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Irregular conformal block, spectral curve and flow equations

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Abstract: Irregular conformal block is motivated by the Argyres-Douglas type of N=2 super conformal gauge theory. We investigate the classical/NS limit of irregular conformal block using the spectral curve on a Riemann surface with irregular punctures, which is equivalent to the loop equation of irregular matrix model. The spectral curve is reduced to the second order (Virasoro symmetry, SU(2) for the gauge theory) and third order (W_3 symmetry, SU(3)) differential equations of a polynomial with finite degree. The conformal and W symmetry generate the flow equations in the spectral curve and determine the irregular conformal block, hence the partition function of the Argyres-Douglas theory ala AGT conjecture.

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Contents

1 Introduction

Irregular conformal block (ICB) is closely related with Argyres-Douglas type (AD) of N=2 super conformal gauge theory in four dimensions [1]. AD type has the non-trivial infrared fixed point on the Coulomb branch and does not allow marginal deformation. Therefore, AD type of gauge theory is considered as a special class of super conformal gauge theory.

According to AGT [2], the Nekrasov partition function [3–5] of the gauge theory is equivalent to the conformal block of Liouville vertex operators in two dimensions. This connection is understood using the twisted compactification of the six dimensional $\mathcal{N} = (2,0)$ theory on a punctured Riemann surface [6–8]. In this context, the Seiberg-Witten curve of the four-dimensional theory is identified with the spectral curve of Hitchin system on the Riemann surface with regular punctures. The Hitchin system has simple poles and the residues are associated with the mass parameter of the gauge theory [9, 10]. On the

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other hand, the AD type theory is characterized in terms of irregular punctures, poles of higher order [11, 12]. Therefore, the irregular puncture is the key point to understand AD type theory.

It is noted that irregular punctures appear when the regular conformal block has the colliding limit [13, 14]. The colliding limit is a fusion of vertex operators so that multiple moments of Liouville charges of the vertex operators are present. It is like the collection of charges distributed over a small region whose collection is viewed as an idealized system of total charge, dipole, quadrupole and multi-poles. As a consequence, the irregular punctures maintain the conformal symmetry but change the conformal state. Note that a regular puncture on the Riemann surface appears due to a primary field, which indicates the primary state, eigen-state of Virasoro generator L_0. Similarly, the irregular puncture of order $(n+1)$ indicates an irregular state of rank n, which is a simultaneous eigen-state of positive Virasoro generators L_k with $n \leq k \leq 2n$. This irregular state is called a Gaiotto state [15] or a Whittaker state [16]. Rank 0 state corresponds to the regular state.

According to AGT, the conformal block provides the partition function of gauge theory. In the same way, the partition function of AD type gauge theory is given by ICB. To find ICB, we will use the property of the Penner-type random matrix model. The Penner-type matrix model is originally introduced to study the topological structure of the punctured Riemann surface [17]. It turns out that the Liouville conformal block is conveniently represented as the Penner-type matrix model, which is an equivalent way of writing of the Selberg integrals [18, 19].

There are a few merits in using the matrix model. The Penner-type matrix model is easy to apply the colliding limit which results in the irregular matrix model (IMM). From IMM one may obtain ICB by properly normalizing the partition function and compensating U(1) factor [20]. In addition, the random matrix model provides the loop equation, Ward-identity representing the conformal symmetry. The loop equation allows one to investigate the detailed structure of the spectral curve corresponding to the Hitchin system. It turns out that the spectral curve contains flow equations corresponding to the conformal and W symmetry and fixes ICB and therefore, the partition function of AD type gauge theory according to AGT. It is remarkable that one may find ICB using only the conformal and W symmetry of the theory. This paper is mainly devoted to elaborating on the relation between ICB and spectral curve through flow equations.

This paper is organized as follows. In section 2, we consider the case with su(2) gauge group which has the Virasoro symmetry. We confine ourselves to the classical/NS (Nekrasov-Shatashivili) limit [21] to present the main features of the system. NS limit is a limit where one of the deformation parameters $\epsilon_{1,2}$ of the guage theory goes to zero and the other (called ϵ) remains to be finite. In the Liouville theory sense, the classical limit is achieved by taking the limit $\epsilon = \hbar Q$ finite where the scale parameter $\hbar$ goes to zero but the background charge Q goes to infinite. One may resort to the classical/NS limit to get simplified spectral curve which still possesses the irregular punctures and the Virasoro symmetry. This spectral curve reduces to the second order differential equation of a polynomial of a finite degree N, size of the random matrix. In addition, we do not have the cut structure on the Riemann surface which usually appears at the large N limit.

However, the concept of the filling fraction (identified with the Coulomb branch parameter in the gauge theory) still remains and plays an important role in ICB. We present the explicit form of the flow equations from the spectral curve and use the flow equations to find the the irregular partition function and ICB. An explicit form of the irregular partition function and ICB are given for lower rank cases.

In section 3 we extend the idea to the case with su(3) gauge group, which has the Virasoro and W_3 symmetry. IMM of A_2 is presented, which is obtained from the Toda field theory and its colliding limit. The spectral curve is obtained and flow equations are found by identifying the Virasoro and W_3 symmetry generator. Using a similar method, we construct partition function of IMM and ICB. In section 4, we briefly summarize the relation between gauge theories and the matrix model approach. Section 5 is the conclusion and outlook. In the appendix, one can find the derivation of the loop equations of A_2 model (appendix A), representation of W_3 currents (appendix B), and perturbative method to find the moments for the flow equation (appendix C).

