

Symmetries of $\mathcal{N} = (1, 0)$ supergravity backgrounds in six dimensions

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ABSTRACT: General $\mathcal{N} = (1, 0)$ supergravity-matter systems in six dimensions may be described using one of the two fully fledged superspace formulations for conformal supergravity: (i) $SU(2)$ superspace; and (ii) conformal superspace. With motivation to develop rigid supersymmetric field theories in curved space, this paper is devoted to the study of the geometric symmetries of supergravity backgrounds. In particular, we introduce the notion of a conformal Killing spinor superfield ϵ^α , which proves to generate extended superconformal transformations. Among its cousins are the conformal Killing vector ξ^a and tensor $\zeta^{a(n)}$ superfields. The former parametrise conformal isometries of supergravity backgrounds, which in turn yield symmetries of every superconformal field theory. Meanwhile, the conformal Killing tensors of a given background are associated with higher symmetries of the hypermultiplet. By studying the higher symmetries of a non-conformal vector multiplet we introduce the concept of a Killing tensor superfield. We also analyse the problem of computing higher symmetries for the conformal d'Alembertian in curved space and demonstrate that, beyond the first-order case, these operators are defined only on a limited class of backgrounds, including all conformally flat ones.

KEYWORDS: Extended Supersymmetry, Higher Spin Symmetry, Supergravity Models, Superspaces

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Dedicated to Jim Gates on the occasion of his 70th birthday

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1 Introduction

The superconformal tensor calculus for $\mathcal{N} = (1, 0)$ supergravity in six dimensions was formulated by Bergshoeff, Sezgin and Van Proeyen in 1986 [1], as a natural generalisation of that for $d = 4$, $\mathcal{N} = 2$ supergravity [2–7]. More recently it was further developed [8, 9], including the construction of the complete off-shell action for minimal Poincaré supergravity [8] and a higher-derivative extension of chiral gauged supergravity [9], see [10] for a pedagogical review.

The tensor calculus [1] has found numerous applications, in particular the explicit construction of off-shell curvature squared supergravity actions [9, 11–13]. It is a powerful approach to formulate supergravity-matter systems. However, similar to its $d = 4$, $\mathcal{N} = 2$ and $d = 5$, $\mathcal{N} = 1$ cousins, it has two limitations. Firstly, it does not offer tools to describe off-shell charged hypermultiplets. Secondly, it is rather impractical from the point of view of constructing nonlinear supergravity actions such as invariants for conformal supergravity, see, e.g., [14] for a related discussion. These limitations are avoided by resorting to superspace techniques. There exist two fully fledged superspace formulations for $\mathcal{N} = (1, 0)$ conformal supergravity and its general off-shell couplings to supersymmetric matter: (i) SU(2) superspace [15]; and (ii) conformal superspace [16]. Both formulations have analogues in $d < 6$ dimensions.

The SU(2) superspace of [15] is a particular $d = 6$ realisation of the general approach to formulate \mathcal{N} -extended conformal supergravity in $3 \leq d \leq 6$ dimensions using the so-called $G_R[d; \mathcal{N}]$ superspace, where $G_R[d; \mathcal{N}]$ is the R -symmetry subgroup of the \mathcal{N} -extended superconformal group in d dimensions.¹ By definition, $G_R[d; \mathcal{N}]$ superspace is a supermanifold $\mathcal{M}^{d|\delta_{\mathcal{N}}}$, with d bosonic and $\delta_{\mathcal{N}}$ fermionic dimensions.² Its structure group is $\text{Spin}(d-1, 1) \times G_R[d; \mathcal{N}]$, where $\text{Spin}(d-1, 1)$ is the double covering group of the connected Lorentz group $\text{SO}_0(d-1, 1)$. This means that the differential geometry of $\mathcal{M}^{d|\delta_{\mathcal{N}}}$ is realised in terms of covariant derivatives of the form

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_{\hat{\alpha}}) = E_A - \Omega_A - \Phi_A . \quad (1.1)$$

Here $E_A = E_A^M \partial / \partial z^M$ denotes the inverse superspace vielbein, $\Omega_A = \frac{1}{2} \Omega_A^{bc} M_{bc}$ is the Lorentz connection, and $\Phi = \Phi_A^I J_I$ the R -symmetry connection. The index $\hat{\alpha}$ of the fermionic operator $\mathcal{D}_{\hat{\alpha}}$ is, in general, composite; it is comprised of a spinor index α and an R -symmetry index. The supergravity gauge group includes a subgroup generated by local transformations

$$\delta_{\mathcal{K}} \mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A], \quad \mathcal{K} := \xi^B \mathcal{D}_B + \frac{1}{2} K^{bc} M_{bc} + K^I J_I, \quad (1.2)$$

where the gauge parameters ξ^A , $K^{bc} = -K^{cb}$ and K^I obey standard reality conditions but are otherwise arbitrary. Given a tensor superfield φ (with suppressed Lorentz and R -symmetry indices), its transformation law under (1.2) is $\delta_{\mathcal{K}} \varphi = \mathcal{K} \varphi$.

¹According to the Nahm classification [17], superconformal algebras exist in spacetime dimensions $d \leq 6$. The $d = 5$ case is truly exceptional, for it allows the existence of the unique superconformal algebra $\mathfrak{F}(4)$.

²Here $\delta_{\mathcal{N}} = 2^{\lfloor d/2 \rfloor} \mathcal{N}$ for $d = 3, 4$ and 6 , and $\delta_{\mathcal{N}} = 8$ for $d = 5$. We denote by $z^M = (x^m, \theta^{\hat{\mu}})$ the local coordinates for $\mathcal{M}^{d|\delta_{\mathcal{N}}}$. Without loss of generality, we assume that the zero section of $\mathcal{M}^{d|\delta_{\mathcal{N}}}$ defined by $\theta^{\hat{\mu}} = 0$ corresponds to the spacetime manifold \mathcal{M}^d .

In order to describe conformal supergravity, the superspace torsion $T_{AB}{}^C$ in

$$[\mathcal{D}_A, \mathcal{D}_B] = -T_{AB}{}^C \mathcal{D}_C - \frac{1}{2} R_{AB}{}^{cd} M_{cd} - R_{AB}{}^I J_I, \quad (1.3)$$

must obey certain algebraic constraints, which may be thought of as generalisations of the torsion-free constraint in gravity. A fundamental requirement on the superspace geometry, in order to describe conformal supergravity, is that the constraints on the torsion be invariant under super-Weyl transformations of the form

$$\delta_\sigma \mathcal{D}_a = \sigma \mathcal{D}_a + \dots, \quad \delta_\sigma \mathcal{D}_{\hat{a}} = \frac{1}{2} \sigma \mathcal{D}_{\hat{a}} + \dots, \quad (1.4)$$

where the scale parameter σ is an arbitrary real superfield. The ellipsis in the expression for $\delta_\sigma \mathcal{D}_a$ includes, in general, a linear combination of the spinor covariant derivatives $\mathcal{D}_{\hat{\beta}}$ and the structure group generators M_{cd} and J_K . The ellipsis in $\delta_\sigma \mathcal{D}_{\hat{a}}$ stands for a linear combination of the generators of the structure group. The resulting curved superspace will be denoted $(\mathcal{M}^{d|\delta\mathcal{N}}, \mathcal{D})$. In many dynamical systems of interest, matter superfields may be chosen to be primary under the super-Weyl group, $\delta_\sigma \varphi = w_\sigma \varphi$, where the parameter w_σ is the super-Weyl weight of φ .

The approach sketched above was pioneered in $d = 4$ by Howe [18, 19] who put forward the concept of $U(\mathcal{N})$ superspace. In particular, he introduced the $U(1)$ and $U(2)$ superspace geometries [19], corresponding to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ conformal supergravity, respectively. Howe's analysis was purely geometric in the sense that he did not address the problem of constructing supergravity-matter actions. The full power of $U(1)$ superspace was revealed in the book [20], which provided a unified description of the off-shell formulations for $\mathcal{N} = 1$ supergravity and their couplings to matter. General off-shell $\mathcal{N} = 2$ supergravity-matter systems in $d = 4$ were constructed in $U(2)$ superspace in [21], building on the concepts of rigid projective superspace [22–24] and superconformal projective multiplets [25, 26]. The four-dimensional results of [21] provided a natural extension of the earlier construction of the $SU(2)$ superspace formalism in five dimensions [27]. The $d = 3$ realisation of $G_R[d; \mathcal{N}]$ superspace is known as $SO(\mathcal{N})$ superspace. Its geometry was developed in [28, 29]. This formalism was used in [29] to construct off-shell supergravity-matter couplings for $\mathcal{N} \leq 4$.

As compared with the $d = 6$, $\mathcal{N} = (1, 0)$ superconformal tensor calculus of [1], the important advantage of the $SU(2)$ superspace approach [15] is that it offered off-shell formulations for general supersymmetric nonlinear σ -models coupled to supergravity.³ This was achieved by making use of the concept of covariant projective supermultiplets.⁴

The superspace formalism of [16] is a particular $d = 6$ realisation of the universal approach to \mathcal{N} -extended conformal supergravity in $d \leq 6$ dimensions, which is based on gauging the entire \mathcal{N} -extended superconformal group, of which $\text{Spin}(d-1, 1) \times G_R[d; \mathcal{N}]$ is a subgroup. This approach, known as conformal superspace, was originally developed for

³The component reduction of these locally supersymmetric σ -models can be carried out using the techniques developed by Butter in the $d = 4$, $\mathcal{N} = 2$ case [30, 31].

⁴The concept of covariant projective supermultiplets was introduced earlier in $d < 6$ dimensions, first in the framework of $d = 5$, $\mathcal{N} = 1$ [27, 32, 33], followed by $d = 4$, $\mathcal{N} = 2$ [21, 34], then in $d = 3$, $\mathcal{N} = 3$ and $\mathcal{N} = 4$ [29], and finally in $d = 2$, $\mathcal{N} = (4, 4)$ supergravity [35, 36].

$\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravity theories in four dimensions by Butter [37, 38]. More recently, it has been extended to the cases of $d = 3$, \mathcal{N} -extended conformal supergravity [39], $d = 5$ conformal supergravity [40], and $d = 6$, $\mathcal{N} = (1, 0)$ conformal supergravity [16]. Conceptually, conformal superspace is a superspace analogue of the famous formulation for conformal (super)gravity as the gauge theory of the (super)conformal group pioneered by Kaku, Townsend and van Nieuwenhuizen [41, 42], and further developed by Kugo and Uehara [43].

One of the important achievements of the conformal superspace approach [16] is that it provided the first ever construction of all the invariants for $\mathcal{N} = (1, 0)$ conformal supergravity in six dimensions.⁵ Several months later, these invariants were reduced to components in [45], which resulted in the first tensor calculus description of the conformal supergravity actions. Conformal superspace has also been used to describe the supersymmetric completion of several curvature-squared invariants for $\mathcal{N} = (1, 0)$ supergravity in six dimensions [46, 47].

Conformal superspace is an ultimate formulation for conformal supergravity in the sense that any different off-shell formulation is either equivalent to it or is obtained from it by partially fixing the gauge freedom. In particular, $G_R[d; \mathcal{N}]$ superspace can be obtained from a partial gauge fixing of conformal superspace, see [37–40] for the technical details. In the case of six dimensions, it was demonstrated in [45] that the $\mathcal{N} = (1, 0)$ superconformal tensor calculus of [1] is a gauged fixed version of the conformal superspace developed in [16].

Recently, local supertwistor formulations for $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (2, 0)$ conformal supergravity in six dimensions have been constructed [48], and analogous formulations have been proposed in diverse dimensions [49]. Ref. [48] offered the first superspace description of the $\mathcal{N} = (2, 0)$ Weyl supermultiplet, which was originally formulated using the superconformal tensor calculus [50]. In accordance with the above discussion, the local supertwistor formulation should be equivalent to conformal superspace.⁶ The latter is at present much more developed and is thus the one favoured in this paper. We should also mention that the harmonic superspace formulation for $\mathcal{N} = (1, 0)$ conformal supergravity was briefly described in [52]. Unfortunately, this approach has not been pursued for over thirty years.

The present work is devoted to new applications of the supergravity formulations [15, 16]. Their fundamental property is that they offer a universal setting to generate off-shell supersymmetric field theories in curved space. In particular, all $\mathcal{N} = (1, 0)$ supersymmetric theories that were originally constructed in terms of ordinary fields, may be read off from a superfield theory upon elimination of the auxiliary fields. In order to develop supersymmetric field theory in a given supergravity background, one needs a formalism to determine the (conformal) isometries of the background superspace. Such a formalism was developed long ago [53] within the framework of $d = 4$, $\mathcal{N} = 1$ old minimal supergravity. The approach described in [53] is universal, for in principle it may be generalised to curved backgrounds associated with any supergravity theory formulated in superspace, see

⁵A simple by-product of the analysis in [16] was the first construction of the locally supersymmetric $F \square F$ action coupled to conformal supergravity. In Minkowski space, the $\mathcal{N} = (1, 0)$ supersymmetric $F \square F$ action was described for the first time in [44] within the harmonic superspace approach.

⁶Both constructions are based on Cartan connections, first discussed in the superspace context in [51].

the discussion in [54]. In particular, this approach has been properly generalised to study supersymmetric backgrounds in $\mathcal{N} = 2$ supergravity in three [55] and four [56] dimensions, and $\mathcal{N} = 1$ supergravity in five dimensions [57]. One of the goals of this paper is to work out the structure of (conformal) isometries of a given $\mathcal{N} = (1, 0)$ supergravity background in six dimensions.

Within the $G_R[d; \mathcal{N}]$ superspace formulation, there exists a universal description of all conformal isometries of a given curved background $(\mathcal{M}^{d|\delta\mathcal{N}}, \mathcal{D})$. Following the discussion in [54], a real supervector field $\xi = \xi^B E_B$ on $(\mathcal{M}^{d|\delta\mathcal{N}}, \mathcal{D})$ is called conformal Killing if

$$(\delta_{\mathcal{K}} + \delta_{\sigma})\mathcal{D}_A = 0, \tag{1.5}$$

for some Lorentz K^{bc} , R -symmetry K^I and super-Weyl σ parameters. For any dimension $3 \leq d \leq 6$ and any conformal supergravity, the following general properties are expected to hold:

- All parameters K^{bc} , K^I and σ are uniquely determined in terms of ξ^B , which allows us to write $K^{bc} = K^{bc}[\xi]$, $K^I = K^I[\xi]$ and $\sigma = \sigma[\xi]$.
- The spinor component $\xi^{\hat{\beta}}$ is uniquely determined in terms of ξ^b .
- The vector component ξ^b obeys a closed equation that contains all information about the conformal Killing supervector field.

The properties have been established for $d < 6$ in several publications [53, 55–57]. The $d = 6$, $\mathcal{N} = (1, 0)$ case will be studied in this paper. By construction, the set of conformal Killing vectors on $(\mathcal{M}^{d|\delta\mathcal{N}}, \mathcal{D})$ is a Lie superalgebra with respect to the standard Lie bracket. This is the superconformal algebra of $(\mathcal{M}^{d|\delta\mathcal{N}}, \mathcal{D})$. One may show that it is finite-dimensional. In the $d = 6$, $\mathcal{N} = (1, 0)$ case, the proof will be given in section 3.

Given a conformal Killing supervector field ξ^A on $(\mathcal{M}^{d|\delta\mathcal{N}}, \mathcal{D})$, the first-order operator $\mathfrak{D}_{\xi}^{(1)} = \mathcal{K}[\xi] + \delta_{\sigma[\xi]}$ is a symmetry of any supersymmetric wave equation $\mathcal{O}\varphi = 0$, where \mathcal{O} is the kinetic operator for some matter supermultiplets φ . For every solution φ of the mass-shell equation, $\mathfrak{D}_{\xi}^{(1)}\varphi$ is also a solution. It is of interest to study higher symmetries of supersymmetric wave equations, n^{th} -order operators $\mathfrak{D}_{\xi}^{(n)}$ taking solutions to solutions, for instance in the context of higher-spin superalgebras [58–62]. Higher symmetries of relativistic wave equations have extensively been studied in the literature, see, e.g., [63–71] and references therein. In the supersymmetric case, however, the program of studying the higher symmetries of the so-called super-Laplacians and related geometric structures in diverse dimensions has been initiated only a few years ago [72–74], mostly in Minkowski superspace $(\mathbb{M}^{d|\delta\mathcal{N}}, \mathcal{D})$. So far there has been only one publication [75] devoted to the higher symmetries of supersymmetric wave equations in curved supergravity backgrounds. The present paper is aimed, in part, at a study of the higher symmetries of several on-shell supermultiplets in a background of $\mathcal{N} = (1, 0)$ conformal supergravity in six dimensions. Their non-supersymmetric analogues are also examined in diverse dimensions, and bring with them new insights for the supersymmetric story.

This paper is organised as follows. Section 2 reviews the SU(2) and conformal superspace formulations for conformal supergravity. The conformal isometries of a fixed superspace are then studied in section 3. In section 4, we introduce the notion of a conformal Killing spinor superfield, which generates extended superconformal transformations. By a systematic study, it is shown that among its cousins are the conformal Killing vectors and tensors, which generate conformal isometries and higher symmetries, respectively. In section 5 we review the higher symmetries of the conformal d’Alembertian and present some new observations pertinent to the supersymmetric story. Following this, we study the higher symmetries of the hypermultiplet and vector multiplet in sections 6 and 7, respectively. Section 8 is devoted to the study of $\mathcal{N} = (1, 0)$ maximally supersymmetric backgrounds. Concluding comments are given in section 9.

The main body of this paper is accompanied by several technical appendices. Appendix A recounts our conventions. We review the conformal Killing supervector fields of $\mathcal{N} = (2, 0)$ Minkowski superspace in appendix B. In appendix C, we detail a formalism for the study of supersymmetric backgrounds from a superspace perspective. Finally, in appendix D, we detail how to ‘degauged’ from conformal to SU(2) superspace.

2 Conformal supergravity in superspace

As discussed in the introduction, there exist two fully fledged superspace formulations for $\mathcal{N} = (1, 0)$ conformal supergravity and its couplings to supersymmetric matter. In the literature they are referred to as (i) SU(2) superspace [15]; and (ii) conformal superspace [16]. Since both approaches will be used in the present paper, in this section we briefly review these formulations.

2.1 SU(2) superspace

We consider a supermanifold $\mathcal{M}^{6|8}$ parametrised by six bosonic (x) and eight fermionic (θ) coordinates $z^M = (x^m, \theta^\mu)$, where $m = 0, 1, \dots, 5$, $\mu = 1, \dots, 4$ and $\iota = \underline{1}, \underline{2}$. The name ‘‘SU(2) superspace’’ derives from the fact that its structure group, $\text{Spin}(5, 1) \times \text{SU}(2)_R$, includes the R -symmetry group $\text{SU}(2)_R$ in addition to the spin group. Therefore, the superspace covariant derivatives, $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha^i)$, have the form

$$\mathcal{D}_A = E_A - \frac{1}{2} \Omega_A{}^{bc} M_{bc} - \Phi_A{}^{jk} J_{jk} . \tag{2.1}$$

Here $E_A = E_A{}^M \partial_M$ is the frame field, with $E_A{}^M$ being the inverse superspace vielbein, $\Omega_A{}^{bc}$ the Lorentz connection, and $\Phi_A{}^{ij}$ the $\text{SU}(2)_R$ connection. The Lorentz (M_{ab}) and the R -symmetry (J^{ij}) generators are defined to act on Weyl spinors, vectors and isospinors as follows:

$$M_\alpha{}^\beta \psi^\gamma = \delta_\alpha^\gamma \psi^\beta - \frac{1}{4} \delta_\alpha^\beta \psi^\gamma , \qquad M_\alpha{}^\beta \psi_\gamma = \frac{1}{4} \delta_\alpha^\beta \psi_\gamma - \delta_\gamma^\beta \psi_\alpha , \tag{2.2a}$$

$$M_{ab} V_c = 2\eta_{c[a} V_{b]} , \qquad J^{ij} \chi^k = \varepsilon^{k(i} \chi^{j)} , \tag{2.2b}$$

where the Lorentz generator with spinor indices, $M_\alpha{}^\beta$, is defined in accordance with the general rule (A.19), $M_\alpha{}^\beta = -\frac{1}{4} (\gamma^{ab})_\alpha{}^\beta M_{ab}$. For further details regarding our spinor conventions we refer the reader to appendix A.1.

The covariant derivatives are characterised by graded commutation relations

$$\{\mathcal{D}_A, \mathcal{D}_B\} = -T_{AB}{}^C \mathcal{D}_C - \frac{1}{2} R_{AB}{}^{cd} M_{cd} - R_{AB}{}^{kl} J_{kl}, \quad (2.3)$$

where $T_{AB}{}^C$ is the torsion, $R_{AB}{}^{cd}$ the Lorentz curvature, and $R_{AB}{}^{kl}$ the R -symmetry curvature. In order to describe conformal supergravity, the torsion must obey certain constraints [15]

$$T_{\alpha\beta}{}^{jc} = 2i\varepsilon^{ij}(\gamma^c)_{\alpha\beta}, \quad (\text{dimension-0}) \quad (2.4a)$$

$$T_{\alpha\beta k}{}^j{}^\gamma = 0, \quad T_{\alpha b}{}^c = 0, \quad (\text{dimension-}\frac{1}{2}) \quad (2.4b)$$

$$T_{ab}{}^c = 0, \quad T_{a\beta(j k)}{}^\beta = 0. \quad (\text{dimension-1}) \quad (2.4c)$$

Their general solution is given by the relations

$$\begin{aligned} \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} &= -2i\varepsilon^{ij}(\gamma^a)_{\alpha\beta} \mathcal{D}_a + 2iC_a^{ij}(\gamma^{abc})_{\alpha\beta} M_{bc} - 2i\varepsilon^{ij} W^{abc}(\gamma_a)_{\alpha\beta} M_{bc} \\ &\quad - 4i\varepsilon^{ij} N^{abc}(\gamma_a)_{\alpha\beta} M_{bc} + 6i\varepsilon^{ij} C_a^{kl}(\gamma^a)_{\alpha\beta} J_{kl} - \frac{8i}{3} N^{abc}(\gamma_{abc})_{\alpha\beta} J^{ij}, \end{aligned} \quad (2.5a)$$

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_\beta^j] &= C^{bj}{}_k(\gamma_{ab})_\beta{}^\delta \mathcal{D}_\delta^k - \frac{1}{4} W_{acd}(\gamma^{cd})_\beta{}^\delta \mathcal{D}_\delta^j - N_{acd}(\gamma^{cd})_\beta{}^\delta \mathcal{D}_\delta^j \\ &\quad - \frac{1}{2} R_{a\beta}{}^{jcd} M_{cd} - R_{a\beta}{}^{jkl} J_{kl}, \end{aligned} \quad (2.5b)$$

where the curvature tensors in the second line of (2.5b) have the following form:

$$\begin{aligned} R_{a\beta}{}^{jcd} &= -\frac{1}{4} \left[(\gamma_a{}^{cd})_{\gamma\delta} \delta_\beta^\rho - 2(\gamma_a)_{\beta\gamma}(\gamma^{cd})_{\delta\rho} \right] \mathcal{W}_\rho{}^{\gamma\delta j} \\ &\quad - \left[(\gamma_a{}^{cd})_{\beta\gamma} + 4\delta_a^{[c}(\gamma^{d]}_{\beta\gamma} \right] \left(\frac{1}{12} \mathcal{W}^{\gamma j} - \mathcal{C}^{\gamma j} \right) \\ &\quad + \left[(\gamma^{cd})_\beta{}^\gamma \delta_a^b + 2\delta_\beta^\gamma \delta_a^{[c} \eta^{d]b} \right] \left(\mathcal{N}_{b\gamma}{}^j - \mathcal{C}_{b\gamma}{}^j \right), \end{aligned} \quad (2.6a)$$

$$\begin{aligned} R_{a\beta}{}^{jkl} &= -(\gamma_a)_{\beta\gamma} \mathcal{C}^{\gamma jkl} - 5(\gamma_a)_{\beta\gamma} \left(\mathcal{C}^{\gamma(k} - \frac{1}{6} \mathcal{W}^{\gamma(k} \right) \varepsilon^{l)j} \\ &\quad - \left(4\mathcal{N}_{a\beta}{}^{(k} - 3\mathcal{C}_{a\beta}{}^{(k} \right) \varepsilon^{l)j}. \end{aligned} \quad (2.6b)$$

The algebra of the covariant derivatives is determined by three dimension-1 real tensors, $W_{abc} = W_{[abc]}$, $N_{abc} = N_{[abc]}$ and $C_a^{ij} = C_a^{ji}$, and their covariant derivatives. The 3-forms N_{abc} and W_{abc} are self-dual and anti-self-dual, respectively,

$$\frac{1}{3!} \varepsilon^{abcdef} N_{def} = N^{abc}, \quad \frac{1}{3!} \varepsilon^{abcdef} W_{def} = -W^{abc}. \quad (2.7)$$

They are equivalently described in terms of the symmetric chiral rank-2 spinors $W^{\alpha\beta} := \frac{1}{6} W_{abc}(\tilde{\gamma}^{abc})^{\alpha\beta}$ and $N_{\alpha\beta} := \frac{1}{6} N_{abc}(\gamma^{abc})_{\alpha\beta}$.

