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Revisiting the classifications of $6d$ SCFTs and LSTs

Lakshya Bhardwaj

Department of Physics, Harvard University,
Cambridge, MA 02138, U.S.A.

E-mail: lbhardwaj@fas.harvard.edu

Abstract: Gauge-theoretic anomaly cancellation predicts the existence of many $6d$ SCFTs and little string theories (LSTs) that have not been given a string theory construction so far. In this paper, we provide an explicit construction of all such "missing" $6d$ SCFTs and LSTs by using the frozen phase of F-theory. We conjecture that the full set of $6d$ SCFTs and LSTs is obtained by combining the set of theories constructed in this paper with the set of theories that have been constructed in earlier literature using the unfrozen phase of F-theory. Along the way, we demonstrate that there exist SCFTs that do not descend from LSTs via an RG flow.

Keywords: F-Theory, Field Theories in Higher Dimensions

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Contents

1 Introduction and conclusions

$6d$ SCFTs and little string theories (LSTs) have been at the focal point of many recent developments in quantum field theory and string theory [11–130]. Many of these developments were inspired by the classifications of these theories carried out in [1–4]. These classifications have taken two different starting points. On one hand are the classifications of [1–3] which study all the $6d$ SCFTs and LSTs which can be constructed by compactifying F-theory on an elliptically fibered Calabi-Yau threefold. These classifications are incomplete, because as pointed out in [5], the F-theory compactifications considered by [1–3] do not include frozen singularities. On the other hand is the classification of [4] which studies all the consistent[1] $6d$ supersymmetric gauge theories that can arise as low energy theories on the tensor branch of a $6d$ SCFT or LST, and conjectures that the corresponding $6d$ SCFTs and LSTs exist. Such a classification is incomplete because there exist $6d$ SCFTs and LSTs that are not described purely by a $6d$ supersymmetric gauge theory on their tensor branch.

To compare the two classifications, one can compare the set of theories obtained in [4] to the subset of those theories in [1–3] that are described purely by a gauge theory on their tensor branch. One finds that some of the theories obtained in [4] are missing from [1–3]. We can divide such theories into two types:

[1]The consistency conditions are based on a version of Green-Schwarz mechanism of anomaly cancellation in the six-dimensional context, which was first discussed in [9].

1. First of all, there are theories which are known to have a field-theoretic inconsistency even though they solve the consistency conditions imposed in [4]. See [11] for an example.

2. Second, there are theories that involve sub-quivers that cannot be constructed in F-theory without frozen singularities, but admit a construction once we allow frozen singularities in F-theory. See [5] for a construction of some of these sub-quivers. It is these theories that will be the main topic of discussion in this paper. It is interesting to note that some, but not all, of these theories are known to admit a brane construction in massive type IIA string theory[2] for around 20 years now [6–8].

This paper is organized as follows. In section 2, we list down all of the possible missing theories that involve sub-quivers that cannot be constructed in F-theory without frozen singularities.[3] We continue in section 3.1 with a brief discussion about the reasons for the omission of such theories from the unfrozen phase of F-theory. Then, in section 3.2, we introduce new constructions of various sub-quivers that we need to construct the theories listed in section 2. Finally, in sections 3.3 and 3.4, we go on to explicitly show how each theory listed in section 2 can be constructed by compactifying F-theory on an elliptically fibered Calabi-Yau threefold involving frozen singularities.

We conjecture that the full list of $6d$ SCFTs and LSTs is obtained by combining the classification of this paper with the earlier classification of [1, 3]. Our conjecture stems from the fact that this combined classification exhausts all the possible tensor branches that can be obtained by putting together gauge theories with known non-gauge theories like the E-string theory and A_1 $(2,0)$ theory. We caution that there is a small set of theories whose F-theory construction was proposed in [1, 3] but a closer look in [12] (see also [13, 14]) revealed an inconsistency in the proposed constructions of those theories. It would be worthwhile to investigate whether such theories can be given a consistent construction in the frozen phase of F-theory. We leave this as an interesting problem for future work.

As a by-product of our work, we demonstrate the existence of SCFTs that do not descend from LSTs via an RG flow. See (2.59), (2.63) and (2.64) for examples of such theories and (3.10), (3.14), (3.15) for their F-theory constructions. Such SCFTs were earlier expected to be inconsistent in [3] because as shown there almost all SCFTs do admit a LST completion. As shown in this paper, this expectation is not correct.

2 Missing theories

We start in section 2.1 by listing down all the sub-quivers appearing in [4] but not admitting a construction in the unfrozen phase of F-theory. We then list down all the possible LSTs and SCFTs containing these sub-quivers[4] in sections 2.2 and 2.3 respectively. In compiling

[2] See [10] for initial work on Hanany-Witten-like brane constructions of six-dimensional theories.

[3] We emphasize that our list also includes those theories that contain non-gauge-theoretic factors like E-string and $\mathcal{N} = (2,0)$ theory. This is unlike [4] where the discussion was entirely restricted to gauge theories.

[4] We slightly enlarge the extent of the classification of [4] by allowing some non-gauge-theoretic factors to appear in the low energy theory on the tensor branch in the form of formal gauge algebras $\mathfrak{sp}(0)$ and $\mathfrak{su}(1)$.

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our list, we discard those theories which involve certain sub-quivers known to have a field theoretic inconsistency [11].

2.1 Missing sub-quivers

- $$\mathsf{S}^2 \text{ —— } \mathfrak{su}(n) \tag{2.1}$$

  which denotes a hyper in two-index symmetric representation S^2 of $\mathfrak{su}(n)$.

- $$\mathfrak{su}(n) \text{ —— } \mathfrak{so}(m) \tag{2.2}$$

  where the edge denotes a hyper in bifundamental of $\mathfrak{su} \oplus \mathfrak{so}$.

- $$\mathfrak{su}(4) \overset{\mathsf{S}}{\text{——}} \mathfrak{so}(7) \tag{2.3}$$

  where the edge decorated by S on one side denotes a hyper in fundamental $\otimes$ spinor of $\mathfrak{su} \oplus \mathfrak{so}$.

- $$\mathfrak{su}(4) \text{ —— } \mathfrak{g}_2 \tag{2.4}$$

  where the edge denotes a hyper in fundamental $\otimes$ $\mathbf{7}$ of $\mathfrak{su} \oplus \mathfrak{g}_2$.

- $$\begin{array}{ccccc} & & \mathfrak{so}(n_2) & & \\ & & | & & \\ \mathfrak{so}(n_1) & \text{—} & \mathfrak{sp}(n) & \text{—} & \mathfrak{so}(n_3) \end{array} \tag{2.5}$$

  where the edge between $\mathfrak{sp}(n)$ and $\mathfrak{so}(n_i)$ denotes a half-hyper in bifundamental of $\mathfrak{sp}(n) \oplus \mathfrak{so}(n_i)$.

- $$\begin{array}{ccccc} & & \mathfrak{so}(n_2) & & \\ & & | & & \\ \mathfrak{so}(n_1) & \text{—} & \mathfrak{sp}(n) & \text{—} & \mathfrak{su}(n_3) \end{array} \tag{2.6}$$

  where the edge between $\mathfrak{sp}(n)$ and $\mathfrak{su}(n_3)$ denotes a hyper in bifundamental of $\mathfrak{sp}(n) \oplus \mathfrak{su}(n_3)$.

- $$\begin{array}{ccccc} & & \mathfrak{so}(n_2) & & \\ & & | & & \\ \mathfrak{so}(n_1) & \text{—} & \mathfrak{sp}(4) & \overset{\mathsf{S}}{\text{—}} & \mathfrak{so}(7) \end{array} \tag{2.7}$$

  where the edge decorated by S on one side denotes a half-hyper in fundamental $\otimes$ spinor of $\mathfrak{sp} \oplus \mathfrak{so}$.

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- $$\begin{array}{ccccc} & & \mathfrak{so}(n_2) & & \\ & & | & & \\ \mathfrak{so}(n_1) & — & \mathfrak{sp}(4) & — & \mathfrak{g}_2 \end{array} \tag{2.8}$$

  where the edge between $\mathfrak{sp}$ and $\mathfrak{g}_2$ denotes a half-hyper in fundamental $\otimes\,\mathbf{7}$ of $\mathfrak{sp} \oplus \mathfrak{g}_2$.

- $$\begin{array}{ccccc} & & \mathfrak{so}(7) & & \\ & & \mathsf{S}| & & \\ \mathfrak{so}(7) & \overset{\mathsf{S}}{—} & \mathfrak{sp}(2) & \overset{\mathsf{S}}{—} & \mathfrak{so}(7) \end{array} \tag{2.9}$$

2.2 Missing LSTs

Let us first list down all the possible LSTs carrying the sub-quivers listed in section 2.1:

- $$\mathsf{S}^2 — \mathfrak{su}(n_0) — \mathfrak{su}(n_1) - \cdots - \mathfrak{su}(n_k) — \mathfrak{sp}(m) \tag{2.10}$$

  where all the edges except the leftmost one denote a hyper in bifundamental. Here $n_i = 2m + 8 + 8(k - i)$ with $m \geq 0$ and $k \geq 0$. The case $m = 0$ corresponds to an E-string theory at the rightmost end of the quiver.

  Its construction is given in (3.16).

- $$\begin{array}{ccccccccccccc} & & & & & & \mathsf{F} & & & & & & \\ & & & & & & | & & & & & & \\ \mathsf{S}^2 & — & \mathfrak{su}(n_0) & — & \mathfrak{su}(n_1) & - \cdots - & \mathfrak{su}(n_j) & - \cdots - & \mathfrak{su}(n_k) & — & \mathfrak{sp}(0) \end{array} \tag{2.11}$$

  where the edge between $\mathfrak{su}(n_j)$ and F denotes a hyper in the fundamental representation F of $\mathfrak{su}(n_j)$. Here $n_i = 9 + 9(k - i)$ for $j \leq i \leq k$ and $n_i = 9 + 9(k - j) + 8(j - i)$ for $0 \leq i \leq j$ with $0 \leq j \leq k$ and $k \geq 0$. $\mathfrak{sp}(0)$ is a shorthand for E-string which allows a neighboring $\mathfrak{su}(n \leq 9)$. Since these theories involve an E-string, they don't appear in [4] but can be obtained by a mild extension of the rules considered there.

  Its construction is given in (3.16).

- $$\mathsf{S}^2 — \mathfrak{su}(n_0) — \mathfrak{su}(n_1) - \cdots - \mathfrak{su}(n_k) — \mathfrak{su}(m) — \Lambda^2 \tag{2.12}$$

  where the rightmost edge denotes a hyper in two-index antisymmetric representation Λ^2 of $\mathfrak{su}(m)$. Here $n_i = m + 8 + 8(k - i)$ with $m \geq 2$ and $k \geq 0$.

  Its construction is given in (3.17).

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- $$\mathsf{S}^2 - \mathfrak{su}(n_0) - \mathfrak{su}(n_1) - \cdots - \overset{\substack{\mathsf{F}\\ |}}{\mathfrak{su}(n_j)} - \cdots - \mathfrak{su}(n_k) - \tfrac{1}{2}\Lambda^3 \tag{2.13}$$

  where the rightmost edge denotes a half-hyper in three-index antisymmetric representation Λ^3 of $\mathfrak{su}(n_k)$. Here $n_i = 6+9(k-i)$ for $j \le i \le k$ and $n_i = 6+9(k-j)+8(j-i)$ for $0 \le i \le j$ with $0 \le j \le k$ and $k \ge 1$.

  Its construction is given in (3.20).

- $$\mathfrak{so}(n_0) - \mathfrak{su}(n_1) - \mathfrak{su}(n_2) - \cdots - \mathfrak{su}(n_k) - \mathfrak{sp}(m) \tag{2.14}$$

  Here $n_i = 2m+8+8(k-i)$ with $m \ge 0$ and $k \ge 1$.

  Its construction is given in (3.23).

- $$\mathfrak{so}(n_0) - \mathfrak{su}(n_1) - \mathfrak{su}(n_2) - \cdots - \overset{\substack{\mathsf{F}\\ |}}{\mathfrak{su}(n_j)} - \cdots - \mathfrak{su}(n_k) - \mathfrak{sp}(0) \tag{2.15}$$

  Here $n_i = 9+9(k-i)$ for $j \le i \le k$ and $n_i = 9+9(k-j)+8(j-i)$ for $0 \le i \le j$ with $1 \le j \le k$ and $k \ge 1$.

  Its construction is given in (3.23).

  For $j = 0$, we obtain

  $$\overset{\substack{\mathsf{F}\\ |}}{\mathfrak{so}(n_0)} - \mathfrak{su}(n_1) - \mathfrak{su}(n_2) - \cdots - \mathfrak{su}(n_k) - \mathfrak{sp}(0) \tag{2.16}$$

  where the edge between $\mathfrak{so}(n_0)$ and F denotes a hyper in the fundamental representation F of $\mathfrak{so}(n_0)$. Here $n_i = 9+9(k-i)$ with $k \ge 1$.

  Its construction is given in (3.23).

- $$\mathfrak{so}(n_0) - \mathfrak{su}(n_1) - \mathfrak{su}(n_2) - \cdots - \mathfrak{su}(n_k) - \mathfrak{su}(m) - \Lambda^2 \tag{2.17}$$

  Here $n_i = m+8+8(k-i)$ with $m \ge 2$ and $k \ge 1$.

  Its construction is given in (3.24).

- $$\mathfrak{so}(n_0) - \mathfrak{su}(n_1) - \mathfrak{su}(n_2) - \cdots - \overset{\substack{\mathsf{F}\\ |}}{\mathfrak{su}(n_j)} - \cdots - \mathfrak{su}(n_k) - \tfrac{1}{2}\Lambda^3 \tag{2.18}$$

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Here $n_i = 6 + 9(k-i)$ for $j \leq i \leq k$ and $n_i = 6 + 9(k-j) + 8(j-i)$ for $1 \leq i \leq j$ with $1 \leq j \leq k$ and $k \geq 2$.

Its construction is given in (3.25).

For $j = 0$, we obtain

$$\begin{array}{c} \mathsf{F} \\ | \\ \mathfrak{so}(n_0) \end{array} — \mathfrak{su}(n_1) — \mathfrak{su}(n_2) - \cdots - \mathfrak{su}(n_k) — \tfrac{1}{2}\Lambda^3 \tag{2.19}$$

Here $n_i = 6 + 9(k-i)$ with $k \geq 2$.

Its construction is given in (3.25).

- $$\mathfrak{su}(n_0) — \mathfrak{so}(n_1) — \mathfrak{sp}(n_2) — \mathfrak{so}(n_3) - \cdots - \mathfrak{sp}(n_{2k}) — \mathfrak{su}(m) \tag{2.20}$$

  where the dots denote an alternating $\mathfrak{sp} - \mathfrak{so}$ chain. We remind the reader that edges between $\mathfrak{so}$ and $\mathfrak{sp}$ correspond to a half-hyper rather than a full hyper in bifundamental. Here $n_{2i+1} = 2n_{2i} = 2m + 16(k-i)$ with $m \geq 2$ and $k \geq 1$.

  Its construction is given in (3.26).

- $$\mathfrak{su}(n_0) — \mathfrak{so}(n_1) — \mathfrak{sp}(n_2) - \cdots - \begin{array}{c} \tfrac{1}{2}\mathsf{F} \\ | \\ \mathfrak{sp}(n_{2j}) \end{array} - \cdots - \mathfrak{sp}(n_{2k}) \xrightarrow{\tfrac{1}{2}\mathsf{F}} \mathfrak{su}(1) \tag{2.21}$$

  where the dots denote alternating $\mathfrak{sp} - \mathfrak{so}$ chains and the edge between $\mathfrak{sp}(n_{2j})$ and $\frac{1}{2}\mathsf{F}$ denotes a half-hyper in fundamental representation F of $\mathfrak{sp}(n_{2j})$. $\mathfrak{su}(1)$ at the rightmost node indicates an unpaired tensor corresponding to A_1 $\mathcal{N} = (2,0)$ theory. The decoration by $\frac{1}{2}\mathsf{F}$ on top of rightmost edge indicates that a half-hyper in fundamental of $\mathfrak{sp}(n_{2k}) = \mathfrak{sp}(1)$ has to be trapped there for the edge between $\mathfrak{sp}(n_{2k}) = \mathfrak{sp}(1)$ and $\mathfrak{su}(1)$ to be consistent.[5] This half-hyper is unlike the half-hyper attached to $\mathfrak{sp}(n_{2j})$ because the latter can move around as we change j but the former must remain attached to $\mathfrak{sp}(n_{2k}) = \mathfrak{sp}(1)$. Here $n_{2i+1} + 1 = 2n_{2i} = 2 + 18(k-i)$ for $j \leq i \leq k$ and $n_{2i+1} = 2n_{2i} = 2 + 18(k-j) + 16(j-i)$ for $0 \leq i \leq j-1$ with $1 \leq j \leq k$ and $k \geq 1$.