2 Irregular conformal block with Virasoso symmetry

2.1 Irregular matrix model and spectral curve

The irregular matrix model with Virasoro symmetry is given as the β deformed random matrix model

$$Z_{(m;n)} = \int \left(\prod_{I=1}^{N} d\lambda_I\right) \prod_{I<J} (\lambda_I - \lambda_J)^{2\beta} e^{\frac{\sqrt{\beta}}{g}\sum_I V(\lambda_I)}\,, \tag{2.1}$$

whose potential has the form

$$V(z) = c_0 \log z - \sum_{k=1}^{n}\left(\frac{c_k}{kz^k}\right) + \sum_{\ell=1}^{m}\left(\frac{c_{-\ell}\, z^{\ell}}{\ell}\right)\,. \tag{2.2}$$

The deformed parameter β is related with the Liouville screening charge $b = i\sqrt{\beta}$. In addition, a small expansion parameter g is introduced in the partition function, which is equivalent to the Liouville scaling parameter $\hbar = -2ig$; $\sqrt{\beta}/g = -2b/\hbar$. It is noted that the parameter c_k in the potential is the one denoted as $\hat{c}_k$ in [22]. We simplify the notation by deleting hat and therefore, c_k stands for old $\hbar c_k$.

The partition function (2.1) is obtained from the colliding limit of regular $(m+n+2)$-point conformal block. As a result, the potential parameter c_k is related with the Liouville charge α_a and the vertex operator position z_a as $c_k = \sum_{r=1}^{n} \hbar\alpha_r(z_r)^k$ with $k > 0$ and $c_{-\ell} = -\sum_{a=1}^{m} \hbar\alpha_a(z_a)^{-\ell}$ with $\ell > 0$. One may regard the partition function as the irregular correlation of two irregular vertex operators of rank m and n, considering the rank-n irregular operator as $I_c^{(n)}(z) = e^{2\Phi(z)}$ with $\Phi(z) = \sum_{k=0}^{n} c_k\partial_k\varphi/k!$ where $\partial_k\varphi$ stands for k-th derivative of the Liouville field. In this sense, the rank-0 irregular operator reduces to the regular vertex operator $V_\alpha(z) = e^{2\alpha\varphi(z)}$. In addition, the two -point irregular correlation is viewed as the inner product of irregular states of rank m and n. Indeed, the two-point correlation is identified with the inner product if one takes care of the proper normalization of the state and U(1) factor which arises at the colliding limit [20]. Multi-point irregular conformal block is also constructed in a straight-forward way.

The partition function has the conformal symmetry and the symmetric property is encoded in the loop equation,

$$4W(z)^2 + 4V'(z)W(z) + 2\epsilon W'(z) = f(z). \tag{2.3}$$

Here $W(z)$ is the resolvent, $W(z) = g\sqrt{\beta}\left\langle \sum_I \frac{1}{z-\lambda_I}\right\rangle$ where the bracket $\langle\cdots\rangle$ denotes the expectation value with respect to the matrix model. $f(z) = 4g\sqrt{\beta}\left\langle \sum_I \frac{V'(z)-V'(\lambda_I)}{z-\lambda_I}\right\rangle$. The loop equation (2.3) is given at the classical/NS limit. Note that the classical/NS limit is obtained when $\hbar \to 0$ and $b \to \infty$ so that $\epsilon = \hbar b$ is finite and multi-point resolvent contribution vanishes [23]. Therefore, we ignore the two-point resolvent in the original loop equation.

From the Liouville field theory point of view, the large b limit is the same as the one with the small b limit because of the duality $b \to 1/b$ is present in of the Liouville theory.[1] Therefore, the same finite $\epsilon = \hbar Q$ is obtained either at the NS limit or at the classical limit ($b \to 0$) of the Liouville theory. It is also noted that the same duality also appears in the corresponding gauge theory. The Ω deformation parameters ϵ_1 and ϵ_2 are identified as $\epsilon_1 = \hbar b$ and $\epsilon_2 = \hbar/b$ according to to AGT conjecture and $\epsilon = \epsilon_1 + \epsilon_2$. The $b \to 1/b$ duality corresponds to $\epsilon_1 \to \epsilon_2$ duality. If one of the deformation parameter vanishes while the other remains finite, the limit is called Nekrasov-Shatashivili (NS) limit, which is the same as the classical limit of the Liouville theory or the matrix model.

The loop equation (2.3) can be put in a more informative form if one uses $x = 2W + V'$,

$$x^2 + \epsilon x' + \xi_2(z) = 0\,, \tag{2.4}$$

which can be regarded as a spectral curve with z the (complex) spectral parameter. The analytic structure of the spectral curve is specified by $\xi_2(z) = -V'^2 + \epsilon V'' - f$ which has a pole of order $2n$ and a zero of order $2m$ on the Riemann surface:

$$\xi_2(z) = \sum_{k=-2m}^{2n} \frac{\Lambda_k}{z^{k+2}} - \sum_{a=-m}^{n-1} \frac{d_a}{z^{a+2}}\,, \tag{2.5}$$

where $f = \sum_{a=-m}^{n-1} \frac{d_a}{z^{a+2}}$ is used. The spectral curve indicates that one may view the mode of $\xi_2(z)$ as the conserved quantity appearing in the integrable theories [9, 10]. Indeed, if one ignores $f(z)$ (putting $d_a = 0$), $\xi_2(z)$ is given in terms of Λ_k which is a constant:

$$\Lambda_k = \epsilon(k+1)c_k - \sum_{r+s=k} c_r c_s\,. \tag{2.6}$$

The new feature in the loop equation (2.4) is that $f(z)$ has the special role in finding the partition function [22]. This role is closely related with the Virasoro symmetry. Note

