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Localization on $AdS_2 \times S^1$

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Abstract: Conformal symmetry relates the metric on $AdS_2 \times S^1$ to that of S^3. This implies that under a suitable choice of boundary conditions for fields on AdS_2 the partition function of conformal field theories on these spaces must agree which makes $AdS_2 \times S^1$ a good testing ground to study localization on non-compact spaces. We study supersymmetry on $AdS_2 \times S^1$ and determine the localizing Lagrangian for $\mathcal{N} = 2$ supersymmetric Chern-Simons theory on $AdS_2 \times S^1$. We evaluate the partition function of $\mathcal{N} = 2$ supersymmetric Chern-Simons theory on $AdS_2 \times S^1$ using localization, where the radius of S^1 is q times that of AdS_2. With boundary conditions on $AdS_2 \times S^1$ which ensure that all the physical fields are normalizable and lie in the space of square integrable wave functions in AdS_2, the result for the partition function precisely agrees with that of the theory on the q-fold covering of S^3.

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Contents

1 Introduction

The method of localization is a powerful technique to evaluate observables in supersymmetric quantum field theories exactly. The method was first introduced in [1] and developed in [2, 3]. It was revived by the work of [4] which has led to the exact computations in supersymmetric quantum field theories in various dimensions and manifolds. Many of these exact computations have been used to provide highly non-trivial checks of the AdS/CFT correspondence. See the review [5] for a comprehensive list of references.

Most of the activity has been focussed on supersymmetric quantum field theory defined on curved but compact spaces. The main reason is because the localizing Lagrangian one adds is exact under the Fermionic symmetry Q up to boundary terms. Therefore all such terms can be neglected on compact spaces without a boundary. Rigid supersymmetric quantum field theories can be defined on curved space which also includes non-compact spaces [6–12]. Quantum field theories on spaces of the form $AdS_n \times S^m$ are relevant in evaluating black hole entropy as well as entanglement entropy across spherical entangling surfaces in conformal field theories. In this context, localization of $\mathcal{N} = 2$ supergravity was studied in a series of work on $AdS_2 \times S^2$ to obtain black hole entropy of extremal black holes [13–18]. Localization of supergravity on AdS_4 was also studied in the context of evaluating the quantum partition function in the bulk for the ABJM theory [19].

Let us examine the second instance where partition functions of quantum field theories on Anti-de Sitter spaces are important. The Rényi entropy of order q of a spherical entangling surface in a d dimensional conformal field theory can be mapped to the evaluation of the partition function of the theory on a q-fold covering of the sphere S^d. This partition function is in turn related to the thermal partition function of the conformal field theory on $AdS_{d-1} \times S^1$ where the radius of S^1 is q times that of AdS_d [20]. This relation between the partition functions on these surfaces offers a situation in which any formulation of localization on non-compact manifolds is more controlled and can be precisely checked. In the context of localization such issues were previously explored in [21, 22].

To be more concrete we focus our attention to conformal field theories in $d = 3$. Consider the following metric on the 3-sphere

$$ds_A^2 = L^2 \left(\cos^2 \phi d\tilde{\tau}^2 + d\phi^2 + \sin^2 \phi d\theta^2\right), \tag{1.1}$$

where the coordinates take values from

$$0 \leq \tilde{\tau} \leq 2\pi q, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \frac{\pi}{2}. \tag{1.2}$$

When q takes values in the set of positive integers, the metric in (1.1) is that of a q-fold covering of S^3 branched on the circle at $\phi = \pi/2$. Lets denote this space with the metric in (1.1) by A_q. Under the transformation

$$\sinh r = \tan \phi, \tag{1.3}$$

the metric in (1.1) is conformal related to

$$ds_B^2 = L^2(d\tilde{\tau}^2 + dr^2 + \sinh^2 r d\theta^2), \tag{1.4}$$

where the coordinates take values from

$$0 \leq \tilde{\tau} \leq 2\pi q, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq r < \infty. \tag{1.5}$$

We denote this space by B_q. The relation between the metrics is given by

$$ds_A^2 = \cos^2 \phi ds_B^2. \tag{1.6}$$

Partition functions of conformal field theories defined on A_q should equal to the partition function of the same theory defined on B_q with suitably chosen boundary conditions. In [23] it was shown that with fields satisfying normalizable boundary conditions in AdS_2 and a suitable regularization, the thermal partition function of of free conformal scalars and free massless fermions on B_q agreed precisely with that on A_q.

In this paper we would like to test this relationship between the partition functions on spaces A_q and B_q for interacting theories. We will restrict our attention to supersymmetric partition functions obtained by the technique of localization. The simplest example of a non-trivial super conformal field theory in 3 dimensions is that of pure $\mathcal{N}=2$ supersymmetric Chern-Simons gauge theory. This theory has the added simplification of the fact that the fermions and the auxillary scalar do not have kinetic terms and therefore the supersymmetric partition function on S^3 is related to the bosonic Chern-Simons on S^3 upto a normalization factor. The supersymmetric partition function of Chern-Simons theory coupled to matter on A_q was evaluated in [24].

We first set up the supersymmetric transformations of the vector multiplet on $AdS_2 \times S^1$, we then determine the localization Lagragian and the partition function on B_q by evaluating the one-loop determinants using the index approach. Then the indices are evaluated by explicitly solving the differential equations and counting the solutions which contribute to the index. We discuss in detail the boundary conditions on B_q which ensure that the space of functions for which the indices are evaluated lie in the space of normalizable functions in AdS_2. We show that with these boundary conditions the supersymmetric partition function on B_q is identical to the partition function on A_q. This in turn ensures that the partition function of super Chern-Simons theory obtained by localization on B_q agrees with the partition function of pure bosonic Chern-Simons theory. Given this agreement, we further consider a family of non-singular 3-manifolds, labelled by a continuous parameter $s \in [0,1]$, which are conformally equivalent to $\mathrm{AdS}_2 \times S^1$ and show that the index and the partition function do not change. These 3-manifolds are defined using the conformal transformation that does not change the asymptotic boundary conditions on the fields.

The organisation of this paper is as follows. In the next section as a warm up we show that the partition function of the Abelian Chern-Simons theory on $AdS_2 \times S^1$ is independent of q and agrees precisely with that on the space A_q which was obtained in [23]. The analysis of this section will show what are the boundary conditions imposed on the fields in AdS_2 which results in the agreement. In section 3 we study $\mathcal{N}=2$ supersymmetry for the gauge multiplet on $AdS_2 \times S^1$. We solve for the Killing spinors on $AdS_2 \times S^1$ and obtain the supersymmetric transformation under which $\mathcal{N}=2$ Chern-Simons Lagrangian is supersymmetric and then obtain the localizing term. In the section 4 we evaluate the supersymmetric partition function on B_q by performing the one loop determinants using the index method. We discuss the boundary conditions of the functions over which the index is evaluated in detail. We show that the result of the partition function coincides with the supersymmetric partition function on A_q. In section 5 we evaluate the expectation value of a supersymmetric Wilson loop operator. In section 6 we consider a family of 3-manifolds which are conformal to $\mathrm{AdS}_2 \times \mathrm{S}^1$ and show that the index does not change. Finally in section 7 we conclude with the discussion of the implications of these results.

2 Abelian Chern-Simons theory on $AdS_2 \times S^1$

As a warm up we begin testing the relationship between the partition functions on theories defined on the space A_q and B_q by first considering the case of the abelian Chern-Simons theory on the space B_q. The action of this theory is given by

$$S_{\mathrm{CS}} = \frac{i\kappa}{4\pi} \int d^3x \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{2.1}$$

Here $\varepsilon^{\mu\nu\rho}$ is a tensor density with $\varepsilon^{\tau r\theta} = 1$ and is related to Levi Civita tensor $\epsilon^{\mu\nu\rho}$ by

$$\epsilon^{\mu\nu\rho} = \frac{1}{\sqrt{g}}\varepsilon^{\mu\nu\rho}, \tag{2.2}$$

where g is the determinant of the metric. The partition function $\mathcal{Z}_q$ of this theory on A_q was evaluated in appendix C of [23] and it was shown that the result is independent of q and is given by

$$\log Z_q = -\frac{1}{2}\log\kappa. \tag{2.3}$$

It is indeed expected that the Chern-Simons partition function is a topological invariant and therefore should be independent of q. However this result is more significant since the space A_q is not smooth.

Our goal is now to reproduce this dependence on κ by evaluating the partition function of the theory on B_q. For convenience we rescale the co-ordinate

$$L\tilde{\tau} = \tau. \tag{2.4}$$

Without losing any generality we also choose

$$q = \frac{1}{L}. \tag{2.5}$$

Then we obtain the metric

$$ds^2 = d\tau^2 + L^2(dr^2 + \sinh^2 r d\theta^2). \tag{2.6}$$

Now the range of τ is given by

$$0 \leq \tau \leq 2\pi\,. \tag{2.7}$$

We use the covariant gauge

$$\nabla^\mu A_\mu = 0. \tag{2.8}$$

The action including the ghosts then become

$$S_{\text{ghost}} = \int d^3x\sqrt{g}\left(-\bar{c}\Box c + ib\nabla^\mu A_\mu\right). \tag{2.9}$$

Here c is the fermionic ghost, while b is the bosonic ghost. The $\Box$ refers to the Laplacian of a massless scalar with the metric in (2.6). The total action including the ghosts is given by

$$S_{\text{total}} = \frac{i\kappa}{4\pi}\int d^3x\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \int d^3x\sqrt{g}\left(-\bar{c}\Box c + ib\nabla^\mu A_\mu\right). \tag{2.10}$$

Substituting the total action into the path integral and performing the integral over both the bosonic and the fermionic ghosts we are left with the following partition function.

$$Z = (\det\Box)\int D[A_\mu]\delta\left[\nabla^\mu A_\mu\right]\exp\left(\frac{i\kappa}{4}\int d^3x\varepsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho)\right). \tag{2.11}$$

Note that now we need to perform the path integral over configurations of gauge fields such that the covariant gauge condition is satisfied. We will find a suitable set of variables

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where this can be carried out. Before we proceed, it is useful to write down the explicit expression for the determinant of the Laplacian on $AdS_2 \times S^1$. The eigen functions of the Laplacian are given by

$$\Box\Phi_{(\lambda,l;n)} = -\left(\tilde{\lambda}^2 + n^2\right)\Phi_{(\lambda,l;n)}, \qquad \tilde{\lambda}^2 = \frac{1}{L^2}\left(\lambda^2 + \frac{1}{4}\right). \tag{2.12}$$

Here λ, l labels the quantum numbers onf AdS_2 and n labels the Kaluza-Klein modes on S^1. The wave function $\Phi_{(\lambda,l;n)}$ are constructed using the the eigen functions of the scalar Laplacian on AdS_2 together with the Fourier mode on S^1. They are defined as

$$\Phi_{(\lambda,l;n)}(r,\theta,\tau) = \frac{1}{L} g_{\lambda,l}(r,\theta)e^{in\tau}, \tag{2.13}$$

and $g_{\lambda,l}(r,\theta)$ are normalizable eigen functions on AdS_2 which are given for example in [25, 26].[1] The eigen value λ takes values from 0 to ∞ while $\{l,n\} \in \mathbb{Z}$. Using the orthonormal properties of $f_{\lambda,l}$ we have

$$\int d^3x\sqrt{g}(\Phi_{(\lambda,l;n)})^*\Phi_{(\lambda',l';n')} = 2\pi\delta(\lambda-\lambda')\delta_{l,-l'}\delta_{n,-n'}. \tag{2.14}$$

These eigen functions satisfy the following properties near the origin and the at boundary of AdS_2.

$$\begin{aligned} \lim_{r\to 0}\Phi_{(\lambda,l;n)}(r\,\theta,\tau) &\sim r^{|l|}e^{i(l\theta+n\tau)}, \\ \lim_{r\to\infty}\Phi_{(\lambda,l;n)}(r\,\theta,\tau) &\sim e^{-\frac{r}{2}\pm i\lambda r}e^{i(l\theta+n\tau)}. \end{aligned} \tag{2.15}$$

Therefore the c ghosts are expanded in terms of normalizable functions in AdS_2. Using these eigen functions we can write down the following expression for the determinant of the Laplacian.

$$\log(\det\Box) = \sum_{n=-\infty}^{n=\infty}\int_0^\infty d\lambda\mu(\lambda)\log\left(\tilde{\lambda}^2+n^2\right). \tag{2.16}$$

Here $\mu(\lambda)$ is the density of states which is given by

$$\mu(\lambda) = \frac{1}{2\pi L^2}\lambda\tanh(\pi\lambda). \tag{2.17}$$

This expression has to be regularized. We can adopt the regularization procedure given in [23], but we will not need it explicitly.

