}{|\underline{q}|^{2+k-(D-2)L}}. \tag{4.45}$$

JHEP03(2022)071

We can have three type of contributions:

Terms with $k = L$ delta-functions. Such contributions, behave as

$$\frac{1}{\hbar|\underline{q}|^{2-(D-3)L}}, \tag{4.46}$$

which is of classical order and given by terms with $r = L-2$ in (4.42). Thus in this case,

$$\mathcal{M}_L(\sigma,|q|)\Big|_{\text{classical}} = \frac{1}{\hbar}\frac{\mathcal{M}_L^{(L-2)}(\sigma,\epsilon)}{|\underline{q}|^{2-(D-3)L}}, \tag{4.47}$$

which implies that for computing the classical part of the amplitude, we can approximate the unitarity delta-function constraint as a velocity cut $\delta((p+\ell)^2-m^2) \sim \delta(2p\cdot\ell)$, which hugely simplifies the integral computation. This classical term receives two kinds of contributions discussed in section 5.2 of [24]: (1) disconnected graphs and (2) connected graphs as in figure 7 and 8 of that paper. As a consequence of the unitarity relations between the classical scattering matrix element and the radial action $\hat{N}$ derived in section 2 of [26], such factorised contributions are cancelled by unitarity and do not contribute to the radial action. We will see an explicit example in section 4.6 below.

Terms with $k < L$ delta-functions. They are of order $\mathcal{O}(\hbar^0)$ and correspond to quantum contributions.

Terms with $k > L$ delta-functions. They correspond to contributions with $-2 \leq r \leq L-3$ in the Laurent expansion (4.42). It was shown in [23, 24] that these contributions to the one-loop and two-loop amplitudes exponentiate and do not contribute to the classical part. These products of unitarity delta-functions are precisely the ones arising from the expansion of the exponential representation of the S-matrix, see ref. [26].

We note that the decomposition of the integrand in (4.43) as a sum of powers of unitarity cut delta-functions realizes the expansion of the exponential representation of the S-matrix given in [26]. The classical part of the amplitude is the radial action N_L. We will now develop this into a new practical tool for computation of post-Minkowskian physics.

4.6 The one-loop radial action

We will in this section illustrate how the considerations from the previous sections are useful for computations of classical contributions from scattering amplitudes. We will start by rederiving the classical contribution from the one-loop amplitude. The two-particle cut along the two graviton lines reads,

$$i\mathcal{M}_1^{\text{cut}}(\sigma,q^2) = \frac{1}{2}\int\frac{d^D\ell_3}{(2\pi)^{2D}}\frac{M_2^{\text{tree}}(p_1,\hat{\ell}_2,\ell_3,-p_1')\times(M_2^{\text{tree}})^\dagger(p_2,-\hat{\ell}_2,-\ell_3,-p_2')}{\hat{\ell}_2^2\ell_3^2}, \tag{4.48}$$

where $\hat{\ell}_2 = -q-\ell_3$.

JHEP03(2022)071

Using the expressions in (4.11) and (4.12) for the tree-level amplitude in the cut, we obtain

$$
\begin{aligned}
i\mathcal{M}_1^{\text{cut}}(\sigma,q^2) = \frac{1}{2}\int \frac{d^D\ell_3}{(2\pi)^{2D-2}} \frac{\delta((p_1+\hat{\ell}_2)^2-m_1^2)\delta((p_2-\hat{\ell}_2)^2-m_2^2)}{\hat{\ell}_2^2\ell_3^2} \\
\times M_1^{\text{tree}(+)}(p_1,\hat{\ell}_2,-p_1-\hat{\ell}_2) M_1^{\text{tree}(+)}(p_1+\hat{\ell}_2,\ell_3,-p_1') \\
\times M_1^{\text{tree}(-)\dagger}(p_2,-\hat{\ell}_2,-p_2+\hat{\ell}_2) M_1^{\text{tree}(-)\dagger}(p_2-\hat{\ell}_2,-\ell_3,-p_2') \\
-\frac{i}{2}\int \frac{d^D\ell_3}{(2\pi)^{2D-1}} \frac{\delta((p_1+\hat{\ell}_2)^2-m_1^2)}{\hat{\ell}_2^2\ell_3^2} \\
\times M_1^{\text{tree}(+)}(p_1,\hat{\ell}_2,-p_1-\hat{\ell}_2) M_1^{\text{tree}(+)}(p_1+\hat{\ell}_2,\ell_3,-p_1') M_2^{\text{tree}(-)\dagger}(p_2,-\hat{\ell}_2,-\ell_3,-p_2') \\
+\frac{i}{2}\int \frac{d^D\ell_3}{(2\pi)^{2D-1}} \frac{\delta((p_2-\hat{\ell}_2)^2-m_2^2)}{\hat{\ell}_2^2\ell_3^2} \\
\times M_2^{\text{tree}(+)}(p_1,\hat{\ell}_2,\ell_3,-p_1') M_1^{\text{tree}(-)\dagger}(p_2,-\hat{\ell}_2,-p_2+\hat{\ell}_2) M_1^{\text{tree}(-)\dagger}(p_2-\hat{\ell}_2,-\ell_3,-p_2') \\
+\frac{1}{2}\int \frac{d^D\ell_3}{(2\pi)^{2D}} \frac{M_2^{\text{tree}(+)}(p_1,\hat{\ell}_2,\ell_3,-p_1') M_2^{\text{tree}(-)\dagger}(p_2,-\hat{\ell}_2,-\ell_3,-p_2')}{\hat{\ell}_2^2\ell_3^2}.
\end{aligned}
\tag{4.49}
$$

In the first three lines we recognize the factorisation of the four-point tree-level amplitudes $\mathcal{M}_0(p_1,p_2,-p_1-\ell_1,-p_2-\ell_1)$ and $\mathcal{M}_0(p_1+\ell_1,p_2+\ell_1,-p_1',-p_2')$ on the massless graviton pole. The contribution proportional to $M_2^{\text{tree}(+)}(p_1,\hat{\ell}_1,\ell_2,-p_1')M_2^{\text{tree}-\dagger}(p_2,-\hat{\ell}_1,-\ell_2,-p_2')$ on the last line can be neglected as it is of quantum order $\frac{\mathcal{O}(|q|^D)\times\mathcal{O}(|q|^0)}{\mathcal{O}(|q|^4)} \sim \mathcal{O}(|q|^{D-4})$ which is $\mathcal{O}(\log(|q|))$ in $D=4$ dimensions. We thus have