  Its construction is given in (3.27) and (3.28).

[5]The existence of this trapped $\frac{1}{2}\mathsf{F}$ can be understood if one views the A_1 $\mathcal{N} = (2,0)$ theory in the $\mathcal{N} = (1,0)$ language. The $\mathcal{N} = (2,0)$ R-symmetry is $\mathfrak{so}(5)$ whose $\mathfrak{so}(4)$ subalgebra decomposes into $\mathfrak{su}(2)$ $\mathcal{N} = (1,0)$ R-symmetry plus an $\mathfrak{su}(2) = \mathfrak{sp}(1)$ flavor symmetry. The $\mathcal{N} = (2,0)$ tensor multiplet decomposes into a $\mathcal{N} = (1,0)$ tensor multiplet plus a $\mathcal{N} = (1,0)$ hypermultiplet such that the hypermultiplet transforms as $\frac{1}{2}\mathsf{F}$ under the flavor $\mathfrak{sp}(1)$. This flavor $\mathfrak{sp}(1)$ is gauged in (2.21) by the gauge algebra $\mathfrak{sp}(n_{2k}) = \mathfrak{sp}(1)$.

For $j = 0$, we obtain

$$\begin{array}{c} \mathsf{F} \\ | \\ \mathfrak{su}(n_0) \end{array} — \mathfrak{so}(n_1) — \mathfrak{sp}(n_2) — \cdots — \mathfrak{sp}(n_{2k}) \overset{\frac{1}{2}\mathsf{F}}{——} \mathfrak{su}(1) \tag{2.22}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1}+1 = 2n_{2i} = 2+18(k-i)$ with $k \geq 1$.

Its construction is given in (3.27).

- $$\mathfrak{su}(n_0) — \mathfrak{so}(n_1) — \mathfrak{sp}(n_2) — \cdots — \mathfrak{sp}(n_{2k}) — \mathfrak{so}(n_{2k+1}) — \mathfrak{sp}(0) — \mathfrak{su}(1) \tag{2.23}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1} = 2n_{2i} = 16+16(k-i)$ with $k \geq 1$. The sub-quiver

$$\mathfrak{sp}(0) — \mathfrak{su}(1) \tag{2.24}$$

formed by the two rightmost nodes denotes a rank two E-string theory.

Its construction is given in (3.29).

For $k = 0$, we obtain

$$\mathfrak{su}(8) — \mathfrak{so}(16) — \mathfrak{sp}(0) — \mathfrak{su}(1) \tag{2.25}$$

Its construction is given in (3.30).

- $$\mathfrak{su}(n_0) — \mathfrak{so}(n_1) — \mathfrak{sp}(n_2) — \mathfrak{so}(n_3) — \cdots — \mathfrak{sp}(n_{2k}) — \mathfrak{so}(n_{2k+1}) — \mathsf{S} \tag{2.26}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain and the rightmost edge denotes a hyper in spinor representation S of $\mathfrak{so}(n_{2k+1})$. Here $n_{2i+1} = 2n_{2i} = 12 + 16(k-i)$ with $k \geq 1$.

Its construction is given in (3.31).

For $k = 0$, we obtain

$$\mathfrak{su}(6) — \mathfrak{so}(12) — \mathsf{S} \tag{2.27}$$

Its construction is given in (3.32).

- $$\mathfrak{su}(n_0) — \mathfrak{so}(n_1) — \mathfrak{sp}(n_2) — \cdots — \begin{array}{c} \frac{1}{2}\mathsf{F} \\ | \\ \mathfrak{sp}(n_{2j}) \end{array} — \cdots — \mathfrak{sp}(n_{2k}) — \mathfrak{so}(n_{2k+1}) — \tfrac{1}{2}\mathsf{S} \tag{2.28}$$

where the dots denote alternating $\mathfrak{sp}-\mathfrak{so}$ chains and the rightmost edge denotes a half-hyper in spinor representation S of $\mathfrak{so}(n_{2k+1})$. Here $n_{2i+1}+1 = 2n_{2i} = 14+18(k-i)$ for $j \leq i \leq k$ and $n_{2i+1} = 2n_{2i} = 14+18(k-j)+16(j-i)$ for $0 \leq i \leq j-1$ with $1 \leq j \leq k$ and $k \geq 1$.

Its construction is given in (3.33).

For $j = 0$, we obtain

$$\begin{array}{c} \mathsf{F} \\ | \\ \mathfrak{su}(n_0) \end{array} \text{ --- } \mathfrak{so}(n_1) \text{ --- } \mathfrak{sp}(n_2) - \cdots - \mathfrak{sp}(n_{2k}) \text{ --- } \mathfrak{so}(n_{2k+1}) \text{ --- } \tfrac{1}{2}\mathsf{S} \tag{2.29}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1}+1 = 2n_{2i} = 14+18(k-i)$ with $k \geq 1$.

Its construction is given in (3.33).

For $k = 0$, we obtain

$$\begin{array}{c} \mathsf{F} \\ | \\ \mathfrak{su}(7) \end{array} \text{ --- } \mathfrak{so}(13) \text{ --- } \tfrac{1}{2}\mathsf{S} \tag{2.30}$$

Its construction is given in (3.34).

- $$\mathfrak{su}(n_0) \text{ --- } \mathfrak{so}(n_1) \text{ --- } \mathfrak{sp}(n_2) \text{ --- } \mathfrak{so}(n_3) - \cdots - \mathfrak{sp}(n_{2k}) \overset{\mathsf{S}}{\text{---}} \mathfrak{so}(7) \tag{2.31}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1} = 2n_{2i} = 8+16(k-i)$ with $k \geq 1$.

Its construction is given in (3.35).

For $k = 0$, we obtain

$$\mathfrak{su}(4) \overset{\mathsf{S}}{\text{---}} \mathfrak{so}(7) \tag{2.32}$$

Its construction is given in (3.36).

- $$\mathfrak{su}(n_0) \text{ --- } \mathfrak{so}(n_1) \text{ --- } \mathfrak{sp}(n_2) - \cdots - \begin{array}{c} \tfrac{1}{2}\mathsf{F} \\ | \\ \mathfrak{sp}(n_{2j}) \end{array} - \cdots - \mathfrak{sp}(n_{2k}) \text{ --- } \mathfrak{g}_2 \tag{2.33}$$

where the dots denote alternating $\mathfrak{sp}-\mathfrak{so}$ chains. Here $n_{2i+1}+1 = 2n_{2i} = 8+18(k-i)$ for $j \leq i \leq k$ and $n_{2i+1} = 2n_{2i} = 8+18(k-j)+16(j-i)$ for $0 \leq i \leq j-1$ with $1 \leq j \leq k$ and $k \geq 1$.

Its construction is given in (3.37).

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For $j = 0$, we obtain

$$\begin{array}{c} \mathsf{F} \\ | \\ \mathfrak{su}(n_0) \end{array} \text{ --- } \mathfrak{so}(n_1) \text{ --- } \mathfrak{sp}(n_2) - \cdots - \mathfrak{sp}(n_{2k}) \text{ --- } \mathfrak{g}_2 \tag{2.34}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1}+1 = 2n_{2i} = 8+18(k-i)$ with $k \geq 1$.

Its construction is given in (3.37).

For $k = 0$, we obtain

$$\begin{array}{c} \mathsf{F} \\ | \\ \mathfrak{su}(4) \end{array} \text{ --- } \mathfrak{g}_2 \tag{2.35}$$

Its construction is given in (3.38).

- $$\mathfrak{su}(n_0) \text{ --- } \mathfrak{so}(n_1) \text{ --- } \mathfrak{sp}(n_2) - \cdots - \mathfrak{sp}(n_{2k}) \text{ --- } \begin{array}{c} \mathfrak{sp}(m) \\ | \\ \mathfrak{so}(n_{2k+1}) \end{array} \text{ --- } \mathfrak{sp}(m) \tag{2.36}$$

where the dots denote an alternating $\mathfrak{sp} - \mathfrak{so}$ chain. Here $n_{2i+1} = 2n_{2i} = 4m + 16 + 16(k-i)$ with $m \geq 0$ and $k \geq 1$. The case $m = 0$ gives rise to two E-string factors at the right end of the quiver.

Its construction is given in (3.39).

For $k = 0$, we obtain

$$\mathfrak{su}(n_0) \text{ --- } \begin{array}{c} \mathfrak{sp}(m) \\ | \\ \mathfrak{so}(n_1) \end{array} \text{ --- } \mathfrak{sp}(m) \tag{2.37}$$

Here $n_1 = 2n_0 = 4m + 16$ with $m \geq 0$.

Its construction is given in (3.40).

- $$\mathfrak{so}(n) \text{ --- } \begin{array}{c} \mathfrak{so}(n) \\ | \\ \mathfrak{sp}(n_0) \end{array} \text{ --- } \mathfrak{so}(n_1) - \cdots - \mathfrak{sp}(n_{2k}) \text{ --- } \mathfrak{su}(m) \tag{2.38}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1} = 2n_{2i} = 2m+16(k-i)$ and $n = m + 8 + 8k$ with $m \geq 2$ and $k \geq 1$.

Its construction is given in (3.41).

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$$
\begin{array}{ccccccccc}
 & \mathfrak{so}(n) & & & \tfrac{1}{2}\mathsf{F} & & & & \\
 & | & & & | & & & & \\
\mathfrak{so}(n) \;—\!\!— & \mathfrak{sp}(n_0) & - \cdots - & & \mathfrak{sp}(n_{2j}) & - \cdots - & \mathfrak{sp}(n_{2k}) & \overset{\frac{1}{2}\mathsf{F}}{——} & \mathfrak{su}(1)
\end{array}
\tag{2.39}
$$

where the dots denote alternating $\mathfrak{sp}-\mathfrak{so}$ chains. Here $n_{2i+1}+1 = 2n_{2i} = 2+18(k-i)$ for $j \leq i \leq k$, $n_{2i+1} = 2n_{2i} = 2+18(k-j)+16(j-i)$ for $0 \leq i \leq j-1$, and $n = 9+9k-j$ with $0 \leq j \leq k$ and $k \geq 1$.

Its construction is given in (3.42) and (3.43).

-

$$
\begin{array}{ccccccccc}
 & & \mathfrak{so}(n) & & & & & & \\
 & & | & & & & & & \\
\mathfrak{so}(n+1) & —\!\!— & \mathfrak{sp}(n_0) & —\!\!— & \mathfrak{so}(n_1) & - \cdots - & \mathfrak{sp}(n_{2k}) & \overset{\frac{1}{2}\mathsf{F}}{——} & \mathfrak{su}(1) \\
| & & & & & & & & \\
\mathsf{F} & & & & & & & &
\end{array}
\tag{2.40}
$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1}+1 = 2n_{2i} = 2+18(k-i)$ and $n = 9+9k$ with $k \geq 1$.

Its construction is given in (3.44).

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$$
\begin{array}{ccccccccc}
 & & \mathfrak{so}(n) & & & & & & \\
 & & | & & & & & & \\
\mathfrak{so}(n) & —\!\!— & \mathfrak{sp}(n_0) & —\!\!— & \mathfrak{so}(n_1) & - \cdots - & \mathfrak{sp}(n_{2k}) & —\!\!— & \mathfrak{su}(1)
\end{array}
\tag{2.41}
$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1} = 2n_{2i} = 16(k-i)$ and $n = 8+8k$ with $k \geq 1$. The two rightmost nodes gives rise to a rank two E-string factor in the low energy theory.

Its construction is given in (3.45).

-

$$
\begin{array}{ccccccccccc}
 & & \mathfrak{so}(n) & & & & & & & & \\
 & & | & & & & & & & & \\
\mathfrak{so}(n) & —\!\!— & \mathfrak{sp}(n_0) & —\!\!— & \mathfrak{so}(n_1) & - \cdots - & \mathfrak{sp}(n_{2k}) & —\!\!— & \mathfrak{so}(n_{2k+1}) & —\!\!— & \mathsf{S}
\end{array}
\tag{2.42}
$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1} = 2n_{2i} = 12+16(k-i)$ and $n = 14+8k$ with $k \geq 0$.

Its construction is given in (3.46).

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- $$\begin{array}{ccccccccccccccc} & & \mathfrak{so}(n) & & & & & & & & & & & & \\ & & | & & & & & & & & & & & & \\ \mathfrak{so}(n) & — & \mathfrak{sp}(n_0) & - & \cdots & - & \mathfrak{sp}(n_{2j}) & - & \cdots & - & \mathfrak{sp}(n_{2k}) & — & \mathfrak{so}(n_{2k+1}) & — & \frac{1}{2}\mathsf{S} \\ & & & & & & | & & & & & & & & \\ & & & & & & \frac{1}{2}\mathsf{F} & & & & & & & & \end{array} \tag{2.43}$$

  where the dots denote alternating $\mathfrak{sp}-\mathfrak{so}$ chains. Here $n_{2i+1}+1 = 2n_{2i} = 14+18(k-i)$ for $j \leq i \leq k$, $n_{2i+1} = 2n_{2i} = 14+18(k-j)+16(j-i)$ for $0 \leq i \leq j-1$ and $n = 15+9k-j$ with $0 \leq j \leq k$ and $k \geq 0$.

  Its construction is given in (3.47).

- $$\begin{array}{ccccccccccc} & & \mathfrak{so}(n) & & & & & & & & \\ & & | & & & & & & & & \\ \mathfrak{so}(n+1) & — & \mathfrak{sp}(n_0) & - & \cdots & - & \mathfrak{sp}(n_{2k}) & — & \mathfrak{so}(n_{2k+1}) & — & \frac{1}{2}\mathsf{S} \\ | & & & & & & & & & & \\ \mathsf{F} & & & & & & & & & & \end{array} \tag{2.44}$$

  where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1}+1 = 2n_{2i} = 14+18(k-i)$ and $n = 15+9k$ with $k \geq 0$.

  Its construction is given in (3.48).

- $$\begin{array}{ccccccccccc} & & \mathfrak{so}(n) & & & & & & & & \\ & & | & & & & & & & & \\ \mathfrak{so}(n) & — & \mathfrak{sp}(n_0) & — & \mathfrak{so}(n_1) & - & \cdots & - & \mathfrak{sp}(n_{2k}) & \overset{\mathsf{S}}{—} & \mathfrak{so}(7) \end{array} \tag{2.45}$$

  where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1} = 2n_{2i} = 8+16(k-i)$ and $n = 12+8k$ with $k \geq 1$.

  Its construction is given in (3.49).

  For $k = 0$, we obtain

  $$\begin{array}{ccccc} & & \mathfrak{so}(12) & & \\ & & | & & \\ \mathfrak{so}(12) & — & \mathfrak{sp}(4) & \overset{\mathsf{S}}{—} & \mathfrak{so}(7) \end{array} \tag{2.46}$$

  Its construction is given in (3.50).

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- $$\begin{tikzcd}[column sep=small] & \mathfrak{so}(n) \arrow[d, dash] & & & & & \\ \mathfrak{so}(n) \arrow[r, dash] & \mathfrak{sp}(n_0) \arrow[r, dash] & \cdots \arrow[r, dash] & \mathfrak{sp}(n_{2j}) \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & \mathfrak{sp}(n_{2k}) \arrow[r, dash] & \mathfrak{g}_2 \\ & & & \frac{1}{2}\mathsf{F} & & & \end{tikzcd} \tag{2.47}$$

  where the dots denote alternating $\mathfrak{sp}-\mathfrak{so}$ chains. Here $n_{2i+1}+1 = 2n_{2i} = 8+18(k-i)$ for $j \le i \le k$, $n_{2i+1} = 2n_{2i} = 8+18(k-j)+16(j-i)$ for $0 \le i \le j-1$ and $n = 12+9k-j$ with $0 \le j \le k$ and $k \ge 1$.

  Its construction is given in (3.51).

  For $k = 0$, we obtain

  $$\begin{tikzcd} & \mathfrak{so}(12) \arrow[d, dash] & \\ \mathfrak{so}(12) \arrow[r, dash] & \mathfrak{sp}(4) \arrow[r, dash] \arrow[d, dash] & \mathfrak{g}_2 \\ & \frac{1}{2}\mathsf{F} & \end{tikzcd} \tag{2.48}$$

  Its construction is given in (3.52).