To impose the delta function in the path integral of (2.11) it is useful to expand the gauge field A_μ in terms of a complete basis. On AdS_2, the vector can be expanded as a gradient of a scalar as well as a transvese component. Furthermore there are discrete modes for the vector on AdS_2 [26].

$$A_m = c_i\partial_m\Phi_i(r,\theta,\tau) + d_i\epsilon_{mn}\partial^n\Phi_i(r,\theta,\tau) + f_j\partial_m\Psi_j(r,\theta,\tau). \tag{2.18}$$

[1] See equation (2.10) of [25].

Here $m \in \{r, \theta\}$ the coordinates of AdS_2. The repeated index over i refers to integration over the AdS_2 eigen value λ and the sum over the angular momentum mode l in AdS_2 together with the sum over the Kaluza-Klein mode n. For example

$$c_i \Phi_i(r, \theta, \tau) = \sum_{l,n} \int_0^\infty d\lambda c_{\{\lambda,l;n\}} \Phi_{(\lambda,l;n)}. \tag{2.19}$$

Note that though there is a sum over the infinite number of angular momentum modes l for each value of λ, it will turn out that the integrand (2.11) is independent of l and therefore we can just sum over the density of states using the measure in (2.17). for each value of λ in evaluating the logarithm of the partition function. From now on we keep track of only the integral over λ and the sum over the Kaluza-Klein modes. The discrete modes in (2.18) will play and important role in our analysis. Let us recall the discrete modes of a vector on AdS_2. There are defined in terms of non-normalizable zero modes of the scalar Laplacian on AdS_2 which are given by

$$\Psi_{\{l,n\}}(r, \theta, \tau) = \frac{1}{\sqrt{2\pi|l|}} \left(\frac{\sinh r}{1 + \cosh r} \right)^{|l|} e^{il\theta} e^{in\tau}, \qquad l = \pm 1, \pm 2, \ldots \tag{2.20}$$

These modes satisfy

$$\Box_{AdS_2} \Psi_{\{l,n\}}(r, \theta, \tau) = 0. \tag{2.21}$$

Note that though these functions are non-normalizable, the gradient of these scalars satisfy the condition

$$\int drd\theta \sqrt{g_{AdS_2}} \nabla_m \Psi_{\{l,n\}} \nabla^m (\Psi_{\{l',n\}})^* = \delta_{l,l'} . \tag{2.22}$$

Thus the summation over j which occurs with the discrete modes in (2.18) refers to the double sum over the angular momentum modes l and the Kaluza-Klein modes n. Since the gauge field A_m is real we have the property

$$c^*_{\{\lambda;n\}} = c_{\{\lambda;-n\}}, \qquad d^*_{\{\lambda;n\}} = d_{\{\lambda;-n\}}, \qquad f^*_{\{l;n\}} = f_{\{-l;-n\}}. \tag{2.23}$$

Finally since the A_τ is a scalar on AdS_2 we can expand it as

$$A_\tau = e_i \Phi_i(r, \theta, \tau). \tag{2.24}$$

From (2.18) and (2.24) we see that the gauge fields satisfy the following boundary conditions in AdS_2.

$$\begin{aligned}
&\lim_{r\to 0} A_r^{(\lambda,l;n)} \sim r^{|l|-1} e^{i(l\theta+n\tau)}, \qquad \lim_{r\to 0} A_\theta^{(\lambda,l;n)} \sim r^{|l|} e^{i(l\theta+n\tau)}, \\
&\lim_{r\to 0} A_\tau^{(\lambda,l;n)} \sim r^{|l|} e^{i(l\theta+n\tau)}, \\
&\lim_{r\to \infty} A_r^{(\lambda,l;n)} \sim e^{-\frac{r}{2} \pm i\lambda r} e^{i(l\theta+n\tau)}, \qquad \lim_{r\to \infty} A_\theta^{(\lambda,l;n)} \sim e^{\frac{r}{2} \pm i\lambda r} e^{i(l\theta+n\tau)}, \\
&\lim_{r\to \infty} A_\tau^{(\lambda,l;n)} \sim e^{-\frac{r}{2} \pm i\lambda r} e^{i(l\theta+n\tau)}.
\end{aligned} \tag{2.25}$$

Note that for the gauge fields in the AdS_2 direction the angular momentum runs from $l \in \{\pm 1, \pm, 2, \cdots\}$.

Let us now find the Jacobian involved in changing the integration over A_μ in (2.11) to the Fourier coefficients $\{c_{\{\lambda;n\}}, d_{\{\lambda;n\}}, e_{\{\lambda;n\}}, f_{\{l;n\}}\}$. We start with the measure over A_μ which is defined with the normalization

$$\int [DA_\mu] \exp\left(-\int d^3x\sqrt{g}A_\mu A^\mu\right) = 1. \tag{2.26}$$

On substituting the expansion given in (2.18) and (2.24) into the exponent we obtain

$$\begin{aligned}\int d^3x\sqrt{g}A_\mu A^\mu = 4\pi\sum_{n=1}^{\infty}\Big[\tilde\lambda^2(c_{\{\lambda;n\}}c_{\{\lambda;-n\}} + d_{\{\lambda;n\}}d_{\{\lambda;-n\}}) + e_{\{\lambda;n\}}e_{\{\lambda;-n\}}\Big] \\ + 2\pi\Big[\tilde\lambda^2(c_{\{\lambda;0\}}c_{\{\lambda;0\}} + d_{\{\lambda;0\}}d_{\{\lambda;0\}}) + e_{\{\lambda;0\}}e_{\{\lambda;0\}}\Big] + \sum_{l,n} f_{l,n}f_{-l-n}.\end{aligned} \tag{2.27}$$

Here we have written down the expression for a given value of λ. Now using this expansion in (2.26) and changing variables we obtain

$$\begin{aligned}&\int\prod_n [dc_{\{\lambda;n\}}d(d_{\{\lambda;n\}})de_{\{\lambda;n\}}]\prod_{n,l}[df_{l,n}]\mathcal{J} \\ &\times\exp\Bigg\{-4\pi\sum_{n=1}^{\infty}\Big[\tilde\lambda^2(c_{\{\lambda;n\}}c_{\{\lambda;-n\}} + d_{\{\lambda;n\}}d_{\{\lambda;-n\}}) + e_{\{\lambda;n\}}e_{\{\lambda;-n\}}\Big] \\ &-2\pi\Big[\tilde\lambda^2(c_{\{\lambda;0\}}c_{\{\lambda;0\}} + d_{\{\lambda;0\}}d_{\{\lambda;0\}}) + e_{\{\lambda;0\}}e_{\{\lambda;0\}}\Big] - \sum_{l,n} f_{l,n}f_{-l-n}\Bigg\} = 1,\end{aligned} \tag{2.28}$$

where $\mathcal{J}$ is the Jacobian involved in the change of integration variables. Again we have written this only for a given value of λ. Peforming the integrations and taking into account of the density of states (2.17) we obtain

$$\log\mathcal{J} = \int d\lambda\mu(\lambda)\left(\log(\tilde\lambda^2) + 2\sum_{n=1}^{\infty}\log(\tilde\lambda^2)\right). \tag{2.29}$$

We have also used the relation (2.23) and the fact that for $n = 0$, the Fourier coefficients are real. Of course one needs to regularize the above expression, in fact the sum over n can be done by using the ζ function regularization. At present we will assume a definite regularization has been chosen and proceed. We now rewrite the delta function in terms of the Fourier coefficients. The divergence on A_μ can be written as

$$\nabla^\mu A_\mu = c_i\Box\Phi_i(r,\theta,\tau) + e_i\partial_\tau\Phi_i(r,\theta,\tau). \tag{2.30}$$

Therefore the delta function which imposes the transversality condition can be written as

$$\delta(\nabla^\mu A_\mu) = \prod_{\lambda;n\neq 0}\Big[\delta(ine_{\lambda;n} - \tilde\lambda^2 c_{\lambda,n})\Big]\prod_\lambda\Big[\delta(-\tilde\lambda^2 c_{\{\lambda,0\}})\Big]. \tag{2.31}$$

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To arrive at this for the $n = 0$ case we have used the fact that there are no normalizable scalars on AdS_2 with zero eigen value for the Laplacian. Now it is easy to see that performing the integration over $c_{\lambda,n}$ results in the Jacobian[2]

$$\log \hat{\mathcal{J}} = -\int d\lambda\mu(\lambda)\left(\log(\tilde{\lambda}^2) + 2\sum_{n=1}^{\infty}\log(\tilde{\lambda}^2)\right). \tag{2.32}$$

Note the factor of 2 for the modes $n \neq 0$ results from the positive and negative Kaluz-Klein modes. Comparing (2.29) and (2.32) we see that the Jacobian resulting from the change of variables precisely cancels on peforming the integral over $c_{\lambda,n}$ using the delta function (2.31) which imposes the transversality condition.

To summarize the set of integration variables left over are $\{e_{\{\lambda,n\}}, d_{\{\lambda,n\}}, f_{\{l,n\}}\}$. The transverse gauge field A_μ is expanded as

$$\begin{aligned}
A_m(x,\tau) &= \frac{in}{\tilde{\lambda}^2} e_{\{\lambda,n\neq 0\}}\partial_m\Phi_{\{\lambda,n\}} + d_{\{\lambda,n\}}\epsilon_{mn}\partial^n\Phi_{\{\lambda,n\}} + f_{\{l,n\}}\partial_m\Psi_i, \\
A_\tau(x,\tau) &= e_{\{\lambda,0\}}\Phi_{\{\lambda,0\}} + e_{\{\lambda,n\neq 0\}}\Phi_{\{\lambda,n\neq 0\}}.
\end{aligned} \tag{2.33}$$

Substituting these modes in the action of the partition function (2.11), we obtain

$$\begin{aligned}
\frac{\kappa}{4\pi}\int d^3x\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho = \kappa\Bigg\{&\tilde{\lambda}^2 d_{\{\lambda,0\}}e_{\{\lambda,0\}} + \sum_{n>0}^{\infty}\left(\tilde{\lambda}^2 + n^2\right)\left(e_{\{\lambda,n\}}d_{\{\lambda,-n\}} - e_{\{\lambda,-n\}}d_{\{\lambda,n\}}\right) \\
&+ \sum_{n,l>0} n(f_{\{l,-n\}}f_{\{-l,n\}} - f_{\{l,n\}}f_{\{-l,-n\}})\Bigg\}.
\end{aligned} \tag{2.34}$$

In the above equation we have consider the modes with fixed λ. In fact there is an integral over λ. Each mode in λ occurs with a density of states given in (2.17). Integrating out $\{e_{\{\lambda,n\}}, d_{\{\lambda,n\}}, f_{\{l,n\}}\}$ we obtain the partition function. As before note that all modes with $n \neq 0$ are complex and we use the relation in (2.23) and the modes $e_{\{\lambda,0\}}, d_{\{\lambda,0\}}$ are real. Performing the gaussian integrations we obtain

$$\log \hat{Z} = -\int_0^\infty d\lambda\mu(\lambda)\left[\log(\kappa\tilde{\lambda}^2) + \sum_{n=1}^{\infty}\log\left(\kappa^2(n^2+\tilde{\lambda}^2)^2\right)\right] - \sum_{n,l=1}^{\infty}\log(\kappa^2 n^2). \tag{2.35}$$

Here $\hat{Z}$ is defined by $Z = (\det\Box)\,\hat{Z}$. After some rearrangements we obtain

$$\begin{aligned}
\log \hat{Z} &= -\int d\lambda\mu(\lambda)\left(\log\kappa + \sum_{n=1}^{\infty}\log(\kappa^2)\right) - \sum_{l,n=1}^{\infty}\log(\kappa^2) \\
&\quad -\int d\lambda\mu(\lambda)\left(\log\tilde{\lambda}^2 + 2\sum_{n=1}^{\infty}(n^2+\tilde{\lambda}^2)\right) - \sum_{n,l=1}^{\infty}\log(n^2).
\end{aligned} \tag{2.36}$$

[2]One can also perform the integration over the variables $e_{\lambda,n}$ and arrive at the same final result.