The curvature tensors (2.5a) and (2.5b) involve several dimension-3/2 descendants of $C_{a ij}$, $N_{\alpha\beta}$ and $W^{\alpha\beta}$, defined by

$$\mathcal{D}_{\gamma k} C_{a ij} = (\gamma_a)_{\gamma\delta} \mathcal{C}_{ijk}^\delta + \varepsilon_{k(i} \mathcal{C}_{a\gamma j)} + \varepsilon_{k(i} (\gamma_a)_{\gamma\delta} \mathcal{C}_{j)}^\delta, \quad (2.8a)$$

$$\mathcal{D}_{\gamma k} N_{\alpha\beta} = \frac{2}{3} \left(\mathcal{D}_{[\gamma}^k N_{\alpha]\beta} + \mathcal{D}_{[\gamma}^k N_{\beta]\alpha} \right) := (\gamma^a)_{\gamma(\alpha} \mathcal{N}_{a\beta)}{}^k, \quad (\gamma^a)_{[\gamma\alpha} \mathcal{N}_{a\beta]}{}^k = 0, \quad (2.8b)$$

$$\mathcal{D}_{\gamma k} W^{\alpha\beta} = \mathcal{W}_{\gamma k}{}^{\alpha\beta} + \delta_\gamma^{(\alpha} \mathcal{W}_k{}^{\beta)}. \quad (2.8c)$$

In accordance with the general discussion in section 1, the curved superspace introduced above will be denoted $(\mathcal{M}^{6|8}, \mathcal{D})$.

In SU(2) superspace, the gauge group of conformal supergravity is generated by three types of local transformations: (i) general coordinate transformations; (ii) structure group transformations; and (iii) super-Weyl transformations. An infinitesimal transformation of the combined type (i) and (ii) acts on the covariant derivatives as

$$\delta_{\mathcal{K}} \mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A], \quad \mathcal{K} = K^C \mathcal{D}_C + \frac{1}{2} K^{cd} M_{cd} + K^{kl} J_{kl}. \quad (2.9)$$

Given a tensor superfield U (with its indices suppressed), its transformation law with respect to (2.9) is

$$\delta_{\mathcal{K}} U = \mathcal{K} U. \quad (2.10)$$

An infinitesimal super-Weyl transformation of the covariant derivative [15] is

$$\delta_{\sigma} \mathcal{D}_{\alpha}^i = \frac{1}{2} \sigma \mathcal{D}_{\alpha}^i - 2(\mathcal{D}_{\beta}^i \sigma) M_{\alpha}^{\beta} - 4(\mathcal{D}_{\alpha j} \sigma) J^{ij}, \quad (2.11a)$$

$$\delta_{\sigma} \mathcal{D}_a = \sigma \mathcal{D}_a - \frac{i}{2} (\tilde{\gamma}_a)^{\alpha\beta} (\mathcal{D}_{\alpha}^k \sigma) \mathcal{D}_{\beta k} - (\mathcal{D}^b \sigma) M_{ab} - \frac{i}{8} (\tilde{\gamma}_a)^{\alpha\beta} (\mathcal{D}_{\alpha}^k \mathcal{D}_{\beta}^l \sigma) J_{kl}, \quad (2.11b)$$

where the real parameter σ is unconstrained. The crucial feature of these transformations is that they preserve the supergravity constraints (2.4).

A tensor superfield U is said to be *primary* of Weyl weight (or dimension) w if it transforms homogeneously under (2.11)

$$\delta_{\sigma} U = w \sigma U. \quad (2.12)$$

The torsion W_{abc} proves to be a primary superfield of dimension +1. It is the $\mathcal{N} = (1, 0)$ supersymmetric extension of the Weyl tensor [15, 76].

In what follows, we will need a finite form of the super-Weyl transformations (2.11). Direct calculations lead to

$$\mathcal{D}'_{\alpha}{}^i = e^{\frac{1}{2}\sigma} \left(\mathcal{D}_{\alpha}^i - 2(\mathcal{D}_{\beta}^i \sigma) M_{\alpha}^{\beta} - 4(\mathcal{D}_{\alpha j} \sigma) J^{ij} \right), \quad (2.13a)$$

$$\begin{aligned} \mathcal{D}'_a = e^{\sigma} & \left(\mathcal{D}_a - \frac{i}{2} (\tilde{\gamma}_a)^{\alpha\beta} (\mathcal{D}_{\alpha}^k \sigma) \mathcal{D}_{\beta k} - (\mathcal{D}^b \sigma) M_{ab} - \frac{i}{8} (\tilde{\gamma}_a)^{\alpha\beta} (\mathcal{D}_{\alpha}^k \mathcal{D}_{\beta}^l \sigma) J_{kl} \right. \\ & \left. - \frac{i}{8} (\tilde{\gamma}_a{}^{cd})^{\alpha\beta} (\mathcal{D}_{\alpha}^k \sigma) (\mathcal{D}_{\beta k} \sigma) M_{cd} - \frac{3i}{4} (\tilde{\gamma}_a)^{\alpha\beta} (\mathcal{D}_{\alpha}^k \sigma) (\mathcal{D}_{\beta}^l \sigma) J_{kl} \right). \end{aligned} \quad (2.13b)$$

Such a transformation acts on the dimension-1 torsion superfields as follows:

$$W'_{abc} = e^{\sigma} W_{abc}, \quad (2.14a)$$

$$N'_{abc} = e^{\sigma} \left(N_{abc} - \frac{i}{32} (\tilde{\gamma}_{abc})^{\alpha\beta} (\mathcal{D}_{\alpha}^k \mathcal{D}_{\beta k} \sigma + 4(\mathcal{D}_{\alpha}^k \sigma) \mathcal{D}_{\beta k} \sigma) \right), \quad (2.14b)$$

$$C'_a{}^{ij} = e^{\sigma} \left(C_a{}^{ij} + \frac{i}{8} (\tilde{\gamma}_a)^{\alpha\beta} (\mathcal{D}_{\alpha}^{(i} \mathcal{D}_{\beta}^{j)} \sigma - 2(\mathcal{D}_{\alpha}^{(i} \sigma) \mathcal{D}_{\beta}^{j)} \sigma) \right). \quad (2.14c)$$

2.2 Conformal superspace

In the conformal superspace approach of [16] (see also [45] for the component analysis) the whole superconformal algebra is gauged in superspace by introducing covariant derivatives $\nabla_A = (\nabla_a, \nabla_\alpha^i)$ of the following form

$$\nabla_A = E_A - \frac{1}{2}\Omega_A{}^{bc}M_{bc} - \Phi_A{}^{kl}J_{kl} - B_A\mathbb{D} - \mathfrak{F}_{AB}K^B. \quad (2.15)$$

The difference compared with the SU(2) superspace covariant derivatives of (2.1) is the presence of dilatation (B_A) and special conformal (\mathfrak{F}_{AB}) connections, where \mathbb{D} is the dilatation generator and $K^A = (K^a, S_i^\alpha)$ are the special conformal generators. The complete list of graded commutation relations defining the $\mathcal{N} = (1, 0)$ superconformal algebra are given in appendix A.2.

To describe the standard $\mathcal{N} = (1, 0)$ Weyl multiplet in conformal superspace, one constrains the algebra of covariant derivatives

$$\begin{aligned} [\nabla_A, \nabla_B] = & -\mathcal{I}_{AB}{}^C\nabla_C - \frac{1}{2}\mathcal{R}(M)_{AB}{}^{cd}M_{cd} - \mathcal{R}(J)_{AB}{}^{kl}J_{kl} \\ & - \mathcal{R}(\mathbb{D})_{AB}\mathbb{D} - \mathcal{R}(K)_{ABC}K^C, \end{aligned} \quad (2.16)$$

to be completely determined in terms of the super-Weyl tensor $W^{\alpha\beta}$

$$K^A W^{\alpha\beta} = 0, \quad \mathbb{D}W^{\alpha\beta} = W^{\alpha\beta}, \quad (2.17)$$

which satisfies the constraints

$$\nabla_\alpha^{(i}\nabla_\beta^{j)}W^{\gamma\delta} = -\delta_{[\alpha}^{(\gamma}\nabla_{\beta]}^{(i}\nabla_\rho^{j)}W^{\delta)\rho}, \quad (2.18a)$$

$$\nabla_\alpha^k\nabla_{\gamma k}W^{\beta\gamma} - \frac{1}{4}\delta_\alpha^\beta\nabla_\gamma^k\nabla_{\delta k}W^{\gamma\delta} = 8i\nabla_{\alpha\gamma}W^{\gamma\beta}. \quad (2.18b)$$

Additionally, we require that the algebra of covariant derivatives resembles a $d = 6$, $\mathcal{N} = (1, 0)$ super Yang-Mills theory

$$\{\nabla_\alpha^i, \nabla_\beta^j\} = -2i\varepsilon^{ij}(\gamma^a)_{\alpha\beta}\nabla_a, \quad [\nabla_a, \nabla_\alpha^i] = (\gamma_a)_{\alpha\beta}\mathscr{W}^{\beta i}. \quad (2.19)$$

Here $\mathscr{W}^{\alpha i}$ is a primary dimension 3/2 operator valued in the superconformal algebra. Moreover, one imposes that the structure group generators act on the covariant derivatives ∇_A precisely as if they were the generators P_A .

By solving the Bianchi identities, one obtains

$$[\nabla_a, \nabla_b] = -\frac{i}{8}(\gamma_{ab})_\alpha{}^\beta\{\nabla_\beta^k, \mathscr{W}_k^\alpha\}, \quad (2.20)$$

and the additional constraints

$$\{\nabla_\alpha^i, \mathscr{W}^{\beta j}\} = \frac{1}{4}\delta_\alpha^\beta\{\nabla_\gamma^i, \mathscr{W}^{\gamma j}\}, \quad \{\nabla_\gamma^k, \mathscr{W}_k^\gamma\} = 0. \quad (2.21)$$

The operator $\mathcal{W}^{\alpha i}$ is then constrained to be

$$\begin{aligned} \mathcal{W}^{\alpha i} &= W^{\alpha\beta} \nabla_{\beta}^i + \nabla_{\gamma}^i W^{\alpha\beta} M_{\beta}^{\gamma} - \frac{1}{4} \nabla_{\gamma}^i W^{\beta\gamma} M_{\beta}^{\alpha} + \frac{1}{2} \nabla_{\beta j} W^{\alpha\beta} J^{ij} + \frac{1}{8} \nabla_{\beta}^i W^{\alpha\beta} \mathbb{D} \\ &\quad - \frac{1}{16} \nabla_{\beta}^j \nabla_{\gamma}^i W^{\alpha\gamma} S_j^{\beta} + \frac{i}{2} \nabla_{\beta\gamma} W^{\gamma\alpha} S^{\beta i} \\ &\quad - \frac{1}{12} (\gamma^{ab})_{\beta}^{\gamma} \nabla_b (\nabla_{\gamma}^i W^{\beta\alpha} - \frac{1}{2} \delta_{\gamma}^{\alpha} \nabla_{\delta}^i W^{\beta\delta}) K_a . \end{aligned} \quad (2.22)$$

Results at mass-dimension higher than $3/2$ can be found in [16].

It is convenient to define the following

$$X^{\alpha i} := -\frac{i}{10} \nabla_{\beta}^i W^{\alpha\beta} , \quad X_{\gamma}^{k\alpha\beta} := -\frac{i}{4} \nabla_{\gamma}^k W^{\alpha\beta} - \delta_{\gamma}^{(\alpha} X^{\beta)k} , \quad (2.23a)$$

$$Y_{\alpha}^{\beta ij} := -\frac{5}{2} \left(\nabla_{\alpha}^{(i} X^{\beta j)} - \frac{1}{4} \delta_{\alpha}^{\beta} \nabla_{\gamma}^{(i} X^{\gamma j)} \right) = -\frac{5}{2} \nabla_{\alpha}^{(i} X^{\beta j)} , \quad (2.23b)$$

$$Y := \frac{1}{4} \nabla_{\gamma}^k X_k^{\gamma} , \quad (2.23c)$$

$$Y_{\alpha\beta}^{\gamma\delta} := \nabla_{(\alpha}^k X_{\beta)k}^{\gamma\delta} - \frac{1}{6} \delta_{\beta}^{(\gamma} \nabla_{\rho}^k X_{\alpha k}^{\delta)\rho} - \frac{1}{6} \delta_{\alpha}^{(\gamma} \nabla_{\rho}^k X_{\beta k}^{\delta)\rho} . \quad (2.23d)$$

Due to the constraints (2.18), these superfields are the only independent descendants of $W^{\alpha\beta}$. As described in detail in [45], the multiplet of superconformal field strengths of the standard Weyl multiplet is described by the $\theta = 0$ projection of the previous superfields. A reduction to components is straightforward and discussed in [45].

In conformal superspace, the gauge group of conformal supergravity, \mathcal{G} , is generated by *covariant general coordinate transformations*, associated with a local superdiffeomorphism parameter ξ^A and *standard superconformal transformations*, associated with the following local superfield parameters: the Lorentz $\Lambda^{ab} = -\Lambda^{ba}$, $SU(2)_R$ $\Lambda^{ij} = \Lambda^{ji}$, dilatation σ , and special conformal transformations $\Lambda_A = (\Lambda_a, \Lambda_{\alpha}^i)$. The covariant derivatives transform as

$$\delta_{\mathfrak{K}} \nabla_A = [\mathfrak{K}, \nabla_A] , \quad \mathfrak{K} = \xi^B \nabla_B + \frac{1}{2} \Lambda^{bc} M_{bc} + \Lambda^{jk} J_{jk} + \sigma \mathbb{D} + \Lambda_B K^B . \quad (2.24)$$

While the transformation law for a tensor superfield U is

$$\delta_{\mathfrak{K}} U = \mathfrak{K} U . \quad (2.25)$$

The superfield U is said to be *primary* and of dimension w if

$$K^A U = 0 , \quad \mathbb{D} U = w U . \quad (2.26)$$

It is important to point out that the dimension of U coincides with its super-Weyl weight (2.12).

We conclude by mentioning that $SU(2)$ superspace is a gauge-fixed version of the conformal superspace geometry and thus their physical multiplets are equivalent. We refer the reader to appendix D for a description of the degauging procedure.

3 Conformal isometries

In this paper a central role is played by the conformal isometries of a given supergravity background $(\mathcal{M}^{6|8}, \mathcal{D})$ and their extensions.⁷ We will say that a real supervector field $\xi = \xi^B E_B$ on $(\mathcal{M}^{6|8}, \mathcal{D})$ is conformal Killing if there exist Lorentz $(K^{bc}[\xi])$, R -symmetry $(K^{jk}[\xi])$ and super-Weyl $(\sigma[\xi])$ parameters such that

$$\delta \mathcal{D}_A = (\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]}) \mathcal{D}_A = \left[\xi^B \mathcal{D}_B + \frac{1}{2} K^{bc}[\xi] M_{bc} + K^{jk}[\xi] J_{jk}, \mathcal{D}_A \right] + \delta_{\sigma[\xi]} \mathcal{D}_A = 0, \quad (3.1)$$

where the super-Weyl transformation $\delta_{\sigma} \mathcal{D}_A$ is defined in (2.11). Such transformations render the superspace geometry invariant, in particular

$$\delta C_a^{ij} = \delta W_{abc} = \delta N_{abc} = 0, \quad (3.2)$$

and thus are said to be superconformal.

3.1 Conformal Killing vector superfields

The solution to (3.1) is:

$$\xi_i^\alpha = \frac{i}{12} \mathcal{D}_{\beta i} \xi^{\beta\alpha} = -\frac{i}{12} (\tilde{\gamma}^a)^{\alpha\beta} \mathcal{D}_{\beta i} \xi_a, \quad (3.3a)$$

$$K_{\alpha}{}^{\beta}[\xi] = \frac{1}{2} \left(\mathcal{D}_{\alpha}^i \xi_i^{\beta} - \frac{1}{4} \delta_{\alpha}^{\beta} \mathcal{D}_{\gamma}^i \xi_i^{\gamma} \right) + \frac{1}{2} \xi^a (W_{abc} + 2N_{abc}) (\gamma^{bc})_{\alpha}{}^{\beta} = -\frac{1}{4} (\gamma^{ab})_{\alpha}{}^{\beta} \mathcal{D}_a \xi_b, \quad (3.3b)$$

$$K_{ij}[\xi] = \frac{1}{4} \mathcal{D}_{\alpha(i} \xi_{j)}^{\alpha} = -\frac{i}{48} (\tilde{\gamma}^a)^{\alpha\beta} \mathcal{D}_{\alpha(i} \mathcal{D}_{\beta j)} \xi_a, \quad (3.3c)$$

$$\sigma[\xi] = \frac{1}{4} \mathcal{D}_{\alpha}^i \xi_i^{\alpha} = \frac{1}{6} \mathcal{D}_a \xi^a, \quad (3.3d)$$

where ξ^a obeys

$$\mathcal{D}_{\alpha}^i \xi^a = -\frac{1}{5} (\gamma^{ab})_{\alpha}{}^{\beta} \mathcal{D}_{\beta}^i \xi_b. \quad (3.4)$$

We have shown that every infinitesimal superconformal transformation, $\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]}$, of $(\mathcal{M}^{6|8}, \mathcal{D})$ is parametrised by the vector superfield ξ^a , and eq. (3.4) is the fundamental constraint defining this transformation. All other conditions are implications of (3.3) and (3.4). For instance, the latter implies the usual conformal Killing equation for the superfield ξ_a :

$$\mathcal{D}_{(a} \xi_{b)} = \frac{1}{6} \eta_{ab} \mathcal{D}_c \xi^c. \quad (3.5)$$

While the analysis above was carried out in the $SU(2)$ superspace setting, equivalent results can be derived using the conformal superspace approach. Here we will say that $\xi = \xi^B E_B$ is conformal Killing if there exist Lorentz $(\Lambda^{bc}[\xi])$, R -symmetry $(\Lambda^{jk}[\xi])$, dilatation $(\sigma[\xi])$ and special conformal $(\Lambda_B[\xi])$ parameters such that

$$\delta_{\mathfrak{K}[\xi]} \nabla_A = [\xi^B \nabla_B + \frac{1}{2} \Lambda^{bc}[\xi] M_{bc} + \Lambda^{jk}[\xi] J_{jk} + \sigma[\xi] \mathbb{D} + \Lambda_B[\xi] K^B, \nabla_A] = 0. \quad (3.6)$$

⁷The conformal isometries of $\mathcal{N} = (1, 0)$ Minkowski superspace in six dimensions were studied in [16, 77].

Since this transformation preserves the superspace geometry, it must also leave the super-Weyl tensor invariant,

$$\delta_{\mathfrak{R}[\xi]} W^{\alpha\beta} = \mathfrak{R}[\xi] W^{\alpha\beta} = 0. \quad (3.7)$$

The solution to (3.6) is:

$$\xi_i^\alpha = \frac{i}{12} \nabla_{\beta i} \xi^{\beta\alpha} = -\frac{i}{12} (\tilde{\gamma}^a)^{\alpha\beta} \nabla_{\beta i} \xi_a, \quad (3.8a)$$

$$\Lambda_\alpha{}^\beta[\xi] = \frac{1}{2} \left(\nabla_\alpha^i \xi_i^\beta - \frac{1}{4} \delta_\alpha^\beta \nabla_\gamma^i \xi_i^\gamma \right) + \xi_{\alpha\gamma} W^{\beta\gamma} = -\frac{1}{4} (\gamma^{ab})_\alpha{}^\beta \nabla_a \xi_b + \xi_{\alpha\gamma} W^{\beta\gamma} \quad (3.8b)$$

$$\Lambda_{ij}[\xi] = \frac{1}{4} \nabla_{\alpha(i} \xi_{j)}^\alpha = -\frac{i}{48} (\tilde{\gamma}^a)^{\alpha\beta} \nabla_{\alpha(i} \nabla_{\beta j)} \xi_a, \quad (3.8c)$$

$$\sigma[\xi] = \frac{1}{4} \nabla_\alpha^i \xi_i^\alpha = \frac{1}{6} \nabla_a \xi^a, \quad (3.8d)$$

$$\Lambda_\alpha^i[\xi] = \frac{1}{2} \nabla_\alpha^i \sigma[\xi] - \frac{1}{16} \xi_{\alpha\beta} \nabla_\gamma^i W^{\beta\gamma} = \frac{1}{12} \nabla_\alpha^i \nabla^a \xi_a - \frac{1}{16} \xi_{\alpha\beta} \nabla_\gamma^i W^{\beta\gamma}, \quad (3.8e)$$

$$\Lambda_a[\xi] = 2(\tilde{\gamma}_a)^{\alpha\beta} \nabla_\alpha^i \Lambda_{\beta i} = \frac{4}{3} \nabla_a \nabla^b \xi_b - \frac{1}{8} (\tilde{\gamma}_a)^{\alpha\beta} \nabla_\alpha^i (\xi_{\beta\gamma} \nabla_{\delta i} W^{\gamma\delta}), \quad (3.8f)$$

where ξ^a obeys the conformal Killing vector equation

$$\nabla_\alpha^i \xi^a = -\frac{1}{5} (\gamma^{ab})_\alpha{}^\beta \nabla_\beta^i \xi_b. \quad (3.9)$$

This equation is conformally invariant provided ξ^a is primary and of dimension -1 ,

$$K^B \xi^a = 0, \quad \mathbb{D} \xi^a = -\xi^a. \quad (3.10)$$

These relations determine the superconformal properties of the parameters in (3.8). An important corollary of (3.9) is

$$\nabla_{(a} \xi_{b)} = \frac{1}{6} \eta_{ab} \nabla_c \xi^c. \quad (3.11)$$

In what follows, we will often make use of the first-order operator

$$\mathfrak{D}_\xi^{(1)} = \xi^b \nabla_b + \xi_i^\alpha \nabla_\alpha^i + \frac{1}{2} \Lambda^{bc}[\xi] M_{bc} + \Lambda^{jk}[\xi] J_{jk} + \sigma[\xi] \mathbb{D} + \Lambda_B[\xi] K^B, \quad (3.12)$$

where ξ^a is characterised by the superconformal properties (3.10), and the remaining parameters are given by (3.8). The operator $\mathfrak{D}_\xi^{(1)}$ is superconformal and of dimension 0 in the sense that it takes every primary superfield U of dimension w to a primary superfield of the same dimension,

$$K^A U = 0, \quad \mathbb{D} U = w U \quad \implies \quad K^A \mathfrak{D}_\xi^{(1)} U = 0, \quad \mathbb{D} \mathfrak{D}_\xi^{(1)} U = w \mathfrak{D}_\xi^{(1)} U. \quad (3.13)$$

If ξ^a is a solution to (3.9), then $\mathfrak{D}_\xi^{(1)}$ generates a conformal isometry,

$$\nabla_\alpha^i \xi^a = -\frac{1}{5} (\gamma^{ab})_\alpha{}^\beta \nabla_\beta^i \xi_b \quad \implies \quad [\mathfrak{D}_\xi^{(1)}, \nabla_A] = 0. \quad (3.14)$$

Since $SU(2)$ superspace is a gauge fixed version of conformal superspace, it must necessarily be true that (3.6) reproduces (3.1) upon degauging. In particular, it is trivial to see that (3.9) degauges to (3.4).

