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Exact finite volume expectation values of $\bar{\Psi}\Psi$ in the massive Thirring model from light-cone lattice correlators

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Abstract: In this paper, using the light-cone lattice regularization, we compute the finite volume expectation values of the composite operator $\bar{\Psi}\Psi$ between pure fermion states in the Massive Thirring Model. In the light-cone regularized picture, this expectation value is related to 2-point functions of lattice spin operators being located at neighboring sites of the lattice. The operator $\bar{\Psi}\Psi$ is proportional to the trace of the stress-energy tensor. This is why the continuum finite volume expectation values can be computed also from the set of non-linear integral equations (NLIE) governing the finite volume spectrum of the theory. Our results for the expectation values coming from the computation of lattice correlators agree with those of the NLIE computations. Previous conjectures for the LeClair-Mussardo-type series representation of the expectation values are also checked.

Keywords: Bethe Ansatz, Integrable Field Theories, Lattice Integrable Models

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Contents

1 Introduction

Finite volume matrix elements of local operators play an important role in several applications of integrable quantum field theories. Namely, they are fundamental building blocks of the form factor perturbation theory [1] and their determination is indispensable for the computation of the string field theory vertex [2] and of the heavy-heavy-light 3-point functions [3] in the planar AdS_5/CFT_4 correspondence.

In the past decade a remarkable progress has been made in the computation of finite volume form factors in integrable quantum field theories [4–11]. Most of the methods use the infinite volume form factors [12] as a starting point and the finite volume form factors are to be determined in the form of a systematic large volume series. As a first step the large volume corrections, that decay with a power of the volume were determined [4, 5] and not much later a method was proposed [6] for computing some special type exponential in volume corrections. These investigations shed light on the fact that the computation of diagonal matrix elements is a much simpler task than that of the non-diagonal ones. Recently, in [13] it has been shown, that the leading term in the large volume series representation of the diagonal form-factors in [5, 15] can be derived from the formulas for the non-diagonal form-factors of [4], by taking the diagonal limit appropriately.

Though the structure of exponentially small in volume corrections for the non-diagonal matrix elements is still unknown, inspired by [14] for the diagonal matrix elements, a nice series representation was proposed in [10, 11]. However the proposal is valid only to purely elastic scattering theories and its extension to non-diagonally scattering theories is still unknown in general.

Recently, in the Massive Thirring (sine-Gordon) model a similar series representation was proposed to describe the finite volume diagonal form factors of the theory [17]. The conjecture was based on the computation of the diagonal solitonic (fermionic) matrix elements of the U(1) current from the light-cone lattice regularization [20] of the theory.

The purpose of this paper is two-fold. On the one hand we would like to demonstrate that the light-cone lattice approach admits an appropriate framework for computing the finite volume form factors of the Massive Thirring (sine-Gordon) model and on the other hand we would like to give further justification for the validity of the LeClair-Mussardo type series representation conjectured in [17].

To do so we compute the diagonal form factors of the composite operator $\bar{\Psi}\Psi$ from the light-cone lattice approach. There are several advantages of the choice of this operator. First of all, this operator is proportional to the trace of the stress-energy tensor. Thus the results of reference [18] imply, that up to a constant factor these expectation values can be computed simply from the non-linear integral equations (NLIE) governing the finite volume spectrum of the model [22]–[31]. This makes possible to check the results coming from the lattice computations against a result coming from a completely different method. Second of all, the operator $\bar{\Psi}\Psi$ is still simple enough not to mix with other operators under renormalization. Nevertheless, contrary to the case of the U(1)-current [17], in this case an infinite renormalization constant arises in accordance with field theoretical computations [19].

In the present paper, using the framework of Quantum Inverse Scattering Method [34]–[61], we compute such special spin-spin 2-point functions on the lattice, in which the spin operators are located at neighboring sites of the lattice. A straightforward computation shows, that the discretized version of the continuum operator $\bar{\Psi}\Psi$ corresponds to the lattice operator: $\sigma^+_n\sigma^-_{n+1} + \sigma^-_n\sigma^+_{n+1}$. We compute the expectation values of these operators between those Bethe eigenstates which correspond to the pure fermion (soliton) states in the continuum theory. Then we show, that in the continuum limit these fermionic expectation values (as expected) are proportional to the fermionic diagonal matrix elements of the trace of the stress-energy tensor. Latter can be computed purely [18] from the NLIE description of the sandwiching states. Our method, by nature accounts for the lattice artifacts, as well.

Our results also show, that in the continuum limit, when the lattice constant tends to zero, the leading order divergence arising in the fermionic expectation values of $\bar{\Psi}\Psi$ is of the same form as that expected from the renormalization group analysis of the Massive Thirring (sine-Gordon) model. Finally, we also checked that the all order conjecture [17] for the systematic large volume series representation of the diagonal fermionic (solitonic) form-factors of the Massive Thirring (sine-Gordon) model is also valid for this operator.

The outline of the paper is as follows: in section 2. we recall the most important properties of the Massive Thirring and sine-Gordon models and their light-cone lattice regularizations. This section contains the pure NLIE computation of the fermionic (solitonic) expectation values of the trace of the stress-energy tensor. In section 3. we summarize the Quantum Inverse Scattering Method framework and the lattice part of the computation of the special spin-spin 2-point functions of interest. The continuum limit procedure is described in section 4. In section 5. we rephrase our results in the form of a systematic large volume series and check the validity of the conjecture of [17]. Our summary and outlook closes the body of the paper in section 6. The paper contains three appendices, as well. In appendix A we rewrite the sums entering the lattice formulas for the two-point functions into integral expressions. In appendix B we describe how to compute the lattice cutoff tend to zero limit within these integral expressions. Finally, appendix C contains the large argument series representations of the convolution integrals being necessary for the computations.

2 Light-cone lattice approach to the massive-Thirring and sine-Gordon models

The Massive Thirring (MT) model is defined by the Lagrangian:

$$\mathcal{L}_{MT} = \bar{\Psi}(i\gamma_\nu\partial^\nu - m_0)\Psi - \frac{g}{2}\bar{\Psi}\gamma^\nu\Psi\bar{\Psi}\gamma_\nu\Psi\,, \tag{2.1}$$

where m_0 and g denotes the bare mass and the coupling constant of the theory, respectively. As usual, γ_μs stand for the γ-matrices. They satisfy the algebraic relations: $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \text{diag}(1,-1)$. Throughout the paper we use the chiral representation for the fermions as follows:

$$\Psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}, \quad \gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad \gamma^5 = \gamma^0\gamma^1 = -\eta = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{2.2}$$

It is well known [32], that this fermion model can be mapped to the sine-Gordon (SG) model:

$$\mathcal{L}_{SG} = \frac{1}{2}\partial_\nu\Phi\partial^\nu\Phi + \alpha_0\left(\cos\left(\beta\Phi\right) - 1\right), \qquad 0 < \beta^2 < 8\pi, \tag{2.3}$$

provided the coupling constants of the two theories are related by the formula:

$$1 + \frac{g}{4\pi} = \frac{4\pi}{\beta^2}. \tag{2.4}$$

A more detailed investigation of this equivalence [33] pointed out, that the two models are identical only in the even topological charge sector of their Hilbert-spaces and they differ in the odd topological charge sector.

The operator we study in this paper is the fermion bilinear $\bar{\Psi}\Psi$ in the MT model. To be more precise, here $\bar{\Psi}\Psi$ means the bare (unrenormalized) fermion bilinear of the model. According to the equivalence [32] it is proportional to the potential of the sine-Gordon model [19]:

$$\bar{\Psi}\Psi \leftrightarrow \frac{1}{\pi a}\cos(\beta\Phi), \tag{2.5}$$

with a being a cutoff in coordinate space. The perturbing operator $\cos(\beta\Phi)$ of the SG model is related to the trace of the stress-energy tensor Θ_T as follows:[1]

$$\Theta_T = 4\pi\alpha_0 \left(1 - \frac{\beta^2}{8\pi}\right) \cos(\beta\,\Phi). \tag{2.6}$$

From (2.5) and (2.6) the fermion bilinear can be expressed in terms of the trace of the stress-energy tensor as follows:

$$\bar{\Psi}\Psi \sim \frac{\beta^2}{4\pi^2\,(1-\beta^2/8\pi)}\,\frac{\Theta_T}{a\,\alpha_0}. \tag{2.7}$$

Due to renormalization effects α_0 scales with the coordinate space cutoff a as $\alpha_0 \sim a^{-\beta^2/4\pi}$ [19], thus

$$\bar{\Psi}\Psi \sim a^{\beta^2/4\pi-1}\,\Theta_T. \tag{2.8}$$

The minimal length a can be thought of as a lattice constant, as well. From (2.8) it can be seen that the matrix elements of $\bar{\Psi}\Psi$ are divergent in the attractive regime ($\beta^2 < 4\pi$) and the operator valued coefficient of the leading order divergence in a is proportional to the trace of the stress-energy tensor.[2] In this paper we show that our light-cone lattice computations account for the scaling behavior (2.8) and up to a constant factor, allow one to compute the diagonal matrix elements of Θ_T.

2.1 The light-cone lattice regularization

The light-cone lattice regularization scheme [20] admits an appropriate lattice approach to the even topological charge sector of the MT model. In this description the space-time is discretized along the light-cone directions: $x_\pm = x \pm t$ with an even number of lattice sites in the spatial direction. The sites of the light-cone lattice correspond to the discretized points of space-time. The left- and right-mover fermion fields live on the left- and right-oriented edges of the lattice. In this manner a left- and a right-mover fermion field can be associated to each site of the lattice (See figure 1). Lattice Fermi operators satisfy the discretized version of the usual anti-commutation relations:

$$\{\psi_{A,n}, \psi_{B,m}\} = 0, \quad \{\psi_{A,n}, \psi^+_{B,m}\} = \delta_{AB}\,\delta_{nm}, \quad A, B = R, L, \quad 1 \le m, n \le N. \tag{2.9}$$

As figure 1 indicates, the chirality of the Fermi operators is related to the parity of the lattice-site index. Namely, left-mover fields live on the odd- and right-mover fields live on the even-edges of the lattice, respectively:

$$\psi_{R,n} = \psi_{2n}, \quad \psi_{L,n} = \psi_{2n-1}, \quad 1 \le n \le \frac{N}{2}. \tag{2.10}$$

[1] In this sine-Gordon — Massive Thirring correspondence, the components of the stress energy tensors of the two models are mapped onto each other.

[2] The trace of the stress-energy tensor is a conserved quantity in the continuum quantum field theory, this is why it is not subjected to multiplicative renormalization. Consequently, its matrix elements are finite in the continuum limit.

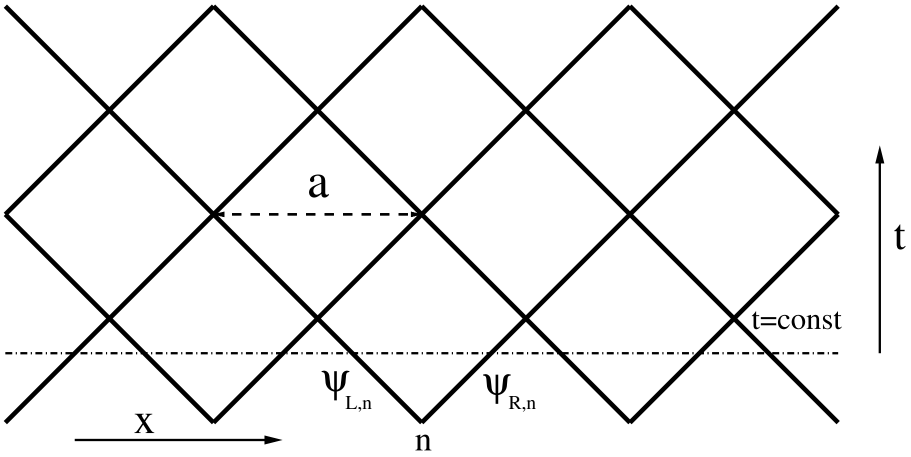

Figure 1. The pictorial representation of the light-cone lattice.

For later purposes it is worth to rewrite the lattice Fermi operators in terms of the spin-operators of the lattice. This can be achieved by a Jordan-Wigner transformation:

$$\psi_n^+ = \sigma_n^+ \prod_{l=1}^{n-1} \sigma_l^z, \qquad \psi_n = \sigma_n^- \prod_{l=1}^{n-1} \sigma_l^z, \tag{2.11}$$

where $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\,\sigma^y)$ with $\sigma^{x,y,z}$ being the Pauli-matrices.

The dynamics of the regularized model is given by light-cone evaluation operators: U_L and U_R. They are given by transfer matrices of an inhomogeneous 6-vertex model [20]:

$$U_L = e^{i\frac{a}{2}(H-P)} = \mathcal{T}(\xi_2|\vec{\xi}), \qquad U_R^+ = e^{-i\frac{a}{2}(H+P)} = \mathcal{T}(\xi_1|\vec{\xi}), \tag{2.12}$$

where $\mathcal{T}$ is the trace of the monodromy matrix over the auxiliary space $V_0 \simeq \mathbb{C}^2$,

$$\mathcal{T}(\lambda|\vec{\xi}) = \mathrm{Tr}_0\, T(\lambda|\vec{\xi}), \qquad \left[\mathcal{T}(\lambda|\vec{\xi}), \mathcal{T}(\lambda'|\vec{\xi})\right] = 0. \tag{2.13}$$

The monodromy matrix is given by the R-matrix of the 6-vertex model in the usual way [40],

$$T(\lambda|\vec{\xi}) = R_{01}(\lambda-\xi_1)\, R_{02}(\lambda-\xi_2)\, \ldots R_{0N}(\lambda-\xi_N) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[0]}, \tag{2.14}$$

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sinh(\lambda)}{\sinh(\lambda-i\gamma)} & \frac{\sinh(-i\gamma)}{\sinh(\lambda-i\gamma)} & 0 \\ 0 & \frac{\sinh(-i\gamma)}{\sinh(\lambda-i\gamma)} & \frac{\sinh(\lambda)}{\sinh(\lambda-i\gamma)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.15}$$

such that ξ_ns denote the inhomogeneities of the model. The entries of the 2×2 monodromy matrix act on the quantum space of the model $\mathcal{H} = \otimes_{i=1}^N V_i$ with $V_i \simeq \mathbb{C}^2$ and they play crucial role in the algebraic Bethe-Ansatz solution of the model. In (2.12) H, P and a denote the Hamiltonian, the momentum and the lattice constant of the model, respectively.

In order to get a massive interacting quantum field theory as the continuum limit of this lattice model, the inhomogeneities of the vertex-model must be chosen as follows [20]:

$$\xi_n = \rho_n - i\frac{\gamma}{2}, \qquad \rho_n = (-1)^n \rho_0, \qquad n = 1, \ldots, N, \tag{2.16}$$

such that the parameter ρ_0 must be tuned with the lattice constant a, or equivalently with the number of lattice sites N according to the formula as follows:

$$\rho_0 = \frac{\gamma}{\pi} \ln \frac{4}{\mathcal{M}\, a} = \frac{\gamma}{\pi} \ln \frac{2\,N}{\mathcal{M}\, L}, \tag{2.17}$$

where $\mathcal{M}$ denotes the physical mass of fermions (solitons) of the MT (SG) model, L stands for the finite volume and N is the number[3] of lattice sites of the 6-vertex model.

The parameters of the regularized lattice model are the inhomogeneities, the number of lattice sites and the anisotropy parameter γ. In (2.16) and (2.17) we described how to choose the inhomogeneities to obtain a massive interacting integrable quantum field theory in the continuum limit. The infinite volume solution of the model[4] shows [21] that this massive continuum quantum field theory is nothing but the MT or SG model, provided the following relation holds between the anisotropy parameter of the vertex model and the coupling constants of the Lagrangians (2.1) and (2.3):

$$\frac{\beta^2}{4\pi} = \frac{1}{1+\frac{g}{4\pi}} = 2\left(1 - \frac{\gamma}{\pi}\right). \tag{2.18}$$

For later purpose it is worth to introduce a new parameterization for the anisotropy parameter:

$$\gamma = \frac{\pi}{p+1}, \quad \text{with} \quad 0 < p < \infty, \quad \text{then:} \quad \frac{\beta^2}{4\pi} = \frac{2p}{p+1}. \tag{2.19}$$

We note, that the regimes $0 < p < 1$ and $1 < p$ correspond to the attractive and repulsive regimes of the quantum field theory, respectively.

The definition (2.12) embeds the light-cone evolution operators of our model into the hierarchy of mutually commuting set of transfer matrices of the 6-vertex model. This implies that the Hamiltonian and the momentum of the model can be diagonalized via the Algebraic Bethe Ansatz method [34].

2.2 Algebraic Bethe Ansatz

In the framework of algebraic Bethe Ansatz, the eigenvectors of the transfer matrix (2.13) are constructed by successive application of creation operators on the bare vacuum of the model. The bare vacuum or reference state $|0\rangle$ is the completely ferromagnetic state with all spins up. The role of creation operators are played by the 12-matrix element of the

[3] In this convention, in the light-cone lattice the number of lattice sites in spatial direction is $\frac{N}{2}$. See figure 1.

[4] The infinite volume solution consists of two steps. First, the $N \to \infty$ limit is taken with a kept finite. Equation (2.17) implies that this means that the inhomogeneity ρ_0 is also kept finite. Then the $a \to 0$ limit is taken by tuning ρ_0 in the large N result according to (2.17).

monodromy matrix (2.14): $T_{12}(\lambda) = B(\lambda)$ which decreases the S_z quantum number of a state by 1. A state constructed in this manner:

$$|\vec{\lambda}\rangle = |\lambda_1, \lambda_2, \dots, \lambda_m\rangle = B(\lambda_1)\, B(\lambda_2)\, \dots B(\lambda_m)\, |0\rangle, \qquad S_z|\vec{\lambda}\rangle = \left(\frac{N}{2} - m\right)|\vec{\lambda}\rangle, \tag{2.20}$$

is an is an eigenvector of the transfer matrix, provided the spectral parameters in the arguments of the creation operators satisfy the Bethe equations:

$$\prod_{i=1}^{N} \frac{\sinh(\lambda_a - \xi_i - i\gamma)}{\sinh(\lambda_a - \xi_i)} \prod_{b=1}^{m} \frac{\sinh(\lambda_a - \lambda_b + i\gamma)}{\sinh(\lambda_a - \lambda_b - i\gamma)} = -1, \qquad a = 1, \dots, m. \tag{2.21}$$

In the Algebraic Bethe Ansatz approach the solutions of the Bethe equations play central role, since all physical quantities can be expressed in terms of these roots. The eigenvalue of the transfer matrix (2.13) on a Bethe-eigenvector (2.20) is given by the formula:

$$\mathcal{T}_{\vec{\lambda}}(\mu|\vec{\xi}) = \prod_{k=1}^{m} \frac{\sinh(\mu - \lambda_k + i\gamma)}{\sinh(\mu - \lambda_k)} + \prod_{i=1}^{N} \frac{\sinh(\mu - \xi_i)}{\sinh(\mu - \xi_i - i\gamma)} \prod_{k=1}^{m} \frac{\sinh(\mu - \lambda_k - i\gamma)}{\sinh(\mu - \lambda_k)}. \tag{2.22}$$

For the cases, when the number of Bethe-roots is large, it is more convenient to reformulate the Bethe-equations (2.21) in their logarithmic form. The central object of this formulation is the so-called counting-function. For the choice of inhomogeneities (2.16) it is defined by the formula [27]:

$$Z_\lambda(\lambda) = \frac{N}{2}\left(\phi_1(\lambda - \rho_0) + \phi_1(\lambda + \rho_0)\right) - \sum_{k=1}^{m} \phi_2(\lambda - \lambda_k), \tag{2.23}$$

where $\phi_\nu(\lambda)$ is an odd function on the whole complex plane with all discontinuities running parallel to the real axis. In its fundamental domain $|\mathrm{Im}\,\lambda| < \nu$, it is given by the analytic formula:

$$\phi_\nu(\lambda) = -i\,\log \frac{\sinh(i\frac{\gamma}{2}\nu - \lambda)}{\sinh(i\frac{\gamma}{2}\nu + \lambda)}, \qquad 0 < \nu, \qquad \phi_\nu(0) = 0, \qquad |\mathrm{Im}\,\lambda| < \nu. \tag{2.24}$$

The counting-function allows one to reformulate the Bethe-equations (2.21) in the form as follows:

$$Z_\lambda(\lambda_a) = 2\pi\, I_a, \qquad I_a \in \mathbb{Z} + \frac{1+\delta}{2}, \qquad \delta = m \ (\mathrm{mod}\, 2), \qquad a = 1, \dots, m. \tag{2.25}$$

In this formulation, depending on the value of δ, an integer or half-integer quantum number I_a can be assigned to each Bethe-root. When one considers states formed by only real Bethe-roots, then all these quantum numbers are different[5] and they characterize the state uniquely.

The true vacuum corresponding to the ground state of the quantum field theory, is the $S_z = 0$, anti-ferromagnetic vacuum with $\delta = 0$. This state is formed by real Bethe-roots such that the quantum numbers of the Bethe-roots fill completely the whole allowed range

[5]Due to the appropriate choice of branch cuts for $\phi_\nu(\lambda)$.

$[Z_\lambda(-\infty)/2\pi, Z_\lambda(\infty)/2\pi]$. The excitations above this sea of real roots are characterized by complex Bethe-roots and holes. In this paper we will consider only hole excitations, since they correspond to fermion or soliton excitations of the continuum quantum field theory [26–31]. The holes are such special real solutions of (2.25), which are not Bethe-roots.[6] Holes can be interpreted as missing Bethe-roots and the quantum numbers of the missing Bethe-roots can be assigned to them:

$$Z_\lambda(h_k) = 2\pi\, I_k, \qquad I_k \in \mathbb{Z} + \frac{1+\delta}{2}, \qquad k = 1, \dots, m_H, \tag{2.26}$$

where h_k denotes the positions of the holes and their number is denoted by m_H.

2.3 NLIE for the finite volume spectrum

When one has to deal with a large number of Bethe-roots, it is worth to rephrase the Bethe-equations (2.21) or equivalently (2.25) in a form of a set of nonlinear-integral equations (NLIE) [22]–[30].

Here we present the equations only for the pure hole sector of the theory [26] and here we will use the rapidity convention for the equations. This means a simple rescaling of the spectral parameter: $\theta = \frac{\pi}{\gamma}\lambda$.

In the pure hole sector, the counting-function in rapidity variable $Z_N(\theta) = Z_\lambda(\frac{\gamma}{\pi}\theta)$ satisfy the nonlinear-integral equations as follows:

$$\begin{aligned} Z_N(\theta) = {} & \frac{N}{2}\left\{\arctan\left[\sinh(\theta-\Theta)\right] + \arctan\left[\sinh(\theta+\Theta)\right]\right\} + \sum_{k=1}^{m_H} \chi(\theta - H_k) \\ & + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi i}\, G(\theta-\theta'-i\eta)\, L_N^{(+)}(\theta'+i\eta) - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi i}\, G(\theta-\theta'+i\eta)\, L_N^{(-)}(\theta'-i\eta), \end{aligned} \tag{2.27}$$

where $\chi(\theta)$ is the soliton-soliton scattering phase and $G(\theta)$ denotes its derivative;

$$\chi(\theta) = 2\int_0^{\infty} d\omega\, \frac{\sin(\omega\,\theta)}{\omega}\, \frac{\sinh(\frac{(p-1)\,\pi\omega}{2})}{2\cosh(\frac{\pi\omega}{2})\,\sinh(\frac{p\,\pi\,\omega}{2})}, \tag{2.28}$$

$$G(\theta) = \frac{d}{d\theta}\chi(\theta) = \int_{-\infty}^{\infty} d\omega\, e^{-i\,\omega\theta}\, \frac{\sinh(\frac{(p-1)\,\pi\omega}{2})}{2\cosh(\frac{\pi\omega}{2})\,\sinh(\frac{p\,\pi\,\omega}{2})}, \tag{2.29}$$

$0 < \eta < \min(p\pi, \pi)$ is an arbitrary positive contour-integral parameter, $\Theta = \ln \frac{2\,N}{\mathcal{M}\,L}$ is the inhomogeneity parameter of the vertex-model and $H_k = \frac{\pi}{\gamma} h_k$ denote the positions of the holes in the rapidity convention. They are subjected to the quantization equations:

$$Z_N(H_k) = 2\pi\, I_k, \qquad I_k \in \mathbb{Z} + \frac{1+\delta}{2}, \qquad k = 1, \dots, m_H. \tag{2.30}$$

The nonlinearity of the equations is encoded into the form of $L_N^{(\pm)}(\theta)$, which takes the form:

$$L_N^{(\pm)}(\theta) = \ln\left(1 + (-1)^\delta\, e^{\pm i\, Z_N(\theta)}\right). \tag{2.31}$$

[6] Namely, they do not enter in the definition of $Z_\lambda(\lambda)$ in (2.23).