- $$\begin{tikzcd}[column sep=small] & \mathfrak{so}(n) \arrow[d, dash] & & & & \\ \mathfrak{so}(n+1) \arrow[r, dash] \arrow[d, dash] & \mathfrak{sp}(n_0) \arrow[r, dash] & \cdots \arrow[r, dash] & \mathfrak{sp}(n_{2k}) \arrow[r, dash] & \mathfrak{g}_2 \\ \mathsf{F} & & & & \end{tikzcd} \tag{2.49}$$

  where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $n_{2i+1}+1 = 2n_{2i} = 8+18(k-i)$ and $n = 12+9k$ with $k \ge 1$.

  Its construction is given in (3.53).

  For $k = 0$, we obtain

  $$\begin{tikzcd} & \mathfrak{so}(12) \arrow[d, dash] & \\ \mathfrak{so}(13) \arrow[r, dash] \arrow[d, dash] & \mathfrak{sp}(4) \arrow[r, dash] & \mathfrak{g}_2 \\ \mathsf{F} & & \end{tikzcd} \tag{2.50}$$

  Its construction is given in (3.54).

- $$\begin{tikzcd}[column sep=small] & \mathfrak{so}(n) \arrow[d, dash] & & & & \mathfrak{sp}(m) \arrow[d, dash] & \\ \mathfrak{so}(n) \arrow[r, dash] & \mathfrak{sp}(n_0) \arrow[r, dash] & \mathfrak{so}(n_1) \arrow[r, dash] & \cdots \arrow[r, dash] & & \mathfrak{so}(n_{2k+1}) \arrow[r, dash] & \mathfrak{sp}(m) \end{tikzcd} \tag{2.51}$$

where the dots denote an alternating $\mathfrak{sp} - \mathfrak{so}$ chain. Here $n_{2i+1} = 2n_{2i} = 4m + 16 + 16(k - i)$ and $n = 2m + 16 + 8k$ with $m \geq 0$ and $k \geq 0$.

Its construction is given in (3.55).

- $$\begin{array}{ccccccccc} & & & & \mathfrak{sp}(m) & & & & \\ & & & & | & & & & \\ & & & & \mathfrak{so}(4m+16) & & & & \\ & & & & | & & & & \\ \mathfrak{sp}(m) & — & \mathfrak{so}(4m+16) & — & \mathfrak{sp}(3m+8) & — & \mathfrak{so}(4m+16) & — & \mathfrak{sp}(m) \end{array} \tag{2.52}$$

  with $m \geq 0$.

  Its construction is given in (3.56).

- $$\begin{array}{ccccc} & & \mathfrak{so}(7) & & \\ & & {\scriptstyle \mathsf{s}}| & & \\ \mathfrak{so}(7) & \overset{\mathsf{s}}{—} & \mathfrak{sp}(2) & \overset{\mathsf{s}}{—} & \mathfrak{so}(7) \end{array} \tag{2.53}$$

  Its construction is given in (3.57).

- $$\begin{array}{ccccccccc} & & & & \frac{1}{2}\mathsf{S} & & & & \\ & & & & | & & & & \\ & & & & \mathfrak{so}(12) & & & & \\ & & & & | & & & & \\ \frac{1}{2}\mathsf{S} & — & \mathfrak{so}(12) & — & \mathfrak{sp}(5) & — & \mathfrak{so}(12) & — & \frac{1}{2}\mathsf{S} \end{array} \tag{2.54}$$

  Its construction is given in (3.58).

2.3 Missing SCFTs

Let us now list down all the possible SCFTs carrying the sub-quivers listed in section 2.1. Our list below will contain SCFTs that do not have an LST parent. These SCFTs are (2.59), (2.63) and (2.64).

- $$\begin{array}{ccccccc} & & m_0\mathsf{F} & & m_1\mathsf{F} & \cdots & m_k\mathsf{F} \\ & & | & & | & \cdots & | \\ \mathsf{S}^2 & — & \mathfrak{su}(n_0) & — & \mathfrak{su}(n_1) & - \cdots - & \mathfrak{su}(n_k) \end{array} \tag{2.55}$$

  where the edge between $\mathfrak{su}(n_i)$ and $m_i\mathsf{F}$ denotes m_i hypers in fundamental of $\mathfrak{su}(n_i)$. Here $m_0 = n_0 - 8 - n_1$ and $m_i = 2n_i - n_{i-1} - n_{i+1}$ for $1 \leq i \leq k$ with $n_{k+1} := 0$ and $k \geq 0$.

  Its construction is given in (3.6).

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- $$\begin{array}{ccccccccc} m_0\mathsf{F} & & m_1\mathsf{F} & & m_2\mathsf{F} & & \cdots & & m_k\mathsf{F} \\ | & & | & & | & & \cdots & & | \\ \mathfrak{so}(n_0) & — & \mathfrak{su}(n_1) & — & \mathfrak{su}(n_2) & - & \cdots & - & \mathfrak{su}(n_k) \end{array} \tag{2.56}$$

where the edge between $\mathfrak{so}(n_0)$ and $m_0\mathsf{F}$ denotes m_0 hypers in vector of $\mathfrak{so}(n_0)$. Here $m_0 = n_0 - 8 - n_1$ and $m_i = 2n_i - n_{i-1} - n_{i+1}$ for $1 \leq i \leq k$ with $n_{k+1} := 0$ and $k \geq 1$.

Its construction is given in (3.7).

- $$\begin{array}{ccccccccc} m_0\mathsf{F} & & m_1\mathsf{F} & & m_2\mathsf{F} & & \cdots & & m_{2k}\mathsf{F} \\ | & & | & & | & & \cdots & & | \\ \mathfrak{su}(n_0) & — & \mathfrak{so}(n_1) & — & \mathfrak{sp}(n_2) & - & \cdots & - & \mathfrak{sp}(n_{2k}) \end{array} \tag{2.57}$$

where the dots denote an alternating $\mathfrak{sp} - \mathfrak{so}$ chain and the edge between $\mathfrak{sp}(n_{2i})$ and $m_{2i}\mathsf{F}$ denotes m_{2i} hypers in fundamental of $\mathfrak{sp}(n_{2i})$. Here $m_0 = 2n_0 - n_1$, $m_{2i-1} = n_{2i-1} - 8 - n_{2i-2} - n_{2i}$ and $m_{2i} = 2n_{2i} + 8 - \frac{n_{2i-1}}{2} - \frac{n_{2i+1}}{2}$ for $1 \leq i \leq k$ with $n_{2k+1} := 0$ and $k \geq 1$. Here n_{2k} can be zero, in which case we obtain an E-string factor at the right end of the quiver.

Its construction is given in (3.8).

- $$\begin{array}{ccccccccc} m\mathsf{F} & & m_0\mathsf{F} & & m_1\mathsf{F} & & \cdots & & m_{2k}\mathsf{F} \\ | & & | & & | & & \cdots & & | \\ \mathfrak{su}(n) & — & \mathfrak{so}(n_0) & — & \mathfrak{sp}(n_1) & - & \cdots & - & \mathfrak{so}(n_{2k}) \end{array} \tag{2.58}$$

where the dots denote an alternating $\mathfrak{sp} - \mathfrak{so}$ chain. Here $m = 2n - n_0$, $m_0 = n_0 - 8 - n_1 - n$, $m_{2i} = n_{2i} - 8 - n_{2i-1} - n_{2i}$ and $m_{2i+1} = 2n_{2i+1} + 8 - \frac{n_{2i}}{2} - \frac{n_{2i+2}}{2}$ for $1 \leq i \leq k$ with $n_{2k+1} := 0$ and $k \geq 0$.

Its construction is given in (3.9).

- $$\begin{array}{ccccccc} m_0\mathsf{F} & & m_1\mathsf{F} & & m_2\mathsf{F} & & m_3\mathsf{F} \\ | & & | & & | & & | \\ \mathfrak{su}(n_0) & — & \mathfrak{su}(n_1) & — & \mathfrak{so}(n_2) & — & \mathfrak{sp}(n_3) \end{array} \tag{2.59}$$

Here $m_0 = 2n_0 - n_1$, $m_1 = 2n_1 - n_0 - n_2$, $m_2 = n_2 - 8 - n_1 - n_3$ and $m_3 = 2n_3 + 8 - \frac{n_2}{2}$.

Its construction is given in (3.10). It was suspected in [3] that this theory is probably not consistent since there is no LST from which it can be obtained by decoupling a tensor multiplet. Our construction in (3.10) demonstrates that this suspicion is not correct, and shows that there exist SCFTs that cannot be obtained via an RG flow starting from a LST.

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$$\begin{array}{ccccccccc} & & m_1\mathsf{F} & & & & & & \\ & & | & & & & & & \\ & & \mathfrak{so}(n_1) & & & & & & \\ & & | & & & & & & \\ \mathfrak{so}(n_0) & — & \mathfrak{sp}(n_2) & — & \mathfrak{so}(n_3) & — & \cdots & — & \mathfrak{sp}(n_{2k}) \\ | & & | & & | & & \cdots & & | \\ m_0\mathsf{F} & & m_2\mathsf{F} & & m_3\mathsf{F} & & \cdots & & m_{2k}\mathsf{F} \end{array} \tag{2.60}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $m_0 = n_0 - 8 - n_2$, $m_1 = n_1 - 8 - n_2$, $m_2 = 2n_2 + 8 - \frac{n_0}{2} - \frac{n_1}{2} - \frac{n_3}{2}$, $m_{2i-1} = n_{2i-1} - 8 - n_{2i} - n_{2i-2}$ and $m_{2i} = 2n_{2i} + 8 - \frac{n_{2i-1}}{2} - \frac{n_{2i+1}}{2}$ for $2 \leq i \leq k$ with $n_{2k+1} := 0$ and $k \geq 2$. Here n_{2k} can be zero, in which case we obtain an E-string factor at the right end of the quiver.

Its construction is given in (3.11).

-

$$\begin{array}{ccccccccc} & & m_1\mathsf{F} & & & & & & \\ & & | & & & & & & \\ & & \mathfrak{so}(n_1) & & & & & & \\ & & | & & & & & & \\ \mathfrak{so}(n_0) & — & \mathfrak{sp}(n_2) & — & \mathfrak{so}(n_3) & — & \cdots & — & \mathfrak{so}(n_{2k+1}) \\ | & & | & & | & & \cdots & & | \\ m_0\mathsf{F} & & m_2\mathsf{F} & & m_3\mathsf{F} & & \cdots & & m_{2k+1}\mathsf{F} \end{array} \tag{2.61}$$

where the dots denote an alternating $\mathfrak{sp}-\mathfrak{so}$ chain. Here $m_0 = n_0 - 8 - n_2$, $m_1 = n_1 - 8 - n_2$, $m_2 = 2n_2 + 8 - \frac{n_0}{2} - \frac{n_1}{2} - \frac{n_3}{2}$, $m_{2i-1} = n_{2i-1} - 8 - n_{2i} - n_{2i-2}$, $m_{2i} = 2n_{2i} + 8 - \frac{n_{2i-1}}{2} - \frac{n_{2i+1}}{2}$ and $m_{2k+1} = n_{2k+1} - 8 - n_{2k}$ for $2 \leq i \leq k$ with $k \geq 1$.

Its construction is given in (3.12).

-

$$\begin{array}{ccccccccc} & & & & m_5\mathsf{F} & & & & \\ & & & & | & & & & \\ & & & & \mathfrak{so}(n_5) & & & & \\ & & & & | & & & & \\ \mathfrak{sp}(n_0) & — & \mathfrak{so}(n_1) & — & \mathfrak{sp}(n_2) & — & \mathfrak{so}(n_3) & — & \mathfrak{sp}(n_4) \\ | & & | & & | & & | & & | \\ m_0\mathsf{F} & & m_1\mathsf{F} & & m_2\mathsf{F} & & m_3\mathsf{F} & & m_4\mathsf{F} \end{array} \tag{2.62}$$

Here $m_0 = 2n_0 + 8 - \frac{n_1}{2}$, $m_1 = n_1 - 8 - n_0 - n_2$, $m_2 = 2n_2 + 8 - \frac{n_1}{2} - \frac{n_3}{2} - \frac{n_5}{2}$, $m_3 = n_3 - 8 - n_4 - n_2$, $m_4 = 2n_4 + 8 - \frac{n_3}{2}$ and $m_5 = n_5 - 8 - n_2$.

Its construction is given in (3.13).

- $$\begin{tikzcd}
 & & m_6\mathsf{F} \arrow[d, dash] & & & \\
 & & \mathfrak{so}(n_6) \arrow[d, dash] & & & \\
\mathfrak{sp}(n_0) \arrow[r, dash] \arrow[d, dash] & \mathfrak{so}(n_1) \arrow[r, dash] \arrow[d, dash] & \mathfrak{sp}(n_2) \arrow[r, dash] \arrow[d, dash] & \mathfrak{so}(n_3) \arrow[r, dash] \arrow[d, dash] & \mathfrak{sp}(n_4) \arrow[r, dash] \arrow[d, dash] & \mathfrak{so}(n_5) \arrow[d, dash] \\
m_0\mathsf{F} & m_1\mathsf{F} & m_2\mathsf{F} & m_3\mathsf{F} & m_4\mathsf{F} & m_5\mathsf{F}
\end{tikzcd} \tag{2.63}$$

  Here $m_0 = 2n_0 + 8 - \frac{n_1}{2}$, $m_1 = n_1 - 8 - n_0 - n_2$, $m_2 = 2n_2 + 8 - \frac{n_1}{2} - \frac{n_3}{2} - \frac{n_6}{2}$, $m_3 = n_3 - 8 - n_4 - n_2$, $m_4 = 2n_4 + 8 - \frac{n_3}{2} - \frac{n_5}{2}$, $m_5 = n_5 - 8 - n_4$ and $m_6 = n_6 - 8 - n_2$.

  Its construction is given in (3.14). Like (2.59), this theory is an example of an SCFT that cannot be obtained from an LST via an RG flow.

- $$\begin{tikzcd}
 & & m_7\mathsf{F} \arrow[d, dash] & & & & \\
 & & \mathfrak{so}(n_7) \arrow[d, dash] & & & & \\
\mathfrak{sp}(n_0) \arrow[r, dash] \arrow[d, dash] & \mathfrak{so}(n_1) \arrow[r, dash] \arrow[d, dash] & \mathfrak{sp}(n_2) \arrow[r, dash] \arrow[d, dash] & \mathfrak{so}(n_3) \arrow[r, dash] \arrow[d, dash] & \mathfrak{sp}(n_4) \arrow[r, dash] \arrow[d, dash] & \mathfrak{so}(n_5) \arrow[r, dash] \arrow[d, dash] & \mathfrak{sp}(n_6) \arrow[d, dash] \\
m_0\mathsf{F} & m_1\mathsf{F} & m_2\mathsf{F} & m_3\mathsf{F} & m_4\mathsf{F} & m_5\mathsf{F} & m_6\mathsf{F}
\end{tikzcd} \tag{2.64}$$

  Here $m_0 = 2n_0 + 8 - \frac{n_1}{2}$, $m_1 = n_1 - 8 - n_0 - n_2$, $m_2 = 2n_2 + 8 - \frac{n_1}{2} - \frac{n_3}{2} - \frac{n_7}{2}$, $m_3 = n_3 - 8 - n_4 - n_2$, $m_4 = 2n_4 + 8 - \frac{n_3}{2} - \frac{n_5}{2}$, $m_5 = n_5 - 8 - n_4 - n_6$, $m_6 = 2n_6 + 8 - \frac{n_5}{2}$ and $m_7 = n_7 - 8 - n_2$.

  Its construction is given in (3.15). Like (2.59) and (2.63), this theory is another example of an SCFT that cannot be obtained from an LST via an RG flow.

3 6*d* SCFTs and LSTs from the frozen phase

3.1 Reasons for missing theories

We now recall the reasons due to which the theories listed in sections 2.2 and 2.3 do not admit a construction in the unfrozen phase of F-theory. These theories can be divided into three types.

The first type of theories involve an $\mathfrak{su}(n)$ gauge algebra with a hyper in S^2 and $n-8$ hypers in F. For such a theory to admit a construction in the unfrozen phase of F-theory, the $\mathfrak{su}(n)$ must arise on a curve C in the base B of the F-theory compactification such that:

1. The arithmetic genus of C must be one.

2. The self-intersection of C in B must be -1.

It was shown in appendix B of [2] that the order of vanishing of (f, g) appearing in the Weierstrass model on such a curve C is at least $(4, 6)$. Such a large order of vanishing of (f, g) on a curve in B is considered to be unphysical. Hence, no such theory can be constructed in the unfrozen phase of F-theory.