[1] Note that before taking the classical/NS limit the parameter ϵ in the loop equation is $\hbar Q = 2g(\sqrt{\beta} - 1/\sqrt{\beta})$ where $Q = b + 1/b$ is the background charge of the Liouville theory. Thus, the original ϵ is invariant under the $b \to 1/b$ duality.

that $\xi_2(z)$ is identified with the expectation value of the energy momentum tensor (Virasoro current)

$$\xi_2(z) = \langle T(z) \rangle = \sum_{k \in \mathbb{Z}} \frac{\langle L_k \rangle}{z^{k+2}} \,. \tag{2.7}$$

Comparing the two, one notes that Λ_k is the eigenvalue of the Virasoro mode L_k for $n \leq k \leq 2n$ and $-2m \leq k \leq -m$. The positive mode L_k with $n \leq k \leq 2n$ applies on the ket so that the ket is the simultaneous eigenstate of L_k $(n \leq k \leq 2n)$. The negative mode applies on the bra since $L_k^\dagger = L_{-k}$ and therefore, the bra is the simultaneous eigenstate of L_k $(-2m \leq k \leq -m)$.

However, the extra mode $\Lambda_a + d_a$ ($-m \leq a \leq n-1$) present in $\xi_2(z)$ is not a simple constant due to d_a and therefore, not an eigenvalue. Instead, the mode represents the expectation value of L_a. It is noted that d_a is directly related to the partition function or free energy $F_{(m;n)} = -\hbar^2 \log Z_{(m;n)}$ [24]:

$$d_a = \begin{cases} v_a(F_{(m;n)}) & \text{if } 0 \leq k \leq n-1 \\ v_a(F_{(m;n)}) + 2\epsilon N c_k & \text{if } -m < a < 0 \\ 2\epsilon N c_{-m} & \text{if } a = -m \,, \end{cases} \tag{2.8}$$

where v_a is given as derivative with respect to the parameters c_k's:

$$v_a = \begin{cases} \sum_{s>0} s \left(c_{s+a} \frac{\partial}{\partial c_s} \right) & \text{if } 0 \leq k \leq n-1 \\ \sum_{s<0} (-s) \left(c_{s+a} \frac{\partial}{\partial c_s} \right) & \text{if } -m < a < 0 \,. \end{cases} \tag{2.9}$$

In fact, v_a in (2.9) is the representation of Virasoro mode L_a on the parameter space and d_a is the vector flow v_a of the partition function. It is, however, noted that v_a behaves differently, depending on the sign of a. When $a \geq 0$, v_a acts on the parameter space $\{c_1, \cdots, c_n\}$ of the right state and satisfies the commutation relation $[v_k, v_\ell] = -(k-\ell) v_{k+\ell}$. On the other hand, v_a with $a < 0$ acts on the parameter space $\{c_{-m}, \cdots, c_{-1}\}$ of the left state and satisfies the commutation relation $[v_k, v_\ell] = (k-\ell) v_{k+\ell}$.

The appearance of left/right representation of L_k brings an unpleasant feature. Suppose we evaluate $Z_{(0;1)}$ whose parameter space is $\{c_1\}$. The vector flow d_0 of v_0 can determine the partition function using (2.8). In fact, the partition function is easily obtained if one scales $\lambda_I \to c_1 \lambda_I$ in (2.1). This shows that the flow equation (2.8) contains the information of the parameter scaling. On the other hand, suppose one evaluates $Z_{(1:0)}$ whose parameter space is $\{c_{-1}\}$. The partition function is found if one scales $\lambda_I \to \lambda_I / c_{-1}$. However, there is no v_{-1} representation in (2.8) when $m = 1$ and we cannot fix the partition function using the flow equation. This unpleasant feature does not raise any problem in evaluating $Z_{(m;n)}$ $(m \neq 0)$ if one scales $\lambda_I^m \to \lambda_I^m / c_{-m}$ first. (See more details in subsection 2.2.)

Nevertheless, this unpleasant feature can be avoided if one takes into account of the conformal invariance of the partition function. Note that conformal transformation $\lambda_I \to 1/\lambda_I$ changes the parameter space $\{c_{-m}, \cdots, c_{-1}, c_1, \cdots, c_n; c_0\}$ into its dual space

$\{\bar{c}_{-n}, \cdots, \bar{c}_{-1}, \bar{c}_1, \cdots, \bar{c}_m; \bar{c}_0\}$ where $\bar{c}_k = -c_{-k}$ $(k \neq 0)$ and $\bar{c}_0 = c_\infty$. Here c_∞ is determined by the neutrality condition

$$c_0 + c_\infty + N\epsilon = \epsilon\,. \tag{2.10}$$

The neutrality condition is hidden in the matrix model but is manifest in the Liouville conformal block. The conformal block is evaluated using the perturbation with the insertion of N screening operators and the neutrality condition is required $\sum_a \alpha_a + Nb = Q$ so that the conformal block is non-vanishing. At the colliding limit, the neutrality condition reduces to $c_0 + c_\infty + N\hbar b = \hbar Q$ where $c_0 = \sum_{r=1}^n \hbar\alpha_r$ is the total Liouville charge (scaled by $\hbar$) at the origin and c_∞ is the charge at infinity. If one has the classical/NS limit, one has the neutrality condition (2.10).