$$
\begin{aligned}
i\mathcal{M}_1^{\text{cut}}(\sigma,q^2) = \frac{i}{2}\int \frac{d^D\ell_2 d^D\ell_3}{(2\pi)^{2D-2}} \delta^{(D)}(\ell_2+\ell_3+q)\delta((p_1+\ell_2)^2-m_1^2)\delta((p_2-\ell_2)^2-m_2^2) \\
\times \mathcal{M}_0(p_1,p_2,-p_1-\ell_2,-p_2-\ell_2)\mathcal{M}_0(p_1+\ell_2,p_2+\ell_2,-p_1',-p_2') \\
-\frac{1}{2}\int \frac{d^D\ell_3}{(2\pi)^{2D-2}} \delta((p_1+\hat{\ell}_1)^2-m_1^2)\times \\
\frac{M_1^{\text{tree}(+)}(p_1,\hat{\ell}_2,-p_1-\hat{\ell}_1) M_1^{\text{tree}(+)}(p_1+\hat{\ell}_2,\ell_3,-p_1') M_2^{\text{tree}(-)\dagger}(p_2,-\hat{\ell}_1,-\ell_3,-p_2')}{\hat{\ell}_1^2\ell_3^2} \\
+\frac{1}{2}\int \frac{d^D\ell_3}{(2\pi)^{2D-2}} \delta((p_2-\hat{\ell}_1)^2-m_2^2)\times \\
\frac{M_1^{\text{tree}(+)}(p_1,\hat{\ell}_1,\ell_3,-p_1') M_1^{\text{tree}(-)\dagger}(p_2,-\hat{\ell}_2,-p_2+\hat{\ell}_1) M_2^{\text{tree}(-)\dagger}(p_2-\hat{\ell}_2,-\ell_3,-p_2')}{\hat{\ell}_1^2\ell_3^2} \\
+\mathcal{O}(|q|^{-2\varepsilon}).
\end{aligned}
\tag{4.50}
$$

This expression matches exactly the expansion in eq. (2.16) of ref. [26]. Using the unitarity relation in eq. (2.10) of [26], we identify the first two lines as the product of tree-level amplitudes from unitarity, while the rest can be associated with the one-loop contribution

JHEP03(2022)071

to the radial action (which is the classical eikonal exponent N_1).

$$
\begin{aligned}
N_1(p_1,p_2,-p_1',-p_2') &= -\frac{1}{2}\int \frac{d^D\ell_2}{(2\pi)^{2D-2}}\delta((p_1+\hat{\ell}_1)^2-m_1^2) \\
&\times \frac{M_1^{\text{tree}(+)}(p_1,\hat{\ell}_1,-p_1-\hat{\ell}_1)M_1^{\text{tree}(+)}(p_1+\hat{\ell}_1,\ell_2,-p_1')M_2^{\text{tree}(-)\,\dagger}(p_2,-\hat{\ell}_1,-\ell_2,-p_2')}{\hat{\ell}_1^2\ell_2^2} \\
&+ \frac{1}{2}\int \frac{d^D\ell_2}{(2\pi)^{2D-2}}\delta((p_2-\hat{\ell}_1)^2-m_2^2) \\
&\times \frac{M_2^{\text{tree}(+)}(p_1,\hat{\ell}_1,\ell_2,-p_1')M_1^{\text{tree}(-)\,\dagger}(p_2,-\hat{\ell}_1,-p_2+\hat{\ell}_1)M_1^{\text{tree}(-)\,\dagger}(p_2-\hat{\ell}_1,-\ell_2,-p_2')}{\hat{\ell}_1^2\ell_2^2} \\
&+ \mathcal{O}(|q|^{-2\varepsilon}).
\end{aligned}
\tag{4.51}
$$

5 Probe radial action

We will now see how the organisation of the integrand of the multi-graviton cut integral gives a direct identification of the integrand of the classical part at L-loop in the probe limit $m_1 \gg m_2$. This follows from the discussion in section 4.5 using that the classical part of the multi-loop amplitude have the symbolic representation

$$
\begin{aligned}
\mathcal{M}_L^{\text{classical}} \sim \int \sum_{n=1}^{L+1} \left(\delta((p+\sum_i \ell_i)^2-m^2)\right)^L \times (M^{\text{tree}(+)})^{L+2-n} \times (M^{\text{tree}(-)\,\dagger})^n \\
\times \delta\left(q+\sum_{i=1}^{L+1}\ell_i\right)\prod_{i=1}^{L+1}\frac{d^D\ell_i}{\ell_i^2}\,.
\end{aligned}
\tag{5.1}
$$

We see that the leading probe contribution arises from the term with $n=1$ in the integrand of the multi-graviton cut in (2.3)

$$
\begin{aligned}
(M_{\text{Left}}M_{\text{Right}}{}^\dagger)\Big|_{\text{probe}} &= M_{L+1}^{\text{tree}(-)\dagger}(p_2,-\hat{\ell}_2,-\ell_3,\ldots,-\ell_{L+1},-p_2') \\
&\times M_1^{\text{tree}(+)}(p_1,\hat{\ell}_2,-p_1-\hat{\ell}_2)\prod_{j=3}^{L}\delta\left((p_1+\hat{\ell}_2+\cdots+\ell_{j-1})^2-m_1^2\right) \\
&M_1^{\text{tree}(+)}(p_1+\hat{\ell}_2+\cdots+\ell_{j-1},\ell_j,-p_1-\hat{\ell}_2-\cdots+\ell_j),
\end{aligned}
\tag{5.2}
$$

evaluated on the cut $\ell_i^2=0$ for $1\le i\le L+1$. This contribution is represented in figure 4. The next-to-probe, contribution arises from the terms with $n=2$ which is represented in figure 5. The mth next-to-probe limit is the sum of the contributions with $n=m+1$.

Beyond the probe regime we need to include other contributions from the soft expansion, as well as other unitarity cuts (like the cut containing self-energy contributions). Including these contributions is beyond the scope of the present paper, but we remark that the multiple soft expansion of the tree amplitude lead to an efficient organisation of the integrand in such cases as well.

JHEP03(2022)071

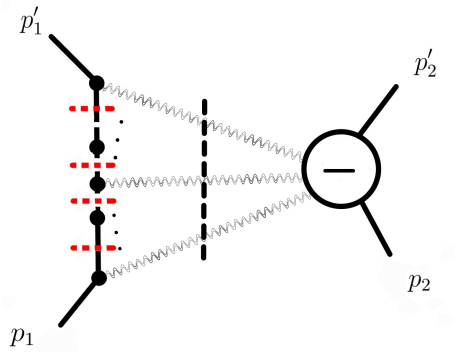


Figure 4. The contribution with $n = 1$ to compute the probe amplitude. The heavy mass m_1 is on the left and the small mass m_2 is on the right side.