The second type of theories involve an $\mathfrak{su}(m \geq 4)$ gauge algebra with $2m$ hypers in F such that a subset of those hypers transform in a representation R of another gauge algebra which is either $\mathfrak{so}(n)$ or $\mathfrak{g}_2$. For such a theory to admit a construction in the unfrozen phase of F-theory, the following conditions must be satisfied:

1. The $\mathfrak{su}(m)$ must arise on a curve C and $\mathfrak{so}(n)$ or $\mathfrak{g}_2$ must arise on a curve D such that $C \cdot D \neq 0$.
2. The $\mathfrak{so}(n)$ or $\mathfrak{g}_2$ algebra must arise from an I_p^* singularity over D.
3. Since $m \geq 4$, $\mathfrak{su}(m)$ must arise from an I_m singularity over C.
4. C must have genus zero and self-intersection -2.

Now, an I_m singularity over such a C cannot consistently intersect an I_p^* singularity. Thus, no such theory can be constructed in the unfrozen phase of F-theory.

The third type of theories involve an $\mathfrak{sp}(m \geq 2)$ gauge algebra with $2m + 8$ hypers in F such that three subsets of those hypers transform respectively in representation R_1, R_2 and R_3 of other gauge algebras $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{h}_3$ such that each $\mathfrak{h}_i$ is either an $\mathfrak{so}$ algebra or a $\mathfrak{g}_2$ algebra. For such a theory to admit a construction in the unfrozen phase of F-theory, the following conditions must be satisfied:

1. The $\mathfrak{sp}(m)$ must arise on a curve C and $\mathfrak{h}_i$ must arise on a curve D_i such that $C \cdot D_i \neq 0$ for each i.
2. The $\mathfrak{h}_i$ must arise from an $\mathrm{I}_{p_i}^*$ singularity over D.
3. Since $m \geq 2$, $\mathfrak{sp}(m)$ must arise from a non-split I_{2m} singularity over C.
4. C must have genus zero and self-intersection -1.

Now, an I_{2m} singularity over such a C cannot consistently intersect three singularities $\mathrm{I}_{p_i}^*$. Thus, no such theory can be constructed in the unfrozen phase of F-theory.

3.2 Ingredients from the frozen phase

3.2.1 New constructions of old ingredients

The frozen phase provides us with novel constructions of some gauge-theoretic ingredients that already admit a construction in the unfrozen phase. We will use the following constructions in this paper:

1. $\mathfrak{sp}(m)$ gauge algebra with $(2m + 8)\mathsf{F}$ can be constructed in the frozen phase by a curve[6] C of self-intersection -4 carrying an $\hat{\mathrm{I}}_{m+4}^*$ singularity where, following the

[6] All of the curves considered in this paper have genus zero.

notation of [5], we add a hat on top of an I_n^* singularity if it carries an algebra of $\mathfrak{sp}$ type[7] rather than $\mathfrak{so}$ type. In type IIB language, an $\hat{\mathrm{I}}_{m+4}^*$ singularity corresponds to a stack of m D7 branes on top of an $\mathrm{O7}^+$ plane.[8]

There are a total of $4m+16$ zeroes of the residual discriminant $\tilde{\Delta}_C$ on C. Each zero carries a $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(m)$ leading to a total of $(2m+8)\mathsf{F}$ of $\mathfrak{sp}(m)$. If all the points on C where $\tilde{\Delta}_C$ vanishes have even multiplicity of zeroes, then the $\hat{\mathrm{I}}_{m+4}^*$ singularity is split. Otherwise, the $\hat{\mathrm{I}}_{m+4}^*$ singularity is non-split.

For future purposes, we define a divisor $F=\sum_i C_i$ where C_i are compact or non-compact curves carrying a singularity of type $\hat{\mathrm{I}}_{n_i}^*$.

2. $\mathfrak{so}(m)$ gauge algebra with $(m-8)\mathsf{F}$ can be constructed in the frozen phase by a curve C of self-intersection -1 carrying a non-split I_m singularity such that $F\cdot C=2$.

   A non-split I_m singularity on a -1 curve corresponds to a stack of m D7 branes intersecting two O7 planes in type IIB language. Since $F\cdot C=2$, both of these O7 planes are $\mathrm{O7}^+$. Hence, the gauge algebra carried by C is $\mathfrak{so}(m)$.

   There are a total of $m+12$ zeroes of $\tilde{\Delta}_C$. 20 of these come from intersections of C with the two $\mathrm{O7}^+$ planes. This is because an $\mathrm{O7}^+$ plane corresponds to a $\hat{\mathrm{I}}_4^*$ singularity over which Δ vanish to order 10. Each remaining zero carries an F of $\mathfrak{so}(m)$, thus leading to a total of $(m-8)\mathsf{F}$ of $\mathfrak{so}(m)$.

   We will also sometimes use a non-split I_{m+1} on C to construct $\mathfrak{so}(m)$ with $(m-8)\mathsf{F}$. This should be viewed as a non-geometric Higgsing of $\mathfrak{so}(m+1)$ living on I_{m+1} down to $\mathfrak{so}(m)$.

3. $\mathfrak{su}(m)$ gauge algebra with $2m\mathsf{F}$ can be constructed in the frozen phase by the following configuration of two curves C and D

$$\begin{array}{ccc} \mathrm{I}_{2m}^{ns} & & \mathrm{I}_m^s \\ 1 & \text{———} & 2 \\ C & & D \end{array} \tag{3.1}$$

   where the numbers displayed over C and D denote the negative of their self-intersections, the edge denotes that $C\cdot D=1$, the singularity over C is non-split I_{2m} and the singularity over D is split I_m. In [5], a *gauge divisor* was associated to every $6d$ gauge algebra. Here the gauge divisor for $\mathfrak{su}(m)$ is $\Sigma=2C+D$ which means that the $6d$ gauge algebra $\mathfrak{su}(m)$ is embedded into the $8d$ gauge algebra $\mathfrak{su}(2m)$ carried by I_{2m} with embedding index 2 and the $8d$ gauge algebra $\mathfrak{su}(m)$ carried by I_m with embedding index 1. We also need $F\cdot\Sigma=2$ for consistency, which is only possible if $F\cdot C=1$ since D cannot intersect any other singularity.

   It is again possible to understand this construction perturbatively. Since $F\cdot C=1$, one of the O7 planes intersecting the stack of $2m$ D7 branes on C is an $\mathrm{O7}^+$ and

[7] Notice that $n\geq 4$ for an $\hat{\mathrm{I}}_n^*$ singularity.

[8] In our notation, a superscript $+$ denotes an O7 plane of positive RR charge and a superscript $-$ denotes an O7 plane of negative RR charge.

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the other is an O7$^-$ plane thus leading to an $\mathfrak{su}(m)$ gauge algebra with embedding index 2 on C. A split I_m singularity on the -2 curve D corresponds simply to a stack of m D7 branes on D leading to another $\mathfrak{su}(m)$ there. Now we can perform a non-geometric Higgsing which combines the two $\mathfrak{su}(m)$ living on C and D.

$\tilde{\Delta}_D$ has no zeroes other than those coming from the intersection with I_{2m}^{ns} singularity on C. $\tilde{\Delta}_C$ has a total of $2m+12$ zeroes. 10 out of these come from the intersection with O7$^+$ and 2 of these come from the intersection with O7$^-$. Each of the remaining zeroes carry $2\mathsf{F}$ of $\mathfrak{su}(m)$, thus leading to a total of $2m\mathsf{F}$ of $\mathfrak{su}(m)$.

For $m=1$ and $m=0$, we obtain new constructions for A_1 $\mathcal{N}=(2,0)$ SCFT.

4. We will need another construction for $\mathfrak{sp}(m)$ gauge algebra with $(2m+8)\mathsf{F}$ which is

$$\begin{array}{ccc} \hat{\mathrm{I}}_{m+4}^{*} & & \mathrm{I}_{2m}^{ns} \\ 4 & \text{———} & 1 \\ C & & D \end{array} \tag{3.2}$$

with no other frozen singularity intersecting either C or D. If a curve carrying a frozen singularity appears in a gauge divisor, then its coefficient in the gauge divisor is the embedding index times an extra factor of half. Thus, the gauge divisor for this configuration is $\Sigma = \frac{1}{2}C + D$.

To understand this construction perturbatively, notice that the other O7 plane intersecting D is an O7$^-$ plane which reduces the gauge algebra on the stack of $2m$ D7 branes on D to $\mathfrak{sp}(m)$. We then combine this $\mathfrak{sp}(m)$ with the $\mathfrak{sp}(m)$ living on C. Unlike the previous case, the O7$^+$ plane carried by C does not induce a further reduction of gauge algebra on D. This makes sense because C and D are part of the same gauge divisor.

$\tilde{\Delta}_C$ has a total of $4m+16$ zeroes out of which $2m$ come from the intersection with the I_{2m}^{ns} singularity living over D. Each other zero carries a $\frac{1}{2}\mathsf{F}$ of the low energy $\mathfrak{sp}(m)$, thus leading to $(m+8)\mathsf{F}$ of $\mathfrak{sp}(m)$ living on C. $\tilde{\Delta}_D$ has a total of $2m+12$ zeroes out of which $m+10$ come from the intersection with the $\hat{\mathrm{I}}_{m+4}^{*}$ singularity living over C. Moreover, 2 other zeroes come from the intersection with the O7$^-$ plane. Each other zero carries an F of the low energy $\mathfrak{sp}(m)$, thus leading to $m\mathsf{F}$ of $\mathfrak{sp}(m)$ living on D. In total, we get $(2m+8)\mathsf{F}$ of $\mathfrak{sp}(m)$.

We will also sometimes use

$$\begin{array}{ccc} \hat{\mathrm{I}}_{m+5}^{*} & & \mathrm{I}_{2m+1}^{ns} \\ 4 & \text{———} & 1 \\ C & & D \end{array} \tag{3.3}$$

with $\Sigma = \frac{1}{2}C + D$ to construct $\mathfrak{sp}(m)$ with $(2m+8)\mathsf{F}$.

5. $\mathfrak{so}(7)$ gauge algebra with $2\mathsf{S}$ can be constructed in the frozen phase by the configuration

$$\begin{array}{ccc} \mathrm{I}_8^{ns} & & \mathrm{I}_2^{*ns} \\ 1 & \text{——} & 3 \\ C & & D \end{array} \tag{3.4}$$

with gauge divisor $\Sigma = 2C + D$ and $F \cdot C = 1$, where we have performed a non-geometric Higgsing to reduce the algebra living over I_2^{*ns} from $\mathfrak{so}(11)$ to $\mathfrak{so}(7)$.

$\tilde{\Delta}_C$ has a total of 20 zeroes. 8 out of these come from the I_2^{*ns} singularity on C. 10 other zeroes come from an intersection with $\mathrm{O7}^+$ plane. The remaining two zeroes each carry an S of $\mathfrak{so}(7)$. We propose that the zeroes of $\tilde{\Delta}_D$ not coming from intersection with I_8^{ns} do not carry any matter content.

6. We will also construct $\mathfrak{sp}(5)$ with $18\mathsf{F}$ via

$$\begin{array}{ccccc} & & \mathrm{I}_{11}^{ns} & & \\ & & 1 & & \\ & & D_3 & & \\ & & | & & \\ \mathrm{I}_{11}^{ns} & & \hat{\mathrm{I}}_9^{*ns} & & \mathrm{I}_{11}^{ns} \\ 1 & \text{——} & 4 & \text{——} & 1 \\ D_1 & & C & & D_2 \end{array} \tag{3.5}$$

with $\Sigma = \frac{1}{2}C + D_1 + D_2 + D_3$ and no other frozen singularity intersects either C or any D_i. Each D_i carries $6\mathsf{F}$ situated at 6 zeroes of residual discriminant on D_i.

3.2.2 A new ingredient

We will also need a gauge-theoretic ingredient arising in the frozen phase that does not admit a construction in the unfrozen phase. This is $\mathfrak{su}(m)$ with $\mathsf{S}^2 + (m-8)\mathsf{F}$ and can be constructed by a curve C of self-intersection -1 carrying an I_m^s singularity with $F \cdot C = 2$. Since the intersection points of F with C are branch points for the monodromy, to obtain a split I_m, F must intersect C tangentially at a single point.

Out of $m+12$ zeroes of $\tilde{\Delta}_C$, 20 come from the tangential intersection with $\mathrm{O7}^+$. The remaining $m-8$ zeroes each carry an F of $\mathfrak{su}(m)$.

3.3 Construction of missing SCFTs

In this subsection, we will show that the frozen phase allows us to construct all the missing SCFTs listed in section 2.3.

- Eq. (2.55) can be constructed via

$$\begin{array}{cccccccc} & & \mathrm{I}_{n_0}^s & & \mathrm{I}_{n_1}^s & & & \mathrm{I}_{n_k}^s \\ \hat{\mathrm{I}}_4^* & \overset{t}{=\!=\!=} & 1 & \text{——} & 2 & \text{——} \cdots & \text{——} & 2 \end{array} \tag{3.6}$$

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where any singularity without a number attached to it denotes a non-compact curve[9] carrying that singularity. The double edge with a tiny t on top of it denotes a tangential intersection between the curve carrying $\hat{\mathrm{I}}^*_4$ and the curve carrying $\mathrm{I}^s_{n_0}$.

- Eq. (2.56) can be constructed via

$$\begin{array}{ccccccccc} & & \mathrm{I}^{ns}_{n_0} & & \mathrm{I}^{s}_{n_1} & & & & \mathrm{I}^{s}_{n_k} \\ \hat{\mathrm{I}}^*_4 & \text{———} & 1 & \text{———} & 2 & \text{———} & \cdots & \text{———} & 2 \\ & & \vert & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & \end{array} \tag{3.7}$$

- Eq. (2.57) can be constructed via

$$\begin{array}{ccccccccccc} \mathrm{I}^{s}_{n_0} & & \mathrm{I}^{ns}_{n_1} & & \hat{\mathrm{I}}^*_{n_2+4} & & \mathrm{I}^{ns}_{n_3} & & & & \hat{\mathrm{I}}^*_{n_{2k}+4} \\ 2 & \text{———} & 1 & \text{———} & 4 & \text{———} & 1 & \text{———} & \cdots & \text{———} & 4 \\ & & \vert & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & \end{array} \tag{3.8}$$

where the dots denote an alternating chain of $\begin{array}{c}\hat{\mathrm{I}}^*_{n_i+4}\\ 4\end{array}$ and $\begin{array}{c}\mathrm{I}^{ns}_{n_{i+1}}\\ 1\end{array}$.

- Eq. (2.58) can be constructed via

$$\begin{array}{ccccccccccccc} \mathrm{I}^{s}_{n} & & \mathrm{I}^{ns}_{n_0} & & \hat{\mathrm{I}}^*_{n_1+4} & & \mathrm{I}^{ns}_{n_2} & & & & \mathrm{I}^{ns}_{n_{2k}} & & \\ 2 & \text{———} & 1 & \text{———} & 4 & \text{———} & 1 & \text{———} & \cdots & \text{———} & 1 & \text{———} & \hat{\mathrm{I}}^*_4 \\ & & \vert & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & \end{array} \tag{3.9}$$

where the dots denote an alternating chain of $\begin{array}{c}\hat{\mathrm{I}}^*_{n_i+4}\\ 4\end{array}$ and $\begin{array}{c}\mathrm{I}^{ns}_{n_{i+1}}\\ 1\end{array}$.

- Eq. (2.59) can be constructed via

$$\begin{array}{ccccccc} \mathrm{I}^{s}_{n_0} & & \mathrm{I}^{s}_{n_1} & & \mathrm{I}^{ns}_{n_2} & & \hat{\mathrm{I}}^*_{n_3+4} \\ 2 & \text{———} & 2 & \text{———} & 1 & \text{———} & 4 \\ & & & & \vert & & \\ & & & & \hat{\mathrm{I}}^*_4 & & \end{array} \tag{3.10}$$

This shows that (2.59) exists even though it does not have any LST parent, thus demonstrating the existence of such SCFTs.

[9]We will only display non-compact curves carrying frozen singularities.