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We use the ζ function regularization to perform the sums involving

$$\sum_{n=1}^{\infty} 1 = -\frac{1}{2}. \tag{2.37}$$

This results in

$$\log \hat{Z} = = -\frac{1}{2}\log\kappa - \int d\lambda\mu(\lambda)\left(\sum_{n=-\infty}^{\infty}\log(\tilde{\lambda}^2+n^2)\right), \tag{2.38}$$

where we have ignored the κ and L independent constant. We can now substitute the partition function $\hat{\mathcal{Z}}$ in (2.11) and use the expression (2.16) for the determinant of the massless scalar in AdS_2 to obtain

$$\log Z = -\frac{1}{2}\log\kappa. \tag{2.39}$$

Thus we obtain the result the the partition function of the Abelian Chern-Simons theory on $AdS_2 \times S^1$ independent of L and depends on the coupling constant κ precisely the same way as that of the Abelian Chern-Simons theory on the space A_q. It is interesting to note that the discrete modes of the vector on AdS_2 played the crucial role in obtaining the dependence on κ. This resulted in the expected relation between the parition function on the two conformally related spaces. It is also important to observe that the space of functions over which we performed the path integral were all normalizable functions on $AdS_2 \times S^1$.

3 Supersymmetry on $AdS_2 \times S^1$

We would like to consider a non-abelian Chern-Simon theory based on a gauge group G on $AdS_2 \times S^1$ and evaluate its partition function. To do this we use the fact that the field content of $\mathcal{N} = 2$ supersymmetric Chern-Simons theory, apart from the gauge field of interest consists of only auxillary fields without kinetic energy terms. Thus the $\mathcal{N} = 2$ supersymmetric Chern-Simons theory is equivalent to the bosonic Chern-Simons theory with the same gauge group G. Therefore we can determine the partition function of the bosonic Chern-Simons theory on $AdS_2 \times S^1$ by evaluating the partition function of the $\mathcal{N} = 2$ supersymmetric Chern-Simons theory on $AdS_2 \times S^1$ using the technique of localization. In this section we first solve for the Killing spinors on $AdS_2 \times S^1$. We use these Killing spinors to construct the supersymmetric variation of the $\mathcal{N} = 2$ gauge multiplet and demonstrate the invariance of the Chern-Simons action. Finally we determine the locallizing term which is exact under the supersymmetric variation.

3.1 Killing spinors on $AdS_2 \times S^1$

The background metric of $AdS_2 \times S^1$ is given by

$$ds^2 = d\tau^2 + L^2(dr^2 + \sinh^2 r\, d\theta^2)\,. \tag{3.1}$$

The vielbein are $e^1 = d\tau$, $e^2 = L\,dr$, $e^3 = L\sinh r\,d\theta$.

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The non vanishing components of Christoffel symbols and spin connections are given by

$$\Gamma^r_{\theta\theta} = -\cosh r \sinh r, \qquad \Gamma^\theta_{r\theta} = \coth r, \qquad \omega^3{}_2 = \cosh r \, d\theta \,. \tag{3.2}$$

The Killing spinors are solutions of the following equations

$$\begin{aligned}(\nabla_\mu - iA_\mu)\,\epsilon &= -\frac{1}{2}H\gamma_\mu\epsilon - iV_\mu\epsilon - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\epsilon, \\ (\nabla_\mu + iA_\mu)\,\tilde{\epsilon} &= -\frac{1}{2}H\gamma_\mu\tilde{\epsilon} + iV_\mu\tilde{\epsilon} + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\tilde{\epsilon}.\end{aligned} \tag{3.3}$$

Here A_μ, H and V_μ are fields in the supergravity multiplet. At the linearised level A_μ couples to R-symmetry current, H couples to string current and V_μ is the dual of graviphoton field strength which couples to central charge current. ϵ and $\tilde{\epsilon}$ are complex spinors parameterizing the supergravity transformations with R-charge $+1$ and -1, respectively. In Lorentzian signature, ϵ and $\tilde{\epsilon}$ are complex conjugate to each other but in Euclidean theory they are independent complex spinors.

It is clear from killing spinor equations (3.3) that the vector field $K^\mu = \tilde{\epsilon}\gamma^\mu\epsilon$ is a Killing vector i.e. it satisfies the Killing vector equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0\,. \tag{3.4}$$

For our metric background the compact isometries are generated by killing vectors of the form $K_\pm = \frac{\partial}{\partial\tau} \pm \frac{\partial}{\partial\theta}$. We make the following choice of killing vector

$$K = \frac{\partial}{\partial\tau} + \frac{1}{L}\frac{\partial}{\partial\theta}\,. \tag{3.5}$$

This choice of Killing vector simplifies the Killing spinor and supergravity background fields which are given by

$$\begin{aligned}\epsilon &= e^{\frac{i\theta}{2}}\begin{pmatrix} i\cosh\left(\frac{r}{2}\right) \\ \sinh\left(\frac{r}{2}\right)\end{pmatrix}, \qquad && \tilde{\epsilon} = e^{-\frac{i\theta}{2}}\begin{pmatrix} \sinh\left(\frac{r}{2}\right) \\ i\cosh\left(\frac{r}{2}\right)\end{pmatrix}, \\ A_\tau &= V_\tau = \frac{1}{L}, && A_{r,\theta} = H = 0\,.\end{aligned} \tag{3.6}$$

For more details on the solutions of Killing spinor equations see the appendix B.

3.2 Supersymmetry of the vector multiplet

Vector multiplet in $\mathcal{N}=2$ theory in Lorentzian signature contains a real scalar σ, gauge field A_μ, an auxiliary real field G and 2 component Weyl fermions λ and $\tilde{\lambda}$. In order to compute partition function we need to analytically continue to Euclidean space. We choose the analytic continuation where the scalar field σ and the auxiliary field G are purely imaginary, the gauge field A_μ is real and the spinors λ and $\tilde{\lambda}$ are two independent complex spinor. As we will see this choice of analytic continuation makes the bosonic part of the Q-deformation in the action positive definite.

The Euclidean supersymmetry transformation of the fields in a vector multiplet is given by

$$
\begin{aligned}
Q\lambda &= -\frac{i}{4}\epsilon\, G - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu}\epsilon - i\gamma^\mu\epsilon\,(iD_\mu\sigma - V_\mu\sigma)\ ,\\
Q\tilde{\lambda} &= \frac{i}{4}\tilde{\epsilon}\, G - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu}\tilde{\epsilon} + i\gamma^\mu\tilde{\epsilon}\,(iD_\mu\sigma + V_\mu\sigma)\ ,\\
QA_\mu &= \frac{1}{2}\left(\epsilon\gamma_\mu\tilde{\lambda} + \tilde{\epsilon}\gamma_\mu\lambda\right)\ ,\\
Q\sigma &= \frac{1}{2}\left(-\epsilon\tilde{\lambda} + \tilde{\epsilon}\lambda\right)\ ,\\
QG &= -2i\left[D_\mu\left(\epsilon\gamma^\mu\tilde{\lambda} - \tilde{\epsilon}\gamma^\mu\lambda\right) - i\left[\sigma, \epsilon\tilde{\lambda} + \tilde{\epsilon}\lambda\right] - iV_\mu\left(\epsilon\gamma^\mu\tilde{\lambda} + \tilde{\epsilon}\gamma^\mu\lambda\right)\right]\ .
\end{aligned}
\tag{3.7}
$$

The square of the susy transformations on vector multiplet fields are given by

$$
\begin{aligned}
Q^2\lambda &= \mathcal{L}_K\lambda + i[\Lambda, \lambda] - \frac{1}{2L}\lambda\,,\\
Q^2\tilde{\lambda} &= \mathcal{L}_K\tilde{\lambda} + i[\Lambda, \tilde{\lambda}] + \frac{1}{2L}\tilde{\lambda}\,,\\
Q^2A_\mu &= \mathcal{L}_K A_\mu + D_\mu\Lambda\,,\\
Q^2\sigma &= \mathcal{L}_K\sigma - iK^\mu[A_\mu, \sigma]\,,\\
Q^2G &= \mathcal{L}_K G + i[\Lambda, G]\,.
\end{aligned}
\tag{3.8}
$$

Here $\Lambda = \tilde{\epsilon}\epsilon\,\sigma - K^\rho A_\rho$.

Using the above supersymmetry transformations we also note that $Q\Lambda = 0$.

Therefore the algebra of supersymmetry transformation is given by

$$
Q^2 = \mathcal{L}_K + \delta_\Lambda^{\text{gauge transf}} + \delta_{\frac{1}{2L}}^{R\text{-symm}}\,. \tag{3.9}
$$

It is equivalent to work with fermion bilinear (Ψ, Ψ_μ) instead of $(\lambda, \tilde{\lambda})$ which are defined as

$$
\Psi = \frac{i}{2}(\tilde{\epsilon}\lambda + \epsilon\tilde{\lambda})\,, \qquad \Psi_\mu = QA_\mu = \frac{1}{2}(\epsilon\gamma_\mu\tilde{\lambda} + \tilde{\epsilon}\gamma_\mu\lambda)\,. \tag{3.10}
$$

The fermion bi-linears are convenient for the evaluation of the index. The inverse of the above relations expresses $(\lambda, \tilde{\lambda})$ in terms of Ψ, Ψ_μ as

$$
\lambda = \frac{1}{\tilde{\epsilon}\epsilon}\left[\gamma^\mu\epsilon\Psi_\mu - i\epsilon\Psi\right], \qquad \tilde{\lambda} = \frac{1}{\epsilon\tilde{\epsilon}}\left[\gamma^\mu\tilde{\epsilon}\Psi_\mu - i\tilde{\epsilon}\Psi\right]\ . \tag{3.11}
$$

The supersymmetry transformation of the bi-linears are

$$
\begin{aligned}
Q\Psi &= \frac{1}{4}(\tilde{\epsilon}\epsilon)G - \frac{i}{2}\left(\tilde{\epsilon}\gamma^{\mu\nu}\epsilon\right)F_{\mu\nu} - \frac{1}{L}\sigma\,,\\
Q\Psi_\mu &= \mathcal{L}_K A_\mu + D_\mu\Lambda\,.
\end{aligned}
\tag{3.12}
$$

3.3 The localizing action

Next we deform the action by a Q-exact term, $t\,QV_{\rm loc}$. According to the principle of supersymmetric localization, the partition function does not depend on the parameter t and the choice of $V_{\rm loc}$. Thus one can take t to infinity. In this limit the path integral receives contribution from the field configurations which are minima of $QV_{\rm loc}$. One convenient choice of $V_{\rm loc}$ is given by

$$V_{\rm loc} = \int d^3x\sqrt{g}\frac{1}{(\tilde{\epsilon}\epsilon)^2}\,{\rm Tr}\left[\Psi^\mu(Q\Psi_\mu)^\dagger + \Psi(Q\Psi)^\dagger\right]\,. \tag{3.13}$$

The bosonic part of the $QV_{\rm loc}$ action is given by

$$\begin{aligned}QV_{\rm loc\{bosonic\}} &= \int d^3x\sqrt{g}\frac{1}{2(\tilde{\epsilon}\epsilon)^2}\,{\rm Tr}\left[(Q\Psi^\mu)(Q\Psi_\mu)^\dagger + (Q\Psi)(Q\Psi)^\dagger\right]\\ &= \int d^3x\sqrt{g}\,{\rm Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\cosh^2 r}D_\mu(\cosh r\,\sigma)D^\mu(\cosh r\,\sigma) - \frac{1}{32}\left(G - \frac{4\sigma}{L\cosh r}\right)^2\right]\,.\end{aligned} \tag{3.14}$$

The minima of $QV_{\rm loc\{bosonic\}}$ are the solutions of the following equations

$$F_{\mu\nu} = 0\,, \qquad D_\mu(\cosh r\,\sigma) = 0\,, \qquad G = \frac{4\sigma}{L\cosh r}\,. \tag{3.15}$$

In order to solve $F_{\mu\nu} = 0$ for the background gauge field A_μ, we choose a gauge $A_r = 0$. In this gauge we get the following equations for A_μ ,

$$F_{\tau r} = 0 \Rightarrow \partial_r A_\tau = 0\,, \tag{3.16}$$

$$F_{r\theta} = 0 \Rightarrow \partial_r A_\theta = 0\,, \tag{3.17}$$

$$F_{\tau\theta} = 0 \Rightarrow \partial_\tau A_\theta - \partial_\theta A_\tau - i\,[A_\tau, A_\theta]\,. \tag{3.18}$$

The first two equations imply that both A_t and A_θ are independent of r. Requiring that the solutions should be normalizable imply that $A_t = 0$. The third equation imply that A_θ is independent of t and thus can only be function of θ. Now requiring further that the solution of localization equation should be smooth near the origin of AdS$_2$ implies that A_θ should approach to zero. Thus the only solution for the localization equation $F_{\mu\nu} = 0$ is trivial and there are no non trivial smooth and normalizable solutions for A_μ. Thus the solution of localization equation upto gauge transformations is given by

$$A_\mu = 0\,, \qquad \sigma = \frac{i\alpha}{\cosh r}\,, \qquad G = \frac{4i\alpha}{L\cosh^2 r}\,. \tag{3.19}$$

Here α is a real constant matrix valued in Lie algebra. Furthermore on this localization background the gauge transformation parameter in supersymmetry algebra reduces to a constant, $\Lambda^{(0)} = i\alpha$.