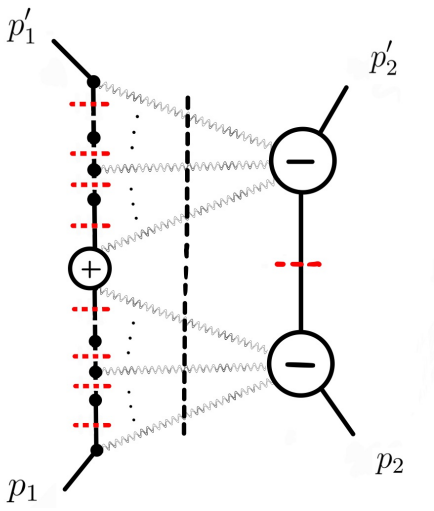


Figure 5. An example of an integrand contribution at the next-to-probe order. The heavy mass m_1 is on the left and the small mass m_2 is on the right side.

5.1 Probe action from the multi-soft limits

Our first practical computational example of the formalism presented will be of the computation of the probe limit of the classical amplitude at various post-Minkowskian orders. In the probe approximation with $m_1 \gg m_2$, we consider the contribution with $n = 1$ in (5.1) which is represented on figure 4. We begin at the second post-Minkowskian order where the part of the integrand N_1 in (4.51), which correspond to the probe mass being m_2 is,

$$\begin{aligned} N_1^{\text{probe}}(\sigma, \vec{q}) = -\frac{(32\pi G_N)^2}{2} \int \frac{d^D\ell_1 d^D\ell_2}{(2\pi)^{2D-2}} \frac{\delta((p_1+\ell_1)^2 - m_1^2)}{\ell_1^2 \ell_2^2} \delta(\ell_1 + \ell_2 + q) \\ \times \sum_{h_1,h_2=\pm 2} M_1^{\text{tree}(+)}(p_1, \ell_1^{h_1}, -p_1 - \ell_1^{h_1}) M_1^{\text{tree}(+)}(p_1 + \ell_1^{h_1}, \ell_2^{h_2}, -p_1') \\ \times M_2^{\text{tree}(-)\,\dagger}(p_2, -\ell_1^{h_1}, -\ell_2^{h_2}, -p_2'). \end{aligned} \tag{5.3}$$

JHEP03(2022)071

We know that the three-point tree-level amplitudes are given by

$$M_1^{\text{tree}(+)}(p_1,\ell_1,-p_1-\ell_1) = 2p_1^\mu p_1^\nu \zeta_{1\,\mu\nu},$$
$$M_1^{\text{tree}(+)}(p_1+\ell_1,\ell_2,-p_1') = 2(p_1^\mu+\ell_1^\mu)(p_1^\nu+\ell_1^\nu)\zeta_{2\,\mu\nu}. \tag{5.4}$$

Now to carry out the polarisation sum for the intermediate gravitons in eq. (5.3) we use the completeness identity

$$\sum_{h=\pm 2} \zeta^h_{\mu\nu}(\zeta^h_{\alpha\beta})^* = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta}\right), \tag{5.5}$$

where we regulate the non-transverse polarization degrees of freedom by taking,

$$\eta_{\mu\nu}\eta^{\mu\nu} = D-2+D_s\,, \tag{5.6}$$

and adjust the state counting parameter D_s to $D_s=2$ to remove the dilaton [64]. In the soft limit we can evaluate the leading soft term of $\mathcal{O}(|q|^0)$ of $M_2^{\text{tree}(-)\,\dagger}$ employing that

$$M_2^{\text{tree}(-)\,\dagger}(p_2,-\ell_1,-\ell_1-q,-p_2')|_{i\varepsilon\to 0,\text{soft}} = M_2^{\text{tree}\,\dagger}(p_2,-\ell_1,-\ell_1-q,-p_2')|_{i\varepsilon\to 0,\text{soft}}. \tag{5.7}$$

Thus using the expression for M_2^{tree} given in (3.8), we obtain the following expression of the amplitude[2]

$$N_1^{\text{probe}}(\sigma,\vec{q}) = 2(8\pi G_N)^2\left(\frac{m_2^4 m_1^4 G_N^2((D-2)\sigma^2-1)^2}{2(D-2)^2}\mathcal{I}_1(2,1,1)\right.$$
$$\left.+\frac{2m_1^4 m_2^2 G_N^2((D-2)\sigma^2-1)}{(D-2)^2}\mathcal{I}_1(0,1,1)+\frac{2m_1^4 G_N^2(D-3)}{(D-2)}\mathcal{I}_1(-2,1,1)\right). \tag{5.8}$$

Where we have introduced a family of integrals on which the L-loop probe amplitude are expanded

$$\mathcal{I}_L(\{a_j\},\{b_{j,k}\},\{c_j\}) \equiv$$
$$\int\prod_{j=1}^{L}\frac{d^D\ell_j\,\delta(2p_1\cdot\ell_j)}{(2\pi)^{D-1}}\frac{1}{\left(2p_2\cdot(\sum_{k=1}^{j}\ell_k)\right)^{a_j}\prod_{k=j}^{L}\left((\sum_{r=j}^{k}\ell_r)^2\right)^{b_{j,k}}\left((q+\sum_{k=1}^{j}\ell_k)^2\right)^{c_j}}. \tag{5.9}$$

This basis involves $3L+\frac{L(L+1)}{2}$ different variables (including the ones in the delta functions). Using `LiteRed` [65] we find that only one master integral contributes to the classical result (we have checked this to four-loop order):

$$I_L(\vec{q}) = \int\frac{d^D\ell_1\cdots d^D\ell_L}{(2\pi)^{L(D-1)}}\frac{\delta(2p_1\cdot\ell_1)\cdots\delta(2p_1\cdot\ell_L)}{\ell_1^2\cdots\ell_L^2(\ell_1+\ell_2+\cdots+\ell_L+q)^2}\,, \tag{5.10}$$

[2]Up to three-loop order, we can directly use the expression of the amplitude derived in the previous section. At higher-loop order, the size of integrand growths, and it is preferable to use the representation in (3.4) and integrate each ordering since it leads to a more convenient generation of the integrand that is easier to automate.