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The number of holes is not independent of the S_z quantum number of the state. The connection between these two quantum numbers is given by the counting-equation[7] [27]:

$$m_H = 2\,S_z - 2\left[\frac{1}{2} + \frac{S_z}{p+1}\right], \tag{2.32}$$

where here $[\ldots]$ stands for integer part. This equation immediately implies that on a lattice with even number of sites, only states with even number of holes can exist.

The main advantage of formulating the spectral problem in terms of the counting function is that it has a well-defined continuum limit. If one keeps the hole quantum numbers fixed, it is just the $N \to \infty$ limit of the lattice counting-function [23–25]:

$$Z(\theta) = \lim_{N\to\infty} Z_N(\theta), \qquad L_\pm(\theta) = \lim_{N\to\infty} L_N^{(\pm)}(\theta) = \ln\left(1 + (-1)^\delta\, e^{\pm i\, Z(\theta)}\right). \tag{2.33}$$

The continuum counting-function satisfy the nonlinear-integral equations as follows [26]–[30]:

$$\begin{aligned} Z(\theta) = \ell \sinh\theta + \sum_{k=1}^{m_H} \chi(\theta - H_k) + \int\limits_{-\infty}^{\infty} \frac{d\theta'}{2\pi i}\, G(\theta-\theta'-i\eta)\, L_+(\theta'+i\eta) \\ - \int\limits_{-\infty}^{\infty} \frac{d\theta'}{2\pi i}\, G(\theta-\theta'+i\eta)\, L_-(\theta'-i\eta), \end{aligned} \tag{2.34}$$

where $\ell = \mathcal{M}\, L$ with L being the volume and $\mathcal{M}$ is the fermion (soliton) mass. The holes formally satisfy exactly the same quantization equations as in the lattice model:

$$Z(H_k) = 2\pi\, I_k, \qquad I_k \in \mathbb{Z} + \frac{1+\delta}{2}, \qquad \delta \in \{0,1\}, \qquad k = 1,\ldots,m_H. \tag{2.35}$$

The energy and momentum of the pure hole states in the continuum theory read as:

$$E = \mathcal{M} \sum_{k=1}^{m_H} \cosh H_k - \frac{\mathcal{M}}{2\pi i} \sum_{\alpha=\pm} \alpha \cdot \int\limits_{-\infty}^{\infty} d\theta\, \sinh(\theta + i\,\alpha\,\eta)\, L_\alpha(\theta + i\,\alpha\,\eta), \tag{2.36}$$

$$P = \mathcal{M} \sum_{k=1}^{m_H} \sinh H_k - \frac{\mathcal{M}}{2\pi i} \sum_{\alpha=\pm} \alpha \cdot \int\limits_{-\infty}^{\infty} d\theta\, \cosh(\theta + i\,\alpha\,\eta)\, L_\alpha(\theta + i\,\alpha\,\eta). \tag{2.37}$$

The counting-equation (2.32) also changes non-trivially in the continuum limit [17, 27]:

$$m_H = Q, \tag{2.38}$$

where Q is the U(1) (topological) charge of the continuum model.

The NLIE (2.34) can be solved iteratively in the large volume limit. From this solution it follows, that the nonlinear terms $L_\pm(\theta \pm i\,\eta)$ are exponentially small in the volume. As a

[7] Here we present the equations without the presence of special objects. For a more detailed description see for example [27, 31].

consequence in (2.36) and (2.37) the integral terms can be dropped in the infinite volume limit and one ends up with the energy and momentum formulas of m_H pieces of fermions (solitons) with rapidities $\{H_j\}_{j=1...m_H}$. This implies that the holes in the sea of real roots describe the fermions (solitons) of the MT (SG) model. This is why in the sequel we will refer to holes as fermions or solitons.

Finally, we note that the actual value[8] of the quantum number δ is important from the point of view of the continuum theory. Its value can make difference between fermions ($\delta = 1$) of the MT model and the solitons ($\delta = 0$) of the SG model in the odd U(1) charge sector of the theory [28]–[30]. In the even charge sector only the $\delta = 0$ value is physical and there is no difference between MT fermions and SG solitons [28]–[30].

From the discussion above it follows that only the even charge sector of the MT and SG models can be regularized by the twistless 6-vertex model. The description of the odd charge sector requires a twisted vertex-model with a twist angle $\omega = \frac{\pi}{2}$ [63]. However, in this paper we restrict ourselves to the twistless case.

2.4 Expectation values of the trace of the stress-energy tensor

In this subsection using the NLIE description of the finite volume spectrum given by (2.34) and (2.36), we compute the fermionic (solitonic) expectation values of the trace of the stress-energy tensor Θ_T. The finite temperature 1-point functions, which correspond to the finite volume vacuum expectation value, has been previously computed and discussed in [70] and [64].

It has been shown in [18] that the diagonal matrix elements of Θ_T can be computed from the volume dependence of the energy of the sandwiching state by the following formula:

$$\langle \Theta_T \rangle = \langle \Theta_T^\infty \rangle + 2\pi\, \mathcal{M} \left(\frac{E(\ell)}{\ell} + \frac{dE(\ell)}{d\ell} \right). \tag{2.39}$$

In the sequel we compute $\langle \Theta_T \rangle$ when the sandwiching state is an m_H-fermion state described by the equations (2.34) and (2.36).

As a starting point, it is worth to compute the infinite volume or in other words the bulk expectation value: $\langle \Theta_T^\infty \rangle$. Using Zamolodchikov's argument [18], it can be expressed in terms of the eigenstate independent bulk energy of the model by the formula:

$$\langle \Theta_T^\infty \rangle = 2\pi\, \mathcal{M} \left(\frac{E_{\rm bulk}(\ell)}{\ell} + \frac{dE_{\rm bulk}(\ell)}{d\ell} \right). \tag{2.40}$$

In the MT (SG) model the bulk energy term is of the form [25]:

$$E_{\rm bulk}(\ell) = -\frac{\mathcal{M}\ell}{4} \tan\left(\frac{p\pi}{2}\right). \tag{2.41}$$

Inserting (2.41) into (2.40) one obtains:

$$\langle \Theta_T^\infty \rangle = -\pi \mathcal{M}^2 \tan\left(\frac{p\pi}{2}\right). \tag{2.42}$$

[8] On the lattice the actual value of δ can be influenced by the parity of $\frac{N}{2}$.

As a next step we express the non-bulk part of $\langle\Theta_T\rangle$ in (2.39) in terms of the solution of the NLIE (2.34). To do so, it is worth to introduce some useful notations. Let $\mathcal{F}_\pm(\theta)$ denote the nonlinear combinations as follows:

$$\mathcal{F}_\pm(\theta) = \frac{(-1)^\delta\, e^{\pm i\, Z(\theta)}}{1+(-1)^\delta\, e^{\pm i\, Z(\theta)}}. \tag{2.43}$$

Then the derivative of $L_\pm(\theta)$ with respect to any parameter $\mathcal{P}$ is given by the formula:

$$\frac{dL_\pm(\theta)}{d\mathcal{P}} = \pm i\, \frac{dZ(\theta)}{d\mathcal{P}}\, \mathcal{F}_\pm(\theta). \tag{2.44}$$

In practice $\mathcal{P}$ can denote one of the parameters of the NLIE equations (2.34). Namely, it can be the dimensionless volume ℓ, the spectral parameter θ or one of the positions of the holes H_j. The second term in the right hand side of (2.39) consists of two terms. The first term $\sim \frac{E(\ell)}{\ell}$ can be expressed in terms of $\frac{dZ(\theta)}{d\theta}$ and of $\mathcal{F}_\pm(\theta)$, while the second term $\sim \frac{dE(\ell)}{d\ell}$ turns out to be the functional of $\frac{dZ(\theta)}{d\ell}$ and of $\mathcal{F}_\pm(\theta)$.

Integrating the right hand side of (2.36) by parts, $\frac{E(\ell)}{\ell}$ can be rephrased as follows:

$$\frac{E(\ell)}{\ell} = \frac{\mathcal{M}}{\ell} \sum_{k=1}^{m_H} \cosh(H_k) X_k^{(d)} + \frac{\mathcal{M}}{\ell} \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh(\theta + i\,\alpha\,\eta)\, \mathcal{G}_d(\theta+i\,\alpha\,\eta)\, \mathcal{F}_\alpha(\theta+i\,\alpha\,\eta), \tag{2.45}$$

where $\mathcal{G}_d(\theta) = Z'(\theta)$, $X_k^{(d)} = \frac{\mathcal{G}_d(H_k)}{Z'(H_k)} = 1$. Differentiating (2.34) with respect to θ, one can show, that they satisfy the set of linear integral equations as follows:

$$\begin{aligned} \mathcal{G}_d(\theta) - \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} G(\theta-\theta'-i\,\alpha\,\eta) \mathcal{G}_d(\theta'+i\,\alpha\,\eta)\, \mathcal{F}_\alpha(\theta'+i\,\alpha\,\eta) = \\ = \ell\, \cosh(\theta) + \sum_{j=1}^{m_H} G(\theta-H_j)\, X_j^{(d)}, \\ X_j^{(d)} = \frac{\mathcal{G}_d(H_j)}{Z'(H_j)}, \qquad j=1,\dots,m_H. \end{aligned} \tag{2.46}$$

Taking the derivative of (2.36) and (2.34) with respect to ℓ, leads the following expression for $\frac{dE(\ell)}{d\ell}$:

$$\frac{dE(\ell)}{d\ell} = -\mathcal{M} \sum_{k=1}^{m_H} \sinh(H_k)\, X_k^{(\ell)} - \mathcal{M} \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \sinh(\theta+i\,\alpha\,\eta)\, \mathcal{G}_\ell(\theta+i\,\alpha\,\eta)\, \mathcal{F}_\alpha(\theta+i\,\alpha\,\eta), \tag{2.47}$$

where $\mathcal{G}_\ell(\theta) = \frac{dZ(\theta)}{d\ell}$, $X_k^{(\ell)} = \frac{\mathcal{G}_\ell(H_k)}{Z'(H_k)} = -H_k'(\ell)$. They are solutions of the set of linear integral equations as follows:[9]

$$\mathcal{G}_\ell(\theta)-\sum_{\alpha=\pm}\int\limits_{-\infty}^{\infty}\frac{d\theta'}{2\pi}G(\theta-\theta'-i\,\alpha\,\eta)\mathcal{G}_\ell(\theta'+i\,\alpha\,\eta)\,\mathcal{F}_\alpha(\theta'+i\,\alpha\,\eta)=$$
$$=\sinh(\theta)+\sum_{j=1}^{m_H}G(\theta-H_j)\,X_j^{(\ell)},\tag{2.48}$$
$$X_j^{(\ell)}=\frac{\mathcal{G}_\ell(H_j)}{Z'(H_j)},\qquad j=1,\dots,m_H.$$

Plugging (2.45) and (2.47) into (2.39) we obtain our final formula for the fermionic (solitonic) expectation values of the trace of the stress-energy tensor:

$$\langle\Theta_T\rangle=\langle\Theta_T^\infty\rangle+2\pi\mathcal{M}^2\sum_{k=1}^{m_H}\left\{\cosh(H_k)\,\frac{X_k^{(d)}}{\ell}-\sinh(H_k)\,X_k^{(\ell)}\right\}+\tag{2.49}$$
$$+\mathcal{M}^2\sum_{\alpha=\pm}\int\limits_{-\infty}^{\infty}d\theta\left[\cosh(\theta+i\,\alpha\,\eta)\,\frac{\mathcal{G}_d(\theta+i\,\alpha\,\eta)}{\ell}-\sinh(\theta+i\,\alpha\,\eta)\,\mathcal{G}_\ell(\theta+i\,\alpha\,\eta)\right]\mathcal{F}_\alpha(\theta+i\,\alpha\,\eta).$$

Representation (2.49) for $\langle\Theta_T\rangle$ should be used as follows. First one has to solve the NLIE equations (2.34) for the sandwiching fermion (soliton) state. Then the linear integral equations (2.46) and (2.48) should be solved. Finally, inserting these solutions into (2.49) gives the required expectation value. Though, this representation for $\langle\Theta_T\rangle$ might seem strange for the first sight, but in the later sections it will turn out, that it fits very well for the structure of the lattice results.

In the rest of the paper our main goal is to reproduce the formula (2.49) from the light-cone lattice computation of the multi-fermion (soliton) expectation values of $\bar{\Psi}\Psi$.

2.5 The lattice counterpart of $\bar{\Psi}\Psi$

We close this section with a short discussion about the lattice counterpart of the operator $\bar{\Psi}\Psi$ in the MT model. A simple Jordan-Wigner transformation (2.11) shows, that certain bilinears of the lattice Fermi operators are simple expressions of the lattice spin operators:

$$\begin{aligned}\psi_n^+\psi_{n+1}&=\sigma_n^+\,\sigma_{n+1}^-,\\ \psi_{n+1}^+\psi_n&=\sigma_n^-\,\sigma_{n+1}^+,\end{aligned}\tag{2.50}$$

where $\sigma_n^\pm$ are the usual spin creation and annihilation operators corresponding to the nth site of the lattice, while ψ_n and ψ_n^+ are the lattice Fermi operators defined by (2.9) and (2.10).

Using the representation (2.2) for γ^0, the following identification can be made for the unrenormalized bare operators on the lattice:

$$\bar{\Psi}\Psi(x)\big|_{x=n\,a}=\Psi_R^+(x)\Psi_L(x)\big|_{x=n\,a}+\Psi_L^+(x)\Psi_R(x)\big|_{x=n\,a}\to\frac{1}{a}\psi_{R,n}^+\psi_{L,n}+\frac{1}{a}\psi_{L,n}^+\psi_{R,n}=$$
$$=\frac{1}{a}\psi_{2n}^+\psi_{2n+1}+\frac{1}{a}\psi_{2n+1}^+\psi_{2n}=\frac{1}{a}\sigma_{2n}^+\,\sigma_{2n+1}^-+\frac{1}{a}\sigma_{2n}^-\,\sigma_{2n+1}^+,\tag{2.51}$$

[9]The $X_k^{(\ell)}=-H_k'(\ell)$ equation can be derived by taking the derivative of the hole quantization equation (2.35) with respect to ℓ.

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where the term $\frac{1}{a}$ is introduced to account for the correct bare dimension of the continuum Fermi field.

A similar computation shows that the pseudo-scalar combination of the Fermi operators correspond to the antisymmetric combination of the spin operators:

$$\bar{\Psi}\gamma^5\Psi(x)\big|_{x=n\,a} \to \frac{1}{a}\psi^+_{2n}\psi_{2n+1} - \frac{1}{a}\psi^+_{2n+1}\psi_{2n} = \frac{1}{a}\sigma^-_{2n}\,\sigma^+_{2n+1} - \frac{1}{a}\sigma^+_{2n}\,\sigma^-_{2n+1}. \tag{2.52}$$

Thus (2.51) and (2.52) implies, that the determination of the expectation values of the bare scalar- and pseudo-scalar fermion bilinears is equivalent to computing the two-point functions of neighboring spin operators. This task is completed in the rest of the paper via the QISM [34]–[61].

We note that beyond the computation of 2-point functions $\langle\sigma^\pm_n\sigma^\mp_{n+1}\rangle$ the 2-point function $\langle e_n\, e_{n+1}\rangle$ with $e_n = \frac{1}{2}(1_n - \sigma^z_n)$ can also be computed with the techniques presented in this paper. This latter 2-point function contains a combination a 4-fermion term, as well: $e_n\, e_{n+1} = (\psi^+_n\psi_n - \frac{1}{2})(\psi^+_{n+1}\psi_{n+1} - \frac{1}{2})$. A usual argument based on the bare dimensions of the operators implies that this operator has a nontrivial mixing under renormalization. This means that the correct implementation of the renormalization process requires the computation of the expectation values of further operators. This investigation is left for future work.

3 Computation of lattice correlators

The strategy of computing the fermionic (solitonic) expectation values of the operators $\bar{\Psi}\Psi$ and $\bar{\Psi}\gamma^5\Psi$ consists of three main steps. First, one has to compute the expectation values of the lattice operators $\sigma^\pm_n\sigma^\mp_{n+1}$ in pure hole states. The second step is to consider the symmetric (2.51) and anti-symmetric (2.52) combinations of these expectation values in order to describe the diagonal form-factors of the operators $\bar{\Psi}\Psi$ and $\bar{\Psi}\gamma^5\Psi$, respectively. Finally, one has to take the continuum limit of the lattice results by sending the number of lattice sites N to infinity such that the inhomogeneity parameter ρ_0 is tuned according to (2.17). In [17] the efficiency of this method has been demonstrated via the computation of the solitonic (fermionic) expectation values the U(1) current of the model. In this section we describe in detail the lattice part of the computations.

Consider a vector of the Hilbert-space obtained by successive actions of creation operators on the bare vacuum:

$$|\vec{\lambda}\rangle = B(\lambda_1)\,B(\lambda_2)\ldots B(\lambda_m)\,|0\rangle. \tag{3.1}$$

Such a state is called Bethe-state if the numbers λ_j are arbitrary and it is called Bethe-eigenstate if the set $\{\lambda_j\}_{j=1,\ldots,m}$ is equal to the set of roots of the Bethe equations (2.21). The corresponding "bra" vector can be defined by acting from the right with the annihilation operators on the "bra" bare vacuum:

$$\langle\vec{\lambda}| = \langle 0|\,C(\lambda_m)\ldots C(\lambda_2)\,C(\lambda_1). \tag{3.2}$$

The determination of the diagonal form-factors of $\bar{\Psi}\Psi$ and $\bar{\Psi}\gamma^5\Psi$ requires the computation of the following two special 2-point functions:

$$\langle \sigma_n^{\pm}\sigma_{n+1}^{\mp}\rangle_\lambda = \frac{\langle \vec{\lambda}|\sigma_n^{\pm}\sigma_{n+1}^{\mp}|\vec{\lambda}\rangle}{\langle\vec{\lambda}|\vec{\lambda}\rangle}, \tag{3.3}$$

where here $|\vec{\lambda}\rangle$ denotes a Bethe-eigenstate.

The determination of these 2-point functions can be achieved in a purely algebraic way [39, 40] within the framework of the QISM [34], such that only the Yang-Baxter algebra relations and the expression of local spin operators in terms of the elements of the monodromy matrix (2.14) of the model are used [39].

The core of the algebraic computations is the relation between the local spin operators and the elements of the Yang-Baxter algebra [39]:

$$E_n^{ab} = \prod_{i=1}^{n-1}(A+D)(\xi_i)\; T_{ab}(\xi_n) \prod_{i=n+1}^{N}(A+D)(\xi_i), \qquad a,b=1,2, \tag{3.4}$$

where the operator E_n is given in terms of local spin operators as follows:

$$E_n^{11} = \frac{1}{2}(1_n+\sigma_n^z), \quad E_n^{12}=\sigma_n^-, \quad E_n^{21}=\sigma_n^+, \quad E_n^{22}=\frac{1}{2}(1_n-\sigma_n^z). \tag{3.5}$$

The formulas (3.4) and (3.5) imply the following representation for the 2-point correlators (3.3) of our interest:

$$\langle \sigma_n^-\,\sigma_{n+1}^+\rangle_\lambda = \frac{\langle\vec{\lambda}|\sigma_n^-\,\sigma_{n+1}^+|\vec{\lambda}\rangle}{\langle\vec{\lambda}|\vec{\lambda}\rangle} = \frac{1}{\mathcal{T}_{\vec{\lambda}}(\xi_n|\vec{\xi})\,\mathcal{T}_{\vec{\lambda}}(\xi_{n+1}|\vec{\xi})}\frac{\langle\vec{\lambda}|B(\xi_n)\,C(\xi_{n+1})|\vec{\lambda}\rangle}{\langle\vec{\lambda}|\vec{\lambda}\rangle}, \tag{3.6}$$

$$\langle \sigma_n^+\,\sigma_{n+1}^-\rangle_\lambda = \frac{\langle\vec{\lambda}|\sigma_n^+\,\sigma_{n+1}^-|\vec{\lambda}\rangle}{\langle\vec{\lambda}|\vec{\lambda}\rangle} = \frac{1}{\mathcal{T}_{\vec{\lambda}}(\xi_n|\vec{\xi})\,\mathcal{T}_{\vec{\lambda}}(\xi_{n+1}|\vec{\xi})}\frac{\langle\vec{\lambda}|C(\xi_n)\,B(\xi_{n+1})|\vec{\lambda}\rangle}{\langle\vec{\lambda}|\vec{\lambda}\rangle}, \tag{3.7}$$

where here $|\vec{\lambda}\rangle$ denotes a Bethe-eigenstate, $\mathcal{T}_{\vec{\lambda}}(\lambda|\vec{\xi})$ denotes the eigenvalue of the transfer matrix (2.13) on the state $|\vec{\lambda}\rangle$ and ξ_n is the inhomogeneity parameter belonging to the nth site of the vertex-model.