The supersymmetric completion of a bosonic Chern-Simons action is given by

$$S_{\rm CS} = \int d^3x\sqrt{g}\,{\rm Tr}\left[i\varepsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \tilde{\lambda}\lambda + \frac{i}{2}G\sigma\right]\,. \tag{3.20}$$

Here $\varepsilon^{\mu\nu\rho} = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu\rho}$, $\epsilon^{\tau\eta\theta} = 1$.

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We note here that the fermions and scalars in the vector multiplet are purely auxiliary fields as they do not have kinetic terms and therefore, one can integrate them out. Thus the supersymmetric Chern-Simons theory is equivalent to a bosonic Chern-Simons theory.

The action in (3.20) is invariant under supersymmetry transformation upto terms which are total derivative

$$QS_{\rm CS} = \int d^3x\sqrt{g}\,\nabla_\rho\,{\rm Tr}\left[\frac{i}{2}\varepsilon^{\rho\mu\nu}A_\mu(\epsilon\gamma_\nu\tilde{\lambda}+\tilde{\epsilon}\gamma_\nu\lambda)+(\epsilon\gamma^\rho\tilde{\lambda}-\tilde{\epsilon}\gamma^\rho\lambda)\sigma\right]. \tag{3.21}$$

In terms of cohomological variable the above boundary terms can be written as

$$QS_{\rm CS} = \int d^3x\sqrt{g}\,\nabla_\rho\,{\rm Tr}\left[\frac{i}{2}\varepsilon^{\rho\mu\nu}A_\mu\Psi_\nu+\frac{2i}{\tilde{\epsilon}\epsilon}(K^\rho\Psi+\varepsilon^{\rho\mu\nu}K_\mu\Psi_\nu)\sigma\right]. \tag{3.22}$$

We will comment more about the boundary terms later when we discuss about the boundary conditions on fields and show that there are no contributions from boundary terms.

4 The one loop determinant

We will now proceed to compute the determinant coming from the quadratic fluctuation of $QV_{\rm loc}$ action. This is done by obtaining the indices of the operators involved in the one loop determinant by explicitly solving for the solutions and counting the ones which contribute to the index. To begin, we simplify our path integral using the gauge invariance which allow us to diagonalize the Lie algebra valued matrix α of the gauge group G. This introduces the Vandermonde determinant in the path integral. Thus our path integral becomes

$$Z = \int d\alpha\,\prod_{\rho>0}(\rho\cdot\alpha)^2\,\exp\left(\frac{\kappa}{4\pi}S_{\rm CS}\right)Z_{\text{1-loop}}(\alpha)\,. \tag{4.1}$$

We will now compute $Z_{\text{1-loop}}(\alpha)$.

4.1 Localization in U(1) Chern-Simons theory

Let us again begin with the warm up example of the one loop determinant about the localizing solution for the case of the Abelian theory for which the evalution of the one loop determinant simplifies considerably. In this case it is very easy to see that the one loop determinant is trivial i.e. independent of the parameter α. The bosonic part of the action given in equation (3.14) at the quadratic order in fluctuations do not have any dependence on α. Therefore, the one loop determinant coming from bosonic fluctuations will not have α dependence. The fermionic part of the $QV_{\rm loc}$-Lagrangian in the Abelian case also does not depend on α. The fermionic part of the Lagrangian is given by

$$\begin{aligned}QV_{\rm loc\{fermionic\}} = {\rm Tr}\Bigg[&-\frac{i}{2L(\tilde{\epsilon}\epsilon)}(\lambda\tilde{\lambda})-\frac{i}{2}V^a(\tilde{\lambda}\gamma_a\lambda)-\frac{1}{2}\tilde{\lambda}\not{D}\lambda-\frac{1}{2}\lambda\not{D}\tilde{\lambda}\\&+\frac{i}{4}\frac{V^a(\epsilon\gamma_a\epsilon)}{\tilde{\epsilon}\epsilon}(\tilde{\lambda}\tilde{\lambda})+\frac{i}{4}\frac{V^a(\tilde{\epsilon}\gamma_a\tilde{\epsilon})}{\tilde{\epsilon}\epsilon}(\lambda\lambda)\Bigg].\end{aligned} \tag{4.2}$$

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Therefore, the one loop determinant coming from fermionic fluctuations will also not have α dependence. Thus upto a normalization constant, the partition function is completely determined by the classical action.

The classical action evaluated at the localizing solution is given by

$$\exp\left(\frac{\kappa}{4\pi}S_{\rm CS}\right) = \exp(-\pi i\alpha^2 L\kappa). \tag{4.3}$$

Thus the partition function of the Abelian Chern-Simons theory is given by

$$Z \sim \int d\alpha \exp(-\pi i\alpha^2 L\kappa) \sim \frac{1}{\sqrt{\kappa L}}. \tag{4.4}$$

As we will show later, a more careful analysis of the determinant shows that the normalization constant do depends on L and in particular it exactly cancel the $\sqrt{L}$ coming from the integral.

4.2 Localization in non-abelian Chern-Simons theory

We will now compute the one loop determinant about the localizing solution for non abelian Chern-Simons gauge theories. For this, first we need to introduce the gauge fixing Lagrangian. This could be achieved by choosing any convenient gauge condition. In our case it turns out that the analysis becomes simpler for the gauge fixing Lagrangian[3]

$$\mathcal{L}_{\rm g.f.} = {\rm Tr}\, Q_B\left[i\tilde{c}\nabla_\mu\left(\frac{1}{\cosh^2 r}a^\mu\right) + \frac{1}{2}\xi\tilde{c}B\right]. \tag{4.5}$$

As we will show below the complete action including the gauge fixing Lagrangian is invariant under BRST transformations on the fields which are given by

$$\begin{aligned} &Q_B a_\mu = D_\mu c, \qquad Q_B\tilde{c} = B, \qquad Q_B c = \frac{i}{2}\{c,c\}, \qquad Q_B\tilde{\lambda} = i\{c,\tilde{\lambda}\} \\ &Q_B\lambda = i\{c,\lambda\}, \qquad Q_B\hat{\sigma} = i[c,\hat{\sigma}], \qquad Q_B\hat{G} = i[c,\hat{G}], \qquad Q_B B = 0. \end{aligned} \tag{4.6}$$

Here a_μ, $\hat{\sigma}$ and $\hat{G}$ are fluctuations away from localizing .

We also define the susy transformations for extra fields

$$Qc = -\Lambda + \Lambda^{(0)}, \qquad QB = \mathcal{L}_K\tilde{c} + i[\Lambda^{(0)},\tilde{c}], \qquad Q\tilde{c} = 0 \tag{4.7}$$

such that the combined transformations generated by $\hat{Q} = Q + Q_B$ satisfy the algebra

$$\hat{Q}^2 = \mathcal{L}_K + \delta^{\text{gauge transf.}}_{\Lambda^{(0)}}. \tag{4.8}$$

To summarize, the complete transformations of fields under $\hat{Q}$ are given by

$$\begin{aligned} &\hat{Q}a_\mu = \Psi_\mu + D_\mu c, && \hat{Q}\hat{\sigma} = Q\hat{\sigma} + i[c,\hat{\sigma}], \\ &\hat{Q}\Psi_\mu = \mathcal{L}_K a_\mu + D_\mu\Lambda + i\{c,\Psi_\mu\}, && \hat{Q}\Psi = \frac{1}{4}(\tilde{\epsilon}\epsilon)\hat{G} - \frac{i}{2}(\tilde{\epsilon}\gamma^{\mu\nu}\epsilon)F_{\mu\nu}(a) - \frac{1}{L}\hat{\sigma} + i\{c,\Psi\}, \\ &\hat{Q}c = -\Lambda + \Lambda^{(0)} + \frac{i}{2}\{c,c\}, && \hat{Q}\tilde{c} = B. \end{aligned} \tag{4.9}$$

[3]As we will show in the subsection 4.3, this gauge choice fixes the gauge completely. It is also possible to show that this gauge choice intersects every gauge orbit.

The gauge fixing Lagrangian (4.5) is not Q closed and therefore we can not use it for the localization problem. But a simple modification of it allows us to use

$$\begin{aligned}\hat{\mathcal{L}}_{\text{g.f.}} &= \operatorname{Tr}\hat{Q}\left[i\tilde{c}\nabla_\mu\left(\frac{1}{\cosh^2 r}a^\mu\right)+\frac{1}{2}\xi\tilde{c}B\right],\\ &= \mathcal{L}_{\text{g.f.}} - i\operatorname{Tr}\tilde{c}\nabla^\mu\left(\frac{1}{\cosh^2 r}\Psi_\mu\right)-\frac{1}{2}\operatorname{Tr}\xi\tilde{c}\left(\mathcal{L}_k\tilde{c}+i[\Lambda^{(0)},\tilde{c}]\right).\end{aligned} \tag{4.10}$$

Clearly $\hat{\mathcal{L}}_{\text{g.f.}}$ is equivalent to $\mathcal{L}_{\text{g.f.}}$ as the rest of the terms do not contribute to the partition function. With $\hat{\mathcal{L}}_{\text{g.f.}}$ our path integral is invariant under $\hat{Q}$ and we will use $\hat{Q}$ for the localization of the path integral.

Boundary conditions. When evaluating partition functions on non compact spaces it is important to specify the choice of boundary conditions obeyed by the fields in the theory. The partition function depends on the boundary conditions chosen. Furthermore to apply the method of supersymmetric localization the boundary conditions must also be invariant under the supersymmetry transformations. In order to achieve this, we begin by imposing the boundary conditions on the fields $(A_\mu, \Psi, c, \tilde{c})$ and then impose the boundary conditions on the rest of the fields following supersymmetry transformations. For example we require that the field $\hat{Q}A_\mu$ has the same boundary condition as that of A_μ.

Looking at the $\hat{Q}V$ Lagrangian we find that in order for the vector field to be square integrable, the components must satisfy the following asymptotic conditions as $r\to\infty$

$$e^{r/2}A_\tau \to 0, \qquad e^{r/2}A_r \to 0, \qquad e^{-r/2}A_\theta \to 0. \tag{4.11}$$

The boundary condition on the ghost field c is determined by using the fact that it is a gauge transformation parameter. It is required that the gauge transformations should not change the asymptotic boundary condition of the gauge field given in (4.11). This forces the ghost field c to have the following asymptotic behaviour as $r\to\infty$

$$c \sim f(\theta) + \tilde{f}(\theta,\tau)\, e^{-r/2} + \dots \tag{4.12}$$

Once we fix the asymptotic behaviour of the allowed gauge transformations, the Faddeev-Popov determinant needs to be computed in this restricted subspace of the gauge transformations. This naturally fixes the $\tilde{c}$ to have the same boundary condition as that of c.[4]

The boundary condition on the field Ψ is determined by the asymptotic behaviour of its $\hat{Q}$ variation (4.9). Given the asymptotic behaviour of the gauge field A_μ and G,[5] we see that $\hat{Q}\Psi$ satisfies the condition $e^{-r/2}\hat{Q}\Psi\to 0$ as $r\to\infty$. We, therefore, require that the field Ψ satisfies the boundary condition $e^{-r/2}\Psi\to 0$.