It is clear from (3.1) that the commutator of superconformal transformations must result in another such transformation

$$[\delta_{\mathcal{K}[\xi_2]} + \delta_{\sigma_2}, \delta_{\mathcal{K}[\xi_1]} + \delta_{\sigma_1}] \mathcal{D}_A = (\delta_{\mathcal{K}[\xi_3]} + \delta_{\sigma_3}) \mathcal{D}_A = 0, \quad (3.15a)$$

$$\mathcal{K}[\xi_3] := [\mathcal{K}[\xi_2], \mathcal{K}[\xi_1]], \quad (3.15b)$$

From this we may extract the form of ξ_3 and σ_3

$$\xi_3^a = \xi_1^b \mathcal{D}_b \xi_2^a - \xi_2^b \mathcal{D}_b \xi_1^a - \frac{i}{48} (\tilde{\gamma}^a)^{\alpha\beta} \mathcal{D}_\alpha^i \xi_1^b \mathcal{D}_{\beta i} \xi_2^c + \frac{i}{48} (\tilde{\gamma}^a{}_{bc})^{\alpha\beta} \mathcal{D}_\alpha^i \xi_1^b \mathcal{D}_{\beta i} \xi_2^c, \quad (3.16a)$$

$$\sigma_3 = \xi_2^A \mathcal{D}_A \sigma_1 - \xi_1^A \mathcal{D}_A \sigma_2. \quad (3.16b)$$

This analysis implies that the conformal Killing supervector fields generate a finite dimensional super Lie algebra. The independent parameters are set of superfields $\Upsilon = (\xi^B, K^{ab}, K^{jk}, \sigma, \mathcal{D}_C \sigma)$ and one can prove that applying any number of covariant derivatives to Υ gives a linear combination of Υ .

The statement above is most easily proven when working in Minkowski superspace, $\mathbb{M}^{6|8}$. Here the superspace covariant derivatives $D_A = (\partial_a, D_\alpha^i)$ take the form

$$\partial_a = \frac{\partial}{\partial x^a}, \quad D_\alpha^i = \frac{\partial}{\partial \theta_i^\alpha} - i(\gamma^a)_{\alpha\beta} \theta^{\beta i} \partial_a, \quad (3.17)$$

and satisfy the algebra

$$\{D_\alpha^i, D_\beta^j\} = -2i\varepsilon^{ij} \partial_{\alpha\beta}, \quad [\partial_a, D_\alpha^i] = 0, \quad [\partial_a, \partial_b] = 0. \quad (3.18)$$

In this context, one may readily derive the following constraints

$$D_\gamma^k K_\alpha{}^\beta[\xi] = 2\delta_\gamma^\beta D_\alpha^k \sigma[\xi] - \frac{1}{2} \delta_\alpha^\beta D_\gamma^k \sigma[\xi], \quad (3.19a)$$

$$D_\alpha^i K^{jk}[\xi] = 4\varepsilon^{i(j} D_\alpha^{k)} \sigma[\xi], \quad (3.19b)$$

$$D_\alpha^i D_\beta^j \sigma[\xi] = -i\varepsilon^{ij} \partial_{\alpha\beta} \sigma[\xi], \quad (3.19c)$$

$$D_\alpha^i \partial_b \sigma[\xi] = 0. \quad (3.19d)$$

Thus, our claim holds for this geometry.

The proof above readily generalises to curved superspace. For example, by analysing the invariance $\delta C_a{}^{ij} = 0$ and $\delta N_{abc} = 0$ one can derive the following relations for the second spinor derivative of σ

$$-\frac{i}{8} (\tilde{\gamma}_a)^{\gamma\delta} \mathcal{D}_\gamma^i \mathcal{D}_\delta^j \sigma = \xi^c \mathcal{D}_c C_a{}^{ij} + \xi_k^\gamma \mathcal{D}_\gamma^k C_a{}^{ij} + K_a{}^b C_b{}^{ij} + 2K^{(i}{}_k C_a{}^{j)k} + \sigma C_a{}^{ij}, \quad (3.20)$$

$$\frac{i}{32} (\tilde{\gamma}^{abc})^{\gamma\delta} \mathcal{D}_\gamma^k \mathcal{D}_{\delta k} \sigma = \xi^d \mathcal{D}_d N_{abc} + \xi_k^\gamma \mathcal{D}_\gamma^k N_{abc} + 3K_{[a}{}^d N_{bc]d} + \sigma N_{abc}. \quad (3.21)$$

Another implication of (3.3) and (3.4) is

$$\mathcal{D}_a \xi_k^\gamma = \frac{i}{2} (\tilde{\gamma}_a)^{\beta\gamma} \mathcal{D}_{\beta k} \sigma + (\gamma_{ab})_\beta{}^\gamma \xi^{\beta j} C^b{}_{jk} + \frac{1}{2} (\gamma^{bc})_\beta{}^\gamma \xi_k^\beta (W_{abc} + 2N_{abc}) - \xi^b T_{ab}{}^\gamma, \quad (3.22)$$

which implies the following expression for the spinor derivative of the super-Weyl parameter

$$\mathcal{D}_{\alpha k}\sigma = -\frac{i}{3}\mathcal{D}_{\alpha\beta}\xi_k^\beta + \frac{4i}{3}\xi^{\delta j}C_{\alpha\delta jk} - \frac{i}{6}(\gamma^{abc})_{\delta\alpha}\xi_k^\delta(W_{abc} + 2N_{abc}) - \frac{i}{3}\xi^b(\gamma^a)_{\alpha\beta}T_{abk}^\beta. \quad (3.23)$$

Note that equation (3.22) plays a fundamental role in the study of supersymmetric space-times.

We also note that by imposing the invariance of the super-Weyl tensor $\delta W_{abc} = 0$ one obtains

$$\xi^d\mathcal{D}_dW_{abc} + \xi_k^\gamma\mathcal{D}_\gamma^k W_{abc} + 3K_{[a}{}^dW_{bc]d} + \sigma W_{abc} = 0, \quad (3.24)$$

which hints at the fact that superspace backgrounds admitting non-trivial conformal isometries are in general constrained.

3.2 Conformally related superspaces

By definition a superspace $(\mathcal{M}^{6|8}, \tilde{\mathcal{D}})$ is said to be conformally related to $(\mathcal{M}^{6|8}, \mathcal{D})$ if the corresponding covariant derivatives $\tilde{\mathcal{D}}_A$ and \mathcal{D}_A are related by a finite super-Weyl transformation,

$$\tilde{\mathcal{D}}_\alpha^i = e^{\frac{1}{2}\rho}\left(\mathcal{D}_\alpha^i - 2(\mathcal{D}_\beta^i\rho)M_\alpha{}^\beta - 4(\mathcal{D}_{\alpha j}\rho)J^{ij}\right), \quad (3.25a)$$

$$\begin{aligned} \tilde{\mathcal{D}}_a = e^\rho\left(\mathcal{D}_a - \frac{i}{2}(\tilde{\gamma}_a)^{\alpha\beta}(\mathcal{D}_\alpha^k\rho)\mathcal{D}_{\beta k} - (\mathcal{D}^b\rho)M_{ab} - \frac{i}{8}(\tilde{\gamma}_a)^{\alpha\beta}(\mathcal{D}_\alpha^k\mathcal{D}_\beta^l\rho)J_{kl} \right. \\ \left. - \frac{i}{8}(\tilde{\gamma}_a{}^{cd})^{\alpha\beta}(\mathcal{D}_\alpha^k\rho)(\mathcal{D}_{\beta k}\rho)M_{cd} - \frac{3i}{4}(\tilde{\gamma}_a)^{\alpha\beta}(\mathcal{D}_\alpha^k\rho)(\mathcal{D}_\beta^l\rho)J_{kl}\right), \end{aligned} \quad (3.25b)$$

for some super-Weyl parameter ρ . The torsion superfields are then mapped from a curved superspace to the other according to (2.14) with σ replaced by ρ . The two superspaces $(\mathcal{M}^{6|8}, \tilde{\mathcal{D}})$ and $(\mathcal{M}^{6|8}, \mathcal{D})$ prove to have the same conformal Killing vector superfields. In fact an efficient way to analyse conformal isometries is by mapping their conformal Killing supervector fields from one superspace to its conformally related one — see for instance the case of conformally flat superspaces. Given such a supervector field $\xi = \xi^A E_A = \tilde{\xi}^A \tilde{E}_A$, it may be shown that

$$\mathcal{K}[\tilde{\xi}] := \tilde{\xi}^B \tilde{\mathcal{D}}_B + \frac{1}{2}K^{bc}[\tilde{\xi}]M_{bc} + K^{kl}[\tilde{\xi}]J_{kl} = \mathcal{K}[\xi], \quad (3.26a)$$

$$\sigma[\tilde{\xi}] = \sigma[\xi] - \xi\rho. \quad (3.26b)$$

3.3 Isometries

In order to describe Poincaré supergravity in $3 \leq d \leq 6$ dimensions, the Weyl multiplet of conformal supergravity has to be coupled to some compensating multiplets Ξ . Two compensators are required for theories with eight supercharges such as $\mathcal{N} = (1, 0)$ supergravity in six dimensions. The conceptual setup is actually universal, which is why it is suitable to start with a general discussion of the situation in d dimensions where conformal supergravity is described using $G_R[d; \mathcal{N}]$ superspace $(\mathcal{M}^{d|\delta_{\mathcal{N}}}, \mathcal{D})$, see section 1.

In general, the compensators are Lorentz scalars, and at least one of them must have a non-zero super-Weyl weight $w_{\Xi} \neq 0$,

$$\delta_{\sigma}\Xi = w_{\Xi}\sigma\Xi . \tag{3.27}$$

They may also transform in some representations of the R -symmetry group. The compensators are required to be nowhere vanishing in the sense that the R -symmetry singlets $|\Xi|^2$ should be strictly positive. Different off-shell supergravity theories correspond to different choices of Ξ . The superspace corresponding to Poincaré supergravity is identified with a triple $(\mathcal{M}^{d|\delta_{\mathcal{N}}}, \mathcal{D}, \Xi)$. The notion of conformally related superspaces, which was introduced in section 3.2, is naturally generalised to the case under consideration. Specifically, two curved superspaces $(\mathcal{M}^{d|\delta_{\mathcal{N}}}, \tilde{\mathcal{D}}, \tilde{\Xi})$ and $(\mathcal{M}^{d|\delta_{\mathcal{N}}}, \mathcal{D}, \Xi)$ are conformally related if their covariant derivatives related to each other according to (3.25), and the compensators $\tilde{\Xi}$ and Ξ are connected by the same finite super-Weyl transformation,

$$\tilde{\Xi} = e^{w_{\Xi}\rho}\Xi . \tag{3.28}$$

Once Ξ has been fixed, the off-shell supergravity multiplet is completely described in terms of the following data: (i) a superspace geometry for conformal supergravity; and (ii) the conformal compensators. Given a supergravity background, its isometries should preserve both of these inputs. This leads us to the concept of Killing supervector fields.

Let $\xi = \xi^B E_B$ be a conformal Killing supervector field on $(\mathcal{M}^{d|\delta_{\mathcal{N}}}, \mathcal{D})$,

$$(\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]})\mathcal{D}_A = 0 , \tag{3.29a}$$

for uniquely determined parameters $K^{bc}[\xi]$, $K^I[\xi]$ and $\sigma[\xi]$. It is called a Killing supervector field on $(\mathcal{M}^{d|\delta_{\mathcal{N}}}, \mathcal{D}, \Xi)$ if the compensators are invariant,

$$(\mathcal{K}[\xi] + w_{\Xi}\sigma[\xi])\Xi = 0 . \tag{3.29b}$$

The set of Killing vectors on $(\mathcal{M}^{d|\delta_{\mathcal{N}}}, \mathcal{D}, \Xi)$ is a Lie superalgebra. The Killing equations (3.29a) and (3.29b) are super-Weyl invariant in the sense that they hold for all conformally related superspace geometries.

Using the compensators Ξ we can always construct a superfield $\Xi = f(\Xi)$ that is a singlet under the structure group and has the properties: (i) it is an algebraic function of Ξ ; (ii) it is nowhere vanishing; and (iii) it has a non-zero super-Weyl weight w_{Ξ} , $\delta_{\sigma}\Xi = w_{\Xi}\sigma\Xi$. It follows from (3.29b) that

$$(\xi^B \mathcal{D}_B + w_{\Xi}\sigma[\xi])\Xi = 0 . \tag{3.30}$$

The super-Weyl invariance may be used to impose the gauge condition

$$\Xi = 1 , \tag{3.31}$$

Then eq. (3.30) reduces to

$$\sigma[\xi] = 0 , \tag{3.32}$$

and the Killing equations (3.29a) and (3.29b) take the following form:

$$[\mathcal{K}[\xi], \mathcal{D}_A] = 0, \tag{3.33a}$$

$$\mathcal{K}[\xi]\Xi = 0. \tag{3.33b}$$

Now we specialise the Killing equations (3.29a) and (3.29b) to the case of $\mathcal{N} = (1, 0)$ supergravity in six dimensions. The equations read

$$\left[\xi^B \mathcal{D}_B + \frac{1}{2} K^{bc}[\xi] M_{bc} + K^{kl} J_{kl}, \mathcal{D}_A \right] + \delta_{\sigma[\xi]} \mathcal{D}_A = 0, \tag{3.34a}$$

$$\left(\xi^B \mathcal{D}_B + K^{kl}[\xi] J_{kl} + w_{\Xi} \sigma[\xi] \right) \Xi = 0. \tag{3.34b}$$

The most convenient set of compensators for $\mathcal{N} = (1, 0)$ Poincaré supergravity [1] consists of a tensor multiplet Φ and a linear multiplet $G^{ij} = G^{ji}$. The former is a primary real scalar of super-Weyl weight $w_{\Phi} = 2$, which obeys the constraint [15, 78, 79]

$$(\mathcal{D}_{\alpha}^{(i} \mathcal{D}^{j)}) \Phi = 0 \tag{3.35}$$

and is nowhere vanishing in the sense that Φ^{-1} exists. The latter is a real SU(2) triplet (that is, $\overline{G}^{ij} = G_{ij} = \varepsilon_{ik} \varepsilon_{jl} G^{kl}$), which is a primary superfield of super-Weyl weight $w_G = 4$ and obeys the constraint [15, 78]

$$\mathcal{D}_{\alpha}^{(i} G^{jk)} = 0. \tag{3.36}$$

The linear compensator is required to be nowhere vanishing in the sense that G^{-1} exists for $G := \sqrt{\frac{1}{2} G^{ij} G_{ij}}$. There are two natural choices for Ξ : either Φ or G .

The above formalism will be employed in section 8 and appendix C to study supersymmetric spacetimes in the superspace setting. Now we will turn to describing the extension of (conformal) Killing vector superfields to the case of (conformal) Killing tensor superfields and higher symmetries of $\mathcal{N} = (1, 0)$ supermultiplets.

4 Conformal Killing spinor superfields and their higher rank cousins

In this section we introduce various cousins of the conformal Killing vector superfields ξ^a , eq. (3.4). Some of them can be used to describe extended superconformal transformations (conformal Killing spinor superfields) and higher symmetries of $\mathcal{N} = (1, 0)$ supermultiplets (conformal Killing tensor superfields).

In SU(2) superspace, a conformal Killing spinor superfield ϵ^{α} is defined to satisfy the constraint

$$\mathcal{D}_{\alpha}^i \epsilon^{\beta} = \frac{1}{4} \delta_{\alpha}^{\beta} \mathcal{D}_{\gamma}^i \epsilon^{\gamma}. \tag{4.1}$$

This equation is super-Weyl invariant provided the super-Weyl transformation of ϵ^{α} is

$$\delta_{\sigma} \epsilon^{\alpha} = -\frac{1}{2} \sigma \epsilon^{\alpha}. \tag{4.2}$$

In conformal superspace, ϵ^α is required to (i) be primary and of dimension $-1/2$; and (ii) obey the constraint obtained from (4.1) by replacing \mathcal{D} 's with ∇ 's.

Equation (4.1) imposes significant restrictions on the component content of ϵ^α . In particular, the following corollary of (4.1)

$$\mathcal{D}_\alpha^i \mathcal{D}_\beta^j \epsilon^{\beta} = -\frac{8i}{3} \epsilon^{ij} \mathcal{D}_{\alpha\beta} \epsilon^\beta + 16i C_{\alpha\beta}^{ij} \epsilon^\beta + 16i \epsilon^{ij} N_{\alpha\beta} \epsilon^\beta, \quad (4.3)$$

implies that $\epsilon^\alpha|_{\theta=0}$ and $\mathcal{D}_\alpha^i \epsilon^\alpha|_{\theta=0}$ are the only independent component fields. Making further use of (4.3) leads to

$$\mathcal{D}_{\alpha\beta} \epsilon^\gamma = -\frac{2}{3} \delta^\gamma_{[\alpha} \mathcal{D}_{\beta]} \epsilon^\delta - \epsilon_{\alpha\beta\sigma\delta} W^{\sigma\gamma} \epsilon^\delta, \quad (4.4a)$$

$$\mathcal{D}_{\alpha\beta} \mathcal{D}_\gamma^i \epsilon^\gamma = \epsilon_{\alpha\beta\gamma\delta} \left[\frac{5i}{12} \mathcal{W}^{\gamma i} - 16 \mathcal{C}^{\gamma i} \right] \epsilon^\delta + 8 [\mathcal{N}_{\alpha\beta,\gamma}^i - \mathcal{C}_{\alpha\beta,\gamma}^i] \epsilon^\gamma, \quad (4.4b)$$

where the torsion superfields on the right-hand side of (4.4b) are defined in (2.8).

Associated with ϵ^α is its conjugate $\bar{\epsilon}^\alpha$ defined by (A.12). The latter is also a conformal Killing spinor superfield. We can combine ϵ^α and $\bar{\epsilon}^\alpha$ into a symplectic Majorana spinor $\epsilon^{\alpha\hat{i}}$ that carries a new $SU(2) \neq SU(2)_R$ index. Such objects naturally arise from an $\mathcal{N} = (2, 0) \rightarrow \mathcal{N} = (1, 0)$ superspace reduction, see appendix B for more details. Given an $\mathcal{N} = (2, 0)$ superconformal theory realised in $\mathcal{N} = (1, 0)$ superspace, $\epsilon^{\alpha\hat{i}}$ describes extended superconformal transformations.

Let ϵ_1^α and ϵ_2^α be two conformal Killing spinor superfields. Associated with them is a vector superfield

$$\xi^a = \epsilon_1^\alpha (\gamma^a)_{\alpha\beta} \epsilon_2^\beta, \quad (4.5)$$

which is primary and of dimension -1 . As follows from (4.1), ξ^a satisfies the conformal Killing vector equation (3.4). As was shown in the previous section, these generate the conformal isometries of superspace.

Given n conformal Killing vector superfields $\xi_1^{a_1}, \dots, \xi_n^{a_n}$, we find that their *symmetric and traceless* product

$$\zeta^{a(n)} = \xi_1^{\{a_1} \dots \xi_n^{a_n\}} := \xi_1^{\{a_1} \dots \xi_n^{a_n\}} - \text{traces} \quad (4.6)$$

has the following super-Weyl transformation law

$$\delta_\sigma \zeta^{a(n)} = -n \sigma \zeta^{a(n)}, \quad (4.7)$$

and satisfies the constraint

$$\mathcal{D}_\alpha^i \zeta^{a(n)} = \frac{n}{n+4} (\gamma^{b(a_1})_\alpha{}^\beta \mathcal{D}_\beta^i \zeta^{a_2 \dots a_n})_b. \quad (4.8)$$

We will say that any solution $\zeta^{a_1 \dots a_n} = \zeta^{\{a_1 \dots a_n\}}$ to (4.8) is a conformal Killing tensor superfield.⁸ It is clear from (4.6) and (4.8) that the symmetric and traceless product of

⁸Our definition is equivalent to the one proposed in [73], although the $SU(2)$ superspace formulation was not used.

two such tensors is also conformal Killing. As will be shown shortly, such tensors generate higher symmetries of the kinetic operators of superconformal field theories with at most two derivatives, in accordance with [72]. An immediate consequence of (4.8) is the usual conformal Killing tensor equation

$$\mathcal{D}_{\{a_1 \zeta_{a_2 \dots a_{n+1}}\}} = 0 . \tag{4.9}$$

The above definition of the conformal Killing tensor superfield can be recast in conformal superspace. A symmetric traceless tensor superfield $\zeta^{a_1 \dots a_n} = \zeta^{\{a_1 \dots a_n\}}$ is called conformal Killing if it has the superconformal properties

$$K^B \zeta^{a(n)} = 0, \quad \mathbb{D} \zeta^{a(n)} = -n \zeta^{a(n)} \tag{4.10}$$

and solves the equation

$$\nabla_{\alpha}^i \zeta^{a(n)} = \frac{n}{n+4} (\gamma^{b(a_1)}_{\alpha}{}^{\beta} \nabla_{\beta}^i \zeta^{a_2 \dots a_n})_b . \tag{4.11}$$

For a given curved superspace, the set of conformal Killing tensor superfields may be endowed with an additional algebraic structure. Let $\zeta_1^{a(m)}$ and $\zeta_2^{a(n)}$ be two such tensors, then

$$\begin{aligned} [\zeta_1, \zeta_2]^{a(m+n-1)} &= m \zeta_1^{\{a_1 \dots a_{m-1} | b\}} \mathcal{D}_b \zeta_2^{a_m \dots a_{m+n-1}} - n \zeta_2^{\{a_1 \dots a_{n-1} | b\}} \mathcal{D}_b \zeta_1^{a_n \dots a_{m+n-1}} \\ &\quad - \frac{imn}{8(m+n+2)} (\tilde{\gamma}^{\{a_1\}}_{\alpha\beta} \mathcal{D}_{\alpha}^i \zeta_1^{a_2 \dots a_m | b\}} \mathcal{D}_{\beta i} \zeta_2^{a_{m+1} \dots a_{m+n-1}})_b \\ &\quad + \frac{imn}{8(m+n+2)} (\tilde{\gamma}^{\{a_1\}}_{bc} \alpha^{\beta} \mathcal{D}_{\alpha}^i \zeta_1^{a_2 \dots a_m | b\}} \mathcal{D}_{\beta i} \zeta_2^{a_{m+1} \dots a_{m+n-1}})_c , \end{aligned} \tag{4.12}$$

is a conformal Killing tensor superfield. This generalises the Lie bracket for conformal Killing vector superfields (3.16a) and coincides with the one presented in [73], where it was called the supersymmetric even Schouten-Nijenhuis bracket.⁹

Having investigated the structure of conformal Killing tensors, we now return to the master equation (4.1). This constraint admits non-trivial generalisations¹⁰

$$\mathcal{D}_{\alpha}^{(i_1} \epsilon^{\beta(n) i_2 \dots i_{m+1}}) = \frac{n}{n+3} \delta_{\alpha}^{(\beta_1} \mathcal{D}_{\gamma}^{(i_1} \epsilon^{\beta_2 \dots \beta_n) \gamma i_2 \dots i_{m+1}}) . \tag{4.13}$$

It is conformally invariant provided

$$\delta_{\sigma} \epsilon^{\beta(n) i(m)} = \left[2m - \frac{n}{2} \right] \sigma \epsilon^{\beta(n) i(m)} . \tag{4.14}$$

Given two solutions $\epsilon^{\beta(n_1) i(m_1)}$ and $\epsilon^{\beta(n_2) i(m_2)}$ to (4.13), one can show that

$$\epsilon^{\beta(n_1+n_2) i(m_1+m_2)} = \epsilon_1^{(\beta_1 \dots \beta_{n_1} (i_1 \dots i_{m_1} \epsilon_2^{\beta_{n_1+1} \dots \beta_{n_1+n_2}) i_{m_1+1} \dots i_{m_1+m_2})} , \tag{4.15}$$

also satisfies this constraint.