To compute (3.6) and (3.7), we need to know, how an operator $B(\xi_n)$ with ξ_n being an inhomogeneity of the vertex model, acts on a "bra"-vector (3.2). This is given by the formula [40]:

$$\begin{aligned}\langle 0|\prod_{k=1}^{M}C(\lambda_k)B(\xi_n) &= \sum_{a=1}^{M}\mathfrak{f}_M^{(0)}(\lambda_a|\xi_n)\\ &\times\left\{\mathfrak{f}_M^{(1)}(\lambda_a|\xi_n)\langle 0|\prod_{\substack{k=1\\k\neq a}}^{M}C(\lambda_k)+\sum_{\substack{b=1\\b\neq a}}^{M}\mathfrak{f}_M^{(2)}(\lambda_a,\lambda_b|\xi_n)\langle 0|\prod_{\substack{k=1\\k\neq a,b}}^{M}C(\lambda_k)C(\xi_n)\right\}.\end{aligned} \tag{3.8}$$

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Here the functions $\mathfrak{f}^{(0)}_M, \mathfrak{f}^{(1)}_M$ and $\mathfrak{f}^{(2)}_M$ are of the form:

$$\begin{aligned}
\mathfrak{f}^{(0)}_M(\lambda_a|\xi) &= \frac{1}{r(\lambda_a)\,\sinh(\lambda_a-\xi)}\,\frac{\prod\limits_{k=1}^{M}\sinh(\lambda_a-\lambda_k-i\,\gamma)}{\prod\limits_{\substack{k=1\\k\neq a}}^{M}\sinh(\lambda_a-\lambda_k)},\\
\mathfrak{f}^{(1)}_M(\lambda_a|\xi) &= \frac{\sinh(\lambda_a-\xi)}{\sinh(\lambda_a-\xi-i\,\gamma)}\prod_{j=1}^{M}\frac{\sinh(\lambda_j-\xi-i\,\gamma)}{\sinh(\lambda_j-\xi)},\\
\mathfrak{f}^{(2)}_M(\lambda_a,\lambda_b|\xi) &= \frac{1}{\sinh(\lambda_a-\lambda_b-i\,\gamma)\,\sinh(\xi-\lambda_b)}\,\frac{\prod\limits_{j=1}^{M}\sinh(\lambda_j-\lambda_b-i\,\gamma)}{\prod\limits_{\substack{j=1\\j\neq a,b}}^{M}\sinh(\lambda_j-\lambda_b)},
\end{aligned} \tag{3.9}$$

where

$$r(\lambda)=\prod_{j=1}^{N}\frac{\sinh(\lambda-\xi_j-i\,\gamma)}{\sinh(\lambda-\xi_j)}. \tag{3.10}$$

From its definition it follows that its inverse becomes zero at the positions of the inhomogeneities of the lattice model:

$$\frac{1}{r(\xi_j)}=0,\quad j=1,\dots,N. \tag{3.11}$$

We note that in general, in (3.8) and (3.9), λ_ks can take any complex value and they do not need to be solutions of the Bethe-equations (2.21). On the other hand it follows from (3.10) and (2.21) that if λ_ks are solutions of (2.21), then $r(\lambda)$ satisfies the identity:

$$\prod_{k=1}^{m} r(\lambda_k)=1. \tag{3.12}$$

Straightforward application of (3.8) to (3.6) and (3.7) lead to the following formulas:

$$\begin{aligned}
\langle\sigma^-_n\,\sigma^+_{n+1}\rangle_\lambda &= \frac{1}{\mathcal{T}_{\vec\lambda}(\xi_n|\vec\xi)\,\mathcal{T}_{\vec\lambda}(\xi_{n+1}|\vec\xi)}\left\{\sum_{a=1}^{m}\mathfrak{f}^{(0)}_m(\lambda_a|\xi_n)\,\mathfrak{f}^{(1)}_m(\lambda_a|\xi_n)\,\frac{\langle\vec\mu^{(a)}(\xi_{n+1})|\vec\lambda\rangle}{\langle\vec\lambda|\vec\lambda\rangle}+\right. \qquad (3.13)\\
&\left.+\sum_{a=1}^{m}\mathfrak{f}^{(0)}_m(\lambda_a|\xi_n)\sum_{\substack{b=1\\b\neq a}}^{m}\mathfrak{f}^{(2)}_m(\lambda_a,\lambda_b|\xi_n)\,\frac{\langle\vec\mu^{(a,b)}(\xi_n,\xi_{n+1})|\vec\lambda\rangle}{\langle\vec\lambda|\vec\lambda\rangle}\right\},\\
\langle\sigma^+_n\,\sigma^-_{n+1}\rangle_\lambda &= \frac{1}{\mathcal{T}_{\vec\lambda}(\xi_n|\vec\xi)\mathcal{T}_{\vec\lambda}(\xi_{n+1}|\vec\xi)}\left\{\sum_{a=1}^{m}\mathfrak{f}^{(0)}_{m+1}(\lambda_a|\xi_{n+1})\sum_{\substack{b=1\\b\neq a}}^{m}\mathfrak{f}^{(2)}_{m+1}(\lambda_a,\lambda_b|\xi_{n+1})\,\frac{\langle\vec\mu^{(a,b)}(\xi_n,\xi_{n+1})|\vec\lambda\rangle}{\langle\vec\lambda|\vec\lambda\rangle}+\right. \qquad (3.14)\\
&\left.+\sum_{a=1}^{m}\mathfrak{f}^{(0)}_{m+1}(\lambda_a|\xi_{n+1})\left[\mathfrak{f}^{(1)}_{m+1}(\lambda_a|\xi_{n+1})\frac{\langle\vec\mu^{(a)}(\xi_n)|\vec\lambda\rangle}{\langle\vec\lambda|\vec\lambda\rangle}+\mathfrak{f}^{(2)}_{m+1}(\lambda_a,\xi_n|\xi_{n+1})\frac{\langle\vec\mu^{(a)}(\xi_{n+1})|\vec\lambda\rangle}{\langle\vec\lambda|\vec\lambda\rangle}\right]\right\},
\end{aligned}$$

where in $\mathfrak{f}^{(0,1,2)}_{m+1}$ defined in (3.9) with $\lambda_{m+1}\equiv\xi_n$, and we introduced the following notation for the states entering the scalar products in (3.13) and (3.14):

- $|\vec\lambda\rangle$ denotes a Bethe-eigenstate (3.1) characterized by m Bethe-roots.

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- $|\vec{\mu}^{(a)}(\xi)\rangle$ denotes a Bethe-state, the difference of which from $|\vec{\lambda}\rangle$ is that a single $\lambda_a \to \xi$ replacement should be done in (3.1).
- $|\vec{\mu}^{(a,b)}(\xi,\xi')\rangle$ denotes a Bethe-state, which differs from $|\vec{\lambda}\rangle$ by the $\lambda_a \to \xi$ and $\lambda_b \to \xi'$ replacements in (3.1).

We note that in (3.13) and (3.14), we exploited that $\mathfrak{f}_M^{(1)}(\xi_n|\xi_{n'}) = 0$ for any values of M and of $n, n' \in \{1, \dots, N\}$.

Formulas in (3.13) and (3.14) imply, that in order to carry out the computation of the necessary special 2-point functions, one needs to know the scalar product of a Bethe-state and a Bethe-eigenstate. This is given by Slavnov's determinant formula [59]. Let $|\vec{\mu}\rangle$ an arbitrary Bethe-state in the sense of (3.1) and $|\vec{\lambda}\rangle$ be a Bethe-eigenstate. Then their scalar product can be determined with the help of the formula [59]:

$$\langle\vec{\mu}|\vec{\lambda}\rangle = \langle\vec{\lambda}|\vec{\mu}\rangle = \prod_{l=1}^{m} \frac{1}{r(\mu_l)} \cdot \frac{\det H(\vec{\mu}|\vec{\lambda})}{\prod\limits_{j>k} \sinh(\mu_k - \mu_j)\, \sinh(\lambda_j - \lambda_k)}, \tag{3.15}$$

where $H(\vec{\mu}|\vec{\lambda})$ is an $m \times m$ matrix with entries:

$$H_{ab}(\vec{\mu}|\vec{\lambda}) = \frac{\sinh(-i\,\gamma)}{\sinh(\lambda_a - \mu_b)} \left(r(\mu_b)\, \frac{\prod\limits_{k=1}^{m} \sinh(\lambda_k - \mu_b - i\,\gamma)}{\sinh(\lambda_a - \mu_b - i\,\gamma)} - \frac{\prod\limits_{k=1}^{m} \sinh(\lambda_k - \mu_b + i\,\gamma)}{\sinh(\lambda_a - \mu_b + i\,\gamma)} \right). \tag{3.16}$$

An important special case of (3.15), when the scalar product of two identical Bethe-eigenstates are considered. This is given by the Gaudin formula [35–37]:

$$\langle\vec{\lambda}|\vec{\lambda}\rangle = \frac{\prod\limits_{j=1}^{m} \prod\limits_{k=1}^{m} \sinh(\lambda_j - \lambda_k - i\,\gamma)}{\prod\limits_{j>k} \sinh(\lambda_k - \lambda_j)\, \sinh(\lambda_j - \lambda_k)} \cdot \det \Phi(\vec{\lambda}), \tag{3.17}$$

where $\Phi(\vec{\lambda})$ is the Gaudin-matrix, which is related to the counting-function (2.23) by the formula:

$$\Phi_{ab}(\vec{\lambda}) = -i\, \frac{\partial}{\partial \lambda_b}\, Z_\lambda(\lambda_a|\vec{\lambda}), \qquad a, b = 1, \dots, m. \tag{3.18}$$

Using the actual form of the counting function (2.23), from (3.18) one obtains the following form for the matrix elements of $\Phi(\vec{\lambda})$:

$$\Phi_{ab}(\vec{\lambda}) = -i\, Z'_\lambda(\lambda_a)\, \delta_{ab} - 2\pi\, i\, K(\lambda_a - \lambda_b|\gamma), \qquad a, b = 1, \dots m, \tag{3.19}$$

where

$$K(\lambda|\gamma) = \frac{1}{2\pi} \frac{\sin(2\,\gamma)}{\sinh(\lambda - i\,\gamma)\, \sinh(\lambda + i\,\gamma)}. \tag{3.20}$$

As it can be seen from (3.13) and (3.14), during the computation of the special 2-point functions considered in this work, such scalar products arise, in which the components of the vector $\vec{\mu}$ take values either from the set of Bethe-roots $\{\lambda_j\}_{j=1,\dots m}$ or from the set of inhomogeneities $\{\xi_k\}_{k=1,\dots N}$ of the model. In these cases the matrix elements of $H(\vec{\mu}|\vec{\lambda})$ remarkably simplify:

$$H_{ab}(\vec{\mu}|\vec{\lambda})\big|_{\mu_b\to\lambda_c} = (-1)^{m-1}\prod_{j=1}^{m}\sinh(\lambda_c-\lambda_j-i\gamma)\,\Phi_{ac}(\vec{\lambda}), \quad a,b,c=1,\dots,m. \tag{3.21}$$

$$\frac{1}{r(\mu_b)}H_{ab}(\vec{\mu}|\vec{\lambda})\big|_{\mu_b\to\xi_c} = \frac{(-1)^m\sinh(-i\gamma)\prod\limits_{j=1}^{m}\sinh(\xi_c-\lambda_j+i\gamma)}{\sinh(\lambda_a-\xi_c)\sinh(\lambda_a-\xi_c-i\gamma)}, \quad a,b=1,\dots,m, \quad c=1,\dots,N. \tag{3.22}$$

These simplifications allow[10] one to compute a typical scalar product arising in the computation of diagonal form factors:

$$\frac{\langle\vec{\mu}^{(a_1,\dots,a_K)}(\xi_{\alpha_1},\dots,\xi_{\alpha_K})|\vec{\lambda}\rangle}{\langle\vec{\lambda}|\vec{\lambda}\rangle} = \det Y\prod_{k=1}^{K}\prod_{j=1}^{m}\frac{\sinh(\xi_{\alpha_k}-\lambda_j+i\gamma)}{\sinh(\lambda_{a_k}-\lambda_j-i\gamma)}\prod_{k>j}^{K}\frac{\sinh(\lambda_{a_k}-\lambda_{a_j})}{\sinh(\xi_{\alpha_k}-\xi_{\alpha_j})}\times$$
$$\times\prod_{k=1}^{K}\prod_{\substack{j=1\\ j\neq a_1,\dots a_K}}^{m}\frac{\sinh(\lambda_{a_k}-\lambda_j)}{\sinh(\xi_{\alpha_k}-\xi_{\alpha_j})}, \qquad K\leq m, \tag{3.23}$$

where $\langle\vec{\mu}^{(a_1,\dots,a_K)}(\xi_{\alpha_1},\dots,\xi_{\alpha_K})|$ denotes a state, which is obtained from $\langle\vec{\lambda}| = \langle\lambda_1,\dots\lambda_m|$ by replacing K pieces of λ_j to certain inhomogeneities of the lattice model:

$$\lambda_{a_k}\to\xi_{\alpha_k}, \qquad a_k\in\{1,\dots,m\}, \qquad \alpha_k\in\{1,\dots,N\}, \qquad 1\leq k\leq K\leq m, \tag{3.24}$$

such that both sets $\{a_k\}$ and $\{\alpha_k\}$ contain distinct numbers. In (3.23) Y denotes a $K\times K$ matrix with entries as follows:

$$Y_{ij} = r(\lambda_{a_i})\,X_{a_i}(\xi_{\alpha_j}), \qquad i,j\in\{1,\dots,K\}, \tag{3.25}$$

where the m-component vector $X_b(\xi)$ is the solution of a set of linear equations:

$$\sum_{b=1}^{m}\Phi_{ab}(\vec{\lambda})\,X_b(\xi) = \mathcal{V}_a(\xi), \qquad a=1,\dots,m, \tag{3.26}$$

with

$$\mathcal{V}_a(\xi) = \frac{-\sinh(-i\gamma)}{\sinh(\lambda_a-\xi)\,\sinh(\lambda_a-\xi-i\gamma)}, \qquad a\in\{1,\dots,m\}. \tag{3.27}$$

In subsection 4.1 of [17] it has been shown, that the discrete linear problem (3.26) can be transformed into a set of linear integral equations. Latter formulation proves to be very convenient, when the continuum limit is taken. For the paper to be self-contained, we recall the derivation of the transformation of (3.26) into linear integral equations.

The actual forms (3.19) of the Gaudin-matrix and the source vector (3.27) suggest, the following Ansatz for the m-component solution vector $X_a(\xi)$ of the linear equations (3.26):

$$X_a(\xi) = X(\lambda_a|\xi), \qquad a=1,\dots,m, \tag{3.28}$$

[10]The main technical steps of the computations are the same as those given in sections 4. of [40] and of [17].

where $X(\lambda|\xi)$ is supposed to be a meromorphic function in λ on the complex plane, such that it is analytic in a small neighborhood of the real axis. Thus our task is the determine the functional form of $X(\lambda|\xi)$. Then using (3.28), (3.19) and (3.20) the linear equations (3.26) take the form:

$$-i\,Z'_\lambda(\lambda_a)\,X(\lambda_a|\xi)-2\pi\,i\sum_{b=1}^{m}K(\lambda_a-\lambda_b|\gamma)\,X(\lambda_b|\xi)=2\pi\,i\,K\left(\lambda_a-\rho(\xi)|\frac{\gamma}{2}\right),\quad a=1,\dots,m, \tag{3.29}$$

where $\rho(\xi)=\xi+i\frac{\gamma}{2}$. In our computations ξ takes value from the set of inhomogeneities of the vertex-model. The actual choice for the inhomogeneities given by (2.16) and (2.17) imply that we can restrict our investigations to the case, when $\rho(\xi)\in\mathbb{R}$. Thus, in the sequel we will assume $\rho(\xi)$ to be real.

To transform (3.29) into integral equations one needs to use the lemma as follows [25, 27]:

Lemma. *Let $\{\lambda_j\}_{j=1,\dots,m}$ solutions of the Bethe-equations (2.21) and let $f(\lambda)$ a meromorphic function, which is continuous and bounded on the real axis. Denote $p^{(f)}$ its pole located the closest to the real axis. Then for $|Im\,\mu|<|Im\,p^{(f)}|$ the following equation holds:*

$$\begin{aligned}\sum_{j=1}^{m}f(\mu-\lambda_j)&=\sum_{j=1}^{m_C}f(\mu-c_j)-\sum_{j=1}^{m_H}f(\mu-h_j)+\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,f(\mu-\lambda)\,Z'_\lambda(\lambda)\\&\quad-\sum_{\alpha=\pm}\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}f(\mu-\lambda-i\,\alpha\,\eta)\,Z'_\lambda(\lambda+i\,\alpha\,\eta)\,\mathcal{F}^{(\lambda)}_\alpha(\lambda+i\,\alpha\,\eta),\end{aligned} \tag{3.30}$$

where h_j and c_j denote the positions of holes and complex Bethe-roots, respectively and $\mathcal{F}^{(\lambda)}_\pm(\lambda)$ is given by

$$\mathcal{F}^{(\lambda)}_\pm(\lambda)=\frac{(-1)^\delta\,e^{\pm i\,Z_\lambda(\lambda)}}{1+(-1)^\delta\,e^{\pm i\,Z_\lambda(\lambda)}}, \tag{3.31}$$

η is a small positive contour-integral parameter which should satisfy the inequalities:

$$0<\eta<min\{|Im\,p^\pm_\lambda|\},\qquad |Im\,\mu\pm\eta|<|Im\,p^{(f)}|, \tag{3.32}$$

where $p^\pm_\lambda$ denotes those complex poles of $\mathcal{F}^{(\lambda)}_\pm(\lambda)$, which are located the closest to the real axis.

The validity of formula (3.30) in μ can be extended to the whole complex plane by an appropriate analytical continuation method [27].

Then one has to apply (3.30) to the discrete sum arising in (3.29). From the actual form of (3.30) one can recognize, that under the integrations a factor $Z'_\lambda(\lambda)$ always arises. To eliminate this factor from the equations, it is worth to formulate the integral form of the discrete set of equations (3.29) in terms of the function:

$$\mathcal{G}(\lambda|\xi)=Z'_\lambda(\lambda)\,X(\lambda|\xi). \tag{3.33}$$

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In the language of this function, the discrete set of linear equations (3.29) take the form of a set of linear integral equations as follows:

$$
\begin{aligned}
&\mathcal{G}(\lambda|\xi) + \int_{-\infty}^{\infty} d\lambda'\, K(\lambda-\lambda'|\gamma)\,\mathcal{G}(\lambda'|\xi) - \\
&-\sum_{\alpha=\pm}\int_{-\infty}^{\infty} d\lambda'\, K(\lambda-\lambda'-i\,\alpha\,\eta|\gamma)\,\mathcal{G}(\lambda'+i\,\alpha\,\eta|\xi)\,\mathcal{F}_{\alpha}^{(\lambda)}(\lambda'+i\,\alpha\,\eta) = \\
&\qquad = -2\pi\, K\left(\lambda-\rho(\xi)|\frac{\gamma}{2}\right) + \sum_{j=1}^{m_H} K(\lambda-h_j|\gamma)\, X(h_j|\xi),
\end{aligned}
\tag{3.34}
$$

where as a consequence of (3.33) the "discrete degrees of freedom" satisfy the equations:

$$
X(h_j|\xi) = \frac{\mathcal{G}(h_j|\xi)}{Z'_{\lambda}(h_j)}, \qquad j=1,\ldots,m_H. \tag{3.35}
$$

To be more precise (3.29) and (3.30) implies that (3.34) holds only at the positions $\{\lambda_a\}_{a=1\ldots m}$. For the pure hole states of our interest all Bethe-roots are real, $\lambda_a \in \mathbb{R}$, and we need the functional form of $\mathcal{G}(\lambda|\xi)$ or equivalently of $X(\lambda|\xi)$ to solve (3.29). It follows that if (3.34) is fulfilled everywhere in an appropriate neighborhood of the real axis, then it will be satisfied at the discrete points $\{\lambda_a\}_{a=1\ldots m}$, as well. Thus, we require $\mathcal{G}(\lambda|\xi)$ to be the solution of the set of linear integral equations (3.34). Finally acting[11] on (3.34) with the inverse of the integral operator[12] $1+K$, one obtains the final form [17] of the linear integral equations satisfied by $\mathcal{G}(\lambda|\xi)$, when only pure hole excitations above the anti-ferromagnetic vacuum are considered:

$$
\begin{aligned}
&\mathcal{G}(\lambda|\xi) - \sum_{\alpha=\pm}\int_{-\infty}^{\infty} d\lambda'\, G_{\lambda}(\lambda-\lambda'-i\,\alpha\,\eta)\,\mathcal{G}(\lambda'+i\,\alpha\,\eta|\xi)\,\mathcal{F}_{\alpha}^{(\lambda)}(\lambda'+i\,\alpha\,\eta) = \\
&\qquad = S_0(\lambda|\xi) + \sum_{j=1}^{m_H} 2\pi\, G_{\lambda}(\lambda-h_j)\, X(h_j|\xi),
\end{aligned}
\tag{3.36}
$$

with

$$
S_0(\lambda|\xi) = -\frac{\pi}{\gamma}\,\frac{1}{\cosh\left(\frac{\pi}{\gamma}(\lambda-\rho(\xi))\right)}, \qquad \rho(\xi) = \xi + i\,\frac{\gamma}{2} \in \mathbb{R}, \tag{3.37}
$$

where h_js denote the positions of the holes, η is a small positive contour-integral parameter, and $G_{\lambda}(\lambda)$ is related to the kernel of NLIE equations (2.29) by:

$$
G_{\lambda}(\lambda) = \frac{1}{2\gamma}\, G\left(\frac{\pi}{\gamma}\lambda\right), \quad \text{with} \quad \gamma = \frac{\pi}{p+1}. \tag{3.38}
$$

[11] This action is necessary in order for the final equations to have well defined continuum limit.

[12] To be more precise the kernel of the integral operator in "lambda" space is given by $\delta(\lambda-\lambda')+K(\lambda-\lambda'|\gamma)$ with $\delta(\lambda)$ being the Dirac-delta function.

Now, we can return to the computation of the 2-point functions of our interest. Inserting (2.22) and (3.23) into (3.13) and (3.14), after some simplification one ends up with the formulas as follows:

$$\langle \sigma_n^- \sigma_{n+1}^+ \rangle_\lambda = -\sum_{a=1}^m \frac{\sinh(\lambda_a - \xi_{n+1})}{\sinh(\lambda_a - \xi_n - i\gamma)} X(\lambda_a|\xi_{n+1}) + \sum_{a,b=1}^m \frac{\sinh(\lambda_a - \xi_{n+1}) \sinh(\lambda_b - \xi_{n+1})}{\sinh(\lambda_a - \lambda_b - i\gamma)} \times$$
$$\times \frac{X(\lambda_a|\xi_n) X(\lambda_b|\xi_{n+1}) - X(\lambda_a|\xi_{n+1}) X(\lambda_b|\xi_n)}{\sinh(\xi_n - \xi_{n+1})}, \tag{3.39}$$

$$\langle \sigma_n^+ \sigma_{n+1}^- \rangle_\lambda = -\frac{\sinh(\xi_n - \xi_{n+1} - i\gamma)}{\sinh(\xi_n - \xi_{n+1})} \sum_{a=1}^m \frac{\sinh(\lambda_a - \xi_n - i\gamma)}{\sinh(\lambda_a - \xi_{n+1} - i\gamma)} X(\lambda_a|\xi_n)$$
$$+ \sum_{a,b=1}^m \frac{\sinh(\lambda_a - \xi_n - i\gamma) \sinh(\lambda_b - \xi_n + i\gamma)}{\sinh(\lambda_a - \lambda_b - i\gamma)} \frac{X(\lambda_a|\xi_n) X(\lambda_b|\xi_{n+1}) - X(\lambda_a|\xi_{n+1}) X(\lambda_b|\xi_n)}{\sinh(\xi_n - \xi_{n+1})}$$
$$+ \frac{\sinh(-i\gamma)}{\sinh(\xi_n - \xi_{n+1})} \sum_{a=1}^m X(\lambda_a|\xi_{n+1}). \tag{3.40}$$

In (3.39) and (3.40) we preserved the determinant structure implied by (3.23). Nevertheless for future computations it is better to shift the anti-symmetrization to the coefficient of the quadratic expression of X. Then the quadratic in X parts of (3.39) and (3.40) can be written as follows:

$$\langle \sigma_n^- \sigma_{n+1}^+ \rangle_\lambda^{\text{quad}} = \frac{1}{\sinh(\xi_n - \xi_{n+1})} \sum_{a,b=1}^m \mathfrak{f}(\lambda_a, \lambda_b|\xi_{n+1}) X(\lambda_a|\xi_n) X(\lambda_b|\xi_{n+1}), \tag{3.41}$$

$$\langle \sigma_n^+ \sigma_{n+1}^- \rangle_\lambda^{\text{quad}} = \frac{1}{\sinh(\xi_n - \xi_{n+1})} \sum_{a,b=1}^m \mathfrak{f}(\lambda_a, \lambda_b|\xi_n) X(\lambda_a|\xi_n) X(\lambda_b|\xi_{n+1}), \tag{3.42}$$

where $\mathfrak{f}(\lambda, \lambda'|\xi)$ is an antisymmetric function given by the formula:

$$\mathfrak{f}(\lambda, \lambda'|\xi) = \cos(\gamma) \frac{\sinh(2(\lambda - \xi)) - \sinh(2(\lambda' - \xi)) - \sinh(2(\lambda - \lambda'))}{\cosh(2(\lambda - \lambda')) - \cos(2\gamma)}. \tag{3.43}$$

In (3.41) and (3.42) the coefficient function $\mathfrak{f}$ has better large λ and λ' asymptotics, than the coefficients of the quadratic terms of (3.39) and (3.40). This property proves to be very useful, when the discrete sums in (3.39) and (3.40) are transformed into integral expressions.

The typical sums arising in (3.39), (3.40), (3.41) and (3.42) are of the form:

$$\Sigma_\lambda^{(1)}[f](\xi) = \sum_{a=1}^m f(\lambda_a) X(\lambda_a|\xi), \tag{3.44}$$

$$\Sigma_\lambda^{(2)}[f](\xi, \xi') = \sum_{a,b=1}^m f(\lambda_a, \lambda_b) X(\lambda_a|\xi) X(\lambda_b|\xi'), \tag{3.45}$$

where in (3.45) $f(\lambda, \lambda')$ is meant to be an antisymmetric function. Formulas (3.44) and (3.45) are not appropriate to take the continuum limit, this is why it is worth to transform these sums into integral expressions with the help of lemma (3.30). The transformation procedure together with the integral representations of (3.44) and (3.45) can be found in appendix A. The final integral representations for (3.44) and (3.45) are given by (A.5) and (A.9) together with the related definitions.