Next we will discuss the smoothness conditions for the fields near $r\to 0$ which play an important role in analysing the space of kernel and cokernel in the next sections. In

[4]The ghost Lagrangian density in our case is given by $\tilde{c}\nabla_\mu\big(\frac{g^{\mu\nu}}{\cosh^2 r}\nabla_\nu c\big)$. Defining $c=\cosh r\,\omega_1$ and $\tilde{c}=\cosh r\,\omega_2$, where $\omega_{1,2}$ are scalar fields, we see that the corresponding operator for $\omega_{1,2}$ is a self adjoint and maps a mode with given asymptotic behaviour to a mode with the same asymptotic behaviour.

[5]From the bosonic part of $\hat{Q}V$ Lagrangian, we see that the field G is square integrable if $e^{r/2}G\to 0$ as $r\to\infty$.

order to find the regularity conditions near $r \to 0$, we expand the field in terms of Fourier mode, $X(\tau, r, \theta) = X^{(n,p)}(r)e^{(in\tau + ip\theta)}$. Near $r \to 0$, the θ-circle shrinks to zero size and therefore, the regular behaviour of the field is determined by integer p. For any scalar field, collectively denoted by Φ, its Fourier mode $\Phi^{(n,p)}$ needs to have $\sim r^p$ behaviour as $r \to 0$. On the other hand the component of a vector field should satisfy the following regularity conditions

$$\begin{aligned} &A_\tau^{(n,p\neq 0)} \sim r^p, && A_r^{(n,p\neq 0)} \sim r^{p-1}, && A_\theta^{(n,p\neq 0)} \sim r^p, \\ &A_\tau^{(n,p=0)} \sim \mathcal{O}(1), && A_r^{(n,p=0)} \sim r, && A_\theta^{(n,p=0)} \sim r^2. \end{aligned} \tag{4.13}$$

Now let us look at the variation of Chern-Simons action under $\hat{Q}$. Compared to (3.22) we obtain extra terms proportional to the ghost field,

$$\hat{Q}S_{\text{CS}} = \int d^3x\sqrt{g}\,\nabla_\rho\,\text{Tr}\left[\frac{i}{2}\varepsilon^{\rho\mu\nu}A_\mu\hat{\Psi}_\nu + \frac{2i}{(\tilde{\epsilon}\epsilon)^2}\varepsilon^{\rho\mu\nu}K_\mu(\hat{\Psi}_\nu - D_\mu c)(\Lambda + K^\mu A_\mu)\right], \tag{4.14}$$

where $\hat{\Psi}_\mu = \hat{Q}A_\mu = \Psi_\mu + D_\mu c$. With the above boundary conditions, both at $r \to 0$ and $r \to \infty$, we see that the $\hat{Q}$ variation of the supersymmetric Chern-Simons action vanishes and thus we can use the techniques of localization to compute the partition function.

To proceed further we change the field variables to $X_0 = (a_\mu)$, $X_1 = (\Psi, c, \tilde{c})$ and $X_0' = \hat{Q}X_0$ and $X_1' = \hat{Q}X_1$. In this notation our set of bosonic fields $(a_\mu, \Lambda, \hat{G}, B)$ are represented by (X_0, X_1') and fermionic fields $(\Psi, \Psi_\mu, c, \tilde{c})$ are represented by (X_1, X_0'). We then rewrite our localization Lagrangian, if necessary integrating by parts. It is important to note that because of the presence of $\frac{1}{\cosh^2 r}$ and the asymptotic boundary conditions on all the fields, there are no boundary terms while integrating by parts. We thus obtain the localization Lagrangian

$$V_{\text{loc}} = \text{Tr}(\hat{Q}X_0\ X_1)\begin{pmatrix} \text{D}_{00} & \text{D}_{01} \\ \text{D}_{10} & \text{D}_{11}\end{pmatrix}\begin{pmatrix} X_0 \\ \hat{Q}X_1\end{pmatrix}. \tag{4.15}$$

Here D_{ij} are various differential operators and we are only keeping the quadratic terms in the fluctuations. By taking $\hat{Q}$ of V_{loc} and assembling bosonic and fermionic terms one can show that the one-loop result is:

$$Z_{\text{1-loop}}(\alpha) = \sqrt{\frac{\text{Det}_{\text{Coker}\,\text{D}_{10}}(\hat{Q}^2)}{\text{Det}_{\text{Ker}\,\text{D}_{10}}(\hat{Q}^2)}}\,. \tag{4.16}$$

Thus for a given eigen value of $H \equiv \hat{Q}^2$, we just need to know the difference in dimensions of kernel and cokernel of D_{10} operator. This difference is encoded in the equivariant index of D_{10} defined as

$$\text{ind}_{\text{equiv}}\,D_{10} = \text{Tr}_{\ker D_{10}}\,e^{iHt} - \text{Tr}_{\text{coker}\,D_{10}}\,e^{iHt} = \sum_{n\in\mathbb{Z}}(m_n^{(0)} - m_n^{(1)})q^n\,. \tag{4.17}$$

Here $m_n^{(0,1)}$ are the dimension of kernel and cokernel for a given eigen value of H labelled by n.

To identify the D_{10} operator, it is convenient to note the following:

$$(\hat{Q}\Psi_\mu)^\dagger = \mathcal{L}_K a_\mu + (D_\mu\Lambda)^\dagger + \cdots = \mathcal{L}_K a_\mu - D_\mu\Lambda - 2D_\mu(K\cdot a) + \ldots, \tag{4.18}$$

$$(\hat{Q}\Psi)^\dagger = -\hat{Q}\Psi - i(\tilde{\epsilon}\gamma^{\mu\nu}\epsilon)F_{\mu\nu} + \ldots \tag{4.19}$$

where dots include terms quadratic in fermionic fields. Since we are only interested in D_{10} the relevant terms are

$$-\frac{\sqrt{g}}{(\tilde{\epsilon}\epsilon)^2}\,\mathrm{Tr}\left[g^{\mu\nu}D_\mu c\,(\mathcal{L}_K a_\nu+[\alpha,a_\nu]-2D_\nu(K\cdot a))+i\Psi(\tilde{\epsilon}\gamma^{\mu\nu}\epsilon)F_{\mu\nu}-i\tilde{c}(\tilde{\epsilon}\epsilon)^2D^\mu\left(\frac{1}{(\tilde{\epsilon}\epsilon)^2}a_\mu\right)\right]. \tag{4.20}$$

Before proceeding for the detailed computations of kernel and cokernel, we notice that the pure constant mode of c and $\tilde{c}$ are zero modes of the $\hat{Q}V$ action. We, therefore, consider the path integral measure such that we don't integrate over the constant zero modes.

Since $\hat{Q}^2$ commutes with ∂_t and ∂_θ, we can study the kernel and cokernel for each Fourier mode in t and θ separately. We, therefore, write X_0 fields as $X_0(r)e^{-i(nt+p\theta)}$ and X_1 fields as $X_1(r)e^{i(nt+p\theta)}$. Note that the eigenvalues of $\hat{Q}^2$ on $X_{0,1}$ in this subspace are $-i\big(n+\frac{p}{L}\pm i\rho\cdot\alpha\big)$ where ρ is a root vector.

4.3 Analysis for kernel and co-kernel

It is convenient to make the following change of variables in each root space ρ:

$$a_t = a(K) - a_\theta/L, \tag{4.21}$$

$$\tilde{c} = \hat{\tilde{c}} + \frac{Ln+p-L\rho\cdot\alpha}{L}c, \tag{4.22}$$

where $a(K) = K\cdot a$. Note that the boundary condition of $\hat{\tilde{c}}$ is the same as $\tilde{c}$ and that of $a(K)$ is the same as a_θ.

By varying (4.20) with respect to c, $\hat{\tilde{c}}$ and Ψ we get the following equations for the kernel:

$$\begin{aligned}
\Delta^{(n,p)}a(K)(r) &= 0\,,\\
i\partial_r\left(\frac{\sinh r}{\cosh^2 r}a_r(r)\right)+\left(\frac{p}{\sinh^2 r}-Ln\right)\frac{\sinh r}{\cosh^2 r}a_\theta(r)+\frac{\sinh r}{\cosh^2 r}L^2na(K) &= 0\,,\\
\partial_r a_\theta(r)+i\left(\frac{p}{\sinh^2 r}-Ln\right)\frac{\sinh^2 r}{\cosh^2 r}a_r(r)-L\tanh^2 r\,\partial_r a(K)(r) &= 0
\end{aligned} \tag{4.23}$$

and by varying $a(K)$, a_r and a_θ the following equations for the co-kernel:

$$\begin{aligned}
\Delta^{(n,p)}c(r)+\frac{1}{2L\sinh r}\partial_r\left(\tanh^2 r\Psi\right)+\frac{n}{4\cosh^2 r}\hat{\tilde{c}} &= 0\,,\\
-2\partial_r\Psi(r)+\left(\frac{p}{\sinh^2 r}-Ln\right)\frac{\sinh r}{\cosh^2 r}\hat{c}(r) &= 0\,,\\
\frac{\sinh r}{\cosh^2 r}\partial_r\hat{c}(r)-2\left(\frac{p}{\sinh^2 r}-Ln\right)\tanh^2 r\,\Psi(r) &= 0
\end{aligned} \tag{4.24}$$

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where $\Delta^{(n,p)} = \nabla^\mu \frac{1}{|\epsilon|^2}\nabla_\mu$ is a second order differential operator restricted to Fourier mode labelled by (n,p). Note that these equations do not have any α-dependence and therefore they are the same in all the root spaces. We make a few remarks on the above equations:

1) The first kernel equation is a decoupled equation for $a(K)$ involving the operator $\Delta^{(n,p)}$. We will show below that it has no smooth solution satisfying the asymptotic boundary condition, except for a constant solution for $(n,p) = (0,0)$. This means that $a(K)$ decouples from the last two kernel equations and they become coupled homogeneous first order differential equations for a_r and a_θ. It is worth noting that the gauge transformation by some function $f(r)e^{i(nt+p\theta)}$ changes the gauge-fixing condition $\nabla^\mu(\frac{1}{\cosh^2 r}A_\mu)$ by $\Delta^{(n,p)}f(r)$. The fact that $\Delta^{(n,p)}f(r) = 0$ has no non-trivial solutions in the allowed space of gauge transformations $f(r)$,[6] shows that our gauge fixing condition indeed fixes the gauge completely.

2) The homogeneous part of the first co-kernel equation for c therefore has also no solution. It may however have particular solution due to the inhomogeneous terms.

3) The last two co-kernel equations are first order differential equations in $\hat{\tilde{c}}$ and Ψ and do not involve c. These equations are identical to the last two kernel equations (after setting $a(K) = 0$) with the identification $a_\theta \to \hat{\tilde{c}}$ and $a_r \to 2i\frac{\cosh^2 r}{\sinh r}\Psi$.

Kernel equations. First we show that $\Delta^{(n,p)}a(K)(r) = 0$ has no allowed solution for $(n,p) \neq (0,0)$. The following argument will also show that for $(n,p) = (0,0)$ there is one solution $a(K)(r) = \text{constant}$. The explicit form of this equation is given by

$$-\sinh^2 r\,\partial_r^2 a(K)(r) - \tanh r(1-\sinh^2 r)\,\partial_r a(K)(r) + (p^2 + n^2L^2\sinh^2 r)a(K)(r) = 0\,. \tag{4.25}$$

This equation has real coefficients, therefore without losing generality, we can assume that the solution $a(K)(r)$ is real in $r \in [0,\infty]$. Multiplying this equation by $a(K)(r)/(\cosh^2 r \cdot \sinh r)$ we obtain:

$$\begin{aligned} L_k \equiv\ & \frac{\sinh r}{\cosh^2 r}\left((\partial_r a(K)(r))^2 + (p^2 + L^2n^2\sinh^2 r)a(K)(r)^2\right) \\ & -\partial_r\left(\frac{\sinh r}{\cosh^2 r}a(K)(r)\partial_r a(K)(r)\right)\,. \end{aligned} \tag{4.26}$$

If $a(K)$ satisfies the equation, then L_k must be zero. Now integrating L_k we obtain the condition

$$\begin{aligned} 0 = \int_0^\infty dr L_k = \int_0^\infty dr \frac{\sinh r}{\cosh^2 r}\left((\partial_r a(K)(r))^2 + (p^2 + L^2n^2\sinh^2 r)a(K)(r)^2\right) \\ -\left(\frac{\sinh r}{\cosh^2 r}a(K)(r)\partial_r a(K)(r)\right)\Bigg|_0^\infty\,. \end{aligned} \tag{4.27}$$

[6] $(n,p) = (0,0)$ and constant f defines gauge transformation by constant which doesn't change the gauge fields.

