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BPS/CFT correspondence: non-perturbative Dyson-Schwinger equations and qq-characters

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Abstract: We study symmetries of quantum field theories involving topologically distinct sectors of the field space. To exhibit these symmetries we define special gauge invariant observables, which we call the qq-characters. In the context of the BPS/CFT correspondence, using these observables, we derive an infinite set of Dyson-Schwinger-type relations. These relations imply that the supersymmetric partition functions in the presence of Ω-deformation and defects obey the Ward identities of two dimensional conformal field theory and its q-deformations. The details will be discussed in the companion papers.

Keywords: Nonperturbative Effects, Supersymmetric gauge theory, D-branes, Quantum Groups

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In memory of Lev Borisovich Okun (1929–2015)

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1 Introduction

1.1 Dyson-Schwinger equations

The correlation functions of Euclidean quantum field theory are defined by the path integral:

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \frac{1}{Z}\int_\Gamma D\Phi\; e^{-\frac{1}{\hbar}S[\Phi]}\;\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\,, \tag{1.1}$$

suitably regularized and renormalized. The classical theory is governed by the Euler-Lagrange equations, which are derived from the variational principle:

$$\delta S[\Phi_{cl}] = 0 \tag{1.2}$$

These equations are modified in the quantum theory: consider an infinitesimal transformation

$$\Phi \longrightarrow \Phi + \delta\Phi \tag{1.3}$$

Assuming (1.3) preserves the measure $D\Phi$ in (1.1) (no anomaly), then

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\delta S[\Phi]\rangle = \hbar\sum_{i=1}^{n}\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_{i-1}(x_{i-1})\delta\mathcal{O}(x_i)\mathcal{O}_{i+1}(x_{i+1})\ldots\mathcal{O}_n(x_n)\rangle \tag{1.4}$$

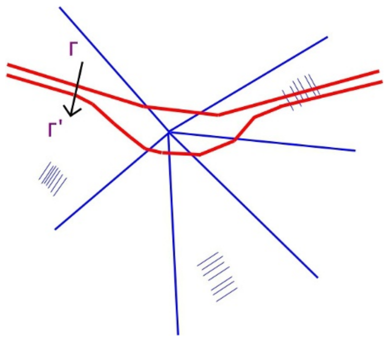

Figure 1. Deformation of the integration contour: $\Gamma \to \Gamma'$.

In (1.3) the change of variables can be also interpreted as a small modification of the integration contour Γ in (1.1), $\Gamma \to \Gamma' = \Gamma + \delta\Gamma$, as in the figure 1.

The small change of contour does not change the integral of a closed form.

The usefulness of the Dyson-Schwinger equations depends on whether one can find a convenient set of observables $\mathcal{O}_i$ in (1.4) and perhaps also take a limit in order to get a closed system of equations. Formally, the loop equations [69, 72] are an example of such a system. Another, related example, is the matrix model, i.e. the zero dimensional gauge theory. The simplest model is the single matrix integral:

$$Z = \int_{\mathrm{LieU}(N)} \left[\frac{\mathcal{D}\Phi}{\mathrm{VolU}(N)} \right] e^{-\frac{1}{g_s}\mathsf{Tr}_N V_{p+1}(\Phi)} \tag{1.5}$$

with the polynomial potential

$$V_{p+1}(x) = \sum_{k=0}^{p} \frac{t_k}{(k+1)!} x^{k+1} \tag{1.6}$$

The convenient observable is

$$\mathbf{Y}(x) = g_s \mathsf{Tr}_N \frac{1}{x - \Phi} - V'(x) \tag{1.7}$$

In the limit $N \to \infty$, $g_s \to 0$, with $\hbar = g_s N$ fixed, the expectation value $Y(x) = \langle\, \mathbf{Y}(x)\, \rangle$ obeys:

$$Y(x)^2 = V'_{p+1}(x)^2 + f_{p-1}(x) \tag{1.8}$$

where $f_{p-1}(x)$ is a degree $p-1$ polynomial of x:

$$f_{p-1}(x) = \left\langle \mathsf{Tr}_N \left(\frac{V'(\Phi) - V'(x)}{\Phi - x} \right) \right\rangle \tag{1.9}$$

whose coefficients encode the expectation values of degree $\le p-1$ Casimirs of Φ. One can reformulate (1.8) somewhat more invariantly by stating that the singularities of $\mathbf{Y}(x)$ disappear in $\langle\, \mathbf{Y}(x)\, \rangle^2$, in the planar limit $N \to \infty$. For finite $N, \hbar$ the Dyson-Schwinger equation has the form

$$\left\langle\, \mathbf{Y}(x)^2 \;-\; g_s\, \partial_x \mathbf{Y}(x)\, \right\rangle = V'_{p+1}(x)^2 + f_{p-1}(x) \tag{1.10}$$

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Although the equation (1.10) is not a closed system of equations per se, it illustrates a principle, which we shall generalize below: given the basic operator $\mathbf{Y}(x)$ which, as a function of the auxiliary parameter x has singularities, one constructs an expression, e.g.

$$\mathbf{T}(x) = \mathbf{Y}(x)^2 - \hbar \partial_x \mathbf{Y}(x) \tag{1.11}$$

whose expectation value has no singularities in x for finite x, cf. (1.10). We shall be able to generalize this procedure for the supersymmetric gauge theories in various spacetime dimensions.

1.2 Non-perturbative Dyson-Schwinger identities

Let us now study the identities, which can be interpreted as the analogs of (1.4), (1.10) corresponding to non-trivial permutations of homology classes $\mathbf{\Gamma} = \sum_a n_a \Gamma_a \longrightarrow \mathbf{\Gamma}' = \sum_a n'_a \Gamma_a$, where (Γ_a) is some basis in the relative homology, cf. [7]

$$H_{\frac{1}{2}\mathrm{dim}}(\mathcal{F}^{\mathbb{C}},\ \mathcal{F}^{\mathbb{C}}_{+})$$

where $\mathcal{F}^{\mathbb{C}}$ is the space of complexified fields, and $\mathcal{F}^{\mathbb{C}}_{+} \subset \mathcal{F}^{\mathbb{C}}$ is the domain, where $\mathrm{Re}S[\Phi] \gg 0$.

H. Nakajima [76] discovered that the cohomology of the moduli spaces of instantons carries representations of the infinite-dimensional algebras (this fact was used in the first strong coupling tests of S-duality of maximally supersymmetric gauge theories [118]). These algebras naturally occur in physics as symmetries of two dimensional conformal theories. This relation suggests the existence of a novel kind of symmetry in quantum field theory which acts via some sort of permutation of the integration regions in the path integral. The transformations of cohomology classes do not, typically, come from the point symmetries of the underlying space. Indeed, the infinitesimal symmetries, e.g. generated by some vector field $v \in \mathsf{Vect}(\mathfrak{X})$ act trivially on the de Rham cohomology $H^*(\mathfrak{X})$ of $\mathfrak{X}$, as closed differential forms change by the exact forms:

$$\delta\omega = \mathrm{Lie}_v \omega = d(\iota_v \omega) \implies [\delta\omega] = 0 \in H^*(\mathfrak{X}) \tag{1.12}$$

The symmetries of the cohomology spaces come, therefore, from the *large* transformations $f : \mathfrak{X} \to \mathfrak{X}$ or, more generally, the correspondences $L \subset \mathfrak{X} \times \mathfrak{X}$:

$$\phi_L = t_* \left(\delta_L \wedge s^* \right) : H^*(\mathfrak{X}) \to H^*(\mathfrak{X}) \tag{1.13}$$

where δ_L is the Poincare dual to L, the maps s, t are the projections

$$\begin{array}{ccc} & \mathfrak{X} \times \mathfrak{X} & \\ {}^{s}\swarrow & & \searrow^{t} \\ \mathfrak{X} & & \mathfrak{X} \end{array} \tag{1.14}$$

and we assume the compactness and smoothness. There exist generalizations which relax these assumptions.

The physical realization of the symmetries generated by (1.14) is yet to be understood. It was proposed in [89, 97, 98] that there are symmetries acting *between* different quantum

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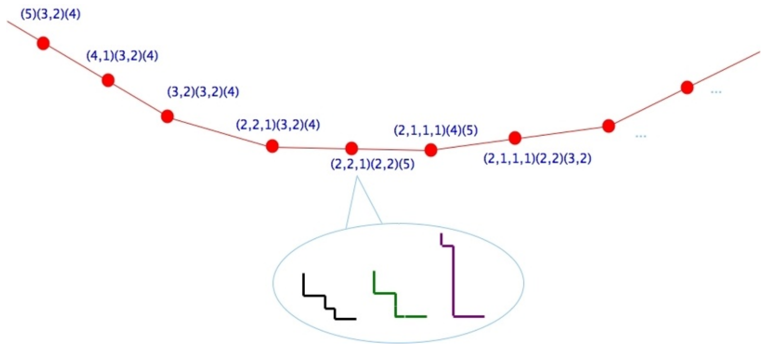

Figure 2. Path integral in U(3) gauge theory in the sector with $k = 14$ instanons. The labels $(\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots)(\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots)(\lambda_1^{(3)}, \lambda_2^{(3)}, \ldots)$ denote various instanton configurations.

field theories, for example changing the gauge groups. Conjecturally [81] supersymmetric domain walls in quantum field theory separating different phases of one theory or even connecting, e.g. in a supersymmetric fashion, two different quantum field theories can be used to generate the generalized symmetries of the sort we discussed earlier. More precisely, one exchanges the spatial and the temporal directions, producing the S-brane [48] version of the domain wall.

This paper deals with another type of *large* symmetries. They are generated by the transformations (1.3) changing the topological sector, i.e. mapping one connected component of the space of fields to another. We shall be concerned with gauge theories, i.e. the Yang-Mills theory on the space-time $\mathcal{N}$,

$$Z = \int DA \, \exp\left(-\frac{1}{4g^2}\int_{\mathcal{N}} \mathsf{Tr}\, F_A \wedge \star F_A + \frac{\mathrm{i}\vartheta}{8\pi^2}\int_{\mathcal{N}} \mathsf{Tr}\, F_A \wedge F_A\right) \tag{1.15}$$

with the gauge group $\mathsf{G}_{\mathbf{g}}$, and its supersymmetric generalizations. The connected components of the space of gauge fields are labeled by the topology types of the principal $\mathsf{G}_{\mathbf{g}}$-bundles, and measured, in particular, by the instanton charge

$$n = -\frac{1}{8\pi^2}\int_{\mathcal{N}} \mathsf{Tr}\, F_A \wedge F_A. \tag{1.16}$$

Gauge theory path integral is the sum over n of the path integrals over the space of fields of fixed topology: see figure 2.

The analog of the contour deformation (1.3) is the discrete deformation, as in the figure 3.

There is no a priori way to deform a connection A_0 on a principal bundle P_0 to a connection A_1 on a principal bundle P_1, which is not isomorphic to P_0. However, imagine that we modify P_0 in a small neighborhood of a point $x \in \mathcal{N}$ so that it becomes isomorphic to P_1. It means that outside a small disk $D_x \subset \mathcal{N}$ there is a gauge transformation, which makes A_0 deformable to A_1. One can loosely call such a modification *adding a point-like instanton at x*

$$A \longrightarrow A + \delta_x^{(1)} A \tag{1.17}$$

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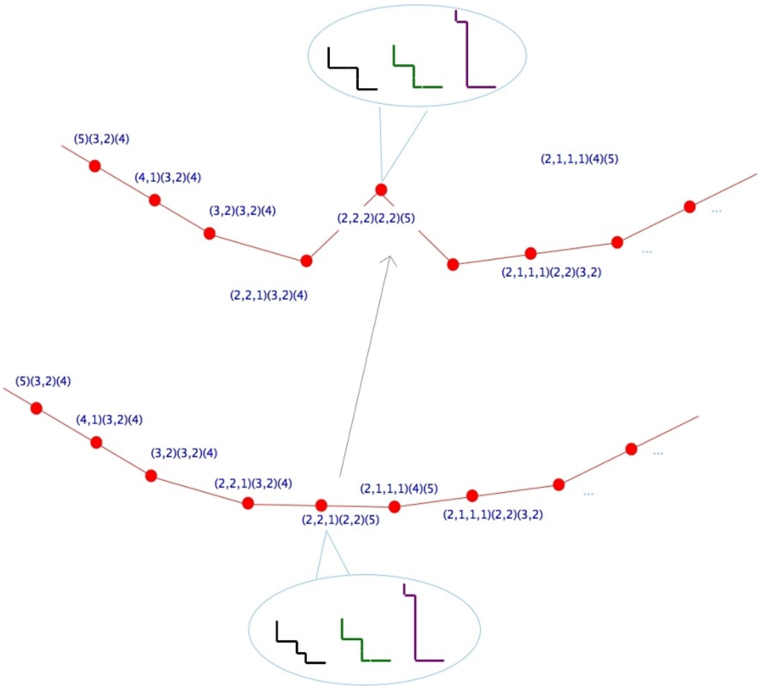

Figure 3. Path integral in U(3) gauge theory in the sector with $k = 14$ instanons, and discrete deformation to account for $k = 15$ instantons.

One can imagine a successive application of the modifications $\delta^{(1)}_{x_1}\delta^{(1)}_{x_2}$, which add point-like instantons at two distinct points $x_1 \neq x_2 \in \mathcal{N}$, or adding two instantons at the same point, $A \longrightarrow A + \delta^{(2)}_x A$, and so on.

The specific realization of such modifications is possible in the string theory context, where the gauge theory instantons are the codimension four D-branes dissolved inside another brane [28]. The modification changing the instanton number is then a transition where, say, a point-like instanton becomes a D0-brane departed from the D4-brane.

1.3 Organization of the presentation

We want to study such modifications in the gauge theory language. Specifically, we shall work in the context of $\mathcal{N} = 2$ supersymmetric gauge theories subject to Ω-deformation. We explore these theories using the special observables $\mathcal{X}$ and $\mathcal{Y}$, which will help us to organize the non-perturbative Dyson-Schwinger identities reflecting the invariance of the path integral with respect to the transformations (1.17). We shall see that these identities are organized in a structure, *the qq-characters*, which suggest a deformation of the q-deformed Kac-Moody symmetry. The latter is familiar from the study of lattice and massive integrable field theories in two dimensions [10, 20–23, 30–33, 35, 36, 38, 39, 53, 60, 109–115].

The qq-characters are local observables, the corresponding operators can be inserted at a point in space-time. One can also define and study non-local observables, which are associated to two-dimensional surfaces in space-time. These will be studied elsewhere.

It is worth revealing at this point that the qq-characters (and the analogous surface operators) can be defined most naturally in the context of string theory, where the gauge

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theory in question arises as a low energy limit of the theory on a stack of $D3$-branes (the 'physical' branes) in some supersymmetric background. The qq-characters in this realization are the low-energy limits of the partition function of the auxiliary theory, which lives on a stack of $D3$-branes intersecting the physical branes transversely at a point. We define the qq-character operators in the presence of the surface operators in [103].

In the companion paper [100] these constructions of gauge theories with and without surface defects, as well as the qq-character operators are given a unified treatment using what we call the *gauge origami*, a generalized gauge theory, which is best thought of a low-energy limit of a theory on a stack of Dp-branes in type II string theory, which span the coordinate $\mathbb{C}^2$-planes inside $\mathbb{C}^4$ times a common flat $2p-4$-dimensional space.

Orbifolding this construction by discrete symmetries, preserving supersymmetry and the Ω-deformation, leads to more examples of qq-observables and defect operators in quiver gauge theories on asymptotically locally Euclidean spaces.

These constructions can be realized mathematically with the help of novel moduli spaces, which we call the crossed instantons and the spiked instantons. The space of *crossed instantons* describes the low-energy modes of open strings connecting k $D(-1)$ instantons and two stacks of $D3$-branes, spanning two transversely intersecting copies of $\mathbb{R}^4$ inside $\mathbb{R}^8$. When one of the two stacks is empty the moduli space coincides with the ADHM moduli space of (noncommutative) instantons on $\mathbb{R}^4$, together with the obstruction bundle, isomorphic, in this case, to the cotangent bundle. The space of *spiked instantons* is the further generalization, describing the low-energy modes of the open strings stretched between the $D(-1)$-instantons and six stacks of $D3$-branes spanning the coordinate complex 2-planes in $\mathbb{C}^4$, a local model of the maximal number of complex surfaces intersecting at a point in a Calabi-Yau fourfold.

In the next section we recall the relevant details about the *BPS* side of the BPS/CFT correspondence, the supersymmetric partition functions of $\mathcal{N}=2$ theories. In this paper we discuss the bulk partition functions, in the companion papers [100, 103] we study the theories with defects. We also give a rough definition of the $\mathcal{X}$ and $\mathcal{Y}$ observables, and some physics behind them. In the section 3 we review quiver gauge theories with unitary gauge groups, which are superconformal in the ultraviolet. The section 4 gives the mathematical expression for the integrals over instanton moduli spaces. The path integrals in the quiver gauge theories under consideration, with and without defects, reduce to those finite dimensional integrals by localization. The section 5 defines the $\mathcal{Y}$-observables in gauge theory, both in the physical theory and in the mathematical problem of integration over the instanton moduli. The section 6 introduces informally the $\mathcal{X}$-observables, the qq-characters, and formulates the main theorem. The section 7 presents the examples of qq-characters. The section 8 defines the qq-characters rigorously, by explicit formulas.

Acknowledgments

I have greatly benefited from discussions with H. Nakajima and A. Okounkov. The suggestion of V. Pestun that the results of [85, 86] must generalize to the case of the general Ω-deformaton was instrumental in pursuing the constructions presented below. Special thanks are to O. Tsymbalyuk who read the preliminary versions of this manuscript and suggested lots of improvements. Part of the work was done while the author visited the Euler International Mathematical Institute in Saint-Petersburg and the Imperial College London.

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The results of the paper, notably the formulae for the qq-characters and their consequences, the equations on the gauge theory correlation functions, were reported at.[1] The preliminary version of this paper was published under the title "Non-Perturbative Schwinger-Dyson Equations: From BPS/CFT Correspondence to the Novel Symmetries of Quantum Field Theory" in the proceedings [47] of the ITEP conference (June 2013) in honor of the 100-th anniversary of Isaac Pomeranchuk. It took us a long time to write up all the details of the story. While the paper was being prepared, several publications have appeared with some degree of overlap. The paper [56] supports the validity of our main theorem in the case of the A-type quivers. The papers [116, 117] contain some discussion of the $(0,4)$-sigma model on our moduli space $\mathfrak{M}(n, \mathtt{w}, k)$ (for $\zeta = 0$). The paper [16] contains the first few instanton checks of some of the results of our paper (for the $\widehat{A}_0$ theory). The paper [41] studies the codimension defects using the superconformal index and sphere partition functions, and the RG flows from vortex constructions, also at the level of the first few instanton checks. The paper [14] discusses the algebra of our Y-observables in the A-type quiver theories.

2 The BPS/CFT correspondence

We start by briefly reviewing the BPS/CFT correspondence [83] between supersymmetric field theories with eight supercharges in four, five, and six dimensions, and conformal

[1] Various conferences in 2013-2015:

- "Facets of Integrability: Random Patterns, Stochastic Processes, Hydrodynamics, Gauge Theories and Condensed Matter Systems", workshop at the SCGP, Jan 21-27, 2013
- Gelfand Centennial Conference: A View of 21st Century Mathematics, MIT, Sept 2013
- ITEP conference in honor of the 100-th anniversary of Isaac Pomeranchuk, June 2013
- MaximFest, IHES, June 2013
- Strings'2014, Princeton, June 2014
- 'Frontiers in Field and String Theory', Yerevan Physics Institute, Sept 2014
- "Gauged sigma-models in two dimensions", workshop at the SCGP, Nov 3-7, 2014
- "Wall Crossing, Quantum Integrable Systems, and TQFT", Nov 17-21, 2014
- "Recent Progress in String Theory and Mirror Symmetry", FRG workshop at Brandeis, Mar 6-7, 2015
- "Resurgence and localization in string theory and quantum field theory", workshop at the SCGP, Mar 16-20, 2015
- "Algebraic geometry and physics", workshop at the Euler Mathematics Institute, Saint-Petersburg, May 2015

seminars in 2013-2015: http://scgp.stonybrook.edu/video_portal/video.php?id=1599.

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and integrable theories in two dimensions. It is based on the observation that the supersymmetric partition functions [94] are the remarkable special functions, which generalize all the known special functions given by the periods, matrix integrals, matrix elements of group, Kac-Moody, and quantum group representations etc. [67, 84, 94]. The particular implementations of this correspondence are well-known under the names of the AGT conjecture [2, 120], and the Bethe/gauge correspondence [89, 96] (see [43, 44, 74] for the prior work). For details the interested reader may consult the references in, e.g. [86].

2.1 $\mathcal{N} = 2$ partition functions

For the definition and some details see [84, 94]. The supersymmetric partition function of $\mathcal{N} = 2$ theory

$$\mathcal{Z}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\boldsymbol{\tau}};\underline{\varepsilon}) = \mathcal{Z}^{\text{tree}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\boldsymbol{\tau}};\underline{\varepsilon})\ \mathcal{Z}^{1-\text{loop}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon})\ \mathcal{Z}^{\text{inst}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\mathfrak{q}};\underline{\varepsilon}) \tag{2.1}$$

depends on the vacuum expectation value $\underline{\mathfrak{a}}$ of the adjoint Higgs field in the vector multiplet, it belongs to the complexified Cartan subalgebra of the gauge group of the theory, the set $\underline{\mathfrak{m}}$ of complex masses of the matter multiplets, and the set $\underline{\boldsymbol{\tau}}$ of the complexified gauge couplings,

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi \mathrm{i}}{g^2}\,,$$

one per simple gauge group factor (we shall not discuss the issue of $\mathrm{SU}(n)$ versus $\mathrm{U}(n)$ gauge factors in this paper). We denote by $\underline{\mathfrak{q}}$ the set of the exponentiated couplings, the instanton factors,

$$\mathfrak{q} = \exp 2\pi \mathrm{i}\boldsymbol{\tau}$$

The non-perturbative factor $\mathcal{Z}^{\text{inst}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\mathfrak{q}};\underline{\varepsilon})$ in (2.1) has the $\underline{\mathfrak{q}}$-expansion, for small $|\underline{\mathfrak{q}}|$:

$$\mathcal{Z}^{\text{inst}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\mathfrak{q}};\underline{\varepsilon}) = \sum_{\underline{\mathbf{k}}} \underline{\mathfrak{q}}^{\underline{\mathbf{k}}}\ \mathcal{Z}_{\underline{\mathbf{k}}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon}) \tag{2.2}$$

Finally, $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathbb{C}^2$ are the complex parameters of the Ω-deformation of the theory [94].

2.1.1 Asymptotics of partition functions

The function (2.1) contains non-trivial information about the theory. For example, the asymptotics at $\underline{\varepsilon} \to (0,0)$, for *generic* $\underline{\mathfrak{a}}$, produces the prepotential [106, 107] of the low-energy effective action of the theory:

$$\mathcal{Z}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\boldsymbol{\tau}};\underline{\varepsilon}) \sim \exp \frac{1}{\varepsilon_1\varepsilon_2}\mathcal{F}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\boldsymbol{\tau}}) + \text{less singular in } \varepsilon_1, \varepsilon_2. \tag{2.3}$$

the low-energy effective action being given by the superspace integral

$$S^{\text{eff}} = \int_{\mathbb{R}^{4|4}} d^4x d^4\vartheta\ \mathcal{F}(\underline{\mathfrak{a}} + \vartheta\psi + \vartheta\vartheta F^- + \ldots; \underline{\mathfrak{m}}; \underline{\boldsymbol{\tau}}) \tag{2.4}$$

The prepotential $\mathcal{F}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\boldsymbol{\tau}})$ as a function of $\underline{\mathfrak{a}}$ determines the special geometry [34] of the moduli space $\mathcal{M}^{\text{vector}}$ of Coulomb vacua:

$$\boldsymbol{d}\begin{pmatrix}\underline{\mathfrak{a}}\\ \\ \frac{\partial\mathcal{F}}{\partial\underline{\mathfrak{a}}}\end{pmatrix} = \text{periods of } \varpi^{\mathbb{C}} \tag{2.5}$$

along the 1-cycles on the abelian variety A_b, the fiber $p^{-1}(b)$ of the Lagrangian projection

$$p : \mathfrak{P} \longrightarrow \mathcal{M}^{\text{vector}} \tag{2.6}$$

of a complex symplectic manifold $(\mathfrak{P}, \varpi^{\mathbb{C}})$, the moduli space of vacua of the same gauge theory, compactified on a circle [108]. The manifold $\mathfrak{P}$ is actually the phase space of an algebraic integrable system [27]. The first example of this relation, the periodic Toda chain for the SU(2) pure super-Yang-Mills theory, was found in [46]. The asymptotics of (2.1) at $\varepsilon_2 \to 0$ with $\varepsilon_1 = \hbar$ fixed, for *generic* $\underline{\mathfrak{a}}$, gives the effective twisted superpotential

$$\mathcal{Z}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\boldsymbol{\tau}};(\hbar,\varepsilon_2)) \sim \exp\frac{1}{\varepsilon_2}\mathcal{W}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\boldsymbol{\tau}};\hbar) + \text{less singular in } \varepsilon_2, \qquad \varepsilon_2 \to 0 \tag{2.7}$$

of a two dimensional effective theory. This function plays an important role in quantization of the symplectic manifold $\mathfrak{P}$ and the Bethe/gauge correspondence [86, 87, 89, 90, 96].

The asymptotics (2.7), (2.3) are modified in an intricate way when the *genericity* assumption on $\underline{\mathfrak{a}}$ is dropped. The interesting non-generic points are where $\underline{\mathfrak{a}}$ and ε_1 (with $\varepsilon_2 \to 0$) are in some integral relation. The behavior near such special points and its rôle in the Bethe/gauge correspondence will be discussed elsewhere.

2.2 Defect operators and lower-dimensional theories

In addition to the $\mathcal{Z}$-functions, which are the partition functions of the theory on $\mathbb{R}^4$, in [103] we also consider the partition functions $\boldsymbol{\Psi}$ of the same gauge theory in the presence of defects preserving some fraction of supersymmetry. These defects could be point-like, or localized along surfaces. We derive the differential equations, which can be used to relate the theory with a surface operator to the theory without one. As a by-product we get the explicit realization of the Bethe/gauge correspondence [89, 96] with an additional bonus: the gauge theory produces not only the equations, characterizing the spectrum of the quantum integrable system, but also gives an expression for the common eigenfunction of the full set of quantum integrals of motion. In particular, we shall show [102] in that a class of surface defect operators in the $\mathcal{N} = 2$ theory with $\mathrm{U}(n)$ gauge group and $2n$ fundamental hypermultiplets solves the Knizhnik-Zamolodchikov (KZ) equation, which is obeyed by the 4-point conformal block of the $\mathrm{SU}(n)$ Wess-Zumino-Witten theory on the sphere; that a class of surface defect operators in the $\mathcal{N} = 2^*$ theory with the $\mathrm{U}(n)$ gauge group solves the Knizhnik-Zamolodchikov-Bernard (KZB) equation, which is obeyed by the 1-point conformal block of the $\mathrm{SU}(n)$ Wess-Zumino-Witten theory on the torus. In the $\varepsilon_2 \to 0$ limit these surface operators become the eigenfunctions of Gaudin Hamiltonian and the elliptic Calogero-Moser Schrödinger equation, respectively, in agreement with the conjectures in [3, 96] and earlier ideas.

In addition to the codimension two defects in four or five dimensional theories we can also consider lower dimensional theories. For example, gauge theory on the AdS_3 space with appropriate boundary conditions can be viewed as the U(1)-orbifold of a four dimensional superconformal gauge theory. We study these cases in [100].

2.3 The $\mathcal{Y}$- and $\mathcal{X}$-observables

The main tools in our analysis are the gauge invariant observables $\mathcal{Y}_{\mathbf{i}}(x)$ and $\mathcal{X}_{\mathbf{i}}(x)$, defined for each simple factor $\mathrm{U}(n_{\mathbf{i}})$ of the gauge group. Here $\mathbf{i}$ belongs to the set Vert_γ, which in our story is the set of vertices of a quiver. The $\mathcal{Y}_{\mathbf{i}}(x)$ are the suitable generalizations of the characteristic polynomials of the adjoint Higgs field,

$$\mathcal{Y}_{\mathbf{i}}(x) \sim \mathtt{Det}(x - \Phi_{\mathbf{i}})\,. \tag{2.8}$$

They are the gauge theory analogues of the matrix model resolvents (1.7). As a function of x, each operator $\mathcal{Y}_{\mathbf{i}}(x)$ has singularities, i.e. the relation (2.8) is modified. This modification is due to the mixing between the adjoint scalar and gluinos, e.g.

$$\Phi_{\mathbf{i}} \sim \Phi_{\mathbf{i}}^{\mathrm{cl}} \;+\; \varepsilon_{\alpha\beta}\varepsilon_{j'j''}\,(d^*_{A_{\mathbf{i}}} d_{A_{\mathbf{i}}})^{-1}\,[\psi_{\mathbf{i}}^{\alpha j'}, \psi_{\mathbf{i}}^{\beta j''}] \tag{2.9}$$

which have zero modes in the presence of gauge instantons, leading to the poles in x, in a way we make much more precise below. The $\mathcal{X}_{\mathbf{i}}(x)$ are composite operators, built out of $\mathcal{Y}$'s. They are Laurent polynomials or series in $\mathcal{Y}_{\mathbf{i}}(x)$'s with shifted arguments and their derivatives. They are the analogues of the matrix model operators (1.11). Their main property is the absence of singularities in $\langle\, \mathcal{X}_{\mathbf{i}}(x)\,\rangle$ for finite x, similarly to the matrix model case, cf. (1.10). In the weak coupling limit $\mathcal{X}_{\mathbf{i}}(x) \to \mathcal{Y}_{\mathbf{i}}(x) \to$ (2.8). We define also the observables $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)$, labelled by the string $\underline{\mathbf{w}} = (\mathtt{w}_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma} \in \mathbb{Z}_{\geq 0}^{\mathsf{Vert}_\gamma}$ of non-negative integers and the string $\underline{\boldsymbol{\nu}} = (\vec{\nu}_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}$, $\vec{\nu}_{\mathbf{i}} \in \mathbb{C}^{\mathtt{w}_{\mathbf{i}}}$ of complex numbers. Here Vert_γ is the set of simple gauge group factors. The expectation values $\langle\, \mathcal{X}_{\underline{\mathbf{w}}}(x)\,\rangle$ also have no singularities, while in the limit of zero gauge couplings $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)$ approach

$$\prod_{\mathbf{i}} \prod_{f=1}^{\mathtt{w}_{\mathbf{i}}} \mathcal{Y}_{\mathbf{i}}(x + \nu_{\mathbf{i},f}).$$

We call $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)$ the *qq-characters*. For $\underline{\mathbf{w}} = (\delta_{\mathbf{i},\mathbf{j}})_{\mathbf{j}\in\mathsf{Vert}_\gamma}$, and $\underline{\boldsymbol{\nu}} = \underline{0}$, we call $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x) =: \mathcal{X}_{\mathbf{i}}(x)$ the *fundamental qq-characters*. The reason for and the meaning of the terms will hopefully become clear in the coming sections.

2.4 The physics of $\mathcal{X}$-observables

The $\mathcal{X}$-observables can be interpreted as the partition functions of the auxiliary gauge theory living on a space, transverse to the space-time of our gauge theory, the "physical space-time". The auxiliary theory has massive degrees of freedom coupled to the degrees of freedom of our gauge theory at some point $p \in \mathbb{R}^4$ in the physical space-time, so that integrating them out induces an operator $\mathcal{O}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},x}(p)$ inserted at p. The data $\underline{\mathbf{w}}$ is the choice of the auxiliary gauge theory while $\underline{\boldsymbol{\nu}}$ and x fix its vacuum.

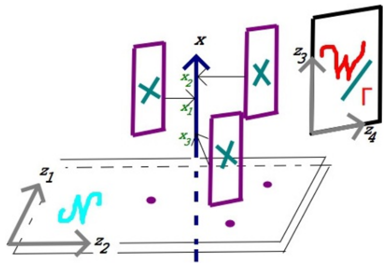

Figure 4. Gauge theory on with observables obtained by integrating out degrees of freedom living on the orbifolded transverse directions.