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- Eq. (2.60) can be constructed via

$$\begin{array}{ccccccccccccc}
 & & & & \hat{\mathrm{I}}^*_4 & & & & & & & & \\
 & & & & | & & & & & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{n_1}}{1} & & & & & & & & \\
 & & & & | & & & & & & & & \\
\hat{\mathrm{I}}^*_4 & \text{---} & \overset{\mathrm{I}^{ns}_{n_0}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_2+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_3}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_4+4}}{4} & \text{---} & \cdots & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_{2k}+4}}{4}
\end{array} \tag{3.11}$$

where the dots denote an alternating chain of $\overset{\hat{\mathrm{I}}^*_{n_i+4}}{4}$ and $\overset{\mathrm{I}^{ns}_{n_{i+1}}}{1}$.

- Eq. (2.61) can be constructed via

$$\begin{array}{ccccccccccccccc}
 & & & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{n_1}}{1} & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
\hat{\mathrm{I}}^*_4 & \text{---} & \overset{\mathrm{I}^{ns}_{n_0}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_2+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_3}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_4+4}}{4} & \text{---} & \cdots & \text{---} & \overset{\mathrm{I}^{ns}_{n_{2k+1}}}{1} & \text{---} & \hat{\mathrm{I}}^*_4
\end{array} \tag{3.12}$$

where the dots denote an alternating chain of $\overset{\hat{\mathrm{I}}^*_{n_i+4}}{4}$ and $\overset{\mathrm{I}^{ns}_{n_{i+1}}}{1}$.

- Eq. (2.62) can be constructed via

$$\begin{array}{ccccccccc}
 & & & & \hat{\mathrm{I}}^*_4 & & & & \\
 & & & & | & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{n_5}}{1} & & & & \\
 & & & & | & & & & \\
\overset{\hat{\mathrm{I}}^*_{n_0+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_1}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_2+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_3}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_4+4}}{4}
\end{array} \tag{3.13}$$

- Eq. (2.63) can be constructed via

$$\begin{array}{ccccccccccccc} & & & & \hat{\mathrm{I}}_4^* & & & & & & & & \\ & & & & | & & & & & & & & \\ & & & & \overset{\mathrm{I}_{n_6}^{ns}}{1} & & & & & & & & \\ & & & & | & & & & & & & & \\ \overset{\hat{\mathrm{I}}^*_{n_0+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_1}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_2+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_3}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_4+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_5}}{1} & \text{---} & \hat{\mathrm{I}}_4^* \end{array} \tag{3.14}$$

  This shows that (2.63) exists even though it does not have any LST parent.

- Eq. (2.64) can be constructed via

$$\begin{array}{ccccccccccccc} & & & & \hat{\mathrm{I}}_4^* & & & & & & & & \\ & & & & | & & & & & & & & \\ & & & & \overset{\mathrm{I}_{n_7}^{ns}}{1} & & & & & & & & \\ & & & & | & & & & & & & & \\ \overset{\hat{\mathrm{I}}^*_{n_0+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_1}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_2+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_3}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_4+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n_5}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_6+4}}{4} \end{array} \tag{3.15}$$

  This shows that (2.64) exists even though it does not have any LST parent.

3.4 Construction of missing LSTs

In this subsection, we will show that the frozen phase allows us to construct all the missing LSTs listed in section 2.2.

- Eq. (2.10) can be constructed via

$$\hat{\mathrm{I}}_4^* \overset{t}{=\!=\!=} \overset{\mathrm{I}^s_{n_0}}{1} \text{---} \overset{\mathrm{I}^s_{n_1}}{2} \text{---} \cdots \text{---} \overset{\mathrm{I}^s_{n_k}}{2} \text{---} \overset{\mathrm{I}^{ns}_{2m}}{1} \tag{3.16}$$

  We substitute $m = 0$ in (3.16) to obtain the construction for (2.11).

- Eq. (2.12) can be constructed via

$$\hat{\mathrm{I}}_4^* \overset{t}{=\!=\!=} \overset{\mathrm{I}^s_{n_0}}{1} \text{---} \overset{\mathrm{I}^s_{n_1}}{2} \text{---} \cdots \text{---} \overset{\mathrm{I}^s_{n_k}}{2} \text{---} \overset{\mathrm{I}^{s}_{m}}{1} \tag{3.17}$$

  The following limit of (3.17)

$$\hat{\mathrm{I}}_4^* \overset{t}{=\!=\!=} \overset{\mathrm{I}^s_{m}}{0} \tag{3.18}$$

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provides a construction for

$$\mathsf{S}^2 \;\text{——}\; \mathfrak{su}(m) \;\text{——}\; \Lambda^2 \tag{3.19}$$

that is dual to the construction provided in [3] using the unfrozen phase of F-theory. Notice that the construction of [3] requires $\mathfrak{su}(m)$ to be realized on a singular curve in B, whereas our construction realizes $\mathfrak{su}(m)$ on a smooth curve in B.

- Eq. (2.13) can be constructed via

$$\hat{\mathrm{I}}_4^* \overset{t}{=\!=\!=} \overset{\mathrm{I}^s_{n_0}}{1} \;\text{——}\; \overset{\mathrm{I}^s_{n_1}}{2} \;\text{——}\; \cdots \;\text{——}\; \overset{\mathrm{I}^s_{n_{k-1}}}{2} \;\text{——}\; \overset{\mathrm{I}^s_{6}}{1} \tag{3.20}$$

where the I^s_6 is tuned to give rise to a $\frac{1}{2}\Lambda^3$.

The following limit of (3.20)

$$\hat{\mathrm{I}}_4^* \overset{t}{=\!=\!=} \overset{\mathrm{I}^s_{6}}{0} \tag{3.21}$$

with a tuned I_6 provides a construction for

$$\begin{array}{ccccc} & & \mathsf{F} & & \\ & & | & & \\ \mathsf{S}^2 & \text{——} & \mathfrak{su}(6) & \text{——} & \frac{1}{2}\Lambda^3 \end{array} \tag{3.22}$$

that is dual to the construction provided in [3] using the unfrozen phase of F-theory. Again, notice that the construction of [3] requires $\mathfrak{su}(6)$ to be realized on a singular curve in B, whereas our construction realizes $\mathfrak{su}(6)$ on a smooth curve in B.

- Eq. (2.14) can be constructed via

$$\begin{array}{ccccccccccc} & & \mathrm{I}^{ns}_{n_0} & & \mathrm{I}^s_{n_1} & & & & \mathrm{I}^s_{n_k} & & \mathrm{I}^{ns}_{2m} \\ \hat{\mathrm{I}}_4^* & \text{——} & 1 & \text{——} & 2 & \text{——} & \cdots & \text{——} & 2 & \text{——} & 1 \\ & & | & & & & & & & & \\ & & \hat{\mathrm{I}}_4^* & & & & & & & & \end{array} \tag{3.23}$$

We substitute $m = 0$ in (3.23) to obtain the constructions for (2.15) and (2.16).

- Eq. (2.17) can be constructed via

$$\begin{array}{ccccccccccc} & & \mathrm{I}^{ns}_{n_0} & & \mathrm{I}^s_{n_1} & & & & \mathrm{I}^s_{n_k} & & \mathrm{I}^{s}_{m} \\ \hat{\mathrm{I}}_4^* & \text{——} & 1 & \text{——} & 2 & \text{——} & \cdots & \text{——} & 2 & \text{——} & 1 \\ & & | & & & & & & & & \\ & & \hat{\mathrm{I}}_4^* & & & & & & & & \end{array} \tag{3.24}$$

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- Eq. (2.18) and (2.19) can be constructed via

$$\begin{array}{ccccccccccc} & & \mathrm{I}^{ns}_{n_0} & & \mathrm{I}^s_{n_1} & & & & \mathrm{I}^s_{n_{k-1}} & & \mathrm{I}^s_6 \\ \hat{\mathrm{I}}^*_4 & \text{——} & 1 & \text{——} & 2 & \text{——} & \cdots & \text{——} & 2 & \text{——} & 1 \\ & & | & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & \end{array} \tag{3.25}$$

where the I^s_6 is tuned to give rise to a $\frac{1}{2}\Lambda^3$.

- Eq. (2.20) can be constructed via

$$\begin{array}{ccccccccccccc} \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^{*s}_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^{*s}_{m_3} & & & & \mathrm{I}^{ns}_{m_{2k}} & & \mathrm{I}^s_m \\ 2 & \text{——} & 1 & \text{——} & 4 & \text{——} & 1 & \text{——} & 4 & \text{——} & \cdots & \text{——} & 1 & \text{——} & 2 \\ & & | & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & \end{array} \tag{3.26}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$ and the dashed ellipse encircling the first two curves indicates that those two curves give rise to a single gauge algebra in $6d$, which in this case is $\mathfrak{su}(n_0)$ as we know from (3.1). Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$.

- Eq. (2.21) for $j < k$ can be constructed via

$$\begin{array}{ccccccccccccc} \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^{*}_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^{*}_{m_3} & & & & \mathrm{I}^{ns}_{m_{2k}} & & \mathrm{II} \\ 2 & \text{——} & 1 & \text{——} & 4 & \text{——} & 1 & \text{——} & 4 & \text{——} & \cdots & \text{——} & 1 & \text{——} & 2 \\ & & | & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & \end{array} \tag{3.27}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$ with $\mathrm{I}^*_{m_{2i-1}}$ singularity being split for $1 \le i \le j$, and $m_{2i} = 2n_{2i}+1$, $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with $\mathrm{I}^*_{m_{2i-1}}$ singularity being non-split for $j+1 \le i \le k$. It is known [1] that the intersection of type II singularity with $\mathrm{I}^{ns}_3 = \mathrm{I}^{ns}_{m_{2k}}$ captures a $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(1) = \mathfrak{sp}(n_{2k})$ as required. The $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(n_{2j})$ is localized at the intersection of $\mathrm{I}^{ns}_{m_{2j}}$ and $\mathrm{I}^{*ns}_{m_{2j+1}}$.

Eq. (2.21) for $j = k$ can be constructed via

$$\begin{array}{ccccccccccccc} \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^{*s}_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^{*s}_{m_3} & & & & \mathrm{I}_{m_{2k}} & & \mathrm{I}_1 \\ 2 & \text{——} & 1 & \text{——} & 4 & \text{——} & 1 & \text{——} & 4 & \text{——} & \cdots & \text{——} & 1 & \text{——} & 2 \\ & & | & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & \end{array} \tag{3.28}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$. It is well-known that the intersection of I_1 with $\mathrm{I}_2 = \mathrm{I}_{m_{2k}}$ captures a full F of $\mathfrak{sp}(1) = \mathfrak{sp}(n_{2k})$, as required.

We substitute $j = 0$ in (3.27) to obtain the construction for (2.22). Here $m_{2i} = 2n_{2i} + 1$, $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with every $\mathrm{I}^*_{m_{2i-1}}$ singularity being non-split.

- Eq. (2.23) can be constructed via

$$\begin{array}{ccccccccccccccccc} \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^{*s}_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^{*s}_{m_3} & & & & \mathrm{I}^{*s}_{m_{2k+1}} & & \mathrm{I}_0 & & \mathrm{I}_0 \\ 2 & \text{———} & 1 & \text{———} & 4 & \text{———} & 1 & \text{———} & 4 & \text{———} & \cdots & \text{———} & 4 & \text{———} & 1 & \text{———} & 2 \\ & & | & & & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & & & \end{array} \tag{3.29}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i+1} = \frac{n_{2i+1}}{2} - 4$.

Eq. (2.25) can be constructed via

$$\begin{array}{ccccccccc} \mathrm{I}^s_8 & & \mathrm{I}^{ns}_{16} & & \mathrm{I}^{*s}_4 & & \mathrm{I}_0 & & \mathrm{I}_0 \\ 2 & \text{———} & 1 & \text{———} & 4 & \text{———} & 1 & \text{———} & 4 \\ & & | & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & \end{array} \tag{3.30}$$

- Eq. (2.26) can be constructed via

$$\begin{array}{ccccccccccccccc} \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^{*s}_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^{*s}_{m_3} & & & & \mathrm{I}^{ns}_{m_{2k}} & & \mathrm{I}^{*s}_{m_{2k+1}} \\ 2 & \text{———} & 1 & \text{———} & 4 & \text{———} & 1 & \text{———} & 4 & \text{———} & \cdots & \text{———} & 1 & \text{———} & 2 \\ & & | & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & \end{array} \tag{3.31}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i+1} = \frac{n_{2i+1}}{2} - 4$.

Eq. (2.27) can be constructed via

$$\begin{array}{ccccc} \mathrm{I}^s_6 & & \mathrm{I}^{ns}_{12} & & \mathrm{I}^{*s}_2 \\ 2 & \text{———} & 1 & \text{———} & 2 \\ & & | & & \\ & & \hat{\mathrm{I}}^*_4 & & \end{array} \tag{3.32}$$

- Eq. (2.28) can be constructed via

$$\begin{array}{ccccccccccccccc} \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^*_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^*_{m_3} & & & & \mathrm{I}^{ns}_{m_{2k}} & & \mathrm{I}^*_{m_{2k+1}} \\ 2 & \text{---} & 1 & \text{---} & 4 & \text{---} & 1 & \text{---} & 4 & \text{---} & \cdots & \text{---} & 1 & \text{---} & 2 \\ & & \vert & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & \end{array} \tag{3.33}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^*_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$ with $\mathrm{I}^*_{m_{2i-1}}$ singularity being split for $1 \le i \le j$, and $m_{2i} = 2n_{2i}+1$, $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with $\mathrm{I}^*_{m_{2i-1}}$ singularity being non-split for $i \ge j+1$. The $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(n_{2j})$ is localized at the intersection of $\mathrm{I}^{ns}_{m_{2j}}$ and $\mathrm{I}^{*ns}_{m_{2j+1}}$.

We substitute $j = 0$ in (3.33) to obtain the construction for (2.29). Here $m_{2i} = 2n_{2i}+1$, $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with every $\mathrm{I}^*_{m_{2i-1}}$ singularity being non-split. The F of $\mathfrak{su}(n_0)$ is localized at the intersection of $\mathrm{I}^{ns}_{2n_0}$ and $\mathrm{I}^{*ns}_{m_1}$.

Eq. (2.30) can be constructed via

$$\begin{array}{ccccc} \mathrm{I}^s_7 & & \mathrm{I}^{ns}_{14} & & \mathrm{I}^{*ns}_3 \\ 2 & \text{---} & 1 & \text{---} & 2 \\ & & \vert & & \\ & & \hat{\mathrm{I}}^*_4 & & \end{array} \tag{3.34}$$

with the F of $\mathfrak{su}(7)$ being localized at the intersection of I^{ns}_{14} and I^{*ns}_3.

- Eq. (2.31) can be constructed via

$$\begin{array}{ccccccccccccccc} \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^{*s}_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^{*s}_{m_3} & & & & \mathrm{I}^{ns}_{m_{2k}} & & \mathrm{I}^{*ss}_0 \\ 2 & \text{---} & 1 & \text{---} & 4 & \text{---} & 1 & \text{---} & 4 & \text{---} & \cdots & \text{---} & 1 & \text{---} & 2 \\ & & \vert & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & \end{array} \tag{3.35}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$.

Eq. (2.32) can be constructed via

$$\begin{array}{ccccc} \mathrm{I}^s_4 & & \mathrm{I}^{ns}_8 & & \mathrm{I}^{*ss}_0 \\ 2 & \text{---} & 1 & \text{---} & 2 \\ & & \vert & & \\ & & \hat{\mathrm{I}}^*_4 & & \end{array} \tag{3.36}$$

- Eq. (2.33) can be constructed via

$$\begin{array}{ccccccccccccccc} \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^*_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^*_{m_3} & & & & \mathrm{I}^{ns}_{m_{2k}} & & \mathrm{I}^{*ns}_0 \\ 2 & \text{——} & 1 & \text{——} & 4 & \text{——} & 1 & \text{——} & 4 & \text{——} & \cdots & \text{——} & 1 & \text{——} & 2 \\ & & | & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & \end{array} \tag{3.37}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^*_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$ with $\mathrm{I}^*_{m_{2i-1}}$ singularity being split for $1 \leq i \leq j$, and $m_{2i} = 2n_{2i}+1$, $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with $\mathrm{I}^*_{m_{2i-1}}$ singularity being non-split for $j+1 \leq i \leq k$. The $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(n_{2j})$ is localized at the intersection of $\mathrm{I}^{ns}_{m_{2j}}$ and $\mathrm{I}^{*ns}_{m_{2j+1}}$ where $\mathrm{I}^{*ns}_{m_{2k+1}} := \mathrm{I}^{*ns}_0$.

We substitute $j = 0$ in (3.37) to obtain the construction for (2.34). Here $m_{2i} = 2n_{2i}+1$, $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with every $\mathrm{I}^*_{m_{2i-1}}$ singularity being non-split. The F of $\mathfrak{su}(n_0)$ is localized at the intersection of $\mathrm{I}^{ns}_{2n_0}$ and $\mathrm{I}^{*ns}_{m_1}$.