Using this conformal symmetry, we have two left representations instead of one left and one right representations: one is the original representation v_a with $a \geq 0$ which applies to original parameter space $\{c_1, \cdots, c_n\}$ with parameters $\{c_{-m}, \cdots, c_{-1}; c_0\}$ intact. The other is $\bar{v}_a$ with $a \geq 0$ which applies to the dual parameter space $\{\bar{c}_{-n}, \cdots, \bar{c}_{-1}\}$ with $\{\bar{c}_1, \cdots, \bar{c}_m; \bar{c}_0\}$ intact. Here the barred representation $\bar{v}_a$ is defined in the same form as in (2.9) but with $\{c_k\}$ replaced with $\{\bar{c}_k\}$ and commutes with v_a; $[v_a, \bar{v}_b] = 0$. The barred partition function is the same as the original one because of the conformal invariance of the partition function. Therefore, one may find the partition function from the flow equation presented in the spectral curve and the existence of the partition function is guaranteed due to the consistency condition of d_a and its dual $\bar{d}_a$:

$$\begin{aligned}
v_a(d_b) - v_b(d_a) &= -(a-b)d_{a+b} \\
\bar{v}_a(\bar{d}_b) - \bar{v}_b(\bar{d}_a) &= -(a-b)\bar{d}_{a+b} \\
v_a(\bar{d}_b) - \bar{v}_b(d_a) &= 0\,.
\end{aligned} \tag{2.11}$$

It is simple to note that if one replaces d_a with Λ_k, then Λ_k also satisfies the consistency condition (2.11).

If one uses the flow equation to construct the partition function, the major step is to find the values of d_a and $\bar{d}_a$ directly from the analytic property of the spectral curve (2.4). In the usual large N approach [25, 26], one expands the spectral curve in powers of $\hbar$ keeping $\hbar N = O(1)$ and assuming x is $O(1)$. In this case, $\hbar Q$ is the sub-dominant order of $\hbar$ [22] and the spectral curve is given as $x = \pm\sqrt{-\xi_2}$ at the leading order. The solution results in $(m+n)$ square-root branch cuts and provides the double covering of the Riemann surface [18, 27]. The contour integral of x over a certain cut becomes an elliptic integral and is identified with the filling fraction. In this way, one can find d_k in terms of the filling fraction and the parameters c_k's. Once d_k is known, the free energy is constructed according to (2.8) and (2.9). This procedure is very interesting from the gauge theory point of view. Since the filling fraction is identified with the Coulomb branch parameter of the gauge theory according to AGT and the spectral curve is the Seiberg-Witten curve of the Hitchin system, Seiberg-Witten curve determines the partition function of the Argyres-Douglas gauge theory with the parameters $\{c_k\}$ which can be rewritten in terms of masses and Coulomb branch expectation value of the gauge theory.

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At the classical/NS limit, the procedure does not change much. Still, there appear important modifications. There are no branch cuts in the spectral curve: only N-number of simple poles are present. This fact can be seen as follows. Let us consider an expectation value

$$P(z) \equiv \left\langle \prod_I (z-\lambda_I) \right\rangle = \sum_{A=0}^{N} P_A z^A = \prod_\alpha (z - z_\alpha) \tag{2.12}$$

which is a polynomial of degree N with $P_N = 1$. z_α's are N-zeros of the polynomial. We assume all zeros are distinct. Note that $P(z)$ is related with the resolvent $W(z)$ at the classical/NS limit since

$$\log \left(\frac{P(z)}{P(z_0)} \right) = \frac{2}{\epsilon} \int_{z_0}^{z} dz' \; W(z') \,. \tag{2.13}$$

Here we use the fact the multi-point resolvent contribution vanishes at the classical/NS limit [23, 28]. Taking the derivative of (2.13), one has $2W(z) = \epsilon P'(z)/P(z)$ which can be put as

$$2W(z) = \epsilon \sum_{\alpha=1}^{N} \frac{1}{z - z_\alpha} \,. \tag{2.14}$$

Therefore, only N-simple poles appear in the spectral curve, which substitute the cuts present at the large N expansion.

The monic polynomial $P(z)$ has the central role in finding d_a. Explicitly, the spectral curve (2.4) reduces to the second order differential equation of $P(z)$: from (2.3) one has

$$\epsilon^2 P''(z) + 2\epsilon V'(z) P'(z) = f(z) P(z) \,. \tag{2.15}$$

One may expand the differential equation in power series of z and find finite number of algebraic relations of P_A's and d_a's since $P(z)$ is the polynomial of degree N. Therefore, one can find d_a's using the algebraic relations only. It should be noted that the solution is not unique. The solution depends on how the zeros of the polynomial $P(z)$ distribute around the stationary point of the potential. Therefore, the filling fraction may be applied to the distribution of the zeros even though there are no branch cuts on the Riemann surface. With the filling fraction one may fix the relevant solution of (2.15). We provide some explicit examples in the next two subsections.

2.2 Partition function $Z_{(0;n)}$

In this section we first consider the case where the potential is given as logarithmic and inverse polynomials only: $c_k = 0$ when $k < 0$. The partition function with this potential is regarded as two point correlation between one regular and one irregular vertex operator. In this case, d_0 is simply obtained if one uses large z expansion of the loop equation:

$$d_0 = 2\epsilon c_0 N + \epsilon^2 N(N-1) \,. \tag{2.16}$$

However, other values of d_k are not easy to find.