The boundary term vanishes at $r = 0$ for smooth $a(K)$ and vanishes at $r = \infty$ for $a(K)$ satisfying the asymptotic behaviour $a(K)e^{-r/2} \to 0$. The integrand on the right hand side is non-negative for all r, therefore for $(n,p) \neq (0,0)$, $a(K)$ must vanish. On the other hand for $(n,p) = (0,0)$, the above condition implies that $\partial_r a(K)(r) = 0$ which allows for a constant solution $a(K)(r) = C_1$.

In either case $a(K)$ disappears from the remaining two kernel equations: in the third kernel equation only $\partial_r a(K)$ appears, while in the second kernel equations $a(K)$ together with a factor of n. It is clear from the structure of the equations that the solutions can be assumed to be such that a_θ is real and a_r is pure imaginary (or vice versa). Multiplying the second kernel equation by $(-i \,\text{sech}\, r \tanh r \, a_r(r))$ and the third kernel equation by $(\text{sech}^2 r \, a_\theta(r))$ and adding them together one obtains:

$$S_k \equiv \frac{\tanh r}{\cosh^2 r} a_\theta(r)^2 + \partial_r \left(\frac{1}{2\cosh^2 r} (a_\theta(r)^2 + \tanh^2 r \; a_r(r)^2) \right) . \tag{4.28}$$

As S_k is a linear combination of the two equations, S_k will be zero on the kernel. The first term in S_k is non-negative for all values of r and the second term is a total derivative in r. This means that if we integrate S_k over r from 0 to ∞

$$0 = \int_0^\infty dr S_k = \int_0^\infty dr \frac{\tanh r}{\cosh^2 r} a_\theta(r)^2 + \left(\frac{1}{2\cosh^2 r} (a_\theta(r)^2 + \tanh^2 \; r a_r(r)^2) \right) \Big|_0^\infty . \tag{4.29}$$

From the smoothness condition at $r = 0$, namely, for $p \neq 0$, $(a_r, a_\theta) \to (r^{|p|-1}, r^{|p|})$ and for $p = 0$, $(a_r, a_\theta) \to (r, r^2)$, we see that the boundary term in S_k, at $r = 0$, vanishes for any smooth configuration. The boundary term at $r = \infty$ vanishes for square integrable gauge fields. This shows that $a_\theta = 0$ and from the second kernel equation, it follows, that there is no solution for a_r which is square integrable. Thus, a_r and a_θ vanish for all n and p. Now $a_t = a(K) - a_\theta/L = a(K)$. For $(n,p) \neq (0,0)$, we have already shown that $a(K) = 0$. For $(n,p) = (0,0)$, there was one solution $a(K) = C_1$. This implies that for $(n,p) = (0,0)$, $a_t = C_1$. However this is not square integrable and therefore $C_1 = 0$. Thus we have shown that in the space of smooth and square-integrable gauge fields, kernel vanishes for all n and p.

Co-kernel equations. As discussed above, the homogeneous part of the first co-kernel equation for $c(r)$ is the same as the kernel equation for $a(K)$ for which we already showed that there is no solution. However there may be a solution for the inhomogeneous equation. In order to find the form of the inhomogeneous terms we need to find the solutions for $\hat{\tilde{c}}$ and Ψ using the last two co-kernel equations. Both the equations involve real coefficients and therefore the solutions for $\hat{\tilde{c}}$ and Ψ can be chosen to be real. Multiplying the third equation by 2Ψ and the second equation by $\frac{\hat{\tilde{c}}}{\sinh r}$ and adding them, one gets:

$$S \equiv \frac{\sinh r}{\cosh^2 r} \hat{\tilde{c}}^2 + \partial_r \left(\frac{1}{2\cosh^2 r} \hat{\tilde{c}}^2 - 2\Psi^2 \right) . \tag{4.30}$$

The second term in S is a total derivative while the first term is non-negative. If the equations are satisfied then S must be zero. Integrating over r one gets:

$$0 = \int_0^\infty dr S = \int_0^\infty dr \frac{\sinh r}{\cosh^2 r} \hat{\tilde{c}}^2 + \left(\frac{1}{2\cosh^2 r} \hat{\tilde{c}}^2 - 2\Psi^2 \right) \Big|_0^\infty . \tag{4.31}$$

The boundary term at $r = 0$ vanishes for $p \neq 0$ as both Ψ and $\hat{\hat{c}}$ must go as $r^{|p|}$. For $p = 0$, however, the boundary term at $r = 0$ may not be zero. Indeed one can study the series solutions of the two co-kernel equations for Ψ and $\hat{\hat{c}}$ and show that the solutions for $p = 0$ go as r^0.

To analyse the boundary term at $r = \infty$, we look for the asymptotic solutions in a series expansion in e^{-r}. For $n \neq 0$, the asymptotic behaviours are $(\hat{\hat{c}}, \Psi) \to (e^{r\gamma_+}, e^{r(\gamma_+ - 1)})$ and $(\hat{\hat{c}}, \Psi) \to (e^{r\gamma_-}, e^{r(\gamma_- - 1)})$ respectively, where

$$\gamma_\pm = \frac{1}{2}(1 \pm \sqrt{1 + 4L^2n^2}). \tag{4.32}$$

Clearly only the second solution satisfies our boundary conditions, and for this the boundary term in S vanishes at $r = \infty$. This proves that there are no acceptable solutions to $(\hat{\hat{c}}, \Psi)$ for n and p both non-zero. This also implies that c must be zero as there are no inhomogeneous terms in the first co-kernel equation.

We already saw for $p = 0$ with $n \neq 0$, that the boundary term in S at $r = 0$ does not vanish for smooth solution. The same happens for $n = 0$ and $p \neq 0$ at $r = \infty$. The asymptotic behaviour of the two solutions, for $n = 0$ and $p \neq 0$, that can be verified by making a series expansion in e^{-r}, are

$$(\hat{\hat{c}}, \Psi) \to (O(1), e^{-3r}) \tag{4.33}$$
$$(\hat{\hat{c}}, \Psi) \to (e^{-r}, O(1)). \tag{4.34}$$

Both are acceptable solutions. While for the first, the boundary term at $r = \infty$ vanishes, for the second the boundary term does not vanish. Thus for $n = 0$ or $p = 0$ (4.31) does not give any useful information. For the co-kernel therefore, we need to study the three special cases: i) $p = 0$, $n \neq 0$, ii) $p \neq 0$ and $n = 0$, iii) $(p, n) = (0, 0)$. Fortunately, for these cases one can find explicit analytic solutions for the co-kernel equations.

i) $p = 0$ and $n \neq 0$. In this case we can solve for Ψ using the third co-kernel equation.

$$\Psi(r) = -\frac{\partial_r \hat{\hat{c}}(r)}{2Ln \sinh r}. \tag{4.35}$$

Substituting this in the second co-kernel equation one gets a second order differential equation which has the general solution:

$$\hat{\hat{c}}(r) = C_1 (\cosh r)^{\gamma_+} + C_2 (\cosh r)^{\gamma_-} \tag{4.36}$$

where $\gamma_\pm$ are defined in (4.32). The acceptable solution is the one with γ_- and for this the complete one-parameter family of solution, after solving the inhomogeneous equation for c, is

$$\hat{\hat{c}}(r) = C_1 (\cosh r)^{\gamma_-} \tag{4.37}$$
$$\Psi(r) = -\frac{\partial_r \hat{\hat{c}}(r)}{2Ln \sinh r} \tag{4.38}$$
$$c(r) = \frac{\hat{\hat{c}}(r)}{2n}. \tag{4.39}$$

ii) $p \neq 0$ and $n = 0$. In this case we can solve for Ψ using the third co-kernel equation.

$$\Psi(r) = \frac{\sinh r \partial_r \hat{\tilde{c}}(r)}{2p} \,. \tag{4.40}$$

Substituting this in the second co-kernel equation one gets a second order differential equation

$$\sinh r \partial_r^2 \hat{\tilde{c}}(r) + \cosh r \partial_r \hat{\tilde{c}}(r) - p^2 \frac{\operatorname{sech}^2 r}{\sinh r} \hat{\tilde{c}}(r) = 0 \,. \tag{4.41}$$

The indicial roots near $r = 0$ are $\pm|p|$. Asymptotically, changing the variable from r to $z = e^{-r}$, the indicial roots at $z = 0$ are 0 and 1. Moreover expanding the two solutions near $z = 0$ shows that there are no logarithmic terms in z. The smooth solution, i.e. the solution that behaves as $r^{|p|}$ at $r = 0$, can be expressed in terms of hypergeometric function:

$$\hat{\tilde{c}}(r) = (\tanh r)^{|p|} {}_2F_1 \left(\frac{1}{2}(|p| + \hat{\gamma}_-), \frac{1}{2}(|p| + \hat{\gamma}_+); 1 + |p|; \tanh^2 r \right) \tag{4.42}$$

where $\hat{\gamma}_\pm = \frac{1}{2}(1 \pm \sqrt{1 + 4p^2})$. This solution, when analytically continued to the asymptotic region, behaves as

$$\frac{\sqrt{\pi}\Gamma(1+|p|)}{\Gamma\left(\frac{1}{2}(1+|p|+\hat{\gamma}_-)\right)\Gamma\left(\frac{1}{2}(1+|p|+\hat{\gamma}_+)\right)} + \frac{4\sqrt{\pi}\Gamma(1+|p|)}{\Gamma\left(\frac{1}{2}(|p|+\hat{\gamma}_-)\right)\Gamma\left(\frac{1}{2}(|p|+\hat{\gamma}_+)\right)} e^{-r} + \dots \tag{4.43}$$

so that it is a linear combination of the two asymptotic solutions that are both acceptable. Particularly note that the coefficient of solution behaving as e^{-r} is non-vanishing. This is consistent with the argument given in (4.34) that shows that the boundary term in (4.31) at $r = \infty$ does not vanish for the solution of $(\hat{\tilde{c}}, \Psi)$ that behaves as $(e^{-r}, O(1))$.

Substituting the above solution for $(\hat{\tilde{c}}(r), \Psi(r))$ in the first co-kernel equation, we can easily find the inhomogeneous solution for c. The complete one-parameter solution for $n = 0$ and $p \neq 0$ is:

$$\hat{\tilde{c}}(r) = C_2 (\tanh r)^{|p|} {}_2F_1 \left(\frac{1}{2}(|p| + \hat{\gamma}_-), \frac{1}{2}(|p| + \hat{\gamma}_+); 1 + |p|; \tanh^2 r \right) \tag{4.44}$$

$$\Psi(r) = -\frac{\sinh r \partial_r \hat{\tilde{c}}(r)}{2p} \tag{4.45}$$

$$c(r) = \frac{L \hat{\tilde{c}}(r)}{2p} \,. \tag{4.46}$$

iii) $(p, n) = (0, 0)$. In this case the solution for the last two co-kernel equations have constant solutions $\hat{\tilde{c}}(r) = C_1$ and $\Psi = C_2$. Plugging this in the first co-kernel equation one finds the solution for c as

$$c(r) = C_3 + C_2 L \log\left(\tanh \frac{r}{2}\right) + C_4 \left(\log\left(\tanh \frac{r}{2}\right) + \cosh r\right) . \tag{4.47}$$

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Smoothness at $r = 0$ and the asymptotic boundary condition on c implies that $C_4 = C_2 = 0$. Thus we have a two paraometer family of solutions:

$$\hat{\tilde{c}}(r) = C_1, \qquad \Psi(r) = 0, \qquad c(r) = C_2\,. \tag{4.48}$$

These are, however, just the constant modes of c and $\tilde{c}$ that, we had argued earlier, should be removed from the path-integral.