Eq. (2.35) can be constructed via

$$\begin{array}{ccccc} \mathrm{I}^s_4 & & \mathrm{I}^{ns}_8 & & \mathrm{I}^{*ns}_0 \\ 2 & \text{——} & 1 & \text{——} & 2 \\ & & | & & \\ & & \hat{\mathrm{I}}^*_4 & & \end{array} \tag{3.38}$$

with the F of $\mathfrak{su}(4)$ being localized at the intersection of I^{ns}_8 and I^{*ns}_0.

- Eq. (2.36) can be constructed via

$$\begin{array}{ccccccccccccccccc} & & & & & & & & & & & & & & \mathrm{I}^{ns}_{2m} & & \\ & & & & & & & & & & & & & & 1 & & \\ & & & & & & & & & & & & & & | & & \\ \mathrm{I}^s_{n_0} & & \mathrm{I}^{ns}_{2n_0} & & \mathrm{I}^{*s}_{m_1} & & \mathrm{I}^{ns}_{m_2} & & \mathrm{I}^{*s}_{m_3} & & & & \mathrm{I}^{ns}_{m_{2k}} & & \mathrm{I}^{*s}_{m_{2k+1}} & & \mathrm{I}^{ns}_{2m} \\ 2 & \text{——} & 1 & \text{——} & 4 & \text{——} & 1 & \text{——} & 4 & \text{——} & \cdots & \text{——} & 1 & \text{——} & 4 & \text{——} & 1 \\ & & | & & & & & & & & & & & & & & \\ & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & & & & & \end{array} \tag{3.39}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$.

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Eq. (2.37) can be constructed via

$$\begin{array}{ccccccc}
 & & & & \overset{\mathrm{I}^{ns}_{2m}}{1} & & \\
 & & & & | & & \\
\overset{\mathrm{I}^{s}_{n_0}}{2} & \text{---} & \overset{\mathrm{I}^{ns}_{2n_0}}{1} & \text{---} & \overset{\mathrm{I}^{*s}_{m_1}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{2m}}{1} \\
 & & | & & & & \\
 & & \hat{\mathrm{I}}^{*}_{4} & & & &
\end{array} \tag{3.40}$$

where $m_1 = \frac{n_1}{2} - 4$.

- Eq. (2.38) can be constructed via

$$\begin{array}{ccccccccccccc}
 & & & & \hat{\mathrm{I}}^{*}_{4} & & & & & & & & \\
 & & & & | & & & & & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{n}}{1} & & & & & & & & \\
 & & & & | & & & & & & & & \\
\hat{\mathrm{I}}^{*}_{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^{*}_{n_0+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{2n_0}}{1} & \text{---} & \overset{\mathrm{I}^{*s}_{m_1}}{4} & \text{---}\cdots\text{---} & \overset{\mathrm{I}^{ns}_{m_{2k}}}{1} & \text{---} & \overset{\mathrm{I}^{s}_{m}}{2}
\end{array} \tag{3.41}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$ and the dashed ellipse encircling the first two curves indicates that those two curves give rise to a single gauge algebra in $6d$, which in this case is $\mathfrak{sp}(n_0)$ as we know from (3.2). Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$.

- Eq. (2.39) for $j < k$ can be constructed via

$$\begin{array}{ccccccccccccc}
 & & & & \hat{\mathrm{I}}^{*}_{4} & & & & & & & & \\
 & & & & | & & & & & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{n}}{1} & & & & & & & & \\
 & & & & | & & & & & & & & \\
\hat{\mathrm{I}}^{*}_{4} & \text{---} & \overset{\mathrm{I}^{ns}_{n}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^{*}_{n_0+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{2n_0}}{1} & \text{---} & \overset{\mathrm{I}^{*}_{m_1}}{4} & \text{---}\cdots\text{---} & \overset{\mathrm{I}^{ns}_{m_{2k}}}{1} & \text{---} & \overset{\mathrm{II}}{2}
\end{array} \tag{3.42}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$ with $\mathrm{I}^{*}_{m_{2i-1}}$ singularity being split for $1 \leq i \leq j$, and $m_{2i} = 2n_{2i}+1$,

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$m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with $\mathrm{I}^*_{m_{2i-1}}$ singularity being non-split for $j+1 \le i \le k$. The $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(1) = \mathfrak{sp}(n_{2k})$ is localized at the intersection of $\mathrm{I}^{ns}_{m_{2k}} = \mathrm{I}^{ns}_3$ and type II singularity. The $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(n_{2j})$ is localized at the intersection of $\mathrm{I}^{ns}_{m_{2j}}$ and $\mathrm{I}^{*ns}_{m_{2j+1}}$.

Eq. (2.39) for $j = k$ can be constructed via

$$\begin{array}{ccccccccccccccc} & & & & \hat{\mathrm{I}}_4^* & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ & & & & \overset{\mathrm{I}_n^{ns}}{1} & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ \hat{\mathrm{I}}_4^* & \text{---} & \overset{\mathrm{I}_n^{ns}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^*_{n_0+4}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{2n_0}}{1} & \text{---} & \overset{\mathrm{I}^{*s}_{m_1}}{4} & \text{---} & \cdots & \text{---} & \overset{\mathrm{I}_{m_{2k}}}{1} & \text{---} & \overset{\mathrm{I}_1}{2} \end{array} \tag{3.43}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$. The $\frac{1}{2}\mathsf{F} + \frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(1) = \mathfrak{sp}(n_{2k})$ is localized at the intersection of $\mathrm{I}_{m_{2k}} = \mathrm{I}_2$ and I_1.

- Eq. (2.40) can be constructed via

$$\begin{array}{ccccccccccccccc} & & & & \hat{\mathrm{I}}_4^* & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ & & & & \overset{\mathrm{I}_{n+1}^{ns}}{1} & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ \hat{\mathrm{I}}_4^* & \text{---} & \overset{\mathrm{I}_{n+2}^{ns}}{1} & \text{---} & \overset{\hat{\mathrm{I}}^{*ns}_{n_0+5}}{4} & \text{---} & \overset{\mathrm{I}^{ns}_{2n_0+1}}{1} & \text{---} & \overset{\mathrm{I}^{*ns}_{m_1}}{4} & \text{---} & \cdots & \text{---} & \overset{\mathrm{I}^{ns}_{m_{2k}}}{1} & \text{---} & \overset{\mathrm{II}}{2} \end{array} \tag{3.44}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*ns}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}+1$ and $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$. The $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(1) = \mathfrak{sp}(n_{2k})$ is localized at the intersection of $\mathrm{I}^{ns}_{m_{2k}} = \mathrm{I}^{ns}_3$ and type II singularity. The $\mathfrak{so}(n+1)$ is realized by I^{ns}_{n+2} and the $\mathfrak{so}(n)$ is realized by I^{ns}_{n+1}. The curves encircled by the dashed ellipse give rise to $\mathfrak{sp}(n_0)$.

- Eq. (2.41) can be constructed via

$$
\begin{array}{ccccccccccccccc}
 & & & & \hat{\mathrm{I}}_4^* & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
 & & & & \overset{\mathrm{I}_n^{ns}}{1} & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
\hat{\mathrm{I}}_4^* & — & \overset{\mathrm{I}_n^{ns}}{1} & — & \overset{\hat{\mathrm{I}}_{n_0+4}^*}{4} & — & \overset{\mathrm{I}_{2n_0}^{ns}}{1} & — & \overset{\mathrm{I}_{m_1}^{*s}}{4} & — & \cdots & — & \overset{\mathrm{I}_{m_{2k}}}{1} & — & \overset{\mathrm{I}_0}{2}
\end{array}
\tag{3.45}
$$

where the dots denote an alternating chain of $\overset{\mathrm{I}_{m_i}^{*s}}{4}$ and $\overset{\mathrm{I}_{m_{i+1}}^{ns}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$.

- Eq. (2.42) can be constructed via

$$
\begin{array}{ccccccccccccccc}
 & & & & \hat{\mathrm{I}}_4^* & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
 & & & & \overset{\mathrm{I}_n^{ns}}{1} & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
\hat{\mathrm{I}}_4^* & — & \overset{\mathrm{I}_n^{ns}}{1} & — & \overset{\hat{\mathrm{I}}_{n_0+4}^*}{4} & — & \overset{\mathrm{I}_{2n_0}^{ns}}{1} & — & \overset{\mathrm{I}_{m_1}^{*s}}{4} & — & \cdots & — & \overset{\mathrm{I}_{m_{2k}}^{ns}}{1} & — & \overset{\mathrm{I}_{m_{2k+1}}^{*s}}{2}
\end{array}
\tag{3.46}
$$

where the dots denote an alternating chain of $\overset{\mathrm{I}_{m_i}^{*s}}{4}$ and $\overset{\mathrm{I}_{m_{i+1}}^{ns}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$.

- Eq. (2.43) can be constructed via

$$
\begin{array}{ccccccccccccccc}
 & & & & \hat{\mathrm{I}}_4^* & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
 & & & & \overset{\mathrm{I}_n^{ns}}{1} & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
\hat{\mathrm{I}}_4^* & — & \overset{\mathrm{I}_n^{ns}}{1} & — & \overset{\hat{\mathrm{I}}_{n_0+4}^*}{4} & — & \overset{\mathrm{I}_{2n_0}^{ns}}{1} & — & \overset{\mathrm{I}_{m_1}^{*}}{4} & — & \cdots & — & \overset{\mathrm{I}_{m_{2k}}^{ns}}{1} & — & \overset{\mathrm{I}_{m_{2k+1}}^{*}}{2}
\end{array}
\tag{3.47}
$$

where the dots denote an alternating chain of $\overset{\mathrm{I}_{m_i}^{*}}{4}$ and $\overset{\mathrm{I}_{m_{i+1}}^{ns}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$ with $\mathrm{I}_{m_{2i-1}}^*$ singularity being split for $1 \le i \le j$, and $m_{2i} = 2n_{2i} + 1$,

$m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with $\mathrm{I}^*_{m_{2i-1}}$ singularity being non-split for $i \geq j+1$. The $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(n_{2j})$ is localized at the intersection of $\mathrm{I}^{ns}_{m_{2j}}$ and $\mathrm{I}^{*ns}_{m_{2j+1}}$.

- Eq. (2.44) can be constructed via

$$
\begin{array}{ccccccccccccccc}
 & & & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{n+1}}{1} & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
\hat{\mathrm{I}}^*_4 & — & \overset{\mathrm{I}^{ns}_{n+2}}{1} & — & \overset{\hat{\mathrm{I}}^{*ns}_{n_0+5}}{4} & — & \overset{\mathrm{I}^{ns}_{2n_0+1}}{1} & — & \overset{\mathrm{I}^{*ns}_{m_1}}{4} & — & \cdots & — & \overset{\mathrm{I}^{ns}_{m_{2k}}}{1} & — & \overset{\mathrm{I}^{*ns}_{m_{2k+1}}}{2}
\end{array}
\tag{3.48}
$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*ns}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}+1$ and $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$. The $\mathfrak{so}(n+1)$ is realized by I^{ns}_{n+2} and the $\mathfrak{so}(n)$ is realized by I^{ns}_{n+1}. The curves encircled by the dashed ellipse give rise to $\mathfrak{sp}(n_0)$.

- Eq. (2.45) can be constructed via

$$
\begin{array}{ccccccccccccccc}
 & & & & \hat{\mathrm{I}}^*_4 & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{n}}{1} & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
\hat{\mathrm{I}}^*_4 & — & \overset{\mathrm{I}^{ns}_{n}}{1} & — & \overset{\hat{\mathrm{I}}^{*}_{n_0+4}}{4} & — & \overset{\mathrm{I}^{ns}_{2n_0}}{1} & — & \overset{\mathrm{I}^{*s}_{m_1}}{4} & — & \cdots & — & \overset{\mathrm{I}^{ns}_{m_{2k}}}{1} & — & \overset{\mathrm{I}^{*ss}_{0}}{2}
\end{array}
\tag{3.49}
$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$.

Eq. (2.46) can be constructed via

$$
\begin{array}{ccccccccc}
 & & & & \hat{\mathrm{I}}^*_4 & & & & \\
 & & & & | & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{12}}{1} & & & & \\
 & & & & | & & & & \\
\hat{\mathrm{I}}^*_4 & — & \overset{\mathrm{I}^{ns}_{12}}{1} & — & \overset{\hat{\mathrm{I}}^{*s}_{8}}{4} & — & \overset{\mathrm{I}^{ns}_{8}}{1} & — & \overset{\mathrm{I}^{*ss}_{0}}{2}
\end{array}
\tag{3.50}
$$

- Eq. (2.47) can be constructed via

$$\begin{tikzcd}[arrows=dash]
 & & \hat{\mathrm{I}}_4^* \arrow[d] & & & & & \\
 & & \overset{\mathrm{I}_n^{ns}}{1} \arrow[d] & & & & & \\
\hat{\mathrm{I}}_4^* \arrow[r] & \overset{\mathrm{I}_n^{ns}}{1} \arrow[r] & \overset{\hat{\mathrm{I}}_{n_0+4}^*}{4} \arrow[r] & \overset{\mathrm{I}_{2n_0}^{ns}}{1} \arrow[r] & \overset{\mathrm{I}_{m_1}^*}{4} \arrow[r] & \cdots \arrow[r] & \overset{\mathrm{I}_{m_{2k}}^{ns}}{1} \arrow[r] & \overset{\mathrm{I}_0^{*ns}}{2}
\end{tikzcd} \tag{3.51}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}_{m_i}^*}{4}$ and $\overset{\mathrm{I}_{m_{i+1}}^{ns}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$ with $\mathrm{I}_{m_{2i-1}}^*$ singularity being split for $1 \leq i \leq j$, and $m_{2i} = 2n_{2i}+1$, $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$ with $\mathrm{I}_{m_{2i-1}}^*$ singularity being non-split for $j+1 \leq i \leq k$. The $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(n_{2j})$ is localized at the intersection of $\mathrm{I}_{m_{2j}}^{ns}$ and $\mathrm{I}_{m_{2j+1}}^{*ns}$ where $\mathrm{I}_{m_{2k+1}}^{*ns} := \mathrm{I}_0^{*ns}$.

Eq. (2.48) can be constructed via

$$\begin{tikzcd}[arrows=dash]
 & & \hat{\mathrm{I}}_4^* \arrow[d] & & \\
 & & \overset{\mathrm{I}_{12}^{ns}}{1} \arrow[d] & & \\
\hat{\mathrm{I}}_4^* \arrow[r] & \overset{\mathrm{I}_{12}^{ns}}{1} \arrow[r] & \overset{\hat{\mathrm{I}}_8^{*s}}{4} \arrow[r] & \overset{\mathrm{I}_8^{ns}}{1} \arrow[r] & \overset{\mathrm{I}_0^{*ns}}{2}
\end{tikzcd} \tag{3.52}$$

with the $\frac{1}{2}\mathsf{F}$ of $\mathfrak{sp}(4)$ being localized at the intersection of I_8^{ns} and I_0^{*ns}.

- Eq. (2.49) can be constructed via

$$\begin{tikzcd}[arrows=dash]
 & & \hat{\mathrm{I}}_4^* \arrow[d] & & & & & \\
 & & \overset{\mathrm{I}_{n+1}^{ns}}{1} \arrow[d] & & & & & \\
\hat{\mathrm{I}}_4^* \arrow[r] & \overset{\mathrm{I}_{n+2}^{ns}}{1} \arrow[r] & \overset{\hat{\mathrm{I}}_{n_0+5}^{*ns}}{4} \arrow[r] & \overset{\mathrm{I}_{2n_0+1}^{ns}}{1} \arrow[r] & \overset{\mathrm{I}_{m_1}^{*ns}}{4} \arrow[r] & \cdots \arrow[r] & \overset{\mathrm{I}_{m_{2k}}^{ns}}{1} \arrow[r] & \overset{\mathrm{I}_0^{*ns}}{2}
\end{tikzcd} \tag{3.53}$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*ns}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}+1$ and $m_{2i-1} = \frac{n_{2i-1}+1}{2} - 4$. The $\mathfrak{so}(n+1)$ is realized by I^{ns}_{n+2} and the $\mathfrak{so}(n)$ is realized by I^{ns}_{n+1}. The curves encircled by the dashed ellipse give rise to $\mathfrak{sp}(n_0)$.