JHEP03(2022)071

where $p_1^2 = m_1^2$, $p_1 \cdot q = 0$ and $q = (0, \vec{q})$. Evaluating the delta functions, the integral can be put in Euclidean form

$$I_L(\vec{q}) = \frac{(-1)^{L+1}}{2^L m_1^L} \int \frac{d^{D-1}\vec{\ell}_1 \cdots d^{D-1}\vec{\ell}_L}{(2\pi)^{L(D-1)}} \frac{1}{\vec{\ell}_1^2 \cdots \vec{\ell}_L^2 (\vec{\ell}_1 + \vec{\ell}_2 + \cdots + \vec{\ell}_L + \vec{q})^2}\,, \tag{5.11}$$

and becomes a $I_L(\vec{q})$ is a massless L-loop sunset integral. This master integral also arises in the metric computation of ref. [21]. The fact that the same master integral arises in the two computations is natural because the probe mass m_2 is evolving the Schwarzschild metric sourced by the mass m_1. The master in (5.11) is readily evaluated using the method of section 2.2 of [21], and remarking that $I_L(\vec{q}) = (-1)^{L+1} J_{(L)}(\vec{q}^2)/(2^L \vec{q}^2 m_1^L)$, we have the result

$$I_L(\vec{q}) = \frac{1}{(\vec{q}^2)^{1-\frac{(D-3)L}{2}}} \times \frac{(-1)^L}{2^{L+1} m_1^L} \frac{\Gamma\left(\frac{D-3}{2}\right)^L \Gamma\left(1 - \frac{(D-3)L}{2}\right)}{(4\pi)^{L\frac{D-1}{2}} \Gamma\left(\frac{(D-3)(L+1)}{2}\right)}\,. \tag{5.12}$$

After reduction with `LiteRed` [65], the probe radial action at L-loop order is

$$N_L^{\text{probe}}(\sigma, \vec{q}) = \frac{(-1)^{L+1} 4}{(L+1)} \frac{(D-3)^L}{(D-2)^{L+1}} \frac{c_L(\sigma, D)}{(\sigma^2-1)^L} I_L(\vec{q})\, m_1^{2(L+1)} m_2^2 (8\pi G_N)^{L+1}. \tag{5.13}$$

At tree-level we have

$$c_0(\sigma, D) = \sigma^2(D-2) - 1. \tag{5.14}$$

Performing the integral reduction with `LiteRed` [65] we find at one-loop order

$$c_1(\sigma, D) = \sigma^4(2D-5)(2D-3) + \sigma^2(30-12D) + 3, \tag{5.15}$$

at two-loop order

$$c_2(\sigma, D) = 2\sigma^6(D-2)(3D-8)(3D-4) - 30\sigma^4(D-2)(3D-8) + 30\sigma^2(3D-8) - 1, \tag{5.16}$$

at three-loop order

$$\begin{aligned} c_3(\sigma, D) = {} & \frac{1}{3}\sigma^8(4D-11)(4D-9)(4D-7)(4D-5) - \frac{28}{3}\sigma^6(4D-11)(4D-9)(4D-7) \\ & + 70\sigma^4(4D-11)(4D-9) + \sigma^2(1540-560D) + 35, \end{aligned} \tag{5.17}$$

and finally four-loop order

$$\begin{aligned} c_4(\sigma, D) = {} & \frac{2}{3}\sigma^{10}(D-2)(5D-14)(5D-12)(5D-8)(5D-6) \\ & - 30\sigma^8(D-2)(5D-14)(5D-12)(5D-8) \\ & + 420\sigma^6(D-2)(5D-14)(5D-12) - 420\sigma^4(5D-14)(5D-12) \\ & + 630\sigma^2(5D-14) - 126. \end{aligned} \tag{5.18}$$

We note that the dimension dependence of the probe coefficients are compatible with the generic form is given in section 9 of [25].

JHEP03(2022)071

We further remark that in $D = 4$ dimensions the coefficients take the simpler factorised form

$$\begin{aligned} c_1(\sigma,4) &= 3(\sigma^2-1)\left(5\sigma^2-1\right), \\ c_2(\sigma,4) &= 2\left(64\sigma^6-120\sigma^4+60\sigma^2-5\right), \\ c_3(\sigma,4) &= 35(\sigma^2-1)^2\left(33\sigma^4-18\sigma^2+1\right), \\ c_4(\sigma,4) &= 6\left(1792\sigma^{10}-5760\sigma^8+6720\sigma^6-3360\sigma^4+630\sigma^2-21\right). \end{aligned} \tag{5.19}$$

These results are in agreement with the probe limit of the two-body scattering amplitude up to fifth post-Minkowskian order in four dimensions derived in [22, 24].

5.2 The probe amplitude from geodesic scattering

In this section, we will discuss how the results derived from scattering amplitudes above can be put into the context of geodesic scattering. We derive the probe amplitude from the geodesic scattering using the Schwarzschild-Tangherlini metric in D dimensions, using the effective-one-body (EOB) formalism of the recent ref. [66]. We will show that the scattering amplitudes obtained this way matches the ones obtained from unitarity.

The Schwarzschild-Tangherlini metric reads in an isotropic coordinate system[3]

$$ds^2 = A(r)dt^2 + B(r)\left(dr^2 + r^2 d^2\Omega_{d-2}\right), \tag{5.20}$$

where $d^2\Omega_{d-2}$ is the metric on the $D-1$ unit sphere where

$$A(r) = \left(\frac{1-\frac{2\pi^{\frac{3-D}{2}}G_N m_1 r^{3-D}\Gamma\left(\frac{D-1}{2}\right)}{D-2}}{1+\frac{2\pi^{\frac{3-D}{2}}G_N m_1 r^{3-D}\Gamma\left(\frac{D-1}{2}\right)}{D-2}}\right)^2, \quad B(r) = \left(1+\frac{2\pi^{\frac{3-D}{2}}G_N m_1 r^{3-D}\Gamma\left(\frac{D-1}{2}\right)}{D-2}\right)^{\frac{4}{D-3}}. \tag{5.21}$$

Using the equation (29) of [66] we deduce the effective potential

$$\frac{V_{\text{eff}}(r)}{p_\infty^2} = 1 - \frac{B(r)}{\sigma^2-1}\left(\frac{\sigma^2}{A(r)}-1\right), \tag{5.22}$$

with $p_\infty^2 = m_2^2(\sigma^2-1)+O(m_2/m_1)$ in the probe limit for $m_2 \ll m_1$. Using this we deduce the scattering angle using the expression given in [15]

$$\chi^{\text{probe}}(\sigma,D) = \sum_{k\geq 1}\frac{2b}{k!}\int_0^\infty du\left(\frac{d}{du^2}\right)^k\left[\frac{1}{u^2+b^2}\left(\frac{V_{\text{eff}}\left(\sqrt{u^2+b^2}\right)(u^2+b^2)}{p_\infty^2}\right)^k\right]. \tag{5.23}$$