⁹The supersymmetric extension of the even Schouten-Nijenhuis bracket was proposed for the first time in the framework of $\mathcal{N} = 1$ AdS supersymmetry in four dimensions [80], although no mention of the even Schouten-Nijenhuis bracket was made.

¹⁰The case $n = 2, m = 0$ describes a conformal Killing-Yano tensor superfield, introduced in a flat superspace context in [74].

Let ϵ_1^α , ϵ_2^α and ϵ_3^α be conformal Killing spinors (4.1). It is clear from the analysis above that their totally symmetric product is a solution to (4.13), while their antisymmetric product is dual to a right-handed spinor

$$\chi_\alpha = \varepsilon_{\alpha\beta\gamma\delta} \epsilon_1^\beta \epsilon_2^\gamma \epsilon_3^\delta, \tag{4.16}$$

which satisfies

$$\mathcal{D}_{(\alpha}^i \chi_{\beta)} = 0. \tag{4.17}$$

This constraint may be immediately generalised

$$\mathcal{D}_{(\alpha_1}^{(i_1} \chi_{\alpha_2 \dots \alpha_{n+1})}^{i_2 \dots i_{m+1})} = 0, \tag{4.18}$$

and is conformally invariant provided

$$\delta_\sigma \chi_{\alpha(n)}^{i(m)} = \left[2m - \frac{3n}{2} \right] \sigma \chi_{\alpha(n)}^{i(m)}. \tag{4.19}$$

If $\chi_{\alpha(n_1)}^{i(m_1)}$ and $\chi_{\alpha(n_2)}^{i(m_2)}$ are solutions to this constraint then

$$\chi_{\alpha(n_1+n_2)}^{i(m_1+m_2)} = \chi_{(\alpha_1 \dots \alpha_{n_1}}^{(i_1 \dots i_{m_1}} \chi_{\alpha_{n_1+1} \dots \alpha_{n_1+n_2})}^{i_{m_1+1} \dots i_{m_1+m_2})}, \tag{4.20}$$

also solves (4.18).

We may also construct a hook field from our three spinors

$$\ell^{\alpha\beta,\gamma} = \frac{1}{2} (\epsilon_1^{[\alpha} \epsilon_2^{\beta]} \epsilon_3^\gamma - \epsilon_1^\gamma \epsilon_2^{[\alpha} \epsilon_3^{\beta]}) = -\ell^{\beta\alpha,\gamma}, \tag{4.21}$$

which satisfies the Young condition

$$\ell^{\alpha\beta,\gamma} + \ell^{\beta\gamma,\alpha} + \ell^{\gamma\alpha,\beta} = 0. \tag{4.22}$$

Further, it satisfies the conformally invariant constraint

$$\begin{aligned} \mathcal{D}_\alpha^i \ell^{\beta\gamma,\delta} &= \frac{1}{3} (\delta_\alpha^\beta \mathcal{D}_\epsilon^i \ell^{\epsilon(\gamma,\delta)} - \delta_\alpha^\gamma \mathcal{D}_\epsilon^i \ell^{\epsilon(\beta,\delta)}) + \frac{1}{5} (\delta_\alpha^\beta \mathcal{D}_\epsilon^i \ell^{\epsilon[\gamma,\delta]} - \delta_\alpha^\gamma \mathcal{D}_\epsilon^i \ell^{\epsilon[\beta,\delta]}) \\ &\quad - \frac{2}{5} \delta_\alpha^\delta \mathcal{D}_\epsilon^i \ell^{\epsilon[\beta,\gamma]}. \end{aligned} \tag{4.23}$$

5 Higher symmetries of the conformal d'Alembertian

There has been extensive study of the higher symmetries of the conformal d'Alembertian in dimensions $d > 2$, including the important publications [67, 69, 71]. Here we will review known results and present some new observations. The outcomes of the non-supersymmetric analysis in this section will guide our study in the next two sections.

Let ϕ be a solution to the conformal wave equation in d dimensions

$$\square \phi = \nabla^a \nabla_a \phi = 0, \tag{5.1}$$

where ∇_a is the conformally covariant derivative (compare with (2.15))

$$\nabla_a = e_a - \frac{1}{2}\omega_a{}^{bc}M_{bc} - b_a\mathbb{D} - f_a{}^bK_b, \tag{5.2}$$

with the commutation relations [16, 39]

$$[\nabla_a, \nabla_b] = -\frac{1}{2}C_{ab}{}^{cd}M_{cd} + \frac{1}{2(d-3)}\nabla_c C_{ab}{}^{cd}K_d. \tag{5.3}$$

Equation (5.1) is known to be conformally invariant if ϕ has the transformation properties

$$K_a\phi = 0, \quad \mathbb{D}\phi = \frac{1}{2}(d-2)\phi. \tag{5.4}$$

A differential operator \mathfrak{D} is called a symmetry of the conformal d'Alembertian, \square , if it obeys the following conditions:

$$\square\mathfrak{D}\phi = 0, \tag{5.5a}$$

$$K_a\mathfrak{D}\phi = 0, \quad \mathbb{D}\mathfrak{D}\phi = \frac{1}{2}(d-2)\mathfrak{D}\phi. \tag{5.5b}$$

Condition (5.5b) means that \mathfrak{D} is a conformal dimension-0 operator. The symmetry operators of \square naturally form an associative algebra.

In the algebra of all symmetry operators of \square , it is natural to introduce the equivalence relation

$$\mathfrak{D}_1 \sim \mathfrak{D}_2 \iff (\mathfrak{D}_1 - \mathfrak{D}_2)\phi = 0. \tag{5.6}$$

Utilising (5.6), it is possible to show that every symmetry operator \mathfrak{D} of order n can be reduced to the canonical form

$$\mathfrak{D}_\zeta^{(n)} = \sum_{k=0}^n \zeta^{a(k)} \nabla_{a_1} \dots \nabla_{a_k}, \quad n \geq 0, \tag{5.7}$$

where the parameters $\zeta^{a(k)}$ are symmetric and traceless. Making use of the condition (5.5a), one observes that $\zeta^{a(n)}$ satisfies the conformal Killing tensor equation

$$\nabla_{\{a_1}\zeta_{a_2\dots a_{n+1}\}} = 0. \tag{5.8}$$

Due to (5.5b), $\zeta^{a(n)}$ is primary, $K_b\zeta^{a(n)} = 0$, and of dimension $-n$.

Let us first study the $n = 0$ and 1 cases in more detail. It is easily seen that zeroth-order symmetry operator $\mathfrak{D}_\zeta^{(0)}$ is a constant,

$$\square\mathfrak{D}_\zeta^{(0)}\phi = 0 \iff \nabla_a\zeta = 0. \tag{5.9}$$

Given a conformal Killing vector field ξ^a , the following first-order operator

$$\mathfrak{D}_\xi^{(1)} = \xi^a\nabla_a + \frac{d-2}{2d}\nabla_a\xi^a, \tag{5.10}$$

is a symmetry of the conformal d'Alembertian,

$$\square \mathfrak{D}_\xi^{(1)} \phi = 0. \tag{5.11}$$

The second term on the right-hand side of (5.10) is uniquely determined by each of the conditions (5.5a) and (5.5b).

Actually, the operator (5.10) is simply a special case of the conformal isometry

$$\mathfrak{D}_\xi^{(1)} = \xi^a \nabla_a + \frac{1}{2} \nabla^a \xi^b M_{ab} + \frac{1}{d} \nabla^a \xi_a \mathbb{D} + \frac{1}{2d} \nabla^a \nabla^b \xi_b K_a, \tag{5.12a}$$

$$[\mathfrak{D}_\xi^{(1)}, \nabla_a] = 0, \tag{5.12b}$$

which reduces to (5.10) when acting on any primary scalar field of dimension $\frac{1}{2}(d-2)$. In addition to (5.12b), the other fundamental property of $\mathfrak{D}_\xi^{(1)}$ is the following:

$$K_a T = 0, \quad \mathbb{D}T = wT \quad \implies \quad K_a \mathfrak{D}_\xi^{(1)} T = 0, \quad \mathbb{D} \mathfrak{D}_\xi^{(1)} T = w \mathfrak{D}_\xi^{(1)} T, \tag{5.13}$$

for every primary tensor field T (with suppressed indices) of dimension w . The relations (5.12b) and (5.13) tell us that $\mathfrak{D}_\xi^{(1)}$ satisfies the conditions (5.5a) and (5.5b), and therefore $\mathfrak{D}_\xi^{(1)}$ is a symmetry of the conformal d'Alembertian. An immediate corollary of the above consideration is that, for any conformal Killing vector fields $\xi_1^a, \xi_2^a, \dots, \xi_n^a$, the operator

$$\mathfrak{D}^{(n)} := \mathfrak{D}_{\xi_1}^{(1)} \mathfrak{D}_{\xi_2}^{(1)} \dots \mathfrak{D}_{\xi_n}^{(1)} \tag{5.14}$$

is a symmetry of the conformal d'Alembertian. Therefore, the algebra of symmetries of \square includes the universal enveloping algebra of the conformal algebra of the background spacetime.

Our consideration above allows for important generalisations. Consider a dynamical system described by primary fields φ^i coupled to conformal gravity. We place this theory on a fixed gravitational background and consider a conformal Killing vector field, ξ^a , on spacetime. Since the operator (5.12a) preserves the background geometry, the matter action $S[\varphi]$ is invariant under the conformal transformation $\delta_\xi \varphi^i = \mathfrak{D}_\xi^{(1)} \varphi^i$. Consequently, $\mathfrak{D}_\xi^{(1)}$ is a symmetry of the corresponding equation of motion, $S_{,i}[\varphi] = 0$.

Let us now return to the general symmetry operator (5.7) for $n > 1$. Similar to the first-order operator (5.10), we would like $\mathfrak{D}_\zeta^{(n)}$ to be determined by its top component, which is the conformal Killing tensor $\zeta^{a(n)}$. Imposing the condition (5.5b) leads to

$$\zeta^{a(k)} = A_k \nabla_{b_1} \dots \nabla_{b_{n-k}} \zeta^{a(k)b(n-k)}, \quad 0 \leq k \leq n, \tag{5.15a}$$

where the constants A_k are given by the solution to the recurrence relation

$$\frac{A_{k-1}}{A_k} = \frac{k(4-2k-d)}{2(k(k+d-3) - n(n+d-1) - d+2)}, \quad A_n = 1. \tag{5.15b}$$

It may be readily shown that the constructed operator is a symmetry of \square in any conformally flat background. However, when the background Weyl tensor is non-vanishing, the

existence of a higher symmetry implies non-trivial restrictions on its structure. To prove this claim it suffices to analyse the $n = 2$ case.

We assert that the second-order operator

$$\mathfrak{D}_\zeta^{(2)} = \zeta^{ab} \nabla_a \nabla_b + \frac{d}{d+2} \nabla_b \zeta^{ab} \nabla_a + \frac{d(d-2)}{4(d+1)(d+2)} \nabla_a \nabla_b \zeta^{ab}, \quad (5.16)$$

only results in a symmetry in backgrounds satisfying

$$C_{abcd} \nabla^c \zeta^{bd} + \frac{d(d-2)}{d-3} \nabla^c C_{abcd} \zeta^{bd} = 0. \quad (5.17)$$

The direct computation necessary to verify (5.17) is tedious, thus here we will present a simpler proof. Consider two conformal Killing vectors ξ_1^a and ξ_2^a and the corresponding first-order symmetry operators $\mathfrak{D}_{\xi_1}^{(1)}$ and $\mathfrak{D}_{\xi_2}^{(1)}$ defined via (5.10). Then their product $\mathfrak{D}^{(2)} := \mathfrak{D}_{\xi_1}^{(1)} \mathfrak{D}_{\xi_2}^{(1)}$ is a second-order symmetry operator. Modulo the equivalence relation (5.6), it may be expressed as a sum of operators of the form (5.7):

$$\mathfrak{D}^{(2)} \sim \mathfrak{D}_\zeta^{(2)} + \frac{1}{2} \mathfrak{D}_{[\xi_1, \xi_2]}^{(1)} - \frac{d-2}{4(d+1)} \mathfrak{D}_{\langle \xi_1, \xi_2 \rangle}^{(0)}, \quad (5.18)$$

where

$$\zeta^{ab} = \xi_1^a \xi_2^b, \quad (5.19a)$$

$$[\xi_1, \xi_2]^a = \xi_1^b \nabla_b \xi_2^a - \xi_2^b \nabla_b \xi_1^a, \quad (5.19b)$$

$$\langle \xi_1, \xi_2 \rangle = \nabla_b \xi_1^a \nabla_a \xi_2^b - \frac{d-2}{d} \nabla_a \xi_1^a \nabla_b \xi_2^b - \frac{2}{d} (\xi_1^a \nabla_a \nabla_b \xi_2^b + \xi_2^a \nabla_a \nabla_b \xi_1^b). \quad (5.19c)$$

As $\mathfrak{D}^{(2)} = \mathfrak{D}_{\xi_1}^{(1)} \mathfrak{D}_{\xi_2}^{(1)}$ is a symmetry operator by construction, we obtain

$$\square \mathfrak{D}^{(2)} \phi \sim \square \mathfrak{D}_\zeta^{(2)} \phi - \frac{d-2}{4(d+1)} \square \mathfrak{D}_{\langle \xi_1, \xi_2 \rangle}^{(0)} \phi = 0, \quad (5.20)$$

which has the immediate consequence

$$\square \mathfrak{D}_\zeta^{(2)} \phi = 0 \iff \nabla_a \langle \xi_1, \xi_2 \rangle = 0. \quad (5.21)$$

Now a direct computation leads to

$$\nabla_a \langle \xi_1, \xi_2 \rangle = -C_{abcd} \nabla^c \zeta^{bd} - \frac{d(d-2)}{d-3} \nabla^c C_{abcd} \zeta^{bd}. \quad (5.22)$$

Hence, $\mathfrak{D}_\zeta^{(2)}$ only results in a symmetry when the right-hand side vanishes. We thus expect that for $n \geq 2$ the operator $\mathfrak{D}_\zeta^{(n)}$ is a symmetry only on a limited class of backgrounds, such as conformally flat ones.

A natural extension of the analysis above is to determine if there exists a symmetry operator of the form

$$\mathfrak{D}^{(2)} = \mathfrak{D}_\zeta^{(2)} + \mathcal{Z}, \quad \nabla_a \mathcal{Z} = -C_{abcd} \nabla^c \zeta^{bd} - \frac{d(d-2)}{d-3} \nabla^c C_{abcd} \zeta^{bd}, \quad (5.23)$$

when the conformal Killing tensor is irreducible ($\zeta^{ab} \neq \xi_1^a \xi_2^b$). This fails since \mathcal{Z} must be primary, of dimension 0 and first order in ζ^{ab} . It is easily verified that no such term exists.

Our analysis in this section has led to several non-trivial results regarding the structure of higher symmetry operators. The most important observation is that, beyond the first-order case, their existence implies previously unknown constraints on the spacetime Weyl tensor.¹¹ It may be shown that our arguments immediately generalise to the higher symmetries of superconformal operators. In particular, the existence of such symmetries should lead to non-trivial restrictions on the super-Weyl tensor $W^{\alpha\beta}$.

6 Higher symmetries of the hypermultiplet

The study of symmetries of relativistic wave equations has a long-standing history in mathematical physics. More recently their supersymmetric generalisations have also been explored. Specifically, in flat superspace it was shown in [72] that the higher symmetries of so-called ‘super-Laplacians’ (superspace differential operators containing the spacetime Laplacian as their highest-dimensional component) are in one-to-one correspondence with conformal Killing tensor superfields. Further, the higher symmetries of the Wess-Zumino operator in curved $d = 4$, $\mathcal{N} = 1$ superspace were analysed in [75]. It is now time to extend this analysis to the hypermultiplet.

The off-shell formulation for a hypermultiplet coupled to conformal supergravity is given in [15]. On the mass shell, the hypermultiplet is described by an isospinor superfield q^i satisfying the equation

$$\nabla_{\alpha}^{(i} q^{j)} = 0 . \tag{6.1}$$

The constraint is conformally invariant provided q^i is a primary superfield of dimension 2

$$K_A q^i = 0, \quad \mathbb{D} q^i = 2q^i . \tag{6.2}$$

Additionally, (6.1) yields the useful corollary

$$\nabla_{\alpha}^i \nabla_{\beta}^j q_j = -4i \nabla_{\alpha\beta} q^i . \tag{6.3}$$

Here we will study the higher symmetries of this model. We will say that a differential operator \mathfrak{D} is a symmetry operator (of the hypermultiplet) if

$$\nabla_{\alpha}^{(i} \mathfrak{D} q^{j)} = 0 . \tag{6.4}$$

It is useful to introduce an equivalence relation on the space of symmetries so that redundant structures can be discarded. Specifically, we say that two symmetry operators \mathfrak{D}_1 and \mathfrak{D}_2 are equivalent if

$$\mathfrak{D}_1 \sim \mathfrak{D}_2 \iff (\mathfrak{D}_1 - \mathfrak{D}_2) q^i = 0 . \tag{6.5}$$

Owing to (6.2), we will also require

$$K_A \mathfrak{D} q^i = 0, \quad \mathbb{D} \mathfrak{D} q^i = 2 \mathfrak{D} q^i , \tag{6.6}$$

which means that \mathfrak{D} is a superconformal dimension-0 operator.

¹¹This is in keeping with the fact that the background geometry generally restricts Killing tensors, see, e.g., [81, 82].

Given a positive integer n , the most general n^{th} -order symmetry operator $\mathfrak{D}^{(n)}$ is

$$\mathfrak{D}^{(n)} = \sum_{k=0}^n \zeta^{A_1 \dots A_k} \nabla_{A_k} \dots \nabla_{A_1} + \sum_{k=0}^{n-1} \zeta^{A_1 \dots A_k, ij} \nabla_{A_k} \dots \nabla_{A_1} J_{ij}, \quad (6.7)$$

where the coefficients may be chosen to be graded-symmetric in their superspace indices

$$\zeta^{A_1 \dots A_m A_{m+1} \dots A_k} = (-1)^{\varepsilon_{A_m} \varepsilon_{A_{m+1}}} \zeta^{A_1 \dots A_{m+1} A_m \dots A_k}, \quad (6.8a)$$

$$\zeta^{A_1 \dots A_m A_{m+1} \dots A_k, ij} = (-1)^{\varepsilon_{A_m} \varepsilon_{A_{m+1}}} \zeta^{A_1 \dots A_{m+1} A_m \dots A_k, ij}. \quad (6.8b)$$

The equivalence relation (6.5) allows us to bring $\mathfrak{D}^{(n)}$ to the canonical form

$$\begin{aligned} \mathfrak{D}^{(n)} = & \sum_{k=0}^n \zeta^{a(k)} \nabla_{a_1} \dots \nabla_{a_k} + \sum_{k=0}^{n-1} \zeta^{a(k)\beta}{}_i \nabla_{a_1} \dots \nabla_{a_k} \nabla_{\beta}^i \\ & + \sum_{k=0}^{n-1} \zeta^{a(k)ij} \nabla_{a_1} \dots \nabla_{a_k} J_{ij}. \end{aligned} \quad (6.9)$$

Here all parameters are symmetric and traceless in their vector indices, $\zeta^{a(k)\beta}{}_i$ is gamma-traceless, $(\gamma_b)_{\alpha\beta} \zeta^{a(k-1)b\beta}{}_i = 0$, and $\zeta^{a(k)ij}$ is symmetric in its isospinor indices.

Equation (6.4) yields numerous constraints on the parameters of $\mathfrak{D}^{(n)}$, including

$$\nabla_{\alpha}^i \zeta^{a(n)} = \frac{n}{n+4} (\gamma^{b(a_1)})_{\alpha}{}^{\beta} \nabla_{\beta}^i \zeta^{a_2 \dots a_n}{}_b, \quad (6.10a)$$

$$\zeta^{a(n-1)\beta i} = \frac{i}{4(n+2)} \nabla_{\alpha}^i \zeta^{a(n-1)b} (\tilde{\gamma}_b)^{\alpha\beta}, \quad (6.10b)$$

$$\zeta^{a(n-1)ij} = -\frac{in(n+1)}{8(n+2)(n+3)} \nabla^{(i} \tilde{\gamma}_b{}^{j)} \zeta^{a(n-1)b}. \quad (6.10c)$$

Hence, we obtain expressions for $\zeta^{a(n-1)\beta i}$ and $\zeta^{a(n-1)ij}$ in terms of $\zeta^{a(n)}$, which is necessarily a conformal Killing tensor (4.10), (4.11). Further, if $\mathfrak{D}^{(n)}$ is completely determined in terms of $\zeta^{a(n)}$, we will denote it by $\mathfrak{D}_{\zeta}^{(n)}$.

If the supergravity background admits a conformal Killing vector superfield ξ^a , it may be shown that the corresponding conformal isometry (3.12) yields the unique first-order symmetry operator

$$\nabla_{\alpha}^{(i} \mathfrak{D}_{\xi}^{(1)} q^{j)} = \nabla_{\alpha}^{(i} \left[\xi^B \nabla_B + \Lambda^{jk} [\xi] J_{jk} + 2\sigma[\xi] \right] q^{j)} = 0. \quad (6.11)$$

Thus, given conformal Killing vector superfields $\xi_1^a, \xi_2^a, \dots, \xi_n^a$, the operator

$$\mathfrak{D}^{(n)} := \mathfrak{D}_{\xi_1}^{(1)} \mathfrak{D}_{\xi_2}^{(1)} \dots \mathfrak{D}_{\xi_n}^{(1)} \quad (6.12)$$

satisfies (6.4). Therefore, the algebra of such symmetries contains the universal enveloping algebra of the conformal algebra of the background superspace. As was discussed in the previous section, (6.12) admits a decomposition as a sum of symmetry operators determined by their top component

$$\mathfrak{D}^{(n)} = \mathfrak{D}_{\zeta}^{(n)} + \dots + \zeta_0, \quad (6.13)$$

if the superspace is conformally-flat. Therefore, it is of interest to construct the symmetry operators $\mathfrak{D}_\zeta^{(n)}$ in backgrounds with vanishing super-Weyl tensor, $W^{\alpha\beta} = 0$.