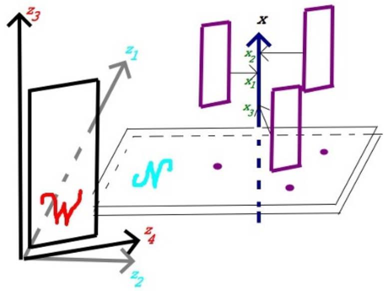

Figure 5. Gauge theory with $\mathcal{X}$-observables in the brane picture.

The dimensionality of the auxiliary gauge theory is a somewhat subtle issue. Most of the theories we study in the paper, such as the $\mathcal{N} = 2$ quiver theories with affine quivers have the $\mathcal{X}$-observables which come from a four dimensional auxiliary theory. The theories with finite quivers can be viewed as a sector in the auxiliary four dimensional theory corresponding to an affine quiver. In fact, the finite quivers of A-type can be realized as a subsector of the $\widehat{A}_\infty$ theory, which corresponds to the orbifold of $\mathbb{C}^2$ by U(1). Gauge theory living on such an orbifold can be viewed either as a three dimensional theory on a manifold with corners (or, conformally, on the AdS_3), or, for the purposes of supersymmetric partition functions, as a two dimensional sigma model [100].

Here is a sketch of the string theory construction. Consider IIB string theory on the ten-dimensional manifold of the form $\mathbb{R}^2_\phi \times \mathcal{N} \times \mathcal{W}/\Gamma$, where $\mathcal{N} = \mathbb{R}^4$, $\mathcal{W} = \mathbb{R}^4$, and Γ a finite subgroup of SU(2) (see [54] for the discussion of IIB string theory on ALE spaces).

Recall [29] that $\mathcal{N} = 2$ quiver gauge theories with affine A, D, E quivers can be realized as the low energy limit of the theory on a stack of n $D3$-branes located at $\varphi \times \mathcal{N} \times 0$, with $\varphi \in \mathbb{R}^2_\phi$ a point, and 0 the tip of the $\mathcal{W}/\Gamma$ singularity, with Γ being the discrete subgroup of SU(2), McKay dual [71] to the corresponding A, D, E simple Lie group.

The worldvolume of these $D3$ branes is a copy of $\mathcal{N}$. Let us now add a stack of $\mathtt{w}$ $D3$-branes located at $x \times \mathbf{0} \times \mathcal{W}/\Gamma$, with the worldvolume being a copy of $\mathcal{W}/\Gamma$. Here $x \in \mathbb{R}^2_\phi \approx \mathbb{C}$ is a complex number, and $\mathbf{0} \in \mathcal{N}$ is a fixed point. Here is the picture:

The low energy configurations in this system of two orthogonal stacks of $D3$ branes are labelled by the separation of branes along $\mathbb{R}^2_\phi$ encoded in $\underline{\boldsymbol{\nu}}$, and by the choice of flat $\mathrm{U}(\mathtt{w})$ connection at infinity $\mathbb{S}^3/\Gamma$ of $\mathcal{W}/\Gamma$, which is equivalent to the choice of the string $\underline{\mathbf{w}}$.

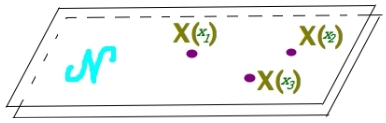

Figure 6. Gauge theory with the observables $\mathcal{X}(x_1), \mathcal{X}(x_2), \mathcal{X}(x_3)$.

The qq-character $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)$ is simply the observable in the original theory on the stack of N $D3$-branes living along $\mathcal{N}$, which is obtained by integrating out the degrees of freedom on the transversal $D3$-branes, in the vacuum corresponding to the particular asymptotic flat connection $\underline{\mathbf{w}}$ and the vacuum expectation values $\underline{\boldsymbol{\nu}}$ of the scalars in the vector multiplets living on $\mathcal{W}/\Gamma$.

The next piece of our construction is the Ω-deformation using a subgroup of the spin cover of the group Spin(8) of rotations of $\mathcal{W} \times \mathcal{N}$ which commutes with Γ, preserves the configuration of branes, and some supersymmetry. This subgroup generically has rank two, which enhances to three for Γ of A type. The parameters of the Ω-deformation are generically two complex numbers $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, and for Γ of A-type there is an additional parameter m. This parameter is the mass of the adjoint hypermultiplet in the $\widehat{A}_0$-case, and the sum of masses of all $k+1$ bi-fundamental hypermultiplets in the $\widehat{A}_k$ case for $k > 0$. It is convenient to introduce four ε-parameters, ε_a, $a = 1, 2, 3, 4$, which sum to zero:

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = 0 \tag{2.10}$$

so that $\varepsilon_3 = m$, $\varepsilon_4 = -m - \varepsilon$, $\varepsilon = \varepsilon_1 + \varepsilon_2$. Together they parametrize the generic SU(4) Ω-deformation.

The K-theoretic and elliptic versions of qq-characters correspond to the five- and six-dimensional theories, which are engineered in the analogous fashion, with $\mathbb{R}^2_\phi$ replaced by $\mathbb{S}^1 \times \mathbb{R}^1$ and $\mathbb{S}^1 \times \mathbb{S}^1$, respectively. In the five dimensional case we use IIA string and the D4 branes wrapped on $\mathbb{S}^1$ instead of D3's, in the six dimensional case we are back in the IIB realm with D5 branes wrapped on $\mathbb{S}^1 \times \mathbb{S}^1$.

The configuration of $D3$ branes which we described above can be generalized, by considering other orbifolds of the ten dimensional Euclidean space $\mathbb{R}^2_\phi \times \mathcal{N} \times \mathcal{W}$. For example, the orbifolds $\mathbb{R}^2_\phi \times \mathcal{N}/\Gamma_{\mathcal{N}} \times \mathcal{W}/\Gamma_{\mathcal{W}}$ with $D3$ branes wrapping $\mathcal{N} \sqcup \mathcal{W}$ define the qq-characters relevant for the $\gamma_{\Gamma_{\mathcal{W}}}$-quiver gauge theory on the ALE space $\widetilde{\mathcal{N}/\Gamma_{\mathcal{N}}}$. The most general orbifold we could employ is by the subgroup $\Gamma = \Gamma_{\mathcal{N}} \times \Gamma_{\Delta} \times \Gamma_{\mathcal{W}} \subset \mathrm{SU}(2)_{\mathcal{N},L} \times \mathrm{SU}(2)_{\Delta} \times \mathrm{SU}(2)_{\mathcal{W},L} \subset \mathrm{Spin}(4)_{\mathcal{N}} \times \mathrm{Spin}(4)_{\mathcal{W}} \subset \mathrm{Spin}(8)$. We explain the rôle of this group and its subgroups in the following sections. Using this construction we also realize various defect operators in various quiver gauge theories on conical spaces.

The final piece of the construction is turning on the appropriate B-field which makes the configurations where the $D(-1)$-instantons are separate from the $D3$-branes non-supersymmetric. The $D(-1)$-instantons bound to the two orthogonal stacks of $D3$ branes give rise to what we call the *crossed instantons*. We can also study the generalization involving six stacks of $D3$ branes spanning complex 2-planes inside $\mathbb{R}^8 \approx \mathbb{C}^4$. The complex coordinate x parametrizes the remaining $\mathbb{R}^2 \approx \mathbb{C}^1$, which is orthogonal to $\mathbb{C}^4$ in the ten dimensional Euclidean space-time of the type IIB string.

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The *main claim*, i.e. the absence of singularities in x of $\langle\, \mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)\,\rangle$, is the statement that the combined system of the intersecting stacks of $D3$ branes has no phase transitions and no runaway flat directions at special values of x, in the presence of Ω-deformation. Mathematically, the argument is the compactness of the moduli space of *crossed* (for two orthogonal stacks of branes) and *spiked instantons* (for six stacks of branes), the supersymmetric configurations of the combined system of branes, with the Ω-deformation and appropriate B-field turned on. We describe the moduli spaces in [101].

One can also apply the orientifold projection (which, unfortunately, would not be consistent with the B-field we are using) to arrive at the theory of crossed instantons for the orthogonal and symplectic groups.

2.5 Hidden symmetries

The IIB string theory on $\mathbb{R}^4/\Gamma \times \mathbb{R}^{1,5}$ contains the non-abelian tensionless strings [119] with the A, D, E tensor symmetry (it becomes the gauge symmetry of the A, D, E type upon compactification on a circle, i.e. when $\Sigma = \mathbb{S}^1 \times \mathbb{R}^1$).

In the limit $\varepsilon_1, \varepsilon_2 \to 0$ our qq-characters approach the ordinary characters for the Kac-Moody group built on the quiver (i.e. the affine Lie group $\widehat{A}, \widehat{D}, \widehat{E}$ for affine quivers, and the simple A, D, E groups for the finite quivers), [85]. The non-abelian tensor symmetry seems to be realized on the worldvolume of $D3$ branes by the *large field deformations* which lead to the non-perturbative Dyson-Schwinger equations we discuss in this paper. The qq-character observables may teach us something important about the nature of the tensor symmetry representation. See [58] for the discussion of the qq-deformed $\mathcal{W}$-algebras and their gauge theory realizations.

2.6 Some notations

2.6.1 Finite sets

We use the following notations for certain finite sets:

$$\begin{aligned} [p] &\equiv \{1, 2, \ldots, p\}, \qquad && p \in \mathbb{Z}_+, \\ [0, q] &\equiv \{0, 1, 2, \ldots, q\}, \qquad && q \in \mathbb{Z}_{\geq 0} \end{aligned} \tag{2.11}$$

and

$$(x_i)_{i\in I} \equiv \{\ x_i \mid i \in I\ \} \tag{2.12}$$

Also for the set $(z_i)_{i\in I}$ of complex numbers indexed by the set I we use the notation

$$z_I = \prod_{i\in I} z_i \tag{2.13}$$

for their product. This is consistent with the notation (2.23).

2.6.2 Roots of unity

$$\mathrm{i} = \sqrt{-1}\,, \tag{2.14}$$

and

$$\varpi_p = \exp \frac{2\pi\mathrm{i}}{p} \tag{2.15}$$

so that $\mathrm{i} = \varpi_4$, $-\mathrm{i} = \varpi_4^3$.

2.6.3 Parameters of Ω-deformations

In four dimensions, we have two parameters $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in \mathbb{C}^2$. We also use their sum

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \,, \tag{2.16}$$

their exponents

$$q_1 = e^{\beta\varepsilon_1} \,, \; q_2 = e^{\beta\varepsilon_2} \,, \; q = e^{\beta\varepsilon} = q_1 q_2 \,, \tag{2.17}$$

and the virtual characters

$$P = (1 - q_1)(1 - q_2) \,, \qquad P^* = (1 - q_1^{-1})(1 - q_2^{-1}) \tag{2.18}$$

The parameter β is the circumference of the circle of compactification of a 4+1 dimensional supersymmetric theory.

In the context of the BPS/CFT correspondence, the parameter

$$b^2 = \varepsilon_1 / \varepsilon_2 \tag{2.19}$$

is useful.

In eight dimensions, or for the theories in four dimensions with adjoint matter, it will be useful to have four parameters

$$\bar{\varepsilon} = (\underline{\varepsilon}, \underline{\tilde{\varepsilon}}) \equiv (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{C}^4 \,, \tag{2.20}$$

which sum to zero:

$$\varepsilon_3 + \varepsilon_4 = -\varepsilon \tag{2.21}$$

We denote $\mathbf{4} = \{1, 2, 3, 4\}$ and

$$\begin{aligned} q_a &= e^{\beta\varepsilon_a}, \qquad P_a = (1 - q_a), \qquad a \in \mathbf{4} \\ q_a^* &= q_a^{-1}, \qquad P_a^* = (1 - q_a^{-1}), \end{aligned} \tag{2.22}$$

For any subset $S \subset \mathbf{4}$ we define $\bar{S} = \mathbf{4} \backslash S$, and:

$$q_S = \prod_{a \in S} q_a, \quad q_S^* = q_{\bar{S}}, \qquad P_S = \prod_{a \in S} P_a, \quad P_S^* = (-1)^{|S|} \, q_{\bar{S}} \, P_S \tag{2.23}$$

so that $q_{\emptyset} = q_{\mathbf{4}} = 1$.

2.6.4 Chern characters and Euler classes

Let $E \to \mathfrak{X}$ be the rank $m = \mathrm{rk}E$ complex vector bundle, and $c_i(E) \in H^{2i}(\mathfrak{X}, \mathbb{Z})$ the corresponding Chern classes. Then $e(E) = c_{\mathrm{top}}(E) = c_m(E)$ is the Euler class of E, and

$$\epsilon_z(E) = e(E) + z c_{m-1}(E) + z^2 c_{m-2}(E) + \ldots z^m \tag{2.24}$$

is the Chern polynomial.

2.6.5 Weights from characters

For a virtual representation R of a Lie group H, the *weights* w are computed using its character as follows:

$$R = R^{+} \ominus R^{-}\,, \quad \mathsf{Tr}_{R^{\pm}}(e^{\theta}) = \sum_{w \in W(R^{\pm})} e^{w(\theta)}, \quad \mathsf{Tr}_{R}(e^{\theta}) = \mathsf{Tr}_{R^{+}}(e^{\theta}) - \mathsf{Tr}_{R^{-}}(e^{\theta})\,, \tag{2.25}$$

where $R^{\pm}$ are the vector spaces, the actual representations of H, and $W(R^{\pm})$ are the sets of the corresponding weights (the linear functions on Lie(H) which take integer values on the root lattice).

2.6.6 Chern polynomials from characters

We denote by $\epsilon_{\theta}(R)$ the following Weyl-invariant rational function on the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of Lie($\mathsf{H}_{\mathbb{C}}$):

$$\epsilon_{\theta}(R) = \frac{\prod_{w \in W(R^{-})} w(\theta)}{\prod_{w \in W(R^{+})} w(\theta)}, \quad \theta \in \mathfrak{h}_{\mathbb{C}}, \tag{2.26}$$

where the weights $w(\theta)$ are given by (2.25). Note that in (2.26) the weights of R^{+} are in the denominator. It follows from the definition (2.25) that

$$\epsilon_{\theta}(R)\epsilon_{\theta}(-R) = 1\,, \tag{2.27}$$

if R does not contain ± 1 as a summand, and, more generally:

$$\epsilon_{\theta}(R_1 \oplus R_2) = \epsilon_{\theta}(R_1)\epsilon_{\theta}(R_2)\,. \tag{2.28}$$

Also,

$$\epsilon_{\theta}(R^{*}) = \epsilon_{-\theta}(R) = (-1)^{\mathsf{d}_R}\, \epsilon_{\theta}(R) \tag{2.29}$$

where

$$\mathsf{d}_R = \dim_{\mathbb{C}} R^{+} - \dim_{\mathbb{C}} R^{-} \tag{2.30}$$

2.6.7 Chern functions from characters

In our story we occasionally encounter the generalizations of the formulas like (2.26) where the representations $R^{\pm}$ are infinite dimensional. In order for (2.26) to make sense in this case we use the

2.6.8 ζ-function regularization

The map from the character (2.25) to $\epsilon_{\theta}(R)$ can be given in the integral form:

$$\epsilon_{\theta}(R) = \exp \left. \frac{d}{ds} \right|_{s=0} \frac{\Lambda^{s}}{\Gamma(s)} \int_{0}^{\infty} \frac{d\beta}{\beta} \beta^{s} \mathsf{Tr}_{R} e^{\beta\theta} \tag{2.31}$$

where one chooses s and θ with the real part in the appropriate domain to ensure the convergence of the integral in the right hand side of (2.31), and then analytically continues. For finite dimensional R the result does not depend on Λ. For infinite dimensional R the left hand side of (2.31) really is defined by the right hand side. More precisely, we assume R is graded,

$$R = \bigoplus_{n=0}^{\infty} R_n \,, \tag{2.32}$$

with the finite dimensional virtual subspaces $R_n = R_n^+ - R_n^-$, $\mathrm{dim}R_n^{\pm} < \infty$, whose superdimensions grow at most polynomially with n. We define

$$\epsilon_\theta(R) = \mathrm{Lim}_{t\to +0} \exp \left. \frac{d}{ds} \right|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{d\beta}{\beta} \beta^s \sum_{n=0}^{\infty} e^{-\beta t(n+1)}\, \mathsf{Tr}_{R_n} e^{\beta\theta} \tag{2.33}$$

where the integral in the right hand side converges for sufficiently large $\Re(t), \Re(s)$, defining an analytic function, whose asymptotics near $s, t = 0$ defines the left hand side.

For example, take $R = \mathbb{C}[z_1, z_2, z_3, \ldots, z_\delta]$ with $\mathsf{H} = (\mathbb{C}^\times)^{\delta+1}$ acting via:

$$t : f \mapsto f^t, \qquad f^t(z) = t_0 f(t_1^{-1} z_1, t_2^{-1} z_2, \ldots, t_\delta^{-1} z_\delta) \tag{2.34}$$

The character $\mathsf{Tr}_R e^{\beta\theta}$

$$\mathsf{Tr}_R e^{\beta\theta} = e^{\beta\theta_0} \prod_{i=1}^{\delta} \frac{1}{1 - e^{\beta\theta_i}} \tag{2.35}$$

and the refined character (where we define the grading by the polynomial degree)

$$\sum_{n=0}^{\infty} e^{-\beta t(n+1)}\, \mathsf{Tr}_{R_n} e^{\beta\theta} = e^{\beta(\theta_0 - t)} \prod_{i=1}^{\delta} \frac{1}{1 - e^{\beta(\theta_i - t)}} \tag{2.36}$$

are easy to compute.

The integral on the right hand side of (2.33) absolutely converges for $\Re t > \max_{i=0}^{\delta} \Re\theta_i$ and $\Re s > \delta$.

2.6.9 Asymptotics of the ζ-regularized ϵ_θ's

Let R be a virtual representation, $\theta \in \mathfrak{h}_{\mathbb{C}}$, and $\chi_{R,\theta}(\beta) = \mathsf{Tr}_R e^{\beta\theta}$. For $\beta \to 0$ the function $\chi_{R,\theta}(\beta)$ has an expansion:

$$\chi_{R,\theta}(\beta) = \sum_{n=-\delta_R}^{+\infty} \beta^n \, \chi_{R,\theta,n} \tag{2.37}$$

where $\chi_{R,\theta,n}$ is a homogeneous rational function of θ of degree n, obeying:

$$\chi_{R^*,\theta,n} = \chi_{R,-\theta,n} = (-1)^n \chi_{R,\theta,n} \tag{2.38}$$

We are interested in the large x asymptotics of

$$\begin{aligned}
\epsilon_{-x+\theta}(R) \equiv \exp \left. \frac{d}{ds} \right|_{s=0} \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{d\beta}{\beta} \beta^s e^{-\beta x} \mathsf{Tr}_R e^{\beta\theta} \sim \\
\sim \exp \left(- \sum_{n=-\delta_R}^{0} \chi_{R,\theta,n} \frac{(-x)^n}{(-n)!} \left(\log\left(\frac{x}{\Lambda}\right) - \sum_{k=1}^{-n} \frac{1}{k} \right) \right) \\
\times \exp \left(\sum_{n=1}^{\infty} \chi_{R,\theta,n} \frac{(n-1)!}{x^n} \right)
\end{aligned} \tag{2.39}$$

(since the variable x shifts the auxiliary variable t used to regularize an infinite trace, we can safely set $t = 0$ in (2.39)). Thus,

$$\epsilon_{x+\theta}(R^*) = \epsilon_{-x+\theta}(R) \;\times\; \exp\left(-\pi\mathrm{i}\sum_{n=-\delta_R}^{0} \chi_{R,\theta,n}\frac{(-x)^n}{(-n)!}\right) \tag{2.40}$$

2.6.10 Flips $\rightsquigarrow$

We shall also use a notation, for a virtual representation $R = R_1 \oplus R_2$,

$$R \rightsquigarrow R_1 \oplus R_2^* \tag{2.41}$$

and similarly for their characters:

$$\mathrm{Tr}_R \rightsquigarrow \mathrm{Tr}_{R_1} + \mathrm{Tr}^*_{R_2} \tag{2.42}$$

where we also use the convention

$$\chi^* = \sum_{w\in W(R)} e^{-w(\theta)}, \qquad \text{for} \qquad \chi = \sum_{w\in W(R)} e^{w(\theta)} \tag{2.43}$$

Sometimes, when the choice of the element $\theta \in \mathfrak{h}_{\mathbb{C}}$ is understood, we denote the trace $\mathrm{Tr}_R(e^\theta)$ in the representation R by the same letter R.

For example

$$(1+q^{-1})(1-q_1) \rightsquigarrow 1 - q_1 + q_1 q_2 - q_2 = P \tag{2.44}$$

The multiplicative measures of the finite dimensional virtual representations R, given by the products (2.26) of weights $w(\theta)$ and their K-theoretic analogues, given by the products of $2\sin\left(\frac{w(\theta)}{2}\right)$ do not change, up to a sign, under the $\rightsquigarrow$ modifications:

$$\epsilon_\theta(R') = (-1)^{d_{R_2}}\,\epsilon_\theta(R), \qquad R \rightsquigarrow R' \tag{2.45}$$

In the infinite dimensional case the multiplicative anomaly of the measure (2.31) follows from (2.40).

2.7 Equivariant virtual Chern polynomials

Let R be a virtual representation as above, and

$$R = \left(\oplus_{w\in W(R^+)} R_w^+\right) \ominus \left(\oplus_{w\in W(R^-)} R_w^-\right) \tag{2.46}$$

be the corresponding weight decomposition. Let E_w, with the weights $w \in W(R^+) \cup W(R^-)$ be some vector bundles over $\mathfrak{X}$, and

$$\mathfrak{E} = \left(\oplus_{w\in W(R^+)} R_w^+ \otimes E_{w^+}\right) \ominus \left(\oplus_{w\in W(R^-)} R_w^- \otimes E_{w^-}\right) \tag{2.47}$$

be the associated virtual bundle over $\mathfrak{X}$. We denote by (cf. (2.24), (2.26))

$$\epsilon_\theta(\mathfrak{E}) = \frac{\prod_{w\in W(R^-)} \epsilon_{w(\theta)}(E_{w^-})}{\prod_{w\in W(R^+)} \epsilon_{w(\theta)}(E_{w^+})} \tag{2.48}$$

the rational function on the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ with values in $H^*(\mathfrak{X}, \mathbb{C})$.

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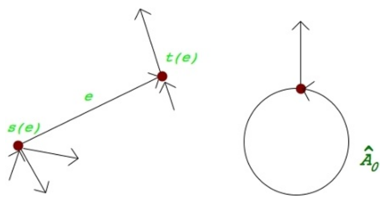

Figure 7. Note that the source and the target of an edge may coincide, as in the $\widehat{A}_0$ example above.

3 Supersymmetric gauge theories

In this section we go back to the gauge theory narrative. Our gauge theories are characterized by a quiver diagram. Let us start by reviewing what we mean by them.

3.1 Quivers

A quiver is an oriented graph γ, with the set Vert_γ of vertices and the set Edges_γ of oriented edges. We have two maps $s, t : \mathsf{Edges}_\gamma \longrightarrow \mathsf{Vert}_\gamma$, sending each edge e to its source $s(e)$ and the target $t(e)$, respectively.

We shall also use an unconventional term *arrow* which is a pair (e, σ), where $e \in \mathsf{Edges}_\gamma$, $\sigma = \pm 1$. The set $\mathsf{Arrows}_\gamma = \mathsf{Edges}_\gamma \times 2^{\mathsf{Edges}_\gamma}$ of arrows is equipped with two maps $\bar{s}, \bar{t} : \mathsf{Arrows}_\gamma \to \mathsf{Vert}_\gamma$, defined by:

$$\bar{s}(e,\sigma) = \begin{cases} s(e), & \text{if} \quad \sigma = +1 \\ \\ t(e), & \text{if} \quad \sigma = -1 \end{cases}$$

$$\bar{t}(e,\sigma) = \begin{cases} t(e), & \text{if} \quad \sigma = +1 \\ \\ s(e), & \text{if} \quad \sigma = -1 \end{cases}$$

3.2 Quivers with colors

In addition to the quiver diagram, the gauge theory is characterized by the vectors $\underline{\mathbf{n}}$, $\underline{\mathbf{m}}$, sometimes called the colorings of the quiver:

$$\underline{\mathbf{n}} = (n_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma} \in \mathbb{Z}_{>0}^{\mathsf{Vert}_\gamma}, \qquad \underline{\mathbf{m}} = (m_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma} \in \mathbb{Z}_{\geq 0}^{\mathsf{Vert}_\gamma} \tag{3.1}$$

to which we associate the vector spaces $N_{\mathbf{i}} = \mathbb{C}^{n_{\mathbf{i}}}$, $M_{\mathbf{i}} = \mathbb{C}^{m_{\mathbf{i}}}$.

3.3 The symmetry groups

3.3.1 The gauge group

The gauge group $\mathsf{G}_{\mathbf{g}}$ of the theory is the product

$$\mathsf{G}_{\mathbf{g}} = \bigtimes_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathrm{U}(n_{\mathbf{i}}) \tag{3.2}$$

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3.3.2 The flavor symmetry

The theory has the global symmetry which is usually called the flavor symmetry. The flavor symmetry group G_{f} is a quotient:

$$\mathsf{G}_{\mathrm{f}} = \left(\bigtimes_{\mathbf{i}\in \mathsf{Vert}_\gamma} \mathrm{U}(m_{\mathbf{i}}) \times \mathrm{U}(1)^{\mathsf{Edges}_\gamma} \right) / \, \mathrm{U}(1)^{\mathsf{Vert}_\gamma} \tag{3.3}$$

where $\mathrm{U}(1)^{\mathsf{Vert}_\gamma}$ acts on

$$\bigtimes_{\mathbf{i}\in \mathsf{Vert}_\gamma} \mathrm{U}(m_{\mathbf{i}}) \times \mathrm{U}(1)^{\mathsf{Edges}_\gamma}$$

as follows:

$$(u_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma} : \left((g_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}, (u_e)_{e\in\mathsf{Edges}_\gamma} \right) \mapsto \left((u_{\mathbf{i}} g_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}, (u_{s(e)} u_e u_{t(e)}^{-1})_{e\in\mathsf{Edges}_\gamma} \right) \tag{3.4}$$

This action is equivalently both left and right, therefore $\mathrm{U}(1)^{\mathsf{Vert}_\gamma}$ is a normal subgroup of $\times_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathrm{U}(m_{\mathbf{i}}) \times \mathrm{U}(1)^{\mathsf{Edges}_\gamma}$. In fact, the flavor group G_{f} occasionally enhances. For example, the $\mathcal{N}=4$ theory, viewed as an $\mathcal{N}=2$ supersymmetric theory, is a particular example of the quiver theory, corresponding to the quiver $\widehat{A}_0$ with one vertex v, and one edge e, connecting this vertex to itself $s(e)=t(e)=v$. The flavor symmetry is enhanced from U(1) to SU(2) in this case. This is a subgroup of the R-symmetry group SU(4) which commutes with the $\mathrm{SU}(2)\times\mathrm{U}(1)_A$ R-symmetry of the particular $\mathcal{N}=2$ subalgebra of the $\mathcal{N}=4$ theory.

3.3.3 Rotational symmetries

Our four dimensional gauge theories, in the absence of defects to be discussed below, are Poincare invariant. In what follows we shall be breaking the translational invariance by deforming the theory in a rotationally covariant way. The spin cover $\mathrm{Spin}(4)_{\mathcal{N}}$ of the group of rotations is the product

$$\mathrm{Spin}(4)_{\mathcal{N}} = \mathrm{SU}(2)_{\mathcal{N},L} \times \mathrm{SU}(2)_{\mathcal{N},R} \tag{3.5}$$

The regularization of the instanton integrals which we employ in [94] and here breaks the $\mathrm{Spin}(4)_{\mathcal{N}}$ invariance down to its subgroup $\mathsf{G}_{\mathrm{rot}} = \mathrm{SU}(2)_{\mathcal{N},L} \times \mathrm{U}(1)_{\mathcal{N},R} \approx \mathrm{U}(2) \subset \mathrm{Spin}(4)_{\mathcal{N}}$ which is the group of rotations of the Euclidean space-time $\mathcal{N}=\mathbb{R}^4$, preserving the identification of the latter with the complex vector space $\mathbb{C}^2$.

Let $S^{\pm}_{\mathcal{N}}$ be the defining two dimensional representations (chiral spinors) of $\mathrm{SU}(2)_{\mathcal{N},L}$ and $\mathrm{SU}(2)_{\mathcal{N},R}$, respectively, so that $\mathcal{N}^{\mathbb{C}} = S^+_{\mathcal{N}} \otimes S^-_{\mathcal{N}}$. Under $\mathsf{G}_{\mathrm{rot}}$, $S^-_{\mathcal{N}}$ splits as $L_{\mathcal{N}} \oplus L_{\mathcal{N}}^{-1}$. Let us denote the two dimensional representation of $\mathsf{G}_{\mathrm{rot}}$ by $Q_{\mathcal{N}} \approx \mathbb{C}^2$. Then

$$Q_{\mathcal{N}} = S^+_{\mathcal{N}} \otimes L_{\mathcal{N}}\,. \tag{3.6}$$

3.4 The parameters of Lagrangian

The field content of the theory is the set of $\mathcal{N}=2$ vector multiplets $\mathbf{\Phi_i} = (\Phi_{\mathbf{i}}, \ldots, A_{\mathbf{i}})$, $\mathbf{i} \in \mathsf{Vert}_\gamma$, transforming in the adjoint representation of $\mathsf{G}_{\mathbf{g}}$, the set $\boldsymbol{Q}_{\mathbf{i}} = (Q_{\mathbf{i}}, \ldots, \tilde{Q}_{\mathbf{i}})$, $\mathbf{i} \in \mathsf{Vert}_\gamma$ of hypermultiplets transforming in the fundamental representation $\mathbb{C}^{n_{\mathbf{i}}}$ of $\mathsf{G}_{\mathbf{g}}$, and the antifundamental representation $\mathbb{C}^{m_{\mathbf{i}}}$ of G_{f}, and the set $\boldsymbol{Q}_e$, $e \in \mathsf{Edges}_\gamma$ of hypermultiplets transforming in the bi-fundamental representation $\left(\overline{\mathbb{C}^{n_{s(e)}}}, \mathbb{C}^{n_{t(e)}}\right)$ of $\mathsf{G}_{\mathbf{g}}$.