[3]The metric in spherical coordinates is given by ref. [67]

$$ds^2 = \left(1-\frac{8\pi G_N m_1\Gamma(D-1)}{(D-2)r^{D-3}}\right)dt^2 - \left(1-\frac{8\pi G_N m_1\Gamma(D-1)}{(D-2)r^{D-3}}\right)^{-1}dr^2 - r^2 d^2\Omega_{D-2}.$$

JHEP03(2022)071

Using that

$$\int_0^\infty \frac{bdu}{(u^2+b^2)^{1+\frac{n(D-3)}{2}}} = \frac{\sqrt{\pi}\Gamma\left(\frac{n(D-3)+1}{2}\right)}{2b^{n(D-3)}\Gamma\left(\frac{n(D-3)+2}{2}\right)}, \tag{5.24}$$

the scattering angle has the post-Minkowskian expansion in D dimensions[4]

$$\chi^{\text{probe}}(\sigma,D) = \sum_{L\geq 0} \frac{c_L(\sigma,D)}{(\sigma^2-1)^{L+1}} \frac{2^{L+2}\Gamma\left(\frac{D-1}{2}\right)^{L+1}\Gamma\left(\frac{(D-3)(L+1)+1}{2}\right)}{(D-2)^{L+1}\pi^{\frac{(D-3)(L+1)-1}{2}}\Gamma\left(\frac{(D-3)(L+1)+2}{2}\right)} \left(\frac{G_N m_1}{b^{D-3}}\right)^{L+1}. \tag{5.25}$$

We can compare with the scattering angle derived from the scattering amplitude in the previous sections. The Fourier transform of the probe radial action in (5.13) to b-space

$$N_L^{\text{probe}}(\sigma,b) = \frac{1}{4m_1m_2\sqrt{\sigma^2-1}} \int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}} N_L^{\text{probe}}(\sigma,\vec{q}) e^{i\vec{q}\cdot\vec{b}}. \tag{5.26}$$

Using the Fourier tranform of the master integral

$$\begin{aligned}\tilde{I}_L(\vec{b}) &= \int_{\mathbb{R}^{D-2}} \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}} \tilde{I}_L(\vec{q}) e^{i\vec{q}\cdot\vec{b}} \\ &= \frac{(-1)^{L-1}}{2^{2+3L}m_1^L} \frac{\Gamma\left(\frac{D-3}{2}\right)^{L+1}\Gamma\left(\frac{(D-3)(L+1)-1}{2}\right)}{\pi^{-\frac{1}{2}+(L+1)\frac{D-1}{2}}\Gamma\left(\frac{(D-3)(L+1)}{2}\right)} \left(\frac{1}{\vec{b}^2}\right)^{-\frac{1}{2}+\frac{(D-3)(L+1)}{2}},\end{aligned} \tag{5.27}$$

we obtain

$$\begin{aligned}N_L^{\text{probe}}(\sigma,b) = m_2\sqrt{\sigma^2-1}\, b \frac{2^{1+L}}{\pi^{\frac{-1+(D-3)(L+1)}{2}}} \\ \times \frac{c_L(\sigma,D)}{(\sigma^2-1)^{L+1}} \frac{\Gamma\left(\frac{D-1}{2}\right)^{L+1}\Gamma\left(\frac{(D-3)(L+1)-1}{2}\right)}{(D-2)^{L+1}\Gamma\left(\frac{(D-3)(L+1)+2}{2}\right)} \left(\frac{G_N m_1}{b^{D-3}}\right)^{L+1}.\end{aligned} \tag{5.28}$$

In the probe limit $m_2 \ll m_1$ the amplitude is related to the scattering angle by the linear relation

$$\chi^{\text{probe}}(\sigma,D) = -\frac{1}{m_2\sqrt{\sigma^2-1}} \frac{\partial N^{\text{probe}}(b,\sigma)}{\partial b}, \tag{5.29}$$

which leads to the angle at the $L+1$ post-Minkowskian order in perfect agreement with the geodesic computation in (5.25).

6 Conclusion

The scattering amplitude approach to the gravitational two-body scattering is a promising avenue for performing post-Minkowskian calculations needed for the construction of waveforms and have already led to a renewed understanding of the connection between quantum

[4]The expression in (5.25) reproduces the results given in the appendix B of [11] to the third Post-Minkowskian order $n=3$. This was guaranteed because the effective potential $V_{\text{eff}}(r)$ in (5.22) is designed to match the scattering angle [66]. We remark that the expression in (5.23) for the scattering angle leads to a much simpler evaluation of the post-Minkowskian expansion than the derivation of the angle obtained by solving Einstein's geodesic equations as in [68, section 101] and in [11].

JHEP03(2022)071

scattering amplitudes and classical observables [5, 23, 31, 32, 44, 45]. Quantum scattering amplitudes contain much more information than their classical parts and thus extracting classical physics from amplitudes becomes more and more challenging at each perturbative order.

In this work, combining unitarity and the concept of velocity cuts introduced in [24], we have identified exactly those elements of integrands that lead to classical physics after integration. Our approach uses an organisation of the integrand of the multi-loop amplitude with unitarity cuts on the massive scalar propagator lines used together with detailed knowledge of the correspondence between the multi-soft graviton expansion and $\hbar \to 0$ limit classical integrand matching the exponential representation of the S-matrix of [26]. In the classical limit, this approach systematically relates the classical part in the scattering amplitude to the matrix elements of the eikonal operator $\hat{N}$, without having to perform the subtractions needed for the exponentiation of the radial action.

We have exemplified our approach by computing the probe amplitude at second, third, fourth, and fifth post-Minkowskian orders. These scattering amplitudes are obtained in the D-dimension. We have verified the agreement with the results obtained by geodesic scattering in the D-dimensional Schwarzschild-Tangherlini metric.

We would like to emphasize that our framework for computation is not restricted to the probe integrands, but can be applied for deriving the complete post-Minkowskian scattering potential. In ref. [25], a heavy mass expansion was used to extract post-Minkowskian physics from amplitudes with applications for computing the probe limits. Although colour-kinematic numerators are also used, numerators in ref. [25] are different from the ones used here. It would be interesting to investigate further the connection between approaches.

We also note that the nature of the external lines plays a very little role in the multi-soft scaling and localisation arguments we make — as expected from the universal behaviour of gravitational interactions [6, 16, 20]. For instance, classical integrands in the post-Minkowskian framework with elementary spinning external particles should be possible to simplify as well using the presented formalism. Although it is interesting to study this question further we will leave it for future research work.