To describe the scalar and pseudo-scalar fermion bilinears, it is worth to introduce the lattice operators O^+_{2n} and O^-_{2n} with the definitions:

$$O^{\pm}_{2n} = \sigma^-_{2n}\,\sigma^+_{2n+1} \pm \sigma^+_{2n}\,\sigma^-_{2n+1}. \tag{3.46}$$

According to (2.51) and (2.52) they correspond to the lattice counterparts of the bare fermion bilinears $\bar\Psi\Psi$ and $\bar\Psi\gamma^5\Psi$, respectively. Using the formulas (3.39), (3.40), (3.41), (3.42), (3.44) and (3.45) the following formal representation can be given for the expectation values of $O^{\pm}_{2n}$:

$$\begin{aligned}\langle O^{\pm}_{2n}\rangle_\lambda = -\Sigma^{(1)}_\lambda[f_3](\xi_-) \pm \frac{\sinh(-i\,\gamma)}{\sinh(2\rho_0)}\,\Sigma^{(1)}_\lambda[f_1](\xi_-) \mp \frac{\sinh(2\rho_0 - i\,\gamma)}{\sinh(2\rho_0)}\,\Sigma^{(1)}_\lambda[f_2](\xi_+)\\ + \frac{1}{\sinh(2\rho_0)}\Sigma^{(2)}_\lambda[f_\pm](\xi_+,\xi_-), \qquad\qquad \xi_\pm = \pm\rho_0 - i\,\frac{\gamma}{2},\end{aligned} \tag{3.47}$$

where the functions f_1, f_2, f_3 and $f_\pm$ are of the form:

$$f_1(\lambda) = 1, \tag{3.48}$$

$$f_2(\lambda) = \frac{\sinh(\lambda - \rho_0 - i\,\frac{\gamma}{2})}{\sinh(\lambda + \rho_0 - i\,\frac{\gamma}{2})}, \tag{3.49}$$

$$f_3(\lambda) = \frac{\sinh(\lambda + \rho_0 + i\,\frac{\gamma}{2})}{\sinh(\lambda - \rho_0 - i\,\frac{\gamma}{2})}, \tag{3.50}$$

$$f_+(\lambda,\lambda') = 2\cos(\gamma)\,\frac{\cosh(2\rho_0)\,[\sinh(2\lambda + i\,\gamma) - \sinh(2\lambda' + i\,\gamma)] - \sinh(2(\lambda-\lambda'))}{\cosh(2(\lambda-\lambda')) - \cos(2\gamma)}, \tag{3.51}$$

$$f_-(\lambda,\lambda') = 2\cos(\gamma)\,\sinh(2\rho_0)\frac{\cosh(2\lambda + i\,\gamma) - \cosh(2\lambda' + i\,\gamma)}{\cosh(2(\lambda-\lambda')) - \cos(2\gamma)}, \tag{3.52}$$

and we exploited the concrete inhomogeneity structure of the model, namely that $\xi_{2n} = \rho_0 - i\,\frac{\gamma}{2}$ and $\xi_{2n+1} = -\rho_0 - i\,\frac{\gamma}{2}$, with ρ_0 given by (2.17).

Using the integral representations (A.5) and (A.9) for $\Sigma^{(1)}_\lambda[f](\xi)$ and $\Sigma^{(2)}_\lambda[f](\xi,\xi')$, respectively, the formula (3.47) can be rephrased as follows:

$$\langle O^{\pm}_{2n}\rangle_\lambda = \mathcal{O}^{\pm}_0 + \mathcal{O}^{\pm}_X + \mathcal{O}^{\pm}_{\mathcal{G}} + \mathcal{O}^{\pm}_{XX} + \mathcal{O}^{\pm}_{\mathcal{G}\mathcal{G}} + \mathcal{O}^{\pm}_{X\mathcal{G}}, \tag{3.53}$$

where

$$\mathcal{O}^{\pm}_0 = -\mathcal{J}_0[f_3](\xi_-) \pm \frac{\sinh(-i\,\gamma)}{\sinh(2\rho_0)}\,\mathcal{J}_0[f_1](\xi_-) \mp \frac{\sinh(2\rho_0 - i\,\gamma)}{\sinh(2\rho_0)}\,\mathcal{J}_0[f_2](\xi_+), \tag{3.54}$$

$$\begin{aligned}\mathcal{O}^{\pm}_X = -\mathcal{S}_X[f_3](\xi_-) \pm \frac{\sinh(-i\,\gamma)}{\sinh(2\rho_0)}\,\mathcal{S}_X[f_1](\xi_-) \mp \frac{\sinh(2\rho_0 - i\,\gamma)}{\sinh(2\rho_0)}\,\mathcal{S}_X[f_2](\xi_+)\\ + \frac{\Sigma_X[f_\pm](\xi_+,\xi_-)}{\sinh(2\rho_0)},\end{aligned} \tag{3.55}$$

$$\begin{aligned}\mathcal{O}^{\pm}_{\mathcal{G}} = -\mathcal{S}_{\mathcal{G}}[f_3](\xi_-) \pm \frac{\sinh(-i\,\gamma)}{\sinh(2\rho_0)}\,\mathcal{S}_{\mathcal{G}}[f_1](\xi_-) \mp \frac{\sinh(2\rho_0 - i\,\gamma)}{\sinh(2\rho_0)}\,\mathcal{S}_{\mathcal{G}}[f_2](\xi_+)\\ + \frac{\Sigma_{\mathcal{G}}[f_\pm](\xi_+,\xi_-)}{\sinh(2\rho_0)},\end{aligned} \tag{3.56}$$

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$$\mathcal{O}^{\pm}_{XX} = \frac{\Sigma_{XX}[f_\pm](\xi_+,\xi_-)}{\sinh(2\rho_0)}, \tag{3.57}$$

$$\mathcal{O}^{\pm}_{X\mathcal{G}} = \frac{\Sigma_{X\mathcal{G}}[f_\pm](\xi_+,\xi_-)}{\sinh(2\rho_0)}, \tag{3.58}$$

$$\mathcal{O}^{\pm}_{\mathcal{G}\mathcal{G}} = \frac{\Sigma_{\mathcal{G}\mathcal{G}}[f_\pm](\xi_+,\xi_-)}{\sinh(2\rho_0)}, \tag{3.59}$$

such that the functionals $\mathcal{J}_0$, $\mathcal{S}_X$, $\mathcal{S}_\mathcal{G}$, Σ_0, Σ_X, $\Sigma_\mathcal{G}$, $\Sigma_{X\mathcal{G}}$, Σ_{XX} and $\Sigma_{\mathcal{G}\mathcal{G}}$, are given by the formulas (A.8), (A.6), (A.7), (A.10), (A.11), (A.12), (A.13), (A.14), (A.15), respectively.

The lower index of the terms in the right hand side of (3.53) carry information about how these terms depend on the dynamical variables $X(\lambda|\xi)$ and $\mathcal{G}(\lambda|\xi)$. Namely:

- $\mathcal{O}^{\pm}_0$ stands for the "bulk" term, which is independent of X and $\mathcal{G}$,
- $\mathcal{O}^{\pm}_X$ linear in X and independent of $\mathcal{G}$,
- $\mathcal{O}^{\pm}_\mathcal{G}$ is linear in $\mathcal{G}$ and independent of X,
- $\mathcal{O}^{\pm}_{X\mathcal{G}}$ is linear in both $\mathcal{G}$ and X,
- $\mathcal{O}^{\pm}_{XX}$ is quadratic in X and independent of $\mathcal{G}$, and finally
- $\mathcal{O}^{\pm}_{\mathcal{G}\mathcal{G}}$ is quadratic in $\mathcal{G}$ and independent of X.

Formula (3.53) together with (3.54)–(3.59) and the integral representations (A.6)–(A.21) given in appendix A constitutes our final result for the lattice expectation values of the scalar and pseudo-scalar fermion bilinears. To get the expectation values of these operators in the continuum theory, the lattice formula (3.53) should be evaluated in the continuum limit. This will be discussed in the next section.

4 Continuum limit

In this section the expectation value formulas (3.53) are evaluated at the continuum limit. This task reduces to the evaluation of the sums and integrals entering (3.53) in the large ρ_0 limit.[13] The ρ_0 dependence of these terms is determined by the ρ_0 dependence of the functions $f_1, f_2, f_3, f_\pm$ given in (3.48)–(3.52), and the ρ_0 dependence of $X(h_j|\xi_\pm)$ and $\mathcal{G}(\lambda|\xi_\pm)$. Latter is governed by the linear integral equation (3.36). First, it is worth to discuss the continuum limit of the variables $X(h_j|\xi_\pm)$ and $\mathcal{G}(\lambda|\xi_\pm)$. They are solutions of the set of equations (3.35)–(3.37). The continuum limit means, that one has to take the number of lattice sites N to infinity, such that the inhomogeneity parameter ρ_0 is tuned with N according to the formula (2.17). This means, that in the continuum limit procedure ρ_0 also tends to infinity, but logarithmically in N or a.

[13] In the rest of the paper, the terms "large ρ_0 limit" and "large N limit" will be equivalently used for the continuum limit procedure.

In this limit the counting-function $Z_\lambda(\lambda)$ (2.23)and its nonlinear combinations $\mathcal{F}_\pm^{(\lambda)}(\lambda)$ (3.31) tend to their (finite) continuum counterparts:

$$\begin{aligned} Z_\lambda(\lambda) &\to Z_{\lambda,c}(\lambda) = Z\left(\frac{\pi}{\gamma}\lambda\right), \\ \mathcal{F}_\pm^{(\lambda)}(\lambda) &\to \mathcal{F}_{\pm,c}^{(\lambda)}(\lambda) = \mathcal{F}_\pm\left(\frac{\pi}{\gamma}\lambda\right), \end{aligned} \tag{4.1}$$

where $Z(\theta)$ is solution of the continuum NLIE (2.34) and $\mathcal{F}_\pm(\theta)$ is given by (2.43). This implies that in the leading order in $\frac{1}{N}$ computations $Z_\lambda(\lambda)$ and $\mathcal{F}_\pm^{(\lambda)}(\lambda)$ can be replaced by their continuum counterparts given by (4.1).

Then equation (3.36) implies, that the large N limit of $\mathcal{G}(\lambda|\xi_\pm)$ is governed by the large ρ_0 expansion of $S_0(\lambda|\xi_\pm)$:

$$S_0(\lambda|\xi_\pm) = -\frac{2\pi}{\gamma}\, e^{\pm\frac{\pi}{\gamma}\lambda}\, e^{-\frac{\pi}{\gamma}\rho_0} + O(e^{-2\,\frac{\pi}{\gamma}\rho_0}). \tag{4.2}$$

According to (2.17) $e^{-\frac{\pi}{\gamma}\rho_0} \sim \frac{1}{N} \sim a$. Since apart from S_0, all terms in the equations (3.35)–(3.37) are proportional to $\mathcal{G}(\lambda|\xi)$ and $X(h_j|\xi)$, (4.2) implies that:

$$\begin{aligned} \mathcal{G}(\lambda|\xi_\pm) &\sim \frac{1}{N} + O\left(\frac{1}{N^2}\right) \sim a + O(a^2), \\ X(\lambda|\xi_\pm) &\sim \frac{1}{N} + O\left(\frac{1}{N^2}\right) \sim a + O(a^2). \end{aligned} \tag{4.3}$$

With the help of (4.2) and (4.3), one can immediately give a rough estimate for the large N magnitude of the different terms arising in the right hand side of (3.53). This is implied by their S_0, X, and $\mathcal{G}$ content:

$$\begin{aligned} \mathcal{O}_0^\pm &\sim S_0 \sim \frac{1}{N} \sim e^{-(1+p)\rho_0} & \mathcal{O}_{XX}^\pm &\sim X\,X \sim \frac{1}{N^2} \sim e^{-2(1+p)\rho_0} \\ \mathcal{O}_X^\pm &\sim X \sim \frac{1}{N} \sim e^{-(1+p)\rho_0} & \mathcal{O}_{X\mathcal{G}}^\pm &\sim X\,\mathcal{G} \sim \frac{1}{N^2} \sim e^{-2(1+p)\rho_0} \\ \mathcal{O}_\mathcal{G}^\pm &\sim \mathcal{G} \sim \frac{1}{N} \sim e^{-(1+p)\rho_0} & \mathcal{O}_{\mathcal{G}\mathcal{G}}^\pm &\sim \mathcal{G}\,\mathcal{G} \sim \frac{1}{N^2} \sim e^{-2(1+p)\rho_0}. \end{aligned} \tag{4.4}$$

This rough estimate implies, that in the continuum limit the terms in (3.53) being quadratic or multilinear in $\mathcal{G}$ and X, (i.e. $O_{XX}^\pm, \mathcal{O}_{\mathcal{G}\mathcal{G}}^\pm, \mathcal{O}_{X\mathcal{G}}^\pm$) become negligible with respect to the constant ($\mathcal{O}_0^\pm$) and linear terms ($\mathcal{O}_\mathcal{G}^\pm, \mathcal{O}_X^\pm$). Thus only the constant and purely linear terms determine the leading order behavior of $\langle O_{2n}^\pm\rangle_\lambda$ in the large[14] N limit.

Though we would like to emphasize, that (4.4) is only a rough and not the exact estimate for the large N behavior for the quantities entering the r.h.s. of (3.53). Its purpose is to give a fast intuitive argument, why the multilinear and quadratic in X and $\mathcal{G}$ terms become negligible in the continuum limit.

The rough estimate (4.4) was derived by neglecting the ρ_0 dependence of the functions $f_1, f_2, f_3, f_\pm$ given by (3.48)–(3.52). For presentational purposes we anticipate the exact

[14] By large N limit, we mean the continuum limit procedure, which means that we consider the $N \to \infty$ limit, such that at the same time ρ_0 is also tuned with N according to the formula (2.17).

result. The careful computations presented in the rest of this section and in appendix B show, that the exact leading order large N or equivalently ρ_0 behavior of the sums and integrals entering (3.53) is given by the formula:

$$\begin{aligned}\mathcal{O}_0^\pm \sim \mathcal{O}_X^\pm \sim \mathcal{O}_\mathcal{G}^\pm \sim\sim e^{-(1+p)\rho_0} \sim e^{(1-p)\rho_0} \sim e^{-2\,p\,\rho_0},\\ \mathcal{O}_{\mathcal{G}\mathcal{G}}^\pm \sim \mathcal{O}_{X\mathcal{G}}^\pm \sim \mathcal{O}_{XX}^\pm \sim e^{-2(1+p)\rho_0} \sim \frac{1}{N^2} \sim a^2.\end{aligned} \tag{4.5}$$

This formula also implies that multilinear and quadratic in X and $\mathcal{G}$ terms are negligible in the continuum limit. This statement is shown in appendix B.1. By comparing (4.5) and (4.4) it can also be seen that the rough estimate came from a simplified train of thoughts is exact at the $p=1$ free fermion point.

Now, our goal is to compute the leading order in N term of $\langle O_{2n}^\pm\rangle_\lambda$ in the continuum limit. To carry out this purpose, it is worth to formulate the problem in terms of the finite parts of the leading order in N terms of $\mathcal{G}(\lambda|\xi_\pm)$ and $X(h_j|\xi_\pm)$. These finite parts are defined by the following large N (or equivalently ρ_0) expansions of $\mathcal{G}(\lambda|\xi_\pm)$ and $X(h_j|\xi_\pm)$:

$$\begin{aligned}\mathcal{G}(\lambda|\xi_\pm) &= -\frac{2\pi}{\gamma}\, e^{-\frac{\pi}{\gamma}\rho_0}\, \mathcal{G}^{(\pm)}(\lambda) + O\left(\frac{1}{N^2}\right),\\ X(h_j|\xi_\pm) &= -\frac{2\pi}{\gamma}\, e^{-\frac{\pi}{\gamma}\rho_0}\, X_j^{(\pm)} + O\left(\frac{1}{N^2}\right), \qquad j=1,\dots,m_H.\end{aligned} \tag{4.6}$$

From (3.36), (3.35) and (4.1) it follows, that the finite parts $\mathcal{G}_\pm$ and $X_j^{(\pm)}$ satisfy the equations:

$$\begin{aligned}&\mathcal{G}^{(\pm)}(\lambda) - \sum_{\alpha=\pm}\int_{-\infty}^{\infty} d\lambda'\, G_\lambda(\lambda-\lambda'-i\,\alpha\,\eta)\,\mathcal{G}^{(\pm)}(\lambda'+i\,\alpha\,\eta)\,\mathcal{F}_{\alpha,c}^{(\lambda)}(\lambda'+i\,\alpha\,\eta) =\\ &= e^{\pm\frac{\pi}{\gamma}\lambda} + \sum_{j=1}^{m_H} 2\pi\, G_\lambda(\lambda-h_j)\, X_j^{(\pm)}, \qquad X_j^{(\pm)} = \frac{\mathcal{G}^{(\pm)}(h_j)}{Z'_{\lambda,c}(h_j)}, \qquad j=1,\dots,m_H.\end{aligned} \tag{4.7}$$

We note, that everywhere in (4.7) the finite continuum limit of the counting-function arises.

Now, we are in the position to determine the leading order term of the expectation value $\langle O_{2n}^\pm\rangle_\lambda$ in the large N limit. As (4.5) implies, only the first three terms from the r.h.s. of (3.53) will contribute at leading order. Namely, $\langle O_{2n}^\pm\rangle_\lambda = \mathcal{O}_0^\pm + \mathcal{O}_X^\pm + \mathcal{O}_\mathcal{G}^\pm +$ "next to leading order terms".

The term $\mathcal{O}_0^\pm$ is independent of the positions of the holes. This means that this contribution is independent of the matrix element of the operator. This is why we will call it the bulk term: $\langle O_{2n}^\pm\rangle_\lambda^{\text{bulk}} \equiv \mathcal{O}_0^\pm$. Formula (3.54) implies, that this bulk term can be represented as follows:

$$\begin{aligned}\langle O_{2n}^\pm\rangle_\lambda^{\text{bulk}} = -\mathcal{J}_0[f_3](\xi_-) \pm \frac{\sinh(-i\,\gamma)}{\sinh(2\rho_0)}\,\mathcal{J}_0[f_1](\xi_-)\\ \mp \frac{\sinh(2\rho_0-i\,\gamma)}{\sinh(2\rho_0)}\,\mathcal{J}_0[f_2](\xi_+) + \frac{\Sigma_0[f_\pm](\xi_+,\xi_-)}{\sinh(2\rho_0)},\end{aligned} \tag{4.8}$$

where $\mathcal{J}_0$ and Σ_0 are given by (A.8) and (A.10), respectively.

On the other hand, with the help of (3.55), (3.56), (A.6)–(A.8), (A.11), (A.12), (A.16) and (A.18), (A.19) the sum $\mathcal{O}_X^\pm + \mathcal{O}_{\mathcal{G}}^\pm$ can be written as follows:

$$\mathcal{O}_X^\pm + \mathcal{O}_{\mathcal{G}}^\pm = \sum_{j=1}^{m_H} \Big[C_+^{(\pm)}(h_j)\, X(h_j|\xi_+) + C_-^{(\pm)}(h_j)\, X(h_j|\xi_-) \Big] + \tag{4.9}$$
$$+ \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, \mathcal{F}_\alpha^{(\lambda)}(\lambda + i\alpha\eta) \Big[\mathcal{G}(\lambda+i\alpha\eta|\xi_+)\, C_+^{(\pm)}(\lambda+i\alpha) + \mathcal{G}(\lambda+i\alpha\eta|\xi_-)\, C_-^{(\pm)}(\lambda+i\alpha) \Big],$$

where

$$C_+^{(\pm)}(\lambda) = \mp \frac{\sinh(2\rho_0 - i\gamma)}{\sinh(2\rho_0)}\, \mathcal{J}_G[f_2](\lambda) + \frac{1}{\sinh(2\rho_0)} \left[J_S[f_\pm](\lambda|\xi_-) - J_{SG}[f_\pm](\lambda|\xi_-) \right], \tag{4.10}$$
$$C_-^{(\pm)}(\lambda) = -\mathcal{J}_G[f_3](\lambda) \pm \frac{\sinh(-i\gamma)}{\sinh(2\rho_0)}\, \mathcal{J}_G[f_1](\lambda) - \frac{1}{\sinh(2\rho_0)} \left[J_S[f_\pm](\lambda|\xi_+) - J_{SG}[f_\pm](\lambda|\xi_+) \right], \tag{4.11}$$

with $\mathcal{J}_G$, J_S and J_{SG} given in (A.8), (A.18) and (A.19), respectively. We are interested in the leading order large N expression for $\mathcal{O}_X^\pm + \mathcal{O}_{\mathcal{G}}^\pm$, this is why from (4.6) the leading order expressions of $X(h_j|\xi_\pm)$ and of $\mathcal{G}(\lambda|\xi_\pm)$ can be replaced into (4.9). Similarly, the $\mathcal{F}_\alpha^{(\lambda)}(\lambda) \to \mathcal{F}_{\alpha,c}^{(\lambda)}(\lambda)$ replacement can also be done at leading order. As a result one obtains:

$$\mathcal{O}_X^\pm + \mathcal{O}_{\mathcal{G}}^\pm = -\frac{2\pi}{\gamma}\, e^{-(1+p)\rho_0} \Bigg\{ \sum_{j=1}^{m_H} \Big[C_+^{(\pm)}(h_j)\, X_j^{(+)} + C_-^{(\pm)}(h_j)\, X_j^{(-)} \Big] + \tag{4.12}$$
$$+ \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, \mathcal{F}_{\alpha,c}^{(\lambda)}(\lambda+i\alpha\eta) \Big[\mathcal{G}^{(+)}(\lambda+i\alpha\eta)\, C_+^{(\pm)}(\lambda+i\alpha) + \mathcal{G}^{(-)}(\lambda+i\alpha\eta)\, C_-^{(\pm)}(\lambda+i\alpha) \Big] \Bigg\} + \ldots,$$

where the dots stand for subleading terms in the large N limit.

The careful evaluation of the functionals (4.8), (4.10) and (4.11) in the large ρ_0 limit, which is presented in appendix B, leads to the following large ρ_0 asymptotics for the bulk term and for $C_\pm^{(\pm)}(\lambda)$:

$$\langle O_{2n}^\pm \rangle_\lambda^{\text{bulk}} = \begin{cases} -\frac{p+1}{\sin\gamma}\, \tan(\frac{p\pi}{2})\, e^{-2p\rho_0} + O(e^{-2\rho_0}), & \text{case: } +, \\ O(e^{-2\rho_0}), & \text{case: } -\,. \end{cases} \tag{4.13}$$

$$C_+^{(\pm)}(\lambda) \overset{\rho_0 \to \infty}{=} \mp \mathcal{K}_+(\lambda|\rho_0) + \ldots, \tag{4.14}$$

$$C_-^{(\pm)}(\lambda) \overset{\rho_0 \to \infty}{=} -\mathcal{K}_-(\lambda|\rho_0) + \ldots, \tag{4.15}$$

where

$$\mathcal{K}_\pm(\lambda|\rho_0) = \frac{p+1}{2\sin\gamma}\, e^{\mp(p+1)\lambda}\, e^{(1-p)\rho_0} + O(e^{-2\rho_0}), \tag{4.16}$$

and the dots stand for terms tending to zero, when $\rho_0 \to \infty$ and $p < 1$.

Putting the results (4.8), (4.13), (4.12), (4.14) together, one obtains the following leading order result for $\langle O^{\pm}_{2n}\rangle_\lambda$ in the attractive regime:

$$\langle O^{\pm}_{2n}\rangle_\lambda = \langle O^{\pm}_{2n}\rangle^{\text{bulk}}_\lambda + \frac{2\pi}{\gamma}\, e^{-(p+1)\rho_0} \Bigg\{ \sum_{j=1}^{m_H} \mathcal{K}_-(h_j|\rho_0)\, X_j^{(-)} \pm \sum_{j=1}^{m_H} \mathcal{K}_+(h_j|\rho_0)\, X_j^{(+)} + \tag{4.17}$$
$$+ \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, \mathcal{F}^{(\lambda)}_{\alpha,c}(\lambda+i\,\alpha\,\eta) \left[\mathcal{G}^{(-)}(\lambda+i\,\alpha\,\eta)\,\mathcal{K}_-(\lambda+i\,\alpha) \pm \mathcal{G}^{(+)}(\lambda+i\,\alpha\,\eta)\,\mathcal{K}_+(\lambda+i\,\alpha)\right] \Bigg\}.$$

Here as a consequence of (4.5), the contributions coming from $\mathcal{O}^{\pm}_{XX}$, $\mathcal{O}^{\pm}_{X\mathcal{G}}$ and $\mathcal{O}^{\pm}_{\mathcal{G}\mathcal{G}}$ were neglected.

It turns out, the leading order expression (4.17) for $\langle O^{\pm}_{2n}\rangle_\lambda$ can be rephrased in terms of the variables $\mathcal{G}_d(\theta), X_j^{(d)}$ of (2.46) and $\mathcal{G}_\ell(\theta)$, $X_j^{(\ell)}$ of (2.48). The reason for this is that $\mathcal{G}^{(\pm)}(\lambda)$ and $X_j^{(\pm)}$ of (4.7) can be simply expressed in terms of $\mathcal{G}_d(\theta), X_j^{(d)}$ and $\mathcal{G}_d(\theta), X_j^{(d)}$.