The Lagrangian $\mathbb{L}$ of the theory is parametrized by the complexified gauge couplings

$$\underline{\tau} = (\tau_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}\,,$$

via

$$\mathbb{L} = -\frac{1}{8\pi^2}\sum_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathrm{iRe}\tau_{\mathbf{i}} \int_{\mathcal{N}} \mathsf{Tr}_{N_{\mathbf{i}}} F_{A_{\mathbf{i}}} \wedge F_{A_{\mathbf{i}}} + \\ + \mathrm{Im}\tau_{\mathbf{i}} \int_{\mathcal{N}} \mathsf{Tr}_{N_{\mathbf{i}}} F_{A_{\mathbf{i}}} \wedge \star F_{A_{\mathbf{i}}} + \mathsf{Tr}_{N_{\mathbf{i}}} D_{A_{\mathbf{i}}}\Phi_{\mathbf{i}} \wedge \star D_{A_{\mathbf{i}}}\bar{\Phi}_{\mathbf{i}} + \mathsf{Tr}_{N_{\mathbf{i}}}[\Phi_{\mathbf{i}}, \bar{\Phi}_{\mathbf{i}}]^2 + \ldots$$

and the masses

$$\underline{\mathtt{m}} = (\mathtt{m}_e)_{e\in\mathsf{Edges}_\gamma} \oplus (\mathtt{m}_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}\,,$$

where

$$\mathtt{m}_e \in \mathbb{C},\ \mathtt{m}_{\mathbf{i}} = \mathsf{diag}(\mathtt{m}_{\mathbf{i},1}, \ldots, \mathtt{m}_{\mathbf{i},m_{\mathbf{i}}}) \in \mathrm{End}(\mathbb{C}^{m_{\mathbf{i}}})\,. \tag{3.7}$$

which enter the superpotential (in the $\mathcal{N}=1$ language)

$$\mathbb{W} = \sum_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathsf{Tr}_{M_{\mathbf{i}}}\left(\mathtt{m}_{\mathbf{i}} Q_{\mathbf{i}}\tilde{Q}_{\mathbf{i}}\right) + \sum_{e\in\mathsf{Edges}_\gamma} m_e \mathsf{Tr}_{N_{s(e)}} \tilde{Q}_e Q_e + \\ \sum_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathsf{Tr}_{M_{\mathbf{i}}}\left(Q_{\mathbf{i}}\Phi_{\mathbf{i}}\tilde{Q}_{\mathbf{i}}\right) + \sum_{e\in\mathsf{Edges}_\gamma} \mathsf{Tr}_{N_{s(e)}}\left(\tilde{Q}_e\Phi_{t(e)}Q_e - \tilde{Q}_e Q_e \Phi_{s(e)}\right),$$

i.e. we view the scalars in the hypermultiplet $\boldsymbol{Q}_{\mathbf{i}}$ as the linear maps, the matrices:

$$Q_{\mathbf{i}} : N_{\mathbf{i}} \to M_{\mathbf{i}}, \qquad \tilde{Q}_{\mathbf{i}} : M_{\mathbf{i}} \to N_{\mathbf{i}},$$

and those in $\boldsymbol{Q}_e$ as

$$Q_e : N_{s(e)} \to N_{t(e)}, \qquad \tilde{Q}_e : N_{t(e)} \to N_{s(e)}\,.$$

The vacua of the theory are parametrized by the Coulomb moduli

$$\underline{\mathfrak{a}} = (\underline{\mathfrak{a}}_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}\,, \qquad \underline{\mathfrak{a}}_{\mathbf{i}} = \mathsf{diag}(\mathfrak{a}_{\mathbf{i},1}, \ldots, \mathfrak{a}_{\mathbf{i},n_{\mathbf{i}}}) \in \mathrm{End}(\mathbb{C}^{n_{\mathbf{i}}})\,, \tag{3.8}$$

so that

$$\langle\ \mathbf{\Phi_i}\ \rangle_{\underline{\mathfrak{a}}} = \underline{\mathfrak{a}}_{\mathbf{i}}\,.$$

It is convenient to package the masses $\mathtt{m}_{\mathbf{i}}$ and the Coulomb moduli $\mathbf{a}_{\mathbf{i}}$ into the polynomials:

$$P_{\mathbf{i}}(x) = \prod_{f=1}^{m_{\mathbf{i}}} (x - \mathtt{m}_{\mathbf{i},f})\,, \qquad A_{\mathbf{i}}(x) = \prod_{\alpha=1}^{n_{\mathbf{i}}} (x - \mathfrak{a}_{\mathbf{i},\alpha}) \tag{3.9}$$

We also use the characters

$$N_{\mathbf{i}} = \sum_{\alpha=1}^{n_{\mathbf{i}}} e^{\beta \mathfrak{a}_{\mathbf{i},\alpha}} \,, \qquad M_{\mathbf{i}} = \sum_{f=1}^{m_{\mathbf{i}}} e^{\beta \mathbb{m}_{\mathbf{i},f}} \tag{3.10}$$

which contain the same information about the masses and Coulomb moduli as the polynomials (3.9).

3.5 The group H

Define

$$\mathsf{H} = \mathsf{G}_{\mathbf{g}} \times \mathsf{G}_{\mathrm{f}} \times \mathsf{G}_{\mathrm{rot}} \,, \tag{3.11}$$

The complexification of the Lie algebra of the maximal torus T_{H} of this group is parameterized by $(\underline{\mathfrak{a}}; \underline{\mathbb{m}}; \underline{\varepsilon})$. It is the domain of definition of the supersymmetric partition functions $\mathcal{Z}_{\underline{\mathbf{k}}}$ in the eq. (2.2).

3.6 Perturbative theory

3.6.1 Perturbative consistency and asymptotic freedom

The theory defined by the quiver data is perturbatively asymptoticaly free if the one-loop beta function of all gauge couplings is not positive. For this to be possible we must restrict the gauge group to be the product of special unitary groups

$$\mathsf{G}_{\mathbf{g}} \longrightarrow \bigtimes_{\mathbf{i} \in \mathsf{Vert}_{\gamma}} \mathrm{SU}(n_{\mathbf{i}}) \tag{3.12}$$

since the abelian factors are not asymptotically free, if there are fields charged under them. For the $\mathrm{SU}(n_{\mathbf{i}})$ gauge coupling the beta function is easy to compute:

$$\beta_{\mathbf{i}} = \mu \frac{d}{d\mu} \tau_{\mathbf{i}} = -2n_{\mathbf{i}} + m_{\mathbf{i}} + \sum_{e \in t^{-1}(\mathbf{i})} n_{s(e)} + \sum_{e \in s^{-1}(\mathbf{i})} n_{t(e)} \tag{3.13}$$

The requirement $\beta_{\mathbf{i}} \leq 0$ for all $\mathbf{i} \in \mathsf{Vert}_{\gamma}$ implies (see [51, 57, 65, 85] for details) that γ is a Dynkin graph of finite or affine type of a simply-laced finite dimensional or affine Lie algebra $\mathfrak{g}_{\gamma}$. In the latter case $\underline{\mathbf{m}} = \underline{0}$ (not to be confused with $\underline{\mathbb{m}} \neq 0$).

3.6.2 Examples

1. The A_r-type quiver γ, with $r \geq 1$, has:

$$\mathsf{Vert}_{\gamma} = [r], \; \mathsf{Edges}_{\gamma} = [r-1], \qquad s(e) = e, \; t(e) = e+1, \qquad e = 1, \ldots, r-1 \,. \tag{3.14}$$

2. The quiver $\widehat{A}_r$, with $r \geq 0$, has (see the figure 6)

$$\mathsf{Vert}_{\gamma} = \mathsf{Edges}_{\gamma} = [r+1], \qquad s(e) = e, \; t(e) = 1 + (e \bmod (r+1)) \,. \tag{3.15}$$

3. The D and E-type quivers have a single tri-valent vertex, see the figure 6.

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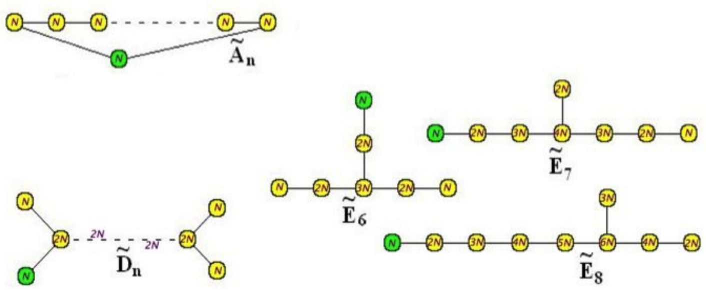

Figure 8. Affine A,D,E quivers with their $\underline{\mathbf{n}}$-coloring. Finite A,D,E quivers are obtained by removing the green node.

3.6.3 Perturbative partition function

The description of the tree level and the perturbative contributions to the partition function (the latter is given exactly by one loop computation) can be found in [86].

Here we just quote the results.

$$\mathcal{Z}_{\gamma}^{\mathrm{tree}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\tau};\underline{\varepsilon}) = \prod_{\mathtt{i}\in\mathsf{Vert}_{\gamma}} \mathfrak{q}_{\mathtt{i}}^{-\frac{1}{2\varepsilon_1\varepsilon_2}\sum_{\alpha=1}^{n_{\mathtt{i}}}\mathfrak{a}_{\mathtt{i},\alpha}^2}, \tag{3.16}$$

and

$$\mathcal{Z}_{\gamma}^{1-\mathrm{loop}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon}) = \epsilon_{\underline{\mathfrak{a}},\underline{\mathfrak{m}},\underline{\varepsilon}}(-\mathcal{T}_{\gamma}^{\mathrm{pert}}) \tag{3.17}$$

where (cf. (3.10)):

$$\mathcal{T}_{\gamma}^{\mathrm{pert}} = \frac{1}{(1-e^{-\beta\varepsilon_1})(1-e^{-\beta\varepsilon_2})}\left(\sum_{\mathtt{i}\in\mathsf{Vert}_{\gamma}}(M_{\mathtt{i}}-N_{\mathtt{i}})\,N_{\mathtt{i}}^{*} + \sum_{e\in\mathsf{Edges}_{\gamma}} e^{\beta\mathfrak{m}_e} N_{t(e)}N_{s(e)}^{*}\right) \tag{3.18}$$

The character (3.18) is not a finite sum of exponents as in (2.25), so the map ϵ from the sums of exponents to the products of weights is defined by analytic continuation, cf. (2.33):

$$\epsilon_{\underline{\mathfrak{a}},\underline{\mathfrak{m}},\underline{\varepsilon}}(-\mathcal{T}_{\gamma}^{\mathrm{pert}}) = -\left.\frac{d}{ds}\right|_{s=0}\frac{\Lambda^s}{\Gamma(s)}\int_0^{\infty}\frac{d\beta}{\beta}\beta^s\,\mathcal{T}_{\gamma}^{\mathrm{pert}} \tag{3.19}$$

There are subtle points of the regularization of (3.19) related to boundary conditions in gauge theory. These will be discussed elsewhere. The ultraviolet, $\Lambda\to\infty$ asymptotics of (3.19), has, a priori, the terms proportional to Λ^2, Λ, $\Lambda^2\log\Lambda$, $\Lambda\log\Lambda$, and $\log\Lambda$. The physically relevant terms are in the last one, they correspond to the one-loop beta-function of $\tau_{\mathtt{i}}$ if the coefficient of $\log\Lambda$ contain the terms proportional to

$$ch_2(N_{\mathtt{i}}) \equiv \sum_{\alpha=1}^{n_{\mathtt{i}}}\mathfrak{a}_{\mathtt{i},\alpha}^2 \tag{3.20}$$

Thus, these terms are absent precisely when (3.13) holds.

3.6.4 Beyond asymptotic freedom

If the asymptotic freedom/conformality conditions are not obeyed, our partition functions are defined as formal power series in the $\mathfrak{q}_{\mathbf{i}}$ couplings, and some additional couplings, which we call the *higher times*.

3.6.5 The extended coupling space

Gauge theory can be deformed, in the ultraviolet, by the irrelevant (higher degree) operators, which preserve $\mathcal{N} = 2$ supersymmetry. One adds to the tree level prepotential the terms of the form:

$$\mathcal{F}^{\text{tree}} = \sum_{\mathbf{i}} \sum_{l=0}^{\infty} \frac{1}{(l+2)!} \tau_{\mathbf{i},l} \operatorname{Tr} \mathbf{\Phi}_{\mathbf{i}}^{l+2} \tag{3.21}$$

The parameters $\tau_{\mathbf{i},l}$ with $l > 0$ are bosonic, in general nilpotent, variables. Actually, for some observables one can make sense of the parameters $\tau_{\mathbf{i},1}$ in a finite domain near zero [70]. One can also add the multi-trace operators $\sim \operatorname{Tr} \mathbf{\Phi}_{\mathbf{i}}^{l'} \operatorname{Tr} \mathbf{\Phi}_{\mathbf{j}}^{l''}$ etc. which can be analyzed with the help of Hubbard-Stratonovich transformation.

3.7 Realizations of quiver theories

3.7.1 Affine quivers and McKay correspondence

For affine quivers γ, the choice of gauge group $\mathsf{G}_{\mathbf{g}}$ is characterized by a single integer N, for the equation $\beta_{\mathbf{i}} = 0$ for all $\mathbf{i} \in \mathsf{Vert}_\gamma$ implies:

$$n_{\mathbf{i}} = N a_{\mathbf{i}} \tag{3.22}$$

where $a_{\mathbf{i}} \geq 1$ solves

$$2a_{\mathbf{i}} = \sum_{e \in t^{-1}(\mathbf{i})} a_{s(e)} + \sum_{e \in s^{-1}(\mathbf{i})} a_{t(e)}$$

with the normalization, that for some $\mathbf{0} \in \mathsf{Vert}_\gamma$, $a_{\mathbf{0}} = 1$. It is well-known, that the numbers $a_{\mathbf{i}} = \dim \mathcal{R}_{\mathbf{i}}$ are the dimensions of the irreducible representations of some finite subgroup $\Gamma \in \mathrm{SU}(2)$.

The A_r-type subgroup of SU(2) is $\mathbb{Z}_{r+1}$, whose generator Ω_{A_r} acts on $\mathbb{C}^2$ via (cf. (2.15)):

$$\Omega_{A_r} : (z_1, z_2) \mapsto \left(\varpi_{r+1}\, z_1, \quad \varpi_{r+1}^{-1}\, z_2 \right), \tag{3.23}$$

so that $\Omega_{A_r}^{r+1} = 1$. The D_r-type subgroup of SU(2) ($r \geq 4$) is the product $\mathbb{Z}_{2(r-2)} \times_{\mathbb{Z}_2} \mathbb{Z}_4$, whose generators Ω_{D_r} and Ξ_{D_r} act on $\mathbb{C}^2$ via:

$$\Omega_{D_r} : (z_1, z_2) \mapsto \left(\varpi_{2(r-2)}\, z_1, \quad \varpi_{2(r-2)}^{-1}\, z_2 \right), \qquad \Xi_{D_r} : (z_1, z_2) \mapsto (z_2, -z_1) \tag{3.24}$$

so that $\Omega_{D_r}^{r-2} = \Xi_{D_r}^2$, $\Xi_{D_r}^4 = 1$.

The $E_{6,7,8}$-type subgroups SU(2) are the binary covers of the symmetry groups of the three platonic solids (and their duals, see [85] for more details): see figure 9.

The quiver γ is associated to Γ as follows: the set Vert_γ is identified with $\Gamma^\vee$, the set of irreducible representations of Γ, $\mathbf{0} \in \mathsf{Vert}_\gamma$ corresponds to the trivial representation

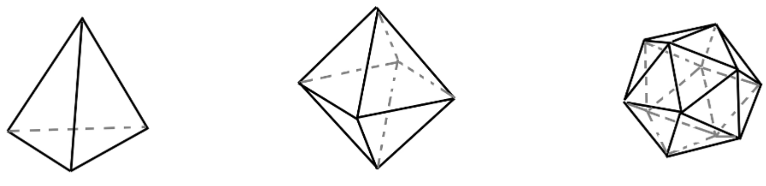

Figure 9. Platonic solids corresponding to E-type subgroups of SU(2).

$\mathcal{R}_{\mathbf{0}} = \mathbb{C}^1$. The set Edges_γ of edges is recovered from the tensor products as follows: define the matrix $A : \mathsf{Vert}_\gamma \times \mathsf{Vert}_\gamma \to \mathbb{Z}_{\geq 0}$ by

$$\mathcal{R}_{\mathbf{i}} \otimes S = \bigoplus_{\mathbf{j} \in \mathsf{Vert}_\gamma} \mathbb{C}^{A_{\mathbf{ij}}} \otimes \mathcal{R}_{\mathbf{j}} \tag{3.25}$$

where $S \approx \mathbb{C}^2$ is the defining two dimensional representation of SU(2). The matrix A is symmetric. There exists another matrix $E : \mathsf{Vert}_\gamma \times \mathsf{Vert}_\gamma \to \mathbb{Z}_{\geq 0}$ such that $E + E^t = A$. Then

$$\mathsf{Edges}_\gamma = \bigsqcup_{(\mathbf{i},\mathbf{j}) \in \mathsf{Vert}_\gamma \times \mathsf{Vert}_\gamma} [E_{\mathbf{i},\mathbf{j}}] \times (\mathbf{i},\mathbf{j}) \quad s\,(k \times (\mathbf{i},\mathbf{j})) = \mathbf{i}, \quad t\,(k \times (\mathbf{i},\mathbf{j})) = \mathbf{j} \tag{3.26}$$

The choice of E given A is the choice of the orientation of edges of γ. Note that this definition associates to $\Gamma = 1$ the quiver $\widehat{A}_0$.

The $\mathcal{N} = 2$ quiver four dimensional gauge theory corresponding to such quiver γ can be described most simply by starting with the $\mathcal{N} = 4$ super-Yang-Mills theory with the gauge group $\mathrm{U}(N|\Gamma|)$, with the fields $A_\mu \in \mathcal{E} \otimes \mathcal{E}^*, \Psi_\alpha \in T \otimes \mathcal{E} \otimes \mathcal{E}^*, \mathbf{\Phi} \in \Lambda^2 T \otimes \mathcal{E} \otimes \mathcal{E}^*$, $\alpha = 1, 2$, $\mu = 0, 1, 2, 3$, with $\mathcal{E} = \mathbb{C}^{N|\Gamma|}$ the defining representation of $\mathrm{U}(N|\Gamma|)$ and $T \approx \mathbb{C}^4$ the defining representation representation of the R-symmetry group SU(4). Now the space of fields is endowed with the action of Γ:

$$T = \mathbb{C}^2 \otimes \mathcal{R}_{\mathbf{0}} \oplus S, \qquad \mathcal{E} = \mathbb{C}^N \otimes \mathbb{C}^\Gamma = \bigoplus_{\mathbf{i} \in \Gamma^\vee} \mathbb{C}^{N a_{\mathbf{i}}} \otimes \mathcal{R}_{\mathbf{i}} \tag{3.27}$$

One then defines the new theory by imposing the Γ-invariance constraint on the fields of the original theory. The $\mathcal{N} = 4$ supersymmetry reduces to $\mathcal{N} = 2$, with $\mathrm{U}(N a_{\mathbf{i}})$-valued vector multiplets $\mathbf{\Phi}_{\mathbf{i}}$ labelled by $\mathbf{i} \in \mathsf{Vert}_\gamma$, and bi-fundamental hypermultiplets labelled by $e \in \mathsf{Edges}_\gamma$. The Lagrangian of the original $\mathcal{N} = 4$ theory can be then deformed, preserving the $\mathcal{N} = 2$ supersymmetry. Since the gauge group $\mathrm{U}(N|\Gamma|)$ becomes the product

$$\mathrm{U}(N|\Gamma|) \longrightarrow \bigtimes_{\mathbf{i}} \mathrm{U}(N a_{\mathbf{i}})\,, \tag{3.28}$$

the gauge couplings $\tau_{\mathbf{i}}$ can be chosen independently:

$$\tau \ \mathsf{Tr}_{\,N|\Gamma|} F^2 \longrightarrow \sum_{\mathbf{i} \in \mathsf{Vert}_\gamma} \tau_{\mathbf{i}} \ \mathsf{Tr}_{\,N a_{\mathbf{i}}} F_{\mathbf{i}}^2$$

We have reviewed this well-known construction here because in what follows we shall use its variants on several occasions.

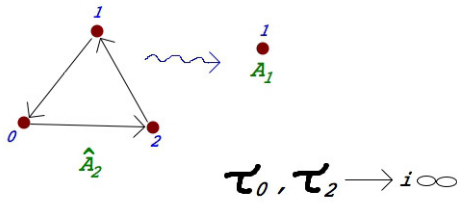

Figure 10. A_1 theory as a limit of $\widehat{A}_2$.

3.7.2 Finite quivers

Some of the finite quiver theories can be obtained as limits of the affine quiver theories. The rest is related to the ones we shall describe below by analytic continuation, sometimes through a strong coupling region.

1. The A_r type theory with $n_1 = \ldots = n_r = N$, and $m_1 = m_r = N$, $m_{\mathbf{i}} = 0$ for $2 \leq \mathbf{i} \leq r-1$, is the limit of the $\widehat{A}_{r+1}$ theory, where one sends $\mathfrak{q}_{\mathbf{0}} \to 0, \mathfrak{q}_{r+1} \to 0$. Then $\mathfrak{a}_{\mathbf{0},\alpha} - \mathfrak{m}_{\mathbf{0}} - \varepsilon$, $\mathfrak{a}_{r+1,\alpha} + \mathfrak{m}_r$ become the masses of the fundamental hypermultiplets, charged under $\mathrm{U}(n_1)$ and $\mathrm{U}(n_r)$, respectively.

2. A particular D_r type theory can be obtained by taking the limit $\mathfrak{q}_{\mathbf{0}} \to 0$ limit of $\widehat{D}_r$ theory. The next-to-last node **2** with $n_{\mathbf{2}} = 2N$ has $m_{\mathbf{2}} = N$.

There are other ways of arriving at the quiver $\mathcal{N} = 2$ theories corresponding to finite quivers.

4 Integration over instanton moduli spaces

In this section we recall the mathematical definition of the instanton partition function $\mathcal{Z}^{\text{inst}}$ of the bulk theory. In [103] we define the defect partition functions $\mathbf{\Psi}^{\text{inst}}$. We give the practical definition first, without actually describing the relevant instanton moduli spaces. In [101] we describe the moduli spaces $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})$ whose contributions dominate the gauge theory path integral, explicitly, via modified ADHM construction. More precisely, the gauge theory path integral localizes to the integral of 1 over the virtual fundamental cycle of degree (dimension) zero $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})$ which is represented, in the perfect obstruction theory language of [11] by a smooth (super)-variety $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c$ (c stands for coarse) and H-equivariant vector bundle $\mathsf{Obs}_\gamma \to \mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c$. The $\underline{\mathbf{k}}$-instanton contribution to the gauge theory partition function is the Euler class

$$\mathcal{Z}^{\text{inst}}_{\underline{\mathbf{k}}} = \int_{\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})} 1 = \int_{\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c} \epsilon(\mathsf{Obs}_\gamma)\,, \tag{4.1}$$

where we omitted the equivariant parameters.

We shall see that for the affine quiver theories (a representative example is the $\mathcal{N} = 2^*$ $\mathrm{U}(n)$ theory) the underlying variety $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c$ is bosonic, while for the finite quiver theories (a representative example is the $\mathrm{U}(n)$ theory with $2n$ fundamental hypermultiplets)

$\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c$ is a split super-manifold, a vector bundle over an ordinary smooth variety with odd fibers.

The same pattern holds for the theories with defects.

4.1 Instanton partition function

4.1.1 The bulk partition function $\mathcal{Z}^{\text{inst}}$

Let $\underline{\mathbf{k}} = (k_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma} \in \mathbb{Z}_+^{\mathsf{Vert}_\gamma}$ be the vector of instanton charges for the gauge group $\mathsf{G}_{\mathbf{g}}$. We denote by $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})$ the moduli space of framed quiver-graded torsion free sheaves $\boldsymbol{E}_\gamma = (E_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}$ on $\mathbb{CP}^2$. More precisely, for each $\mathbf{i} \in \mathsf{Vert}_\gamma$, $E_{\mathbf{i}}$ is a torsion free sheaf on $\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1_\infty$, with the charge $\mathrm{ch}_2(E_{\mathbf{i}}) = k_{\mathbf{i}}$, and the framing at infinity:

$$E_{\mathbf{i}}\,|_{\mathbb{CP}^1_\infty} \xrightarrow{\sim} N_{\mathbf{i}} \tag{4.2}$$

Set theoretically,

$$\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c = \bigtimes_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathcal{M}(n_{\mathbf{i}}, k_{\mathbf{i}}) \tag{4.3}$$

is the product of ADHM moduli spaces of $\mathrm{U}(n_{\mathbf{i}})$ instantons of charge $k_{\mathbf{i}}$.

Let $\mathcal{E}_{\mathbf{i}}$ be the universal $\mathbf{i}$'th sheaf over $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c \times \mathbb{CP}^2$, and $\pi : \mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c \times \mathbb{CP}^2 \longrightarrow \mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c$ the projection onto the first factor. Define the *obstruction sheaf* Obs_γ over

$$\mathcal{M}_\gamma(\underline{\mathbf{n}})^c = \bigsqcup_{\underline{\mathbf{k}}} \mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})^c \tag{4.4}$$

by:

$$\mathsf{Obs}_\gamma = R\pi_* \bigoplus_{e\in\mathsf{Edges}_\gamma} \mathrm{Hom}(\mathcal{E}_{s(e)}, \mathcal{E}_{t(e)}) \oplus \bigoplus_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathrm{Hom}(\mathcal{E}_{\mathbf{i}}, M_{\mathbf{i}}) \tag{4.5}$$

The sheaves above are all $\mathsf{H}_{\mathbb{C}}$-equivariant, where H was defined in (3.11).

The complexification of $\mathsf{G}_{\mathbf{g}}$ acts on the isomorphisms $\mathcal{E}_i|_{\mathbb{CP}^1_\infty} \xrightarrow{\sim} N_i$, the complexification of G_{f} acts on the fibers of (4.5) in the natural way, the complexification $GL(2, \mathbb{C})$ of $\mathsf{G}_{\mathrm{rot}}$ acts by the symmetries of $\mathbb{CP}^2$, with the fixed point $0 \in \mathbb{C}^2 = \mathbb{CP}^2 \backslash \mathbb{CP}^1_\infty$. Let $T_{\mathsf{H}} \subset \mathsf{H}$, $T_{\mathsf{H}}^{\mathbb{C}}$ denote the maximal torus of H and its complexification, respectively.

The Coulomb moduli $\underline{\mathfrak{a}}$ belong to $\mathrm{Lie}T^{\mathbb{C}}_{\mathsf{G}_{\mathbf{g}}}$, the masses $\underline{\mathfrak{m}}$ belong to $\mathrm{Lie}T^{\mathbb{C}}_{\mathsf{G}_{\mathrm{f}}}$. The Ω-deformed theory has two additional complex parameters $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ which belong to the Cartan subalgebra of ${\mathsf{G}_{\mathrm{rot}}}^{\mathbb{C}}$, $\underline{\varepsilon} \in \mathrm{Lie}T^{\mathbb{C}}_{\mathsf{G}_{\mathrm{rot}}} \approx \mathbb{C}^2$.

In [101] we shall discuss the modification of the ADHM construction [8] producing the moduli spaces $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})$ (cf. [77, 88]) and the obstruction sheaf. More precisely, there is a moduli space of solutions to a system of matrix equations, determined by the quiver data, which depends on the choice of the Fayet-Illiopoulos (stability) parameters $\vec{\zeta} \in \mathbb{R}^{\mathsf{Vert}_\gamma}$. It is the choice of these Fayet-Illiopoulos parameters which breaks the rotation symmetry from $\mathrm{Spin}(4)$ down to $\mathsf{G}_{\mathrm{rot}}$. When $\vec{\zeta}$ is in certain chamber $\mathcal{C} \subset \mathbb{R}^{\mathsf{Vert}_\gamma}$ the space of solutions to this system of equations coincides with $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})$. The linearization of the equations at the particular solution defines the obstruction sheaf, as the space of solutions to the dual linear system.

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The instanton factor in the partition function can be shown to reduce to the generating function of the equivariant integrals

$$\mathcal{Z}^{\mathrm{inst}}_{\gamma}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\mathfrak{q}};\underline{\varepsilon}) = \sum_{\underline{\mathbf{k}}} \underline{\mathfrak{q}}^{\underline{\mathbf{k}}} \int_{\mathcal{M}_{\gamma}(\underline{\mathbf{n}},\underline{\mathbf{k}})^{c}} \epsilon_{\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon}}\left(\mathsf{Obs}_{\gamma}\right), \tag{4.6}$$

with

$$\underline{\mathfrak{q}}^{\underline{\mathbf{k}}} = \prod_{\mathbf{i}\in\mathsf{Vert}_{\gamma}} \mathfrak{q}_{\mathbf{i}}^{k_{\mathbf{i}}}$$

Mathematically (4.6) is just a definition of the left hand side. Each term of the $\underline{\mathfrak{q}}$-expansion is a rational function on $\mathrm{Lie}(\mathsf{H}_{\mathbb{C}})$, of negative degree of homogeneity for the asymptotically free theories, and degree zero (i.e. they are homogeneous functions) for the asymptotically conformal theories.

4.1.2 Localization and fixed points

The fixed points $\mathcal{M}(\underline{\mathbf{n}},\underline{\mathbf{k}})^{\mathsf{H}}$ of the T_{H}-action on $\mathcal{M}(\underline{\mathbf{n}},\underline{\mathbf{k}})$ are the sheaves which split as direct sums of monomial ideals:

$$\boldsymbol{\mathcal{E}} \in \mathcal{M}_{\gamma}(\underline{\mathbf{n}},\underline{\mathbf{k}})^{T_{\mathsf{H}}} \Leftrightarrow E_{\mathbf{i}} = \bigoplus_{\alpha=1}^{n_{\mathbf{i}}} \mathcal{I}_{\mathbf{i},\alpha}, \tag{4.7}$$

where $\mathcal{I}_{\mathbf{i},\alpha} = I_{\lambda^{(\mathbf{i},\alpha)}}$, Thus, the set of fixed points $\mathcal{M}_{\gamma}(\underline{\mathbf{n}},\underline{\mathbf{k}})^{T_{\mathsf{H}}}$ is in one-to-one correspondence with the set of *quiver* $\underline{\mathbf{n}}$*-colored partitions*:

$$E_{\underline{\boldsymbol{\lambda}}} \leftrightarrow \underline{\boldsymbol{\lambda}} = \left\{ \lambda^{(\mathbf{i},\alpha)} \;\middle|\; \mathbf{i}\in\mathsf{Vert}_{\gamma}, \alpha\in[n_{\mathbf{i}}],\ \lambda^{(\mathbf{i},\alpha)} \text{ is a partition, } \sum_{\alpha=1}^{n_{\mathbf{i}}} |\lambda^{(\mathbf{i},\alpha)}| = k_{\mathbf{i}} \right\} \tag{4.8}$$

These points are also the fixed points of the action of T_{H} on the moduli space of Γ-invariant instantons. The fixed point formula expresses the gauge theory path integral as the sum over the set of quiver $\underline{\mathbf{n}}$-colored partitions. We shall present the explicit formula in the next section.

Now that the path integration is reduced to a finite sum, the non-perturbative field redefinitions involving adding a point-like instanton can be discussed rigorously.

4.2 Characters, tangent spaces

The contribution of a given fixed point to the partition function can be conveniently expressed using the characters of various vector spaces involved in the local analysis of the path integral measure. The instanton partition function can be then written as:

$$\mathcal{Z}^{\mathrm{inst}}_{\gamma}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon};\underline{\mathfrak{q}}) = \sum_{\underline{\boldsymbol{\lambda}}} \underline{\mathfrak{q}}^{\underline{\boldsymbol{\lambda}}}\, \boldsymbol{\mu}_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon}) \tag{4.9}$$

where

$$\boldsymbol{\mu}_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon}) = \epsilon_{\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon}}\left(-\mathcal{T}_{\underline{\boldsymbol{\lambda}}}\right), \tag{4.10}$$

and

$$\underline{\boldsymbol{\lambda}} = \left(\lambda^{(\mathbf{i},\alpha)}\right)_{\mathbf{i}\in\mathsf{Vert}_{\gamma}}^{\alpha\in[n_{\mathbf{i}}]}, \tag{4.11}$$

and

$$\underline{\mathfrak{q}}^{\underline{\lambda}} = \prod_{\mathbf{i}\in\mathsf{Vert}_\gamma}\prod_{\alpha=1}^{n_\mathbf{i}} \mathfrak{q}_\mathbf{i}^{|\lambda^{(\mathbf{i},\alpha)}|} \quad , \tag{4.12}$$

and

$$\begin{aligned}\mathcal{T}_{\underline{\lambda}} = &\left(\sum_{\mathbf{i}\in\mathsf{Vert}_\gamma} \left(N_\mathbf{i} K_\mathbf{i}^* + N_\mathbf{i}^* K_\mathbf{i} q - P K_\mathbf{i} K_\mathbf{i}^*\right)\right) \\ &- \left(\sum_{\mathbf{i}\in\mathsf{Vert}_\gamma} M_\mathbf{i}^* K_\mathbf{i} + \sum_{e\in\mathsf{Edges}_\gamma} e^{\beta\mathfrak{m}_e}\left(N_{t(e)}K^*_{s(e)} + N^*_{s(e)}K_{t(e)} q - P K_{t(e)} K^*_{s(e)}\right)\right) .\end{aligned} \tag{4.13}$$

In writing (4.13) we adopted a convention where the characters of the vector spaces are denoted by the same letters as the vector spaces themselves. We are thus using the notations (3.10) and

$$K_\mathbf{i} = \sum_{\alpha=1}^{n_\mathbf{i}} \left(e^{\beta\mathfrak{a}_{\mathbf{i},\alpha}} \sum_{\square\in\lambda^{(\mathbf{i},\alpha)}} e^{\beta c_\square} \right) \tag{4.14}$$

In (4.14) we use the convention (2.43).