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JHEP03(2022)071

A Soft scaling from momentum kernel

We will here outline the soft scaling behaviour at generic multiplicity using the momentum kernel given in eq. (2.20) of [63].

$$M_L^{\text{tree}}(p_1,\ell_2,\ldots,\ell_{L+1},-p_1') = (-1)^{L-1}\sum_{\sigma,\gamma\in\mathfrak{S}_L}\mathcal{S}(\sigma,\gamma)_{\ell_{L+1}} \times A_L(p_1,\sigma(2,\ldots,L),L+1,-p_1')A_L(p_1,L+1,\gamma(2,\ldots,L),-p_1')\,, \tag{A.1}$$

where

$$\mathcal{S}(\sigma,\gamma)_{\ell_{L+1}} \equiv i\prod_{t=2}^{L}(2\ell_{\gamma(t)}\cdot\ell_{(L+1)} + \sum_{q<t}\theta(\gamma(q),\gamma(t))2\ell_{\gamma(t)}\cdot\ell_{\gamma(q)}), \tag{A.2}$$

is the momentum kernel and where $\theta(i_t,i_q)$ equals 1 if the ordering of the legs i_t and i_q is opposite in the sets $\{i_1,\ldots,i_k\}$ and $\{j_1,\ldots,j_k\}$, and 0 if the ordering is the same. This representation is convenient as no massive momenta enter the momentum kernel.

The colour-ordered Yang-Mills amplitudes are A_L. Following the flipping convention, we take $\hat{\ell}_{L+1} = -q-\sum_{j=2}^{L}\ell_j$, and know using the arguments of section 4 that the colour-ordered amplitudes (we have only $p_1\cdot k + i\epsilon$ propagators) satisfy

$$A_L(p_1,\sigma(2,\ldots,L),\widehat{L+1},-p_1') = A_L^+(p_1,\sigma(2,\ldots,L),\widehat{L+1},-p_1')\,, \tag{A.3}$$

while

$$A_L(p_1,\widehat{L+1},\sigma(2,\ldots,L),-p_1') = \sum_{k=1}^{L-1}\left(\prod_{\sum_{j=1}^{k+1}i_j=L}A_{i_j}^+\right)(\delta(\ldots))^k + A_L^+(p_1,\widehat{L+1},\sigma(2,\ldots,L),-p_1'), \tag{A.4}$$

where the product in the sum contains k tree-level amplitudes A_i^+. Hence we can write the generic M_L^{tree} amplitude in the form

$$M_L^{\text{tree}}(p_1,\ell_2,\ldots,\widehat{\ell_{L+1}},-p_1') = M_L^{\text{tree}(+)}(p_1,\ell_2,\ldots,\widehat{\ell_{L+1}},-p_1') - (-1)^L\sum_{\sigma,\gamma\in\mathfrak{S}_{L-1}}\mathcal{S}(\sigma,\gamma)_{\ell_{L+1}}\left(\sum_{k=1}^{L-1}(\delta(\ldots))^k\prod_{\sum_{j=1}^{k+1}i_j=L}A_{i_j}^+\right)A_L^+(p_1,\sigma(2,\ldots,L),\widehat{L+1},-p_1')\,, \tag{A.5}$$

where

$$M_L^{\text{tree}(+)}(p_1,\ell_2,\ldots,\hat{\ell}_{L+1},-p_1') = (-1)^{L-1}\sum_{\sigma,\gamma\in\mathfrak{S}_{L-1}}\mathcal{S}(\sigma,\gamma)_{\ell_{L+1}} \times A_L^+(p_1,\sigma(2,\ldots,L),\widehat{L+1},-p_1')A_L^+(p_1,\widehat{L+1},\gamma(2,\ldots,L),-p_1')\,. \tag{A.6}$$

Now considering the expression in (A.5) and picking the term

$$A_L(p_1,\widehat{L+1},L,\ldots,2,-p_1')\times A_L(p_1,2,\ldots,L,\widehat{L+1},-p_1')\,, \tag{A.7}$$

JHEP03(2022)071

we have propagator products such as

$$\frac{1}{(p_1+\hat{\ell}_{L+1}+\ell_L+\cdots+\ell_3)^2-m_1^2+i\epsilon}\times\frac{1}{(p_1+\ell_2)^2-m_1^2+i\epsilon}\,. \tag{A.8}$$

After flipping we have contributions with delta functions

$$-\frac{2\pi i\delta((p_1-\ell_2-q)^2-m_1^2)}{(p_1+\ell_2)^2-m_1^2+i\epsilon}+\frac{1}{(p_1-\ell_2-q)^2-m_1^2+i\epsilon}\frac{1}{(p_1+\ell_2)^2-m_1^2+i\epsilon}\,, \tag{A.9}$$

and the following soft scaling

$$\lim_{|\vec{q}|\to 0}\frac{\delta((p_1-|\vec{q}|\tilde{\ell}_2-q)^2-m_1^2)}{(p_1+|\vec{q}|\tilde{\ell}_2)^2-m_1^2+i\epsilon}\sim\frac{\delta(2p_1\cdot\tilde{\ell}_2-q\cdot\hat{\ell}_2)}{|\vec{q}|^2 2p_1\cdot\tilde{\ell}_2}\sim\mathcal{O}\left(\frac{1}{|\vec{q}|^2}\right)\,, \tag{A.10}$$

so that in the multi-soft limit $\ell_i=|q|\hat{\ell}_i$ and $|q|\to 0$, each delta function adds a $\frac{1}{|\vec{q}|}$ factor in the soft expansion. Consequently, each term of the previous sum scales as

$$\mathcal{S}(\sigma,\gamma)_{\ell_{L+1}}(\delta(\dots))^k\prod_{\sum_{j=1}^{k+1}i_j=L}A^+_{i_j}A^+_L\sim|\vec{q}|^{2L-2}\frac{1}{|\vec{q}|^{2k}}\frac{1}{|\vec{q}|^{L-1-k}}\frac{1}{|\vec{q}|^{L-1}}\sim\frac{1}{|\vec{q}|^k}, \tag{A.11}$$

and the multiple soft behaviour for $|q|\to 0$ of the plus amplitude is (for $k=0$)

$$\begin{aligned}
&\lim_{|q|\to 0}M_L^{\text{tree}(+)}(p_1,\ell_2,\dots,\hat{\ell}_{L+1},-p_1')\\
&\quad=(-1)^{L-1}\sum_{\sigma,\gamma\in\mathfrak{S}_{L-1}}\mathcal{S}(\sigma,\gamma)_{\ell_{L+1}}\,A_L^+(p_1,\sigma(2,\dots,L),\widehat{L+1},-p_1')\Big|_{|\vec{q}|\to 0}\\
&\qquad\qquad\times A_L^+(p_1,\widehat{L+1},\gamma(2,\dots,L),-p_1')\Big|_{|\vec{q}|\to 0}\\
&\quad\sim|\vec{q}|^0\,.
\end{aligned} \tag{A.12}$$

We arrive to the conclusion that the leading soft contribution of $M^{\text{tree}(+)}$ is of order $|\vec{q}|^0$ and can be expressed as a product of leading soft Yang-Mills amplitudes. It means that $M^{\text{tree}(+)}$ will have the same universality properties as the soft Yang-Mills amplitudes. $M^{\text{tree}(-)}$ sharing the same property, we conclude that the universality property will be transmitted to all classical integrands.