Here we will restrict our attention to the evaluation of $\mathfrak{D}_\zeta^{(2)}$. When acting on the hypermultiplet it takes the form

$$\begin{aligned} \mathfrak{D}_\zeta^{(2)} q^i &= \zeta^{ab} \nabla_a \nabla_b q^i - \frac{1}{2} \zeta^{a\alpha i} \nabla_a \nabla_{\alpha j} q^j + \zeta^{ai}{}_j \nabla_a q^j + \zeta^a \nabla_a q^i - \frac{1}{2} \zeta^{\alpha i} \nabla_{\alpha j} q^j \\ &\quad + \zeta^i{}_j q^j + \zeta q^i. \end{aligned} \quad (6.14)$$

The unique solution compatible with (6.4) and (6.6) is

$$\zeta^{a\alpha i} = -\frac{i}{16} (\tilde{\gamma}_b)^{\alpha\beta} \nabla_\beta^i \zeta^{ab}, \quad (6.15a)$$

$$\zeta^{aij} = -\frac{3}{80} i \nabla^{(i} \tilde{\gamma}_b \nabla^{j)} \zeta^{ab}, \quad (6.15b)$$

$$\zeta^a = \frac{3}{4} \nabla_b \zeta^{ab}, \quad (6.15c)$$

$$\zeta^{\alpha i} = -\frac{i}{20} (\tilde{\gamma}_a)^{\alpha\beta} \nabla_\beta^i \nabla_b \zeta^{ab} + \frac{1}{800} (\tilde{\gamma}_a)^{\alpha\beta} (\tilde{\gamma}_b)^{\gamma\delta} \nabla_\gamma^i \nabla_{(\beta}^j \nabla_{\delta)j} \zeta^{ab}, \quad (6.15d)$$

$$\zeta^{ij} = -\frac{i}{80} \nabla^{(i} \tilde{\gamma}_a \nabla^{j)} \nabla_b \zeta^{ab}, \quad (6.15e)$$

$$\zeta = -\frac{3}{20} \nabla_a \nabla_b \zeta^{ab} + \frac{1}{800} (\tilde{\gamma}_a)^{\alpha\beta} (\tilde{\gamma}_b)^{\gamma\delta} \nabla_{(\alpha}^i \nabla_{\beta)i} \nabla_{(\gamma}^j \nabla_{\delta)j} \zeta^{ab}. \quad (6.15f)$$

In particular, we find that all parameters are expressed solely in terms of the conformal Killing tensor ζ^{ab} .

For completeness, we also present the SU(2) superspace form of $\mathfrak{D}_\zeta^{(2)}$. A routine de-gauging leads to

$$\begin{aligned} \mathfrak{D}_\zeta^{(2)} q^i &= \zeta^{ab} \mathcal{D}_a \mathcal{D}_b q^i - \frac{1}{2} \hat{\zeta}^{a\alpha i} \mathcal{D}_a \mathcal{D}_{\alpha j} q^j + \hat{\zeta}^{ai}{}_j \mathcal{D}_a q^j + \hat{\zeta}^a \mathcal{D}_a q^i - \frac{1}{2} \hat{\zeta}^{\alpha i} \mathcal{D}_{\alpha j} q^j \\ &\quad + \hat{\zeta}^i{}_j q^j + \hat{\zeta} q^i, \end{aligned} \quad (6.16)$$

where we have employed the definitions

$$\hat{\zeta}^{a\alpha i} = -\frac{i}{16} (\tilde{\gamma}_b)^{\alpha\beta} \mathcal{D}_\beta^i \zeta^{ab}, \quad (6.17a)$$

$$\hat{\zeta}^{aij} = -\frac{3}{80} i \mathcal{D}^{(i} \tilde{\gamma}_b \mathcal{D}^{j)} \zeta^{ab} - \frac{2}{5} C_b^{ij} \zeta^{ab}, \quad (6.17b)$$

$$\hat{\zeta}^a = \frac{3}{4} \mathcal{D}_b \zeta^{ab}, \quad (6.17c)$$

$$\begin{aligned} \hat{\zeta}^{\alpha i} &= -\frac{i}{20} (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{D}_\beta^i \mathcal{D}_b \zeta^{ab} + \frac{1}{800} (\tilde{\gamma}_a)^{\alpha\beta} (\tilde{\gamma}_b)^{\gamma\delta} \mathcal{D}_\gamma^i \mathcal{D}_{(\beta}^j \mathcal{D}_{\delta)j} \zeta^{ab} \\ &\quad + \frac{i}{25} (\tilde{\gamma}_a)^{\alpha\beta} C_b^{ij} \mathcal{D}_{\beta j} \zeta^{ab} - \frac{11}{50} i (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{N}_{b\beta}{}^i \zeta^{ab} + \frac{i}{10} (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{C}_{b\beta}{}^i \zeta^{ab}, \end{aligned} \quad (6.17d)$$

$$\begin{aligned} \hat{\zeta}^{ij} &= -\frac{i}{80} \mathcal{D}^{(i} \tilde{\gamma}_a \mathcal{D}^{j)} \mathcal{D}_b \zeta^{ab} - \frac{i}{80} (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{C}_{b\alpha}{}^{(i} \mathcal{D}_{\beta}^{j)} \zeta^{ab} - \frac{3}{80} i (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{N}_{b\alpha}{}^{(i} \mathcal{D}_{\beta}^{j)} \zeta^{ab} \\ &\quad + \frac{i}{80} C_a^{k(i} \mathcal{D}^{j)} \tilde{\gamma}_b \mathcal{D}^k \zeta^{ab} - \frac{i}{10} (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{D}_\alpha^{(i} \mathcal{C}_{b\beta}^{j)} \zeta^{ab} + \frac{i}{10} (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{D}_\alpha^{(i} \mathcal{N}_{b\beta}^{j)} \zeta^{ab} \\ &\quad + \frac{5}{16} i (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{C}_{b\alpha}{}^{(i} \mathcal{D}_{\beta}^{j)} \zeta^{ab} - \frac{5}{16} i (\tilde{\gamma}_a)^{\alpha\beta} \mathcal{N}_{b\alpha}{}^{(i} \mathcal{D}_{\beta}^{j)} \zeta^{ab} + \mathcal{D}_a C_b^{ij} \zeta^{ab}, \end{aligned}$$

$$\begin{aligned}
\hat{\zeta} = & -\frac{3}{20}\mathcal{D}_a\mathcal{D}_b\zeta^{ab} + \frac{1}{800}(\tilde{\gamma}_a)^{\alpha\beta}(\tilde{\gamma}_b)^{\gamma\delta}\mathcal{D}_{(\alpha}^i\mathcal{D}_{\beta)i}\mathcal{D}_{(\gamma}^j\mathcal{D}_{\delta)j}\zeta^{ab} + \frac{73}{800}iC_a^{ij}\mathcal{D}_{(i}\tilde{\gamma}_b\mathcal{D}_{j)}\zeta^{ab} \\
& -\frac{i}{10}(\tilde{\gamma}_a)^{\alpha\beta}(\tilde{\gamma}_b)^{\gamma\delta}N_{\alpha\gamma}\mathcal{D}_{(\beta}^i\mathcal{D}_{\delta)i}\zeta^{ab} + \frac{i}{50}(\tilde{\gamma}_a)^{\alpha\beta}\mathcal{D}_\alpha^i\mathcal{N}_{b\beta}^i\zeta^{ab} + \frac{i}{10}(\tilde{\gamma}_a)^{\alpha\beta}\mathcal{D}_\alpha^i\mathcal{C}_{b\beta}^i\zeta^{ab} \\
& +\frac{81}{10}C_a^{ij}C_{bij}\zeta^{ab} - \frac{12}{5}(\tilde{\gamma}_a)^{\alpha\beta}(\tilde{\gamma}_b)^{\gamma\delta}N_{\alpha\gamma}N_{\beta\delta}\zeta^{ab} + \frac{91}{400}i(\tilde{\gamma}_a)^{\alpha\beta}\mathcal{C}_{b\alpha}^i\mathcal{D}_{\beta i}\zeta^{ab} \\
& -\frac{47}{80}i(\tilde{\gamma}_a)^{\alpha\beta}\mathcal{N}_{b\alpha}^i\mathcal{D}_{\beta i}\zeta^{ab} .
\end{aligned} \tag{6.17e}$$

It may be verified that (6.16) is a superconformal dimension-0 operator

$$\delta_\sigma\mathcal{D}_\zeta^{(2)}q^i = 2\sigma\mathcal{D}_\zeta^{(2)}q^i , \tag{6.18}$$

and yields a symmetry on conformally-flat superspace backgrounds

$$\mathcal{D}_\alpha^{(i}\mathcal{D}_\zeta^{(2)}q^{j)} = 0 . \tag{6.19}$$

As a result, we have shown the existence of the higher symmetry operator $\mathcal{D}_\zeta^{(2)}$, which is completely determined in terms of ζ^{ab} . We expect that, in conformally-flat superspaces, this is true for symmetries of all orders; every $\mathcal{D}_\zeta^{(n)}$ is uniquely determined in terms of its top component, the conformal Killing tensor superfield $\zeta^{a(n)}$, as was shown for the non-supersymmetric case in section 5.

7 Higher symmetries of the vector multiplet

The higher symmetry operators (6.7) belong to a broader family of symmetry operators acting on tensor superfields of arbitrary index structure. Here we will generalise these operators by adding Lorentz dependent terms via an analysis of the higher symmetries of the vector multiplet.

7.1 Superconformal vector multiplet

Consider a vector multiplet coupled to conformal supergravity. Its dynamics is described by the higher-derivative action constructed in [16], which is a locally supersymmetric extension of $F\Box F$. The vector multiplet can be realised in terms of the field strength $F^{\alpha i}$ subject to the Bianchi identities [16, 78, 83]

$$\nabla_\alpha^{(i}F^{\beta j)} - \frac{1}{4}\delta_\alpha^\beta\nabla_\gamma^{(i}F^{\gamma j)} = 0, \quad \nabla_\alpha^i F_i^\alpha = 0 . \tag{7.1}$$

The field strength $F^{\alpha i}$ is a primary superfield of dimension $\frac{3}{2}$,

$$S_j^\beta F^{\alpha i} = 0, \quad \mathbb{D}F^{\alpha i} = \frac{3}{2}F^{\alpha i} . \tag{7.2}$$

The equation of motion for the superconformal vector multiplet [84] is

$$\mathcal{G}^{ij} := \nabla^a\nabla_a X^{ij} - 2Y_\alpha^{\beta ij}F_\beta^\alpha + \frac{5}{2}X^\alpha(i\overleftrightarrow{\nabla}_{\alpha\beta}F^{\beta j}) = 0, \tag{7.3}$$

where we have defined $S \overleftrightarrow{\nabla}_a T := S \nabla_a T - (\nabla_a S) T$, for arbitrary superfields S and T , and introduced the following descendants of F_i^α :

$$X^{ij} := \frac{i}{4} \nabla_\gamma^{(i} F^{\gamma j)}, \quad F_\alpha^\beta := -\frac{i}{4} \left(\nabla_\alpha^k F_k^\beta - \frac{1}{4} \delta_\alpha^\beta \nabla_\gamma^k F_k^\gamma \right) = -\frac{i}{4} \nabla_\alpha^k F_k^\beta. \quad (7.4)$$

The equation of motion (7.3) involves the torsion superfields $Y_\alpha^{\beta ij}$ and $X^{\alpha i}$ which are defined according to (2.23).

Let $F^{\alpha i}$ be a solution of the equations (7.1) and (7.3). A superconformal dimension-0 operator \mathfrak{D} is called a symmetry of these equations if $\mathfrak{D} F^{\alpha i}$ is also a solution. Given a conformal Killing vector superfield ξ^a , the first-order operator $\mathfrak{D}_\xi^{(1)}$ defined by (3.12) is a symmetry. Higher-order symmetries of the equations (7.1) and (7.3) may be generated by considering products of the first-order symmetries,

$$\mathfrak{D}^{(n)} := \mathfrak{D}_{\xi_1}^{(1)} \mathfrak{D}_{\xi_2}^{(1)} \dots \mathfrak{D}_{\xi_n}^{(1)}. \quad (7.5)$$

7.2 Supersymmetric Maxwell theory

In SU(2) superspace, the off-shell vector multiplet is described by a superfield $F^{\alpha i}$ subject to the constraints

$$\mathcal{D}_\alpha^{(i} F^{\beta j)} - \frac{1}{4} \delta_\alpha^\beta \mathcal{D}_\gamma^{(i} F^{\gamma j)} = 0, \quad \mathcal{D}_\alpha^i F_i^\alpha = 0. \quad (7.6)$$

These constraints are super-Weyl invariant provided $F^{\alpha i}$ is a primary superfield with the super-Weyl transformation

$$\delta_\sigma F^{\alpha i} = \frac{3}{2} \sigma F^{\alpha i}. \quad (7.7)$$

When the vector multiplet is placed on-shell, it obeys the additional equation

$$\mathcal{D}_\alpha^{(i} F^{\alpha j)} = 0 \quad \implies \quad \mathcal{D}_\alpha^{(i} F^{\beta j)} = 0 \quad (7.8)$$

It is important to note that the equation (7.8) is not super-Weyl invariant, since the super-Weyl invariance has been fixed by imposing an appropriate gauge condition. In the superconformal setting to Poincaré supergravity, the vector multiplet couples to the tensor compensator Φ introduced in section 3.3. The compensator appears in the superspace action for the vector multiplet [15], and the action is super-Weyl invariant. The corresponding equation of motion for the vector multiplet is

$$\frac{1}{4} \Phi \mathcal{D}_\alpha^{(i} F^{\alpha j)} + \mathcal{D}_\alpha^{(i} \Phi F^{\alpha j)} = 0. \quad (7.9)$$

Choosing the super-Weyl gauge

$$\Phi = 1 \quad (7.10)$$

reducing the equation of motion to (7.8). In this subsection we make use of the super-Weyl gauge (7.10).

We now turn to the analysis of the higher symmetries of this theory. The operator \mathfrak{D} is said to be a symmetry of the (on-shell) vector multiplet if

$$\mathcal{D}_\alpha^{(i} \mathfrak{D} F^{\beta j)} = 0, \quad \mathcal{D}_\alpha^i \mathfrak{D} F_i^\alpha = 0. \quad (7.11)$$

The most general n^{th} -order symmetry operator for the vector multiplet is

$$\begin{aligned} \mathfrak{D}^{(n)} = & \sum_{k=0}^n \zeta^{a(k)} \mathcal{D}_{a_1} \dots \mathcal{D}_{a_k} + \sum_{k=0}^{n-1} \zeta^{a(k)\beta}{}_i \mathcal{D}_{a_1} \dots \mathcal{D}_{a_k} \mathcal{D}_\beta^i \\ & + \sum_{k=0}^{n-1} \zeta^{a(k)ij} \mathcal{D}_{a_1} \dots \mathcal{D}_{a_k} J_{ij} + \frac{1}{2} \sum_{k=0}^{n-1} \zeta^{a(k),bc} \mathcal{D}_{a_1} \dots \mathcal{D}_{a_k} M_{bc} \\ & + \frac{1}{2} \sum_{k=0}^{n-1} \zeta^{a(k-1)\alpha, bc}{}_i \mathcal{D}_{a_1} \dots \mathcal{D}_{a_{k-1}} \mathcal{D}_\alpha^i M_{bc} + \frac{1}{2} \sum_{k=0}^{n-2} \zeta^{a(k),bcij} \mathcal{D}_{a_1} \dots \mathcal{D}_{a_k} M_{bc} J_{ij}, \end{aligned} \quad (7.12)$$

where $\zeta^{a(k)}$ is a conformal Killing tensor (4.8). When $n = 1$, its action on $F^{\alpha i}$ reduces to

$$\mathfrak{D}_\xi^{(1)} F^{\alpha i} = \delta_{\mathcal{K}[\xi]} F^{\alpha i} = (\xi^B \mathcal{D}_B + K_{jk}[\xi] J^{jk} + \frac{1}{2} K^{bc}[\xi] M_{bc}) F^{\alpha i}. \quad (7.13)$$

As the procedure to compute (7.12), say for $n = 2$, is analogous to that of (6.9) for the hypermultiplet, we will not pursue such analysis here. Instead, we will extract some non-trivial information regarding the structure of this operator for general n via (7.11). Our analysis reveals the following restrictions on its parameters:

$$\zeta^{a(n-1)\beta i} = \frac{i}{4(n+2)} \nabla_\alpha^i \zeta^{a(n-1)b} (\tilde{\gamma}_b)^{\alpha\beta}, \quad (7.14)$$

$$\zeta^{a(n-1)ij} = -\frac{in(n+1)}{8(n+2)(n+3)} \nabla^{(i} \tilde{\gamma}_b \nabla^{j)} \zeta^{a(n-1)b}, \quad (7.15)$$

$$\begin{aligned} \zeta^{a(n-1)}{}_\alpha{}^\beta = & -\frac{i}{8(n+2)} \left((\tilde{\gamma}_b)^{\beta\gamma} \mathcal{D}_\alpha^i \mathcal{D}_{\gamma i} - \frac{1}{4} \delta_\alpha^\beta \mathcal{D}^i \tilde{\gamma}_b \mathcal{D}_i \right) \zeta^{a(n-1)b} \\ & + n \zeta^{a(n-1)b} N_{bcd} (\gamma^{cd})_\alpha{}^\beta, \end{aligned} \quad (7.16)$$

as well as the Killing condition for tensor superfields

$$\mathcal{D}^b \zeta_{a(n-1)b} = 0 \quad \implies \quad \mathcal{D}_{(a_1} \zeta_{a_2 \dots a_{n+1})} = 0. \quad (7.17)$$

Equations (4.8) and (7.17) define $\mathcal{N} = (1, 0)$ Killing tensor superfields in six dimensions.¹² Given two Killing tensor superfields $\zeta_1^{a(m)}$ and $\zeta_2^{b(n)}$, it may be shown that their bracket, defined by (4.12), is also Killing

$$\mathcal{D}^b [\zeta_1, \zeta_2]_{a(m+n-2)b} = 0. \quad (7.18)$$

¹²The concept of a Killing tensor superfield was introduced for the first time in [80] in the framework of $\mathcal{N} = 1$ AdS supersymmetry in four dimensions.

8 Maximally supersymmetric backgrounds

The existence of (conformal) Killing vector and tensor superfields places non-trivial restrictions on the superspace geometry. So far we have not examined the constraints imposed by such conditions. In this section the case of Killing vectors (where eq. (3.32) is imposed) is further elaborated on. More results are given in appendix C, where we discuss how to obtain component results from superspace.

Here we restrict ourselves to the case of eight supercharges, i.e. maximally $\mathcal{N} = (1, 0)$ supersymmetric backgrounds, and derive constraints on the superspace geometry. By a similar analysis¹³ to [57], in such backgrounds it may be shown that (C.1) implies

$$\mathcal{D}_\alpha^i C_a{}^{kl} = 0, \quad \mathcal{D}_\alpha^i W_{abc} = 0, \quad \mathcal{D}_\alpha^i N_{abc} = 0. \quad (8.1)$$

Additionally, the Killing spinor equation (3.22) reduces to

$$\mathcal{D}_a \xi_k^\gamma = (\gamma_{ab})_\beta{}^\gamma \xi^{\beta j} C_{jk}^b + \frac{1}{2} (\gamma^{bc})_\beta{}^\gamma \xi_k^\beta (W_{abc} + 2N_{abc}). \quad (8.2)$$

Equation (8.1) leads to severe restrictions on the background superspace geometry. In particular, the integrability conditions $\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} C_a{}^{kl} = 0$, $\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} W_{abc} = 0$, and $\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} N_{abc} = 0$ imply the following differential equations

$$\mathcal{D}_d W_{abc} = -6(W_{d[a}{}^e + 2N_{d[a}{}^e])W_{bc]e}, \quad (8.3a)$$

$$\mathcal{D}_d N_{abc} = -6(W_{d[a}{}^e + 2N_{d[a}{}^e])N_{bc]e}, \quad (8.3b)$$

$$\mathcal{D}_d C_a{}^{kl} = -2(W_{dab} + 2N_{dab})C^{bkl} - 6C_d{}^{p(k} C_{ap}{}^{l)}, \quad (8.3c)$$

together with the algebraic conditions

$$(\gamma_{[a}{}^{de})_{\alpha\beta} W_{bc]d} C_e{}^{ij} = 0, \quad (8.4a)$$

$$(\gamma_{[a}{}^{de})_{\alpha\beta} N_{bc]d} C_e{}^{ij} = 0, \quad (8.4b)$$

$$(\gamma_{abc})_{\alpha\beta} C^{bil} C^{cjk} = -\frac{2}{3} N^{bcd} (\gamma_{bcd})_{\alpha\beta} (\varepsilon^{j(i} C_a{}^{l)k} + \varepsilon^{k(i} C_a{}^{j)l}). \quad (8.4c)$$

Note that (8.3) can be compactly rewritten as

$$\tilde{\mathcal{D}}_d W_{abc} = 0, \quad \tilde{\mathcal{D}}_d N_{abc} = 0, \quad \tilde{\mathcal{D}}_d C_a{}^{kl} = 0, \quad (8.5)$$

where we have defined

$$\tilde{\mathcal{D}}_a := \mathcal{D}_a - 3C_a{}^{kl} J_{kl} + (W_{abc} + 2N_{abc}) M^{bc}. \quad (8.6)$$

A lengthy, though straightforward, analysis of the consistency of (8.3) and (8.4) together with the superspace Bianchi identities leads to the following algebraic constraints

$$C_a{}^{kl} = C_a C^{kl}, \quad C^{ij} C_{ij} = 2, \quad (8.7a)$$

$$W_{abc} C_d = 0, \quad (8.7b)$$

$$N_{abc} C_d = 0, \quad (8.7c)$$

$$W^{\alpha\gamma} N_{\beta\gamma} = \frac{1}{4} \delta_\beta^\alpha W^{\gamma\delta} N_{\gamma\delta}, \quad (8.7d)$$

¹³For any background admitting eight supercharges, if there is a tensor superfield T such that its bar-projection vanishes, $T| = 0$, and this condition is supersymmetric, then the entire superfield is zero, $T = 0$. See [57] for a more detailed discussion.

as well as the conditions

$$\mathcal{D}_A C_a = 0, \quad \mathcal{D}_A C^{kl} = 0, \quad \mathcal{D}_A N_{bcd} = 0, \quad \mathcal{D}_A W_{bcd} = 0. \quad (8.8)$$

It should be emphasised that, due to (8.7b) and (8.7c), two branches of solutions exist, defined by: (i) $C_a = 0$; and (ii) $W_{abc} = N_{abc} = 0$.

Note that the algebraic constraint (8.7d), due to the (anti-)self-duality conditions on W_{abc} and N_{abc} , is equivalent to any of the following relations

$$N_{[a}{}^{de}W_{b]de} = 0 \iff N_{a[b}{}^eW_{cd]e} = 0 \iff W_{a[b}{}^eN_{cd]e} = 0 \iff N_{[ab}{}^eW_{cd]e} = 0, \quad (8.9)$$

while the following relations hold identically

$$W_{[a}{}^{de}W_{b]de} = 0 \iff W_{a[b}{}^eW_{cd]e} = 0 \iff W_{[ab}{}^eW_{cd]e} = 0 \iff W_{[abc}W_{def]} = 0, \quad (8.10a)$$

$$N_{[a}{}^{de}N_{b]de} = 0 \iff N_{a[b}{}^eN_{cd]e} = 0 \iff N_{[ab}{}^eN_{cd]e} = 0 \iff N_{[abc}N_{def]} = 0. \quad (8.10b)$$

By a routine calculation it may be shown that

$$N_{abc} = \alpha \left(\omega_{[a}^{(0)} \omega_b^{(1)} \omega_c^{(2)} + \omega_{[a}^{(3)} \omega_b^{(4)} \omega_c^{(5)} \right), \quad (8.11a)$$

$$W_{abc} = \beta \left(\omega_{[a}^{(0)} \omega_b^{(1)} \omega_c^{(2)} - \omega_{[a}^{(3)} \omega_b^{(4)} \omega_c^{(5)} \right), \quad (8.11b)$$

is a solution to (8.7d), provided that $\omega_a^{(i)}$, $0 \leq i \leq 5$, are orthogonal one-forms, that is,

$$\omega_a^{(i)} \omega^{(j)a} = 0, \quad (i \neq j). \quad (8.12)$$

This result was originally derived in [85] (see also [86]). These one-forms may be normalised to constitute an orthonormal basis, and then the expressions for N_{abc} and W_{abc} will, in general, involve overall factors α and β as in (8.11).