Note that the eqs. (4.14) identify $N_\mathbf{i}, K_\mathbf{i}, M_\mathbf{i}$ with representations of T_H. While $N_\mathbf{i}, M_\mathbf{i}$ are Weyl-invariant, and correspond to representations of H, the spaces $K_\mathbf{i}$ do not, in general, carry a representation of H.

4.3 Integral representation

The measure (4.9) can be also given an integral representation:

$$\mathcal{Z}_\gamma^{\text{inst}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\mathfrak{q}};\underline{\varepsilon}) = \sum_{\underline{\mathbf{k}}} \frac{\underline{\mathfrak{q}}^{\underline{\mathbf{k}}}}{\underline{\mathbf{k}}!} \oint_{\Gamma_\gamma} \prod_{\mathbf{i}\in\mathsf{Vert}_\gamma} \Upsilon_\mathbf{i} \prod_{e\in\mathsf{Edges}_\gamma} \Upsilon_e \, , \tag{4.15}$$

where

$$\Upsilon_\mathbf{i} = \bigwedge_{\alpha=1}^{n_\mathbf{i}} \frac{\varepsilon}{\varepsilon_1\varepsilon_2} \frac{d\phi_{\mathbf{i},\alpha} P_\mathbf{i}(\phi_{\mathbf{i},\alpha})}{\mathcal{A}_\mathbf{i}(\phi_{\mathbf{i},\alpha})\mathcal{A}_\mathbf{i}(\phi_{\mathbf{i},\alpha}+\varepsilon)} \prod_{\alpha'\neq\alpha''} \frac{1}{\mathbf{S}(\phi_{\mathbf{i},\alpha'}-\phi_{\mathbf{i},\alpha''})} \, , \tag{4.16}$$

$$\Upsilon_e = \prod_{\alpha'=1}^{n_{s(e)}} \mathcal{A}_{t(e)}(\phi_{s(e),\alpha'} - \mathfrak{m}_e) \prod_{\alpha''=1}^{n_{t(e)}} \mathcal{A}_{s(e)}(\phi_{t(e),\alpha''} + \varepsilon + \mathfrak{m}_e) \prod_{\alpha'=1}^{n_{s(e)}} \prod_{\alpha''=1}^{n_{t(e)}} \mathbf{S}(\phi_{t(e),\alpha''} - \phi_{s(e),\alpha'} + \mathfrak{m}_e)$$

where

$$\mathbf{S}(x) = 1 + \frac{\varepsilon_1\varepsilon_2}{x(x+\varepsilon)} \tag{4.17}$$

and the choice of the contour

$$\Gamma_\gamma \approx \bigtimes_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathbb{R}^{n_\mathbf{i}} \tag{4.18}$$

will be discussed elsewhere.

For generic values of the parameters $\mathfrak{a}_\mathbf{i}$, $\underline{\varepsilon}$ etc. the fixed point formula (4.9) can be used. This is equivalent to the statement that (4.15) can be evaluated by computing the

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residues at simple poles. The contributions of the particular toric instanton configuration $\underline{\boldsymbol{\lambda}}$ (4.11) are rational functions with lots of poles. These poles lead to potential divergencies of the instanton partition function. For example, whenever the ratio (2.19) is a positive rational number, $b^2 \in \mathbb{Q}_+$ some of the individual terms $\boldsymbol{\mu}_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon})$ blow up. However the divergencies cancel between several terms. The contour integral representation is more convenient in this case, as it remains finite, as long as the contour Γ_γ does not get pinched between two approaching poles.

Let us briefly explain the reason why these apparent poles occur, and why they potentially cancel between themselves. A rational relation between the Coulomb parameters and the Ω-deformation parameters means that the symmetry group used in the equivariant localization is strictly smaller then the maximal torus of H. Reduction of the symmetry group means a potential enhancement of the fixed point locus. For example, instead of a set of isolated points one may find a copy of $\mathbb{CP}^1$ or a more complicated positive dimension subvariety. Each component of the fixed point locus contributes an integral to the instanton partition function. This contribution is finite if the component is compact. In the extreme case $\mathfrak{a} = \underline{\varepsilon} = 0$, the symmetry group is trivial. The fixed point locus in this case is the whole original moduli space $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})$, and the integral diverges.

4.4 Full partition functions

The full partition functions are the products of the instanton partition functions and the tree and one-loop partition functions. They are given by the product of (3.16), (3.17) and (4.1) leading to the following simple formulas

$$\mathcal{Z}_\gamma(\underline{\mathfrak{a}},\underline{\mathfrak{m}},\underline{\varepsilon};\underline{\mathfrak{q}}) = \sum_{\underline{\boldsymbol{\lambda}}} \mathcal{Q}(\mathcal{T}_{\underline{\boldsymbol{\lambda}}})\epsilon_{\underline{\mathfrak{a}},\underline{\mathfrak{m}},\underline{\varepsilon}}(-\mathcal{T}[\underline{\boldsymbol{\lambda}}]) \tag{4.19}$$

where

$$\mathcal{T}[\underline{\boldsymbol{\lambda}}] = \frac{1}{(1-e^{-\beta\varepsilon_1})(1-e^{-\beta\varepsilon_2})}\left(\sum_{\mathbf{i}\in\mathsf{Vert}_\gamma}(M_{\mathbf{i}} - S_{\mathbf{i}}[\underline{\boldsymbol{\lambda}}])\,S^*_{\mathbf{i}}[\underline{\boldsymbol{\lambda}}] + \sum_{e\in\mathsf{Edges}_\gamma} e^{\beta\mathfrak{m}_e} S_{t(e)}[\underline{\boldsymbol{\lambda}}]S^*_{s(e)}[\underline{\boldsymbol{\lambda}}]\right) \tag{4.20}$$

and

$$\mathcal{Q}(\mathcal{T}[\underline{\boldsymbol{\lambda}}]) = \prod_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{-\frac{1}{\varepsilon_1\varepsilon_2}\mathrm{ch}_2(S_{\mathbf{i}}[\underline{\boldsymbol{\lambda}}])} = \left(\prod_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{-\frac{1}{2\varepsilon_1\varepsilon_2}\sum_{\alpha=1}^{n_{\mathbf{i}}}\mathfrak{a}^2_{\mathbf{i},\alpha}}\right) \times \underline{\mathfrak{q}}^{\underline{\boldsymbol{\lambda}}} \tag{4.21}$$

where

$$\mathrm{ch}_2(S_{\mathbf{i}}[\underline{\boldsymbol{\lambda}}]) = \frac{1}{2}\sum_{\alpha=1}^{n_{\mathbf{i}}} \mathfrak{a}^2_{\mathbf{i},\alpha} - \varepsilon_1\varepsilon_2 k_{\mathbf{i}}[\underline{\boldsymbol{\lambda}}] \tag{4.22}$$

5 The $\mathcal{Y}$-observables

The measures (4.9) $\boldsymbol{\mu}_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}},\underline{\mathfrak{m}};\underline{\varepsilon})$ define the complexified statistical models which can be studied without a reference to the original gauge theory. To any function $F = F[\underline{\boldsymbol{\lambda}}]$ on the

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space of quiver $\underline{\mathbf{n}}$-colored partitions one associates its normalized expectation value:

$$\langle\, F \,\rangle_\gamma = \frac{1}{\mathcal{Z}_\gamma^{\text{inst}}} \sum_{\underline{\boldsymbol{\lambda}}} \underline{\mathfrak{q}}^{\underline{\boldsymbol{\lambda}}}\, \boldsymbol{\mu}_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}; \underline{\varepsilon})\, F[\underline{\boldsymbol{\lambda}}] \tag{5.1}$$

Sometimes we shall also use the un-normalized expectation value

$$\langle\!\langle F \rangle\!\rangle_\gamma = \mathcal{Z}_\gamma^{\text{tree}} \mathcal{Z}_\gamma^{1-\text{loop}} \sum_{\underline{\boldsymbol{\lambda}}} \underline{\mathfrak{q}}^{\underline{\boldsymbol{\lambda}}}\, \boldsymbol{\mu}_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}}, \underline{\mathfrak{m}}; \underline{\varepsilon})\; F[\underline{\boldsymbol{\lambda}}] = \mathcal{Z}_\gamma \,\langle\, F \,\rangle_\gamma \tag{5.2}$$

Sometimes, in what follows we shall view such a function F as an *operator*, acting in the infinite-dimensional vector space $\mathbb{H}$ with the basis $e_{\underline{\boldsymbol{\lambda}}}$ labelled by the quiver $\underline{\mathbf{n}}$-colored partitions,

$$F e_{\underline{\boldsymbol{\lambda}}} = F[\underline{\boldsymbol{\lambda}}] e_{\underline{\boldsymbol{\lambda}}} \tag{5.3}$$

The rôle of $\mathbb{H}$ in gauge theory will be discussed elsewhere.

The functions F which do come from gauge theory will be called *observables*. An example of observable is the $\mathbf{i}$'th instanton charge:

$$k_{\mathbf{i}}[\underline{\boldsymbol{\lambda}}] = \sum_{\alpha=1}^{n_{\mathbf{i}}} \left| \lambda^{(\mathbf{i},\alpha)} \right| \tag{5.4}$$

5.1 The bulk $\mathcal{Y}$-observables

The important observables are the characteristic polynomials of the adjoint Higgs fields:

$$\mathcal{Y}_{\mathbf{i}}(x) = x^{n_{\mathbf{i}}} \exp \sum_{l=1}^{\infty} -\frac{1}{l x^l} \operatorname{Tr} \left(\Phi_{\mathbf{i}}|_0\right)^l \tag{5.5}$$

Here we denote by $\Phi_{\mathbf{i}}|_0$ the lowest component of the vector multiplet $\boldsymbol{\Phi}_{\mathbf{i}}$ corresponding to the node $\mathbf{i} \in \mathsf{Vert}_\gamma$, evaluated at the specific point $0 \in \mathbb{C}^2$ in the Euclidean space-time. This is the fixed point of the rotational symmetry $\mathrm{Spin}(4)_{\mathrm{N}}$ of which the maximal torus $\mathrm{U}(1) \times \mathrm{U}(1)$ is generated by the rotations in the two orthogonal two-planes.

In the $\mathcal{N} = 2$ theory the gauge-invariant polynomials of the scalar components of the vector multiplets, i.e.

$$\mathcal{O}_l(\mathbf{x}) = \operatorname{Tr}\, \Phi_{\mathbf{i}}^l(\mathbf{x})\,, \tag{5.6}$$

for $\mathbf{x} \in \mathbb{R}^4$ are invariant under some supersymmetry transformations, which are nilpotent on the physical states. Moreover, the $\mathbf{x}$-variation of such operators is in itself a supersymmetry variation. Therefore, in the cohomology of such a supercharge, the observable $\mathcal{O}_l(\mathbf{x})$ is $\mathbf{x}$-independent. The supersymmetry of the Ω-deformed $\mathcal{N} = 2$ gauge theory is such that the operator $\mathcal{O}_l(\mathbf{x})$ is invariant only at $\mathbf{x} = 0$, i.e. at the fixed point of the rotations.

Classically, i.e. for the ordinary matrix-valued function $\Phi_{\mathbf{i}}(\mathbf{x})$ the exponential (5.5) evaluates to the characteristic polynomial of this matrix (cf. (3.9)):

$$\mathcal{Y}_{\mathbf{i}}(x)^{\text{tree}} = \mathrm{Det}_{n_{\mathbf{i}}}(x - \Phi_{\mathbf{i}}|_0) = \mathcal{A}_{\mathbf{i}}(x) \tag{5.7}$$

5.1.1 $\mathcal{Y}$-observables from sheaves

Mathematically $\mathcal{Y}_{\mathbf{i}}(x)$ is defined using the virtual Chern polynomials of the universal sheaves, localized at the point $0 \in \mathbb{C}^2$:

$$\mathcal{Y}_{\mathbf{i}}(x) = c_x(R\pi_* \left[E_{\mathbf{i}} \to E_{\mathbf{i}} \otimes T_{\mathbb{P}^2} \to E_{\mathbf{i}} \otimes \wedge^2 T_{\mathbb{P}^2} \right]) \tag{5.8}$$

Here we used the Koszul resolution of the skyscrape sheaf $\mathcal{S}_0$ supported at $0 \in \mathbb{C}^2$:

$$0 \to \wedge^2 \mathcal{T}^*_{\mathbb{P}^2} \to \mathcal{T}^*_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{S}_0 \tag{5.9}$$

where the second and the third maps are the contraction with the Euler vector field $z_1 \partial_{z_1} + z_2 \partial_{z_2}$.

5.1.2 $\mathcal{Y}$-observables from noncommutative gauge fields

The proper physical definition of the observable (5.5) is also subtler then the naive expression (5.7). In computing the instanton partition function one uses the non-commutative deformation of the gauge theory, in order to make the instanton moduli space smooth with isolated fixed points [88]. In the noncommutative world, the notion of a particular point $\mathbf{x} = 0$ in the space-time $\mathbb{R}^4_\theta$ makes no sense. The gauge fields and the adjoint scalar $\Phi_{\mathbf{i}}$ are the operators in the Hilbert space $\mathcal{H}$,

$$\mathcal{H} = \bigoplus_{\mathbf{i} \in \mathsf{Vert}_\gamma} N_{\mathbf{i}} \otimes \mathfrak{H}$$

where $\mathfrak{H}$ is the 2-oscillator Fock space representation of the algebra of functions on $\mathbb{R}^4_\theta$, generated by $\widehat{x}^\mu$, $\mu = 1, \ldots, 4$, obeying the Heisenberg algebra $[\widehat{x}^\mu, \widehat{x}^\nu] = \mathrm{i}\vartheta^{\mu\nu}$, with constant antisymmetric (non-degenerate) matrix θ:

$$A_{\mathbf{i},\mu}(x) \mapsto \mathbf{X}^\mu_{\mathbf{i}} = \widehat{x}^\mu + \vartheta^{\mu\nu} A_{\mathbf{i},\nu}(\widehat{x})$$
$$\Phi_{\mathbf{i}} \mapsto \mathbf{\Phi}_{\mathbf{i}} = \frac{1}{2} \mathfrak{h}_{\mu\nu} \mathbf{X}^\mu_{\mathbf{i}} \mathbf{X}^\nu_{\mathbf{i}} + \phi_{\mathbf{i}}(\widehat{x})$$

where $\mathfrak{h}_{\mu\nu}$ is the symmetric matrix, obeying

$$\mathfrak{h}_{\mu\nu} \vartheta^{\nu\alpha} = \Omega^\alpha_\mu$$

with Ω being the matrix of infinitesimal rotation of $\mathbb{R}^4$, preserving both the metric and ϑ. In the vacuum

$$\mathbf{\Phi}_{\mathbf{i}} = \mathsf{diag}(\mathfrak{a}_{\mathbf{i},1}, \ldots, \mathfrak{a}_{\mathbf{i},n_{\mathbf{i}}}) \otimes \mathbf{1}_{\mathfrak{H}} + \mathbf{1}_{N_{\mathbf{i}}} \otimes (\varepsilon_1 \widehat{n}_1 + \varepsilon_2 \widehat{n}_2)$$

where $\widehat{n}_\xi = a^+_\xi a_\xi$ are the oscillator number operators in $\mathfrak{H}$, $\xi = 1, 2$. The observables like (5.6) are defined, cf. [84], as the ratio of infinite dimensional determinants,

$$\mathcal{Y}_{\mathbf{i}}(x) = \frac{\mathtt{Det}_{\mathcal{H}}(x - \mathbf{\Phi}_{\mathbf{i}})\ \mathtt{Det}_{\mathcal{H}}(x - \mathbf{\Phi}_{\mathbf{i}} - \varepsilon_1 - \varepsilon_2)}{\mathtt{Det}_{\mathcal{H}}(x - \mathbf{\Phi}_{\mathbf{i}} - \varepsilon_1)\ \mathtt{Det}_{\mathcal{H}}(x - \mathbf{\Phi}_{\mathbf{i}} - \varepsilon_2)} \tag{5.10}$$

or, equivalently, via a limiting procedure involving the regularized traces

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-t\mathbf{\Phi}_{\mathbf{i}}}$$

partly explaining the non-triviality of what follows. Without going into detail, let us quote the results which we shall need in this paper.

5.1.3 $\mathcal{Y}$-observables from Chern classes

Another, equivalent, definition of $\mathcal{Y}_{\mathbf{i}}(x)$ is the following. We have the vector bundles $N_{\mathbf{i}}$, $K_{\mathbf{i}}$, $\mathbf{i} \in \mathsf{Vert}_\gamma$, over $\mathcal{M}_\gamma(\underline{\mathbf{n}}, \underline{\mathbf{k}})$. Topologically $N_{\mathbf{i}}$ are trivial bundles, while $K_{\mathbf{i}}$ are, in general, not. These bundles are H-equivariant. Then:

$$\mathcal{Y}_{\mathbf{i}}(x) = \epsilon_x(N_{\mathbf{i}}^*) \frac{\epsilon_{x-\varepsilon_1}(K_{\mathbf{i}}^*)\epsilon_{x-\varepsilon_2}(K_{\mathbf{i}}^*)}{\epsilon_x(K_{\mathbf{i}}^*)\epsilon_{x-\varepsilon}(K_{\mathbf{i}}^*)} \tag{5.11}$$

5.1.4 $\mathcal{Y}$-observables on toric instantons

For our calculations, we need the fixed point expression, i.e. the value $\mathcal{Y}_{\mathbf{i}}(x)[\underline{\boldsymbol{\lambda}}]$ of the observable $\mathcal{Y}_{\mathbf{i}}(x)$ on the special instanton configuration $\mathcal{E}_{\underline{\boldsymbol{\lambda}}}$:

$$\begin{aligned} \mathcal{Y}_{\mathbf{i}}(x)[\underline{\boldsymbol{\lambda}}] &= \prod_{\alpha=1}^{n_{\mathbf{i}}} \left((x - \mathfrak{a}_{\mathbf{i},\alpha}) \prod_{\square \in \lambda^{(\mathbf{i},\alpha)}} \frac{(x - \mathfrak{a}_{\mathbf{i},\alpha} - c_\square - \varepsilon_1)(x - \mathfrak{a}_{\mathbf{i},\alpha} - c_\square - \varepsilon_2)}{(x - \mathfrak{a}_{\mathbf{i},\alpha} - c_\square)(x - \mathfrak{a}_{\mathbf{i},\alpha} - c_\square - \varepsilon)} \right) = \\ &= \prod_{\alpha=1}^{n_{\mathbf{i}}} \frac{\prod_{{\color{green}\blacksquare} \in \partial_+ \lambda^{(\mathbf{i},\alpha)}} (x - \mathfrak{a}_{\mathbf{i},\alpha} - c_{\color{green}\blacksquare})}{\prod_{{\color{magenta}\blacksquare} \in \partial_- \lambda^{(\mathbf{i},\alpha)}} (x - \mathfrak{a}_{\mathbf{i},\alpha} - \varepsilon - c_{\color{magenta}\blacksquare})} \end{aligned} \tag{5.12}$$

where for a monomial ideal I_λ, corresponding to the partition λ the *outer boundary* $\partial_+\lambda$ and the *inner boundary* $\partial_-\lambda$ are the monomials corresponding to the generators, and the relations (divided by the factor $z_1 z_2$) of the ideal, cf. figure 11. Explicitly, given the character χ_λ of the quotient $\mathbb{C}[z_1, z_2]/I_\lambda$ which is the same thing as the character of the partition λ, the contents of the inner and the outer boundaries can be read off the character of the *tautological sheaf*:

$$S_\lambda = 1 - P\chi_\lambda = \sum_{\square \in \partial_+\lambda} e^{\beta c_\square} - q \sum_{\blacksquare \in \partial_-\lambda} e^{\beta c_\blacksquare} \tag{5.13}$$

Note that

$$S_\lambda = \mathsf{Tr}_{S_\lambda^+}\, \widehat{q} - q \mathsf{Tr}_{S_\lambda^-} \widehat{q} \tag{5.14}$$

where $S_\lambda^\pm$ are the fibers over $\mathcal{I}_\lambda \in \mathsf{Hilb}^{[|\lambda|]}(\mathbb{C}^2)$ of the vector bundles $S^\pm$ which we study in more detail in [101]. It is easy to see from the picture of the Young diagram λ, to which stratum $\mathsf{HM}_{k,l} \subset \mathsf{Hilb}^{[|\lambda|]}(\mathbb{C}^2)$ it belongs:

$$\lambda \in \mathsf{HM}_{k,l} \quad \Leftrightarrow \quad k = |\lambda|, \qquad l = \ell(\lambda) = \#\partial_-\lambda = \#\partial_+\lambda - 1\,. \tag{5.15}$$

The $\mathcal{Y}$-observable $\mathcal{Y}_{\mathbf{i}}(x)$ is essentially the character of the localized tautological complex $\mathcal{S}_{\mathbf{i}}$, which is the cohomology (along $\mathbb{CP}^2$) of the complex

$$\mathcal{S}_{\mathbf{i}} = \left(\mathcal{E}_{\mathbf{i}} \to \mathcal{E}_{\mathbf{i}} \otimes \mathcal{T}_{\mathbb{P}^2} \to \mathcal{E}_{\mathbf{i}} \otimes \wedge^2 \mathcal{T}_{\mathbb{P}^2}\right)[-1] \tag{5.16}$$

which is the dual Koszul complex tensored with the universal sheaf $\mathcal{E}_{\mathbf{i}}$. The relevant character $S_{\mathbf{i}}$ is easy to calculate:

$$S_{\mathbf{i}} = N_{\mathbf{i}} - PK_{\mathbf{i}} = \sum_{\alpha=1}^{n_{\mathbf{i}}} e^{\beta \mathfrak{a}_{\mathbf{i},\alpha}} S_{\lambda_{\mathbf{i},\alpha}} \tag{5.17}$$

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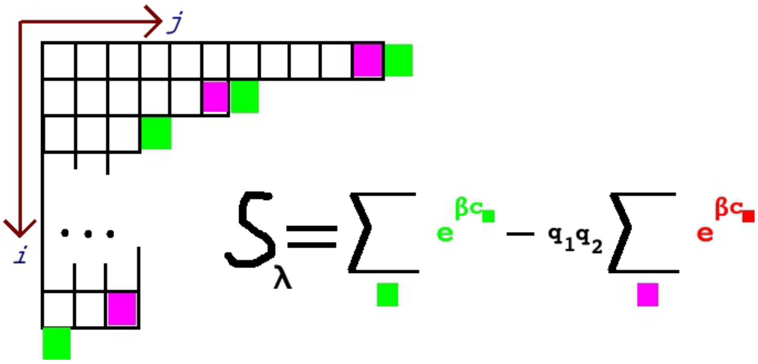

Figure 11. Generators ■ and relations ■ of a monomial ideal I_λ. The character of the tautological sheaf $S_\lambda = 1 - P\chi_\lambda$.

The previous formulae can be succinctly written as:

$$\mathscr{Y}_{\mathbf{i}}(x)[\underline{\boldsymbol{\lambda}}] = \beta^{-n_{\mathbf{i}}}\, \epsilon[e^{\beta x} S^*_{\mathbf{i}}] \tag{5.18}$$

or, in more detail, cf. the notation (10.1)

$$\mathscr{Y}_{\mathbf{i}}(x)[\underline{\boldsymbol{\lambda}}] = x^{n_{\mathbf{i}}} \exp\left(-\sum_{l=1}^{\infty} \frac{1}{l x^l}[\beta^l] S_{\mathbf{i}}\right) \tag{5.19}$$

For large x the observable $\mathscr{Y}_{\mathbf{i}}(x)$ can be expanded as:

$$\mathscr{Y}_{\mathbf{i}}(x) = \mathscr{A}_{\mathbf{i}}(x)\left(1 + \frac{\varepsilon_1\varepsilon_2}{x^2} k_{\mathbf{i}} + \dots\right) \tag{5.20}$$

5.1.5 The importance of $\mathscr{Y}$-observables

The observables $\mathscr{Y}_{\mathbf{i}}(x)$ and the characters $S_{\mathbf{i}}$ are used in the analysis of the non-perturbative Schwinger-Dyson equations. The large field redefinitions (1.17) we shall employ involve adding a point-like instanton at the **i**'th gauge factor, or, conversely, removing a point-like instanton of the **i**'th type. This transformation maps one allowed quiver $\underline{\mathbf{n}}$-colored partition $\underline{\boldsymbol{\lambda}}$ to another one $\underline{\widetilde{\boldsymbol{\lambda}}}$, with modified instanton charge

$$k_{\mathbf{j}}[\underline{\widetilde{\boldsymbol{\lambda}}}] = k_{\mathbf{j}}[\underline{\boldsymbol{\lambda}}] \pm \delta_{\mathbf{i},\mathbf{j}}\,. \tag{5.21}$$

An inspection of the picture figure 6 easily shows that the modifications of the indicated type consist of either adding a box ■ $\in \partial_+\lambda^{(\mathbf{i},\alpha')}$ for some $\alpha' = 1,\dots,n_{\mathbf{i}}$, or removing a box ■ $\in \partial_-\lambda^{(\mathbf{i},\alpha'')}$ for some $\alpha'' = 1,\dots,n_{\mathbf{i}}$. In other words, the allowed modifications of $\underline{\boldsymbol{\lambda}}$ at the **i**'th node correspond to the zeroes and poles of $\mathscr{Y}_{\mathbf{i}}(x)[\underline{\boldsymbol{\lambda}}]$.

The measures $\boldsymbol{\mu}_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}};\underline{\mathtt{m}};\underline{\varepsilon})$ and $\boldsymbol{\mu}_{\underline{\widetilde{\boldsymbol{\lambda}}}}(\underline{\mathfrak{a}};\underline{\mathtt{m}};\underline{\varepsilon})$ are related to each other in a simple manner. Indeed, the character $\mathcal{T}_{\underline{\boldsymbol{\lambda}}}$ is quadratic in $K_{\mathbf{i}}$, more precisely, it is sesquilinear. The variation $\mathcal{T}_{\underline{\widetilde{\boldsymbol{\lambda}}}} - \mathcal{T}_{\underline{\boldsymbol{\lambda}}}$ is, therefore, linear in $K_{\mathbf{i}}$ and $K^*_{\mathbf{i}}$. In fact, it is linear in $S_{\mathbf{j}}$'s and $S^*_{\mathbf{j}}$'s. For the modification $\underline{\boldsymbol{\lambda}} \to \underline{\widetilde{\boldsymbol{\lambda}}}$ consisting of adding a box ■ $\in \partial_+\lambda^{(\mathbf{i},\alpha)}$ for some $\alpha = 1,\dots,n_{\mathbf{i}}$:

$$\mathcal{T}_{\underline{\widetilde{\boldsymbol{\lambda}}}} - \mathcal{T}_{\underline{\boldsymbol{\lambda}}} = S_{\mathbf{i}}[\underline{\boldsymbol{\lambda}}]\xi^{-1} + S^*_{\mathbf{i}}[\underline{\widetilde{\boldsymbol{\lambda}}}]q\xi - M^*_{\mathbf{i}}\xi - \sum_{e\in t^{-1}(\mathbf{i})} S^*_{s(e)}[\underline{\boldsymbol{\lambda}}]\xi q e^{\beta \mathtt{m}_e} - \sum_{e\in s^{-1}(\mathbf{i})} e^{\beta\mathtt{m}_e}\xi^{-1} S_{t(e)}[\underline{\boldsymbol{\lambda}}] + \sum_{e\in s^{-1}(\mathbf{i})\cap t^{-1}(\mathbf{i})} e^{\beta\mathtt{m}_e} P \tag{5.22}$$

where

$$K_{\mathbf{j}}[\widetilde{\boldsymbol{\lambda}}] = K_{\mathbf{j}}[\boldsymbol{\lambda}] + \delta_{\mathbf{i},\mathbf{j}}\xi, \qquad \xi = e^{\beta(\mathfrak{a}_{\mathbf{i},\alpha}+c_{\blacksquare})} \tag{5.23}$$

The ratio of the measures can be, therefore, expressed as a product of the values and residues of various functions $\mathcal{Y}_{\mathbf{j}}(x)[\boldsymbol{\lambda}]$ in the variable x, for example, as

$$\begin{aligned}\frac{\boldsymbol{\mu}_{\widetilde{\boldsymbol{\lambda}}}(\mathfrak{a};\mathfrak{m};\varepsilon)}{\boldsymbol{\mu}_{\boldsymbol{\lambda}}(\mathfrak{a};\mathfrak{m};\varepsilon)} = (-1)^{\kappa_{\mathbf{i}}}\ \mathfrak{q}_{\mathbf{i}} \frac{\varepsilon}{\varepsilon_1\varepsilon_2} \frac{P_{\mathbf{i}}(x)}{\mathcal{Y}_{\mathbf{i}}(x+\varepsilon)[\boldsymbol{\lambda}]\mathcal{Y}_{\mathbf{i}}'(x)[\boldsymbol{\lambda}]} \times \\ \prod_{e\in t^{-1}(\mathbf{i})} \mathcal{Y}_{s(e)}(x+\varepsilon+\mathfrak{m}_e)[\boldsymbol{\lambda}] \prod_{e\in s^{-1}(\mathbf{i})} \mathcal{Y}_{t(e)}(x-\mathfrak{m}_e)[\boldsymbol{\lambda}]\times \\ \prod_{e\in s^{-1}(\mathbf{i})\cap t^{-1}(\mathbf{i})} \frac{(\mathfrak{m}_e+\varepsilon_1)(\mathfrak{m}_e+\varepsilon_2)}{\mathfrak{m}_e(\mathfrak{m}_e+\varepsilon)} \\ x = \mathfrak{a}_{\mathbf{i},\alpha} + c_{\blacksquare}\end{aligned} \tag{5.24}$$

where

$$\kappa_{\mathbf{i}} = n_{\mathbf{i}} - 1 + \sum_{e\in s^{-1}(\mathbf{i})} n_{t(e)} \tag{5.25}$$

Note the identity:

$$\mathrm{res}_{x=\mathfrak{a}_{\mathbf{i},\alpha}+c_{\blacksquare}}\ \mathcal{Y}_{\mathbf{i}}(x+\varepsilon)[\widetilde{\boldsymbol{\lambda}}] = \frac{\varepsilon_1\varepsilon_2}{\varepsilon}\ \mathcal{Y}_{\mathbf{i}}(\mathfrak{a}_{\mathbf{i},\alpha}+c_{\blacksquare}+\varepsilon)[\boldsymbol{\lambda}] \tag{5.26}$$

5.2 $\mathcal{Q}$-observables

The inspection of the eq. (5.12) shows that $\mathcal{Y}_{\mathbf{i}}(x)[\boldsymbol{\lambda}]$ can be represented as a ratio of two entire functions, in two ways:

$$\mathcal{Y}_{\mathbf{i}}(x) = \frac{\mathcal{Q}_{\mathbf{i}}^{(1,2)}(x)}{\mathcal{Q}_{\mathbf{i}}^{(1,2)}(x-\varepsilon_2)} = \frac{\mathcal{Q}_{\mathbf{i}}^{(2,1)}(x)}{\mathcal{Q}_{\mathbf{i}}^{(2,1)}(x-\varepsilon_1)}\,, \tag{5.27}$$

where

$$\mathcal{Q}_{\mathbf{i}}^{(a,b)}(x)[\boldsymbol{\lambda}] = \prod_{\alpha=1}^{n_{\mathbf{i}}} \left(\frac{(-\varepsilon_b)^{\left(\frac{x-\mathfrak{a}_{\mathbf{i},\alpha}}{\varepsilon_b}\right)}}{\Gamma\left(-\frac{x-\mathfrak{a}_{\mathbf{i},\alpha}}{\varepsilon_b}\right)} \prod_{\square\in\lambda^{(\mathbf{i},\alpha)}} \frac{x-\mathfrak{a}_{\mathbf{i},\alpha}-c_\square-\varepsilon_a}{x-\mathfrak{a}_{\mathbf{i},\alpha}-c_\square} \right), \quad (a,b)=(1,2) \text{ or } (2,1) \tag{5.28}$$

The rôle of these observables will be revealed in [100, 103]. In the limit $\varepsilon_2 \to 0$ with ε_1-fixed the observables $\mathcal{Q}_{\mathbf{i}}^{(2,1)}$ tend to the so-called Baxter operators of the quantum integrable system, which is Bethe/gauge-dual [97] to the gauge theory under consideration [86].