Suppose one evaluates the simplest partition function, rank $n = 1$. In this case, d_0 is enough to find the partition function if one uses the flow equation (2.8). $Z_{(0;1)} =$

$c_1^{-d_0/\hbar^2}\mathcal{N}_{(0;1)}$ where $\mathcal{N}_{(0;1)}$ is the normalization independent of c_1. The same result is obtained if one rescales λ_I in the partition function (2.1). In addition, one can find $P(z)$ explicitly since the mode expansion gives a recursive relation: $P_A = \xi_{A+1}P_{A+1}$ where $\xi_{A+1} = 2(A+1)\epsilon c_1/(d_0 - 2\epsilon c_0 A - \epsilon^2 A(A-1))$. Here $P_{N+1} = 0$ and $\xi_{N+1} = 0$ are used for notational simplicity and $P_N = 1$. The recursion relation shows that for $0 \leq A < N$

$$P_A = \prod_{\alpha=A+1}^{N} \xi_\alpha \,. \tag{2.17}$$

Suppose $N = 1$, one has $P(z) = z - z_1$ and $P_0 = \xi_1 = c_1/c_0 = -z_1$. The zero z_1 corresponds to the stationary point of the potential V. When $N > 1$, the solution P_A provides the information of N zeros around the stationary point since P_A is written as the the polynomial of N-zeros in a permutation invariant form: $P_A = \sum_{\{\alpha_i\}} z_{\alpha_1} \cdots z_{\alpha_{N-A}}$ where the index sum is ordered.

If the rank is greater than one, the solutions P_A and d_k are more complicated. To get the idea on this solution, let us consider the rank two with $N = 1$. In this case we need d_0 and d_1. d_0 is trivially given: $d_0 = 2\epsilon c_0$. On the other hand, one has $d_1 = 2\epsilon c_1 - d_0 P_0$ and d_1 is fixed by a quadratic equation:

$$d_1^2 - 2\epsilon c_1 d_1 + 2\epsilon c_2 d_0 = 0 \,. \tag{2.18}$$

One has two solutions: $d_1^{\pm} = \epsilon c_1 \left(1 \pm \sqrt{1-\eta}\right)$ where $\eta = 4c_2c_0/c_1^2$. Note that two solutions shows that $P_0^- \sim c_1/c_0$ and $P_0^+ \sim c_2/c_1$, each of which lies near one of two stationary points of the potential. This two different solutions are due to the fact that zeros of the polynomial (or the poles of the resolvent) distribute around two different stationary points of the potential. One may classify the solutions in terms of the filling fraction. Therefore, if one integrates the flow equation (2.8), one has the different free energy depending on the filling fraction,

$$F^{\mp}_{(0;2)} = \epsilon c_0 \left(\log c_2 - \frac{1}{8} \int^{\eta} \frac{1 \mp \sqrt{1-x}}{x^2} dx \right) . \tag{2.19}$$

For general N, one can easily convince that P_{N-1} has $N+1$ solutions. The solutions correspond to the zero distribution of $P(z)$ so that $N = N_1 + N_2$ where N_1 zeros at one stationary point and N_2 at the other. However, it is not easy to find the exact form of d_a when N is large and one may resort to perturbative expansion. One way to find the solution is using the power series of ϵ with ϵ small. Note that $W(z) = O(\epsilon)$ and $f(z) = O(1)$ while $V(z) = O(1)$ from the definition. In this case, we may apply the ϵ expansion to the loop equation (2.3) directly. Denoting $W(z) = \sum_{k\geq 1} \epsilon^k W^{(k)}(z)$ and $f(z) = \sum_{k\geq 1} \epsilon^k f^{(k)}(z)$, one has the leading order contribution

$$2W^{(1)}(z) = \frac{f^{(1)}}{2V'} \,. \tag{2.20}$$

This shows that the poles of the resolvent is located closely to the stationary points of the potential at the leading order. This is consistent with the expectation that zeros of the polynomial $P(z)$ are accumulated around the stationary points. Denoting N_k for the filling fraction around the stationary point ξ_k of the potential, one obtains the identity

$$N_k = \oint_{\mathcal{A}_k} \frac{f^{(1)}}{2V'} dz \tag{2.21}$$

where $\mathcal{A}_k$ is the contour encircling ξ_k only. Since the number of stationary point is the same as the number of d_a, one can fix d_a at $O(\epsilon^1)$. At the second order of the loop equation, one has the identity

$$0 = \oint_{\mathcal{A}_k} dz \left\{ \frac{f^{(2)}}{2V'} - \frac{(f^{(1)})'}{4(V')^2} + \frac{(2V'' - f^{(1)})f^{(1)}}{8(V')^3} \right\} . \tag{2.22}$$

Here the identity assumes that the contour $\mathcal{A}_k$ encircles all the poles of the resolvent corresponding to the fractional number N_k. In this way, one finds d_a order by order in ϵ and therefore, the partition function.

For the case rank 2 we have d_1:

$$\begin{aligned} d_1 =& -2\epsilon c_0(N_2\xi_1 + N_1\xi_2) \\ &+ \epsilon^2 \frac{\xi_1\xi_2}{(\xi_1-\xi_2)^2} \left[N_2(N_2-1)(2\xi_1-\xi_2) - N_1(N_1-1)(\xi_1-2\xi_2) - 2N_1N_2(\xi_1+\xi_2)\right] + \mathcal{O}(\epsilon^3) \end{aligned} \tag{2.23}$$

where ξ_1, ξ_2 are two stationary points satisfying $c_0z^2 + c_1z + c_2 = c_0(z-\xi_1)(z-\xi_2)$. Using the flow equations $d_0 = v_0\left(F_{(0:2)}\right)$ and $d_1 = v_1\left(F_{(0:2)}\right)$, we get the free energy $F_{(0:2)}$. The partition function is given up to $\mathcal{O}(\epsilon^2)$,