In accordance with our analysis, for every maximally supersymmetric $\mathcal{N} = (1, 0)$ background the algebra of covariant derivatives is given by the following graded commutation relations:

$$\begin{aligned} \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} &= -2i\varepsilon^{ij}(\gamma^a)_{\alpha\beta} \mathcal{D}_a + \left(-2i\varepsilon^{ij}(\gamma^a)_{\alpha\beta} (W^{acd} + 2N^{acd}) + 2i(\gamma^{acd})_{\alpha\beta} C_a C^{ij} \right) M_{cd} \\ &\quad + \left(6i\varepsilon^{ij}(\gamma^a)_{\alpha\beta} C_a C^{kl} + \frac{8i}{3}(\gamma^{abc})_{\alpha\beta} N_{abc} \varepsilon^{i(k} \varepsilon^{l)j} \right) J_{kl}, \end{aligned} \quad (8.13a)$$

$$[\mathcal{D}_a, \mathcal{D}_\gamma^k] = \left((\gamma_{ab})_\gamma{}^\delta C^b C^k{}_l - \frac{1}{2}(W_{abc} + 2N_{abc})(\gamma^{bc})_\gamma{}^\delta \delta_l^k \right) \mathcal{D}_\delta^l, \quad (8.13b)$$

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_b] &= \left(8\delta_{[a}^{[c} C_{b]} C^{d]} - 4\delta_{[a}^{[c} \delta_{b]}^{d]} C^e C_e \right) M_{cd} \\ &\quad + \left(8\delta_{[a}^{[c} N^{d]ef} N_{b]ef} + 2\delta_{[a}^{[c} W^{d]ef} W_{b]ef} + 16N_{[a}{}^{e[c} W_{b]e}{}^{d]} \right) M_{cd}. \end{aligned} \quad (8.13c)$$

The algebra is determined by the four tensors C_a , C^{kl} , N_{bcd} and W_{abc} , which are covariantly constant, eq. (8.8), and obey the algebraic constraints (2.7) and (8.7). In conjunction with the Lorentz and R -symmetry commutation relations, $[M_{cd}, \mathcal{D}_A]$ and $[J_{kl}, \mathcal{D}_A]$, the graded commutation relations (8.13) define the most general superalgebras with eight

supercharges, which are associated with the maximally supersymmetric backgrounds of $\mathcal{N} = (1, 0)$ Poincaré supergravity. These superalgebras were derived two years ago [87] using sophisticated algebraic techniques. Here we have demonstrated that the superspace techniques allow one to derive these superalgebras via a simple calculation.

In accordance with the discussion in appendix C, the commutation relation (8.13c) is equivalent to that of the spacetime covariant derivatives. Therefore we can immediately read off the Ricci tensor, the scalar curvature and the Weyl tensor:

$$R_{ab} = -16C_a C_b + 16\eta_{ab} C^c C_c - 16N_a{}^{cd} N_{bcd} - 4W_a{}^{cd} W_{bcd} - \frac{8}{3}\eta_{ab} N^{cde} W_{cde}, \quad (8.14a)$$

$$R = 80C^c C_c - 16N^{cde} W_{cde}, \quad (8.14b)$$

$$\begin{aligned} C_{abcd} &= R_{abcd} - \frac{1}{2}(\eta_{a[c} R_{d]b} - \eta_{b[c} R_{d]a}) + \frac{1}{10} R \eta_{a[c} \eta_{d]b}, \\ &= 16N_{a[c}{}^e W_{d]be} - 16N_{b[c}{}^e W_{d]ae} + \frac{16}{15} \eta_{a[c} \eta_{d]b} N^{efg} W_{efg}. \end{aligned} \quad (8.14c)$$

Note that the condition $W_{abc} = 0$ implies that the superspace is conformally flat [16]. As a result of (8.14c) it is clear that $W_{abc} = 0$ implies $C_{abcd} = 0$, though the reverse is not true in general.

It should be remarked that the algebra of covariant derivatives (8.13) takes a particularly simple form if the torsion-free covariant derivative \mathcal{D}_a is replaced with the torsionful one defined by (8.6). One obtains

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} = -2i\varepsilon^{ij}(\gamma^a)_{\alpha\beta} \tilde{\mathcal{D}}_a + 2i(\gamma^{acd})_{\alpha\beta} C_a C^{ij} M_{cd} - \frac{8i}{3}(\gamma^{abc})_{\alpha\beta} N_{abc} J^{ij}, \quad (8.15a)$$

$$[\tilde{\mathcal{D}}_a, \mathcal{D}_\gamma^k] = \left\{ [(\gamma_{ab})_\gamma{}^\delta C^b - 3C_a \delta_\gamma^b] C^k{}_l - (W_{abc} + 2N_{abc})(\gamma^{bc})_\gamma{}^\delta \delta_l^k \right\} \mathcal{D}_\delta^l, \quad (8.15b)$$

$$[\tilde{\mathcal{D}}_a, \tilde{\mathcal{D}}_b] = 4(W_{ab}{}^d + 2N_{ab}{}^d) \tilde{\mathcal{D}}_d + \left(8\delta_{[a}^{[c} C_{b]} C^d - 4\delta_{[a}^{[c} \delta_{b]}^d C^e C_e \right) M_{cd}. \quad (8.15c)$$

We see that the R -symmetry curvature vanishes if $N_{abc} = 0$. The graded commutation relations take a remarkably simple form if $N_{abc} = 0$ and $C_a = 0$; the bosonic body of this superspace is a conformally flat $\text{AdS}_3 \times S^3$ spacetime or a pp-wave, see later.

We now employ the above analysis to identify all possible maximally supersymmetric spacetimes, which are the bosonic bodies of the superspaces with geometry (8.13), or equivalently (8.15). The most obvious solution is Minkowski space, $\mathbb{M}^6 \equiv \mathbb{R}^{5,1}$, which corresponds to the choice $C_a = 0$ and $N_{abc} = W_{abc} = 0$. When this is not the case, it follows from (8.7b) and (8.7c) that there are two disconnected branches of solutions, defined by $C_a \neq 0$ or $C_a = 0$.

Solutions belonging to the branch $C_a \neq 0$, which necessarily have $N_{abc} = W_{abc} = 0$, are characterised by the existence of a parallel, nowhere vanishing vector field. Thus, since $C^2 = C^a C_a$ is constant, the possible backgrounds are locally equivalent to the following three cases, $\mathbb{R} \times S^5$ for $C^2 < 0$, $\text{AdS}_5 \times \mathbb{R}$ when $C^2 > 0$ and a pp-wave spacetime if $C^2 = 0$ [87].

When $C_a = 0$, the corresponding geometries are described by the covariantly constant three-forms N_{abc} and W_{abc} , which decompose as the sum of two orthogonal simple forms (8.11). If either of the corresponding simple forms are null, the background is a

pp-wave spacetime [88]. When this is not the case, it follows that locally the spacetime decomposes into the product of two three-dimensional symmetric spaces. This can be inferred by the structure of the three-form fluxes given in (8.11). In particular, the possible solutions are [87]: (i) $\text{AdS}_3 \times S^3$; (ii) $\text{AdS}_3 \times \mathbb{R}^3$; and (iii) $\mathbb{R}^{2,1} \times S^3$.

In general, for backgrounds belonging to (i) the radii of the AdS_3 and S^3 do not necessarily coincide — in particular, (ii) and (iii) are degenerate cases of (i). Additionally, if $N_{abc} = 0$ and $W_{abc} \neq 0$, their radii must be equal (proportional to β in (8.11)). This background is one of the well-known solutions to minimal $\mathcal{N} = (1, 0)$ supergravity in six dimensions [89]. It is also an example of a superspace which is not superconformally flat, $W_{abc} \neq 0$, though its bosonic body is conformally flat, $C_{abcd} \equiv 0$.¹⁴

So far we have not specified any conformal compensators Ξ . We have worked in the super-Weyl gauge (3.31), where Ξ is a descendant of the compensators Ξ which is a singlet under the structure group and has the properties: (i) it is an algebraic function of Ξ ; (ii) it is nowhere vanishing; and (iii) it has a non-zero super-Weyl weight w_{Ξ} , $\delta_{\sigma}\Xi = w_{\Xi}\sigma\Xi$. Additional constraints on supergravity backgrounds often occur once a specific choice of compensators is made.

Let us analyse the case of the compensators introduced in section 3.3, specifically: the tensor multiplet Φ and the linear multiplet $G^{ij} = G^{ji}$. Then it is possible to identify Ξ with Φ . In the super-Weyl gauge (7.10), the tensor multiplet constraint (3.35) reduces to

$$C_a^{ij} = 0 . \tag{8.16}$$

Every Killing supervector field ξ^B must leave the linear compensator G^{ij} invariant,

$$\left(\xi^B \mathcal{D}_B + K^{kl}[\xi] J_{kl}\right) G^{ij} = 0 , \tag{8.17}$$

in accordance with (3.34b). In the case of a maximally supersymmetric background, this equation implies that G^{ij} is annihilated by the spinor covariant derivatives,

$$\mathcal{D}_{\alpha}^i G^{jk} = 0 . \tag{8.18}$$

Now, the integrability condition $\{\mathcal{D}_{\alpha}^i, \mathcal{D}_{\beta}^j\} G^{kl} = 0$ leads to the constraint

$$N^{abc} (\gamma_{abc})_{\alpha\beta} (\varepsilon^{k(i} G^{j)l} + \varepsilon^{l(i} G^{j)k}) = 0 , \tag{8.19}$$

which is solved by

$$N_{abc} = 0 . \tag{8.20}$$

We have shown that the conditions (8.16) and (8.20) hold for all maximally supersymmetric backgrounds of Poincaré supergravity with the tensor and linear compensators.

Instead of identifying $\Xi = \Phi$ as has been done above, we can instead choose $\Xi = G$. Next, we impose the super-Weyl gauge

$$G^2 = 1 \iff G^i_k G^k_j = -\delta_j^i . \tag{8.21}$$

¹⁴The reader can consult [90] for an interesting discussion of superconformal flatness of $\text{AdS}_p \times S^q$ superspaces based on coset constructions.

Then the analyticity constraint (3.36) and the super-Weyl gauge condition (8.21) tell us that G^{ij} is annihilated by the spinor covariant derivatives, eq. (8.18), and thus the integrability condition $\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\}G^{kl} = 0$ must hold. The latter contains nontrivial information, in accordance with the anti-commutation relation (2.5a). Specifically, the integrability condition tells us that the condition (8.20) holds. Every Killing supervector field ξ^B must leave the tensor compensator Φ invariant,

$$\xi^B \mathcal{D}_B \Phi = 0, \tag{8.22}$$

in accordance with (3.34b). In the case of a maximally supersymmetric background, this equation implies that Φ is annihilated by the spinor covariant derivatives, and therefore

$$\Phi = \text{const} . \tag{8.23}$$

As a result, the tensor multiplet constraint (3.35) reduces to (8.16).

We have discussed the two possible choices: (i) $\Xi = \Phi$; and (ii) $\Xi = G$. Both of them lead to the same maximally supersymmetric backgrounds, which are characterised by the conditions (8.16) and (8.20). The superspace torsion is determined by the super-Weyl tensor W_{abc} which is covariantly constant. Such a superspaces are the only maximally supersymmetric solutions of Poincaré supergravity. Let us discuss this point in more detail.

The equations of motion for Poincaré supergravity have the simplest form in conformal superspace. In this setting, the tensor compensator Φ and the linear compensator G^{ij} obey the constraints

$$\nabla_\alpha^{(i} \nabla_\beta^{j)} \Phi = 0, \tag{8.24a}$$

$$\nabla_\alpha^{(i} G^{jk)} = 0 . \tag{8.24b}$$

For more details, including the Poincaré supergravity action, we refer the reader to [47, 84, 91]. The superfield equations of motion for $\mathcal{N} = (1, 0)$ Poincaré supergravity were derived in [84]. They have the form

$$\mathbb{W}^{\alpha i} = 0, \quad \nabla_\alpha^{(i} \nabla_\beta^{j)} \left(\frac{G}{\Phi} \right) = 0, \tag{8.25}$$

where $\mathbb{W}^{\alpha i}$ is the field strength of a composite vector multiplet

$$\begin{aligned} \mathbb{W}^{\alpha i} = & \frac{1}{G} \nabla^{\alpha\beta} \Upsilon_\beta^i + \frac{4}{G} (W^{\alpha\beta} \Upsilon_\beta^i + 10i X_j^\alpha G^{ij}) - \frac{1}{2G^3} G_{jk} (\nabla^{\alpha\beta} G^{ij}) \Upsilon_\beta^k \\ & + \frac{1}{2G^3} G^{ij} F^{\alpha\beta} \Upsilon_{\beta j} + \frac{i}{16G^5} \varepsilon^{\alpha\beta\gamma\delta} \Upsilon_{\beta j} \Upsilon_{\gamma k} \Upsilon_{\delta l} G^{ij} G^{kl}, \end{aligned} \tag{8.26}$$

with $\Upsilon_\alpha^i := \frac{2}{3} \nabla_{\alpha j} G^{ij}$ and $F_{\alpha\beta} := \frac{i}{4} \nabla_{[\alpha}^k \Upsilon_{\beta]k}$. To make contact with our previous results, we now degauge to SU(2) superspace. Upon degauging, the tensor multiplet constraint (8.24a) turns into (3.35), while the linear multiplet constraint (8.24b) takes the form (8.18). The second equation of motion for Poincaré supergravity in (8.25) becomes

$$(\mathcal{D}_\alpha^{(i} \mathcal{D}_\beta^{j)} + 4i C_{\alpha\beta}^{ij}) \left(\frac{G}{\Phi} \right) = 0 . \tag{8.27}$$

The analysis given above tells us that the following properties hold for all maximally supersymmetric backgrounds: (i) both compensators Φ and G^{ij} are covariantly constant; and (ii) the conditions (8.16) and (8.20) hold. Now eq. (8.27) is satisfied. The first equation of motion in (8.25), $\mathbb{W}^{\alpha i} = 0$, is also satisfied, since all maximally supersymmetric backgrounds have no covariant background spinor superfields.

9 Conclusion

To conclude, we summarise the main results of this paper and outline some interesting areas for future work. Our main outcomes are as follows:

- We have described the structure of (conformal) isometries of $\mathcal{N} = (1, 0)$ supergravity backgrounds within the $SU(2)$ and conformal superspace formulations. In the infinitesimal case they were shown to form a closed algebra on any fixed supergravity background. Further, we detailed how these may be utilised to trivially read off the (conformal) Killing spinor equation at the component level. Its solutions may be uplifted to a unique (conformal) Killing vector superfield on $\mathcal{M}^{6|8}$.
- The conformal Killing spinor superfields ϵ^α , which generate extended conformal supersymmetries, were introduced. In addition, their relation to the conformal Killing vector ξ^a and tensor $\zeta^{a(n)}$ superfields was shown. The former parametrise the conformal isometries of superspace, while the latter are associated with the higher symmetries of the kinetic operators of on-shell multiplets. Additionally, it was proven that the conformal Killing tensors of a fixed superspace form a superalgebra with respect to the bracket (4.12).
- We studied the higher symmetries of three on-shell models in curved backgrounds, namely: (i) the conformal scalar field; (ii) the hypermultiplet; and (iii) the non-conformal vector multiplet. In our analysis of (i) we have, for the first time, derived the explicit form of every higher symmetry operator on curved backgrounds. For (ii), it was proven that the conformal Killing tensor superfields $\zeta^{a(n)}$ generate all (non-trivial) symmetries of their kinetic operators. Finally, in the case of (iii), we deduced that its higher symmetries are parametrised by Killing tensor superfields, which were also introduced in this work (7.17).
- The maximally supersymmetric backgrounds of $\mathcal{N} = (1, 0)$ supergravity in six dimensions were classified. Our analysis leads to the superalgebra (8.13), or equivalently (8.15), which contains three distinct branches. Further, their corresponding space-time backgrounds are derived, reproducing the results of [87–89].

Interesting open problems include the following:

- Our approach to the higher symmetries of the conformal d'Alembertian in section 5 may be immediately generalised to the study of more complex conformal field theories. In particular, it would be interesting to extend this analysis to Maxwell electrodynamics in four dimensions.

- We believe that, as was shown for the conformal d'Alembertian, every higher symmetry operator for the hypermultiplet and vector multiplet is uniquely determined in terms of its top component. It would be interesting to prove this explicitly.
- As an extension of our analysis of the higher symmetries of the (massless) hypermultiplet in section 6, it would be interesting to study the higher symmetries of the massive hypermultiplet on $d = 4, \mathcal{N} = 2$ and $d = 5, \mathcal{N} = 1$ supergravity backgrounds.

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A Conventions

A.1 Spinors in six dimensions

Our 6D notation and conventions are similar to those of [15], with a few minor modifications. All relevant details are summarized here.

The Minkowski metric is $\eta_{ab} = \text{diag}(-1, 1, 1, 1, 1, 1)$, the Levi-Civita tensor ε_{abcdef} is normalised by $\varepsilon_{012345} = -\varepsilon^{012345} = 1$, and the Levi-Civita tensor with world indices is given by $\varepsilon^{mnpqrs} := \varepsilon^{abcdef} e_a^m e_b^n e_c^p e_d^q e_e^r e_f^s$.

We exclusively use four component spinors in the body of the paper, but it is useful to relate these to eight component spinor conventions. The Dirac 8×8 matrices Γ^a and the charge conjugation matrix C obey the relations

$$\begin{aligned} \{\Gamma_a, \Gamma_b\} &= -2\eta_{ab}\mathbb{1}, & (\Gamma^a)^\dagger &= -\Gamma_a, & C\Gamma_a C^{-1} &= -\Gamma_a^T, \\ C^\dagger C &= \mathbb{1}, & C &= C^T = C^* . \end{aligned} \tag{A.1}$$

In particular, $\Gamma_a C^{-1}$ is antisymmetric. The chirality matrix Γ_* is defined by

$$\Gamma_{[a}\Gamma_b\Gamma_c\Gamma_d\Gamma_e\Gamma_f] = \varepsilon_{abcdef}\Gamma_* . \tag{A.2}$$

As a consequence of the above conditions, one can show that

$$\Gamma^a = B(\Gamma^a)^* B^{-1}, \quad B = \Gamma_* \Gamma_0 C^{-1} . \tag{A.3}$$

¹⁵However the entire responsibility of the publication belongs to the owners of the publication. The financial support received from TÜBITAK does not mean that the content of the publication is approved in a scientific sense by TÜBITAK.

The charge conjugate Ψ^c of a Dirac spinor is conventionally defined by

$$\bar{\Psi} \equiv \Psi^\dagger \Gamma_0 =: (\Psi^c)^T C \quad \Longrightarrow \quad \Psi^c = -\Gamma_0 C^{-1} \Psi^* = -\Gamma_* B \Psi^* . \quad (\text{A.4})$$

Because $B^* B = -\mathbb{1}$, charge conjugation is an involution only for objects with an even number of spinor indices, so it is not possible to have Majorana spinors in six dimensions. One can instead have a symplectic Majorana condition when the spinors possess an $SU(2)$ index. Conventionally this is denoted

$$(\Psi_i)^c = \Psi^i \quad \Longrightarrow \quad \Psi^i = -\Gamma_0 C^{-1} (\Psi_i)^* = -\Gamma_* B (\Psi_i)^* \quad (\text{A.5})$$

for a spinor of either chirality. We raise and lower $SU(2)$ indices $i = \underline{1}, \underline{2}$ using the conventions

$$\Psi^i = \varepsilon^{ij} \Psi_j, \quad \Psi_i = \varepsilon_{ij} \Psi^j, \quad \varepsilon^{\underline{1}\underline{2}} = \varepsilon_{\underline{2}\underline{1}} = 1 . \quad (\text{A.6})$$

We employ a Weyl basis for the gamma matrices so that an eight-component Dirac spinor Ψ decomposes into a four-component left-handed Weyl spinor ψ^α and a four-component right-handed spinor χ_α so that

$$\Psi = \begin{pmatrix} \psi^\alpha \\ \chi_\alpha \end{pmatrix}, \quad \Gamma_* = \begin{pmatrix} \delta^\alpha_\beta & 0 \\ 0 & -\delta_\alpha^\beta \end{pmatrix}, \quad \alpha = 1, \dots, 4 . \quad (\text{A.7})$$

The spinors ψ^α and χ_α are valued in the two inequivalent fundamental representations of $\mathfrak{su}^*(4) \cong \mathfrak{so}(5, 1)$. We further take

$$\Gamma^a = \begin{pmatrix} 0 & (\tilde{\gamma}^a)^{\alpha\beta} \\ (\gamma^a)_{\alpha\beta} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & \delta_\alpha^\beta \\ \delta^\alpha_\beta & 0 \end{pmatrix} . \quad (\text{A.8})$$

The Pauli-type 4×4 matrices $(\gamma^a)_{\alpha\beta}$ and $(\tilde{\gamma}^a)^{\alpha\beta}$ are antisymmetric and related by

$$(\tilde{\gamma}^a)^{\alpha\beta} = \frac{1}{2} \varepsilon^{\alpha\beta\gamma\delta} (\gamma^a)_{\gamma\delta}, \quad (\gamma^a)^* = \tilde{\gamma}_a, \quad (\text{A.9})$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the canonical antisymmetric symbol of $\mathfrak{su}^*(4)$. They obey

$$(\gamma^a)_{\alpha\beta} (\tilde{\gamma}^b)^{\beta\gamma} + (\gamma^b)_{\alpha\beta} (\tilde{\gamma}^a)^{\beta\gamma} = -2\eta^{ab} \delta_\alpha^\gamma, \quad (\text{A.10a})$$

$$(\tilde{\gamma}^a)^{\alpha\beta} (\gamma^b)_{\beta\gamma} + (\tilde{\gamma}^b)^{\alpha\beta} (\gamma^a)_{\beta\gamma} = -2\eta^{ab} \delta_\gamma^\alpha, \quad (\text{A.10b})$$

and as a consequence of (A.3),

$$(\gamma^a)_{\alpha\beta} = B_\alpha^{\dot{\gamma}} B_\beta^{\dot{\delta}} ((\gamma^a)_{\gamma\delta})^*, \quad (\tilde{\gamma}^a)^{\alpha\beta} = B^{\alpha\dot{\gamma}} B^{\beta\dot{\delta}} ((\tilde{\gamma}^a)^{\gamma\delta})^*, \quad B = \begin{pmatrix} B^{\alpha\dot{\beta}} & 0 \\ 0 & B_\alpha^{\dot{\beta}} \end{pmatrix} . \quad (\text{A.11})$$

A dotted index denotes the complex conjugate representation in $\mathfrak{su}^*(4)$. It is natural to use the B matrix to define bar conjugation on a four component spinor via

$$\bar{\psi}^\alpha = B^{\alpha\dot{\beta}} (\psi^\beta)^*, \quad \bar{\chi}_\alpha = B_\alpha^{\dot{\beta}} (\chi_\beta)^*, \quad (\text{A.12})$$

with the obvious extension to any object with multiple spinor indices. For example, $\overline{(\gamma^a)_{\alpha\beta}} = (\gamma^a)_{\alpha\beta}$ using (A.11) and similarly for $\tilde{\gamma}^a$. We also note that, as a consequence of $B^*B = -\mathbb{1}$,

$$\overline{\overline{\psi^\alpha}} = -\psi^\alpha, \tag{A.13}$$

with the natural extension to any tensor carrying an odd number of spinor indices. A symplectic Majorana spinor Ψ_i , decomposed as in (A.7) and obeying (A.5), has Weyl components that obey

$$\overline{\psi^{\alpha i}} = \psi_i^\alpha, \quad \overline{\chi^{\alpha i}} = \chi_\alpha^i. \tag{A.14}$$

The Grassmann coordinates θ_i^α and the parameters η_α^i of S -supersymmetries are both symplectic Majorana-Weyl using this definition.