6 Enter the qq-characters

Remarkably, the Dyson-Schwinger relations based on (5.24) can be summarized in the following proposition:

6.1 The main theorem

For any γ-graded vector space

$$\boldsymbol{W} = \bigoplus_{\mathbf{i}\in \mathsf{Vert}_\gamma} W_{\mathbf{i}}\,, \tag{6.1}$$

with the corresponding dimension vector $\underline{\mathbf{w}} \in \mathbb{Z}_{\geq 0}^{\mathsf{Vert}_\gamma}$, $W_{\mathbf{i}} = \mathbb{C}^{\mathtt{w}_{\mathbf{i}}}$, *and a choice of ℓ-weights* $\underline{\boldsymbol{\nu}} = (\nu_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}$, $\nu_{\mathbf{i}} = \mathsf{diag}\,(\nu_{\mathbf{i},1},\ldots,\nu_{\mathbf{i},\mathtt{w}_{\mathbf{i}}}) \in \mathrm{End}(W_{\mathbf{i}})$, *there is a Laurent polynomial (Laurent power series for affine γ) in $\mathcal{Y}_{\mathbf{j}}(x+\xi_{\mathbf{j},\kappa})$'s , i.e. in $\mathcal{Y}_{\mathbf{j}}$'s with possibly shifted arguments, including the nilpotent shifts (i.e. a finite number of derivatives in x applied to $\mathcal{Y}_{\mathbf{j}}$)*

$$\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}\,(\mathcal{Y}(x+\ldots)) = \prod_{\mathbf{i}\in\mathsf{Vert}_\gamma}\prod_{l=1}^{\mathtt{w}_{\mathbf{i}}} \mathcal{Y}_{\mathbf{i}}(x+\nu_{\mathbf{i},l}+\varepsilon) \;+\; O(\mathfrak{q}) \tag{6.2}$$

such that its expectation value in the γ-quiver gauge theory:

$$\left\langle\, \mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}\,(\mathcal{Y})\,\right\rangle_\gamma \equiv \frac{1}{\mathcal{Z}_\gamma^{\mathrm{inst}}}\sum_{\underline{\boldsymbol{\lambda}}} \mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}\,(\mathcal{Y}[\underline{\boldsymbol{\lambda}}])\;\underline{\mathfrak{q}}^{\boldsymbol{\lambda}}\,\boldsymbol{\mu}_{\underline{\boldsymbol{\lambda}}}(\underline{\mathfrak{a}};\underline{\mathfrak{m}};\underline{\varepsilon}) = T_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x), \tag{6.3}$$

is a polynomial in x. More specifically, $T_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)$ is a polynomial in x of degree

$$\deg T_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x) = \underline{\mathbf{w}}\cdot\underline{\mathbf{n}} = \sum_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathtt{w}_{\mathbf{i}} n_{\mathbf{i}}\,. \tag{6.4}$$

We call $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)$ the Yangian qq-character of $Y(\mathfrak{g}_\gamma)$. For $\underline{\mathbf{w}} = (\delta_{\mathbf{j},\mathbf{i}})_{\mathbf{j}\in\mathsf{Vert}_\gamma}$ and $\underline{\boldsymbol{\nu}} = 0$ the corresponding qq-character will be denoted by $\chi_{\mathbf{i}}(x)$, the $\mathbf{i}$'th *fundamental qq-character.*

Remark. The qq-characters are the gauge theory generalizations of the matrix model expression $\mathbf{T}(x)$ (1.11).

In the limit $\varepsilon_2 \to 0$ $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)$ reduces to the Yangian q-characters of finite-dimensional representations of the Yangian $Y(\mathfrak{g}_\gamma)$, constructed for finite γ in [60]. In [38] the q-characters for the quantum affine algebras $U_q(\mathfrak{g}_\gamma)$ for finite γ's and in [49] for affine γ's are constructed. These correspond to the K-theoretic version of our story in the limit $q_2 \to 1$, $q_1 = q$ finite, which was discussed in [86].

The K-theoretic version of our story with general (q_1, q_2) produces the qq-characters, corresponding to $U_q(\mathfrak{g}_\gamma)$. The physical applications of the qq-characters are the five dimensional supersymmetric gauge theories compactified on a circle [82]. We shall give the definition and the formulae below, without going into much detail.

7 Examples of qq-characters

In this section we prepare the reader by giving a few explicit examples of the qq-characters, before unveiling the general formula in the next section.

7.1 A-type theories: one factor gauge group

Let us start with a couple of examples for the theories with a single factor gauge group, i.e. either the A_1 theory or the $\widehat{A}_0$ theory.

7.1.1 The A_1 case

The A_1 theory is the $U(n)$ gauge theory with $N_f = 2n$ fundamental hypermultiplets. The theory is characterized by the gauge coupling $\mathfrak{q}$ and $2n$ masses $\underline{\mathtt{m}} = (\mathtt{m}_1, \ldots, \mathtt{m}_{2n})$, which are encoded in the polynomial

$$P(x) = \prod_{\mathfrak{f}=1}^{2n} (x - \mathtt{m}_{\mathfrak{f}})$$

Since the quiver consists of a single vertex, we omit the subscript $\mathbf{i}$ in $\mathcal{Y}(x)$ and $P(x)$.

The fundamental A_1 qq-character is equal to

$$\mathcal{X}_{1,0}(x) = \mathcal{Y}(x+\varepsilon) + \mathfrak{q}\, \frac{P(x)}{\mathcal{Y}(x)} \tag{7.1}$$

The general A_1 qq-character depends on a w-tuple $\underline{\boldsymbol{\nu}}$ of complex numbers, $\underline{\boldsymbol{\nu}} = (\nu_1, \ldots, \nu_{\mathrm{w}}) \in \mathbb{C}^{\mathrm{w}}$. It is given by:

$$\mathcal{X}_{\mathrm{w},\underline{\boldsymbol{\nu}}}(x) = \sum_{[\mathrm{w}]=I\sqcup J} \mathfrak{q}^{|J|} \prod_{i\in I, j\in J} \mathbf{S}(\nu_i - \nu_j) \prod_{j\in J} \frac{P(x+\nu_j)}{\mathcal{Y}(x+\nu_j)} \prod_{i\in I} \mathcal{Y}(x+\varepsilon+\nu_i) \tag{7.2}$$

It has potential poles in ν's, when $\nu_i = \nu_j$ or $\nu_i = \nu_j + \varepsilon$, for $i \neq j$.

The expression (7.2) is actually non-singular at the diagonals $\nu_i = \nu_j$. The limit contains, however, the derivatives $\partial_x \mathcal{Y}$. For example, for $\mathtt{w} = 2$, $\nu_1 = \nu_2 = 0$ the qq-character is equal to:

$$\begin{aligned}\mathcal{X}_{2,(0,0)}(x) = \mathcal{Y}(x+\varepsilon)^2 \left(1 - \mathfrak{q}\frac{\varepsilon_1\varepsilon_2}{\varepsilon}\partial_x \left(\frac{P(x)}{\mathcal{Y}(x)\mathcal{Y}(x+\varepsilon)}\right)\right) + \\ + 2\mathfrak{q}P(x)\frac{\mathcal{Y}(x+\varepsilon)}{\mathcal{Y}(x)}\left(1 - \frac{\varepsilon_1\varepsilon_2}{\varepsilon^2}\right) + \mathfrak{q}^2 \frac{P(x)^2}{\mathcal{Y}(x)^2}\end{aligned} \tag{7.3}$$

The expression (7.2) has a first order pole at the hypersurfaces where $\nu_i = \nu_j + \varepsilon$ for some pair $i \neq j$. The residue of $\mathcal{X}_{\mathrm{w},\boldsymbol{\nu}}$ is equal to the qq-character $\mathcal{X}_{\mathrm{w}-2,\boldsymbol{\nu}\backslash\{\nu_i,\nu_j\}}$, times the polynomial in x factor

$$\prod_{k\neq i,j} \mathbf{S}(\nu_k - \nu_j) P(x+\nu_k)\,. \tag{7.4}$$

The finite part $\mathcal{X}^{\mathrm{fin}}_{\mathrm{w},\boldsymbol{\nu}}$ of the expansion of $\mathcal{X}_{\mathrm{w},\boldsymbol{\nu}}$ in ν_i near $\nu_i = \nu_j + \varepsilon$ is the properly defined qq-character for the arrangement of weights $\boldsymbol{\nu}$ landing on the hypersurface $\nu_i = \nu_j + \varepsilon$. It involves the terms with the derivative $\partial_x \mathcal{Y}$. For example

$$\begin{aligned}\mathcal{X}^{\mathrm{fin}}_{2,(-\varepsilon,0)} = {} & \mathcal{Y}(x+\varepsilon)\mathcal{Y}(x) + \\ & + \mathfrak{q}\left(1 + \frac{\varepsilon_1\varepsilon_2}{2\varepsilon^2}\right) P(x-\varepsilon)\frac{\mathcal{Y}(x+\varepsilon)}{\mathcal{Y}(x-\varepsilon)} + \mathfrak{q}P(x)\left(1 - \frac{\varepsilon_1\varepsilon_2}{\varepsilon}\frac{\partial_x\mathcal{Y}(x)}{\mathcal{Y}(x)}\right) + \\ & + \mathfrak{q}^2 \frac{P(x)P(x-\varepsilon)}{\mathcal{Y}(x)\mathcal{Y}(x-\varepsilon)}\end{aligned} \tag{7.5}$$

7.1.2 The $\widehat{A}_0$ theory

The $\widehat{A}_0$ theory (also known as the $\mathcal{N}=2^*$ theory) is characterized by one mass parameter $\mathtt{m}$, the mass of the adjoint hypermultiplet, and the gauge coupling $\mathfrak{q}$.

Here we give the expression for the fundamental character $\mathcal{X}_1(x) \equiv \mathcal{X}_{1,0}(x)$:

$$\begin{aligned}
\mathcal{X}_1(x) &= \sum_{\lambda} \mathfrak{q}^{|\lambda|} \prod_{\square\in\lambda} \mathbf{S}(\mathtt{m}h_\square + \varepsilon a_\square) \cdot \frac{\prod_{\square\in\partial_+\lambda} \mathcal{Y}(x+\sigma_\square+\varepsilon)}{\prod_{\square\in\partial_-\lambda} \mathcal{Y}(x+\sigma_\square)} = \\
&= \mathcal{Y}(x+\varepsilon) \sum_{\lambda} \mathfrak{q}^{|\lambda|} \prod_{\square\in\lambda} \mathbf{S}(\mathtt{m}h_\square + \varepsilon a_\square) \cdot \prod_{\square\in\lambda} \frac{\mathcal{Y}(x+\sigma_\square-\mathtt{m})\mathcal{Y}(x+\sigma_\square+\mathtt{m}+\varepsilon)}{\mathcal{Y}(x+\sigma_\square)\mathcal{Y}(x+\sigma_\square+\varepsilon)} = \\
&= \mathcal{Y}(x+\varepsilon) + \ \mathfrak{q}\ \mathbf{S}(\mathtt{m}) \frac{\mathcal{Y}(x-\mathtt{m})\,\mathcal{Y}(x+\varepsilon+\mathtt{m})}{\mathcal{Y}(x)} + \ldots
\end{aligned} \tag{7.6}$$

Here

$$\sigma_\square = \mathtt{m}(i-j) + \varepsilon(1-j) \tag{7.7}$$

is the content of $\square$ defined relative to the pair of weights $(\mathtt{m}, -\mathtt{m}-\varepsilon)$. It is not too difficult to write an expression for the general $\widehat{A}_0$ qq-character $\mathcal{X}_{\mathtt{w},\boldsymbol{\nu}}$, in terms of an infinite sum over the $\mathtt{w}$-tuples of partitions, but we feel it is not very illuminating.

Note that the expression (7.6) has apparent singularities when $\mathtt{m}$ and $-(\mathtt{m}+\varepsilon)$ are in a positive congruence, i.e. if

$$\mathtt{m}(p-q) = \varepsilon q \tag{7.8}$$

for some positive integers $p, q > 0$. In fact, the limit of the expression (7.6) is finite, but it involves not only the ratios of shifted $\mathcal{Y}$'s, but also its derivatives. The most efficient way to study this asymptotics is to use the geometric expression to be discussed below. The geometric expression also leads to the contour integral representation of the qq-characters.

7.2 A-type theories: linear quiver theories

Let us now present the formulas for the general A_r theories, assuming

$$m_1 = m_r = n_1 = \ldots = n_r = N, \ m_2 = m_3 = \ldots = m_{r-1} = 0\,. \tag{7.9}$$

We treat the general A_r case $m_i = 2n_i - n_{i-1} - n_{i+1}$, $n_0 = n_{r+1} = 0$ in the section below. We have r observables $\mathcal{Y}_{\mathbf{i}}(x)$, and couplings $\mathfrak{q}_{\mathbf{i}}$, for $\mathbf{i} = 1, \ldots, r$. Define $r+1$ complex numbers z_i, $i = 0, 1, \ldots, r$ by:

$$z_i = z_0\, \mathfrak{q}_1 \ldots \mathfrak{q}_i, \qquad i = 1, \ldots, r \tag{7.10}$$

and define $r+1$ functions $\Lambda_i(x)$, $i = 0, \ldots, r$, by:

$$\Lambda_i(x) = z_i \frac{\mathcal{Y}_{i+1}(x+\varepsilon)}{\mathcal{Y}_i(x)}, \tag{7.11}$$

where we set $\mathcal{Y}_0(x) = P_1(x)$, $\mathcal{Y}_{r+1}(x) = P_r(x)$, in other words the masses $\mathtt{m}_{1,f}$, $\mathtt{m}_{r,f}$ of fundamentals are denoted as $\mathfrak{a}_{0,f}$, $\mathfrak{a}_{r+1,f}$, respectively. We also choose the normalization $m_e = -\varepsilon$.

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7.2.1 The height functions

For a *finite* set $I \subset \mathbb{R}$, we define the *height function*:

$$h_I : I \to [p], \qquad p = |I| \equiv \#I\,, \qquad h_I(i) = \#\{\, i' \,|\, i' \in I,\ i' < i\,\} \tag{7.12}$$

In other words, for $I = \{i_1, \dots, i_p\}$ with $i_1 < i_2 < \dots < i_p$, $h_{i_b} = b - 1$, $1 \le b \le p$.

7.2.2 Pre-character

Define the l'th fundamental *qq pre-character* by (cf. (7.12)):

$$\chi_l(x) = \sum_{I \subset [0,r],\, |I|=l} \prod_{i \in I} \Lambda_i \left(x + (h_I(i) + 1 - l)\,\varepsilon\right) \tag{7.13}$$

7.2.3 Fundamental *qq*-character of type A_r

Then l'th *fundamental qq-character* is given by the properly normalized $\chi_l(x)$:

$$\mathcal{X}_l(x) = \mathcal{Y}_0\left(x + (1-l)\,\varepsilon\right) \frac{\chi_l(x)}{z_0 z_1 \dots z_{l-1}} = \mathcal{Y}_l(x+\varepsilon) + \mathfrak{q}_l \frac{\mathcal{Y}_{l-1}(x)\mathcal{Y}_{l+1}(x+\varepsilon)}{\mathcal{Y}_l(x)} + \dots \tag{7.14}$$

7.2.4 Comparison to the *q*-characters of E. Frenkel and N. Reshetikhin

Our formula (7.13) looks similar to the formula in the section 11.1 of [37] (adapted to the Yangian $Y(\mathfrak{sl}_{r+1})$, of course). The similarity is a little bit misleading, as shows the example of the D-type theories.

7.2.5 The main property of the *qq*-characters of the *A*-type

The relations (5.24) suffice to prove that the expectation values of the *qq*-characters of the A-type theories do not have singularities as the functions of x. It suffices to check the cancellation of residues between the poles related by adding or removing one square in one of the Young diagrams in $\underline{\boldsymbol{\lambda}}$. It would take more elaborate arguments to prove the analogous claim for all $\mathcal{N} = 2$ theories.

7.3 The *D*-type theories

7.3.1 The D_4 theory

This is the theory with four gauge groups. The quiver is the graph with $\mathsf{Vert}_\gamma = \{1,2,3,4\}$, and $\mathsf{Edges}_\gamma = \{1,3,4\}$ with $s(e) = e$, $t(e) = 2$ for all $e \in \mathsf{Edges}_\gamma$. The graph has the obvious $\mathcal{S}_3$ symmetry, permuting the vertices $1,3,4$. We present the formula for the asymptotically conformal theory with $n_1 = n_3 = n_4 = N = m_2$, $n_2 = 2N$, $m_1 = m_3 = m_4 = 0$, leaving an obvious extension to the general case to the interested reader as an exercise in the deciphering the general formula of the next section (essentially one replaces $\mathfrak{q}_i \mapsto \mathfrak{q}_i P_i(x + a\varepsilon)$ for $i = 1,3,4$ with some integers a).

In what follows we use the short-hand notation:

$$Y_{\mathtt{i},a} = \mathcal{Y}_{\mathtt{i}}(x + a\varepsilon),\ Y_{\mathtt{i}} = \mathcal{Y}_{\mathtt{i}}(x),\ P_a = P_2(x + a\varepsilon),\ P = P_2(x) \tag{7.15}$$

There are four fundamental qq-characters, three of which are permuted by the $\mathcal{S}_3$. The qq-character $\mathcal{X}_{1,0}$ is given by the sum of 8 terms (as would be the case for the characters of $\mathbf{8_v}, \mathbf{8_s}, \mathbf{8_c}$ of vector or spinor representations of Spin(8)):

$$\begin{aligned}\mathcal{X}_{1,0} = &\, Y_{1,1} + \mathfrak{q}_1 Y_1^{-1} Y_2 + \mathfrak{q}_1\mathfrak{q}_2 P_{-1} Y_{2,-1}^{-1} Y_3 Y_4 + \\ &+ \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 P_{-1} Y_{3,-1}^{-1} Y_4 + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_4 P_{-1} Y_{4,-1}^{-1} Y_3 + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4 P_{-1} Y_{3,-1}^{-1} Y_{4,-1}^{-1} Y_{2,-1} + \\ &+ \mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4 P_{-1} P_{-2} Y_{2,-2}^{-1} Y_{1,-1} + \mathfrak{q}_1^2\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4 P_{-1} P_{-2} Y_{1,-2}^{-1}\end{aligned} \tag{7.16}$$

The formulae for $\mathcal{X}_{3,0}, \mathcal{X}_{4,0}$ are obtained by the cyclic permutation of the indices $1, 3, 4$. The qq-character $\mathcal{X}_{2,0}$ reveals a surprising structure,

$$\mathcal{X}_{2,0} = \mathcal{X}_2^{+} + \mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4 P P_{-1} \mathcal{X}_2^{-} \tag{7.17}$$

and contains the derivatives of $\mathcal{Y}_{\mathbf{i}}$'s. Explicitly,

$$\begin{aligned}\mathcal{X}_2^{+} = &\, Y_{2,1} + \mathfrak{q}_2 P Y_2^{-1} Y_{1,1} Y_{3,1} Y_{4,1} + \mathfrak{q}_1\mathfrak{q}_2 P Y_1^{-1} Y_{3,1} Y_{4,1} + \mathfrak{q}_3\mathfrak{q}_2 P Y_3^{-1} Y_{1,1} Y_{4,1} + \\ &+ \mathfrak{q}_4\mathfrak{q}_2 P Y_4^{-1} Y_{1,1} Y_{3,1} + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 P Y_1^{-1} Y_3^{-1} Y_2 Y_{4,1} + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_4 P Y_1^{-1} Y_4^{-1} Y_2 Y_{3,1} + \\ &+ \mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4 P Y_3^{-1} Y_4^{-1} Y_2 Y_{1,1} + \mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_3 P P_{-1} Y_{2,-1}^{-1} Y_4 Y_{4,-1} + \mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_4 P P_{-1} Y_{2,-1}^{-1} Y_3 Y_{3,-1} + \\ &+ \mathfrak{q}_3\mathfrak{q}_2^2\mathfrak{q}_4 P P_{-1} Y_{2,-1}^{-1} Y_1 Y_{1,-1} + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4 P Y_1^{-1} Y_3^{-1} Y_4^{-1} Y_2^2\end{aligned} \tag{7.18}$$

$$\mathcal{X}_2^{-} = \mathcal{X}_2^{-,0} + \mathcal{X}_2^{--}, \tag{7.19}$$

$$\begin{aligned}\mathcal{X}_2^{-,0} = &\, \frac{Y_2}{Y_{2,-1}} \left(2\left(1 - \frac{\varepsilon_1\varepsilon_2}{\varepsilon^2}\right) + \frac{\varepsilon_1\varepsilon_2}{\varepsilon} \partial_x \log\left(\frac{Y_2 Y_{2,-1}}{P_{-1} Y_1 Y_3 Y_4}\right)\right) + \\ &+ \left(1 + \frac{\varepsilon_1\varepsilon_2}{2\varepsilon^2}\right)\left(Y_{1,-1}^{-1} Y_{1,1} + Y_{3,-1}^{-1} Y_{3,1} + Y_{4,-1}^{-1} Y_{4,1}\right)\end{aligned} \tag{7.20}$$

$$\begin{aligned}\mathcal{X}_2^{--} = &\, \mathfrak{q}_4 Y_4^{-1} Y_{4,-1}^{-1} Y_2 + \mathfrak{q}_3 Y_3^{-1} Y_{3,-1}^{-1} Y_2 + \mathfrak{q}_1 Y_1^{-1} Y_{1,-1}^{-1} Y_2 + \mathfrak{q}_2 P_{-1} Y_{2,-1}^{-2} Y_1 Y_3 Y_4 + \\ &\mathfrak{q}_2\mathfrak{q}_4 P_{-1} Y_{2,-1}^{-1} Y_{4,-1}^{-1} Y_1 Y_3 + \mathfrak{q}_2\mathfrak{q}_3 P_{-1} Y_{2,-1}^{-1} Y_{3,-1}^{-1} Y_1 Y_4 + \mathfrak{q}_2\mathfrak{q}_1 P_{-1} Y_{2,-1}^{-1} Y_{1,-1}^{-1} Y_3 Y_4 + \\ &+ \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_4 P_{-1} Y_{1,-1}^{-1} Y_{4,-1}^{-1} Y_3 + \mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4 P_{-1} Y_{3,-1}^{-1} Y_{4,-1}^{-1} Y_1 + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3 P_{-1} Y_{1,-1}^{-1} Y_{3,-1}^{-1} Y_4 + \\ &+ \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3\mathfrak{q}_4 P_{-1} Y_{1,-1}^{-1} Y_{3,-1}^{-1} Y_{4,-1}^{-1} Y_{2,-1} + \\ &+ \mathfrak{q}_1\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_4 P_{-1} P_{-2} Y_{2,-2}^{-1}\end{aligned} \tag{7.21}$$

8 The qq-character formula

In order to write the general formula for the qq-character we shall use an auxiliary geometric object, the quiver variety $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$, which we presently define.

8.1 Nakajima quiver variety

Given a quiver γ, two dimension vectors

$$\underline{\mathbf{v}} = (v_{\mathbf{i}})_{\mathbf{i} \in \mathsf{Vert}_\gamma}, \ \underline{\mathbf{w}} = (w_{\mathbf{i}})_{\mathbf{i} \in \mathsf{Vert}_\gamma} \in \mathbb{Z}_{\geq 0}^{\mathsf{Vert}_\gamma} \tag{8.1}$$

and a choice of *stability parameter* $\underline{\boldsymbol{\zeta}} \in \mathbb{R}^{\mathsf{Vert}_\gamma}$ H. Nakajima [76] defines the quiver variety as the hyperkahler quotient:

$$\mathbb{M}_{\gamma, \underline{\boldsymbol{\zeta}}}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) = \mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(\underline{\boldsymbol{\zeta}}) / G_{\underline{\mathbf{v}}} \tag{8.2}$$

where

$$G_{\underline{\mathbf{v}}} = \bigtimes_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathrm{U}(V_{\mathbf{i}})\,, \tag{8.3}$$

and $\mu_{\mathbb{C}}$, $\mu_{\mathbb{R}}$ are the quadratic maps $\mathcal{H}_\gamma \to \mathrm{Lie}G^*_{\underline{\mathbf{v}}} \otimes \mathbb{C}, \mathbb{R}$, respectively, with $\mathcal{H}_\gamma$ the vector space

$$\mathcal{H}_\gamma = T^* \left(\bigoplus_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathrm{Hom}(V_{\mathbf{i}}, W_{\mathbf{i}}) \oplus \bigoplus_{e\in\mathsf{Edges}_\gamma} \mathrm{Hom}(V_{s(e)}, V_{t(e)}) \right) \tag{8.4}$$

of linear operators (matrices) $(\tilde{I}_{\mathbf{i}}, \tilde{J}_{\mathbf{i}}, B_{e,\pm})$:

$$\begin{aligned} \tilde{I}_{\mathbf{i}}:&\quad W_{\mathbf{i}} \to V_{\mathbf{i}}, & \tilde{J}_{\mathbf{i}}:&\quad V_{\mathbf{i}} \to W_{\mathbf{i}} \\ B_{e,+}:&\quad V_{s(e)} \to V_{t(e)}, & B_{e,-}:&\quad V_{t(e)} \to V_{s(e)} \end{aligned} \tag{8.5}$$

Explicitly: $\mu_{\mathbb{R},\mathbb{C}} = (\mu^{\mathbb{R}}_{\mathbf{i}}, \mu^{\mathbb{C}}_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}$, with

$$\begin{aligned} \mu^{\mathbb{C}}_{\mathbf{i}} &= \tilde{I}_{\mathbf{i}}\tilde{J}_{\mathbf{i}} + \sum_{e\in s^{-1}(\mathbf{i})} B_{e,-}B_{e,+} - \sum_{e\in t^{-1}(\mathbf{i})} B_{e,+}B_{e,-} \\ \mu^{\mathbb{R}}_{\mathbf{i}} &= \tilde{I}_{\mathbf{i}}\tilde{I}^{\dagger}_{\mathbf{i}} - \tilde{J}^{\dagger}_{\mathbf{i}}\tilde{J}_{\mathbf{i}} + \sum_{e\in s^{-1}(\mathbf{i})} B_{e,-}B^{\dagger}_{e,-} - B^{\dagger}_{e,+}B_{e,+} + \\ &\quad + \sum_{e\in t^{-1}(\mathbf{i})} B_{e,+}B^{\dagger}_{e,+} - B^{\dagger}_{e,-}B_{e,-} \end{aligned} \tag{8.6}$$

The definition (8.2) translates to the set of equations:

$$\begin{aligned} \mu^{\mathbb{R}}_{\mathbf{i}} &= \zeta_{\mathbf{i}}\, \mathbf{1}_{V_{\mathbf{i}}}\,, \\ \mu^{\mathbb{C}}_{\mathbf{i}} &= 0, \qquad \mathbf{i} \in \mathsf{Vert}_\gamma \end{aligned} \tag{8.7}$$

with the identification of solutions related by the $G_{\underline{\mathbf{v}}}$ transformations:

$$(B_{e,+}, B_{e,-}, \tilde{I}_{\mathbf{i}}, \tilde{J}_{\mathbf{i}}) \mapsto (h_{t(e)}B_{e,+}h^{-1}_{s(e)}, h_{s(e)}B_{e,-}h^{-1}_{t(e)}, h_{\mathbf{i}}\tilde{I}_{\mathbf{i}}, \tilde{J}_{\mathbf{i}}h^{-1}_{\mathbf{i}}), \qquad h_{\mathbf{i}} \in \mathrm{U}(V_{\mathbf{i}}) \tag{8.8}$$

8.1.1 Stability parameters

Solving the real moment map equations (the first line in the eq. (8.7)) and dividing by $G_{\underline{\mathbf{v}}}$ can be replaced by dividing the set of *stable* solutions to the complex moment map equations (the second line in the eq. (8.7)) by the action of the complexified group

$$G^{\mathbb{C}}_{\underline{\mathbf{v}}} = \times_{\mathbf{i}}\, \mathrm{GL}(V_{\mathbf{i}})\,. \tag{8.9}$$

The notion of stability depends on the choice of $\underline{\boldsymbol{\zeta}}$. In this paper we assume $\zeta_{\mathbf{i}} > 0$ for all $\mathbf{i} \in \mathsf{Vert}_\gamma$. The solution $(B_{e,\pm}, \tilde{I}_{\mathbf{i}}, \tilde{J}_{\mathbf{i}})$ is stable iff any collection $(V'_{\mathbf{i}})_{\mathbf{i}\in\mathsf{Vert}_\gamma}$ of subspaces $V'_{\mathbf{i}} \subset V_{\mathbf{i}}$, such that

$$\begin{aligned} &(1) \qquad \tilde{I}_{\mathbf{i}}(W_{\mathbf{i}}) \subset V'_{\mathbf{i}} \qquad \text{for all} \qquad \mathbf{i} \in \mathsf{Vert}_\gamma \quad \text{and} \\ &(2) \qquad \mathcal{B}_p(V'_{s(p)}) \subset V'_{t(p)}, \quad \text{and} \quad \tilde{\mathcal{B}}_p(V'_{t(p)}) \subset V'_{s(p)} \quad \text{for all} \quad p \in \mathsf{Paths}_\gamma \end{aligned} \tag{8.10}$$

is such that $V'_{\mathbf{i}} = V_{\mathbf{i}}$ for all $\mathbf{i} \in \mathsf{Vert}_\gamma$. Here $p \in \mathsf{Paths}_\gamma$ denotes a sequence $(e_i, \sigma_i) \in \mathsf{Arrows}_\gamma$, $i = 1, \ldots, m$, such that $\bar{s}(e_1, \sigma_1) = s(p)$, $\bar{t}(e_1, \sigma_1) = \bar{s}(e_2, \sigma_2)$, ..., $\bar{t}(e_i, \sigma_i) = \bar{s}(e_{i+1}, \sigma_{i+1})$, ..., $\bar{t}(e_m, \sigma_m) = t(p)$.