Eq. (2.50) can be constructed via

$$
\begin{array}{ccccccccc}
 & & & & \hat{\mathrm{I}}^{*}_{4} & & & & \\
 & & & & | & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{13}}{1} & & & & \\
 & & & & | & & & & \\
\hat{\mathrm{I}}^{*}_{4} & \text{——} & \overset{\mathrm{I}^{ns}_{14}}{1} & \text{——} & \overset{\hat{\mathrm{I}}^{*ns}_{9}}{4} & \text{——} & \overset{\mathrm{I}^{ns}_{9}}{1} & \text{——} & \overset{\mathrm{I}^{*ns}_{0}}{2}
\end{array}
\tag{3.54}
$$

The $\mathfrak{so}(13)$ is realized by I^{ns}_{14} and the $\mathfrak{so}(12)$ is realized by I^{ns}_{13}.

- Eq. (2.51) can be constructed via

$$
\begin{array}{ccccccccccccccc}
 & & & & \hat{\mathrm{I}}^{*}_{4} & & & & & & & & & & \\
 & & & & | & & & & & & & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{n}}{1} & & & & & & & & \overset{\mathrm{I}^{ns}_{2m}}{1} & & \\
 & & & & | & & & & & & & & | & & \\
\hat{\mathrm{I}}^{*}_{4} & \text{——} & \overset{\mathrm{I}^{ns}_{n}}{1} & \text{——} & \overset{\hat{\mathrm{I}}^{*}_{n_0+4}}{4} & \text{——} & \overset{\mathrm{I}^{ns}_{2n_0}}{1} & \text{——} & \overset{\mathrm{I}^{*s}_{m_1}}{4} & \text{——} \cdots \text{——} & \overset{\mathrm{I}^{ns}_{m_{2k}}}{1} & \text{——} & \overset{\mathrm{I}^{*s}_{m_{2k+1}}}{4} & \text{——} & \overset{\mathrm{I}^{ns}_{2m}}{1}
\end{array}
\tag{3.55}
$$

where the dots denote an alternating chain of $\overset{\mathrm{I}^{*s}_{m_i}}{4}$ and $\overset{\mathrm{I}^{ns}_{m_{i+1}}}{1}$. Here $m_{2i} = 2n_{2i}$ and $m_{2i-1} = \frac{n_{2i-1}}{2} - 4$.

- Eq. (2.52) can be constructed via

$$
\begin{array}{ccccccccc}
 & & & & \overset{\hat{\mathrm{I}}^{*}_{m+4}}{4} & & & & \\
 & & & & | & & & & \\
 & & & & \overset{\mathrm{I}^{ns}_{4m+16}}{1} & & & & \\
 & & & & | & & & & \\
\overset{\hat{\mathrm{I}}^{*}_{m+4}}{4} & \text{——} & \overset{\mathrm{I}^{ns}_{4m+16}}{1} & \text{——} & \overset{\hat{\mathrm{I}}^{*}_{3m+12}}{4} & \text{——} & \overset{\mathrm{I}^{ns}_{4m+16}}{1} & \text{——} & \overset{\hat{\mathrm{I}}^{*}_{m+4}}{4}
\end{array}
\tag{3.56}
$$

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- Eq. (2.53) can be constructed via

$$\begin{array}{ccccccccc} & & & & \overset{\mathrm{I}_2^{*ns}}{3} & & & & \\ & & & & | & & & & \\ & & & & \overset{\mathrm{I}_8^{ns}}{1} & & & & \\ & & & & | & & & & \\ \overset{\mathrm{I}_2^{*ns}}{3} & \text{---} & \overset{\mathrm{I}_8^{ns}}{1} & \text{---} & \overset{\hat{\mathrm{I}}_6^{*s}}{4} & \text{---} & \overset{\mathrm{I}_8^{ns}}{1} & \text{---} & \overset{\mathrm{I}_2^{*ns}}{3} \end{array} \tag{3.57}$$

  where the curves encircled by each dashed ellipse give rise to an $\mathfrak{so}(7)$ with $2\mathsf{S}$ as we suggested in (3.4).

- Eq. (2.54) can be constructed via

$$\begin{array}{ccccccccc} & & & & \overset{\mathrm{I}_2^{*s}}{3} & & & & \\ & & & & | & & & & \\ & & & & \overset{\mathrm{I}_{11}^{ns}}{1} & & & & \\ & & & & | & & & & \\ \overset{\mathrm{I}_2^{*s}}{3} & \text{---} & \overset{\mathrm{I}_{11}^{ns}}{1} & \text{---} & \overset{\hat{\mathrm{I}}_9^{*ns}}{4} & \text{---} & \overset{\mathrm{I}_{11}^{ns}}{1} & \text{---} & \overset{\mathrm{I}_2^{*s}}{3} \end{array} \tag{3.58}$$

  where the four curves encircled by the dashed circle give rise to an $\mathfrak{sp}(5)$ with $18\mathsf{F}$ as we suggested in (3.5).

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References

[1] J.J. Heckman, D.R. Morrison, T. Rudelius and C. Vafa, *Atomic Classification of* 6*D SCFTs*, *Fortsch. Phys.* **63** (2015) 468 [arXiv:1502.05405] [INSPIRE].

[2] J.J. Heckman, D.R. Morrison and C. Vafa, *On the Classification of 6D SCFTs and Generalized ADE Orbifolds*, *JHEP* **05** (2014) 028 [*Erratum ibid.* **1506** (2015) 017] [arXiv:1312.5746] [INSPIRE].

[3] L. Bhardwaj, M. Del Zotto, J.J. Heckman, D.R. Morrison, T. Rudelius and C. Vafa, *F-theory and the Classification of Little Strings*, *Phys. Rev.* **D 93** (2016) 086002 [*Erratum ibid.* **D 100** (2019) 029901] [arXiv:1511.05565] [INSPIRE].

[4] L. Bhardwaj, *Classification of* 6d $\mathcal{N} = (1, 0)$ *gauge theories*, *JHEP* **11** (2015) 002 [arXiv:1502.06594] [INSPIRE].

[5] L. Bhardwaj, D.R. Morrison, Y. Tachikawa and A. Tomasiello, *The frozen phase of F-theory*, *JHEP* **08** (2018) 138 [arXiv:1805.09070] [INSPIRE].

[6] I. Brunner and A. Karch, *Branes at orbifolds versus Hanany Witten in six-dimensions*, *JHEP* **03** (1998) 003 [hep-th/9712143] [INSPIRE].

[7] A. Hanany and A. Zaffaroni, *Branes and six-dimensional supersymmetric theories*, *Nucl. Phys.* **B 529** (1998) 180 [hep-th/9712145] [INSPIRE].

[8] A. Hanany and A. Zaffaroni, *Issues on orientifolds: On the brane construction of gauge theories with* SO(2N) *global symmetry*, *JHEP* **07** (1999) 009 [hep-th/9903242] [INSPIRE].

[9] A. Sagnotti, *A Note on the Green-Schwarz mechanism in open string theories*, *Phys. Lett.* **B 294** (1992) 196 [hep-th/9210127] [INSPIRE].

[10] I. Brunner and A. Karch, *Branes and six-dimensional fixed points*, *Phys. Lett.* **B 409** (1997) 109 [hep-th/9705022] [INSPIRE].

[11] K. Ohmori, H. Shimizu, Y. Tachikawa and K. Yonekura, 6*d* $\mathcal{N} = (1,\ 0)$ *theories on* $S^1\ /T^2$ *and class S theories: part II*, *JHEP* **12** (2015) 131 [arXiv:1508.00915] [INSPIRE].

[12] P.R. Merkx, *Classifying Global Symmetries of* 6*D SCFTs*, *JHEP* **03** (2018) 163 [arXiv:1711.05155] [INSPIRE].

[13] M. Bertolini, P.R. Merkx and D.R. Morrison, *On the global symmetries of* 6*D superconformal field theories*, *JHEP* **07** (2016) 005 [arXiv:1510.08056] [INSPIRE].

[14] D.R. Morrison and T. Rudelius, *F-theory and Unpaired Tensors in* 6*D SCFTs and LSTs*, *Fortsch. Phys.* **64** (2016) 645 [arXiv:1605.08045] [INSPIRE].

[15] L. Bhardwaj and P. Jefferson, *Classifying* 5d *SCFTs via* 6d *SCFTs: Arbitrary rank*, *JHEP* **10** (2019) 282 [arXiv:1811.10616] [INSPIRE].

[16] L. Bhardwaj and P. Jefferson, *Classifying* 5d *SCFTs via* 6d *SCFTs: Rank one*, *JHEP* **07** (2019) 178 [arXiv:1809.01650] [INSPIRE].

[17] D. Gaiotto and A. Tomasiello, *Holography for* (1, 0) *theories in six dimensions*, *JHEP* **12** (2014) 003 [arXiv:1404.0711] [INSPIRE].

[18] K. Ohmori, H. Shimizu and Y. Tachikawa, *Anomaly polynomial of E-string theories*, *JHEP* **08** (2014) 002 [arXiv:1404.3887] [INSPIRE].

[19] M. Del Zotto, J.J. Heckman, A. Tomasiello and C. Vafa, 6d *Conformal Matter*, *JHEP* **02** (2015) 054 [arXiv:1407.6359] [INSPIRE].

JHEP03(2020)171

[20] J.J. Heckman, *More on the Matter of 6D SCFTs*, *Phys. Lett.* **B 747** (2015) 73 [arXiv:1408.0006] [INSPIRE].

[21] K. Sakai, *A reduced BPS index of E-strings*, *JHEP* **12** (2014) 047 [arXiv:1408.3619] [INSPIRE].

[22] K. Ohmori, H. Shimizu, Y. Tachikawa and K. Yonekura, *Anomaly polynomial of general 6d SCFTs*, *PTEP* **2014** (2014) 103B07 [arXiv:1408.5572] [INSPIRE].

[23] K. Intriligator, *6d, $\mathcal{N} = (1, 0)$ Coulomb branch anomaly matching*, *JHEP* **10** (2014) 162 [arXiv:1408.6745] [INSPIRE].

[24] P. Karndumri, *Noncompact gauging of $N = 2$ 7D supergravity and AdS/CFT holography*, *JHEP* **02** (2015) 034 [arXiv:1411.4542] [INSPIRE].

[25] B. Haghighat, A. Klemm, G. Lockhart and C. Vafa, *Strings of Minimal 6d SCFTs*, *Fortsch. Phys.* **63** (2015) 294 [arXiv:1412.3152] [INSPIRE].

[26] M. Del Zotto, J.J. Heckman, D.R. Morrison and D.S. Park, *6D SCFTs and Gravity*, *JHEP* **06** (2015) 158 [arXiv:1412.6526] [INSPIRE].

[27] F. Apruzzi, M. Fazzi, A. Passias and A. Tomasiello, *Supersymmetric AdS_5 solutions of massive IIA supergravity*, *JHEP* **06** (2015) 195 [arXiv:1502.06620] [INSPIRE].

[28] M. Del Zotto, J.J. Heckman, D.S. Park and T. Rudelius, *On the Defect Group of a 6D SCFT*, *Lett. Math. Phys.* **106** (2016) 765 [arXiv:1503.04806] [INSPIRE].

[29] P. Karndumri, *RG flows from $(1,0)$ 6D SCFTs to $N = 1$ SCFTs in four and three dimensions*, *JHEP* **06** (2015) 027 [arXiv:1503.04997] [INSPIRE].

[30] D. Gaiotto and S.S. Razamat, *$\mathcal{N} = 1$ theories of class $\mathcal{S}_k$*, *JHEP* **07** (2015) 073 [arXiv:1503.05159] [INSPIRE].

[31] K. Ohmori, H. Shimizu, Y. Tachikawa and K. Yonekura, *6d $\mathcal{N} = (1,0)$ theories on T^2 and class S theories: Part I*, *JHEP* **07** (2015) 014 [arXiv:1503.06217] [INSPIRE].

[32] S. Hohenegger, A. Iqbal and S.-J. Rey, *M-strings, monopole strings and modular forms*, *Phys. Rev.* **D 92** (2015) 066005 [arXiv:1503.06983] [INSPIRE].

[33] S.-S. Kim, M. Taki and F. Yagi, *Tao Probing the End of the World*, *PTEP* **2015** (2015) 083B02 [arXiv:1504.03672] [INSPIRE].

[34] A. Gadde, B. Haghighat, J. Kim, S. Kim, G. Lockhart and C. Vafa, *6d String Chains*, *JHEP* **02** (2018) 143 [arXiv:1504.04614] [INSPIRE].

[35] M. Del Zotto, C. Vafa and D. Xie, *Geometric engineering, mirror symmetry and $6\mathrm{d}_{(1,0)} \to 4\mathrm{d}_{(\mathcal{N}=2)}$*, *JHEP* **11** (2015) 123 [arXiv:1504.08348] [INSPIRE].

[36] J.J. Heckman, D.R. Morrison, T. Rudelius and C. Vafa, *Geometry of 6D RG Flows*, *JHEP* **09** (2015) 052 [arXiv:1505.00009] [INSPIRE].

[37] H. Hayashi, S.-S. Kim, K. Lee, M. Taki and F. Yagi, *A new 5d description of 6d D-type minimal conformal matter*, *JHEP* **08** (2015) 097 [arXiv:1505.04439] [INSPIRE].

[38] K. Yonekura, *Instanton operators and symmetry enhancement in 5d supersymmetric quiver gauge theories*, *JHEP* **07** (2015) 167 [arXiv:1505.04743] [INSPIRE].

[39] C. Cordova, T.T. Dumitrescu and K. Intriligator, *Anomalies, renormalization group flows and the a-theorem in six-dimensional $(1,0)$ theories*, *JHEP* **10** (2016) 080 [arXiv:1506.03807] [INSPIRE].

[40] A. Passias, A. Rota and A. Tomasiello, *Universal consistent truncation for* 6d/7d *gauge/gravity duals*, *JHEP* **10** (2015) 187 [arXiv:1506.05462] [INSPIRE].

[41] J.J. Heckman and T. Rudelius, *Evidence for C-theorems in* 6*D SCFTs*, *JHEP* **09** (2015) 218 [arXiv:1506.06753] [INSPIRE].

[42] N. Bobev, M. Bullimore and H.-C. Kim, *Supersymmetric Casimir Energy and the Anomaly Polynomial*, *JHEP* **09** (2015) 142 [arXiv:1507.08553] [INSPIRE].

[43] G. Zafrir, *Brane webs,* 5*d gauge theories and* 6d $\mathcal{N} = (1, 0)$ *SCFT's*, *JHEP* **12** (2015) 157 [arXiv:1509.02016] [INSPIRE].

[44] K. Ohmori and H. Shimizu, S^1/T^2 *compactifications of* 6d $\mathcal{N} = (1,\ 0)$ *theories and brane webs*, *JHEP* **03** (2016) 024 [arXiv:1509.03195] [INSPIRE].

[45] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, 6*d SCFTs,* 5*d Dualities and Tao Web Diagrams*, *JHEP* **05** (2019) 203 [arXiv:1509.03300] [INSPIRE].

[46] J. Kim, S. Kim and K. Lee, *Higgsing towards E-strings*, arXiv:1510.03128 [INSPIRE].

[47] S. Hohenegger, A. Iqbal and S.-J. Rey, *Instanton-monopole correspondence from M-branes on* $\mathbb{S}^1$ *and little string theory*, *Phys. Rev.* **D 93** (2016) 066016 [arXiv:1511.02787] [INSPIRE].

[48] L.B. Anderson, J. Gray, N. Raghuram and W. Taylor, *Matter in transition*, *JHEP* **04** (2016) 080 [arXiv:1512.05791] [INSPIRE].

[49] J.J. Heckman, T. Rudelius and A. Tomasiello, 6*D RG Flows and Nilpotent Hierarchies*, *JHEP* **07** (2016) 082 [arXiv:1601.04078] [INSPIRE].

[50] F. Apruzzi, G. Dibitetto and L. Tizzano, *A new* 6d *fixed point from holography*, *JHEP* **11** (2016) 126 [arXiv:1603.06576] [INSPIRE].

[51] A. Font, I. García-Etxebarria, D. Lüst, S. Massai and C. Mayrhofer, *Heterotic T-fects,* 6*D SCFTs and F-theory*, *JHEP* **08** (2016) 175 [arXiv:1603.09361] [INSPIRE].

[52] D.R. Morrison and C. Vafa, *F-theory and* $\mathcal{N} = 1$ *SCFTs in four dimensions*, *JHEP* **08** (2016) 070 [arXiv:1604.03560] [INSPIRE].