Consider the following linear combinations of $\mathcal{G}^{(\pm)}(\lambda)$ and $X_j^{(\pm)}$:

$$\begin{aligned} \hat{\mathcal{G}}_d(\theta) &= \frac{\ell}{2}\left[\mathcal{G}^{(+)}\left(\frac{\gamma}{\pi}\theta\right) + \mathcal{G}^{(-)}\left(\frac{\gamma}{\pi}\theta\right)\right], \quad \hat{X}_j^{(d)} = \frac{\ell}{2}\frac{\pi}{\gamma}\left(X_j^{(+)} + X_j^{(-)}\right), \\ \hat{\mathcal{G}}_\ell(\theta) &= \frac{1}{2}\left[\mathcal{G}^{(+)}\left(\frac{\gamma}{\pi}\theta\right) - \mathcal{G}^{(-)}\left(\frac{\gamma}{\pi}\theta\right)\right], \quad \hat{X}_j^{(\ell)} = \frac{1}{2}\frac{\pi}{\gamma}\left(X_j^{(+)} - X_j^{(-)}\right). \end{aligned} \tag{4.18}$$

As a consequence of the linearity of (4.7), it can be shown that the new variables $\hat{\mathcal{G}}_d(\theta), \hat{\mathcal{G}}_\ell(\theta)$, $\hat{X}_j^{(d)}, \hat{X}_j^{(\ell)}$ satisfy the linear integral equations as follows:

$$\hat{\mathcal{G}}_d(\theta) - \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} G(\theta-\theta'-i\,\alpha\,\eta)\hat{\mathcal{G}}_d(\theta'+i\,\alpha\,\eta)\,\mathcal{F}_\alpha(\theta'+i\,\alpha\,\eta) = \tag{4.19}$$
$$= \ell\,\cosh(\theta) + \sum_{j=1}^{m_H} G(\theta-H_j)\,\hat{X}_j^{(d)}, \qquad \hat{X}_j^{(d)} = \frac{\hat{\mathcal{G}}_d(H_j)}{Z'(H_j)}, \qquad j=1,\dots,m_H.$$

$$\hat{\mathcal{G}}_\ell(\theta) - \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} G(\theta-\theta'-i\,\alpha\,\eta)\hat{\mathcal{G}}_\ell(\theta'+i\,\alpha\,\eta)\,\mathcal{F}_\alpha(\theta'+i\,\alpha\,\eta) = \tag{4.20}$$
$$= \sinh(\theta) + \sum_{j=1}^{m_H} G(\theta-H_j)\,\hat{X}_j^{(\ell)}, \qquad \hat{X}_j^{(\ell)} = \frac{\hat{\mathcal{G}}_\ell(H_j)}{Z'(H_j)}, \qquad j=1,\dots,m_H,$$

where $Z(\theta)$ and H_j are the counting function and the positions of the holes in rapidity convention. They are solutions of the equations (2.34) and (2.35).

Comparing (4.19) and (4.20) to (2.46) and (2.48) one can recognize that

$$\begin{aligned} \hat{\mathcal{G}}_d(\theta) &= \mathcal{G}_d(\theta), \quad \hat{X}_j^{(d)} = X_j^{(d)}, \quad j=1,\dots,m_H, \\ \hat{\mathcal{G}}_\ell(\theta) &= \mathcal{G}_\ell(\theta), \quad \hat{X}_j^{(\ell)} = X_j^{(\ell)}, \quad j=1,\dots,m_H. \end{aligned} \tag{4.21}$$

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Using (4.21) and substituting the inverse relation of (4.18) into (4.17) together with a change of integrating variables from λ to θ one ends up with the final result:

$$\langle O^+_{2n}\rangle_\lambda = \frac{2\,(p+1)\,e^{-2p\rho_0}}{\sin\gamma}\left\{-\frac{1}{2}\tan\left(\frac{p\pi}{2}\right)+\sum_{k=1}^{m_H}\left\{\cosh(H_k)\,\frac{X_k^{(d)}}{\ell}-\sinh(H_k)\,X_k^{(\ell)}\right\}+\right. \tag{4.22}$$

$$\left.+\sum_{\alpha=\pm}\int_{-\infty}^{\infty}d\theta\left[\cosh(\theta+i\,\alpha\,\eta)\,\frac{\mathcal{G}_d(\theta+i\,\alpha\,\eta)}{\ell}-\sinh(\theta+i\,\alpha\,\eta)\,\mathcal{G}_\ell(\theta+i\,\alpha\,\eta)\right]\mathcal{F}_\alpha(\theta+i\,\alpha\,\eta)+\ldots\right\},$$

$$\langle O^-_{2n}\rangle_\lambda = \frac{2\,(p+1)\,e^{-2p\rho_0}}{\sin\gamma}\left\{\sum_{k=1}^{m_H}\left(\sinh(H_k)\,\frac{X_k^{(d)}}{\ell}-\cosh(H_k)\,X_k^{(\ell)}\right)+\right. \tag{4.23}$$

$$\left.+\sum_{\alpha=\pm}\int_{-\infty}^{\infty}d\theta\left[\sinh(\theta+i\,\alpha\,\eta)\,\frac{\mathcal{G}_d(\theta+i\,\alpha\,\eta)}{\ell}-\cosh(\theta+i\,\alpha\,\eta)\,\mathcal{G}_\ell(\theta+i\,\alpha\,\eta)\right]\mathcal{F}_\alpha(\theta+i\,\alpha\,\eta)+\ldots\right\},$$

where dots mean next to leading order terms in the N tends to infinity limit of the attractive regime. Comparing (2.49) and (4.22) one can easily recognize the proportionality of $\langle O^+_{2n}\rangle_\lambda$ and $\langle\Theta_T\rangle$:

$$\langle O^+_{2n}\rangle_\lambda = \frac{2\,(p+1)\,e^{-2\,p\,\rho_0}}{\sin\gamma}\,\frac{\langle\Theta_T\rangle}{2\pi\mathcal{M}^2}+\ldots. \tag{4.24}$$

According to (2.51) the expectation value for the bare fermion bilinear is given by:

$$\langle\bar\Psi\Psi\rangle = \frac{1}{a}\,\langle O^+_{2n}\rangle_\lambda = \frac{\mathcal{M}\,(p+1)}{2\sin\gamma}\left(\frac{4}{\mathcal{M}a}\right)^{\frac{1-p}{p+1}}\frac{\langle\Theta_T\rangle}{2\pi\mathcal{M}^2}+\ldots, \tag{4.25}$$

where we exploited the relation (2.17) between the lattice constant a and the inhomogeneity parameter ρ_0. Using the relation between p and β in (2.19), one can see that $\langle\bar\Psi\Psi\rangle$ is proportional to the expectation value of the stress energy tensor, and it scales as $a^{\beta^2/4\pi-1}$ as it is expected from (2.8) obtained via purely field theoretical considerations.

5 Large volume expansion

In this section we rephrase the leading order terms in the large N expansions of (4.22) and (4.23) in the form of a systematic large volume series. To get rid of the unnecessary constants, we consider the following quantities:

$$\mathcal{O}^+ = \sum_{\alpha=\pm}\int_{-\infty}^{\infty}d\theta\left[\cosh(\theta+i\,\alpha\,\eta)\,\frac{\mathcal{G}_d(\theta+i\,\alpha\,\eta)}{\ell}-\sinh(\theta+i\,\alpha\,\eta)\,\mathcal{G}_\ell(\theta+i\,\alpha\,\eta)\right]\mathcal{F}_\alpha(\theta+i\,\alpha\,\eta)+$$
$$+\sum_{k=1}^{m_H}\left\{\cosh(H_k)\,\frac{X_k^{(d)}}{\ell}-\sinh(H_k)\,X_k^{(\ell)}\right\}, \tag{5.1}$$

$$\mathcal{O}^- = \sum_{\alpha=\pm} \int_{-\infty}^{\infty} d\theta \left[\sinh(\theta+i\,\alpha\,\eta)\, \frac{\mathcal{G}_d(\theta+i\,\alpha\,\eta)}{\ell} - \cosh(\theta+i\,\alpha\,\eta)\,\mathcal{G}_\ell(\theta+i\,\alpha\,\eta) \right] \mathcal{F}_\alpha(\theta+i\,\alpha\,\eta) + \\ + \sum_{k=1}^{m_H} \left\{ \sinh(H_k)\, \frac{X_k^{(d)}}{\ell} - \cosh(H_k)\, X_k^{(\ell)} \right\}, \tag{5.2}$$

where $\mathcal{G}_d(\theta)$, $X^{(d)}$ and $\mathcal{G}_\ell(\theta)$, $X^{(\ell)}$ are defined by the equations (2.46) and (2.48) respectively. From (2.49) and (4.22) it can be seen that $\mathcal{O}^+$ is simply related to the fermionic expectation value of the trace of the stress-energy tensor:

$$\mathcal{O}^+ = \frac{\langle \Theta_T \rangle - \langle \Theta_T^\infty \rangle}{2\pi\, \mathcal{M}^2}. \tag{5.3}$$

On the other hand $\mathcal{O}^-$ is proportional to the fermionic expectation value of the renormalized pseudo-scalar fermion bilinear $\langle \bar{\Psi}\gamma^5\Psi \rangle$. Here we do not care about the actual value of the proportionality factor, since it will turn out, that this expectation value is zero between multi-fermion states.

The process of the evaluation of (5.1) and (5.2) in the large volume limit is very similar to the method used for computing the diagonal matrix elements of the trace of the stress-energy tensor in purely elastic scattering theories [11]. The reason for this is that formally the NLIE equations (2.34) are very similar to the Thermodynamic Bethe Ansatz (TBA) equations of a purely diagonally scattering theory of two types of particles. This analogy, the actual form of the large volume series of the U(1) current of the theory [17] together with the all order large volume series conjectures for the diagonal form factors of purely elastic scattering theories [10, 11, 14–16], led to the following large volume series conjecture for the diagonal multi-fermion (soliton) expectation values of local operators in the MT (SG) models [17]:

Conjecture. *For any local operator $\mathcal{O}(x)$ in the MT (SG) model the expectation value in an n-fermion (soliton) state with rapidities $\{H_1, H_2, \ldots, H_n\}$ can be written as:*

$$\langle H_1, \ldots, H_n | \mathcal{O}(x) | H_1, \ldots, H_n \rangle = \frac{1}{\rho(H_1, \ldots, H_n)} \\ \times \sum_{\{H_+\}\cup\{H_-\}} \mathcal{D}^{\mathcal{O}}(\{H_+\})\, \rho(\{H_-\}|\{H_+\}), \tag{5.4}$$

where $\rho(\vec{H})$ is the determinant of the exact Gaudin-matrix:

$$\rho(H_1, \ldots, H_n) = det\,\hat{\Phi}(\vec{H}), \qquad \hat{\Phi}_{kj}(\vec{H}) = \frac{d}{dH_j} Z(H_k|\vec{H}), \qquad j,k = 1, \ldots, m_H, \tag{5.5}$$

the sum in (5.4) runs for all bipartite partitions of the rapidities of the sandwiching state: $\{H_1, \ldots, H_n\} = \{H_+\} \cup \{H_-\}$, such that

$$\rho(\{H_+\}|\{H_-\} = det\,\hat{\Phi}_+(\vec{H}), \tag{5.6}$$

with $\hat{\Phi}_+(\vec{H})$ being the sub-matrix of $\hat{\Phi}(\vec{H})$ corresponding to the subset $\{H_+\}$. The quantity $\mathcal{D}^{\mathcal{O}}(\{H\})$ in (5.4) is called the dressed form-factor [11] and it is given by an infinite sum in terms of the connected diagonal form-factors of the theory:

$$\mathcal{D}^{\mathcal{O}}(\{H_1,\dots,H_n\}) = \sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty}\frac{1}{n_+!\,n_-!}\int_{-\infty}^{\infty}\prod_{i=1}^{n_++n_-}\frac{d\theta_i}{2\pi}\prod_{i=1}^{n_+}\mathcal{F}_+(\theta_i+i\,\eta)\prod_{i=n_++1}^{n_++n_-}\mathcal{F}_-(\theta_i-i\,\eta)$$
$$\times F_c^{\mathcal{O}}(H_1,H_2,\dots,H_n,\theta_1+i\,\eta,\dots,\theta_{n_+}+i\,\eta,\theta_{n_++1}-i\,\eta,\dots,\theta_{n_++n_-}-i\,\eta), \tag{5.7}$$

where $F_c^{\mathcal{O}}$ denotes the connected diagonal form factors of $O(x)$ in pure fermion (soliton) states, $0<\eta<min(p\pi,\pi)$ is a small contour deformation parameter and $\mathcal{F}_\pm(\theta)$ are the nonlinear expressions of the counting function given by (2.43).

We note that the structure (5.4) is the same for the purely elastic scattering theories and for the MT (SG) model. The difference arises in the concrete form of the exact Gaudin-matrix[15] and in the actual form of the dressed form factors. However up to exponentially small in volume corrections the formulas of purely elastic scattering theories are also appropriate to describe the multi-fermion (soliton) expectation values of local operators [9, 17].

So far, conjecture (5.4)–(5.7) has been checked against the diagonal fermionic (solitonic) form factors of the U(1) current of the theory [17] and now by rephrasing $\mathcal{O}^+$ as a large volume series, we will show that this conjecture remains valid in the case of the trace of the stress energy tensor, too. Thus, our purpose is to bring $\mathcal{O}^\pm$ into the form of (5.4) and check whether the coefficients of $\rho(\{H_-\}|\{H_+\})$ agrees with $\mathcal{D}^{\mathcal{O}}(\{H_+\})$ given by (5.7).

In [11], starting from the Thermodynamic Bethe Ansatz (TBA) equations, the analog formulas of (5.4)–(5.7) were derived for the diagonal matrix elements of the trace of the stress energy tensor in purely elastic scattering theories. The computation we present below is an appropriate adaptation of the derivation given in section 3 of ref. [11].

The first step of the computation is to rewrite $\mathcal{G}_d(\theta)$, $X^{(d)}$ and $\mathcal{G}_\ell(\theta)\,X^{(\ell)}$ in terms of the solutions of some "elementary" linear problems. For any function f, let $f^{[\pm]}(\theta)=f(\theta\pm i\eta)$, then by definition an "elementary" solution indexed by A satisfy the linear equations as follows:

$$\mathcal{G}_A^{[\alpha]}(\theta)-\sum_{\beta=\pm}\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\psi_{\alpha\beta}(\theta-\theta')\,\mathcal{G}_A^{[\beta]}(\theta')\,\mathcal{F}_\beta^{[\beta]}(\theta')=f_A^{[\alpha]}(\theta),\qquad \alpha=\pm, \tag{5.8}$$

where the symmetric kernel $\psi_{\alpha\beta}(\theta)$ is given by:

$$\psi_{\alpha\beta}(\theta)=G(\theta+i\,(\alpha-\beta)\,\eta),\qquad \alpha,\beta=\pm, \tag{5.9}$$

and $f_A(\theta)$ is the source term specifying $\mathcal{G}_A^{[\alpha]}(\theta)$. An "elementary" solution with unshifted argument satisfies the equations as follows:

$$\mathcal{G}_A(\theta)-\sum_{\beta=\pm}\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\,\psi_{\alpha\beta}(\theta-\theta'-i\,\alpha\,\eta)\,\mathcal{G}_A^{[\beta]}(\theta')\,\mathcal{F}_\beta^{[\beta]}(\theta')=f_A(\theta),\qquad \alpha=\pm. \tag{5.10}$$

[15] In general, the Gaudin-matrix is the derivative of the exact Bethe-equations with respect to the particle rapidities.

In our problem the index A can take values from the set $\mathfrak{I} = \{s, c, 1, 2, \ldots, m_H\}$, such that the source functions $f_A(\theta)$ in (5.8) take the form:

$$f_s(\theta) = \sinh(\theta), \quad f_c(\theta) = \cosh(\theta), \quad f_j(\theta) = -G(\theta - H_j), \qquad j = 1, \ldots, m_H. \tag{5.11}$$

From the defining linear equations (5.8) it can be shown, that the "elementary" solutions satisfy the following identities:

$$\sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, f_A^{[\alpha]}(\theta)\, \mathcal{G}_B^{[\alpha]}(\theta)\, \mathcal{F}_\alpha^{[\alpha]}(\theta) = \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, f_B^{[\alpha]}(\theta)\, \mathcal{G}_A^{[\alpha]}(\theta)\, \mathcal{F}_\alpha^{[\alpha]}(\theta), \qquad A, B \in \mathfrak{I}, \tag{5.12}$$

$$\sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, f_A^{[\alpha]}(\theta)\, \mathcal{G}_j^{[\alpha]}(\theta)\, \mathcal{F}_\alpha^{[\alpha]}(\theta) = f_A(H_j) - \mathcal{G}_A(H_j), \quad j = 1, \ldots, m_H, \qquad A \in \mathfrak{I}. \tag{5.13}$$

With the help of the linear equations (2.46), (2.48) and (5.8) with (5.11), the pairs of quantities $\mathcal{G}_d(\theta)$, $X^{(d)}$ and $\mathcal{G}_\ell(\theta)$, $X^{(\ell)}$ can be expressed in terms of the elementary solutions as follows:

$$\mathcal{G}_d(\theta) = \ell\, \mathcal{G}_c(\theta) - \sum_{j=1}^{m_H} \mathcal{G}_j(\theta)\, X_j^{(d)}, \quad X_k^{(d)} = 1, \qquad k = 1, \ldots, m_H, \tag{5.14}$$

$$\mathcal{G}_\ell(\theta) = \mathcal{G}_s(\theta) - \sum_{j=1}^{m_H} \mathcal{G}_j(\theta)\, X_j^{(\ell)}, \qquad X_k^{(\ell)} = \sum_{j=1}^{m_H} \hat{\Phi}_{kj}^{-1}(\vec{H})\, \mathcal{G}_s(H_j), \quad k = 1, \ldots, m_H, \tag{5.15}$$

where $\hat{\Phi}_{kj}(\vec{H})$ is the exact Gaudin-matrix defined by the formula:

$$\hat{\Phi}_{kj}(\vec{H}) = \frac{d}{dH_j} Z(H_k|\vec{H}) = Z'(H_k)\, \delta_{jk} + \mathcal{G}_j(H_k), \qquad j, k = 1, \ldots, m_H. \tag{5.16}$$

Using the formulas (5.12)–(5.16), $\hat{\Phi}_{kj}(\vec{H})$ and $\mathcal{O}^\pm$ can be expressed in terms of the elementary solutions of (5.8) as follows:

$$\hat{\Phi}_{kj}(\vec{H}) = \left(\ell\, \mathcal{G}_c(H_k) - \sum_{k'=1}^{m_H} \mathcal{G}_{k'}(H_k) \right) \delta_{kj} + \mathcal{G}_j(H_k), \qquad j, k = 1, \ldots, m_H, \tag{5.17}$$

$$\begin{aligned} \mathcal{O}^+ &= \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \mathcal{F}_\alpha^{[\alpha]}(\theta) \left[f_c^{[\alpha]}(\theta)\, \mathcal{G}_c^{[\alpha]}(\theta) - f_s^{[\alpha]}(\theta)\, \mathcal{G}_s^{[\alpha]}(\theta) \right] \\ &\quad + \frac{1}{\ell} \sum_{j=1}^{m_H} \mathcal{G}_c(H_j) - \sum_{j,k=1}^{m_H} \mathcal{G}_s(H_k)\, \hat{\Phi}_{kj}^{-1}(\vec{H})\, \mathcal{G}_s(H_j), \end{aligned} \tag{5.18}$$

$$\begin{aligned} \mathcal{O}^- &= \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \mathcal{F}_\alpha^{[\alpha]}(\theta) \left[f_c^{[\alpha]}(\theta)\, \mathcal{G}_s^{[\alpha]}(\theta) - f_s^{[\alpha]}(\theta)\, \mathcal{G}_c^{[\alpha]}(\theta) \right] \\ &\quad - \frac{1}{\ell} \sum_{j=1}^{m_H} \mathcal{G}_s(H_j) + \sum_{j,k=1}^{m_H} \mathcal{G}_s(H_k)\, \hat{\Phi}_{kj}^{-1}(\vec{H})\, \mathcal{G}_c(H_j). \end{aligned} \tag{5.19}$$

To bring (5.18) and (5.19) into the form of (5.4) one needs to use two theorems.

Theorem 1. *The inverse of the Gaudin-matrix can be expressed in terms of its principal minors and sequences of its matrix elements [11, 62] by the formula as follows:*

$$\hat{\Phi}^{-1}_{ij} = \frac{\mathcal{C}_{ij}}{det\,\hat{\Phi}}, \qquad i,j = 1,\dots,m_H, \tag{5.20}$$

with $\mathcal{C}_{ij}$ *being the co-factor matrix with entries:*

$$\mathcal{C}_{ij} = \begin{cases} det\,\hat{\Phi}(\{i\}), & i = j, \\ \sum\limits_{n=0}^{m_H-2} \sum\limits_{\{\alpha\}} (-1)^{n+1}\,\hat{\Phi}_{i\alpha_1}\,\hat{\Phi}_{\alpha_1\alpha_2}\dots\hat{\Phi}_{\alpha_n j}\; det\,\hat{\Phi}(\{j,i,\alpha_1,\dots,\alpha_n\}), & i \neq j, \end{cases} \tag{5.21}$$

where $\{\alpha\} = \{1,2,\dots,m_H\} \setminus \{i,j\}$ *and* $\hat{\Phi}(\{\mathcal{I}\})$ *denotes the matrix obtained by omitting from* $\hat{\Phi}$ *the rows and columns indexed by the set* $\{\mathcal{I}\}$.