The proof is simple. Let $P_{\mathbf{i}}$ be the orthogonal projection of $V_{\mathbf{i}}$ onto the orthogonal complement $(V'_{\mathbf{i}})^{\perp}$ of the "invariant" subspace $V'_{\mathbf{i}} \subset V_{\mathbf{i}}$. We have $P_{\mathbf{i}}\tilde{I}_{\mathbf{i}} = 0$ and $P_{t(e)}B_{e,+}(1 - P_{s(e)}) = 0$, $P_{s(e)}B_{e,-}(1 - P_{t(e)}) = 0$. Now compute

$$\begin{aligned} 0 \le \sum_{\mathbf{i}} \zeta_{\mathbf{i}} \dim (V'_{\mathbf{i}})^{\perp} &= \sum_{\mathbf{i}} \mathsf{Tr}\, P_{\mathbf{i}} \mu^{\mathbb{R}}_{\mathbf{i}} P_{\mathbf{i}} = -\sum_{\mathbf{i}} \mathsf{Tr} \left(P_{\mathbf{i}} \tilde{J}^{\dagger}_{\mathbf{i}} \tilde{J}_{\mathbf{i}} P_{\mathbf{i}} \right) + \sum_{e} \mathsf{Tr}\, P_{s(e)} B_{e,-} B^{\dagger}_{e,-} P_{s(e)} \\ &+ \mathsf{Tr}\, P_{t(e)} B_{e,+} B^{\dagger}_{e,+} P_{t(e)} - \mathsf{Tr}\, P_{t(e)} B^{\dagger}_{e,-} B_{e,-} P_{t(e)} - \mathsf{Tr}\, P_{s(e)} B^{\dagger}_{e,+} B_{e,+} P_{s(e)} = \\ &- \sum_{\mathbf{i}} \| \tilde{J}_{\mathbf{i}} P_{\mathbf{i}} \|^2 - \sum_{e} \left(\|(1 - P_{s(e)}) B_{e,-} P_{t(e)}\|^2 + \|(1 - P_{t(e)}) B_{e,+} P_{s(e)}\|^2 \right) \le 0 \end{aligned} \tag{8.11}$$

which implies $V'_{\mathbf{i}} = V_{\mathbf{i}}$ for all $\mathbf{i}$. The stability condition is equivalent to the condition that the path operators $\mathcal{B}_p$ and $\tilde{\mathcal{B}}_p$ for $p \in \mathsf{Paths}_\gamma$ acting on $\tilde{I}_{\mathbf{i}'}(W_{\mathbf{i}'})$ generate $V_{\mathbf{i}''}$:

$$V_{\mathbf{i}} = \sum_{p \in t^{-1}(\mathbf{i})} \mathbb{C}\mathcal{B}_p\, \tilde{I}_{s(p)}(W_{s(p)}) + \sum_{p \in s^{-1}(\mathbf{i})} \mathbb{C}\tilde{\mathcal{B}}_p\, \tilde{I}_{t(p)}(W_{t(p)}) \tag{8.12}$$

Conversely, in order to establish that the stability condition implies that the $G^{\mathbb{C}}_{\underline{\mathbf{v}}}$-orbit of $(B_{e,\pm}, \tilde{I}_{\mathbf{i}}, \tilde{J}_{\mathbf{i}})$ solving the $\mu^{\mathbb{C}} = 0$ equations in (8.7) passes through the solution of the $\underline{\mu}^{\mathbb{R}} = \underline{\boldsymbol{\zeta}}$ equations in (8.7), we use the standard method: consider the Morse-Bott function

$$f = \sum_{\mathbf{i} \in \mathsf{Vert}_\gamma} \| \mu^{\mathbb{R}}_{\mathbf{i}} - \zeta_{\mathbf{i}} \mathbf{1}_{V_{\mathbf{i}}} \|^2 \tag{8.13}$$

The trajectory of the gradient flow

$$\frac{d}{dt}(B_{e,\pm}, \tilde{I}_{\mathbf{i}}, \tilde{J}_{\mathbf{i}}) = -(\nabla_{B^{\dagger}_{e,\pm}} f, \nabla_{\tilde{I}^{\dagger}_{\mathbf{i}}} f, \nabla_{\tilde{J}^{\dagger}_{\mathbf{i}}} f) \tag{8.14}$$

belongs to the $G^{\mathbb{C}}_{\underline{\mathbf{v}}}$-orbit. Indeed, (8.14) exponentiates to the transformation:

$$\exp t\mu_{\mathbf{i}} \in \mathrm{GL}(V_{\mathbf{i}}) \tag{8.15}$$

The function f decreases along the flow. In the limit $t \to \infty$ the value of f either tends to its absolute minimum, i.e. $f = 0$, which is the locus of solutions to the equations (8.7), or it stops at another critical point with the critical value $f_* > 0$. Now, the critical points with $f_* > 0$ are the configurations $(B_{e,\pm}, \tilde{I}_{\mathbf{i}}, \tilde{J}_{\mathbf{i}})$ for which the real moment map $(\mu_{\mathbf{i}})_{\mathbf{i} \in \mathsf{Vert}_\gamma}$ viewed as an element of the Lie algebra of $G^{\mathbb{C}}_{\underline{\mathbf{v}}}$ (more precisely, it is in $\mathrm{i}\,\mathrm{Lie}G_{\underline{\mathbf{v}}} \subset \mathrm{Lie}G^{\mathbb{C}}_{\underline{\mathbf{v}}}$), is a non-trivial infinitesimal symmetry, i.e.:

$$\begin{aligned} &\mu_{\mathbf{i}} \tilde{I}_{\mathbf{i}} = \zeta_{\mathbf{i}} \tilde{I}_{\mathbf{i}}, && \tilde{J}_{\mathbf{i}} \mu_{\mathbf{i}} = \tilde{J}_{\mathbf{i}} \zeta_{\mathbf{i}} \\ &\mu_{s(e)} B_{e,-} - B_{e,-} \mu_{t(e)} = (\zeta_{s(e)} - \zeta_{t(e)}) B_{e,-}, && \mu_{t(e)} B_{e,+} - B_{e,+} \mu_{s(e)} = (\zeta_{t(e)} - \zeta_{s(e)}) B_{e,+} \end{aligned} \tag{8.16}$$

Define $V'_{\mathbf{i}} = \ker(\mu_{\mathbf{i}} - \zeta_{\mathbf{i}} \mathbf{1}_{V_{\mathbf{i}}}) \subset V_{\mathbf{i}}$ for all $\mathbf{i} \in \mathsf{Vert}_\gamma$. By (8.16) these subspaces obey all the conditions of (8.10), therefore $\mu_{\mathbf{i}} = \zeta_{\mathbf{i}} \mathbf{1}_{V_{\mathbf{i}}}$ for all $\mathbf{i} \in \mathsf{Vert}_\gamma$.

In what follows we omit the subscripts γ and $\boldsymbol{\zeta}$ in the notations for the quiver variety:

$$\mathbb{M}_{\gamma, \underline{\boldsymbol{\zeta}}}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) \quad \longrightarrow \quad \mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$$

8.1.2 Symmetries of $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$

The group $\mathsf{H}_{\underline{\mathbf{w}}} = G_{\underline{\mathbf{w}}} \times \mathrm{U}(1)^{b_*(\gamma)}$ acts on $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$ by isometries. Here

$$G_{\underline{\mathbf{w}}} = \bigtimes_{\mathbf{i} \in \mathsf{Vert}_\gamma} \mathrm{U}(W_\mathbf{i}) \tag{8.17}$$

acts on the $\tilde{I}, \tilde{J}$ maps:

$$(g_\mathbf{i})_{\mathbf{i} \in \mathsf{Vert}_\gamma} : (\tilde{I}_\mathbf{i}, \tilde{J}_\mathbf{i})_{\mathbf{i} \in \mathsf{Vert}_\gamma} \mapsto (\tilde{I}_\mathbf{i} g_\mathbf{i}, g_\mathbf{i}^{-1} \tilde{J}_\mathbf{i})_{\mathbf{i} \in \mathsf{Vert}_\gamma} \tag{8.18}$$

The $\mathrm{U}(1)^{b_0(\gamma)} = \mathrm{U}(1)$-factor acts by rotating all of the $\tilde{I}_\mathbf{i}, B_{e,-}$'s while keeping $\tilde{J}_\mathbf{i}, B_{e,+}$'s intact (this definition can be extended to the disconnected quivers in a trivial fashion: rotate $\tilde{I}_\mathbf{i}, B_{e,-}$ belonging to a given connected component):

$$(\tilde{I}_\mathbf{i}, \tilde{J}_\mathbf{i}, B_{e,+}, B_{e,-}) \mapsto (u\tilde{I}_\mathbf{i}, \tilde{J}_\mathbf{i}, uB_{e,+}, B_{e,-}) \tag{8.19}$$

The group $\mathrm{U}(1)^{b_1(\gamma)} \approx \mathrm{U}(1)^{\mathsf{Edges}_\gamma} / \mathrm{U}(1)^{\mathsf{Vert}_\gamma}$ acts on the $G_{\underline{\mathbf{v}}}$-equivalence classes of the $(B_{e,\pm})$ maps:

$$(u_e)_{e \in \mathsf{Edges}_\gamma} : (B_{e,+},\ B_{e,-})_{e \in \mathsf{Edges}_\gamma} \mapsto (u_e B_{e,+},\ u_e^{-1} B_{e,-})_{e \in \mathsf{Edges}_\gamma} \tag{8.20}$$

so that the normal subgroup $\mathrm{U}(1)^{\mathsf{Vert}_\gamma}$ acts by the $G_{\underline{\mathbf{v}}}$-transformations

$$(u_\mathbf{i})_{\mathbf{i} \in \mathsf{Vert}_\gamma} : (B_{e,+},\ B_{e,-})_{e \in \mathsf{Edges}_\gamma} \mapsto (u_{s(e)}^{-1} u_{t(e)} B_{e,+},\ u_{s(e)} u_{t(e)}^{-1} B_{e,-})_{e \in \mathsf{Edges}_\gamma} \tag{8.21}$$

8.1.3 The canonical complexes and bundles

For each $\mathbf{i} \in \mathsf{Vert}_\gamma$ the vector space $V_\mathbf{i}$ descends to $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$ as a vector bundle. In addition, there are also the canonical complexes of bundles over $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$:

$$\mathcal{C}_\mathbf{i} = \left[0 \to V_\mathbf{i} \xrightarrow{d_2} W_\mathbf{i} \bigoplus_{e \in t^{-1}(\mathbf{i})} V_{s(e)} \bigoplus_{e \in s^{-1}(\mathbf{i})} V_{t(e)} \xrightarrow{d_1} V_\mathbf{i} \to 0 \right] \tag{8.22}$$

where the first and the second maps are given by:

$$d_2 = \tilde{J}_\mathbf{i} \bigoplus_{e \in t^{-1}(\mathbf{i})} (-B_{e,-}) \bigoplus_{e \in s^{-1}(\mathbf{i})} B_{e,+}\,, \qquad d_1 = \tilde{I}_\mathbf{i} \bigoplus_{e \in t^{-1}(\mathbf{i})} B_{e,+} \bigoplus_{e \in s^{-1}(\mathbf{i})} B_{e,-} \tag{8.23}$$

The moment map equation (8.7), $\mu_\mathbf{i}^\mathbb{C} = 0$, implies $d_1 \circ d_2 = 0$, hence $\mathcal{C}_\mathbf{i}$ is a complex. We set the leftmost term $V_\mathbf{i}$ in (8.22) to be in degree zero.

8.2 The bi-observables

Let

$$\mathcal{G}_x = e^{\beta x} \sum_{\mathbf{i} \in \mathsf{Vert}_\gamma} (q S_\mathbf{i}^* \mathcal{C}_\mathbf{i} + M_\mathbf{i}^* V_\mathbf{i}) \tag{8.24}$$

denote the Chern character of the $\mathsf{H}_{\underline{\mathbf{w}}}$-equivariant complex of vector bundles over $\mathcal{M}(\underline{\mathbf{n}}, \underline{\mathbf{k}}) \times \mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$:

$$\mathcal{G}_x = e^{\beta x}\, \mathrm{Ch} \bigoplus_{\mathbf{i} \in \mathsf{Vert}_\gamma} (q\, [\, S_\mathbf{i} \to \mathcal{C}_\mathbf{i}\,] \oplus [\, M_\mathbf{i} \to V_\mathbf{i}\,])$$

We identify the equivariant parameters of the $G_{\underline{\mathbf{w}}}$ group with $\underline{\boldsymbol{\nu}}$, the equivariant parameters for the U(1) factor with ε, the equivariant parameters for the $\mathrm{U}(1)^{\mathsf{Edges}_\gamma}$-group with $\mathtt{m}_e+\varepsilon$. Explicitly, the equivariant Chern character of the complex $\mathcal{C}_{\mathbf{i}}$ is equal to:

$$\mathrm{Ch}\,\mathcal{C}_{\mathbf{i}} = W_{\mathbf{i}} - V_{\mathbf{i}} - q^{-1}V_{\mathbf{i}} + \sum_{e\in t^{-1}(\mathbf{i})} q^{-1}e^{-\beta\mathtt{m}_e}V_{s(e)} + \sum_{e\in s^{-1}(\mathbf{i})} e^{\beta\mathtt{m}_e}V_{t(e)} \tag{8.25}$$

where $W_{\mathbf{i}}$ is a pure c-number character (the sum of exponents of $\underline{\boldsymbol{\nu}}$-components), while $V_{\mathbf{j}}$'s are the Chern characters of $\mathsf{H}_{\underline{\mathbf{w}}}$-equivariant bundles, i.e. may have components of positive degrees cohomology classes.

8.3 The formula

Finally, we can present the formula for $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}$. There are several ways to write it.

8.3.1 Integral over the quiver variety

$$\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}} = \sum_{\underline{\mathbf{v}}} \mathfrak{q}^{\underline{\mathbf{v}}} \int_{\mathbb{M}(\underline{\mathbf{w}},\underline{\mathbf{v}})} \epsilon_{\varepsilon_2}(T\mathbb{M}(\underline{\mathbf{w}},\underline{\mathbf{v}}))\epsilon_x(\mathcal{G}) \tag{8.26}$$

$$\mathfrak{q}^{\underline{\mathbf{v}}} = \prod_{\mathbf{i}\in\mathsf{Vert}_\gamma} \mathfrak{q}_{\mathbf{i}}^{v_{\mathbf{i}}},$$

$\epsilon_x(\mathcal{G})$ is understood as the $\mathsf{H}_{\underline{\mathbf{w}}}$-equivariant cohomology class of $\mathbb{M}(\underline{\mathbf{w}},\underline{\mathbf{v}})$: represent $\mathcal{C}_{\mathbf{i}}$ as the virtual bundle $\mathcal{C}_{\mathbf{i}}^+ - \mathcal{C}_{\mathbf{i}}^-$ over $\mathbb{M}(\underline{\mathbf{w}},\underline{\mathbf{v}})$, where $\mathcal{C}_{\mathbf{i}}^+$, $\mathcal{C}_{\mathbf{i}}^-$ are the actual bundles, with the formal Chern roots $\xi^{\pm}_{\mathbf{i},\kappa_\pm}$. Then

$$\epsilon_x(\mathcal{G}) = \prod_{\mathbf{i}\in\mathsf{Vert}_\gamma}\left(\frac{\prod_{\kappa_+}\mathcal{Y}_{\mathbf{i}}(x+\xi^+_{\mathbf{i},\kappa_+})}{\prod_{\kappa_-}\mathcal{Y}_{\mathbf{i}}(x+\xi^-_{\mathbf{i},\kappa_-})}\prod_{\kappa=1}^{v_{\mathbf{i}}} P_{\mathbf{i}}(x+\eta_{\mathbf{i},\kappa})\right) \tag{8.27}$$

where $\eta_{\mathbf{i},\kappa}$ are the formal Chern roots of $V_{\mathbf{i}}$.

For the A_1, $\widehat{A}_0$ examples we considered so far the quiver varieties $\mathbb{M}(\underline{\mathbf{w}},\underline{\mathbf{v}})$ are the cotangent bundle $T^*\mathrm{Gr}(\mathtt{v},\mathtt{w})$ to the Grassmanian of $\mathtt{v}$-planes in $\mathbb{C}^{\mathtt{w}}$ and the Hilbert scheme $\mathsf{Hilb}^{[\mathtt{v}]}(\mathbb{C}^2)$ of $\mathtt{v}$ points on $\mathbb{C}^2$, respectively.

8.3.2 Contour integral representations

Equivalently, one can write a contour integral representation for (8.26) which has the advantage of being explicit, albeit less concise:

$$\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x) = \sum_{\underline{\mathbf{v}}}\prod_{\mathbf{i}\in\mathsf{Vert}_\gamma}\frac{1}{\mathtt{v}_{\mathbf{i}}!}\left(\frac{\varepsilon\,\mathfrak{q}_{\mathbf{i}}}{2\pi\sqrt{-1}\,\varepsilon_1\varepsilon_2}\right)^{\mathtt{v}_{\mathbf{i}}}\prod_{j=1}^{\mathtt{w}_{\mathbf{i}}}\mathcal{Y}_{\mathbf{i}}(x+\varepsilon+\nu_{\mathbf{i},j})\oint_{\Gamma_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}}}}\Upsilon_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}}}(x)$$

$$\Upsilon_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}}}(x) = \prod_{e\in\mathsf{Edges}_\gamma}\Upsilon_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}};e}(x)\prod_{\mathbf{i}\in\mathsf{Vert}_\gamma}\Upsilon_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}};\mathbf{i}}(x),$$

$$\Upsilon_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}};e}(x) = \prod_{\kappa=1}^{\mathtt{v}_{t(e)}}\mathcal{Y}_{s(e)}(x+\varepsilon+\mathtt{m}_e+\phi_\kappa^{(t(e))})\prod_{\ell=1}^{\mathtt{v}_{s(e)}}\mathcal{Y}_{t(e)}(x-\mathtt{m}_e+\phi_\ell^{(s(e))}),$$

$$\Upsilon_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}};\mathbf{i}}(x) = \prod_{\kappa=1}^{\mathtt{v}_{\mathbf{i}}}\left(\frac{d\phi_\kappa^{(\mathbf{i})}\,P_{\mathbf{i}}(x+\phi_\kappa^{(\mathbf{i})})}{\mathcal{Y}_{\mathbf{i}}(x+\varepsilon+\phi_\kappa^{(\mathbf{i})})\mathcal{Y}_{\mathbf{i}}(x+\phi_\kappa^{(\mathbf{i})})}\prod_{\ell\neq\kappa}\mathbf{S}(\phi_\kappa^{(\mathbf{i})}-\phi_\ell^{(\mathbf{i})})\prod_{j=1}^{\mathtt{w}_{\mathbf{i}}}\mathbf{S}(\phi_\kappa^{(\mathbf{i})}-\nu_{\mathbf{i},j})\right). \tag{8.28}$$

The contour $\Gamma_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}}}$ is chosen in such a fashion, so as to ignore the poles coming from the zeroes of the $\mathcal{Y}$-functions in the denominator of (8.28) or the poles of $\mathcal{Y}$-functions in the numerator there. Let us assume that $\nu_{\mathbf{i},j}$ are all real, and that $\varepsilon_1, \varepsilon_2$ have positive imaginary part. We also assume that zeroes and poles of $\mathcal{Y}(x+z)$ in z are far away from the real axis. Then the contour $\Gamma_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}}} \approx \mathbb{R}^{|\underline{\mathbf{v}}|}$, i.e. all $\phi_\kappa^{(\mathbf{i})}$ are real. Now deform $\nu_{\mathbf{i},j}$ and $\varepsilon_1, \varepsilon_2$ to whatever values we desire, all the while deforming the contour $\Gamma_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}}}$ in a such a way, that the poles of (8.28) in $\phi_\kappa^{(\mathbf{i})}$'s do not cross $\Gamma_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}},\underline{\mathbf{v}}}$.

The technique to arrive from (8.28) to (8.26) is well-known, see, e.g. [74]

8.4 Five dimensional theory

The gauge theories we studied so far in four dimensions canonically lift to five dimensions, with the vector multiplets lifting to vector multiplets. The complex scalars $\mathfrak{a}_{\mathbf{i},\alpha}$ in the vector multiplet in four dimensions come from a real scalar in five dimensions and the fifth component of the gauge field. Now we compactify the theory on a circle of circumference β, and impose the twisted boundary conditions, rotating the space $\mathcal{N}$ by the angles $(-\mathrm{i}\beta\varepsilon_1, -\mathrm{i}\beta\varepsilon_2)$ in the two orthogonal two-planes $\mathbf{R}^2$ in $\mathcal{N} = \mathbf{R}^4$. In addition we perform the SU(2) R-symmetry rotation

$$\exp \frac{\mathrm{i}\beta\varepsilon}{2}\sigma_3$$

and the constant gauge transformation

$$e^{\beta\mathfrak{a}_{\mathbf{i}}} = \mathsf{diag}(e^{\beta\mathfrak{a}_{\mathbf{i},\alpha}})_{\alpha=1,\ldots,\underline{\mathbf{v}}_{\mathbf{i}}}\,.$$

The observables $\mathcal{Y}_{\mathbf{i}}(x)$ generalize to:

$$\begin{aligned}\mathcal{Y}_{\mathbf{i}}(z) &= z^{n_{\mathbf{i}}} \exp\left(-\sum_{k=1}^{\infty} \frac{1}{kz^k}\,\mathrm{Ch}\,\psi^k \mathcal{S}_{\mathbf{i}}\right) = \\ &= \mathrm{Det}\left(z - e^{\beta\Phi_{\mathbf{i}}|_0}\right)\end{aligned} \tag{8.29}$$

Again, as in the four dimensional theory, the non-perturbative effects make the naive polynomial in the right hand side of (8.29) a rational function. In particular, on the $\mathrm{U}(1)\times\mathrm{U}(1)$ invariant instanton configuration $\underline{\boldsymbol{\lambda}}$ the observable $\mathcal{Y}_{\mathbf{i}}(z)$ evaluates to:

$$\mathcal{Y}_{\mathbf{i}}(z)[\underline{\boldsymbol{\lambda}}] = \prod_{\alpha=1}^{n_{\mathbf{i}}} \frac{\prod\limits_{{\color{green}\blacksquare}\in\partial_+\lambda^{(\mathbf{i},\alpha)}} \left(z - e^{\beta(\mathfrak{a}_{\mathbf{i},\alpha}+c_{\color{green}\blacksquare})}\right)}{\prod\limits_{{\color{magenta}\blacksquare}\in\partial_-\lambda^{(\mathbf{i},\alpha)}} \left(z - qe^{\beta(\mathfrak{a}_{\mathbf{i},\alpha}+c_{\color{magenta}\blacksquare})}\right)}\,. \tag{8.30}$$

The K-theoretic version of the qq-characters is defined in a similar fashion, one should use the $\chi_{q_2^{-1}}$-genus instead of the Chern polynomial and to use push forwards in equivariant K-theory instead of the equivariant integrals. The formula (8.26) generalizes to:

$$\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(z) = \sum_{\underline{\mathbf{v}},j} \mathfrak{q}^{\underline{\mathbf{v}}} q_1^{-\widehat{n}}(-q_2)^{\widehat{n}-j} \int_{\mathbb{M}(\underline{\mathbf{w}},\underline{\mathbf{v}})} Td_{T\mathbb{M}(\underline{\mathbf{w}},\underline{\mathbf{v}})}\,\mathrm{Ch}\left(\bigwedge^j T\mathbb{M}(\underline{\mathbf{w}},\underline{\mathbf{v}})\right)\Xi_z[\mathcal{F}_{\underline{\mathbf{v}}}] \tag{8.31}$$

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where $\widehat{n} = \dim_{\mathbb{C}} \mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$

$$\Xi_z[\mathcal{F}_{\underline{\mathbf{v}}}] = \prod_{\mathbf{i} \in \mathsf{Vert}_\gamma} \left(\prod_{\kappa} P_{\mathbf{i}}(z e^{\eta_{\mathbf{i},\kappa}}) \frac{\prod_{\kappa_+} \mathcal{Y}_{\mathbf{i}}(z e^{\xi^+_{\mathbf{i},\kappa_+}})}{\prod_{\kappa_-} \mathcal{Y}_{\mathbf{i}}(z e^{\xi^-_{\mathbf{i},\kappa_-}})} \right) \tag{8.32}$$

where

$$\begin{aligned} \mathrm{Ch}(\mathcal{C}^\pm_{\mathbf{i}}) &= \sum_{\kappa^\pm} e^{\xi^\pm_{\mathbf{i},\kappa_\pm}} \\ \mathrm{Ch}(V_{\mathbf{i}}) &= \sum_{\kappa} e^{\eta_{\mathbf{i},\kappa}} \end{aligned} \tag{8.33}$$

8.5 The symmetry. $\varepsilon_1 \leftrightarrow \varepsilon_2$, $q_1 \leftrightarrow q_2$

The formulas (8.26), (8.28), (8.31) are not obviously symmetric with respect to the exchange $\varepsilon_1 \leftrightarrow \varepsilon_2$, $q_1 \leftrightarrow q_2$. However, the symmetry becomes clear once we recall that $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$ is a holomorphic symplectic manifold. Its tangent bundle is isomorphic to the cotangent bundle, the isomorphism being provided by the holomorphic symplectic form $\omega^{\mathbb{C}}$, which descends from the canonical symplectic form on $\mathcal{H}_\gamma$:

$$\sum_{e \in \mathsf{Edges}_\gamma} \mathsf{Tr}\, \delta B_e \wedge \delta \tilde{B}_e + \sum_{\mathbf{i} \in \mathsf{Vert}_\gamma} \mathsf{Tr}\, \delta \tilde{I}_{\mathbf{i}} \wedge \delta \tilde{J}_{\mathbf{i}} \tag{8.34}$$

Since the symplectic form $\omega^{\mathbb{C}}$ is scaled as $\omega^{\mathbb{C}} \to q^{-1}\omega^{\mathbb{C}}$ by the action of $\mathsf{H}_{\underline{\mathbf{w}}}$, the equivariant Chern character

$$\sum_{j=0}^{\widehat{n}} (-q_2)^{-j} \mathrm{Ch}\left(\bigwedge^j T\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) \right) = \prod_{l=1}^{\widehat{n}} (1 - e^{x_l} q_2^{-1}) = (q_1/q_2)^{\widehat{n}/2} \prod_{l=1}^{\widehat{n}} (1 - e^{-x_l} q_2) \tag{8.35}$$

which is equal to $(q_1/q_2)^{\widehat{n}/2} \prod_{l=1}^{\widehat{n}} (1 - e^{x_l} q_1^{-1})$ since every equivariant virtual Chern root x_l is paired with another equivariant virtual Chern root $\beta(\varepsilon_1 + \varepsilon_2) - x_l$. Thus,

$$\sum_{j=0}^{\widehat{n}} q_1^{-\widehat{n}} (-q_2)^{\widehat{n}-j}\, \mathrm{Ch}\left(\bigwedge^j T\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) \right) = \sum_{j=0}^{\widehat{n}} q_2^{-\widehat{n}} (-q_1)^{\widehat{n}-j}\, \mathrm{Ch}\left(\bigwedge^j T\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) \right)$$

8.6 Convergence of the integrals

The integrals (8.31) may be divergent. Indeed, the quiver varieties $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$ are non-compact. We understand the integrals (8.31) as the integrals in $\mathsf{H}_{\underline{\mathbf{w}}}$-equivariant cohomology. Practically this means that the differential form representative for the integrand in (8.31) contains a factor:

$$\exp D\left(\mathbf{g}(\cdot, V(\bar{\xi}))\right) = e^{-\mathbf{g}(V(\xi), V(\bar{\xi}))} \times (1 + \ldots) \tag{8.36}$$

Here,

$$D = \mathrm{d} + \iota_{V(\xi)} \tag{8.37}$$

is the equivariant de Rham differential, $\xi, \bar{\xi} \in \mathrm{Lie}(\mathsf{H}^{\mathbb{C}}_{\underline{\mathbf{w}}})$, ξ is the collective notation for $(\underline{\boldsymbol{\nu}}, \varepsilon)$, the equivariant parameters, $V : \mathrm{Lie}(\mathsf{H}_{\underline{\mathbf{w}}}) \to \mathrm{Vect}(\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}}))$ is the infinitesimal action of $\mathsf{H}_{\underline{\mathbf{w}}}$ on $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$, and $\mathbf{g}$ is any $\mathsf{H}_{\underline{\mathbf{w}}}$-invariant metric on $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$, in which $\mathbf{g}(V(\xi), V(\bar{\xi}))$ grows for generic ξ and $\bar{\xi} \approx \xi^*$ sufficiently fast at "infinity" of $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$.

With this convergence factor understood the integrals over $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$ converge. Moreover, the D-exactness of the exponential in (8.36) means that small variations of $\bar{\xi}$ or $\mathbf{g}$ with fixed ξ do not change the integral. The result does, however, depend on ξ. Indeed, for special values of $\xi = (\underline{\boldsymbol{\nu}}, \varepsilon)$ it diverges, as we saw in the examples (7.2). Our point is that it converges for all values of x. We shall establish this fact in full generality in [101] and [100].

8.7 Reduction to the fixed loci

The integrals (8.31), (8.26) can be computed by localization with respect to the $\mathsf{H}_{\underline{\mathbf{w}}}$-action on $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$. The isolated fixed points contribute rational expressions in $\mathcal{Y}$'s with shifted arguments, while positive dimension components of the fixed locus contribute terms with derivatives of $\mathcal{Y}$'s.

The character of the virtual tangent bundle to the quiver variety can be transformed to

$$T^{\mathrm{virt}}\mathbb{M}_{\gamma}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) \rightsquigarrow P\left(\sum_{\mathbf{i}\in\mathsf{Vert}_{\gamma}} (W_{\mathbf{i}} - V_{\mathbf{i}})V_{\mathbf{i}}^{*} + \sum_{e\in\mathsf{Edges}_{\gamma}} e^{\beta m_e} V_{t(e)} V_{s(e)}^{*}\right) \tag{8.38}$$

Indeed, the tangent bundle to $\mathbb{M}_{\gamma}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$ is equal to:

$$\begin{aligned} T\mathbb{M}_{\gamma}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) = &\sum_{\mathbf{i}\in\mathsf{Vert}_{\gamma}} \left((W_{\mathbf{i}} - V_{\mathbf{i}})V_{\mathbf{i}}^{*} + q^{-1}(W_{\mathbf{i}} - V_{\mathbf{i}})^{*}V_{\mathbf{i}}\right) \\ &+ \sum_{e\in\mathsf{Edges}_{\gamma}} \left(e^{\beta m_e} V_{t(e)} V_{s(e)}^{*} + q^{-1} e^{-\beta m_e} V_{s(e)} V_{t(e)}^{*}\right) \end{aligned} \tag{8.39}$$

in the $G_{\underline{\mathbf{w}}} \times \mathrm{U}(1)^{b_*(\gamma)}$-equivariant K-theory of $\mathbb{M}_{\gamma}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$. The virtual tangent bundle is equal to

$$T^{\mathrm{virt}}\mathbb{M}_{\gamma}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) = (1 - q_1) T\mathbb{M}_{\gamma}(\underline{\mathbf{w}}, \underline{\mathbf{v}})\,. \tag{8.40}$$

Now, dualize the terms in (8.39) proportional to q^{-1}:

$$(1 - q_1)q^{-1}T \rightsquigarrow (1 - q_1^{-1})qT^{*} = -(1 - q_1)q_2 T^{*} \tag{8.41}$$

to arrive at (8.38).

Let $\mathbf{T}_{\gamma,\underline{\mathbf{w}}}$ denote the maximal torus in $G_{\underline{\mathbf{w}}} \times \mathrm{U}(1)^{b_*(\gamma)}$. The set of $\mathbf{T}_{\gamma,\underline{\mathbf{w}}}$-fixed points $\mathbb{M}_{\gamma}(\underline{\mathbf{w}}, \underline{\mathbf{v}})^{\mathbf{T}_{\gamma,\underline{\mathbf{w}}}}$ is a union

$$\mathbb{M}_{\gamma}(\underline{\mathbf{w}}, \underline{\mathbf{v}})^{\mathbf{T}_{\gamma,\underline{\mathbf{w}}}} = \bigcup_{\underline{\mathbf{c}}} \mathbb{M}_{\gamma,\underline{\mathbf{c}}}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) \tag{8.42}$$

of connected components. Each component is a product

$$\mathbb{M}_{\gamma,\underline{\mathbf{c}}}(\underline{\mathbf{w}}, \underline{\mathbf{v}}) = \bigtimes_{\mathbf{i}\in\mathsf{Vert}_{\gamma}} \bigtimes_{\beta=1}^{\mathbf{w}_{\mathbf{i}}} \bigtimes_{n\in L} \mathbb{M}_{\gamma,\mathbf{i},c_{\mathbf{i},\beta}}(\mathbf{e}_{\mathbf{i}}, \underline{\mathbf{v}}_{\mathbf{i},\beta,n}) \tag{8.43}$$

for some $\underline{\mathbf{v}}_{\mathbf{i},\beta,n} \in \mathbb{Z}_{\geq 0}^{\mathsf{Vert}_\gamma}$ such that

$$\sum_{\mathbf{i}\in\mathsf{Vert}_\gamma} \sum_{\beta=1}^{\mathtt{w}_\mathbf{i}} \sum_{n\in L} \underline{\mathbf{v}}_{\mathbf{i},\beta,n} = \underline{\mathbf{v}} \tag{8.44}$$

Here we used a notation

$$\mathbb{M}_{\gamma,\mathbf{i},c}(\mathbf{e_i}, \underline{\mathbf{v}}) \subset \mathbb{M}_\gamma(\mathbf{e_i}, \underline{\mathbf{v}}), \quad c \in \mathsf{C}_{\mathbf{i},\underline{\mathbf{v}}} \tag{8.45}$$

for a connected component of the set of $\mathbf{T}_{\gamma,\mathbf{e_i}}$-fixed points of Nakajima quiver variety $\mathbb{M}_\gamma(\underline{\mathbf{w}}, \underline{\mathbf{v}})$ with $\underline{\mathbf{w}} = \mathbf{e_i}$:

$$\mathbb{M}_\gamma(\mathbf{e_i}, \underline{\mathbf{v}})^{\mathbf{T}_{\gamma,\mathbf{e_i}}} = \bigsqcup_{c\in\mathsf{C}_{\mathbf{i},\underline{\mathbf{v}}}} \mathbb{M}_{\gamma,\mathbf{i},c}(\mathbf{e_i}, \underline{\mathbf{v}}) \tag{8.46}$$

The vector bundles $W_\mathbf{i}$, $V_\mathbf{i}$, restricted onto each connected component $\mathbb{M}_{\gamma,\mathbf{i},c}(\mathbf{e_i}, \underline{\mathbf{v}})$ of the fixed point set split, as a sum of $\mathbf{T}_{\gamma,\underline{\mathbf{w}}}$-equivariant vector bundles:

$$\begin{aligned} W_\mathbf{i} &= \bigoplus_{\beta=1}^{w_\mathbf{i}} e^{\nu_{\mathbf{i},\beta}} \\ V_\mathbf{i} &= \bigoplus_{\beta=1}^{w_\mathbf{i}} \bigoplus_n e^{\nu_{\mathbf{i},\beta}} \otimes q^{-n}\, V_{\mathbf{i},\beta,n} \end{aligned} \tag{8.47}$$

Here n runs over the lattice of representations of $\mathrm{U}(1)^{b_*(\gamma)}$, and q^n stands for the corresponding character. In all cases except for the affine A-type quivers, q is literally $q = q_1 q_2$, and n runs through some lattice $L \subset \mathbb{Z}$. In the $\widehat{A}$-case, $q^{-n} = (q_1 q_2 e^{\mathtt{m}})^{-n_1} e^{n_2 \mathtt{m}}$ for $(n_1, n_2) \in L \subset \mathbb{Z} \oplus \mathbb{Z}$.