B Yang-Mills amplitudes and numerator factors

Following the construction presented in [53] we can construct symmetric numerators for scalar-gluon tree-level amplitudes.[5]

B.1 The three-point amplitude and numerator factors

For three-point scalar graviton amplitudes, we have,

$$N_1(p,\ell_2,-p')=i\sqrt{2}\,\zeta_2\cdot p,\quad A_1(p,\ell_2,-p')=N_1(p,\ell_2,-p'), \tag{B.1}$$

[5] An alternative construction of the numerators can be done using the tree-level BCJ master numerators derived from 10D pure-spinor formalism [69].

JHEP03(2022)071

B.2 The four-point amplitude and numerator factors

For the four-point amplitude and numerator factors we have

$$N_2(p,\ell_2,\ell_3,-p') = \frac{i}{2}\Big(s_{2\,p}(\zeta_2\cdot\zeta_3) - 4(\zeta_2\cdot p)\zeta_3\cdot(p+\ell_2)\Big)\,, \tag{B.2}$$

$$\begin{aligned} A_2(p,\ell_2,\ell_3,-p') &= \frac{N_2(p,\ell_2,\ell_3,-p')}{s_{2p}} + \frac{N_2(p,\ell_2,\ell_3,-p') - N_2(p,\ell_3,\ell_2,-p')}{s_{23}} \\ &= \frac{N_2(p,\ell_2,\ell_3,-p')}{s_{2p}} + \frac{N_2(p,[2,3],-p')}{s_{23}}. \end{aligned} \tag{B.3}$$

inspired by the compact notation of [70], and where we have defined $s_{ip} \equiv (p+\ell_i)^2 - m^2$, $s_{ij} \equiv (\ell_i+\ell_j)^2$, and

$$\begin{aligned} N_2(p,[2,3],-p') &\equiv N_2(p,\ell_2,\ell_3,-p') - N_2(p,\ell_3,\ell_2,-p') \\ &= \frac{1}{2}i((s_{2p}-s_{3p})\zeta_2\cdot\zeta_3 - 4(\zeta_3\cdot\ell_2)(\zeta_2\cdot p) + 4(\zeta_2\cdot\ell_3)(\zeta_3\cdot p))\,. \end{aligned} \tag{B.4}$$

B.3 The five-point amplitude and numerator factors

For the five-point amplitude and numerator factors we have

$$\begin{aligned} N_3(p,\ell_2,\ell_3,\ell_4,-p') = \frac{-i}{\sqrt{2}}\Big(&\frac{1}{3}(\zeta_3\cdot\zeta_4)(3(s_{23}+s_{3p})(\zeta_2\cdot p) - 2s_{2p}(\zeta_2\cdot\ell_3)) \\ &+ \frac{1}{3}s_{2p}(\zeta_2\cdot\zeta_4)(2(\zeta_3\cdot\ell_2)+3(\zeta_3\cdot p)) + \frac{1}{3}s_{2p}(\zeta_2\cdot\zeta_3)(2(\zeta_4\cdot\ell_2)+4(\zeta_4\cdot\ell_3)+3(\zeta_4\cdot p)) \\ &- 2(\zeta_2\cdot p)(\zeta_3\cdot(\ell_2+p))(\zeta_4\cdot(\ell_2+\ell_3+p))\Big)\,, \end{aligned} \tag{B.5}$$

with the colour-ordered amplitude

$$\mathcal{A}_3(p,\ell_2,\ell_3,\ell_4,-p') = \frac{{N_3}^{2,3,4}}{s_{2p}s_{23p}} + \frac{{N_3}^{[2,3],4}}{s_{23}s_{23p}} + \frac{{N_3}^{2,[3,4]}}{s_{34}s_{2p}} + \frac{{N_3}^{[2,[3,4]]}}{s_{34}s_{234}} + \frac{{N_3}^{[[2,3],4]}}{s_{23}s_{234}}\,, \tag{B.6}$$

where we have used the shorthand notation $N_3(p,\ell_2,\ell_3,\ell_4,-p') \equiv N_3^{2,3,4}$ and ${N_3}^{[2,3],4} \equiv {N_3}^{2,3,4} - {N_3}^{3,2,4}$, ${N_3}^{2,[3,4]} \equiv {N_3}^{2,3,4} - {N_3}^{2,4,3}$, ${N_3}^{[[2,3],4]} \equiv {N_3}^{[2,3],4} - {N_3}^{4,[2,3]}$ and ${N_3}^{[2,[3,4]]} \equiv {N_3}^{2,[3,4]} - {N_3}^{[3,4],2}$. We have defined $s_{i,\ldots,j,p} \equiv (p+\ell_i+\cdots+\ell_j)^2 - m^2$, $s_{i,\ldots,j} \equiv (\ell_i+\cdots+\ell_j)^2$.

B.4 The six-point amplitude and numerator factors

We give the expression for the colour-ordered six-point amplitude (emission of four gluons from a massive scalar) (the numerator factors are given in [71])

$$\begin{aligned} &\mathcal{A}_4(p,\ell_2,\ell_3,\ell_4,\ell_5,-p') \\ &= \frac{{N_4}^{2,3,4,5)}}{s_{2p}s_{23p}s_{234p}} + \frac{{N_4}^{2,3,[4,5]}}{s_{45}s_{2p}s_{23p}} + \frac{{N_4}^{2,[3,4],5)}}{s_{34}s_{2p}s_{234p}} + \frac{{N_4}^{[2,3],4,5)}}{s_{23}s_{23p}s_{234p}} \\ &\quad + \frac{{N_4}^{[2,3],[4,5]}}{s_{23}s_{45}s_{23p}} + \frac{{N_4}^{[[2,3],4,5)}}{s_{23}s_{234}s_{234p}} + \frac{{N_4}^{[2,[3,4]],5)}}{s_{34}s_{234}s_{234p}} + \frac{{N_4}^{2,[[3,4],5]}}{s_{34}s_{345}s_{2p}} + \frac{{N_4}^{2,[3,[4,5]]}}{s_{45}s_{345}s_{2p}} \\ &\quad + \frac{{N_4}^{[[2,3],[4,5]]}}{s_{23}s_{45}s_{2345}} + \frac{{N_4}^{[[[2,3],4],5]}}{s_{23}s_{234}s_{2345}} + \frac{{N_4}^{[[2,[3,4]],5]}}{s_{34}s_{234}s_{2345}} + \frac{{N_4}^{[2,[[3,4],5]]}}{s_{34}s_{345}s_{2345}} + \frac{{N_4}^{[2,[3,[4,5]]]}}{s_{45}s_{345}s_{2345}}\,. \end{aligned} \tag{B.7}$$