4.4 Summary and result for the one loop determinant

Let us summarize here the results. $\mathrm{Dim}(\ker(D_{10})) = 0$ for all n and p. $\mathrm{Dim}(\mathrm{coker}(D_{10})) = 0$ for $n \neq 0$ and $p \neq 0$, $\mathrm{Dim}(\mathrm{coker}(D_{10})) = 1$ for $p = 0$ and $n \neq 0$ as well as for $n = 0$ and $p \neq 0$ and $\mathrm{Dim}(\mathrm{coker}(D_{10})) = 2$ for $n = p = 0$. Therefore the index of D_{10} operator is

1. $\mathrm{ind}\, D_{10} = 0$ for $n \neq 0$ and $p \neq 0$,
2. $\mathrm{ind}\, D_{10} = -1$ for $n \neq 0$ and $p = 0$,
3. $\mathrm{ind}\, D_{10} = -1$ for $n = 0$ and $p \neq 0$,
4. $\mathrm{ind}\, D_{10} = -2$ for $n = 0$ and $p = 0$.

Now we note that for the case $(n = 0,\, p = 0)$ we have two constant ghost modes (4.48). As we argued earlier these are zero modes and we take the path integral measure such that there are no integrations over constant mode for ghost fields. We therefore, neglect these zero modes and their contribution to index.[7]

Combining all the results above we get the final answer for the 1-loop super-determinant around the saddle point:

$$Z_{\text{1-loop}}(\alpha) = \prod_{\rho} \sqrt{\prod_{n\neq 0}(n - i\rho\cdot\alpha)\prod_{p\neq 0}\left(\frac{p}{L} - i\rho\cdot\alpha\right)}. \tag{4.49}$$

Substituting the above in (4.1), we obtain the matrix model

$$\begin{aligned} Z &= \int d\alpha\;\exp(-\pi i L\kappa\,\mathrm{Tr}\,\alpha^2)\prod_{\rho>0}(\rho\cdot\alpha)^2\prod_{\rho}\sqrt{\prod_{n\neq 0}(n - i\rho\cdot\alpha)\prod_{p\neq 0}\left(\frac{p}{L} - i\rho\cdot\alpha\right)} \\ &= \mathcal{N}\int d\alpha\;\exp(-\pi i L\kappa\,\mathrm{Tr}\,\alpha^2)\prod_{\rho>0}\sinh(\pi\rho\cdot\alpha)\sinh(\pi L\rho\cdot\alpha). \end{aligned} \tag{4.50}$$

Here $\mathcal{N}$ is some constant which depends only on L. In order to compute the L dependence in $\mathcal{N}$, we need to include also the contribution to the determinant from the Cartan part[8] $\prod_{i=1}^{\mathrm{rank}} \sqrt{\prod_{n\neq 0} n \prod_{p\neq 0}\left(\frac{p}{L}\right)}$. Therefore, we need to regularize the infinite product of L coming from both the Cartan and non-Cartan part of the determinant. Using the zeta

[7] One could absorb these zero modes by introducing two bosonic ghosts of ghost which would give a contribution of +2 to the index. It would be interesting to check this explicitly following the analysis in [5].

[8] The Cartan part of the determinant does not contain any α dependence and hence are not useful for most of the analysis.

function regularization one finds that $\mathcal{N} \propto L^{\frac{r}{2}}$ where r is the rank of the Lie algebra. Thus we find that the L dependence in $\mathcal{N}$ exactly cancel the $\sqrt{L}$ dependence in (4.4).[9] In the non abelian case there is another potential source of L in the partition function coming from the L dependence in one of the sinh function. We do this integral explicitly for a general gauge group in the appendix C and find that total L dependence in the partition function Z coming from the index computation is just a pure phase. The rest of the integral equals that of the partition function of the bosonic Chern-Simons theory on S^3.[10] Finally we mention the result in (4.50) is also equal to the result of the partition function of Chern-Simons theory of a q fold cover of the sphere S^3 where $q = \frac{1}{L}$ obtained in [24]. This can be seen by a simple rescaling of the integral in (4.50).

5 Wilson loop

In this section we determine the expectation value of a supersymmetric Wilson loop in $AdS_2 \times S^1$. We consider the following Wilson loop operator in the representation R of the gauge group

$$W_R = \frac{1}{\dim R} \mathrm{Tr}_R\, P \exp\left[i \oint dt (A_\mu \dot{x}^\mu - \sigma |\dot{x}|)\right] . \tag{5.1}$$

The susy transformation of the above Wilson loop is given by

$$\delta W_R = \frac{1}{\dim R} \mathrm{Tr}_R\, P \left[i \oint dt (\delta A_\mu \dot{x}^\mu - \delta\sigma |\dot{x}|)\right] \exp\left[i \oint dt (A_\mu \dot{x}^\mu - \sigma |\dot{x}|)\right] . \tag{5.2}$$

Using susy transformation of the vector multiplet, we write the above expression as

$$\delta W_R = \frac{1}{\dim R} \mathrm{Tr}_R\, P \oint \left\{\frac{i}{2}(\epsilon\gamma_\mu \tilde{\lambda} + \tilde{\gamma}_\mu \lambda)\dot{x}^\mu + \frac{i}{2}(\epsilon\tilde{\lambda} - \tilde{\epsilon}\lambda)|\dot{x}|\right\} \exp\left[i \oint dt (A_\mu \dot{x}^\mu - \sigma |\dot{x}|)\right] . \tag{5.3}$$

Thus the Wilson loop preserve susy if the following relations hold

$$(\gamma_\mu \dot{x}^\mu - |\dot{x}|)\epsilon = 0, \qquad (\gamma_\mu \dot{x}^\mu + |\dot{x}|)\tilde{\epsilon} = 0 . \tag{5.4}$$

Now if we choose the Wilson loop wrapping the τ-direction i.e. $\dot{x}^\mu = h\, e^\mu_1$, h is some constant, then we find the following condition on the killing spinor

$$(\gamma_1 - 1)\epsilon = 0, \qquad (\gamma + 1)\tilde{\epsilon} = 0 . \tag{5.5}$$

Using the explicit form of the killing spinor (3.6), we see that this condition is satisfied if $\sinh \frac{r}{2} = 0$. Thus the Wilson loop preserve the killing spinor if it wraps the circle at the origin of the AdS_2. The expectation value of this Wilson loop is given by

$$\langle W_R \rangle = \frac{1}{\tilde{Z} \dim R} \int d\alpha\, \mathrm{Tr}_R \exp[2\pi h \alpha] \exp(-\pi i L \kappa\, \mathrm{Tr}\, \alpha^2) \prod_{\rho>0} \sinh(\pi\rho\cdot\alpha) \sinh(\pi L \rho\cdot\alpha) \tag{5.6}$$

where $\tilde{Z}$ is the partition function without $\mathcal{N}$.

[9] We note here that there might be a L dependence in the Jacobian in the path integral measure coming from the change of variables to cohomological variables.

[10] Note that going from first line to second line in (4.50), converting infinite product to the hyperbolic functions, we have ignored infinite products of (-1)'s under square root. Carefully treating the sign in the square root using the gauge invariant regularization as in [27], one recovers the usual shift in κ.

6 General conformal transformations

In this section we will consider a family of manifolds which are conformally equivalent to $\text{AdS}_2 \times S^1$ and show that the index and the partition function do not change. We begin with the following metric

$$ds^2 = f_s^2(r)(d\tau^2 + L^2(dr^2 + \sinh^2 r\, d\theta^2))\,. \tag{6.1}$$

The metric is conformally equivalent to $\text{AdS}_2 \times S^1$ and we choose the conformal factor $f_s(r)$ such that it does not change the asymptotic behaviour of all fields. This will ensure that the space of functions over which we will calculate the index does not change drastically. This implies that we take $f_s(r)$ such that it approaches $\sim \mathcal{O}(1)$ as $r \to 0$ and ∞. Also the family of conformally equivalent manifolds, we will consider here are labelled by parameter s such that $f_{s=0}(r) = 1$ which corresponds to $\text{AdS}_2 \times S^1$. An example of a such function which we will use in our computations is[11]

$$f_s(r) = 1 - s + s\,\text{sech}\, r\,. \tag{6.2}$$

We see that for $s = 1$ the metric (6.1) is that of branched S^3 and any other value of $s \leq 1$ corresponds to the metric which is non singular and asymptotically $\text{AdS}_2 \times S^1$.

The metric (6.1) admits Killing spinors. Following a similar analysis presented in appendix B, we obtain the following Killing spinors

$$\epsilon = e^{\frac{i\theta}{2}}\sqrt{f_s(r)}\begin{pmatrix} i\cosh\left(\frac{r}{2}\right) \\ \sinh\left(\frac{r}{2}\right)\end{pmatrix}, \qquad \tilde{\epsilon} = e^{-\frac{i\theta}{2}}\sqrt{f_s(r)}\begin{pmatrix} \sinh\left(\frac{r}{2}\right) \\ i\cosh\left(\frac{r}{2}\right)\end{pmatrix}. \tag{6.3}$$

These Killing spinors correspond to the Killing vector ($K^\mu = \tilde{\epsilon}\gamma^\mu\epsilon$)

$$K = \frac{\partial}{\partial\tau} + \frac{1}{L}\frac{\partial}{\partial\theta}\,. \tag{6.4}$$

However in the present case the background supergravity fields A_μ, V_μ and H acquire non trivial dependence on the function $f_s(r)$ to satisfy the Killing spinor equations. Their explicit forms are

$$A_\tau = \frac{2f_s(r) + 3\coth r\,\partial_r f_s(r)}{2Lf_s(r)}\,, \qquad V_\tau = \frac{f_s(r) + \coth r\,\partial_r f_s(r)}{2Lf_s(r)}\,, \qquad A_{r,\theta} = 0\,,$$
$$V_{r,\theta} = 0, \qquad H = \frac{i\partial_r f_s(r)}{Lf_s^2(r)\sinh r}\,. \tag{6.5}$$

Thus we can use this background to compute the partition function using the localization technique. We begin with the QV action. The bosonic part of the QV action in the present case is given by

$$QV_{\text{loc}\{\text{bosonic}\}} = \int d^3x\sqrt{g}\,\text{Tr}\Bigg[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2f_s^2(r)\cosh^2 r}D_\mu(f_s(r)\cosh r\,\sigma)D^\mu(f_s(r)\cosh r\,\sigma)$$
$$-\frac{1}{32}\left(G - \frac{4}{Lf_s(r)\cosh r}\sigma\right)^2\Bigg]\,. \tag{6.6}$$

[11] It would be very interesting to explore the other possible choices of function $f_s(r)$ and prove that the index does not change.

The minima of the above action are the solutions to the following equations

$$F_{\mu\nu} = 0\,, \qquad D_\mu(f_s(r)\cosh r\,\sigma) = 0\,, \qquad G = \frac{4\sigma}{Lf_s(r)\cosh r}\,. \tag{6.7}$$

As we described earlier, the solution of the above equation upto a gauge transformation is given by

$$A_\nu = 0, \qquad \sigma = \frac{i\alpha}{f_s(r)\cosh r}, \qquad G = \frac{i\alpha}{L(f_s(r)\cosh r)^2}\,. \tag{6.8}$$

And thus we see that the Chern-Simons action evaluated on the above background remains unchanged

$$\exp\left(\frac{\kappa}{4\pi}S_{\rm CS}\right) = \exp(-\pi i L\kappa\,{\rm Tr}\,\alpha^2)\,. \tag{6.9}$$

Next we look for D_{10} operator. It is not very hard to convince oneself that the relevant terms in the $\hat{Q}V$ action needed for D_{10} operator are the same as given in (4.20). Since the killing spinors now depend on the function $f_s(r)$, the explicit form of the kernel and cokernel equations will also depend on the function $f_s(r)$ through the killing spinors. We will not present here the details of these equations. To solve these equations we will follow exactly the same analysis presented in the subsection 4.3. Since we have presented the analysis for $s = 0$ in details, we will not repeat here the same analysis and therefore, just state the results. We find that for $s < 1$, the spaces of kernel and cokernel of the D_{10} operator remain unchanged compared to $s = 0$ case and thus the index of D_{10} is again given by 1–4. The situation becomes interesting for the $s = 1$ case. In this case we find that the spaces of kernel and cokernel of the D_{10} operator are different than the one for $s = 0$ case. In particular the kernel is one dimensional for every (n, p) satisfying $(n > 0, p > 0)$ and $(n < 0, p < 0)$. On the other hand the space of cokernel is also one dimensional for every combination of (n, p) satisfying $(n > 0, p > 0)$, $(n < 0, p < 0)$, $(n = 0, p \neq 0)$ and $(n \neq 0, p = 0)$. The cokernel is 2 dimensional for $(n = 0, p = 0)$ which are pure constant modes for ghost fields and we do not integrate over these modes. Thus we see that although the spaces of kernel and cokernel for $s = 1$ case are very different than the $s \neq 1$, the index remains same.