$$\begin{aligned} Z_{(0;2)} =& c_2^{-\frac{\epsilon}{\hbar^2}c_0(N_1+N_2) - \frac{\epsilon^2}{\hbar^2}\frac{3}{4}(N_1(N_1-1)+N_2(N_2-1))} \left(c_1^2 - 4c_0c_2\right)^{\frac{\epsilon^2}{4\hbar^2}(N_1(N_1-1)+N_2(N_2-1)-4N_1N_2)} \\ &\times \left(\frac{c_1 + \sqrt{c_1^2 - 4c_0c_2}}{c_1 - \sqrt{c_1^2 - 4c_0c_2}}\right)^{-\frac{\epsilon}{\hbar^2}c_0(N_1-N_2) - \frac{3}{4}\frac{\epsilon^2}{\hbar^2}(N_1(N_1-1)+N_2(N_2-1))} \\ &\times e^{-\frac{\epsilon}{\hbar^2}\frac{c1}{2c_2}\left(\sqrt{c_1^2-4c_0c_2}(N_2-N_1)+c_1(N_1+N_2)\right)} \end{aligned} \tag{2.24}$$

For the rank n with N zeros, one has $\frac{(N+n-1)!}{N!(n-1)!}$ solutions. One may view this solutions as the zero distribution with $N = \sum_{i=1}^{n} N_i$ with N_i zeros around each stationary point of the potential. Considering this zero distribution, one may expect that the solutions can be obtained perturbatively using parameter ratio set $\{\frac{c_1}{c_0}, \frac{c_2}{c_1}, \cdots, \frac{c_n}{c_{n-1}}\}$. Each ratio stands for each stationary point. The same conclusion also holds for $Z_{(n;0)} = \bar{Z}_{(0;n)}$ if one works with the barred notations.

2.3 Partition function $Z_{(m;n)}$ and ICB

If one considers the case rank $(m;n)$, the potential contains finite Laurent series. One may evaluate the flow equations by finding d_a's. The special feature of the partition function

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$Z_{(m;n)}$ is that one may evaluate ICB of irregular vertex of rank m and n. In this case, ICB is identified with the inner product $\langle I^{(m)}|I^{(n)}\rangle$ of two irregular states of rank m and n. The relation of ICB with $Z_{(m;n)}$ is given in [20].

$$\mathcal{F}_\Delta^{(m:n)}(\{c_{-k}\}:\{c_k\}) = \frac{e^{\zeta_{(m:n)}} Z_{(m;n)}(c_0;\{c_\ell\})}{Z_{(0:n)}(c_0;\{c_k\}) Z_{(0:m)}(\bar{c}_0;\{\bar{c}_k\}))}\,. \tag{2.25}$$

$Z_{(m;n)}$ is divided by $Z_{(0;n)}$ and $Z_{(0;m)}$ to give the proper normalization. The extra factor $e^{\zeta_{(m:n)}}$ comes when m vertex operators are put at infinity and n operators at the origin. Explicit result is given as $\hbar^2\zeta_{(m;n)} = -\sum_k^{\min(m,n)} 2(c_k c_{-k})/k$. Therefore, it is obvious that ICB is exponentiated and the exponent should be inversely proportional to $\hbar^2$ [29], considering the $\hbar$ dependence of the free energies and $\zeta_{(m;n)}$.

For example, ICB $\mathcal{F}_\Delta^{(1;1)}$ is obtained directly using the relation (2.25). $Z_{(0;1)}(c_0;c_1)$ can be easily obtained from the flow equation $v_0\left(F_{(0;1)}\right) = d_0$ and its dual $Z_{(0;1)}(\bar{c}_0;\bar{c}_1)$:

$$Z_{(0;1)} Z_{(0;1)} = c_{-1}^{(\epsilon N(\epsilon N+2c_0)-\epsilon^2 N)/\hbar^2}\, \eta_0^{-(\epsilon N_0(\epsilon N_0+2c_0)-\epsilon^2 N_0)/\hbar^2} \tag{2.26}$$

where $\eta_0 \equiv c_1 c_{-1}$. To evaluate $Z_{(1:1)}$, we first obtain c_{-1} dependence by rescaling $\lambda_I \to \lambda_I/c_{-1}$: $Z_{(1;1)} = c_{-1}^{(\epsilon N(\epsilon N+2c_0)-\epsilon^2 N)/\hbar^2} \widetilde{Z}_{(1;1)}$. $\widetilde{Z}_{(1:1)}$ is the partition function with the potential $\widetilde{V}(z) = c_0 \log z + z - \eta_0/z$, which is to be evaluated from the flow equation using $d_0 = v_0\left(-\hbar^2 \log \widetilde{Z}_{(1;1)}\right)$. We use the ϵ expansion to obtain the parameter dependence of d_0:

$$d_0 = -2\epsilon(N_0\xi_1 + N_\infty\xi_2) + \epsilon^2 \frac{N_0(N_0-1)\xi_1^2 - 4N_0N_\infty\xi_1\xi_2 + N_\infty(N_\infty-1)\xi_2^2}{(\xi_1-\xi_2)^2} + \mathcal{O}(\epsilon^3) \tag{2.27}$$

where ξ_1 and ξ_2 are stationary points of $\widetilde{V}(z)$. The flow equation shows that