[53] S. Hohenegger, A. Iqbal and S.-J. Rey, *Self-Duality and Self-Similarity of Little String Orbifolds*, *Phys. Rev.* **D 94** (2016) 046006 [arXiv:1605.02591] [INSPIRE].

[54] S.B. Johnson and W. Taylor, *Enhanced gauge symmetry in* 6*D F-theory models and tuned elliptic Calabi-Yau threefolds*, *Fortsch. Phys.* **64** (2016) 581 [arXiv:1605.08052] [INSPIRE].

[55] M. Buican, J. Hayling and C. Papageorgakis, *Aspects of Superconformal Multiplets in* $D > 4$, *JHEP* **11** (2016) 091 [arXiv:1606.00810] [INSPIRE].

[56] Y. Yun, *Testing* 5d*-*6d *dualities with fractional D-branes*, *JHEP* **12** (2016) 016 [arXiv:1607.07615] [INSPIRE].

[57] B. Haghighat and W. Yan, *M-strings in thermodynamic limit: Seiberg-Witten geometry*, arXiv:1607.07873 [INSPIRE].

[58] H.-C. Kim, S. Kim and J. Park, 6d *strings from new chiral gauge theories*, arXiv:1608.03919 [INSPIRE].

[59] H. Shimizu and Y. Tachikawa, *Anomaly of strings of* 6d $\mathcal{N} = (1, 0)$ *theories*, *JHEP* **11** (2016) 165 [arXiv:1608.05894] [INSPIRE].

JHEP03(2020)171

[60] M. Del Zotto and G. Lockhart, *On Exceptional Instanton Strings*, *JHEP* **09** (2017) 081 [arXiv:1609.00310] [INSPIRE].

[61] J.J. Heckman, P. Jefferson, T. Rudelius and C. Vafa, *Punctures for theories of class $\mathcal{S}_\Gamma$*, *JHEP* **03** (2017) 171 [arXiv:1609.01281] [INSPIRE].

[62] F. Apruzzi, F. Hassler, J.J. Heckman and I.V. Melnikov, *From 6D SCFTs to Dynamic GLSMs*, *Phys. Rev.* **D 96** (2017) 066015 [arXiv:1610.00718] [INSPIRE].

[63] S. Hohenegger, A. Iqbal and S.-J. Rey, *Dual Little Strings from F-theory and Flop Transitions*, *JHEP* **07** (2017) 112 [arXiv:1610.07916] [INSPIRE].

[64] S.S. Razamat, C. Vafa and G. Zafrir, *4d $\mathcal{N}=1$ from 6d $(1,0)$*, *JHEP* **04** (2017) 064 [arXiv:1610.09178] [INSPIRE].

[65] N. Haouzi and C. Schmid, *Little String Defects and Bala-Carter Theory*, arXiv:1612.02008 [INSPIRE].

[66] C. Lawrie, S. Schäfer-Nameki and T. Weigand, *Chiral 2d theories from $N=4$ SYM with varying coupling*, *JHEP* **04** (2017) 111 [arXiv:1612.05640] [INSPIRE].

[67] N. Bobev, G. Dibitetto, F.F. Gautason and B. Truijen, *Holography, Brane Intersections and Six-dimensional SCFTs*, *JHEP* **02** (2017) 116 [arXiv:1612.06324] [INSPIRE].

[68] N. Mekareeya, T. Rudelius and A. Tomasiello, *T-branes, Anomalies and Moduli Spaces in 6D SCFTs*, *JHEP* **10** (2017) 158 [arXiv:1612.06399] [INSPIRE].

[69] J. Gu, M.-x. Huang, A.-K. Kashani-Poor and A. Klemm, *Refined BPS invariants of 6d SCFTs from anomalies and modularity*, *JHEP* **05** (2017) 130 [arXiv:1701.00764] [INSPIRE].

[70] J. Kim and K. Lee, *Little strings on D_n orbifolds*, *JHEP* **10** (2017) 045 [arXiv:1702.03116] [INSPIRE].

[71] S. Yankielowicz and Y. Zhou, *Supersymmetric Rényi entropy and Anomalies in 6d $(1,0)$ SCFTs*, *JHEP* **04** (2017) 128 [arXiv:1702.03518] [INSPIRE].

[72] H. Hayashi and K. Ohmori, *5d/6d DE instantons from trivalent gluing of web diagrams*, *JHEP* **06** (2017) 078 [arXiv:1702.07263] [INSPIRE].

[73] M. Del Zotto, J.J. Heckman and D.R. Morrison, *6D SCFTs and Phases of 5D Theories*, *JHEP* **09** (2017) 147 [arXiv:1703.02981] [INSPIRE].

[74] I. Bah, A. Passias and A. Tomasiello, *AdS_5 compactifications with punctures in massive IIA supergravity*, *JHEP* **11** (2017) 050 [arXiv:1704.07389] [INSPIRE].

[75] O. Chacaltana, J. Distler, A. Trimm and Y. Zhu, *Tinkertoys for the E_7 theory*, *JHEP* **05** (2018) 031 [arXiv:1704.07890] [INSPIRE].

[76] C. Couzens, C. Lawrie, D. Martelli, S. Schäfer-Nameki and J.-M. Wong, *F-theory and AdS_3/CFT_2*, *JHEP* **08** (2017) 043 [arXiv:1705.04679] [INSPIRE].

[77] B. Haghighat, W. Yan and S.-T. Yau, *ADE String Chains and Mirror Symmetry*, *JHEP* **01** (2018) 043 [arXiv:1705.05199] [INSPIRE].

[78] C.-M. Chang and Y.-H. Lin, *Carving Out the End of the World or (Superconformal Bootstrap in Six Dimensions)*, *JHEP* **08** (2017) 128 [arXiv:1705.05392] [INSPIRE].

[79] P. Jefferson, H.-C. Kim, C. Vafa and G. Zafrir, *Towards Classification of 5d SCFTs: Single Gauge Node*, arXiv:1705.05836 [INSPIRE].

[80] K.-S. Choi and S.-J. Rey, *E(lementary) Strings in Six-Dimensional Heterotic F-theory*, JHEP **09** (2017) 092 [arXiv:1706.05353] [INSPIRE].

[81] B. Bastian and S. Hohenegger, *Five-Brane Webs and Highest Weight Representations*, JHEP **12** (2017) 020 [arXiv:1706.08750] [INSPIRE].

[82] N. Mekareeya, K. Ohmori, Y. Tachikawa and G. Zafrir, *E_8 instantons on type-A ALE spaces and supersymmetric field theories*, JHEP **09** (2017) 144 [arXiv:1707.04370] [INSPIRE].

[83] N. Mekareeya, K. Ohmori, H. Shimizu and A. Tomasiello, *Small instanton transitions for M5 fractions*, JHEP **10** (2017) 055 [arXiv:1707.05785] [INSPIRE].

[84] G. Dibitetto and N. Petri, *6d surface defects from massive type IIA*, JHEP **01** (2018) 039 [arXiv:1707.06154] [INSPIRE].

[85] F. Apruzzi, J.J. Heckman and T. Rudelius, *Green-Schwarz Automorphisms and 6D SCFTs*, JHEP **02** (2018) 157 [arXiv:1707.06242] [INSPIRE].

[86] J.J. Heckman and L. Tizzano, *6D Fractional Quantum Hall Effect*, JHEP **05** (2018) 120 [arXiv:1708.02250] [INSPIRE].

[87] A. Font and C. Mayrhofer, *Non-geometric vacua of the $Spin(\mathbf{32})/\mathbb{Z}_2$ heterotic string and little string theories*, JHEP **11** (2017) 064 [arXiv:1708.05428] [INSPIRE].

[88] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, *E-String Theory on Riemann Surfaces*, Fortsch. Phys. **66** (2018) 1700074 [arXiv:1709.02496] [INSPIRE].

[89] B. Bastian, S. Hohenegger, A. Iqbal and S.-J. Rey, *Dual little strings and their partition functions*, Phys. Rev. **D 97** (2018) 106004 [arXiv:1710.02455] [INSPIRE].

[90] F. Hassler and J.J. Heckman, *Punctures and Dynamical Systems*, Lett. Math. Phys. **109** (2019) 449 [arXiv:1711.03973] [INSPIRE].

[91] B. Bastian, S. Hohenegger, A. Iqbal and S.-J. Rey, *Triality in Little String Theories*, Phys. Rev. **D 97** (2018) 046004 [arXiv:1711.07921] [INSPIRE].

[92] N. Haouzi and C. Kozçaz, *The ABCDEFG of Little Strings*, arXiv:1711.11065 [INSPIRE].

[93] T. Bourton and E. Pomoni, *Instanton counting in Class $\mathcal{S}_k$*, arXiv:1712.01288 [INSPIRE].

[94] F. Apruzzi and M. Fazzi, *AdS_7/CFT_6 with orientifolds*, JHEP **01** (2018) 124 [arXiv:1712.03235] [INSPIRE].

[95] M. Del Zotto, J. Gu, M.-X. Huang, A.-K. Kashani-Poor, A. Klemm and G. Lockhart, *Topological Strings on Singular Elliptic Calabi-Yau 3-folds and Minimal 6d SCFTs*, JHEP **03** (2018) 156 [arXiv:1712.07017] [INSPIRE].

[96] A. Font, I. Garcia-Etxebarria, D. Lüst, S. Massai and C. Mayrhofer, *Non-geometric heterotic backgrounds and 6D SCFTs/LSTs*, PoS(CORFU2016)123 (2017) [arXiv:1712.07083] [INSPIRE].

[97] R.-D. Zhu, *An Elliptic Vertex of Awata-Feigin-Shiraishi type for M-strings*, JHEP **08** (2018) 050 [arXiv:1712.10255] [INSPIRE].

[98] B. Nazzal and S.S. Razamat, *Surface Defects in E-String Compactifications and the van Diejen Model*, SIGMA **14** (2018) 036 [arXiv:1801.00960] [INSPIRE].

[99] H.-C. Kim, J. Kim, S. Kim, K.-H. Lee and J. Park, *6d strings and exceptional instantons*, arXiv:1801.03579 [INSPIRE].

[100] P. Jefferson, S. Katz, H.-C. Kim and C. Vafa, *On Geometric Classification of* 5d *SCFTs*, *JHEP* **04** (2018) 103 [arXiv:1801.04036] [INSPIRE].

[101] L.B. Anderson, A. Grassi, J. Gray and P.-K. Oehlmann, *F-theory on Quotient Threefolds with (2,0) Discrete Superconformal Matter*, *JHEP* **06** (2018) 098 [arXiv:1801.08658] [INSPIRE].

[102] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, *D-type Conformal Matter and* SU/USp *Quivers*, *JHEP* **06** (2018) 058 [arXiv:1802.00620] [INSPIRE].

[103] I. Bah and E. Nardoni, *Structure of Anomalies of* 4d *SCFTs from M*5*-branes and Anomaly Inflow*, *JHEP* **03** (2019) 024 [arXiv:1803.00136] [INSPIRE].

[104] F. Apruzzi, J.J. Heckman, D.R. Morrison and L. Tizzano, 4*D Gauge Theories with Conformal Matter*, *JHEP* **09** (2018) 088 [arXiv:1803.00582] [INSPIRE].

[105] S.-J. Lee, D. Regalado and T. Weigand, 6*d SCFTs and* U(1) *Flavour Symmetries*, *JHEP* **11** (2018) 147 [arXiv:1803.07998] [INSPIRE].

[106] M. Dierigl, P.-K. Oehlmann and F. Ruehle, *Global Tensor-Matter Transitions in F-Theory*, *Fortsch. Phys.* **66** (2018) 1800037 [arXiv:1804.07386] [INSPIRE].

[107] A. Hanany and G. Zafrir, *Discrete Gauging in Six Dimensions*, *JHEP* **07** (2018) 168 [arXiv:1804.08857] [INSPIRE].

[108] M. Del Zotto and G. Lockhart, *Universal Features of BPS Strings in Six-dimensional SCFTs*, *JHEP* **08** (2018) 173 [arXiv:1804.09694] [INSPIRE].

[109] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, *Compactifications of ADE conformal matter on a torus*, *JHEP* **09** (2018) 110 [arXiv:1806.07620] [INSPIRE].

[110] B. Haghighat, J. Kim, W. Yan and S.-T. Yau, *D-type fiber-base duality*, *JHEP* **09** (2018) 060 [arXiv:1806.10335] [INSPIRE].

[111] M. Cvetič, J.J. Heckman and L. Lin, *Towards Exotic Matter and Discrete Non-Abelian Symmetries in F-theory*, *JHEP* **11** (2018) 001 [arXiv:1806.10594] [INSPIRE].

[112] J.J. Heckman, T. Rudelius and A. Tomasiello, *Fission, Fusion and* 6*D RG Flows*, *JHEP* **02** (2019) 167 [arXiv:1807.10274] [INSPIRE].

[113] S.S. Razamat, O. Sela and G. Zafrir, *Curious patterns of IR symmetry enhancement*, *JHEP* **10** (2018) 163 [arXiv:1809.00541] [INSPIRE].

[114] G. Zafrir, *On the torus compactifications of* Z_2 *orbifolds of E-string theories*, *JHEP* **10** (2019) 040 [arXiv:1809.04260] [INSPIRE].

[115] U. Naseer, $(1,0)$ *gauge theories on the six-sphere*, *SciPost Phys.* **6** (2019) 002 [arXiv:1809.06272] [INSPIRE].

[116] Z. Duan, J. Gu and A.-K. Kashani-Poor, *Computing the elliptic genus of higher rank E-strings from genus 0 GW invariants*, *JHEP* **03** (2019) 078 [arXiv:1810.01280] [INSPIRE].

[117] B. Bastian, S. Hohenegger, A. Iqbal and S.-J. Rey, *Five-Dimensional Gauge Theories from Shifted Web Diagrams*, *Phys. Rev.* **D 99** (2019) 046012 [arXiv:1810.05109] [INSPIRE].

[118] S. Kachru, A. Tripathy and M. Zimet, *K*3 *metrics from little string theory*, arXiv:1810.10540 [INSPIRE].

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[119] J. Gu, B. Haghighat, K. Sun and X. Wang, *Blowup Equations for* 6d *SCFTs. I*, *JHEP* **03** (2019) 002 [arXiv:1811.02577] [INSPIRE].

[120] J. Tian and Y.-N. Wang, *E-string spectrum and typical F-theory geometry*, arXiv:1811.02837 [INSPIRE].

[121] B. Bastian and S. Hohenegger, *Dihedral Symmetries of Gauge Theories from Dual Calabi-Yau Threefolds*, *Phys. Rev.* **D 99** (2019) 066013 [arXiv:1811.03387] [INSPIRE].

[122] D.D. Frey and T. Rudelius, 6*D SCFTs and the Classification of Homomorphisms* $\Gamma_{ADE} \to E_8$, arXiv:1811.04921 [INSPIRE].

[123] B. Haghighat and R. Sun, *M5 branes and Theta Functions*, *JHEP* **10** (2019) 192 [arXiv:1811.04938] [INSPIRE].

[124] C. Córdova, G.B. De Luca and A. Tomasiello, AdS_8 *Solutions in Type II Supergravity*, *JHEP* **07** (2019) 127 [arXiv:1811.06987] [INSPIRE].

[125] S. Gukov, D. Pei, P. Putrov and C. Vafa, 4*-manifolds and topological modular forms*, arXiv:1811.07884 [INSPIRE].

[126] F. Apruzzi, L. Lin and C. Mayrhofer, *Phases of* 5d *SCFTs from M-/F-theory on Non-Flat Fibrations*, *JHEP* **05** (2019) 187 [arXiv:1811.12400] [INSPIRE].

[127] K. Ohmori, Y. Tachikawa and G. Zafrir, *Compactifications of* 6d $N = (1,0)$ *SCFTs with non-trivial Stiefel-Whitney classes*, *JHEP* **04** (2019) 006 [arXiv:1812.04637] [INSPIRE].

[128] K. Filippas, C. Núñez and J. Van Gorsel, *Integrability and holographic aspects of six-dimensional* $\mathcal{N} = (1,\ 0)$ *superconformal field theories*, *JHEP* **06** (2019) 069 [arXiv:1901.08598] [INSPIRE].

[129] P.R. Merkx, *Pairing* 6*D SCFTs*, arXiv:1903.00079 [INSPIRE].

[130] C. Núñez, J.M. Penín, D. Roychowdhury and J. Van Gorsel, *The non-Integrability of Strings in Massive Type IIA and their Holographic duals*, *JHEP* **06** (2018) 078 [arXiv:1802.04269] [INSPIRE].