We define the antisymmetric products of two or three Pauli-type matrices as

$$\gamma_{ab} := \gamma_{[a}\tilde{\gamma}b] := \frac{1}{2}(\gamma_a\tilde{\gamma}b - \gamma_b\tilde{\gamma}a), \quad \tilde{\gamma}_{ab} := \tilde{\gamma}_{[a}\gamma b] = -(\gamma_{ab})^T, \tag{A.15a}$$

$$\gamma_{abc} := \gamma_{[a}\tilde{\gamma}b\gamma c], \quad \tilde{\gamma}_{abc} := \tilde{\gamma}_{[a}\gamma b\tilde{\gamma}c]. \tag{A.15b}$$

Note that γ_{ab} and $\tilde{\gamma}_{ab}$ are traceless, whereas γ_{abc} and $\tilde{\gamma}_{abc}$ are symmetric. Further antisymmetric products obey

$$\gamma_{abc} = -\frac{1}{3!}\varepsilon_{abcdef}\gamma^{def}, \quad \tilde{\gamma}_{abc} = \frac{1}{3!}\varepsilon_{abcdef}\tilde{\gamma}^{def}, \tag{A.16a}$$

$$\gamma_{abcd} = \frac{1}{2}\varepsilon_{abcdef}\gamma^{ef}, \quad \tilde{\gamma}_{abcd} = -\frac{1}{2}\varepsilon_{abcdef}\tilde{\gamma}^{ef}, \tag{A.16b}$$

$$\gamma_{abcde} = \varepsilon_{abcdef}\gamma^f, \quad \tilde{\gamma}_{abcde} = -\varepsilon_{abcdef}\tilde{\gamma}^f, \tag{A.16c}$$

$$\gamma_{abcdef} = -\varepsilon_{abcdef}, \quad \tilde{\gamma}_{abcdef} = \varepsilon_{abcdef}. \tag{A.16d}$$

Making use of the completeness relations

$$(\gamma^a)_{\alpha\beta}(\tilde{\gamma}^a)^{\gamma\delta} = 4\delta_{[\alpha}^{\gamma}\delta_{\beta]}^{\delta}, \tag{A.17a}$$

$$(\gamma^{ab})_{\alpha\beta}(\gamma_{ab})_{\gamma\delta} = -8\delta_\alpha^\delta\delta_\gamma^\beta + 2\delta_\alpha^\beta\delta_\gamma^\delta, \tag{A.17b}$$

$$(\gamma^{abc})_{\alpha\beta}(\tilde{\gamma}_{abc})^{\gamma\delta} = 48\delta_{[\alpha}^{\gamma}\delta_{\beta]}^{\delta}, \tag{A.17c}$$

$$(\gamma^{abc})_{\alpha\beta}(\tilde{\gamma}_{abc})_{\gamma\delta} = (\gamma^{abc})^{\alpha\beta}(\tilde{\gamma}_{abc})^{\gamma\delta} = 0, \tag{A.17d}$$

it is straightforward to establish natural isomorphisms between tensors of $\mathfrak{so}(5,1)$ and matrix representations of $\mathfrak{su}^*(4)$. Vectors V^a and antisymmetric matrices $V_{\alpha\beta} = -V_{\beta\alpha}$ are related by

$$V_{\alpha\beta} := (\gamma^a)_{\alpha\beta}V_a \iff V_a = \frac{1}{4}(\tilde{\gamma}^a)^{\alpha\beta}V_{\alpha\beta}. \tag{A.18}$$

Antisymmetric rank-two tensors F_{ab} are related to traceless matrices F_α^β via

$$F_\alpha^\beta := -\frac{1}{4}(\gamma^{ab})_{\alpha\beta}F_{ab}, \quad F_\alpha^\alpha = 0 \iff F_{ab} = \frac{1}{2}(\gamma_{ab})_{\beta\alpha}F_\alpha^\beta = -F_{ba}. \tag{A.19}$$

Self-dual and anti-self-dual rank-three antisymmetric tensors $T_{abc}^{(\pm)}$,

$$\frac{1}{3!}\varepsilon^{abcdef}T_{def}^{(\pm)} = \pm T^{(\pm)abc}, \tag{A.20}$$

are related to symmetric matrices $T_{\alpha\beta}$ and $T^{\alpha\beta}$ via

$$T_{\alpha\beta} := \frac{1}{3!}(\gamma^{abc})_{\alpha\beta}T_{abc} = T_{\beta\alpha} \iff T_{abc}^{(+)} = \frac{1}{8}(\tilde{\gamma}^{abc})^{\alpha\beta}T_{\alpha\beta}, \quad (\text{A.21a})$$

$$T^{\alpha\beta} := \frac{1}{3!}(\tilde{\gamma}^{abc})^{\alpha\beta}T_{abc} = T^{\beta\alpha} \iff T_{abc}^{(-)} = \frac{1}{8}(\gamma_{abc})_{\alpha\beta}T^{\alpha\beta}. \quad (\text{A.21b})$$

A.2 The $\mathcal{N} = (1, 0)$ superconformal algebra

The bosonic sector of the $\mathcal{N} = (1, 0)$ superconformal algebra contains the translation (P_a), Lorentz (M_{ab}), special conformal (K_a), dilatation (\mathbb{D}) and $\text{SU}(2)$ generators (J_{ij}), where $a, b = 0, 1, 2, 3, 4, 5$ and $i, j = \underline{1}, \underline{2}$. Their algebra is

$$[M_{ab}, M_{cd}] = 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c}, \quad (\text{A.22a})$$

$$[M_{ab}, P_c] = 2\eta_{c[a}P_{b]}, \quad [\mathbb{D}, P_a] = P_a, \quad (\text{A.22b})$$

$$[M_{ab}, K_c] = 2\eta_{c[a}K_{b]}, \quad [\mathbb{D}, K_a] = -K_a, \quad (\text{A.22c})$$

$$[K_a, P_b] = 2\eta_{ab}\mathbb{D} + 2M_{ab}, \quad (\text{A.22d})$$

$$[J^{ij}, J^{kl}] = \varepsilon^{k(i}J^{j)l} + \varepsilon^{l(i}J^{j)k}, \quad (\text{A.22e})$$

with all other commutators vanishing.

Its superconformal generalisation is obtained by extending the translation generator to $P_A = (P_a, Q_\alpha^i)$ and the special conformal generator to $K^A = (K^a, S_i^\alpha)$. The fermionic generator Q_α^i obeys the algebra

$$\{Q_\alpha^i, Q_\beta^j\} = -2i\varepsilon^{ij}(\gamma^c)_{\alpha\beta}P_c, \quad [Q_\alpha^i, P_a] = 0, \quad [\mathbb{D}, Q_\alpha^i] = \frac{1}{2}Q_\alpha^i, \quad (\text{A.23a})$$

$$[M_{ab}, Q_\gamma^k] = -\frac{1}{2}(\gamma_{ab})_\gamma{}^\delta Q_\delta^k, \quad [J^{ij}, Q_\alpha^k] = \varepsilon^{k(i}Q_\alpha^{j)}, \quad (\text{A.23b})$$

while the generator S_i^α obeys the algebra

$$\{S_i^\alpha, S_j^\beta\} = -2i\varepsilon_{ij}(\tilde{\gamma}^c)^{\alpha\beta}K_c, \quad [S_i^\alpha, K_a] = 0, \quad [\mathbb{D}, S_i^\alpha] = -\frac{1}{2}S_i^\alpha, \quad (\text{A.24a})$$

$$[M_{ab}, S_k^\gamma] = \frac{1}{2}(\gamma_{ab})_\delta{}^\gamma S_k^\delta, \quad [J^{ij}, S_k^\alpha] = \delta_k^{(i}S_\alpha^{j)}, \quad (\text{A.24b})$$

Finally, the (anti-)commutators of K^A and P_B are

$$[K_a, Q_\alpha^i] = -i(\gamma_a)_{\alpha\beta}S^{\beta i}, \quad [S_i^\alpha, P_a] = -i(\tilde{\gamma}_a)^{\alpha\beta}Q_{\beta i}, \quad (\text{A.25a})$$

$$\{S_i^\alpha, Q_\beta^j\} = 2\delta_\beta^\alpha\delta_i^j\mathbb{D} - 4\delta_i^jM_\beta{}^\alpha + 8\delta_\beta^\alpha J_i{}^j. \quad (\text{A.25b})$$

B The conformal Killing supervector fields of $\mathbb{M}^{6|16}$

The aim of this appendix is to study the structure of conformal Killing supervector fields of $\mathcal{N} = (2, 0)$ Minkowski superspace in six dimensions. Such analyses were previously conducted in [77, 92]. By employing this construction, we will explicitly prove our earlier claim that the proposed conformal Killing spinor superfields (4.1) naturally arise from an $\mathcal{N} = (2, 0) \longrightarrow (1, 0)$ superspace reduction.

We recall that $\mathcal{N} = (2, 0)$ Minkowski superspace, $\mathbb{M}^{6|16}$, is parametrised by the coordinates $z^A = (x^a, \theta_I^\alpha)$, where $a = 0, 1, \dots, 5$, $\alpha = 1, \dots, 4$ and $I = \underline{1}, \dots, \underline{4}$. Its covariant derivatives take the form

$$\partial_a = \frac{\partial}{\partial x^a}, \quad D_\alpha^I = \frac{\partial}{\partial \theta_I^\alpha} - i(\gamma^a)_{\alpha\beta} \theta^{\beta I} \partial_a, \quad (\text{B.1})$$

and satisfy the algebra

$$\{D_\alpha^I, D_\beta^J\} = -2i\Omega^{IJ} \partial_{\alpha\beta}, \quad [\partial_a, D_\alpha^I] = 0, \quad [\partial_a, \partial_b] = 0. \quad (\text{B.2})$$

Here $\Omega^{IJ} = -\Omega^{JI}$ is an invariant tensor of the $\mathcal{N} = (2, 0)$ R-symmetry group $\text{USp}(4)$. It is convenient to choose a basis for Ω^{IJ} such that it takes the form

$$\Omega^{IJ} = \begin{pmatrix} \varepsilon^{ij} & 0 \\ 0 & \varepsilon^{\hat{i}\hat{j}} \end{pmatrix}, \quad i, \hat{i} = \underline{1}, \underline{2}. \quad (\text{B.3})$$

We will say that the real supervector field

$$\xi = \bar{\xi} = \xi^a \partial_a + \xi_I^\alpha D_\alpha^I, \quad (\text{B.4})$$

is conformal Killing if it satisfies

$$[\xi, D_\alpha^I] = -(D_\alpha^I \xi_J^\beta) D_\beta^J. \quad (\text{B.5})$$

This constraint implies the fundamental equation

$$D_\alpha^I \xi^a = -2i(\gamma^a)_{\alpha\beta} \xi^{\beta I}, \quad (\text{B.6})$$

which yields

$$\xi_I^\alpha = -\frac{i}{12} (\tilde{\gamma}_a)^{\alpha\beta} D_{\beta I} \xi^a. \quad (\text{B.7})$$

By a routine computation, we may bring (B.5) to the form

$$[\xi, D_\alpha^I] = -\omega_\alpha^\beta [\xi] D_\beta^I + \Lambda^I{}_J [\xi] D_\alpha^J - \frac{1}{2} \sigma [\xi] D_\alpha^I, \quad (\text{B.8})$$

where we have made use of the definitions

$$\begin{aligned} \omega_\alpha^\beta [\xi] &= -\frac{1}{4} (\gamma^{ab})_{\alpha\beta} \partial_a \xi_b, \\ \Lambda^I{}_J [\xi] &= -\frac{1}{4} \left[D_\alpha^I \xi_J^\alpha - \frac{1}{4} \delta_J^I D_\alpha^K \xi_K^\alpha \right], \\ \sigma [\xi] &= \frac{1}{6} \partial^a \xi_a. \end{aligned} \quad (\text{B.9})$$

It is clear that the above parameters generate Lorentz, R-symmetry and scaling transformations, respectively.

We now briefly consider the problem of performing a reduction to $\mathcal{N} = (1, 0)$ Minkowski superspace. Without loss of generality, we will assume that this coincides with the section

of $\mathbb{M}^{6|16}$ defined by $\theta_I^\alpha = (\theta_i^\alpha, \theta_{\hat{i}}^\alpha) = (\theta_i^\alpha, 0)$. It then follows that, upon such a reduction, every solution to (B.5) decomposes into an $\mathcal{N} = (1, 0)$ conformal Killing supervector field, a spinor superfield and an additional triplet of scalar superfields defined by

$$\epsilon_{\hat{i}}^\alpha = \xi_{\hat{i}}^\alpha|_{\theta_{\hat{i}}^\alpha=0}, \quad \lambda^{\hat{i}\hat{j}} := \frac{1}{4} D_\alpha^{(\hat{i}} \xi^{\alpha\hat{j})}|_{\theta_{\hat{i}}^\alpha=0}. \quad (\text{B.10})$$

The spinor $\epsilon_{\hat{i}}^\alpha$ and $\lambda^{\hat{i}\hat{j}}$ generate non-manifest extended superconformal symmetries in $\mathbb{M}^{6|8}$, where $\lambda^{\hat{i}\hat{j}}$ generates the hidden $\text{SU}(2)$ R-symmetry within $\text{USp}(4)$. By making use of the fundamental equation (B.6), one may show that $\epsilon_{\hat{i}}^\alpha$ satisfies

$$D_\alpha^i \epsilon_{\hat{j}}^\beta = \frac{1}{4} \delta_\alpha^\beta D_\gamma^i \epsilon_{\hat{j}}^\gamma, \quad (\text{B.11})$$

which implies that $\epsilon_{\hat{1}}^\alpha$ and $\epsilon_{\hat{2}}^\alpha$ are conformal Killing spinor superfields (4.1). As a result, we have proven our original claim.

C Bosonic backgrounds

In this appendix, we introduce a formalism to study 6D $\mathcal{N} = (1, 0)$ supersymmetric backgrounds starting from a superspace perspective. Our analysis will be restricted to bosonic backgrounds, meaning that the following conditions hold

$$\mathcal{D}_\alpha^i C_a{}^{kl}| = 0, \quad \mathcal{D}_\alpha^i W_{abc}| = 0, \quad \mathcal{D}_\alpha^i N_{abc}| = 0. \quad (\text{C.1})$$

Following [20, 93, 94], the bar-projection of a superfield is defined as usual:

$$U| := U(x, \theta)|_{\theta=0}, \quad (\text{C.2})$$

for any superfield $U(z) = U(x, \theta)$. The coordinates x^m parametrise a curved spacetime \mathcal{M}^6 , the bosonic body of the superspace $\mathcal{M}^{6|8}$. The bar-projection of the superspace covariant derivatives is defined similarly by:

$$\mathcal{D}_A| = E_A{}^M| \partial_M - \frac{1}{2} \Omega_A{}^{bc}| M_{bc} - \Phi_A{}^{kl}| J_{kl}. \quad (\text{C.3})$$

Due to (C.1), one can completely gauge away the gravitini such that the projection of the vector covariant derivatives takes the simple form

$$\mathcal{D}_a| = \mathbf{D}_a \iff \psi_{m_i}^\alpha = 0, \quad (\text{C.4})$$

where

$$\mathbf{D}_a = e_a - \frac{1}{2} \omega_a{}^{bc} M_{bc} - \phi_a{}^{kl} J_{kl}, \quad e_a := e_a{}^m \partial_m \quad (\text{C.5})$$

is a spacetime covariant derivative with Lorentz ($\omega_a{}^{bc}$) and $\text{SU}(2)_R$ ($\phi_a{}^{kl}$) connections. In what follows, the gauge (C.4) will be assumed. The covariant derivatives \mathbf{D}_a obey

$$[\mathbf{D}_a, \mathbf{D}_b] = -\frac{1}{2} \mathcal{R}_{ab}{}^{cd} M_{cd} - \mathcal{R}_{ab}{}^{kl} J_{kl}, \quad \mathcal{R}_{ab}{}^{cd} = R_{ab}{}^{cd}|, \quad \mathcal{R}_{ab}{}^{kl} = R_{ab}{}^{kl}|. \quad (\text{C.6})$$

For convenience, we also make the definitions

$$c_a^{kl} = C_a^{kl}, \quad w_{abc} = W_{abc}, \quad n_{abc} = N_{abc}. \quad (\text{C.7})$$

An important feature of such backgrounds (C.1) is that every conformal Killing vector superfield (3.4) can be uniquely decomposed as a sum of even and odd ones. We will say that the conformal Killing supervector ξ^A is even if

$$v^a := \xi^a| \neq 0, \quad \xi_i^\alpha| = 0. \quad (\text{C.8})$$

or odd if

$$\xi^a| = 0, \quad \epsilon_i^\alpha := \xi_i^\alpha| \neq 0. \quad (\text{C.9})$$

The fields v^a and ϵ_i^α encode complete information about the parent conformal Killing vector superfield. This appendix is devoted to the study of symmetries they induce at the spacetime level.

C.1 Conformal Killing vectors

Let ξ^A be an even conformal Killing supervector field (C.8). By bar-projecting (3.5) we obtain

$$\mathbf{D}_a v_b = k_{ab}[v] + \eta_{ab} \omega[v], \quad (\text{C.10})$$

where we have defined

$$k_{ab}[v] := K_{ab}[\xi]| = \mathbf{D}_{[a} v_{b]}, \quad \omega[v] := \sigma[\xi]| = \frac{1}{6} \mathbf{D}^a v_a. \quad (\text{C.11})$$

In particular, it follows from (C.10) that v^a is a conformal Killing vector field

$$\mathbf{D}_{(a} v_{b)} = \frac{1}{6} \eta_{ab} \mathbf{D}_c v^c. \quad (\text{C.12})$$

By employing the results of section 3, it may be shown that every conformal Killing vector field on \mathcal{M}^6 may be lifted to a unique even conformal Killing vector superfield on $\mathcal{M}^{6|8}$.

We also note that, at the component level, $\text{SU}(2)_R$ transformations are generated by

$$k^{ij}[v] := K^{ij}[\xi]|, \quad (\text{C.13})$$

which satisfies the differential equation

$$\mathbf{D}_a k^{ij}[v] = \mathcal{R}_{ab}{}^{ij} v^b + v^b \mathbf{D}_b c_a{}^{ij} + k_a{}^b[v] c_b{}^{ij} + 2k^{(i}{}_k[v] c_a{}^{j)k} + \omega[v] c_a{}^{ij}. \quad (\text{C.14})$$

It should be remarked that in the case of Poincaré supergravity, we must supplement the constraints above with

$$\sigma[\xi] = 0 \implies \omega[v]| = 0, \quad (\text{C.15})$$

which implies that v^a is a Killing vector field

$$\mathbf{D}_{(a} v_{b)} = 0. \quad (\text{C.16})$$

Further, by making use of (3.20) equation (C.14) reduces to

$$\mathbf{D}_a k^{ij}[v] = \mathcal{R}_{ab}{}^{ij} v^b. \quad (\text{C.17})$$

C.2 Conformal Killing spinors

Our analysis in this subsection will be restricted to those backgrounds which admit at least one conformal supersymmetry. Such spacetimes are associated with a superspace possessing an odd conformal Killing supervector field ξ^A (C.9), from which one can identify a conformal Killing spinor ϵ_i^α as the bar projection of ξ_i^α . The spinor ϵ_i^α generates a Q -supersymmetry transformation, while S -supersymmetry transformations are parametrised by

$$\eta_\alpha^i := \mathcal{D}_\alpha^i \sigma[\xi] . \quad (\text{C.18})$$

With the previous assumptions at hand, bar-projecting equation (3.22) gives

$$\mathbf{D}_a \epsilon_k^\gamma = (\gamma_{ab})_\beta{}^\gamma \epsilon^{\beta j} c^b{}_{jk} + \frac{1}{2} (\gamma^{bc})_\beta{}^\gamma \epsilon_k^\beta (w_{abc} + 2n_{abc}) + \frac{i}{2} (\tilde{\gamma}_a)^{\beta\gamma} \eta_{\beta k} . \quad (\text{C.19})$$

Moreover, eq. (3.23) implies

$$\eta_{\beta k} = -\frac{i}{3} \mathbf{D}_{\beta\delta} \epsilon_k^\delta + \frac{4i}{3} \epsilon^{\delta j} c_{\beta\delta jk} - \frac{i}{6} (\gamma^{abc})_{\delta\beta} \epsilon_k^\delta (w_{abc} + 2n_{abc}) , \quad (\text{C.20})$$

and (C.19) becomes

$$\mathbf{D}_a \epsilon_k^\gamma - c_{ak}{}^j \epsilon_j^\gamma - \frac{1}{2} (w_{abc} + 2n_{abc}) (\gamma^{bc})_\beta{}^\gamma \epsilon_k^\beta = \frac{i}{2} (\tilde{\gamma}_a)^{\beta\gamma} (\eta_{\beta k} - 2i (\gamma_b)_{\beta\delta} \epsilon^{\delta j} c^b{}_{jk}) , \quad (\text{C.21})$$

or, equivalently,

$$\hat{\mathbf{D}}_a \epsilon_k^\gamma = \frac{i}{2} (\tilde{\gamma}_a)^{\beta\gamma} (\eta_{\beta k} - 2i (\gamma_b)_{\beta\delta} \epsilon^{\delta j} c^b{}_{jk}) \equiv \frac{i}{2} (\tilde{\gamma}_a)^{\beta\gamma} \hat{\eta}_{\beta k} , \quad (\text{C.22})$$

where we have defined

$$\hat{\mathbf{D}}_a := \mathbf{D}_a + c_a{}^{kl} J_{kl} - (w_{abc} + 2n_{abc}) M^{bc} , \quad (\text{C.23})$$

and

$$\hat{\eta}_{\alpha k} := -\frac{i}{3} (\gamma^a)_{\alpha\beta} \hat{\mathbf{D}}_a \epsilon_k^\beta . \quad (\text{C.24})$$

The conformal Killing spinor equation then takes the particularly simple form

$$\hat{\mathbf{D}}_a \epsilon_k^\gamma = -\frac{1}{6} (\tilde{\gamma}_a)^{\gamma\beta} (\gamma^b)_{\beta\delta} \hat{\mathbf{D}}_b \epsilon_k^\delta \iff \hat{\mathbf{D}}_{\alpha\beta} \epsilon_k^\gamma = -\frac{2}{3} \delta_{[\alpha}{}^\gamma \hat{\mathbf{D}}_{\beta]\delta} \epsilon_k^\delta \quad (\text{C.25})$$

In particular, we see that the gamma-traceless part of $\hat{\mathbf{D}}_a \epsilon_k^\gamma$ is identically zero.