Theorem 2. *The determinant of the Gaudin-matrix can be expressed in terms of its principal minors and sequences of its matrix elements [11] by the formula:*

$$\begin{aligned} det\,\hat{\Phi} = \ell\,\mathcal{G}_c(H_i)\; det\,\hat{\Phi}(\{i\}) + \sum_{n=1}^{m_H-1} \sum_{\{\alpha\}} (-1)^n \hat{\Phi}_{i\alpha_1}\,\hat{\Phi}_{\alpha_1\alpha_2}\dots \\ \hat{\Phi}_{\alpha_{n-1}\alpha_n}\,\ell\,\mathcal{G}_c(H_{\alpha_n})\; det\,\hat{\Phi}(\{i,\alpha_1,\dots,\alpha_n\}). \end{aligned} \tag{5.22}$$

Theorems 1 and 2 allow one to rewrite the double and single sums respectively in (5.18) and (5.19) into a more convenient form. Using (5.22), the single sums of (5.18) and (5.19) can be represented as:

$$\begin{aligned} &\sum_{i=1}^{m_H} \frac{1}{\ell}\mathcal{G}_A(H_i) = \sum_{i=1}^{m_H} \mathcal{G}_A(H_i)\,\mathcal{G}_c(H_i)\,\frac{\det\hat{\Phi}(\{i\})}{\det\hat{\Phi}} + \\ &+\sum_{\substack{i,j=1\\ i\neq j}}^{m_H} \mathcal{G}_A(H_i)\mathcal{G}_c(H_j) \sum_{n=0}^{m_H-2}\sum_{\{\alpha\}} (-1)^{n+1}\mathcal{G}_{\alpha_1}(H_i)\mathcal{G}_{\alpha_2}(H_{\alpha_1})\dots\mathcal{G}_j(H_{\alpha_n})\frac{\det\hat{\Phi}(\{i,j,\alpha_1,\dots,\alpha_n\})}{\det\hat{\Phi}}, \end{aligned} \tag{5.23}$$

with $A \in \{s,c\}$. The double sums of (5.18) and (5.19) can be represented in a very similar form:

$$\begin{aligned} &\sum_{i,j=1}^{m_H} \mathcal{G}_A(H_i)\,\hat{\Phi}^{-1}_{ij}(\vec{H})\,\mathcal{G}_B(H_j) = \sum_{i=1}^{m_H} \mathcal{G}_A(H_i)\,\mathcal{G}_B(H_i)\,\frac{\det\hat{\Phi}(\{i\})}{\det\hat{\Phi}} + \\ &+\sum_{\substack{i,j=1\\ i\neq j}}^{m_H} \mathcal{G}_A(H_i)\mathcal{G}_B(H_j) \sum_{n=0}^{m_H-2}\sum_{\{\alpha\}} (-1)^{n+1}\mathcal{G}_{\alpha_1}(H_i)\mathcal{G}_{\alpha_2}(H_{\alpha_1})\dots\mathcal{G}_j(H_{\alpha_n})\frac{\det\hat{\Phi}(\{i,j,\alpha_1,\dots,\alpha_n\})}{\det\hat{\Phi}}, \end{aligned} \tag{5.24}$$

with $A, B \in \{s,c\}$. It is convenient to determine first the large volume series expansion of the first terms in the right hand sides of (5.18) and (5.19). They are called the vacuum contributions [10, 11] since they correspond to the $\{H_+\} = \emptyset$ case. These terms can be rephrased as an infinite series similar to that of LeClair and Mussardo [14–16]. To get this series representation, first one has to construct the all order large volume solution of (5.8)

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for $A = s, c$. This can be obtained by a simple iterative solution of the equations. Then one has to insert these large volume series into (5.18), (5.19). At the end of this process one gets a bulky sum of terms, such that a lot of terms are identical under certain permutations of the integrating variables. Taking into account these permutational symmetries by appropriate symmetry factors, one ends up with the formula for the vacuum contributions as follows:

$$\mathcal{O}^{\pm}\big|_{vac} = \sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty} \frac{1}{n_+!\,n_-!} \int_{-\infty}^{\infty} \prod_{i=1}^{n_++n_-} \frac{d\theta_i}{2\pi} \prod_{i=1}^{n_+} \mathcal{F}_+(\theta_i + i\,\eta) \prod_{i=n_++1}^{n_++n_-} \mathcal{F}_-(\theta_i - i\,\eta) \\ \times F^{O^{\pm}}_{n_++n_-,c}(\theta_1 + i\,\eta, \ldots, \theta_{n_+} + i\,\eta, \theta_{n_++1} - i\,\eta, \ldots, \theta_{n_++n_-} - i\,\eta), \tag{5.25}$$

where the "connected" diagonal form factors of $O^{\pm}$ are given by the definitions:

$$F^{O^+}_{n,c}(\theta_1, \theta_2, \ldots, \theta_n) = \sum_{\sigma \in S_n} \big(\cosh(\theta_{\sigma(1)})\,\cosh(\theta_{\sigma(n)}) - \sinh(\theta_{\sigma(1)})\,\sinh(\theta_{\sigma(n)})\big) \\ \times \prod_{j=1}^{n-1} G(\theta_{\sigma(j)} - \theta_{\sigma(j+1)}), \tag{5.26}$$

$$F^{O^-}_{n,c}(\theta_1, \theta_2, \ldots, \theta_n) = \sum_{\sigma \in S_n} \big(\cosh(\theta_{\sigma(1)})\,\sinh(\theta_{\sigma(n)}) - \sinh(\theta_{\sigma(1)})\,\cosh(\theta_{\sigma(n)})\big) \\ \times \prod_{j=1}^{n-1} G(\theta_{\sigma(j)} - \theta_{\sigma(j+1)}), \tag{5.27}$$

where σ denotes the elements of the symmetric group S_n. From the permutation symmetry of the summand in (5.27), it follows that $F^{O^-}_{n,c}(\theta_1, \theta_2, \ldots, \theta_n) = 0$, and consequently: $\mathcal{O}^-\big|_{vac} = 0$. The next step is to determine[16] the large volume series representations for the 2nd and 3rd terms in (5.18) and (5.19). First one has to rewrite them with the help of the right hand sides of (5.23) and (5.24) taken at appropriate values of the indexes A and B. Then the all order large volume series representation of the solution of (5.8) must be inserted into the result. Finally, the careful bookkeeping of terms being identical under certain permutations of the variables leads to the final formula:

$$\mathcal{O}^{\pm} = \frac{1}{\rho(H_1, \ldots, H_{m_H})} \sum_{\{H_+\}\cup\{H_-\}} \mathcal{D}^{O^{\pm}}(\{H_+\})\ \rho(\{H_-\}|\{H_+\}), \tag{5.28}$$

where the so-called dressed form factors take the form:

$$\mathcal{D}^{O^{\pm}}(\{H_1, \ldots, H_n\}) = \sum_{n_+=0}^{\infty}\sum_{n_-=0}^{\infty} \frac{1}{n_+!\,n_-!} \int_{-\infty}^{\infty} \prod_{i=1}^{n_++n_-} \frac{d\theta_i}{2\pi} \prod_{i=1}^{n_+} \mathcal{F}_+(\theta_i + i\,\eta) \prod_{i=n_++1}^{n_++n_-} \mathcal{F}_-(\theta_i - i\,\eta) \\ \times F^{O^{\pm}}_{n+n_++n_-,c}(H_1, H_2, \ldots, H_n, \theta_1 + i\,\eta, \ldots, \theta_{n_+} + i\,\eta, \theta_{n_++1} - i\,\eta, \ldots, \theta_{n_++n_-} - i\,\eta). \tag{5.29}$$

Now we can discuss the results. First we discuss the case of $\mathcal{O}^-$. (5.29) and (5.27) implies that $\mathcal{O}^- = 0$ exactly. Taking into account the connection between $\mathcal{O}^-$ and the fermionic

[16] The necessary computations are almost literally the same as those presented in section 3. of ref. [11].

expectation values of the pseudo-scalar fermion bilinear:

$$\langle \bar{\Psi}\gamma^5\Psi \rangle = \frac{1}{a}\langle O^-_{2n}\rangle_\lambda \sim a^{\frac{p-1}{p+1}}\,\mathcal{O}^- + \dots ,$$

one[17] can conclude that the fermionic expectation values of $\bar{\Psi}\gamma^5\Psi$ are zero.

Next we discuss the result obtained for $\mathcal{O}^+$. According to (5.3), $\mathcal{O}^+$ is proportional to the fermionic (solitonic) expectation values of Θ_T. This operator belongs to a conserved current, this is why its connected diagonal form-factors between pure fermion (soliton) states can be determined by using the arguments of references [14] and [15]. The actual computations lead to the following simple result:

$$F^{\Theta_T}_{n,c}(\theta_1,\theta_2,\dots,\theta_n) = 2\pi\mathcal{M}^2\, F^{O^+}_{n,c}(\theta_1,\theta_2,\dots,\theta_n). \tag{5.30}$$

Then, (5.30) together with (5.3), (5.28) and (5.29) imply that the conjecture (5.4) of ref. [17] is valid for the diagonal fermionic (solitonic) matrix elements of the trace of the stress-energy tensor, too.

6 Summary, outlook and discussion

In this paper, using the light-cone lattice regularization, we computed the finite volume expectation values of the composite operators $\bar{\Psi}\Psi$ and $\bar{\Psi}\gamma^5\Psi$ between multi-fermion (soliton) states in the Massive Thirring (sine-Gordon) model. In the light-cone regularized picture, these expectation values are related to such 2-point functions of the lattice spin operators, in which the operators are located at neighboring sites of the lattice. The operator $\bar{\Psi}\Psi$ is a particularly interesting operator, since it is proportional to the trace of the stress-energy tensor. Thus, its continuum finite volume expectation values can be computed [18] also from the set of non-linear integral equations (NLIE) governing the finite volume spectrum of the theory. The final result, which was obtained after a lengthy computation of the spin-spin 2-point functions of neighboring operators, reproduced the pure NLIE result.

In general the finite volume matrix elements of local operators are computed via their large volume series. Thinking in this framework, previously in [17], an all order large volume series representation similar to that of [10, 11, 14] was conjectured for the finite volume diagonal matrix elements of local operators between multi-fermion (soliton) states. To check the conjecture of [17] we rephrased the diagonal multi-fermion (soliton) matrix elements of the trace of the stress-energy tensor as a large volume series. The form of the series was conform to the conjecture of [17].

Nevertheless, one has to note that, so far the large volume series conjecture of [17] for the finite volume diagonal matrix elements of local operators between pure fermion (soliton) states, has been checked in cases when the local operator belongs to a conserved quantity of the theory. This is why it would be interesting to test the conjecture for such operators, which do not belong to the conserved quantities of the model. The results of [7, 8] indicate that the truncated conformal space approach could be an appropriate method for such investigations.

[17] Here dots stand for terms tending to zero when $a \to 0$ in the attractive $0 < p < 1$ regime of the model.

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Beyond the results of [17] and the present paper, several questions are still open. The light-cone lattice approach gives access to all eigenstates of the Hamiltonian. Thus, diagonal matrix elements between non-pure fermion or soliton states can also be computed in this framework. It would be interesting to see, how the large volume series representation of [17] should be modified in that case. Finally, a much more difficult but still open problem is the determination of non-diagonal finite volume form factors.

Beyond our approach, there is another approach to the form-factors of the SG model in cylindrical geometry [65–71]. We close the paper with some discussion concerning the comparison of our method with that of the series of papers [65–71]. In [65–71] the hidden Grassmannian structure of the XXZ model was exploited to determine the finite temperature 1-point functions [70] and ratios of infinite volume form-factors [71] of the local operators of the SG theory. In this approach the compactified direction is time and the compactification length corresponds to the inverse temperature. The 1-point functions and the form-factors are computed as the continuum limit of appropriate partition functions of the 6-vertex model.

Our approach is more conventional. We consider the inhomogeneous 6-vertex model as a lattice regularization of the MT (SG) model and the operators we consider is the set of composite operators of Fermi fields and their derivatives.[18] In our case the compactified direction is space which makes easy to consider matrix elements of operators between excited states of the model. The Fermi fields are expressed in terms of local spin operators and the form-factors are given by such correlation functions of the spins, in which the spins are located at neighboring positions on the lattice. The correlators are evaluated by usual Algebraic Bethe Ansatz methods [39]–[43].

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A Integral representation of some typical sums

In this appendix we summarize the integral representations of the typical sums (3.44), (3.45) arising in the computation of the expectation values (3.6) and (3.7). For the sake of completeness we recall them in this appendix, too:

$$\Sigma_\lambda^{(1)}[f](\xi) = \sum_{a=1}^{m} f(\lambda_a)\, X(\lambda_a|\xi), \tag{A.1}$$

$$\Sigma_\lambda^{(2)}[f](\xi,\xi') = \sum_{a,b=1}^{m} f(\lambda_a,\lambda_b)\, X(\lambda_a|\xi)\, X(\lambda_b|\xi'). \tag{A.2}$$

[18] Or equivalently their bosonized counterparts in the SG model.

In order to be able to transform these sums into integral expressions, we require that in each of its arguments f should not have worse than constant asymptotics at infinity. In (A.2) we also require for f to be an antisymmetric function of its arguments.

First, let us consider the single sum (A.1). The straightforward application of (3.30) to (A.1) gives the integral representation as follows:

$$\Sigma^{(1)}_\lambda[f](\xi) = -\sum_{j=1}^{m_H} f(h_j)\, X(h_j|\xi) + \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, f(\lambda)\, \mathcal{G}(\lambda|\xi) \\ - \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, f(\lambda + i\,\alpha\,\eta)\, \mathcal{G}(\lambda + i\,\alpha\,\eta|\xi)\, \mathcal{F}^{(\lambda)}_\alpha(\lambda + i\,\alpha\,\eta). \tag{A.3}$$

However the form (A.3) is still not appropriate for our purposes, since it has no well defined continuum limit in the cases of our interest, when $\xi = \xi_\pm$. If one tries to take the continuum limit in (A.3), it becomes immediately obvious that the first integral term in (A.3) will diverge in the continuum limit. The reason is as follows. On the one hand, as a consequence of (4.6) and (4.7), it follows that in the continuum limit $\mathcal{G}(\lambda|\xi_\pm) \sim e^{\pm\frac{\pi}{\gamma}\lambda}$ at large λ. On the other hand, as it was mentioned in section 3, the concrete functions[19] $f(\lambda)$ for which we should apply (A.3) have constant asymptotics at infinity. Thus, the integrand in $\int\limits_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, f(\lambda)\, \mathcal{G}(\lambda|\xi)$ in the r.h.s. of (A.3) blows up exponentially at infinity in the continuum limit, which implies that the integral itself diverges.

However on the lattice each term of (A.3) is well defined and convergent, because as a consequence of (3.36) and (3.37) on the lattice $\mathcal{G}(\lambda|\xi)$ decays exponentially at large λ. This means, that in order to be able to define the continuum limit, further transformations of (A.3) are required. By exploiting (3.36) one can make the following replacement into the first integral term of (A.3):

$$\mathcal{G}(\lambda|\xi) \to -\sum_{\alpha=\pm} \int_{-\infty}^{\infty} d\lambda'\, G_\lambda(\lambda - \lambda' - i\,\alpha\,\eta)\, \mathcal{G}(\lambda' + i\,\alpha\,\eta|\xi)\, \mathcal{F}^{(\lambda)}_\alpha(\lambda' + i\,\alpha\,\eta) + \\ + S_0(\lambda|\xi) + \sum_{j=1}^{m_H} 2\pi\, G_\lambda(\lambda - h_j)\, X_j(\xi), \tag{A.4}$$

Making the replacement (A.4) into the first integral term of (A.3), one ends up with the formula as follows for $\Sigma^{(1)}_\lambda[f](\xi)$:

$$\Sigma^{(1)}_\lambda[f](\xi) = \mathcal{J}_0[f](\xi) + \mathcal{S}_X[f](\xi) + \mathcal{S}_\mathcal{G}[f](\xi), \tag{A.5}$$

[19] We just recall, that in our actual computations f can be f_1, f_2, or f_3 given by the formulas (3.48)–(3.50).

where $\mathcal{J}_0[f](\xi)$, $\mathcal{S}_X[f](\xi)$ and $\mathcal{S}_\mathcal{G}[f](\xi)$ are functionals of f and are of the form:

$$\mathcal{S}_X[f](\xi) = \sum_{j=1}^{m_H} X(h_j|\xi)\, \mathcal{J}_G[f](h_j) \tag{A.6}$$

$$\mathcal{S}_\mathcal{G}[f](\xi) = \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, \mathcal{F}_\alpha^{(\lambda)}(\lambda+i\,\alpha\,\eta)\, \mathcal{G}(\lambda+i\,\alpha\,\eta|\xi)\; \mathcal{J}_G[f](\lambda+i\,\alpha\,\eta), \tag{A.7}$$

$$\mathcal{J}_0[f](\xi) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, f(\lambda)\, S_0(\lambda|\xi), \qquad\qquad \mathcal{J}_G[f](\lambda) = (f\star G_\lambda)(\lambda) - f(\lambda). \tag{A.8}$$

Here $\star$ denotes convolution with conventions given by (B.3). In this final representation each term has a well defined continuum limit.[20]

Similar, but more tedious computations lead to the following formula for $\Sigma_\lambda^{(2)}[f](\xi,\xi')$. It is composed of six terms:

$$\begin{aligned}\Sigma_\lambda^{(2)}[f](\xi,\xi') = \Sigma_0[f](\xi,\xi') + \Sigma_X[f](\xi,\xi') + \Sigma_\mathcal{G}[f](\xi,\xi') + \Sigma_{XX}[f](\xi,\xi') \\ + \Sigma_{\mathcal{G}\mathcal{G}}[f](\xi,\xi') + \Sigma_{X\mathcal{G}}[f](\xi,\xi').\end{aligned} \tag{A.9}$$

The lower index of each term on the right hand side of (A.9) refers to the internal structure of the expression as it becomes clear from their explicit form:

$$\Sigma_0[f](\xi,\xi') = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda'}{2\pi} S_0(\lambda'|\xi)\, f(\lambda',\lambda)\, S_0(\lambda|\xi'), \tag{A.10}$$

$$\Sigma_X[f](\xi,\xi') = \sum_{j=1}^{m_H} X(h_j|\xi)\, F_X[f](h_j|\xi') - \sum_{j=1}^{m_H} X(h_j|\xi')\, F_X[f](h_j|\xi), \tag{A.11}$$

$$\begin{aligned}\Sigma_\mathcal{G}[f](\xi,\xi') = \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, \mathcal{F}_\alpha^{(\lambda)}(\lambda+i\,\alpha\,\eta)\, \big\{\mathcal{G}(\lambda+i\,\alpha\,\eta|\xi)\, F_X[f](\lambda+i\,\alpha\,\eta|\xi') \\ - \mathcal{G}(\lambda+i\,\alpha\,\eta|\xi')\, F_X[f](\lambda+i\,\alpha\,\eta|\xi)\big\}\,,\end{aligned} \tag{A.12}$$

$$\begin{aligned}\Sigma_{X\mathcal{G}}[f](\xi,\xi') = \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, \mathcal{F}_\alpha^{(\lambda)}(\lambda+i\,\alpha\,\eta) \sum_{j=1}^{m_H} F_{XX}[f](h_j,\lambda+i\,\alpha\,\eta) \times \\ \times \big\{X(h_j|\xi')\, \mathcal{G}(\lambda+i\,\alpha\,\eta|\xi) - X(h_j|\xi)\, \mathcal{G}(\lambda+i\,\alpha\,\eta|\xi')\big\}\,,\end{aligned} \tag{A.13}$$

$$\Sigma_{XX}[f](\xi,\xi') = \sum_{j=1}^{m_H}\sum_{k=1}^{m_H} X(h_j|\xi')\, F_{XX}[f](h_j,h_k)\, X(h_k|\xi), \tag{A.14}$$

$$\begin{aligned}\Sigma_{\mathcal{G}\mathcal{G}}[f](\xi,\xi') = \sum_{\alpha,\beta=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda'}{2\pi} \mathcal{F}_\alpha^{(\lambda)}(\lambda+i\,\alpha\,\eta)\, \mathcal{F}_\beta^{(\lambda)}(\lambda'+i\,\beta\,\eta) \times \\ \times \mathcal{G}(\lambda+i\,\alpha\,\eta|\xi')\, \mathcal{G}(\lambda'+i\,\beta\,\eta|\xi)\, F_{XX}[f](\lambda+i\,\alpha\,\eta, \lambda'+i\,\beta\,\eta),\end{aligned} \tag{A.15}$$

[20] In the terms containing $S_0(\lambda|\xi)$ the continuum limit can be taken after evaluating the integral, because the naive expansion (4.2) of $S_0(\lambda|\xi)$ under the integral leads to incorrect results. The careful computations can be found in appendix B.

where the two functions F_X and F_{XX}, which are functionals of f, take the form:

$$F_X[f](\lambda|\xi) = J_S[f](\lambda|\xi) - J_{SG}[f](\lambda|\xi), \tag{A.16}$$
$$F_{XX}[f](\lambda,\lambda') = f(\lambda',\lambda) + J_G[f](\lambda,\lambda') - J_G[f](\lambda',\lambda) - J_{GG}[f](\lambda,\lambda'), \tag{A.17}$$

with the “elementary” functionals:

$$J_S[f](\lambda|\xi) = \int_{-\infty}^{\infty} \frac{d\lambda'}{2\pi} S_0(\lambda'|\xi)\, f(\lambda',\lambda), \tag{A.18}$$
$$J_{SG}[f](\lambda|\xi) = \int_{-\infty}^{\infty} \frac{d\lambda'}{2\pi} \int_{-\infty}^{\infty} d\lambda''\, S_0(\lambda'|\xi)\, f(\lambda',\lambda'')\, G_\lambda(\lambda''-\lambda), \tag{A.19}$$
$$J_G[f](\lambda,\lambda') = \int_{-\infty}^{\infty} d\lambda'' G_\lambda(\lambda-\lambda'')\, f(\lambda'',\lambda'), \tag{A.20}$$
$$J_{GG}[f](\lambda,\lambda') = \int_{-\infty}^{\infty} d\lambda'' \int_{-\infty}^{\infty} d\lambda''' G_\lambda(\lambda-\lambda'')\, f(\lambda'',\lambda''')\, G_\lambda(\lambda'''-\lambda'). \tag{A.21}$$

Once again, we note, that at the derivation of (A.5) and (A.9), it was important to bring the sums into a sum of such integral expressions which contain $\mathcal{G}(\lambda)$ only in the combination $\mathcal{G}(\lambda)\,\mathcal{F}_\alpha^{(\lambda)}(\lambda)$. The reason for the preference of such a form is, that in the continuum limit, this combination is integrable along the lines $\lambda \pm i\,\eta$ with $\lambda \in \mathbb{R}$ and with η being a small positive contour deformation parameter. This convenient form could be derived by eliminating the single $\mathcal{G}(\lambda)$ terms with the help of (3.36).

B Large ρ_0 expansions

In this appendix we summarize, how one can obtain the coefficient functions $\mathcal{K}_\pm(\lambda|\rho_0)$ (4.16) and the bulk term (4.13) of (4.17) via the computation of the large ρ_0 (2.17) limit of the functionals (4.8), (4.10) and (4.11). The key point of the computations is that one should work in Fourier space. This is why as a first step we fix our conventions for the Fourier-transformations. The Fourier-transform of a function f is given by:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} dx\, e^{i\omega x}\, f(x). \tag{B.1}$$

The inverse transformation reads as:

$$f(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x}\, \tilde{f}(\omega). \tag{B.2}$$

The Fourier-transform of the convolution of two functions f and g is given by the product of individual Fourier-transforms:[21]

$$(f\star g)(x)=\int\limits_{-\infty}^{\infty}dy\, f(x-y)\, g(y), \qquad \widetilde{(f\star g)}(\omega)=\tilde{f}(\omega)\,\tilde{g}(\omega). \tag{B.3}$$

Formulas (4.8), (4.10) and (4.11) imply that in order to derive the formulas (4.14), (4.16) and (4.13), the following functionals should be computed in the large ρ_0 limit:

- $\mathcal{J}_G[f](\lambda)$ of (A.8) taken at the functions $f_1, f_2, \text{and}\, f_3$ given by (3.48), (3.49) and (3.50).
- $\mathcal{J}_0[f_1](\xi_-)$, $\mathcal{J}_0[f_2](\xi_+)$, and $\mathcal{J}_0[f_3](\xi_-)$ defined by (A.8) with $\xi_\pm=\pm\rho_0-i\frac{\gamma}{2}$.
- $F_X[f_\pm](\lambda|\xi_\pm)$ of (A.16) with $f_\pm$ given by (3.51) and (3.52).
- $\Sigma_0[f_\pm](\xi_-,\xi_+)$ of (A.10).

The strategy of the large ρ_0 evaluation of the above listed functionals is as follows. One can write them in a very special form, namely as a linear combination of convolutions, such that ρ_0 appears in the argument of the convolutions. This means, that the large ρ_0 expansion of the functionals of our interest is equivalent to determine the large argument series expansion of the convolutions appearing in them. It is worth to represent these convolutions in Fourier-space, since by using the property (B.3) they can be written as a single Fourier-integral. The large argument series expansion of these Fourier-integral expressions can be computed by using the residue theorem. Thus the positions of the poles of the integrand will determine the large argument decay of the convolutions of our interest.

To complete the concrete large ρ_0 computations, first one has to define some functions, which constitute the elementary building blocks of the calculations:

$$F_c(\lambda)=\frac{1}{2\gamma\cosh(\frac{\pi}{\gamma}\lambda)}, \qquad \tilde{F}_c(\omega)=\frac{1}{2\cosh(\frac{\gamma}{2}\omega)}, \tag{B.4}$$

$$F_c^{\pm}(\lambda)=e^{\pm 2\lambda}F_c(\lambda), \qquad \text{in case } p>1:\ \tilde{F}_c^{\pm}(\omega)=\tilde{F}_c(\omega\mp 2i), \tag{B.5}$$

$$g(\lambda)=\frac{1}{\cosh(2\lambda)-\cos(2\gamma)}, \qquad \tilde{g}(\omega)=\frac{\pi}{\sin(2\gamma)}\frac{\sinh(\frac{\omega}{2}(\pi-2\gamma))}{\sinh(\frac{\pi}{2}\omega)}, \tag{B.6}$$

$$\widetilde{g^2}(\omega)=\frac{\pi}{2\sin^2(2\gamma)}\left\{\frac{2\cot(2\gamma)\sinh(\frac{\omega}{2}(\pi-2\gamma))}{\sinh(\frac{\pi}{2}\omega)}+\frac{\omega\cosh(\frac{\omega}{2}(\pi-2\gamma))}{\sinh(\frac{\pi}{2}\omega)}\right\}, \tag{B.7}$$

$$g_\alpha(\lambda)=\frac{1}{\sinh^2(\lambda-i\alpha)}, \qquad \tilde{g}_\alpha(\omega)=-\pi\,\omega\frac{e^{(\frac{\pi}{2}-\alpha)\omega}}{\sinh(\frac{\pi}{2}\omega)}, \quad \alpha\in(0,\pi), \tag{B.8}$$

$$\psi(\lambda)=\frac{\sinh(2\lambda)}{\cosh(2\lambda)-\cos(2\gamma)}, \qquad \tilde{\psi}'(\omega)=\frac{\pi\omega}{\sinh(\frac{\pi}{2}\omega)}\cosh\left(\frac{\omega}{2}(\pi-2\gamma)\right), \tag{B.9}$$

$$G_\lambda(\lambda)=\int\limits_{-\infty}^{\infty}\frac{dx}{2\pi}e^{-i\omega x}\,\tilde{G}_\lambda(\omega), \qquad \tilde{G}_\lambda(\omega)=\frac{1}{2}\frac{\sinh(\frac{\pi\omega}{2}(1-\frac{2\gamma}{\pi}))}{\cosh(\frac{\gamma\omega}{2})\sinh(\frac{\pi\omega}{2}(1-\frac{\gamma}{\pi}))}, \tag{B.10}$$

[21] Provided they exist.