8.7.1 Example: the A_1 case

Recall the expression (7.3) for the A_1 qq-character corresponding to $\mathtt{w} = 2$, $\boldsymbol{\nu} = (0, 0)$. The corresponding quiver varieties $\mathbb{M}_{A_1}(2, \mathtt{v})$, with $\mathtt{v} = 0, 1, 2$ are the point, $T^*\mathbb{CP}^1$, and another point, respectively. The contribution of $\mathtt{v} = \mathtt{1}$, i.e. the integral over $T^*\mathbb{CP}^1$ reduces, by the fixed point formula, to the integral over $F = \mathbb{CP}^1$.

The vector bundle V reduces to $L^{-1} \approx \mathcal{O}(-1)$, The character-bundle (8.38) specifies to:

$$T^{\mathrm{virt}}\mathbb{M} = P(W - V)V^* = P(2L - 1)$$

the tangent bundle to ℓ is equal to

$$TF = 2L - 1$$

(check: L has two sections, while TF has three), while the complex $\mathcal{C}$ becomes $q(2 - L^{-1}) - L^{-1}$. The contribution of F to the formula (8.26) is given by:

$$\begin{aligned} &\mathfrak{q}\mathcal{Y}(x+\varepsilon)^2 \frac{\varepsilon}{\varepsilon_1\varepsilon_2} \int_\ell \left(\frac{(\omega + \varepsilon_1)(\omega + \varepsilon_2)}{\omega + \varepsilon} \right)^2 \frac{P(x - \omega)}{\mathcal{Y}(x + \varepsilon - \omega)\mathcal{Y}(x - \omega)} = \\ &\quad - \mathfrak{q}\mathcal{Y}(x+\varepsilon)^2 \frac{\varepsilon_1\varepsilon_2}{\varepsilon} \partial_x \left(\frac{P(x)}{\mathcal{Y}(x)\mathcal{Y}(x+\varepsilon)} \right) + \\ &\quad + 2\mathfrak{q}P(x) \frac{\mathcal{Y}(x+\varepsilon)}{\mathcal{Y}(x)} \left(1 - \frac{\varepsilon_1\varepsilon_2}{\varepsilon^2} \right) \end{aligned} \tag{8.48}$$

where $\omega = c_1(L)$, $\int_F \omega = 1$. It is evident that (8.48) reproduces the $\mathfrak{q}^1$ term in (7.3).

8.7.2 Example: the D_4 case

The D_4 fundamental character $\mathcal{X}_2$ provides a representative example.

Let $\underline{\mathbf{w}} = (0,1,0,0)$, $\underline{\mathbf{v}} = (1,2,1,1)$, with $\mathsf{Vert}_\gamma = \{1,2,3,4\}$. In this subsection $\mathbb{M}_4 = \mathbb{M}_{D_4}(\underline{\mathbf{w}},\underline{\mathbf{v}})$.

The character (8.38) specifies to

$$T^{\mathrm{virt}}\mathbb{M}_4 \rightsquigarrow P\left((1-V_2)V_2^* - 3 + V_2(V_1^* + V_3^* + V_4^*)\right) \tag{8.49}$$

Now, the three terms in the second line of (7.20) are coming from the isolated fixed points p_i, $i = 1,3,4$ in $\mathbb{M}_4$, with $W_2 = 1, V_2 = 1 + q^{-1}, V_i = q^{-1}$, $V_j = 1$, $j \neq i$,$j = 1,3,4$. This gives $T^{\mathrm{virt}}_{p_i}\mathbb{M}_4 \rightsquigarrow Pq = q + q^2 - qq_1 - qq_2$, which translates to the factor

$$\frac{(\varepsilon+\varepsilon_1)(\varepsilon+\varepsilon_2)}{\varepsilon\cdot 2\varepsilon} = 1 + \frac{\varepsilon_1\varepsilon_2}{2\varepsilon^2}$$

in the second line of (7.20).

The first line in (7.20) is the contribution of the non-isolated component of the fixed point set, the fixed projective line line $F = \mathbb{P}^1 \subset \mathbb{M}_4$, with $W_2 = V_1 = V_3 = V_4 = 1$, $V_2 = 1 + q^{-1}L$, where $L \approx \mathcal{O}(-1)$ is a non-trivial line bundle over F.

The corresponding complexes $\mathcal{C}_\mathbf{i}$ are given by:

$$\begin{aligned} \mathcal{C}_i &= V_2 - (1+q^{-1})V_i, \qquad i = 1,3,4 \\ \mathcal{C}_2 &= W_2 \oplus q^{-1}\left(V_1 + V_3 + V_4\right) - \left(1 + q^{-1}\right)V_2 \end{aligned} \tag{8.50}$$

For F this gives:

$$T^{\mathrm{virt}}\mathbb{M}_4|_F = P\left(2q^{-1}L - 1\right) = 2(q^{-1} + 1 - q_1^{-1} - q_2^{-1})L - 1 + q_1 + q_2 - q \tag{8.51}$$

The restriction of $\mathcal{C}_i$ onto F is given by:

$$\begin{aligned} \mathcal{C}_i &= L - 1, \qquad i = 1,3,4 \\ \mathcal{C}_2 &= 2 - (1+q^{-1})L \end{aligned} \tag{8.52}$$

The tangent bundle to F is given by

$$TF = 2L - 1 \tag{8.53}$$

The corresponding contribution to $\mathcal{X}_{2,0}$ is the integral over F of the equivariant Euler class of $T\mathbb{M}_4|_F$ with the equivariant parameter ε_1, divided by the equivariant Euler class of the virtual normal bundle $N^{\mathrm{virt}}_{F\subset\mathbb{M}_4} = T^{\mathrm{virt}}\mathbb{M}_4|_F - TF$, which is equal to

$$N^{\mathrm{virt}}_{F\subset\mathbb{M}_4} \rightsquigarrow (1-q_1)T\mathbb{M}_4 - TF \rightsquigarrow 2(q^{-1} - q_1^{-1} - q_2^{-1})L + q_1 + q_2 - q \tag{8.54}$$

times the product of $\mathcal{Y}$-observables:

$$\begin{aligned} &\int_F \frac{\varepsilon}{\varepsilon_1\varepsilon_2}\left(\frac{(\omega+\varepsilon_1)(\omega+\varepsilon_2)}{\omega+\varepsilon}\right)^2 \frac{P(x)P(x-\varepsilon-\omega)\mathcal{Y}_2(x)^2}{\mathcal{Y}_2(x-\omega)\mathcal{Y}_2(x-\omega-\varepsilon)} \prod_{i=1,3,4}\frac{\mathcal{Y}_i(x-\omega)}{\mathcal{Y}_i(x)} = \\ &PP_{-1}\frac{\mathcal{Y}_2}{\mathcal{Y}_{2,-1}}\left(2\left(1 - \frac{\varepsilon_1\varepsilon_2}{\varepsilon^2}\right) + \frac{\varepsilon_1\varepsilon_2}{\varepsilon}\partial_x \log\left(\frac{\mathcal{Y}_2\mathcal{Y}_{2,-1}}{P_{-1}\mathcal{Y}_1\mathcal{Y}_3\mathcal{Y}_4}\right)\right) \end{aligned} \tag{8.55}$$

9 More on the Physics of q-characters

Let G be a Lie group, and (R, π) its representation, i.e. π is a group homomorphism $\pi : G \to \mathrm{End}(R)$. The character $\chi_R(g)$ is a (generalized) function on G, given by the trace of the matrix $\pi(g)$ in the representation R:

$$\chi_R(g) = \mathrm{Trace}_R\, \pi(g) \tag{9.1}$$

By definition χ_R is an adjoint-invariant function, i.e. the function on the space of conjugacy classes:

$$\chi_R(g) = \chi_R(h^{-1}gh), \qquad \text{for any } h \in G \tag{9.2}$$

For the compact Lie group G the space of conjugacy classes $G/Ad(G) = T/W$ is the quotient of the maximal torus $T \subset G$ by the action of discrete group, the Weyl group.

9.1 Characters from supersymmetric quantum mechanics

A familiar realization of a character $\chi_R(e^h)$ in quantum mechanics as the partition functions of a quantum mechanical system with G-symmetry, whose space of states is the representation R and the Hamiltonian is a realization $\pi(h)$ of an element $h \in t$ of the Lie algebra $t = \mathrm{Lie}T$ of the maximal torus T.

For example, if G is a compact Lie group, and R is a unitary representation, corresponding to the highest weight $\lambda \in t^*$, then the *geometric quantization* program associates R to the symplectic manifold $\mathfrak{X} = G/K_\lambda \subset \mathbf{g}^*$, the coadjoint orbit of λ, with the canonical Kirillov-Kostant symplectic form $\omega_{\mathfrak{X}}/\hbar$. The geometric quantization realizes R as the space of holomorphic sections of the pre-quantization line bundle L over $\mathfrak{X}$ (which is a Kähler manifold), such that $c_1(L) = [\frac{\omega_{\mathfrak{X}}}{2\pi\hbar}] \in H^2(\mathfrak{X}, \mathbb{Z})$.

$$R = H^0(\mathfrak{X}, L) \tag{9.3}$$

This correspondence extends, with some friction, to a wider class of groups and representations [59].

There are various explicit formulas for the character χ_R, due to Harish-Chandra, Weyl, Kirillov, and Kac [55, 104]. For the dominant weight λ the line bundle L has vanishing higher degree cohomology, so that

$$\mathrm{Trace}_R = \sum_i (-1)^i \mathrm{Trace}_{H^i(\mathfrak{X}, L)} \tag{9.4}$$

One interpretation of the Kac-Weyl character formula is the equivariant Riemann-Roch-Grothendieck formula applied to (9.4).

Physically, one takes $(\mathfrak{X}, \omega_{\mathfrak{X}})$ as the phase space of the mechanical system. For the Hamiltonian one takes the function h defined as: for $x \in \mathfrak{X}$, $h(x) = \langle i(x), t \rangle$, where $i : \mathfrak{X} \to \mathfrak{g}^*$ is the embedding, and t is some fixed element of $t \subset \mathfrak{g}$. Then the character can be realized as the path integral [4, 5]:

$$\chi_R(g) = \int D\mathfrak{X}\, \exp \frac{\mathrm{i}}{\hbar} \oint \left(d^{-1}\omega_{\mathfrak{X}} - h(x(t))dt \right) \tag{9.5}$$

where the integral is taken over the space $L\mathfrak{X}$ of parametrized loops $x : S^1 \to \mathfrak{X}$. The loop space $L\mathfrak{X}$ is acted upon by the torus $T \times \mathrm{U}(1)$, where T acts pointwise on $\mathfrak{X}$, and U(1) acts by the loop rotations: $e^{2\pi \mathrm{i} s} \cdot x(t) = x(t+s)$.

The integral (9.5) can be evaluated exactly by the infinite-dimensional version of the Duistermaat-Heckman formula. The loop space $L\mathfrak{X}$ is viewed as the symplectic manifold with the symplectic form being the integral ("a point-wise sum")

$$\Omega = \frac{1}{2} \int_{S^1} dt \ \omega_{\mu\nu} \psi^{\mu} \psi^{\nu} \tag{9.6}$$

Then the action $\oint \left(d^{-1}\omega_{\mathfrak{X}} - h(x(t))dt \right)$ is interpreted as the Hamiltonian, generating a one-parametric subgroup in $T \times \mathrm{U}(1)$.

The character formula can be also interpreted with the help of the supersymmetric quantum mechanics on $\mathfrak{X}$, [6, 50]. Instead of X one takes the supermanifold $\mathfrak{Y} = \Pi T\mathfrak{X} \otimes T^{*}\mathfrak{X}$ (the total space of the sum of the cotangent bundle and the tangent bundle with fermionic fibers over $\mathfrak{X}$). $\mathfrak{Y}$ is endowed with the even symplectic form (as opposed to the BV formalism, where the symplectic form is odd):

$$\omega_{\mathfrak{Y}} = dp_{\mu} \wedge dx^{\mu} + g_{\mu\nu} d\psi^{\mu} \wedge d\psi^{\nu} \tag{9.7}$$

where $g_{\mu\nu}$ is a metric on $\mathfrak{X}$. We study the quantum mechanics on $\mathfrak{Y}$ with the Hamiltonian

$$H_{\mathfrak{Y}} = \frac{1}{2} g^{\mu\nu} \left(p_{\mu} - \Gamma^{\tilde{\kappa}}_{\mu\tilde{\nu}} g_{\tilde{\kappa}\tilde{\lambda}} \psi^{\tilde{\nu}} \psi^{\tilde{\lambda}} \right) \left(p_{\nu} - \Gamma^{\widehat{\kappa}}_{\nu\widehat{\nu}} g_{\widehat{\kappa}\widehat{\lambda}} \psi^{\widehat{\nu}} \psi^{\widehat{\lambda}} \right) \tag{9.8}$$

The supersymmetry is generated by the odd function

$$\mathcal{Q} = \psi^{\mu} \left(p_{\mu} - \Gamma^{\tilde{\kappa}}_{\mu\tilde{\nu}} g_{\tilde{\kappa}\tilde{\lambda}} \psi^{\tilde{\nu}} \psi^{\tilde{\lambda}} \right) \tag{9.9}$$

If the target space $\mathfrak{X}$ itself is a moduli space of solutions to some partial differential equations involving gauge fields on a d-dimensional space B^d, e.g. vortices in $d = 2$, monopoles in $d = 3$, or instantons in $d = 4$, then the quantum mechanics on $\mathfrak{X}$ is a low energy approximation to the $d+1$-dimensional gauge theory. The character (9.5) would be then given by the path integral in the theory on $\mathbb{S}^1 \times B^d$. The parameters $t \in \mathbf{g}$ in the limit of shrinking $\mathbb{S}^1$ would be interpreted as the (twisted) mass parameters for the flavor symmetry G acting on the moduli space $\mathfrak{X}$. If $d = 3$, then using the three dimensional mirror symmetry we can exchange these parameters for the Fayet-Illiopoulos parameters for the dual target space $\mathfrak{X}^{\vee}$. Then, the weight subspaces would identify with the contribution of components of fixed topology. This is almost as good as the statement of our theorem.

9.2 Gauge theory realization of the qq-characters

In this section we expand on the interpretation of qq-characters we sketched in the section 2.4 using the intersecting branes, or its string dual.

We shall mainly consider the case of affine γ. The theories corresponding to the finite dimensional A, D, E-type quivers can be viewed as limits of the affine ones, by sending some of the gauge couplings to zero. For example, the A_1 theory is a limit of $\widehat{A}_2$ theory with two out of three gauge couplings sent to zero.

As we recalled above the $\mathcal{N}=2$ quiver gauge theories with affine A, D, E quivers can be realized as the low energy limit of the theory on a stack of n $D3$-branes placed at the tip of the $\mathbb{C}^2/\Gamma$ singularity, with $\Gamma \subset \mathrm{SU}(2)$ the McKay dual finite group.

For definiteness, let $\mathcal{N} \approx \mathbb{R}^4$ denote the worldvolume of the stack of the 'physical' $D3$ branes. The six dimensional transversal slice splits as a product $\mathcal{W}/\Gamma \times \mathbb{R}^2_{\phi}$. Here $\mathcal{W} \approx \mathbb{R}^4$ is the Euclidean cover of the singularity. The fluctuations along $\mathbb{R}^2_{\phi}$ are represented, in the $D3$ theory, by the adjoint complex scalar in the vector multiplet.

We want to subject the theory to the Ω-deformation. To this purpose we choose a point $\mathbf{0} \in \mathcal{N}$, and use the symmetry of rotations about $\mathbf{0}$.

The superconformal vacuum of the theory on $D3$-branes corresponds to the branes located at the origin in $\mathcal{W}$, fixed by the Γ action and at some point p in Σ. Let us identify $\Sigma \approx \mathbb{C}$ and p with $0 \in \mathbb{C}$.

Let us now add a stack of $\mathtt{w}$ $D3$-branes located at $0 \times x \in \mathcal{N} \times \Sigma$, with the worldvolume being a copy of $\mathcal{W}/\Gamma$. Here $x \in \mathbb{C}$ is a complex number.

The low energy configurations in the combined system of $n + \mathtt{w}$ $D3$ branes split into two orthogonal stacks are labelled by some continuous and discrete parameters, such as the separation of branes along Σ and the choice of flat $\mathrm{U}(\mathtt{w})$ connection at infinity $\mathbb{S}^3/\Gamma$ of $\mathcal{W}/\Gamma$, i.e. a homomorphism $\rho : \Gamma \to \mathrm{U}(\mathtt{w})$.

When $\Gamma = 1$ is trivial, the supersymmetry of the combined system of branes is consistent with Ω-deformation, which uses the subgroup $\mathrm{SU}(2)_{\mathcal{N},L} \times \mathrm{SU}(2)_{\Delta} \times \mathrm{SU}(2)_{\mathcal{W},L}$ of the group

$$\mathrm{Spin}(4)_{\mathcal{N}} \times \mathrm{Spin}(4)_{\mathcal{W}} = \mathrm{SU}(2)_{\mathcal{N},L} \times \mathrm{SU}(2)_{\mathcal{N},R} \times \mathrm{SU}(2)_{\mathcal{W},L} \times \mathrm{SU}(2)_{\mathcal{W},R}$$

of rotations of the two orthogonal $\mathbb{R}^4$'s.

The open string Hilbert space splits as a sum of the spaces corresponding to the strings stretched between different types of D-branes. We have the $(-1)-(-1)$ strings connecting the D(-1)-instantons, we have the $(-1)-3$ strings connecting the D(-1)-instantons to the stack of n D3-branes, we have the $(-1)-3'$ strings connecting the D(-1)-instantons to the stack of $\mathtt{w}$ D3 branes. There are also the $3-3'$ open strings. In [101] we define the moduli space using the low-energy modes of these open strings.

The qq-character $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}(x)$ is simply the observable in the original theory on the stack of n $D3$-branes living along $\mathcal{N}$, which is obtained by integrating out the degrees of freedom on the transversal $\mathtt{w}$ $D3$-branes, in the vacuum corresponding to the particular $\underline{\mathbf{w}}$ and the vacuum expectation values $\underline{\boldsymbol{\nu}}$ of the scalars in the vector multiplets living on $\mathcal{W}/\Gamma$.

The integral (8.31) can be interpreted (cf. [82]) as the partition function of the supersymmetric quantum mechanics on the moduli space of Yang-Mills instantons on the ALE gravitational instantons $\widetilde{\mathbb{R}^4/\Gamma_{\gamma}}$, constructed in [64]. Here $\Gamma_{\gamma} \subset \mathrm{SU}(2)$ is the MacKay dual to γ discrete group, whose representation theory is encoded in the quiver γ: $\mathsf{Vert}_{\gamma} = \Gamma^{\vee}_{\gamma}$. The gauge group $\mathrm{U}(\mathtt{w})$, with

$$\mathtt{w} = \sum_{\mathbf{i} \in \Gamma^{\vee}_{\gamma}} \mathtt{w}_{\mathbf{i}} \dim \mathcal{R}_{\mathbf{i}}$$

is broken at infinity, by the choice of flat connection $\Gamma_\gamma \to \mathrm{U}(\mathtt{w})$, to a subgroup

$$\mathrm{U}(\mathtt{w}) \longrightarrow \bigtimes_{\mathbf{i}\in \mathsf{Vert}_\gamma} \mathrm{U}(\mathtt{w}_\mathbf{i})$$

The dimensions $\underline{\mathbf{v}}$ encode the magnetic fluxes through the exceptional two-spheres in the resolved orbifold $\widetilde{\mathbb{R}^4/\Gamma_\gamma}$, and the ordinary instanton charge

$$v_{\mathrm{tot}} = \sum_{\mathbf{i}\in\Gamma_\gamma^\vee} \mathrm{v}_\mathbf{i} \mathrm{dim}\mathcal{R}_\mathbf{i}$$

In the supersymmetric quantum mechanics the sum over $\underline{\mathbf{v}}$ is not natural, as it adds the partition functions of different Hilbert spaces. However, in the $4+1$ dimensional gauge theory on $\widetilde{\mathbb{R}^4/\Gamma_\gamma} \times \mathbb{S}^1$ the sum over $\underline{\mathbf{v}}$ is just the sum over various topological sectors which is enforced anyway by the cluster decomposition. A more careful look at (8.31) reveals that we are dealing with the maximally supersymmetric Yang-Mills theory subject to the Ω-deformation (more on it below) and coupled to a point-like source localized along the circle $\mathbb{S}^1 \times \tilde{0}$, where $\tilde{0}$ is the exceptional variety, the joint of the exceptional spheres, which arose in the resolution of singularities.

The coupling $\mathfrak{q}^{\underline{\mathbf{v}}}$ comes naturally from the usual Chern-Simons couplings on the $D4$-branes

$$\int C_1 \wedge \mathsf{Tr}\,(F - B_{NS}) \wedge (F - B_{NS}) + \int C_3 \wedge \mathsf{Tr}\,(F - B_{NS}) \tag{9.10}$$

In the IIB picture these would translate to the couplings to

$$\int_{\mathbb{S}^2_i} B_{NS} + \tau B_{RR} = \tau_i$$

as described, e.g. in [65].

Here is the general construction (the reader is invited to consult [91, 95] for details). Consider the maximal supersymmetric super-Yang-Mills theory in eight dimensions, on a noncommutative $\mathbb{R}^8 \approx \mathbb{C}^4$. One can view this theory as a particular background in the IKKT matrix model [52] with an dimensional theory with an infinite dimensional gauge group, the group of unitary operators in the Hilbert space. This theory can also be lifted to $8+1$ and $9+1$ dimensions, (also known as the Matrix theory [9] and the matrix string theory [26], respectively).

Recall that the gauge fields on the noncommutative Euclidean space $\mathbb{R}^n_\theta$ with the coordinates $\widehat{x}^\mu$, $\mu = 1, \ldots, n$, obeying

$$[\widehat{x}^\mu, \widehat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}\cdot 1$$

can be described, more conveniently, as operators

$$\widehat{X}^\mu = \widehat{x}^\mu + \theta^{\mu\nu} A_\nu(\widehat{x})$$

In the vacuum, $A_\nu = 0$ and $\widehat{X}^\mu = \widehat{x}^\mu$. The equations of motion of Yang-Mills theory on $\mathbb{R}^n_\theta$ translate to the relations on the commutators of $\widehat{X}^\mu$'s:

$$G_{\mu'\mu''}[\widehat{X}^{\mu'}, [\widehat{X}^{\mu''}, \widehat{X}^\mu]] = 0 \tag{9.11}$$

In the form (9.11) the equations of motion do not distinguish between the gauge fields and adjoint scalars, and are equally applicable both to the n-dimensional theory and to its dimensional reductions to lower dimensions. Everything is hidden in the nature of the operators $\widehat{X}^{\mu}$.

Let us now take $n = 10$, choose an identification $\mathbb{R}^{10} \approx \mathbb{C}^4 \times \mathbb{C}$, assume the metric to be Euclidean, $G_{\mu\nu} = \delta_{\mu\nu}$, and the Poisson tensor $\theta^{\mu\nu}$ to be of the $(1,1)$-type. We assume it vanishes on the last $\mathbb{C}$ factor. Define $Z^i = \widehat{X}^{2i-1} + \mathrm{i}\widehat{X}^{2i}$, $i = 1,2,3,4$, $\mathbf{\Phi} = \widehat{X}^9 + \mathrm{i}\widehat{X}^{10}$. We are interested in the supersymmetric field configurations, the generalized instantons. In the present case the relevant equations are (cf. [73]):

$$[Z^i, Z^j] + \varepsilon^{ijkl}[Z^k, Z^l]^{\dagger} = 0, \qquad i,j = 1,2,3,4 \tag{9.12}$$

$$\sum_{i=1}^{4}[Z^i, Z^{i\dagger}] = -2\theta \cdot \mathbf{1}_{\mathcal{H}} \tag{9.13}$$

and

$$[\mathbf{\Phi}, Z^i] = [\mathbf{\Phi}, Z^{i\dagger}] = 0 \tag{9.14}$$

The equations (9.12), (9.13), (9.14) imply (9.11). The Ω-deformation modifies the equations (9.14) to:

$$[\mathbf{\Phi}, Z^i] + \varepsilon_i Z^i = 0, \qquad [\mathbf{\Phi}, Z^{i\dagger}] - \varepsilon_i Z^{i\dagger} = 0 \tag{9.15}$$

The equation (9.12) involves the $(4,0)$-form $\frac{1}{4!}\varepsilon_{ijkl}dz^i \wedge dz^j \wedge dz^k \wedge dz^l$ which is only invariant under the SU(4) rotations, forcing the constraint (2.10).

Let us denote by $\mathfrak{H}$ a copy of the two-oscillator Fock space:

$$\mathfrak{H} = \bigoplus_{n_1,n_2=0}^{\infty} \mathbb{C}|\vec{n}\rangle, \tag{9.16}$$

acted upon by the creation and the annihilation operators:

$$\mathbf{A}_i^{\dagger}|\vec{n}\rangle = \sqrt{n_i+1}|\vec{n}+\mathbf{e}_i\rangle, \qquad \mathbf{A}_i|\vec{n}\rangle = \sqrt{n_i}|\vec{n}-\mathbf{e}_i\rangle, \qquad i = 1,2 \tag{9.17}$$

with $\vec{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2$, $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$. The Hilbert space $\mathfrak{H}$ is the irreducible representation of the 2-oscillators Heisenberg algebra

$$[\mathbf{A}_i, \mathbf{A}_j^{\dagger}] = \delta_{ij}, \ [\mathbf{A}_1, \mathbf{A}_2] = 0, \ i,j = 1,2$$

A simple solution to the eqs. (9.12), (9.13), (9.14) describing a stack of n parallel $D3$-branes stretched in the $\mathbb{R}^4_{1234}$ direction is given by: identify $\mathcal{H} = \mathfrak{H} \otimes N$, with N the fintie dimensional complex vector space of dimension n:

$$Z^1 = \sqrt{\theta}\mathbf{A}_1^{\dagger} \otimes \mathbf{1}_N, \qquad Z^2 = \sqrt{\theta}\mathbf{A}_2^{\dagger} \otimes \mathbf{1}_N \tag{9.18}$$

while for $i = 3,4$, $Z^i = \mathbf{1}_{\mathfrak{H}} \otimes \mathrm{diag}(z^i_{(1)}, \ldots, z^i_{(n)})$, where $z^i_{(a)}, \mathfrak{a}_a \in \mathbb{C}$, $a = 1,\ldots,n$, $i = 3,4$. The scalar $\mathbf{\Phi}$ is equal to $\mathbf{1}_{\mathfrak{H}} \otimes \mathrm{diag}(\mathfrak{a}_1,\ldots,\mathfrak{a}_n)$ in the absence of Ω-deformation, and to

$$\mathbf{\Phi} = \left(\varepsilon_1\mathbf{A}_1^{\dagger}\mathbf{A}_1 + \varepsilon_2\mathbf{A}_2^{\dagger}\mathbf{A}_2\right) \otimes \mathbf{1}_N + \mathbf{1}_{\mathfrak{H}} \otimes \mathrm{diag}(\mathfrak{a}_1,\ldots,\mathfrak{a}_n) \tag{9.19}$$

when the Ω-deformation is turned on.

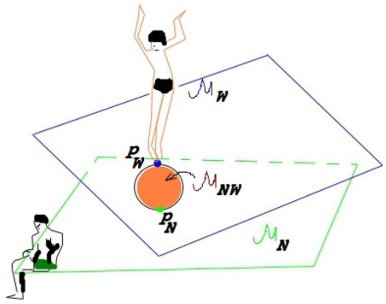

Figure 12. Charge one abelian crossed instanton moduli space.

The solution which preserves less supersymmetry has $\mathcal{H} = H_{12} \oplus H_{34}$, $H_{12} = \mathfrak{H} \otimes N$, $H_{34} = \mathfrak{H} \otimes W$, with two vector spaces N and W, of dimensions n and w, respectively and:

$$\begin{aligned} Z^i &= \sqrt{\theta}\, P_{12}\mathbf{A}_i^\dagger \otimes \mathbf{1}_N P_{12}\,, \qquad && i = 1,2 \\ Z^i &= \sqrt{\theta}\, P_{34}\mathbf{A}_{i-2}^\dagger \otimes \mathbf{1}_W P_{34}\,, \qquad && i = 3,4 \end{aligned} \tag{9.20}$$

where $P_{ij} : \mathcal{H} \to H_{ij}$ is the orthogonal projection, $P_{ij}^2 = P_{ij} = P_{ij}^\dagger$, $P_{12}P_{34} = P_{34}P_{12} = 0$, $\mathbf{1}_{\mathcal{H}} = P_{12} + P_{34}$. The scalar $\mathbf{\Phi}$ is given by

$$\mathbf{\Phi} = P_{12}\mathbf{1}_{\mathfrak{H}} \otimes \mathrm{diag}(\mathfrak{a}_1, \dots, \mathfrak{a}_n)P_{12} + P_{34}\mathbf{1}_{\mathfrak{H}} \otimes \mathrm{diag}(\nu_1, \dots, \nu_w)P_{34}$$

without Ω-deformation, and by

$$\begin{aligned} \mathbf{\Phi} = P_{12} &\left(\left(\varepsilon_1 \mathbf{A}_1^\dagger \mathbf{A}_1 + \varepsilon_2 \mathbf{A}_2^\dagger \mathbf{A}_2 \right) \otimes \mathbf{1}_N + \mathbf{1}_{\mathfrak{H}} \otimes \mathrm{diag}\left(\mathfrak{a}_1, \dots, \mathfrak{a}_n\right) \right) P_{12} + \\ P_{34} &\left(\left(\varepsilon_3 \mathbf{A}_1^\dagger \mathbf{A}_1 + \varepsilon_4 \mathbf{A}_2^\dagger \mathbf{A}_2 \right) \otimes \mathbf{1}_W + \mathbf{1}_{\mathfrak{H}} \otimes \mathrm{diag}\left(\nu_1, \dots, \nu_w\right) \right) P_{34} \end{aligned} \tag{9.21}$$

with the Ω-deformation corresponding to the generic $SU(4)$ rotation.

We are mostly interested in the solutions, which asymptotically tend to the $4+4$-dimensional background, corresponding to the intersecting branes solution (the asymptotics does not allow shifting Z^i by a constant). Let us describe the so-called 1-instanton solutions in the "abelian" case $n = w = 1$. Let e_N and e_W denote the orthonormal bases in N and W, respectively. The space of solutions has three components: $\mathcal{M}_N \cup \mathcal{M}_{NW} \cup \mathcal{M}_W$.