Associated with a non-zero *commuting* spinor ϵ_i^α is the 6-vector

$$V_a = (\gamma_a)_{\alpha\beta} \epsilon_i^\alpha \epsilon_j^\beta , \quad (\text{C.26})$$

which proves to be a conformal Killing vector field when ϵ_i^α is a solution to (C.19):

$$\mathbf{D}_{(a} V_{b)} = \frac{1}{6} \eta_{ab} \mathbf{D}^c V_c . \quad (\text{C.27})$$

Furthermore, it is a null vector

$$V^2 := V^a V_a = 0 . \quad (\text{C.28})$$

By construction, the following identities hold

$$\delta(\mathcal{D}_\alpha^i C_a^{jk}) = 0, \quad \delta(\mathcal{D}_\alpha^i W_{abc}) = 0, \quad \delta(\mathcal{D}_\alpha^i N_{abc}) = 0, \quad (\text{C.29})$$

which implies the conditions (C.1) are superconformal. The bar-projection of the above conditions imply the following constraints

$$\begin{aligned} (\tilde{\gamma}_a)^{\beta\gamma} \mathcal{D}_\alpha^i \mathcal{D}_\beta^j \mathcal{D}_\gamma^k \sigma &= 4i\epsilon_l^\beta \left(-[\mathcal{D}_\alpha^i, \mathcal{D}_\beta^l] C_a^{jk} - 2i\epsilon^{il} \mathbf{D}_{\alpha\beta} C_a^{jk} - 4i(\gamma_{abc})_{\alpha\beta} c^{bil} c^{cjk} \right. \\ &\quad + 4i\epsilon^{il} (\gamma^b)_{\alpha\beta} (w_{abc} + 2n_{abc}) c^{cjk} - 12i\epsilon^{il} c_{\alpha\beta}{}^p (c_{ap}{}^k) \\ &\quad \left. - \frac{8i}{3} N^{bcd} (\gamma_{bcd})_{\alpha\beta} (\epsilon^{j(i} c_a^{l)k} + \epsilon^{k(i} c_a^{i)j}) \right) \\ &\quad + 8i(\gamma_{ab})_\alpha{}^\gamma \eta_\gamma^i c^{bjk} - 32i\eta_{\alpha l} (\epsilon^{j(i} c_a^{l)k} + \epsilon^{k(i} c_a^{l)j}) + 8i\eta_\alpha^i c_a^{jk}, \end{aligned} \quad (\text{C.30a})$$

$$\begin{aligned} 0 &= \epsilon_j^\beta \left(-\frac{1}{2} [\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j] W_{abc} - i\epsilon^{ij} \mathbf{D}_{\alpha\beta} w_{abc} + 6i(\gamma_{[a}{}^{de})_{\alpha\beta} w_{bc]d} c_e{}^{ij} \right. \\ &\quad \left. + 6i\epsilon^{ij} (\gamma_d)_{\alpha\beta} (w^{de}{}_{[a} + 2n^{de}{}_{[a} w_{bc]e}) \right. \\ &\quad \left. - 3\eta_\beta^i (\gamma^d{}_{[a})_\alpha{}^\beta w_{bc]d} + \eta_\alpha^i w_{abc}, \right) \end{aligned} \quad (\text{C.30b})$$

$$\begin{aligned} \frac{i}{32} (\tilde{\gamma}_{abc})^{\gamma\delta} \mathcal{D}_\alpha^i \mathcal{D}_\gamma^j \mathcal{D}_\delta^k \sigma &= \epsilon_j^\beta \left(-\frac{1}{2} [\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j] N_{abc} - i\epsilon^{ij} \mathbf{D}_{\alpha\beta} n_{abc} + 6i(\gamma_{[a}{}^{de})_{\alpha\beta} n_{bc]d} c_e{}^{ij} \right. \\ &\quad \left. + 6i\epsilon^{ij} (\gamma_d)_{\alpha\beta} (w^{de}{}_{[a} + 2n^{de}{}_{[a} n_{bc]e}) \right. \\ &\quad \left. - 3\eta_\beta^i (\gamma^d{}_{[a})_\alpha{}^\beta n_{bc]d} + \eta_\alpha^i n_{abc}. \right) \end{aligned} \quad (\text{C.30c})$$

Restrictions on higher mass-dimension component parameters may be obtained from the invariance of higher-order spinor derivatives of C_a^{jk} , W_{abc} and N_{abc} . These results exemplify how results in components can be efficiently obtained from a superspace setting.

In the case of Poincaré supergravities, the equations given above must be supplemented by the additional condition

$$\sigma[\xi] = 0 \implies \eta_\alpha^i = 0, \quad (\text{C.31})$$

which is a consequence of (3.32). The conformal Killing spinor equation (C.19) becomes

$$\hat{\mathbf{D}}_a \epsilon_k^\gamma = (\tilde{\gamma}_a)^{\beta\gamma} (\gamma_b)_{\beta\rho} \epsilon^{\rho j} c^b{}_{jk}, \quad (\text{C.32})$$

which implies

$$\hat{\mathbf{D}}_{\delta\rho} \epsilon_k^\rho = 6(\gamma_b)_{\delta\rho} \epsilon^{\rho j} c^b{}_{jk}, \quad (\text{C.33})$$

and

$$\epsilon^{\delta k} \hat{\mathbf{D}}_{\delta\rho} \epsilon_k^\rho = 0. \quad (\text{C.34})$$

This implies that

$$\hat{\mathbf{D}}^a V_a = 0 = \mathbf{D}^a V_a, \quad (\text{C.35})$$

thus V_a is a Killing vector field

$$\mathbf{D}_{(a} V_{b)} = 0. \quad (\text{C.36})$$

D From conformal to SU(2) superspace

As is well known, SU(2) superspace exists as a gauge-fixed version of conformal superspace. The process of moving from the latter to the former is known as ‘degauging’ and we outline it here extending previous analysis in $3 \leq D \leq 5$, see [37–40].

The first step in this procedure is to eliminate the dilatation connection. Under an infinitesimal special conformal gauge transformation the one-form $B = E^a B_a + E_i^\alpha B_\alpha^i$ transforms as

$$\delta_K(\Lambda)B = -2E^a \Lambda_a - 2E_i^\alpha \Lambda_\alpha^i . \quad (\text{D.1})$$

Thus, in exchange for a loss of unconstrained special conformal gauge freedom,¹⁶ one can gauge away B .

$$B_A = 0 . \quad (\text{D.2})$$

As a result, the special conformal connection becomes auxiliary and must be manually extracted from ∇_A .

The *degauged* covariant derivatives are given by

$$\mathcal{D}_A := \nabla_A + \mathfrak{F}_{AB} K^B = E_A - \frac{1}{2} \Omega_A^{bc} M_{bc} - \Phi_A^{ij} J_{ij} . \quad (\text{D.3})$$

Since their structure group is $\text{SO}(5, 1) \times \text{SU}(2)_R$ it is clear that they are SU(2) superspace covariant derivatives. They satisfy the algebra

$$[\mathcal{D}_A, \mathcal{D}_B] = -\mathbf{T}_{AB}{}^C \mathcal{D}_C - \frac{1}{2} \mathbf{R}_{AB}{}^{cd} M_{cd} - \mathbf{R}_{AB}{}^{kl} J_{kl} . \quad (\text{D.4})$$

The degauged special conformal connections $\mathfrak{F}_A{}^B$ provide new contributions to the torsion, and by extension to the other curvatures.

We use different symbols for the degauged derivatives and the SU(2) ones of section 2.1 since, as we will see, they satisfy slightly different torsion constraints. Since the vielbein, Lorentz, and SU(2) connections are exactly those of conformal superspace, it is easy to give expressions for the new torsion and curvature tensors in terms of their conformal counterparts. This can be done by using the expression of the conformal superspace torsion and curvature two-forms in terms of the vielbein and connection superfields [16]

$$\mathcal{T}^a = dE^a + E^b \wedge \Omega_b{}^a + E^a \wedge B , \quad (\text{D.5a})$$

$$\mathcal{T}_i^\alpha = dE_i^\alpha + E_i^\beta \wedge \Omega_{\beta}{}^\alpha + \frac{1}{2} E_i^\alpha \wedge B - E^{\alpha j} \wedge \Phi_{ji} - i E^c \wedge \mathfrak{F}_{\beta i} (\tilde{\gamma}_c)^{\alpha\beta} , \quad (\text{D.5b})$$

$$\mathcal{R}(\mathbb{D}) = dB + 2E^a \wedge \mathfrak{F}_a + 2E_i^\alpha \wedge \mathfrak{F}_\alpha^i , \quad (\text{D.5c})$$

$$\mathcal{R}(M)^{ab} = d\Omega^{ab} + \Omega^{ac} \wedge \Omega_c{}^b - 4E^{[a} \wedge \mathfrak{F}^{b]} + 2E_j^\alpha \wedge \mathfrak{F}_\beta^j (\gamma^{ab})_\alpha{}^\beta , \quad (\text{D.5d})$$

$$\mathcal{R}(J)^{ij} = d\Phi^{ij} - \Phi^{k(i} \wedge \Phi^{j)k} - 8E^{\alpha(i} \wedge \mathfrak{F}_\alpha^{j)} , \quad (\text{D.5e})$$

$$\mathcal{R}(K)^a = d\mathfrak{F}^a + \mathfrak{F}^b \wedge \Omega_b{}^a - \mathfrak{F}^a \wedge B - i \mathfrak{F}_\alpha^k \wedge \mathfrak{F}_{\beta k} (\tilde{\gamma}^a)^{\alpha\beta} , \quad (\text{D.5f})$$

$$\mathcal{R}(S)_\alpha^i = d\mathfrak{F}_\alpha^i - \mathfrak{F}_\beta^i \wedge \Omega_\alpha{}^\beta - \frac{1}{2} \mathfrak{F}_\alpha^i \wedge B - \mathfrak{F}_\alpha^j \wedge \Phi_j{}^i - i E^{\beta i} \wedge \mathfrak{F}^c (\gamma_c)_{\alpha\beta} . \quad (\text{D.5g})$$

¹⁶There exists a class of combined local dilatations and special conformal transformations preserving the gauge $B = 0$. These exactly reproduce the super-Weyl transformations (2.11), see e.g. [29, 37, 95].

For example, in the gauge $B = E^A B_A \equiv 0$, one finds the torsion tensors are related by

$$\mathbf{T}^a = \mathcal{T}^a, \quad \mathbf{T}_i^\alpha = \mathcal{T}_i^\alpha + iE^c \wedge \mathfrak{F}_{\beta i}(\tilde{\gamma}_c)^{\alpha\beta}. \quad (\text{D.6})$$

By investigating (D.6), one can extract the structure of the torsion constraints in the degauged geometry. We find that these are all the same as for the covariant derivatives \mathcal{D}_A , except that

$$\mathbf{T}_{a\beta(j^{\beta}k)} \neq 0, \quad \mathbf{T}_{ab}{}^c \neq 0. \quad (\text{D.7})$$

In the SU(2) superspace geometry of section 2.1, both of these torsions are required to vanish. As we will see, these conditions can be satisfied by redefining the degauged vector covariant derivative. Then, the resulting geometry exactly reproduces the SU(2) superspace geometry of section 2.1.

To elaborate further, we must analyse the additional superfields introduced by the special conformal connections $\mathfrak{F}_A{}^B$. In the gauge (D.2) the dilatation curvature, eq. (D.5c), is given by

$$\mathcal{R}(\mathbb{D})_{AB} = 2\mathfrak{F}_{AB} - 2\mathfrak{F}_{BA}(-1)^{\varepsilon_A\varepsilon_B}. \quad (\text{D.8})$$

The vanishing of the dilatation curvature at dimension-1, see (2.19), constrains the special conformal connection as

$$\mathfrak{F}_{\alpha\beta}{}^{ij} = -\mathfrak{F}_{\beta\alpha}{}^{ji} = -\frac{i}{4}A_{\alpha\beta}{}^{ij} + i\varepsilon^{ij}Y_{\alpha\beta}, \quad (\text{D.9})$$

where the superfields $A_{\alpha\beta}{}^{ij}$, and $Y_{\alpha\beta}$ satisfy

$$A_{\alpha\beta}{}^{ij} = (\gamma^a)_{\alpha\beta}A_a{}^{ij} = A_{\alpha\beta}{}^{ji} = -A_{\beta\alpha}{}^{ji}, \quad Y_{\alpha\beta} = Y_{\beta\alpha} = \frac{1}{6}(\gamma^{abc})_{\alpha\beta}Y_{abc}. \quad (\text{D.10})$$

At this point it is possible to derive the degauged algebra of covariant derivatives. An efficient way to do this is to consider a weight-zero primary superfield U_0 transforming as a tensor in some representation of the remainder of the superconformal algebra. For example, to determine the anti-commutator of spinor derivatives we consider

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\}U_0 = \{\nabla_\alpha^i, \nabla_\beta^j\}U_0 + \mathfrak{F}_{\alpha C}^i[K^C, \nabla_\beta^j]U_0 + \mathfrak{F}_{\beta C}^j[K^C, \nabla_\alpha^i]U_0. \quad (\text{D.11})$$

The resulting algebra is

$$\begin{aligned} \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} &= -2i\varepsilon^{ij}(\gamma^a)_{\alpha\beta}\mathcal{D}_a + 4i\varepsilon^{ij}Y^{bcd}(\gamma_b)_{\alpha\beta}M_{cd} + 2i\varepsilon^{ij}(\gamma^a)_{\alpha\beta}A_a{}^{kl}J_{kl} \\ &\quad + \frac{i}{2}A_b{}^{ij}(\gamma^{bcd})_{\alpha\beta}M_{cd} + \frac{8i}{3}(\gamma^{abc})_{\alpha\beta}Y_{abc}J^{ij}. \end{aligned} \quad (\text{D.12})$$

To match (2.5a) it is necessary to make the following identifications

$$A_a{}^{ij} = 4C_a{}^{ij}, \quad Y_{abc} = -N_{abc}, \quad (\text{D.13})$$

and

$$\mathcal{D}_\alpha^i = \mathcal{D}_\alpha^i, \quad \mathcal{D}_a = \mathcal{D}_a + W_{abc}M^{bc} + C_a{}^{kl}J_{kl}. \quad (\text{D.14})$$

Next, we compute $[\mathcal{D}_a, \mathcal{D}_\beta^j]$ at dimension-1. One finds

$$[\mathcal{D}_a, \mathcal{D}_\beta^j]U_0 = [\nabla_a, \nabla_\beta^j]U_0 + \mathfrak{F}_{aC}[K^C, \nabla_\beta^j]U_0 - \mathfrak{F}_{\beta C}^j[K^C, \nabla_a]U_0,$$

which implies

$$[\mathcal{D}_a, \mathcal{D}_\beta^j] = C_a^j{}_k \mathcal{D}_\beta^k + C^{bj}{}_k (\gamma_{ab})_\beta^\delta \mathcal{D}_\delta^k - W_{acd} (\gamma^{cd})_\gamma^\delta \mathcal{D}_\delta^j - N_{acd} (\gamma^{cd})_\beta^\delta \mathcal{D}_\delta^j + \dots$$

Finally, we turn to $\mathfrak{F}_{a\beta}^j$. Since at mass dimension-3/2, (D.5c) implies

$$\mathcal{R}(\mathbb{D})_{a\beta}^j = 2\mathfrak{F}_{a\beta}^j - 2\mathfrak{F}_{\beta a}^j, \quad (\text{D.15})$$

by employing the dilatation curvatures of eq. (2.22), one obtains

$$\mathfrak{F}_{\beta a}^j = \mathfrak{F}_{a\beta}^j + \frac{1}{16} \mathcal{D}_\beta^i W^{\alpha\beta} = \mathfrak{F}_{a\beta}^j + \frac{5i}{8} X^{\beta i}. \quad (\text{D.16a})$$

By examining the expressions for the conformal superspace torsion and curvatures of eq. (D.5), one can obtain

$$\mathbf{T}_{AB}{}^c = \mathcal{T}_{AB}{}^c, \quad (\text{D.17a})$$

$$\mathbf{T}_{AB}{}^\gamma_k = \mathcal{T}_{AB}{}^\gamma_k - i\delta_A^c (\tilde{\gamma}_c)^{\gamma\delta} \mathfrak{F}_{B,\delta k} + i\delta_B^c (\tilde{\gamma}_c)^{\gamma\delta} \mathfrak{F}_{A,\delta k}, \quad (\text{D.17b})$$

$$\begin{aligned} \mathbf{R}_{AB}{}^{cd} &= \mathcal{R}(M)_{AB}{}^{cd} - 4\delta_A^{[c} \mathfrak{F}_{B,}{}^{d]} + 4\delta_B^{[c} \mathfrak{F}_{A,}{}^{d]} + 2\delta_{A_k}^\gamma (\gamma^{cd})_\gamma^\delta \mathfrak{F}_{B,\delta}^k (-1)^{\varepsilon B} \\ &\quad - 2\delta_{B_k}^\gamma (\gamma^{cd})_\gamma^\delta \mathfrak{F}_{A,\delta}^k, \end{aligned} \quad (\text{D.17c})$$

$$\mathbf{R}_{AB}{}^{kl} = \mathcal{R}(J)_{AB}{}^{kl} - 8\delta_{A\rho}^\rho \mathfrak{F}_{B,\rho}^{(k\varepsilon^l)p} (-1)^{\varepsilon B} + 8\delta_{B\rho}^\rho \mathfrak{F}_{A,\rho}^{(k\varepsilon^l)p}, \quad (\text{D.17d})$$

as well as the following conditions on the special conformal connections

$$\begin{aligned} \mathcal{R}(S)_{AB}{}^k{}_\gamma &= 2\mathcal{D}_{[A} \mathfrak{F}_{B)\gamma}{}^k + \mathbf{T}_{AB}{}^D \mathfrak{F}_{D,\gamma}{}^k + i\varepsilon^{kl} \delta_{A_l}^\delta (\gamma_c)_{\gamma\delta} \mathfrak{F}_{B,}{}^c (-1)^{\varepsilon B} \\ &\quad - i\varepsilon^{kl} \delta_{B_l}^\delta (\gamma_c)_{\gamma\delta} \mathfrak{F}_{A,}{}^c, \end{aligned} \quad (\text{D.18a})$$

$$\begin{aligned} \mathcal{R}(K)_{AB}{}^c &= 2\mathcal{D}_{[A} \mathfrak{F}_{B)}{}^c + \mathbf{T}_{AB}{}^D \mathfrak{F}_{D,}{}^c + i\mathfrak{F}_{A,\gamma}{}^k \mathfrak{F}_{B,\delta k} (\tilde{\gamma}^c)^{\gamma\delta} (-1)^{\varepsilon B} \\ &\quad - i\mathfrak{F}_{B,\gamma}{}^k \mathfrak{F}_{A,\delta k} (\tilde{\gamma}^c)^{\gamma\delta} (-1)^{\varepsilon A + \varepsilon B}. \end{aligned} \quad (\text{D.18b})$$

At dimension-3/2 only the S -curvature equation, eq. (D.18a), with $A = \alpha^i$ and $B = \beta^j$ gives nontrivial constraints. In particular, by using $\mathcal{R}(S)_{\alpha\beta\gamma}^{ijk} = 0$ and (D.16a), one obtains

$$\begin{aligned} 0 &= \mathcal{D}_\alpha^i \mathfrak{F}_{\beta\gamma}{}^j{}^k + \mathcal{D}_\beta^j \mathfrak{F}_{\alpha\gamma}{}^i{}^k + 2i\varepsilon^{ij} (\gamma^a)_{\alpha\beta} \mathfrak{F}_{a\gamma}{}^k - i\varepsilon^{ki} (\gamma^a)_{\gamma\alpha} \mathfrak{F}_{a\beta}{}^j - i\varepsilon^{kj} (\gamma^a)_{\gamma\beta} \mathfrak{F}_{a\alpha}{}^i \\ &\quad + \frac{i}{8} \varepsilon^{ij} \varepsilon_{\alpha\beta\gamma\delta} \mathcal{D}_\rho^k W^{\delta\rho}. \end{aligned} \quad (\text{D.19})$$

Its solution implies the differential constraints

$$\mathcal{D}_{(\alpha}^i C_{\beta)\gamma}{}^{jk} = 0, \quad \mathcal{D}_{(\alpha}^i N_{\beta\gamma)} = 0, \quad (\text{D.20})$$

which indicates that the decomposition into irreducible and nontrivial tensors of the spinor derivatives of dimension-1 torsions is

$$\mathcal{D}_\gamma^k C_a{}^{ij} = (\gamma_a)_{\gamma\delta} \mathcal{C}^{\delta k ij} - \varepsilon^{k(i} \mathcal{C}_{a\gamma}{}^{j)} - \varepsilon^{k(i} (\gamma_a)_{\gamma\delta} \mathcal{C}^{\delta j)}, \quad (\text{D.21a})$$

$$\mathcal{D}_\gamma^k N_{abc} = -\frac{3}{4} (\gamma_{[ab})_\gamma^\beta \mathcal{N}_{c]\beta}{}^k, \quad (\text{D.21b})$$

$$\mathcal{D}_\gamma^k W_{abc} = i(\gamma_{abc})_{\alpha\beta} X_\gamma{}^{\alpha\beta k} + i(\gamma_{abc})_{\gamma\delta} X^{\delta k}. \quad (\text{D.21c})$$

Equation (D.19) then implies

$$\mathfrak{F}_{a\beta}^j = -\frac{5i}{24}(\gamma_a)_{\beta\delta}X^{\delta j} - \frac{1}{2}(\gamma_a)_{\beta\delta}\mathcal{C}^{\delta j} + \frac{1}{2}\mathcal{C}_{a\beta}^j - \frac{1}{2}\mathcal{N}_{a\beta}^j. \quad (\text{D.22})$$

To conclude the analysis at dimension-3/2 we derive the corresponding torsion and curvatures. For the dimension-3/2 torsion it holds that

$$\mathbf{T}_{abk}^\gamma = \mathcal{T}_{abk}^\gamma - 2i(\tilde{\gamma}_{[a})^{\gamma\delta}\mathfrak{F}_{b]\delta k}, \quad (\text{D.23})$$

which leads to

$$\begin{aligned} \mathbf{T}_{abk}^\gamma &= (\gamma_{ab})_\beta{}^\alpha \left(X_\alpha{}^{\beta\gamma}{}_k - \frac{3}{4}\delta_\alpha^\gamma X_k^\beta \right) + i(\tilde{\gamma}_{[a})^{\gamma\delta} \left(\mathcal{N}_{b]\delta k} - \mathcal{C}_{b]\delta k} \right) \\ &\quad + (\gamma_{ab})_\rho{}^\gamma \left(\frac{5}{12}X_k^\rho - i\mathcal{C}_k^\rho \right). \end{aligned} \quad (\text{D.24})$$

The dimension-3/2 Lorentz curvature can be computed by using

$$\mathbf{R}_{a\beta}{}^{jcd} = \mathcal{R}(M)_{a\beta}{}^{jcd} - 4\delta_a^{[c}\mathfrak{F}_{\beta}{}^{j]d} - 2\delta_\beta^{\gamma j}\delta_k^c(\gamma^{cd})_{\gamma}{}^\delta\mathfrak{F}_{a\delta}{}^k, \quad (\text{D.25})$$

which becomes

$$\begin{aligned} \mathbf{R}_{a\beta}{}^{jcd} &= 2i(\gamma_a)_{\beta\gamma}(\gamma^{cd})_{\delta\rho}X_\rho{}^{\gamma\delta j} + (\gamma_a{}^{cd})_{\beta\gamma} \left(\frac{2i}{3}X^{\gamma j} + \mathcal{C}^{\gamma j} \right) + \delta_a^{[c}(\gamma^{d])}_{\beta\gamma} \left(-\frac{4i}{3}X^{\gamma j} + 4\mathcal{C}^{\gamma j} \right) \\ &\quad + 2\delta_a^{[c} \left(\mathcal{N}^{d]}_{\beta}{}^{jj} - \mathcal{C}^{d]}_{\beta}{}^{jj} \right) + (\gamma^{cd})_{\beta}{}^\gamma \left(\mathcal{N}_{a\gamma}{}^j - \mathcal{C}_{a\gamma}{}^j \right). \end{aligned} \quad (\text{D.26})$$

Finally, the $SU(2)_R$ curvature derives from

$$\mathbf{R}_{a\beta}{}^{jkl} = \mathcal{R}(J)_{a\beta}{}^{jkl} + 8\delta_\beta^\rho\delta_p^j\mathfrak{F}_{a\rho}{}^{(k}\varepsilon^{l)p} \quad (\text{D.27})$$

which implies

$$\mathbf{R}_{a\beta}{}^{jkl} = \left\{ (\gamma_a)_{\beta\gamma} \left(\frac{10i}{3}X^{\gamma(k} - 4\mathcal{C}^{\gamma(k)} \right) - 4 \left(\mathcal{N}_{a\beta}{}^{(k} - \mathcal{C}_{a\beta}{}^{(k)} \right) \right\} \varepsilon^{l)j}. \quad (\text{D.28})$$

One can then prove that these results coincide with the dimension-3/2 results of section (2.1) upon using (D.14) and identifying

$$X_{\gamma k}{}^{\alpha\beta} = -\frac{i}{4}\mathcal{W}_{\gamma k}{}^{\alpha\beta}, \quad X_k^\beta = -\frac{i}{4}\mathcal{W}_k^\beta. \quad (\text{D.29})$$

It is straightforward to continue the degauging procedure and obtain results at dimensions higher than 3/2. We will not pursue such an analysis here.

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