7 Conclusions

We have used the method of localization to evaluate the partition function of Chern-Simons theory on the non-compact space $AdS_2 \times S^1$. The radius of AdS_2 is $L = 1/q$ times that of the S^1. The partition function agrees precisely with that on the q fold cover of S^3 as expected from the conformal symmetry which relates the partition function on these spaces. Furthermore since the theory is topological, this partition function is equal to that on S^3 by a pure phase which depends on L, upto some L dependent factor coming from the Jacobian in the path integral measure. This constitutes a non-trivial check of the method of localization developed for $AdS_2 \times S^1$ in this paper. Though this paper focuses on the $\mathcal{N} = 2$ vector multiplet, the method can be generalised to matter multiplets and to theories with higher supersymmetry. We expect the equality between partition functions of conformal fields theories on $AdS_2 \times S^1$ and S^3 to hold for general super conformal field theories in 3 dimensions.

In our analysis we showed that the relation between the partition function on $AdS_2 \times S^1$ and S^3 was obtained by considering the usual space of square integrable wave functions on $AdS_2 \times S^1$. The localizing Lagrangian in particular did not develop any boundary terms in any steps which involved a total derivative. The fields satisfied boundary conditions to ensure that total derivative terms vanished at the origin and the boundary of AdS_2.

Finally we mention that this method of localization developed for $AdS_2 \times S^1$ can be generalized to higher dimensions. The space $AdS_2 \times S^2$ is particularly an interesting one. One can extend the approach of this paper and address localization of 4 dimensional supersymmetric field theories in non-compact space. There is an added benefit of studying localization in this space. $AdS_2 \times S^2$ is the near horizon geometry of supersymmetric black holes in 4 dimensions. Developing localization in this space will lead to a better understanding of black hole microstates from the bulk. We hope to address some of these aspects in the future.

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A Conventions

The covariant derivative of a fermion is given by

$$\nabla_\mu \psi = \left(\partial_\mu + \frac{i}{4}\omega_{\mu\, ab}\varepsilon^{abc}\gamma_c\right)\psi, \qquad \varepsilon^{123} = 1. \tag{A.1}$$

Our choice of gamma matrices are

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{A.2}$$

They satisfy gamma matrices algebra

$$\gamma^a \gamma^b = \delta^{ab} + i\varepsilon^{abc}\gamma_c\,, \tag{A.3}$$

$$\gamma^{aT} = -C\gamma^a C^{-1}, \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad C^T = -C = C^{-1}\,. \tag{A.4}$$

In Lorentzian space ψ and $\bar{\psi}$ are complex conjugate to each other but in Euclidean space fermions ψ and $\bar{\psi}$ are independent two component complex spinor. The product of two fermions ϵ and ψ is defined through charge conjugation matrix

$$\epsilon\psi = \epsilon^T C \psi\,. \tag{A.5}$$

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B Solving Killing spinor equations

The Killing spinor equations are given by

$$\begin{aligned}(\nabla_\mu - iA_\mu)\,\epsilon &= -\frac{1}{2}H\gamma_\mu\epsilon - iV_\mu\epsilon - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\epsilon\\ (\nabla_\mu + iA_\mu)\,\tilde\epsilon &= -\frac{1}{2}H\gamma_\mu\tilde\epsilon + iV_\mu\tilde\epsilon + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\tilde\epsilon\,.\end{aligned} \tag{B.1}$$

Here $\varepsilon^{\mu\nu\rho} = \frac{1}{\sqrt{g}}\epsilon^{\mu\nu\rho}$, $\epsilon^{\tau\eta\theta} = 1$.

In order to solve the above equations we make the following ansatz

$$\epsilon(\tau, r, \theta) = e^{\frac{i\theta}{2}}\begin{pmatrix}\epsilon_1(r)\\ \epsilon_2(r)\end{pmatrix}, \qquad \tilde\epsilon(\tau, r, \theta) = e^{-\frac{i\theta}{2}}\begin{pmatrix}\tilde\epsilon_1(r)\\ \tilde\epsilon_2(r)\end{pmatrix}, \qquad V_r = V_\theta = 0\,. \tag{B.2}$$

In particular the ansatz for Killing spinor does not depend on the τ-coordinate. Solving the τ component equations, one finds

$$A_\tau = V_\tau, \qquad H = 0\,. \tag{B.3}$$

The θ-component equation is given by

$$\begin{pmatrix}1 - \cosh r - 2A_\theta & -i\sinh r\, V_\tau L\\ -i\sinh r\, V_\tau L & 1 - 2A_\theta + \cosh r\end{pmatrix}\begin{pmatrix}\epsilon_1(r)\\ \epsilon_2(r)\end{pmatrix} = 0\,. \tag{B.4}$$

Similarly for $\tilde\xi$. Requiring the existence of a non trivial solution for ξ determines A_θ in terms of V_τ as

$$A_\theta = \frac{1}{2}\left(1 \pm \sqrt{1 + (1 - L^2V_\tau^2)\sinh^2 r}\right)\,. \tag{B.5}$$

Now we look at the r-component equations,

$$\begin{aligned}\partial_r\epsilon_1(r) - iA_r\epsilon_1(r) - \frac{i}{2}LV_\tau\epsilon_2(r) &= 0\\ \partial_r\epsilon_2(r) - iA_r\epsilon_2(r) + \frac{i}{2}LV_\tau\epsilon_1(r) &= 0\\ \partial_r\tilde\epsilon_1(r) + iA_r\tilde\epsilon_1(r) + \frac{i}{2}LV_\tau\tilde\epsilon_2(r) &= 0\\ \partial_r\tilde\epsilon_2(r) + iA_r\tilde\epsilon_2(r) - \frac{i}{2}LV_\tau\tilde\epsilon_1(r) &= 0\,.\end{aligned} \tag{B.6}$$

One finds that if we define $R = \frac{\epsilon_2(r)}{\epsilon_1(r)}$ and $\tilde R = \frac{\tilde\epsilon_1(r)}{\tilde\epsilon_2(r)}$, then from above set of equations

$$\partial_r R = -\frac{iL}{2}V_\tau(1 + R^2), \qquad \partial_r\tilde R = -\frac{iL}{2}V_\tau(1 + \tilde R^2)\,. \tag{B.7}$$

Now let us look at the form of the Killing vector.

$$K^\mu = \tilde\epsilon\gamma^\mu\epsilon = (a, 0, b)\,, \tag{B.8}$$

$$\tilde\epsilon_1(r)\epsilon_2(r) + \tilde\epsilon_2(r)\epsilon_1(r) = -a, \qquad \tilde\epsilon_1(r)\epsilon_1(r) = \tilde\epsilon_2(r)\epsilon_2(r) = \frac{ibL}{2}\sinh r\,. \tag{B.9}$$

Using the above equations it is very simple to determine R which is given by

$$R = \frac{\epsilon_2(r)}{\epsilon_1(r)} = i\frac{a \pm \sqrt{a^2 + b^2L^2\sinh^2 r}}{bL\sinh r}\,. \tag{B.10}$$

Substituting above in (B.7), we determine V_τ as

$$V_\tau = \mp\frac{b\cosh r}{\sqrt{a^2 + b^2L^2\sinh^2 r}}\,. \tag{B.11}$$

Substituting the expression of V_τ in (B.5), we obtain

$$A_\theta = \frac{1}{2}\left[1 \pm \frac{a\cosh r}{\sqrt{a^2 + b^2L^2\sinh^2 r}}\right]. \tag{B.12}$$

Thus susy preserving backgrounds are labelled by two real parametrs. Our choice of susy transformation parameters (3.6) satisfy above equations with the choice $a = 1$, $b = \frac{1}{L}$.

C L dependence in the partition function

To demonstrate that the L dependence in the partition function is just a pure phase we evaluate the integral

$$\tilde{Z} = \int \prod_{i=1}^{r} d\alpha_i\, e^{-\pi i L\kappa \operatorname{Tr}\alpha^2} \prod_{\rho>0} \sinh(\pi\rho\cdot\alpha)\sinh(\pi L\rho\cdot\alpha)\,. \tag{C.1}$$

This matrix model has been studied using the method of orthogonal polynomials in [28]. We follow the steps given in [29], first we will use Weyl denominator formula

$$\sum_{w\in\mathcal{W}} \epsilon(w)\, e^{w(\delta)\cdot\alpha} = \prod_{\rho>0} 2\sinh\left(\frac{\rho\cdot\alpha}{2}\right). \tag{C.2}$$

Here δ is the sum over positive roots

$$\delta = \frac{1}{2}\sum_{\rho>0}\rho\,. \tag{C.3}$$

We get

$$\begin{aligned}\tilde{Z} &= \frac{1}{L^{r/2}}\int \prod_{i=1}^{r} d\mu_i\, e^{-\frac{1}{2g_s}\operatorname{Tr}\mu^2} \prod_{\rho>0}\sinh\left(\frac{\pi\rho\cdot\mu}{\sqrt{L}}\right)\sinh(\pi\sqrt{L}\rho\cdot\mu)\,,\\ &= \frac{1}{2^{2\Delta_+}L^{r/2}}\int \prod_{i=1}^{r} d\mu_i\, e^{-\frac{1}{2g_s}\operatorname{Tr}\mu^2} \sum_{w,w'\in\mathcal{W}} \epsilon(w'')\, e^{w(\delta)\cdot\frac{\pi\mu}{\sqrt{L}}} e^{w'(\delta)\cdot\pi\mu\sqrt{L}}\,. \end{aligned}\tag{C.4}$$

Here $\sqrt{L}\alpha = \mu$, $\frac{1}{2g_s} = \pi i \kappa$, $w'' = w \cdot w'$ and $\Delta_+ =$ total number of positive roots. Now we can explicitly do the above Gaussian integral.

$$\begin{aligned}
\tilde{Z} &= \frac{(\det C)^{1/2}}{2^{2\Delta_+}} \left(\frac{2\pi g_s}{L}\right)^{r/2} \sum_{w,w' \in \mathcal{W}} \epsilon(w'') e^{\frac{g_s \pi^2}{2}\left(\frac{w(\delta)}{\sqrt{L}} + w'(\delta)\sqrt{L}\right)\cdot\left(\frac{w(\delta)}{\sqrt{L}} + w'(\delta)\sqrt{L}\right)}, \\
&= \frac{(\det C)^{1/2}}{2^{2\Delta_+}} \left(\frac{2\pi g_s}{L}\right)^{r/2} |\mathcal{W}| \sum_{w'' \in \mathcal{W}} \epsilon(w'') e^{\frac{g_s \pi^2}{2}(\delta\cdot\delta)\left(\frac{1}{L}+L\right)} e^{g_s \pi^2 (\delta \cdot w''(\delta))}, \\
&= \frac{(\det C)^{1/2}}{2^{2\Delta_+}} \left(\frac{2\pi g_s}{L}\right)^{r/2} |\mathcal{W}| e^{\frac{g_s \pi^2}{2}(\delta\cdot\delta)\left(\frac{1}{L}+L\right)} \prod_{\rho>0} 2\sinh\left(\frac{g_s \pi^2 \rho\cdot\delta}{2}\right). \qquad \text{(C.5)}
\end{aligned}$$

Here C and $|\mathcal{W}|$ are the inverse of Cartan matrix and the order of the Weyl group, respectively. Now using Freudenthal de Vries formula

$$(\delta \cdot \delta) = \frac{d_{\mathrm{G}}\, y}{12}, \qquad \text{(C.6)}$$

where d_{G} is the dimension of the group and y is the dual Coxeter number, we get

$$\tilde{Z} = \frac{(\det C)^{1/2}}{2^{\Delta_+}} \left(\frac{2\pi g_s}{L}\right)^{r/2} |\mathcal{W}| e^{-\frac{i\pi\Delta_+}{2}} e^{-i\frac{\pi}{48\kappa} d_{\mathrm{G}} y \left(\frac{1}{L}+L\right)} \prod_{\rho>0} \sin\left(\frac{\pi}{4\kappa}\rho\cdot\delta\right). \qquad \text{(C.7)}$$

Thus we see that the entire L dependence is a phase plus overall $\frac{1}{L^{r/2}}$. This together with the $L^{r/2}$ contribution in $\mathcal{N}$ in (4.50) leaves behind a partition function which depends on the metric through a pure phase.

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