JHEP03(2022)071

B.5 The seven-point amplitude and numerator factors

We give the expression for the colour-ordered seven-point amplitude (emission of five gluons from a massive scalar) (the numerator factors are given in [71])

$$
\begin{aligned}
&\mathcal{A}_5(p,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6,-p') = \\
&\frac{N_5^{2,3,4,5,6}}{s_{2p}s_{23p}s_{234p}s_{2345p}} + \frac{N_5^{[2,3],4,5,6}}{s_{23}s_{23p}s_{234p}s_{2345p}} + \frac{N_5^{2,[3,4],5,6}}{s_{2p}s_{34}s_{234p}s_{2345p}} + \frac{N_5^{2,3,[4,5],6}}{s_{2p}s_{45}s_{23p}s_{2345p}} \\
&+ \frac{N_5^{2,3,4,[5,6]}}{s_{2p}s_{56}s_{23p}s_{234p}} + \frac{N_5^{[2,3],4,[5,6]}}{s_{23}s_{56}s_{23p}s_{234p}} + \frac{N_5^{2,[3,4],[5,6]}}{s_{2p}s_{34}s_{56}s_{234p}} + \frac{N_5^{2,3,[[4,5],6]}}{s_{2p}s_{45}s_{23p}s_{456}} \\
&+ \frac{N_5^{2,3,[4,[5,6]]}}{s_{2p}s_{56}s_{23p}s_{456}} + \frac{N_5^{[2,3],[4,5],6}}{s_{23}s_{45}s_{23p}s_{2345p}} + \frac{N_5^{[[2,3],4],5,6}}{s_{23}s_{234}s_{234p}s_{2345p}} + \frac{N_5^{[2,[3,4]],5,6}}{s_{34}s_{234}s_{234p}s_{2345p}} \\
&+ \frac{N_5^{2,[[3,4],5],6}}{s_{2p}s_{34}s_{345}s_{2345p}} + \frac{N_5^{2,[3,[4,5]],6}}{s_{2p}s_{45}s_{345}s_{2345p}} + \frac{N_5^{2,[[3,4],[5,6]}}{s_{2p}s_{34}s_{56}s_{3456}} + \frac{N_5^{[[2,3],[4,5]],6}}{s_{23}s_{45}s_{2345}s_{2345p}} \\
&+ \frac{N_5^{[2,3],[6,[5,4]]}}{s_{23}s_{45}s_{23p}s_{456}} + \frac{N_5^{[2,3],[4,[5,6]]}}{s_{23}s_{56}s_{23p}s_{456}} + \frac{N_5^{[[2,3],4],[5,6]}}{s_{23}s_{56}s_{234}s_{234p}} + \frac{N_5^{[2,[3,4]],[5,6]}}{s_{34}s_{56}s_{234}s_{234p}} \\
&+ \frac{N_5^{[[[2,3],4],5],6}}{s_{23}s_{234}s_{2345}s_{2345p}} + \frac{N_5^{[[2,[3,4]],5],6}}{s_{34}s_{234}s_{2345}s_{2345p}} + \frac{N_5^{[2,[[3,4],5]],6}}{s_{34}s_{345}s_{2345}s_{2345p}} + \frac{N_5^{[2,[3,[4,5]]],6}}{s_{45}s_{345}s_{2345}s_{2345p}} \\
&+ \frac{N_5^{2,[[[3,4],5],6]}}{s_{2p}s_{34}s_{345}s_{3456}} + \frac{N_5^{2,[[3,[4,5]],6]}}{s_{2p}s_{45}s_{345}s_{3456}} + \frac{N_5^{2,[3,[[4,5],6]]}}{s_{2p}s_{45}s_{456}s_{3456}} + \frac{N_5^{2,[3,[4,[5,6]]]}}{s_{2p}s_{56}s_{456}s_{3456}} \\
&+ \frac{N_5^{[[2,3],[[4,5],6]]}}{s_{23}s_{45}s_{456}s_{23456}} + \frac{N_5^{[[2,3],[4,[5,6]]]}}{s_{23}s_{56}s_{456}s_{23456}} + \frac{N_5^{[[[2,3],4],[5,6]]}}{s_{23}s_{56}s_{234}s_{23456}} + \frac{N_5^{[[2,[3,4]],[5,6]}}{s_{34}s_{56}s_{234}s_{23456}} \\
&+ \frac{N_5^{[[[[2,3],4],5],6]}}{s_{23}s_{45}s_{2345}s_{23456}} + \frac{N_5^{[[2,[[3,4],5],6]]}}{s_{34}s_{56}s_{3456}s_{23456}} + \frac{N_5^{[[[[2,3],4],5],6]}}{s_{23}s_{234}s_{2345}s_{23456}} + \frac{N_5^{[[[2,[3,4]],5],6]}}{s_{34}s_{234}s_{2345}s_{23456}} \\
&+ \frac{N_5^{[[2,[[3,4],5]],6]}}{s_{34}s_{345}s_{2345}s_{23456}} + \frac{N_5^{[[2,[3,[4,5]]],6]}}{s_{45}s_{345}s_{2345}s_{23456}} + \frac{N_5^{[2,[[[3,4],5],6]]}}{s_{34}s_{345}s_{3456}s_{23456}} + \frac{N_5^{[2,[[3,[4,5]],6]]}}{s_{45}s_{345}s_{3456}s_{23456}} \\
&+ \frac{N_5^{[2,[3,[[4,5],6]]]}}{s_{45}s_{456}s_{3456}s_{23456}} + \frac{N_5^{[2,[3,[4,[5,6]]]]}}{s_{56}s_{456}s_{3456}s_{23456}} \,. \qquad \text{(B.8)}
\end{aligned}
$$

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JHEP03(2022)071

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JHEP03(2022)071