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$$G^{\pm}_{\lambda}(\lambda) = e^{\pm 2\lambda} G_{\lambda}(\lambda), \qquad \text{in case } p>1: \quad \tilde{G}^{\pm}_{\lambda}(\omega) = \tilde{G}_{\lambda}(\omega \mp 2i), \tag{B.11}$$

$$\chi_F(\lambda) = \int_0^{\lambda} d\lambda' F_c(\lambda') = \frac{1}{\pi} \arctan\left[\tanh\left(\frac{\pi\lambda}{2\gamma}\right)\right], \qquad \tilde{\chi}_F(\omega) = i\,\tilde{F}_c(\omega)\,\mathfrak{r}(\omega), \tag{B.12}$$

$$\chi(\lambda) = \int_0^{\lambda} d\lambda' G_{\lambda}(\lambda'), \quad \tilde{\chi}(\omega) = i\,\tilde{G}_{\lambda}(\omega)\,\mathfrak{r}(\omega), \quad \text{with} \quad \mathfrak{r}(\omega) = \frac{1}{2}\left(\frac{1}{\omega + i0} + \frac{1}{\omega - i0}\right), \tag{B.13}$$

where the $\pm i\,0$ prescription in $\mathfrak{r}(\omega)$ ensures the correct treatment of the $1/\omega$ singularity of the Fourier integral representations of (B.12) and (B.13).

Let us start with the computation of the bulk (4.8) or in other words the global constant in rapidity term. The representation of the building blocks of (4.8) as linear combinations of convolutions read as:

$$\mathcal{J}_0[f_1](\xi_-) = -\frac{1}{2}, \tag{B.14}$$

$$\mathcal{J}_0[f_2](\xi_+) = -\sinh(2\rho_0)\,(g_{\gamma/2} \star \chi_F)(2\rho_0) - \frac{1}{2}\cosh(2\rho_0), \tag{B.15}$$

$$\mathcal{J}_0[f_3](\xi_-) = \sinh(2\rho_0 + i\,\gamma)\,(g_{\gamma/2} \star \chi_F)(-2\rho_0) - \frac{1}{2}\cosh(2\rho_0 + i\,\gamma), \tag{B.16}$$

$$\Sigma_0[f_+](\xi_+,\xi_-) = \frac{\cos\gamma}{2\gamma^2}\left[\cosh(2\rho_0)\,\mathcal{T}_{\gamma}(\rho_0) - I(\rho_0)\right], \tag{B.17}$$

$$\Sigma_0[f_-](\xi_+,\xi_-) = i\,\frac{\cos\gamma}{2\gamma^2}\cosh(2\rho_0)\,\sin\gamma\,\mathcal{T}_0(\rho_0), \tag{B.18}$$

where $\mathcal{T}_{\alpha}(\rho_0)$ and $I(\rho_0)$ are given by:

$$I(\rho_0) = 4\gamma^2\,(F_c \star \psi' \star \chi_F)(2\rho_0), \tag{B.19}$$

$$\mathcal{T}_{\alpha}(\rho_0) = \cos\alpha\,\mathcal{T}_0(\rho_0) = 4\gamma^2\cos\alpha\left[e^{2\rho_0}\,(F_c^- \star g \star F_c)(2\rho_0) - e^{-2\rho_0}\,(F_c^+ \star g \star F_c)(2\rho_0)\right]. \tag{B.20}$$

The large argument expansions of appendix C lead to the following leading order large ρ_0 expressions for these building blocks in the attractive regime:

$$\mathcal{J}_0[f_2](\xi_+) = -\frac{e^{-i\frac{p\pi}{2}}}{2}\,\frac{1+p}{\cos(\frac{p\pi}{2})}\,e^{-2p\rho_0} + O(e^{-2\rho_0}), \tag{B.21}$$

$$\mathcal{J}_0[f_3](\xi_-) = -\frac{e^{i\gamma + i\frac{p\pi}{2}}}{2}\,\frac{1+p}{\cos(\frac{p\pi}{2})}\,e^{-2p\rho_0} + O(e^{-2\rho_0}), \tag{B.22}$$

$$\Sigma_0[f_+](\xi_+,\xi_-) = \frac{\pi}{2\gamma}\cos\gamma\left[\cot\gamma\,\cot\frac{\pi^2}{2\gamma} - 1\right] e^{2(1-p)\rho_0} + O(e^{-2\rho_0}), \tag{B.23}$$

$$\Sigma_0[f_-](\xi_+,\xi_-) = \frac{i}{2}\sin\gamma + i\sin(2\gamma)\frac{\pi}{4\gamma}\left[\frac{1}{\sin\gamma}\cot\frac{\pi^2}{2\gamma} - \frac{1}{\cos\gamma}\right] e^{2(1-p)\rho_0} + O(e^{-2\rho_0}). \tag{B.24}$$

Finally, inserting (B.21), (B.22), (B.23) and (B.24) into (4.8) one ends up with the final result given by (4.13).

Now, we can continue with the computation of the coefficient functions $C^{\pm}_{\pm}(\lambda)$ given in (4.10) and (4.11). These functions can also be represented as appropriate linear combinations of some convolutions. Such convolution type representations of the elementary building blocks of $C^{\pm}_{\pm}(\lambda)$ are given by the following formulas:

$$\mathcal{J}_G[f_1](\lambda) = -\frac{p+1}{2p}, \tag{B.25}$$

$$\mathcal{J}_G[f_2](\lambda) = 2\cosh(2\rho_0)\,\chi(\infty)+\sinh(2\rho_0)\,(g_{\gamma/2}\star\chi)(\lambda+\rho_0)-f_2(\lambda|\rho_0), \tag{B.26}$$

$$\mathcal{J}_G[f_3](\lambda) = 2\cosh(2\rho_0+i\gamma)\,\chi(\infty)-\sinh(2\rho_0+i\gamma)\,(g_{\gamma/2}\star\chi)(\lambda-\rho_0)-f_3(\lambda|\rho_0), \tag{B.27}$$

$$J_S[f_+](\lambda|\xi_\pm) = \cos\gamma\cosh(2\rho_0)\Big\{e^{i\gamma}\Big[e^{2\lambda}(g\star F_c)(\lambda\mp\rho_0)-e^{\pm 2\rho_0}(g\star F_c^+)(\lambda\mp\rho_0)\Big] \tag{B.28}$$
$$-\,e^{-i\gamma}\Big[e^{-2\lambda}(g\star F_c)(\lambda\mp\rho_0)-e^{\mp 2\rho_0}(g\star F_c^-)(\lambda\mp\rho_0)\Big]\Big\}-2\cos\gamma\,(\psi'\star\chi_F)(\lambda\mp\rho_0),$$

$$\begin{aligned}J_{SG}[f_+](\lambda|\xi_\pm) = {}&\cos\gamma\cosh(2\rho_0)\Big\{e^{i\gamma}\Big[e^{2\lambda}(G^-\star g\star F_c)(\lambda\mp\rho_0)-e^{\pm 2\rho_0}(G\star g\star F_c^+)(\lambda\mp\rho_0)\Big]\\ &-e^{-i\gamma}\Big[e^{-2\lambda}(G^+\star g\star F_c)(\lambda\mp\rho_0)-e^{\mp 2\rho_0}(G\star g\star F_c^-)(\lambda\mp\rho_0)\Big]\Big\}\\ &-2\cos\gamma(G\star\psi'\star\chi_F)(\lambda\mp\rho_0).\end{aligned} \tag{B.29}$$

$$\begin{aligned}J_S[f_-](\lambda|\xi_\pm) = {}&\cos\gamma\sinh(2\rho_0)\Big\{e^{i\gamma}\Big[e^{2\lambda}(g\star F_c)(\lambda\mp\rho_0)-e^{\pm 2\rho_0}(g\star F_c^+)(\lambda\mp\rho_0)\Big]-\\ &+e^{-i\gamma}\Big[e^{-2\lambda}(g\star F_c)(\lambda\mp\rho_0)-e^{\mp 2\rho_0}(g\star F_c^-)(\lambda\mp\rho_0)\Big]\Big\},\end{aligned} \tag{B.30}$$

$$\begin{aligned}J_{SG}[f_-](\lambda|\xi_\pm) = {}&\cos\gamma\sinh(2\rho_0)\Big\{e^{i\gamma}\Big[e^{2\lambda}(G^-\star g\star F_c)(\lambda\mp\rho_0)-e^{\pm 2\rho_0}(G\star g\star F_c^+)(\lambda\mp\rho_0)\Big]\\ &+e^{-i\gamma}\Big[e^{-2\lambda}(G^+\star g\star F_c)(\lambda\mp\rho_0)-e^{\mp 2\rho_0}(G\star g\star F_c^-)(\lambda\mp\rho_0)\Big]\Big\}.\end{aligned} \tag{B.31}$$

The large argument expansions presented in appendix C lead to the following leading order large ρ_0 forms for the functionals (B.25)–(B.31):

$$\mathcal{J}_G[f_2](\lambda) = -e^{(1-p)\rho_0}e^{-(1+p)\lambda}\,e^{-i\frac{p\pi}{2}}\,\frac{1+p}{2\sin(\frac{p\pi}{2})}+O(e^{-2\rho_0}), \tag{B.32}$$

$$\mathcal{J}_G[f_3](\lambda) = -e^{(1-p)\rho_0}e^{(1+p)\lambda}\,e^{i\gamma+i\frac{p\pi}{2}}\,\frac{1+p}{2\sin(\frac{p\pi}{2})}+O(e^{-2\rho_0}), \tag{B.33}$$

$$\frac{J_S[f_+](\lambda|\xi_\pm)}{\sinh(2\rho_0)} = \mp\cos\gamma\,e^{\pm(2\lambda+i\gamma)}+O(e^{-2\rho_0}), \tag{B.34}$$

$$\begin{aligned}\frac{J_{SG}[f_+](\lambda|\xi_\pm)}{\sinh(2\rho_0)} = {}&\pm e^{\pm i\gamma}\,\frac{\pi}{2\gamma}\left[\tan\frac{\pi^2}{2\gamma}-\cot\gamma\right]\,e^{\pm(1+p)\lambda}\,e^{(1-p)\rho_0}\mp\cos\gamma\,e^{\pm(2\lambda+i\gamma)}\\ &+O(e^{-2\rho_0}),\end{aligned} \tag{B.35}$$

$$\frac{J_S[f_-](\lambda|\xi_\pm)}{\sinh(2\,\rho_0)} = -\cos\gamma\,e^{\pm(2\lambda+i\gamma)}+O(e^{-2\rho_0}), \tag{B.36}$$

$$\begin{aligned}\frac{J_{SG}[f_-](\lambda|\xi_\pm)}{\sinh(2\rho_0)} = {}&e^{\pm i\gamma}\,\frac{\pi}{2\gamma}\left[\tan\frac{\pi^2}{2\gamma}-\cot\gamma\right]\,e^{\pm(1+p)\lambda}\,e^{(1-p)\rho_0}-\cos\gamma\,e^{\pm(2\lambda+i\gamma)}\\ &+O(e^{-2\rho_0}).\end{aligned} \tag{B.37}$$

Finally inserting these leading order expressions into (4.10) and (4.11), one ends up with (4.14) with $\mathcal{K}_\pm(\lambda|\rho_0)$ given by (4.16).

B.1 The large N magnitude of the terms $\mathcal{O}^{\pm}_{XX}$, $\mathcal{O}^{\pm}_{X\mathcal{G}}$, and $\mathcal{O}^{\pm}_{\mathcal{G}\mathcal{G}}$

At the derivation of (4.17) we omitted the contributions of the terms $\mathcal{O}^{\pm}_{XX}$, $\mathcal{O}^{\pm}_{X\mathcal{G}}$ and $\mathcal{O}^{\pm}_{\mathcal{G}\mathcal{G}}$ from (3.53). The reason for this was that, according to the anticipated result (4.5), these terms are next to leading order ones with respect to $\mathcal{O}_0^\pm$, $\mathcal{O}_X^\pm$, and $\mathcal{O}_{\mathcal{G}}^\pm$ in the continuum limit. In this subsection we present the proof of the second line of (4.5). Namely, we show , that the multilinear or quadratic in $\mathcal{G}$ and X terms are indeed of order $\frac{1}{N^2}$ in the large N limit and so they are really negligible with respect to the constant and purely linear terms.

Formulas (3.57), (3.58) and (3.59) together with (A.13)–(A.15) imply the following representations for $\mathcal{O}_{XX}^\pm$, $\mathcal{O}_{X\mathcal{G}}^\pm$, and $\mathcal{O}_{\mathcal{G}\mathcal{G}}^\pm$:

$$\begin{aligned}
\mathcal{O}_{X\mathcal{G}}^\pm &= \frac{1}{\sinh(2\rho_0)} \sum_{\alpha=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, \mathcal{F}_\alpha^{(\lambda)}(\lambda + i\,\alpha\,\eta) \sum_{j=1}^{m_H} F_{XX}[f_\pm](h_j, \lambda + i\,\alpha\,\eta) \times \\
&\quad \times \left\{ X(h_j|\xi_-)\, \mathcal{G}(\lambda + i\,\alpha\,\eta|\xi_+) - X(h_j|\xi_+)\, \mathcal{G}(\lambda + i\,\alpha\,\eta|\xi_-) \right\}, \qquad \text{(B.38)} \\
\mathcal{O}_{XX}^\pm &= \frac{1}{\sinh(2\rho_0)} \sum_{j=1}^{m_H} \sum_{k=1}^{m_H} X(h_j|\xi_-)\, F_{XX}[f_\pm](h_j, h_k)\, X(h_k|\xi_+), \qquad \text{(B.39)} \\
\mathcal{O}_{\mathcal{G}\mathcal{G}}^\pm &= \frac{1}{\sinh(2\rho_0)} \sum_{\alpha,\beta=\pm} \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda'}{2\pi} \mathcal{F}_\alpha^{(\lambda)}(\lambda + i\,\alpha\,\eta)\, \mathcal{F}_\beta^{(\lambda)}(\lambda' + i\,\beta\,\eta) \times \\
&\quad \times \mathcal{G}(\lambda + i\,\alpha\,\eta|\xi_-)\, \mathcal{G}(\lambda' + i\,\beta\,\eta|\xi_+)\, F_{XX}[f_\pm](\lambda + i\,\alpha\,\eta, \lambda' + i\,\beta\,\eta), \qquad \text{(B.40)}
\end{aligned}$$

where according to (A.17), (A.20) and (A.21):

$$\begin{aligned}
F_{XX}[f_\pm](\lambda, \lambda') &= f_\pm(\lambda', \lambda) + \int_{-\infty}^{\infty} d\lambda'' G_\lambda(\lambda - \lambda'')\, f(\lambda'', \lambda') \\
&- \int_{-\infty}^{\infty} d\lambda'' G_\lambda(\lambda' - \lambda'')\, f(\lambda'', \lambda) - \int_{-\infty}^{\infty} d\lambda'' \int_{-\infty}^{\infty} d\lambda''' G_\lambda(\lambda - \lambda'')\, f(\lambda'', \lambda''')\, G_\lambda(\lambda''' - \lambda'). \qquad \text{(B.41)}
\end{aligned}$$

The function $G_\lambda(\lambda)$ is given by (B.10) and $f_\pm(\lambda, \lambda')$ is defined in (3.51) and (3.52). To make apparent the ρ_0 dependence of $f_\pm$, we rephrase them as follows:

$$\begin{aligned}
f_+(\lambda, \lambda') &= \cosh(2\rho_0)\, f_+^{(1)}(\lambda, \lambda') + f_+^{(2)}(\lambda, \lambda'), \qquad \text{(B.42)} \\
f_-(\lambda, \lambda') &= \sinh(2\rho_0)\, f_-^{(1)}(\lambda, \lambda'), \qquad \text{(B.43)}
\end{aligned}$$

with

$$\begin{aligned}
f_+^{(1)}(\lambda, \lambda') &= 2\cos(\gamma)\, \frac{[\sinh(2\lambda + i\,\gamma) - \sinh(2\lambda' + i\,\gamma)]}{\cosh(2(\lambda - \lambda')) - \cos(2\gamma)}, \qquad \text{(B.44)} \\
f_+^{(2)}(\lambda, \lambda') &= -2\cos(\gamma)\, \frac{\sinh(2(\lambda - \lambda'))}{\cosh(2(\lambda - \lambda')) - \cos(2\gamma)}, \qquad \text{(B.45)} \\
f_-^{(1)}(\lambda, \lambda') &= 2\cos(\gamma)\, \frac{\cosh(2\lambda + i\,\gamma) - \cosh(2\lambda' + i\,\gamma)}{\cosh(2(\lambda - \lambda')) - \cos(2\gamma)}. \qquad \text{(B.46)}
\end{aligned}$$

The point in the representations (B.42) and (B.43) is that the ρ_0 dependence is lifted as a prefactor, and the coefficient functions $f_\pm^{(1)}$ and $f_+^{(2)}$ are ρ_0 independent.

To determine the magnitude of $\mathcal{O}_{XX}^\pm$, $\mathcal{O}_{X\mathcal{G}}^\pm$, and $\mathcal{O}_{\mathcal{G}\mathcal{G}}^\pm$, first one has to compute the large N magnitudes of each building blocks of the formulas (B.38), (B.39) and (B.40).

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In all the three quantities the functional $F_{XX}[f_\pm](\lambda,\lambda')$ given by (B.41) arises. Now we show , that it is of order $e^{2\rho_0}$ at large ρ_0. Formulas (B.44), (B.45) and ((B.46)) for the functions $f_\pm^{(1)}$, $f_+^{(2)}$, imply that these functions have constant asymptotics at infinity in each of their variables. On the other hand (B.10) implies that $G_\lambda(\lambda)$ is ρ_0 independent and has the large λ asymptotics:

$$G_\lambda(\lambda) \sim e^{-\alpha_G |\lambda|}, \qquad \text{with:} \quad \alpha_G = 1 + \text{Min}\left(p, 1+\frac{2}{p}\right), \qquad 0<p, \tag{B.47}$$

with p being the coupling constant defined by (2.19). These large λ and λ' asymptotics ensure, that all integrals will converge in (B.41). Furthermore, if one uses the representations (B.42) and (B.43) in (B.41), then it becomes obvious that the ρ_0 dependence is only given by the trivial factors $\cosh(2\rho_0)$ or $\sinh(2\rho_0)$ of (B.42) and (B.43), respectively, such that these factors can be lifted in front of the convergent ρ_0 independent integrals. This implies, that $F_{XX}[f_\pm] \sim e^{2\rho_0}$ in the large ρ_0 limit.

The next common building block in (B.38), (B.39) and (B.40) is a trivial ρ_0 dependent prefactor: $\frac{1}{\sinh(2\rho_0)}$ which is of order $e^{-2\rho_0}$, when $\rho_0 \to \infty$.

Formulas (B.38), (B.39) and (B.40) are multilinear in $\mathcal{G}(\lambda|\xi_\pm)$ and $X(\lambda|\xi_\pm)$. Their large N magnitudes can be read off from (4.3) and they both turn to be of order $\frac{1}{N}$ in the large N limit.

The amount of information provided so far is enough to give the large N estimate for $\mathcal{O}^\pm_{XX}$ given by (B.39). Multiplying the magnitudes of the building blocks immediately leads to the large N estimate: $\mathcal{O}^\pm_{XX} \sim \frac{1}{N^2}$.

To prove that $\mathcal{O}^\pm_{X\mathcal{G}}$ and $\mathcal{O}^\pm_{\mathcal{G}\mathcal{G}}$ are also of order $\frac{1}{N^2}$, one should deal with the large N limit of $\mathcal{F}^{(\lambda)}_\pm(\lambda \pm i\,\eta)$, too. This function is defined in (3.31) and due to (4.1) it becomes of order one in the large N limit. This is why, at leading order its continuum counterpart $\mathcal{F}^{(\lambda)}_{\pm,c}(\lambda \pm i\,\eta)$ can be substituted into the formulas (B.38) and (B.40). Equation (2.34) implies, that at large λ this function decays as: $\mathcal{F}^{(\lambda)}_{\pm,c}(\lambda \pm i\,\eta) \sim e^{-\ell \sinh \frac{\pi}{\gamma}(\lambda \pm i\eta)}$. This extremely rapid decay at infinity ensures the convergence of all integrals entering the expressions occurring in (B.38) and (B.40). This implies that, the large N magnitudes of $\mathcal{O}^\pm_{X\mathcal{G}}$ and $\mathcal{O}^\pm_{\mathcal{G}\mathcal{G}}$ are given by the product of the magnitudes of the basic building blocks determined above.