The components $\mathcal{M}_N, \mathcal{M}_W$ are both isomorphic to $\mathbb{C}^2$, while the component $\mathcal{M}_{NW} \approx \mathbb{CP}^1$ is compact. Moreover these components intersect, at two points: $\mathcal{M}_N \cap \mathcal{M}_{NW} = p_N$, $\mathcal{M}_W \cap \mathcal{M}_{NW} = p_W$ (it is tempting to call p_N the North pole, and p_W the Western pole, unfortunately we couldn't place the latter on the map, even on the Google map).

Explicitly, $\mathcal{M}_N$ parametrizes the solutions where the pair (Z^1, Z^2) in (9.20) is replaced by the one-instanton solution of [88] (see also [93] for more details), while $(Z^3, Z^4, \mathbf{\Phi})$ are intact. Likewise, $\mathcal{M}_W$ parametrizes the solutions where the pair (Z^3, Z^4) in (9.20) is replaced by the one-instanton solution. Recall [40, 93, 99] that these solutions make use of Murray-von Neumann partial isometries $S : \mathcal{H} \to \mathcal{H}$, which obey:

$$SS^\dagger = \mathbf{1}_{\mathcal{H}}, \qquad S^\dagger S = \mathbf{1}_{\mathcal{H}} - P_K \tag{9.22}$$

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with P_K an orthogonal projection onto a finite-dimensional subspace $K \subset \mathfrak{H}$. For the one-instanton solutions in the components $\mathcal{M}_{N,W}$ the subspace K is one-dimensional, $K = \mathbb{C}e^{u\mathbf{A}_1^\dagger + v\mathbf{A}_2^\dagger}|0,0\rangle \otimes e_{N,W}$, with $(u,v) \in \mathcal{M}_{N,W} \approx \mathbb{C}^2$ being the instanton modulus. The solutions corresponding to the component $\mathcal{M}_{NW}$ have $K = \mathbb{C}e^{\perp}$, where

$$e^{\perp} = \bar{\alpha}|0,0\rangle \otimes e_N + \bar{\beta}|0,0\rangle \otimes e_W, \tag{9.23}$$

with some $\alpha, \beta \in \mathbb{C}$,

$$|\alpha|^2 + |\beta|^2 = 1 \tag{9.24}$$

Define

$$e = \beta|0,0\rangle \otimes e_N - \alpha|0,0\rangle \otimes e_W, \qquad e^\dagger K = 0 \tag{9.25}$$

and

$$\mathfrak{H}' = \bigoplus_{n_1+n_2>0} \mathbb{C}\,|\vec{n}\rangle \subset \mathfrak{H} \tag{9.26}$$

and

$$\mathcal{H}' = \mathbb{C}e \oplus \left(\mathfrak{H}' \otimes N\right) \oplus \left(\mathfrak{H}' \otimes W\right) \tag{9.27}$$

the orthogonal complement to K. Define:

$$\begin{aligned}
\tilde{Z}^i\,|\vec{n}\rangle \otimes e_N &= \sqrt{\theta}\,g_{n+1}^{-1}g_n\sqrt{n_i+1}\,|\vec{n}+\mathbf{e}_i\rangle \otimes e_N, \qquad && i = 1,2\\
\tilde{Z}^i\,|\vec{n}\rangle \otimes e_W &= \sqrt{\theta}\,\tilde{g}_{n+1}^{-1}\tilde{g}_n\sqrt{n_{i-2}+1}\,|\vec{n}+\mathbf{e}_{i-2}\rangle \otimes e_W, \qquad && i = 3,4\\
\tilde{Z}^{1,2}\,|\vec{n}\rangle \otimes e_W &= 0, && \tilde{Z}^{3,4}\,|\vec{n}\rangle \otimes e_N = 0\\
n &= n_1+n_2 > 0, && g_n, \tilde{g}_n \in \mathbb{C}\\
\tilde{Z}^1 e &= \sqrt{\theta}\,\gamma_{12}\,|1,0\rangle \otimes e_N, && \tilde{Z}^2 e = \sqrt{\theta}\,\gamma_{12}\,|0,1\rangle \otimes e_N,\\
\tilde{Z}^3 e &= \sqrt{\theta}\,\gamma_{34}\,|1,0\rangle \otimes e_W, && \tilde{Z}^4 e = \sqrt{\theta}\,\gamma_{34}\,|0,1\rangle \otimes e_W,\\
\tilde{Z}^{i\dagger} e &= 0, && i = 1,2,3,4
\end{aligned} \tag{9.28}$$

with

$$\gamma_{12}, \gamma_{34} \in \mathbb{C}, \qquad |\gamma_{12}|^2 + |\gamma_{34}|^2 = 1, \qquad (\gamma_{12} : \gamma_{34}) \in \mathcal{M}_{NW} \tag{9.29}$$

Now define:

$$Z^i = SZ^iS^\dagger \tag{9.30}$$

where $S^\dagger$ maps $\mathcal{H}$ onto $\mathcal{H}'$ isometrically (use the Hilbert hotel construction) obeying (9.22).

The diagonal matrices $g_n, \tilde{g}_n$ are fixed, up to the unitary gauge transformations, by the equation (9.13),

$$|g_n|^2 = \frac{(n+1)!n!}{(n+\kappa_{12})!(n+1-\kappa_{12})!}, \qquad |\tilde{g}_n|^2 = \frac{(n+1)!n!}{(n+\kappa_{34})!(n+1-\kappa_{34})!} \tag{9.31}$$

where

$$\kappa_{ij}(1-\kappa_{ij}) = 2(|\gamma_{ij}|^2 - 1) \tag{9.32}$$

In the limiting cases $(\gamma_{12} : \gamma_{34}) \to (1:0) = p_W$ or $(\gamma_{12} : \gamma_{34}) \to (0:1) = p_N$ the solution (9.28) approaches the direct sum of the vacuum solution for H_{34} and one-instanton solution on H_{12} or the direct sum of the one-instanton solution for H_{34} and the vacuum solution on H_{12}, respectively.

In [100] we shall consider more general intersecting brane solutions. Let $\mathbf{6} = \binom{\mathbf{4}}{2}$, the set of 2-element subsets of $\mathbf{4}$. Fix 6 vector spaces N_A. We take the Hilbert space to be the sum

$$\mathcal{H} = \bigoplus_{A \in \mathbf{6}} H_A, \qquad H_A = \mathfrak{H} \otimes N_A \tag{9.33}$$

Define

$$Z_0^a = \sum_{A, a \in A} \mathbf{A}^\dagger_{h_A(a)+1} \otimes \mathbf{1}_{N_A} \tag{9.34}$$

so that $Z_0^a|_{H_B} = 0$ whenever $a \notin B$. This is the reference solution for the generalized instantons in the theory we call the *gauge origami* in [100, 101].

9.3 Other realizations of $\mathfrak{X}$-observables

A natural question is what is the meaning of the $\mathfrak{X}_{\mathbf{i}}(x)$ observables on the `CFT` side of the BPS/CFT-correspondence?

It might seem natural, e.g. in the AGT setup [1, 2] to assign the $\mathfrak{X}_{\mathbf{i}}(x)$-observables to the non-intersecting loops on the curve C on which one compactifies the A_1 $(0,2)$-theory, which define the α-coordinates in the system of Darboux coordinates on the moduli space of SL_2 local systems [87].

One systematic way to derive such a representation would be to start with the type IIB ten-dimensional background whose geometry is a rank 4 complex vector bundle E over a flat two-torus with an SU(4)-flat connection. One can add up to six stacks of $D5$ branes, wrapping the base torus and one of the complex rank two sub-bundles of E, invariant under the action of the product of the maximal torus $T \subset \mathrm{SU}(4)$ and the two-torus translating the base. This symmetry can be used to T-dualize the configuration of branes leading to various equivalent realizations. This direction will be explored elsewhere.

However, one may try to address the question directly within the realm of the two-dimensional conformal field theory. We know the $\mathcal{N} = 2^*$ SU(n) theory corresponds to the A_{n-1} Toda theory [120] on a two-torus $\mathbb{C}^\times/\mathfrak{q}^{\mathbb{Z}}$ with an insertion of a special vertex operator. The coupling constant b^2 of the Toda theory is determined by the ratio $\varepsilon_2/\varepsilon_1$ of two equivariant parameters. The qq-character $\mathcal{X}_{\underline{\mathbf{w}},\underline{\boldsymbol{\nu}}}$ of the $\widehat{A}_0$-theory is generated by the auxiliary $\mathcal{N} = 2^*$ theory on the transverse $\mathbb{R}^4$ with the equivariant parameters $\varepsilon_3, \varepsilon_4$. It corresponds to its own A_{w-1} Toda theory with the coupling constant $\tilde{b}^2 = \varepsilon_4/\varepsilon_3$. It would be interesting to work out the coupling between these theories generating the x-dependent contributions to the qq-character.

10 The first applications

10.1 Expansion coefficients

For the formal Laurent series $f(z^{-1})$ near $z = \infty$ we denote by $[z^n]f(z)$ the z^n coefficient:

$$[z^n]f(z) \equiv \mathrm{Coeff}_{z^n} f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\infty} f(z) \frac{dz}{z^{n+1}}\,, \tag{10.1}$$

the latter equality holding for actual functions $f(z)$.

10.2 Effective prepotentials and superpotentials

One obvious application of our formalism is the solution of the low-energy theories. Indeed, in the limit $\varepsilon_2 \to 0$ the integrals (8.26) simplify (the Chern polynomial of the tangent bundle drops out). In addition, the sum over all quiver $\underline{\mathbf{n}}$-colored partitions (6.3) is dominated by a single *limit shape* [86] which maps (8.26) to a system of difference equations for the $\mathcal{Y}_{\mathbf{i}}$-functions. These equations were studied in [86]. In this case it suffices to study the equations for the dimension vectors $\underline{\mathbf{w}}$ corresponding to the fundamental weights of $\mathfrak{g}_\gamma$.

10.3 Instanton fusion

In the quantum case where $\varepsilon_1, \varepsilon_2$ are finite we need all $\underline{\mathbf{w}}$. The equations (8.26) can be viewed as a system of Hirota difference equations, which should fix $\mathcal{Z}$ uniquely. This direction is currently investigated. Note that for the finite A-type quivers with the special choice of $\underline{\mathbf{n}}$ the related equations were found in [56] by somewhat different methods, although we couldn't match them with our equations for specific $\underline{\mathbf{w}}$. It would be very interesting to relate the algebraic structure found in [56] to the one we exhibited here.

10.4 Undressing the U(1) legs

Another application of our formalism is the reduction formula, which allows to relate the partition functions of the gauge theories with U(1) gauge factors to the partition functions of the gauge theories with these U(1)'s being treated as global symmetries. We assume the asymptotic freedom condition $\beta_{\mathbf{i}} \leq 0$ (3.13) is obeyed.

Let $\mathbf{i} \in \mathsf{Vert}_\gamma$ be the node with $n_{\mathbf{i}} = 1$. Shift the argument of $\mathcal{Y}_{\mathbf{i}}(x)$ so as to set $\mathfrak{a}_{\mathbf{i}} = 0$. Then, we have the expansion (cf. (5.20)):

$$\mathcal{Y}_{\mathbf{i}}(x) = x + \frac{\varepsilon_1 \varepsilon_2}{x} k_{\mathbf{i}} + \ldots, \qquad x \to \infty \tag{10.2}$$

There are two possibilities: either the node $\mathbf{i}$ is connected to itself by an edge $e \in s^{-1}(\mathbf{i}) \cap t^{-1}(\mathbf{i})$, or $s^{-1}(\mathbf{i}) \cap t^{-1}(\mathbf{i})$ is empty.

10.4.1 The $\widehat{A}_0$ theory

In the first case the theory is the $\mathcal{N} = 2^*$ theory. In the U(1) case it is characterized by the mass $\mathtt{m}$ of the adjoint hypermultiplet, the complexified coupling $\mathfrak{q}$ and the Ω-background parameters $(\varepsilon_1, \varepsilon_2)$. The partition function $\mathcal{Z}_{\widehat{A}_0}$ is a homogeneous function of $\mathtt{m}, \varepsilon_1, \varepsilon_2$, symmetric in $\varepsilon_1, \varepsilon_2$ and invariant under $\mathtt{m} \to -\mathtt{m} - \varepsilon$:

$$\mathcal{Z}(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) = \sum_{\lambda} \mathfrak{q}^{|\lambda|} \prod_{\square \in \lambda} \left(1 + \frac{\mathtt{m}(\mathtt{m}+\varepsilon)}{c_\square^\vee (\varepsilon - c_\square^\vee)} \right) , \tag{10.3}$$

where $c_\square^\vee = \varepsilon_1(l_\square + 1) - \varepsilon_2 a_\square$. Since for $\lambda \neq \emptyset$ there always exists a locally most south-east box $\square$ for which $l_\square = a_\square = 0$, the partition function $\mathcal{Z}(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) = 1$ for $\mathtt{m} = -\varepsilon_1$ or $\mathtt{m} = -\varepsilon_2$:

$$\mathcal{Z}(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) = 1 + \frac{(\mathtt{m}+\varepsilon_1)(\mathtt{m}+\varepsilon_2)}{\varepsilon_1 \varepsilon_2} \tilde{\mathcal{Z}}(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) \tag{10.4}$$

The normalization

$$\mathcal{Z}(\mathfrak{q}; 0 : \varepsilon_1 : \varepsilon_2) = \mathcal{Z}(\mathfrak{q}; -\varepsilon : \varepsilon_1 : \varepsilon_2) = \phi(\mathfrak{q})^{-1} \tag{10.5}$$

follows trivially from (10.3). Let us expand the character $\mathcal{X}_{1,0}(x)$ (7.6) in x near $x = \infty$:

$$\mathcal{X}_{1,0}(x) = \sum_{\lambda} \mathfrak{q}^{|\lambda|} \prod_{\square \in \lambda} \mathbf{S}(\mathtt{m} h_\square + \varepsilon a_\square) \left(x + \varepsilon + \frac{\varepsilon_1 \varepsilon_2}{x} k + \dots \right) \left(1 - \frac{\mathtt{m}(\mathtt{m}+\varepsilon)}{x^2} |\lambda| + \dots \right) \tag{10.6}$$

Recall that the formula above gives the x-expansion of an observable. It has the form $\mathcal{X}_{1,0}(x) = \mathcal{X}_{1,0}^{(0)}(x) + \mathcal{X}_{1,0}^{(1)}(x) k + \dots$, where k is our familiar observable (5.4).

Thus

$$[x^{-1}] \mathcal{X}_{1,0}(x) = \varepsilon_1 \varepsilon_2 \mathcal{Z}(\mathfrak{q}; \varepsilon_1 : -\mathtt{m} - \varepsilon : \mathtt{m}) k - \mathtt{m}(\mathtt{m}+\varepsilon) \mathfrak{q} \frac{d}{d\mathfrak{q}} \mathcal{Z}(\mathfrak{q}; \varepsilon_1 : -\mathtt{m} - \varepsilon : \mathtt{m}) \tag{10.7}$$

and the consequence of our equations (6.3) reads

$$\begin{aligned} 0 = \langle \langle \, [x^{-1}] \, \mathcal{X}_{1,0}(x) \, \rangle \rangle_{\mathfrak{q}; \mathtt{m}, \varepsilon_1, \varepsilon_2} = \\ \varepsilon_1 \varepsilon_2 \mathcal{Z}(\mathfrak{q}; \varepsilon_1 : -\mathtt{m} - \varepsilon : \mathtt{m}) \mathfrak{q} \frac{d}{d\mathfrak{q}} \mathcal{Z}(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) - \\ \mathtt{m}(\mathtt{m}+\varepsilon) \mathcal{Z}(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) \mathfrak{q} \frac{d}{d\mathfrak{q}} \mathcal{Z}(\mathfrak{q}; \varepsilon_1 : -\mathtt{m} - \varepsilon : \mathtt{m}) \end{aligned} \tag{10.8}$$

Introduce:

$$\Phi(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) = \frac{\varepsilon_1 \varepsilon_2}{(\mathtt{m}+\varepsilon_1)(\mathtt{m}+\varepsilon_2)} \log \mathcal{Z}(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) \tag{10.9}$$

For fixed $\mathfrak{q}$ it is a priori a meromorphic function on $\mathbb{CP}^2$, with possible singularities at $\varepsilon_2/\varepsilon_1 \in \mathbb{Q}_{\geq 0}$ (but not at $\mathtt{m} = -\varepsilon_1$, $\mathtt{m} = -\varepsilon_2$, cf. (10.4)). For $\mathfrak{q} = 0$, $\Phi = 0$. Then (10.8) implies:

$$\Phi(\mathfrak{q};\, \mathtt{m} : \varepsilon_1 : \varepsilon_2) = \Phi(\mathfrak{q};\, \varepsilon_1 : -\mathtt{m} - \varepsilon : \mathtt{m}) \tag{10.10}$$

which shows that Φ has no singularities in $(\mathtt{m} : \varepsilon_1 : \varepsilon_2)$ for fixed $\mathfrak{q}$, i.e. it is a constant. The normalization (10.5) then implies that $\Phi(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) = \log \phi(\mathfrak{q})$, i.e.

$$\mathcal{Z}(\mathfrak{q}; \mathtt{m} : \varepsilon_1 : \varepsilon_2) = \phi(\mathfrak{q})^{-\frac{(\mathtt{m}+\varepsilon_1)(\mathtt{m}+\varepsilon_2)}{\varepsilon_1 \varepsilon_2}} \tag{10.11}$$

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10.4.2 Other theories

In this case the node $\mathbf{i}$ is such that $s^{-1}(\mathbf{i}) \cap t^{-1}(\mathbf{i})$ is empty. The fundamental qq-character $\mathcal{X}_{\mathbf{i}}(x)$ has the following structure:

$$\mathcal{X}_{\mathbf{i}}(x) = \mathcal{Y}_{\mathbf{i}}(x+\varepsilon) + \mathfrak{q}_{\mathbf{i}}\Gamma_2(x)\mathcal{Y}_{\mathbf{i}}^{-1}(x) + \mathfrak{q}_{\mathbf{i}}\Gamma_1(x) \tag{10.12}$$

where $\Gamma_1(x), \Gamma_2(x)$ are built out of $\mathcal{Y}_{\mathbf{j}}, \mathfrak{q}_{\mathbf{j}}$ with $\mathbf{j} \neq \mathbf{i}$. For large x the functions $\Gamma_a(x)$ behave as $x^a(1+O(1/x))$, for $a = 1, 2$. The expansion in x near $x = \infty$ gives:

$$0 = [x^{-1}]\ \langle\ \mathcal{X}_{\mathbf{i}}(x)\ \rangle = \varepsilon_1\varepsilon_2(1-\mathfrak{q}_{\mathbf{i}})\mathfrak{q}_{\mathbf{i}}\frac{d}{d\mathfrak{q}_{\mathbf{i}}}\mathcal{Z}^{\text{inst}} + \mathfrak{q}_{\mathbf{i}}\mathcal{D}\mathcal{Z}^{\text{inst}} \tag{10.13}$$

where $\mathcal{D}$ is the first order differential operator in $\mathfrak{q}_{\mathbf{j}}$, with $\mathbf{j} \neq \mathbf{i}$. The equation (10.13) is the quasilinear partial differential equation of the first order, which can be solved using the method of characteristics. The solution is unique given the initial condition, which can be set at $\mathfrak{q}_{\mathbf{i}} = 0$, where the $\mathrm{U}(1)_{\mathbf{i}}$ gauge factor becomes a flavor group.

10.4.3 The linear quiver abelian theories

As an example of the application of this technique, consider the Type I theories with the A_r-type quiver, with $m_1 = n_1 = n_2 = \ldots = n_r = m_r = 1$, $m_{\mathbf{j}} = 0, 1 < \mathbf{j} < r$. The theory is characterized by the masses $\mathtt{m}_1, \mathtt{m}_r$ of the fundamental hypermultiplets, the Coulomb moduli $\underline{\mathfrak{a}} = (\mathfrak{a}_{\mathbf{i}})_{\mathbf{i}=1}^r$ (which could be traded for the masses of the bi-fundamental hypermultiplets, cf. [85]) and the couplings $\underline{\mathfrak{q}} = (\mathfrak{q}_{\mathbf{j}})_{\mathbf{j}=1}^r$. Let us introduce the "momenta" $p_i^{\pm}$, $i = 0, \ldots, r$:

$$p_i^+ = \varepsilon + \mathfrak{a}_i - \mathfrak{a}_{i+1}, \qquad p_i^- = \mathfrak{a}_i - \mathfrak{a}_{i+1}, \tag{10.14}$$

Using (7.14), (7.13), we derive:

$$\begin{aligned} 0 &= -[x^{-1}]\,\langle\ \mathcal{X}_l(x)\ \rangle = \\ &= \sum_{I\subset[0,r],\,|I|=l} z_I \left(\varepsilon_1\varepsilon_2\sum_{i\in I}\nabla_i^z \log\mathcal{Z}_{A_r}^{\text{inst}} + \sum_{i\in I, j\in[0,r]\setminus I, j<i} p_i^+ p_j^-\right). \end{aligned} \tag{10.15}$$

The solution to (10.15) is given by the simple "free-field formula":

$$\mathcal{Z}_{A_r}^{\text{inst}}(\underline{\mathfrak{a}}, \underline{\mathfrak{q}}) = \prod_{0\le j<i\le r}(1 - z_i/z_j)^{-\frac{p_i^+ p_j^-}{\varepsilon_1\varepsilon_2}} \tag{10.16}$$

Thus, we have derived the formulas conjectured in the sections C.1 and C.2 of [2]. Our derivation here differs from that in [18].

For $r = 1$ we get:

$$\mathcal{Z}_{A_1}^{\text{inst}} = (1-\mathfrak{q})^{\frac{(\varepsilon+\mathfrak{a}_0-\mathfrak{a}_1)(\mathfrak{a}_2-\mathfrak{a}_1)}{\varepsilon_1\varepsilon_2}}$$

10.4.4 The D-type theories

Let us present now an example of the D_4-type theory. This is the theory with four gauge group factors, which we shall label by $\mathbf{i} = 0, 1, 2, 3$, with the assignments: $n_0 = 2$, $n_1 = n_2 = n_3 = m_0 = 1$. The theory is characterized by four couplings $\underline{\mathfrak{q}} = (\mathfrak{q}_0, \mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3)$, and six Coulomb and mass parameters: the Coulomb parameters

$$\underline{\mathfrak{a}} = (\mathfrak{a}_{0,1} = a_1, \mathfrak{a}_{0,2} = a_2, \mathfrak{a}_{1,1} = \mathtt{m}_1, \mathfrak{a}_{2,1} = \mathtt{m}_2, \mathfrak{a}_{3,1} = \mathtt{m}_3)$$

two for the U(2) gauge group factor, and three for three U(1) factors, and the mass $\mathtt{m}_4$.

By computing $[x^{-1}]\mathfrak{X}_{\mathbf{i},0}(x)$ in (7.16) using (5.20) for $\mathbf{i} = 1, 3, 4$ we derive three first order differential equations, whose solution give:

$$\begin{aligned} \mathcal{Z}_{D_4}\left(\underline{\mathfrak{a}}; \mathtt{m}_4; \underline{\mathfrak{q}}\right) = \mathcal{Z}_{A_1}&\left(a_1, a_2; \mathtt{m}_1, \mathtt{m}_2, \mathtt{m}_3, \mathtt{m}_4; \mathfrak{q}\right) \times \\ &(1-\mathfrak{q}_1)^{\mu_1}(1-\mathfrak{q}_2)^{\mu_2}(1-\mathfrak{q}_3)^{\mu_3}(1-\mathfrak{q}_0^2\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3)^{\mu_4}\times \\ &(1-\mathfrak{q}_0\mathfrak{q}_1)^{\nu_1}(1-\mathfrak{q}_0\mathfrak{q}_2)^{\nu_2}(1-\mathfrak{q}_0\mathfrak{q}_3)^{\nu_3}(1-\mathfrak{q}_0\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3)^{\nu_4}\times \\ &(1-\mathfrak{q}_0\mathfrak{q}_1\mathfrak{q}_2)^{\kappa_3}(1-\mathfrak{q}_0\mathfrak{q}_1\mathfrak{q}_3)^{\kappa_2}(1-\mathfrak{q}_0\mathfrak{q}_2\mathfrak{q}_3)^{\kappa_1} \end{aligned} \tag{10.17}$$

where

$$\begin{aligned} \varepsilon_1\varepsilon_2\mu_j &= (\mathtt{m}_j - a_1)(\mathtt{m}_j - a_2)\,, \\ \varepsilon_1\varepsilon_2\nu_j &= (a_1+a_2+\varepsilon)(a_1+a_2+\varepsilon+\mathtt{m}_j-\mathtt{m}) - a_1a_2 - \varepsilon\mathtt{m}_4 + \mathtt{m}_j(\mathtt{m}_j-\mathtt{m}) + \sum_{1\le i<k\le 4}\mathtt{m}_i\mathtt{m}_k\,, \\ \varepsilon_1\varepsilon_2\kappa_j &= (a_1+a_2+\varepsilon-\mathtt{m}_j-\mathtt{m}_4)\left(a_1+a_2+\varepsilon+\mathtt{m}_j-\mathtt{m}\right) \\ & j = 1,\ldots,4\,, \quad \mathtt{m} = \mathtt{m}_1+\mathtt{m}_2+\mathtt{m}_3 \end{aligned} \tag{10.18}$$

and

$$\mathfrak{q} = \mathfrak{q}_0\frac{(1-\mathfrak{q}_1)(1-\mathfrak{q}_2)(1-\mathfrak{q}_3)(1-\mathfrak{q}_0^2\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3)}{(1-\mathfrak{q}_0\mathfrak{q}_1)(1-\mathfrak{q}_0\mathfrak{q}_2)(1-\mathfrak{q}_0\mathfrak{q}_3)(1-\mathfrak{q}_0\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3)} \tag{10.19}$$

10.5 Fractional instantons and quantum differential equations

The equations of the schematic form:

$$\kappa\frac{\partial}{\partial\tau_a}\mathbf{\Psi} = \widehat{H}_a(\tau)\cdot\mathbf{\Psi} \tag{10.20}$$

where a label the set of couplings and the operators $\widehat{H}_a$ on the right hand side are κ-independent, show up in mathematical physics on several occasions (Knizhnik-Zamolodchikov connection [61, 105], t-part of tt^*-connection [19], Gauss-Manin connection for exponential periods [66, 68], λ-connection associated to the solution of the WDVV equations [45, 62], and more recently, e.g. [15]). The consistency of (10.20), i.e. the flatness of the corresponding connection for any value of κ, is equivalent to two sets of equations:

$$\begin{aligned} [\widehat{H}_a, \widehat{H}_b] &= 0, \\ \frac{\partial}{\partial\tau_a}\widehat{H}_b - \frac{\partial}{\partial\tau_b}\widehat{H}_a &= 0 \end{aligned} \tag{10.21}$$

The first set of equations imply that at each value of τ one has a quantum integrable system (if the number of the operators $\widehat{H}_a$ is maximal in the appropriate sense).

In the present case the meaning of these equations is the following. We have a quantum field theory with some set of couplings τ_a in which we study a codimension two defect, which has its own couplings $\tilde{\tau}_{a,\omega}$. Differentiating the partition function of the theory with defect brings down the corresponding observable $\mathcal{O}_a$, deforming the Lagrangian. Integration over the positions of $\mathcal{O}_a$'s has a contribution of the region where $\mathcal{O}_a$ approaches the defect. When $\mathcal{O}_a$ hits the defect, it fractionalizes, and splits into the observables of the defect theory:

$$\mathcal{O}_a \sim \sum_{\omega} f^{(1)}_{a,\omega}(\tau,\tilde{\tau})\tilde{\mathcal{O}}_{a,\omega} + \sum_{\omega',\omega''} f^{(2)}_{a,\omega',\omega''}(\tau,\tilde{\tau})\tilde{\mathcal{O}}_{a,\omega'}\tilde{\mathcal{O}}_{a,\omega''} + \dots \tag{10.22}$$

The equation we derive in [102] is an example of such a relation, where the bulk operator $\mathcal{O}_a$ is, in fact, the familiar $\mathsf{Tr}\,\mathbf{\Phi}^2$, and its supersymmetric descendents (which are all equal up to the powers of ε_2 in cohomology of the Ω-deformed supersymmetry). What about other operators, such as $\mathsf{Tr}\,\Phi^k$ for $k > 2$?

The operators deforming the gauge Lagrangian by

$$\delta_{\tau}\mathbb{L} = \sum_{k>2} \frac{\tau_k}{k!} \int d^4x d^4\vartheta \mathsf{Tr}\,\Phi^k \sim \frac{1}{(k-2)!}\int \mathsf{Tr}\,\Phi^{k-2}F^2 + \dots \tag{10.23}$$

are irrelevant and lead to non-renormalizable theories. We can, nevertheless, study them by treating τ_k as formal variables (i.e. assuming some power $\tau_k^{n_k}$ of τ_k to vanish). The qq-characters are modified by the introduction of the higher times. In the A_1 case, for example, the qq-character modifies to

$$\mathcal{Y}(x+\varepsilon) + \mathcal{Y}(x)^{-1}\mathfrak{q}P(x)\exp\sum_{l=1}^{\infty}\frac{1}{l!}\tau_l x^l \tag{10.24}$$

In this way we get the realization of the W-algebra and its qq-deformation in gauge theory. We also get a new perspective on the rôle of Whitham hierarchies [46, 63] and their quantum and qq-deformations in gauge theory.

See also [13] for more applications of qq-characters in the U(1) case.

11 Discussion and open questions

Of course, the most interesting question is to extend our formalism of non-perturbative Dyson-Schwinger equations beyond the BPS limit, even beyond the realm of supersymmetric theories.

However, even in the world of moderately supersymmetric theories our approach seems to be useful. It appears that the exact computations of [24, 25] of effective superpotentials of $\mathcal{N} = 1$ theories and their gravitational descendands can be cast in the form of the non-perturbative Dyson-Schwinger identities. The precise definition of the qq-characters in $\mathcal{N} = 1$ theories will be discussed elsewhere.

Another exciting problem is to find the string theory analogue of our qq-characters and the stringy version of the large field redefinitions (see [42] for a discussion of stringy symmetries).

The considerations of this paper and its companions are local, they describe gauge theories in the vicinity of a fixed point of rotational symmetry. In [92] four dimensional $\mathcal{N} = 2$ gauge theories on the smooth toric surfaces were studied. It was found that the partition function of the gauge theory on a toric surface S has the *topological vertex* structure:

$$\mathcal{Z}_S \sim \sum_{\text{lattice}} \prod_{v \in S} \mathcal{Z}_{\mathbb{R}^4}(\text{local Coulomb parameters}, \text{local } \Omega - \text{ parameters}) \tag{11.1}$$

where the sum goes over the lattice of magnetic fluxes $H^2(S, \Lambda_{\mathrm{w}})$, the product is over the fixed points of the two-torus action on S (see [12] for the recent progress in this direction). Our generalized gauge theories involving intersecting four dimensional spacetimes naturally live on Calabi-Yau fourfolds. They describe generalized complex surfaces which may have several components with different multiplicities. It would be interesting to apply these ideas to topological strings and to topological gravity.

On a more mathematical note, let us discuss the relation of our qq-characters to the t-deformation of q-characters of [38], introduced by H. Nakajima in [75, 78–80]. His definition is basically the weighted sum of the Poincare polynomials of the $H_{\underline{\mathbf{w}},\gamma}$-fixed loci on $\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})$. Let us observe that if in the formula (8.31) we pull the $\mathcal{Y}_{\mathbf{i}}$'s and $P_{\mathbf{i}}$'s out of the integral, with some clever choice of the arguments replacing those in (8.32), the remaining integral, for each $\underline{\mathbf{v}}$ would compute

$$\sum_j (-q_2)^{-j} \chi(\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}}), \Omega^j_{\mathbb{M}(\underline{\mathbf{w}}, \underline{\mathbf{v}})}) \tag{11.2}$$

i.e. the holomorphic Poincare polynomial. In other words, if the qq-operator is viewed as the difference-differential operator on the functions $\mathcal{Y}_{\mathbf{i}}$, then the t-deformed q-character looks like its symbol. It would be interesting to develop some kind of deformation quantization scheme, allowing to compute our qq-characters using the knowledge of the t-deformed q-characters [80], and to apply them to rederive the results of [17]. The paper [58] is a step in this direction.

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