$$\begin{aligned} Z_{(1;1)} = {}& c_{-1}^{\frac{1}{\hbar^2}(\epsilon N(\epsilon N+2c_0)-\epsilon^2 N)}\, \eta_0^{-\frac{\epsilon^2}{2\hbar^2}(N_0^2-N_0+N_\infty^2-N_\infty)} \left(\frac{c_0+\sqrt{c_0^2-4\eta_0}}{c_0-\sqrt{c_0^2-4\eta_0}}\right)^{-\frac{\epsilon^2}{2\hbar^2}(N_\infty^2-N_\infty-N_0^2+N_0)} \\ & \times \left(c_0-\sqrt{c_0^2-4\eta_0}\right)^{-(2\epsilon c_0 N_0)/\hbar^2} \left(c_0+\sqrt{c_0^2-4\eta_0}\right)^{-(2\epsilon c_0 N_\infty)/\hbar^2} \\ & \times \left(c_0^2-4\eta_0\right)^{-\frac{\epsilon^2}{4\hbar^2}(N_0^2-N_0-N_\infty^2+N_\infty+4N_0N_\infty)}\, e^{-\frac{2\epsilon}{\hbar^2}(N_0-N_\infty)\sqrt{c_0^2-4\eta_0}}\,, \end{aligned} \tag{2.28}$$

up to $\mathcal{O}(\epsilon^2)$. From the result, one can obtain $\mathcal{F}_\Delta^{(1;1)}$. We provide $\mathcal{F}_\Delta^{(1:1)}$ in powers of η_0 and ϵ:

$$\begin{aligned} \mathcal{F}_\Delta^{(1;1)} = {}& \left[1 - 2\frac{\eta_0}{\hbar^2} + 2\frac{\eta_0^2}{\hbar^4} + \mathcal{O}(\eta_0^3)\right] + \epsilon\left[\frac{2(N_0-N_\infty)}{c_0}\frac{\eta_0}{\hbar^2} - \frac{(4c_0^2-\hbar^2)(N_0-N_\infty)}{c_0^3}\frac{\eta_0^2}{\hbar^4} + \mathcal{O}(\eta_0^3)\right] \\ & - \epsilon^2\left[\frac{2(N_0^2-N_0-2N_0N_\infty)}{c_0^2}\frac{\eta_0}{\hbar^2} + \left\{\frac{\hbar^2\left(7N_0(N_0-1)+N_\infty(N_\infty-1)-16N_0N_\infty\right)}{2c_0^4}\right.\right. \\ & \left.\left. - \frac{2\left(N_0(3N_0-2)+N_\infty^2-6N_0N_\infty\right)}{c_0^2}\right\}\frac{\eta_0^2}{\hbar^4} + \mathcal{O}(\eta_0^3)\right] + \mathcal{O}(\epsilon^3) \end{aligned} \tag{2.29}$$

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One may also evaluate ICB directly using the perturbative approach of IMM as noted in [20]. This is because the relation (2.25) shows that one needs $Z_{(m;n)}$ with proper normalization and compensates by the U(1) factor. For the perturbative approach one may divide the potential matrix with N eigenvalues into the one with N_0 eigenvalues and the other with N_∞ so that $N = N_0 + N_\infty$. The normalization $Z_{(0;n)}$ is obtained if one uses $V_{(0:n)}$ instead of $V_{(m:n)}$ with N_0 eigenvalues in $Z_{(m;n)}$. Therefore, one may consider the potential $V_{(m;n)}$ with N_0 eigenvalues composed of $V_{(0:n)}$ and the rest so that $V_{(0:n)}$ is treated as the reference potential V_0 and the rest as the perturbative one ΔV_0:

$$V_0 = \sum_{I=1}^{N_0}\left(c_0 \log\lambda_I - \sum_{k=1}^{n}\frac{c_k}{k\lambda_I^k}\right), \qquad \Delta V_0 = \sum_{I=1}^{N_0}\left(\sum_{\ell=1}^{n}\frac{c_{-\ell}}{\ell}\lambda_I^\ell\right). \tag{2.30}$$

Likewise, the potential $V_{(m;n)}$ with N_∞ eigenvalues composed of $V_{(m;0)}$ as the reference potential V_∞ and the perturbation ΔV_∞ The partition function may be rewritten using the conformal transformation $\lambda_J \to 1/\mu_J$ so that one can use the dual form

$$V_\infty = \sum_{J=1}^{N_\infty}\left(\bar{c}_0 \log\mu_J - \sum_{\ell=1}^{m}\frac{\bar{c}_\ell}{\ell\mu_J^\ell}\right), \qquad \Delta V_\infty = \sum_{J=1}^{N_\infty}\left(\sum_{k=1}^{n}\frac{\bar{c}_{-k}}{k}\mu_J^k\right). \tag{2.31}$$

Once the normalization is done, the cross term $\prod_{I,J}(1-\lambda_I\mu_J)^{2\beta}$ remains from the Vandermonde determinant. Therefore, the perturbative approach is to evaluate the expectation value $\left\langle \prod_{I,J}(1-\lambda_I\mu_J)^{2\beta}\exp\left(\frac{\sqrt{\beta}}{g}(\Delta V_0(\lambda_I)+\Delta V_\infty(\mu_J))\right)\right\rangle_0$ where $\langle\cdots\rangle_0$ refers to the expectation value with respect to the reference potential. Practically, one may evaluate the expectation values from the loop equations of the partition functions $Z_{(0:n)}$ and $Z_{(0:m)}$ by using the large z expansion of the resolvents. However, this perturbative approach needs additional identities for multi-point resolvent correlations even at the classical/NS limit [22, 35]. We have $\mathcal{F}_\Delta^{(1;1)}$ at the classical/NS limit from the result obtained in [20]:

$$\mathcal{F}_\Delta^{(1;1)} = 1 + \left[\frac{\ell_1\ell_{-1}}{2\hbar^2\Delta}\right] + \left[\frac{\ell_1^2\ell_{-1}^2(2\Delta-\hbar^2)}{16\hbar^4\Delta^3} + \frac{(3\ell_1^2-4\Delta\ell_2)(3\ell_{-1}^2-4\Delta\ell_{-2})}{16\hbar^2\Delta^2(4\Delta+3\epsilon^2)}\right] + \mathcal{O}(\eta_0^3). \tag{2.32}$$