For completeness we summarize below the magnitudes of the important building blocks of $\mathcal{O}^\pm_{XX}$, $\mathcal{O}^\pm_{X\mathcal{G}}$, and $\mathcal{O}^\pm_{\mathcal{G}\mathcal{G}}$ together with the relation between ρ_0 and N given by (2.17):

$$\begin{aligned} \frac{1}{N} \sim e^{-(1+p)\rho_0}, \quad \frac{1}{\sinh(2\rho_0)} \sim e^{-2\rho_0} \qquad & F_{XX}[f_\pm](\lambda,\lambda') \sim f_\pm(\lambda,\lambda') \sim e^{2\rho_0}, \\ \mathcal{G}(\lambda|\xi_\pm) \sim e^{-(1+p)\rho_0}, \quad X(\lambda|\xi_\pm) \sim e^{-(1+p)\rho_0}. & \end{aligned} \tag{B.48}$$

We emphasize, that these ρ_0 dependences are not entangled with the λ dependence of the quantities they belong to, thus they can be lifted in front of ρ_0 independent convergent sums and integrals entering the formulas (B.38), (B.39) and (B.40) after replacing (B.41) with the representations (B.42) and (B.43) into them. This implies that the magnitude of the individual building blocks listed in (B.48) determine that large N magnitudes of $\mathcal{O}^\pm_{XX}$,

$\mathcal{O}^{\pm}_{X\mathcal{G}}$ and $\mathcal{O}^{\pm}_{\mathcal{G}\mathcal{G}}$:

$$\begin{aligned}
\mathcal{O}^{\pm}_{XX} &\sim \frac{1}{\sinh(2\rho_0)}\; X\; X\; F_{XX}[f_\pm] \sim e^{-2\,(1+p)\rho_0} \sim \frac{1}{N^2},\\
\mathcal{O}^{\pm}_{X\mathcal{G}} &\sim \frac{1}{\sinh(2\rho_0)}\; X\; \mathcal{G}\; F_{XX}[f_\pm] \sim e^{-2\,(1+p)\rho_0} \sim \frac{1}{N^2},\\
\mathcal{O}^{\pm}_{\mathcal{G}\mathcal{G}} &\sim \frac{1}{\sinh(2\rho_0)}\; \mathcal{G}\; \mathcal{G}\; F_{XX}[f_\pm] \sim e^{-2\,(1+p)\rho_0} \sim \frac{1}{N^2}.
\end{aligned} \tag{B.49}$$

One can see from (B.49) that each expression is of order $\frac{1}{N^2} \sim a^2$ in the large N limit, consequently they are negligible in the continuum limit

C Large argument series representations

In this appendix we list the large argument expansions of the convolutions being necessary for the explicit computations presented in appendix B. As it was mentioned in appendix B, the series representations listed below, can be obtained by evaluating the Fourier-representations of the convolutions with the help of the residue theorem. The constituent functions of the relevant convolutions together with their Fourier-transforms are listed in (B.4)–(B.13). The definitions of the necessary convolutions together with their large argument expansion read as follows:

$$\begin{aligned}
(G_\lambda \star g \star F_c^{+})(\lambda) \overset{\lambda\to\pm\infty}{=} \sum_{k=0}^{\infty} \Big\{ & U^{(\pm)}_{1,k}\, e^{\mp 2(1+k)\lambda} + U^{(\pm)}_{2,k}\, e^{\mp(1+2k)\frac{\pi}{\gamma}\lambda} + U^{(\pm)}_{3,k}\, e^{2\lambda\mp(1+2k)\frac{\pi}{\gamma}\lambda} \\
&+ U^{(\pm)}_{4,k}\, e^{\mp 2(1+k)\frac{\pi}{\pi-\gamma}\lambda} \Big\}, \qquad \text{with:}
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
U^{(+)}_{1,k} &= \csc(2\gamma)\sin(2\gamma(k+1))\sec(\gamma(k+2)),\\
U^{(+)}_{2,k} &= \frac{\pi(-1)^{-k}\csc^2(\gamma)\sec(\gamma)\tan\left(\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma},\\
U^{(+)}_{3,k} &= \frac{\pi\csc^2(\gamma)\sec(\gamma)\sin^2\left(\frac{(\pi-2\gamma)(-2\gamma+2\pi k+\pi)}{2\gamma}\right)\csc\left(\frac{(\pi-\gamma)(-2\gamma+2\pi k+\pi)}{2\gamma}\right)\sec\left(\frac{\pi(\gamma+2\pi k+\pi)}{2\gamma}\right)}{4\gamma},\\
U^{(+)}_{4,k} &= \frac{\pi(-1)^{-k}\csc(2\gamma)\sin^2\left(\frac{\pi(\pi-2\gamma)(k+1)}{\pi-\gamma}\right)\csc\left(\frac{\pi^2(k+1)}{\pi-\gamma}\right)\sec\left(\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)\sec\left(\gamma+\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)}{2(\pi-\gamma)},\\
U^{(-)}_{1,k} &= \csc(2\gamma)\sin(2\gamma(k+1))\sec(\gamma k),\\
U^{(-)}_{2,k} &= -\frac{\pi(-1)^{k}\csc^2(\gamma)\sec(\gamma)\tan\left(\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma},\\
U^{(-)}_{3,k} &= \frac{\pi\csc^2(\gamma)\sec(\gamma)\sin^2\left(\frac{(\pi-2\gamma)(2\gamma+2\pi k+\pi)}{2\gamma}\right)\csc\left(\frac{(\pi-\gamma)(2\gamma+2\pi k+\pi)}{2\gamma}\right)\sec\left(\frac{\pi(-\gamma+2\pi k+\pi)}{2\gamma}\right)}{4\gamma},\\
U^{(-)}_{4,k} &= \frac{\pi(-1)^{k}\csc(\gamma)\sec(\gamma)\sin^2\left(\frac{\pi(\pi-2\gamma)(k+1)}{\pi-\gamma}\right)\csc\left(\frac{\pi^2(k+1)}{\pi-\gamma}\right)\sec\left(\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)\sec\left(\frac{\gamma(\gamma+\pi k)}{\pi-\gamma}\right)}{4\pi-4\gamma}.
\end{aligned} \tag{C.2}$$

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$$(G_\lambda \star g \star F_c^-)(\lambda) \overset{\lambda\to\pm\infty}{=} \sum_{k=0}^{\infty} \Big\{ Z_{1,k}^{(\pm)}\, e^{\mp 2(1+k)\lambda} + Z_{2,k}^{(\pm)}\, e^{\mp(1+2k)\frac{\pi}{\gamma}\lambda} + Z_{3,k}^{(\pm)}\, e^{-2\lambda\mp(1+2k)\frac{\pi}{\gamma}\lambda} + Z_{4,k}^{(\pm)}\, e^{\mp 2(1+k)\frac{\pi}{\pi-\gamma}\lambda} \Big\}, \qquad \text{with:} \tag{C.3}$$

$$\begin{aligned}
Z_{1,k}^{(+)} &= \csc(2\gamma)\sin(2\gamma(k+1))\sec(\gamma k),\\
Z_{2,k}^{(+)} &= \frac{\pi(-1)^{-k}\csc^2(\gamma)\sec(\gamma)\cot\left(\frac{\pi(\gamma+2\pi k+\pi)}{2\gamma}\right)}{4\gamma},\\
Z_{3,k}^{(+)} &= -\frac{\pi(-1)^{-k}\csc^2(\gamma)\sec(\gamma)\sin^2\left(\frac{\pi^2(2k+1)}{2\gamma}-2\gamma\right)\sec\left(\frac{\pi^2(2k+1)}{2\gamma}-\gamma\right)\sec\left(\frac{\pi(\gamma+2\pi k+\pi)}{2\gamma}\right)}{4\gamma},\\
Z_{4,k}^{(+)} &= \frac{\pi(-1)^{-k}\csc(\gamma)\sec(\gamma)\sin^2\left(\frac{\pi(\pi-2\gamma)(k+1)}{\pi-\gamma}\right)\csc\left(\frac{\pi^2(k+1)}{\pi-\gamma}\right)\sec\left(\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)\sec\left(\frac{\gamma(\gamma+\pi k)}{\pi-\gamma}\right)}{4\pi-4\gamma},\\
Z_{1,k}^{(-)} &= \csc(2\gamma)\sin(2\gamma(k+1))\sec(\gamma(k+2)),\\
Z_{2,k}^{(-)} &= \frac{\pi(-1)^{k}\csc^2(\gamma)\sec(\gamma)\tan\left(\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma}\\
Z_{3,k}^{(-)} &= -\frac{\pi(-1)^{k}\csc^2(\gamma)\sec(\gamma)\sin^2\left(2\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)\sec\left(\frac{\pi(-\gamma+2\pi k+\pi)}{2\gamma}\right)\sec\left(\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma},\\
Z_{4,k}^{(-)} &= \frac{\pi(-1)^{k}\csc(2\gamma)\sin^2\left(\frac{\pi(\pi-2\gamma)(k+1)}{\pi-\gamma}\right)\csc\left(\frac{\pi^2(k+1)}{\pi-\gamma}\right)\sec\left(\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)\sec\left(\gamma+\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)}{2(\pi-\gamma)},
\end{aligned}$$

$$(G_\lambda^+ \star g \star F_c)(\lambda) \overset{\lambda\to\pm\infty}{=} \sum_{k=0}^{\infty} \Big\{ V_{1,k}^{(\pm)}\, e^{\mp 2(1+k)\lambda} + V_{2,k}^{(\pm)}\, e^{\mp(1+2k)\frac{\pi}{\gamma}\lambda} + V_{3,k}^{(\pm)}\, e^{2\lambda\mp(1+2k)\frac{\pi}{\gamma}\lambda} + V_{4,k}^{(\pm)}\, e^{2\lambda\mp 2(1+k)\frac{\pi}{\pi-\gamma}\lambda} \Big\}, \qquad \text{with:} \tag{C.4}$$

$$\begin{aligned}
V_{1,k}^{(+)} &= \csc(\gamma)\sec(\gamma)\sin(\gamma(k+1)),\\
V_{2,k}^{(+)} &= \frac{\pi(-1)^{-k}\csc^2(\gamma)\sec(\gamma)\sin\left(\frac{\pi^2(2k+1)}{2\gamma}-2\gamma\right)\sec\left(\frac{\pi^2(2k+1)}{2\gamma}-\gamma\right)}{4\gamma},\\
V_{3,k}^{(+)} &= -\frac{\pi(-1)^{-k}\csc^2(\gamma)\sec(\gamma)\sin\left(2\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)\csc\left(\frac{\pi(\gamma+2\pi k+\pi)}{2\gamma}\right)}{4\gamma},\\
V_{4,k}^{(+)} &= -\frac{\pi(-1)^{-k}\csc(\gamma)\sec(\gamma)\sin\left(\frac{\pi(\pi-2\gamma)(k+1)}{\pi-\gamma}\right)\sin\left(\frac{(\pi-2\gamma)(\gamma+\pi k)}{\pi-\gamma}\right)}{4(\pi-\gamma)\,\sin\left(\frac{\pi^2(k+1)}{\pi-\gamma}\right)\,\cos\left(\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)\,\cos\left(\frac{\gamma(\gamma+\pi k)}{\pi-\gamma}\right)},\\
V_{1,k}^{(-)} &= \csc(\gamma)\sec(\gamma)\sin(\gamma(k+1)) - \delta_{k,0}\,\frac{\pi\sec(\gamma)}{2\pi-2\gamma},\\
V_{2,k}^{(-)} &= -\frac{\pi(-1)^{k}\csc^2(\gamma)\sec(\gamma)\sin\left(2\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)\sec\left(\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma},
\end{aligned}$$

$$V_{3,k}^{(-)} = \frac{\pi(-1)^k \csc^2(\gamma)\sec(\gamma)\sin\left(\frac{\pi^2(2k+1)}{2\gamma} - 2\gamma\right)\sec\left(\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma},$$

$$V_{4,k}^{(-)} = -\frac{\pi(-1)^k \csc(\gamma)\sec(\gamma)\sin\left(\frac{\pi(\pi-2\gamma)(k+1)}{\pi-\gamma}\right)\sin\left(\frac{(\pi-2\gamma)(\pi(k+2)-\gamma)}{\pi-\gamma}\right)}{4(\pi-\gamma)\,\sin\left(\frac{\pi^2(k+1)}{\pi-\gamma}\right)\cos\left(\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)\cos\left(\frac{\gamma(\pi(k+2)-\gamma)}{\pi-\gamma}\right)}.$$

$$(G_\lambda^- \star g \star F_c)(\lambda) \overset{\lambda\to\pm\infty}{=} \sum_{k=0}^{\infty}\Big\{ J_{1,k}^{(\pm)}\, e^{\mp 2(1+k)\lambda} + J_{2,k}^{(\pm)}\, e^{\mp(1+2k)\frac{\pi}{\gamma}\lambda} + J_{3,k}^{(\pm)}\, e^{-2\lambda\mp(1+2k)\frac{\pi}{\gamma}\lambda} + J_{4,k}^{(\pm)}\, e^{-2\lambda\mp 2(1+k)\frac{\pi}{\pi-\gamma}\lambda}\Big\}, \qquad \text{with:} \tag{C.5}$$

$$J_{1,k}^{(+)} = \csc(\gamma)\sec(\gamma)\sin(\gamma(k+1)) - \delta_{k,0}\,\frac{\pi\sec(\gamma)}{2\pi-2\gamma},$$

$$J_{2,k}^{(+)} = -\frac{\pi(-1)^{-k}\csc^2(\gamma)\sec(\gamma)\sin\left(2\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)\sec\left(\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma},$$

$$J_{3,k}^{(+)} = \frac{\pi(-1)^{-k}\csc^2(\gamma)\sec(\gamma)\sin\left(\frac{\pi^2(2k+1)}{2\gamma}-2\gamma\right)\sec\left(\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma}$$

$$J_{4,k}^{(+)} = \frac{\pi(-1)^{-k}\csc(2\gamma)\sin\left(\frac{\pi(\pi-2\gamma)(k+1)}{\pi-\gamma}-2\gamma\right)\sin\left(\frac{\pi(\pi-2\gamma)(k+1)}{\pi-\gamma}\right)}{2(\pi-\gamma)\,\sin\left(\frac{\pi(\gamma+\pi(-k-2))}{\pi-\gamma}\right)\cos\left(\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)\cos\left(\frac{\gamma(\pi(k+2)-\gamma)}{\pi-\gamma}\right)},$$

$$J_{1,k}^{(-)} = \csc(\gamma)\sec(\gamma)\sin(\gamma k),$$

$$J_{2,k}^{(-)} = \frac{\pi(-1)^k\csc^2(\gamma)\sec(\gamma)\sin\left(\frac{\pi^2(2k+1)}{2\gamma}-2\gamma\right)\sec\left(\frac{\pi^2(2k+1)}{2\gamma}-\gamma\right)}{4\gamma},$$

$$J_{3,k}^{(-)} = -\frac{\pi(-1)^k\csc^2(\gamma)\sec(\gamma)\sin\left(2\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)\sec\left(\frac{\pi^2(2k+1)}{2\gamma}\right)}{4\gamma},$$

$$J_{4,k}^{(-)} = \frac{\pi(-1)^k\csc(\gamma)\sec(\gamma)\sin\left(\frac{(\pi-2\gamma)(\gamma+\pi k)}{\pi-\gamma}\right)\sin\left(\frac{\pi(\gamma+2\gamma k-\pi k)}{\pi-\gamma}\right)}{4(\pi-\gamma)\,\sin\left(\frac{\pi(\gamma+\pi k)}{\pi-\gamma}\right)\cos\left(\frac{\pi\gamma(k+1)}{\pi-\gamma}\right)\cos\left(\frac{\gamma(\gamma+\pi k)}{\pi-\gamma}\right)}.$$

$$(G_\lambda \star \psi' \star \chi_F)(\lambda) \overset{\lambda\to\pm\infty}{=} \sum_{k=1}^{\infty}\Big\{\left(2\lambda\,\hat{I}_{1,k}^{(\pm)} + I_{1,k}^{(\pm)}\right) e^{\mp(2k-1)\frac{\pi}{\gamma}\lambda} + I_{2,k}^{(\pm)}\, e^{\mp 2k\lambda} + I_{3,k}^{(\pm)}\, e^{\mp 2k\frac{\pi}{\pi-\gamma}\lambda}\Big\}, \tag{C.6}$$

with:

$$\hat{I}_{1,k}^{(+)} = \frac{(-1)^{k+1}(p+1)^2}{\pi},$$

$$I_{1,k}^{(+)} = \frac{1}{2}(-1)^k(p+1)(3\cos(\pi(1-2k)p)-1)\csc(\pi(1-2k)p),$$

$$I_{2,k}^{(+)} = \frac{1}{4}(-1)^k\sin\left(\frac{2\pi k(p-1)}{p+1}\right)\csc\left(\frac{\pi kp}{p+1}\right)\sec^2\left(\frac{\pi k}{p+1}\right),$$

$$I_{3,k}^{(+)} = \frac{(p+1)\sec(\pi k)\sin\left(\frac{2\pi k(p-1)}{p}\right)\csc\left(\frac{\pi kp+\pi k}{p}\right)\sec^2\left(\frac{\pi k}{p}\right)}{4p},$$

$$\hat{I}_{1,k}^{(-)} = \frac{(-1)^{k+1}(p+1)^2}{\pi},$$

$$I_{1,k}^{(-)} = -\frac{1}{2}(-1)^{k+1}(p+1)(3\cos(\pi(2(k+1)+1)p)-1)\csc(\pi(2(k+1)+1)p),$$

$$I_{2,k}^{(-)} = -\frac{1}{4}(-1)^k\sin\left(\frac{2\pi k(p-1)}{p+1}\right)\csc\left(\frac{\pi kp}{p+1}\right)\sec^2\left(\frac{\pi k}{p+1}\right),$$

$$I_{3,k}^{(-)} = \frac{(p+1)\sec\left(\frac{\pi k}{p}\right)}{2p}.$$

$$(g\star F_c)(\lambda) \overset{\lambda\to\pm\infty}{=} \sum_{k=0}^{\infty}\left\{H_{1,k}^{(\pm)}\,e^{\mp 2(k+1)\lambda} + H_{2,k}^{(\pm)}\,e^{\mp(2k+1)\frac{\pi}{\gamma}\lambda}\right\}, \qquad \text{with:} \tag{C.7}$$

$$H_{1,k}^{(+)} = \csc(\gamma)\sec(\gamma)\sin(\gamma k),$$

$$H_{2,k}^{(+)} = -\frac{\pi(-1)^k\csc(2\gamma)}{\gamma},$$

$$H_{1,k}^{(-)} = \csc(\gamma)\sec(\gamma)\sin(\gamma k),$$

$$H_{2,k}^{(-)} = -\frac{\pi(-1)^k\csc(2\gamma)}{\gamma},$$

$$(g\star F_c^+)(\lambda) \overset{\lambda\to\pm\infty}{=} \sum_{k=0}^{\infty}\left\{K_{1,k}^{(\pm)}\,e^{\mp 2(k+1)\lambda} + K_{2,k}^{(\pm)}\,e^{2\lambda\mp(2k+1)\frac{\pi}{\gamma}\lambda}\right\}, \qquad \text{with:} \tag{C.8}$$

$$K_{1,k}^{(+)} = \csc(2\gamma)\sin(2\gamma k)\sec(\gamma+\gamma k),$$

$$K_{2,k}^{(+)} = -\frac{\pi(-1)^{-k}\csc(2\gamma)\sin\left(\frac{(\pi-2\gamma)(2\gamma-2\pi(k+1)+\pi)}{2\gamma}\right)\sec\left(\frac{\pi(\gamma+2\pi k+\pi)}{2\gamma}\right)}{\gamma},$$

$$K_{1,k}^{(-)} = \csc(2\gamma)\sin(2\gamma k)\sec(\gamma-\gamma k),$$

$$K_{2,k}^{(-)} = -\frac{\pi(-1)^k\csc(2\gamma)\sin\left(\frac{(\pi-2\gamma)(2\gamma+2\pi k+\pi)}{2\gamma}\right)\sec\left(\frac{\pi(-\gamma+2\pi k+\pi)}{2\gamma}\right)}{\gamma}.$$

$$(g\star F_c^-)(\lambda) \overset{\lambda\to\pm\infty}{=} \sum_{k=0}^{\infty}\left\{L_{1,k}^{(\pm)}\,e^{\mp 2(k+1)\lambda} + L_{2,k}^{(\pm)}\,e^{-2\lambda\mp(2k+1)\frac{\pi}{\gamma}\lambda}\right\}, \qquad \text{with:} \tag{C.9}$$

$$L_{1,k}^{(+)} = \csc(2\gamma)\sin(2\gamma k)\sec(\gamma-\gamma k),$$

$$L_{2,k}^{(+)} = -\frac{\pi(-1)^k\csc(2\gamma)\sin\left(\frac{(\pi-2\gamma)(2\gamma+2\pi k+\pi)}{2\gamma}\right)\sec\left(\frac{\pi(-\gamma+2\pi k+\pi)}{2\gamma}\right)}{\gamma},$$

$$L_{1,k}^{(-)} = \csc(2\gamma)\sin(2\gamma k)\sec(\gamma+\gamma k),$$

$$L_{2,k}^{(-)} = -\frac{\pi(-1)^k\csc(2\gamma)\sin\left(2\gamma+\frac{\pi^2(2k+1)}{2\gamma}\right)\sec\left(\frac{\pi(-\gamma+2\pi k+\pi)}{2\gamma}\right)}{\gamma}.$$

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$$(g_{\gamma/2}\star\chi)(\lambda) \overset{\lambda\to\pm\infty}{=} \mp 2\chi(\infty) + \sum_{k=1}^{\infty}\left\{e^{(\pm)}_{1,k}\, e^{\mp 2k\lambda} + e^{(\pm)}_{2,k}\, e^{(1\mp 2k)\frac{\pi}{\gamma}\lambda} + e^{(\pm)}_{3,k}\, e^{\mp\frac{2k\pi}{\pi-\gamma}\lambda}\right\}, \quad \text{(C.10)}$$

$$\begin{aligned}
e^{(+)}_{1,k} &= -2e^{i\gamma k}, & e^{(-)}_{1,k} &= -e^{(+)}_{1,-k},\\
e^{(+)}_{2,k} &= -\frac{i\pi(-1)^k e^{-\frac{i\pi^2(2k-1)}{2\gamma}}\sec\left(\frac{\pi^2(2k-1)}{2\gamma}\right)}{\gamma}, & e^{(-)}_{2,k} &= -e^{(+)}_{2,-k},\\
e^{(+)}_{3,k} &= \frac{\pi\sec\left(\frac{\pi\gamma k}{\pi-\gamma}\right)}{\pi-\gamma}, & e^{(-)}_{3,k} &= -e^{(+)}_{3,-k}.
\end{aligned}$$

Then

$$\begin{aligned}
(f_2\star G)(\lambda) &= 2\cosh(2\rho_0)\,\chi(\infty) + \sinh(2\rho_0)\,(g_{\gamma/2}\star\chi)(\lambda+\rho_0),\\
f_2(\lambda|\rho_0) &= e^{-2\rho_0} + \sinh(2\rho_0)\sum_{k=1}^{\infty} e^{(+)}_{1,k}\, e^{-2k(\lambda+\rho_0)}, &\text{(C.11)}\\
(f_3\star G)(\lambda) &= 2\cosh(2\rho_0+i\gamma)\,\chi(\infty) - \sinh(2\rho_0+i\gamma)\,(g_{\gamma/2}\star\chi)(\lambda-\rho_0),\\
f_3(\lambda|\rho_0) &= e^{-2\rho_0-i\gamma} - \sinh(2\rho_0+i\gamma)\sum_{k=1}^{\infty} e^{(-)}_{1,k}\, e^{2k(\lambda-\rho_0)}. &\text{(C.12)}
\end{aligned}$$

For $\alpha\in(0,\pi)$:

$$(g_\alpha\star\chi_F)(\pm 2\rho_0) = \mp\frac{1}{2} + \sum_{k=0}^{\infty}\hat{a}^{+}_k e^{\mp\frac{2\pi}{\gamma}(1+2k)\rho_0} + \sum_{k=1}^{\infty}\hat{b}^{+}_k e^{\mp 4k\rho_0}, \quad \text{(C.13)}$$

$$\begin{aligned}
\hat{a}^{(+)}_k &= \frac{\pi(-1)^k e^{-\frac{i\pi(\pi-2\alpha)(2k-3)}{2\gamma}}\sec\left(\frac{\pi(\gamma+\pi(2k-3))}{2\gamma}\right)}{\gamma}, & \hat{a}^{-}_k &= -(\hat{a}^{(+)}_k)^*,\\
\hat{b}^{(+)}_k &= -e^{2\,i\,\alpha\,k}\sec(\gamma k), & \hat{b}^{-}_k &= -(\hat{b}^{(+)}_k)^*,
\end{aligned}$$

where here $*$ denotes complex conjugation. Finally, we close this appendix with the large ρ_0 series representation of $I(\rho_0)$ and $\mathcal{T}_0(\rho_0)$ given by (B.19) and (B.20), respectively.

$$\begin{aligned}
I(\rho_0) &= \gamma^2 + \sum_{k=0}^{\infty}\hat{I}_{1,k}\, e^{-4\,k\,\rho_0} + \sum_{k=0}^{\infty}\left\{(\hat{I}_{2,k}+\hat{I}_{3,k}\,\rho_0)\, e^{-\frac{2\pi}{\gamma}\,(1+2k)\rho_0}\right\}, &\text{(C.14)}\\
\hat{I}_{1,k} &= -2\gamma^2\left(\tan^2(\gamma k)-1\right),\\
\hat{I}_{2,k} &= 2\pi\left(\pi\csc^2\left(\frac{\pi^2(2k+1)}{2\gamma}\right) - 2\gamma\right),\\
\hat{I}_{3,k} &= 8\pi\cot\left(\frac{\pi^2(2k+1)}{2\gamma}\right).\\
\mathcal{T}_0(\rho_0) &= e^{-2\rho_0}\, t_0 + \sum_{k=0}^{\infty}\Big\{t_{1,k}\, e^{2\rho_0-\frac{2\pi}{\gamma}(1+2k)\,\rho_0} + t_{2,k}\, e^{-2\rho_0-\frac{2\pi}{\gamma}(1+2k)\,\rho_0}\\
&\qquad + t_{3,k}\, e^{-2\,(k+2)\,\rho_0}\Big\}, &\text{(C.15)}
\end{aligned}$$

with:

$$
\begin{aligned}
t_0 &= 2\gamma^2 \sec(\gamma), \\
t_{1,k} &= 2\pi\gamma \left(\csc(\gamma)\cot\left(\frac{\pi^2(2k+1)}{2\gamma}\right) - \sec(\gamma)\right), \\
t_{2,k} &= -2\pi\gamma \left(\sec(\gamma) + \csc(\gamma)\cot\left(\frac{\pi^2(2k+1)}{2\gamma}\right)\right), \\
t_{3,k} &= 2\gamma^2 \left[2\sec(\gamma) - \sec(\gamma(k+1))\sec(\gamma(k+2))\right].
\end{aligned}
$$

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