where $\Delta = (c_0+\epsilon N_0)(\epsilon-(c_0+\epsilon N_0))$, $\ell_1 = 2c_1(\epsilon-c_0)$ and $\ell_2 = -c_1^2$. ℓ_{-1} and ℓ_{-2} are its duals. It is noted that Δ and ℓ_k correspond to the modes appearing in $\xi_2(z)$ and can be represented as the expectation values between regular and irregular state: $\Delta = \langle\Delta|L_0|I^{(1)}\rangle/\langle\Delta|I^{(1)}\rangle$ where $\langle\Delta|I^{(1)}\rangle$ is $Z_{(0:1)}$. $\ell_{+k} = \langle\Delta|L_{+k}|I^{(1)}\rangle/\langle\Delta|I^{(1)}\rangle$, $\ell_{-k} = \langle I^{(1)}|L_{-k}|\Delta\rangle/\langle I^{(1)}|\Delta\rangle$ for $k = 1, 2$ and $\langle I^{(1)}|\Delta\rangle$ is $\bar{Z}_{(0:1)}$. This identification clearly shows that $\mathcal{F}_\Delta^{(1;1)}$ satisfies the dual symmetry observed in section 2.1.

The result (2.32) is given in power of $\eta_0 = c_1c_{-1}$; each squared bracket in (2.32) corresponds to each order of η_0. One can easily check that the squared bracket term is consistent with the one appeared in [37] if one uses the series expansion in $1/\hbar^2$ at each order of η_0^k and ignores higher order than $1/\hbar^{2k}$. This shows that $\mathcal{F}_\Delta$ in powers of $1/\hbar^2$ should be considered as the formal series. This fact is already suggested by Zamolodchikov and Zamolodchikov [38] for the regular conformal block and investigated fully in [29] for the irregular conformal block: at the classical/NS limit, the (ir)-regular conformal block should behave as $\mathcal{F}_\Delta \to \exp(-F/\hbar^2)$ where the free energy F is finite as $\hbar \to 0$.

3 Irregular conformal block with W_3 symmetry

3.1 Irregular matrix model and loop equation

The irregular matrix model with W_3 symmetry can be derived from Toda field theories at the colliding limit [30, 36]. The simplest matrix model is obtained from A_2 Toda theory. The A_2 irregular matrix model is two matrix model with potential V_1 and V_2:

$$\mathcal{Z}_{(m;n)} = \int \prod_{i=1}^{N}\prod_{j=1}^{M} dx_i dy_j \Delta(x)^{2\beta}\Delta(y)^{2\beta}\Delta(x,y)^{-\beta} e^{\frac{\sqrt{\beta}}{g}\left[\sum_{i=1}^{N} V_1(x_i)+\sum_{j=1}^{M} V_2(y_j)\right]}, \tag{3.1}$$

where $\Delta(x)=\prod_{i<k}(x_i-x_k)$ and $\Delta(x,y)=\prod_{i,j}(x_i-y_j)$ are Vandermonde determinants. β is the deformed parameter and conveniently put $\sqrt{\beta} = -ib$ as in the Virasoro case. When $\beta = 1$, the model reduces to hermitian two-matrix model and the powers of the Vandermonde determinant correspond to the A_2 Dynkin index. The potential with rank $(m;n)$ has the explicit form

$$\begin{aligned} V_1(z) &= b_0 \log z - \sum_{k=1}^{n}\frac{b_k}{kz^k} + \sum_{k=1}^{m}\frac{b_{-k}z^k}{k}, \\ V_2(z) &= a_0 \log z - \sum_{k=1}^{n}\frac{a_k}{kz^k} + \sum_{k=1}^{m}\frac{a_{-k}z^k}{k}. \end{aligned} \tag{3.2}$$

To obtain the matrix model we work with the primary operator $V_a(z_a) = e^{\vec{\alpha}_a\cdot\vec{\varphi}(z_a)}$ where Toda field $\vec{\varphi}$ has the orthogonal components $\vec{\varphi} = \varphi_1\frac{(1,1,-2)}{\sqrt{6}} + \varphi_2\frac{(1,-1,0)}{\sqrt{2}}$ and satisfy the free field correlation $\varphi_i(z,\bar{z})\varphi_j(w,\bar{w}) \sim -\delta_{ij}\log|z-w|^2$. The conformal block with $n+m+2$ primary operators are considered with N screening operators of the type $e^{b\vec{e}_1\cdot\vec{\varphi}(x_i)}$ and M of type $e^{b\vec{e}_1\cdot\vec{\varphi}(x_i)}$. Here $\vec{e}_1$ and $\vec{e}_2$ are two root vectors of A_2. The conformal block is non-vanishing if the neutrality condition holds [39]:

$$\vec{\alpha}_\infty + \sum_a \vec{\alpha}_a + bN\vec{e}_1 + bM\vec{e}_2 = 2Q\vec{\rho}, \tag{3.3}$$

where $Q = b + 1/b$ is the background charge and $\vec{\rho} = \vec{e}_1 + \vec{e}_2$ is the Weyl vector.

The correlation between screening operators provides the Vandermonde determinant with powers related to the Dynkin index of the two roots. The potential is obtained if one has the colliding limit of the colliding limit of the primary operators. If the Toda momentum of the primary operator is presented as $\vec{\alpha}_a = \alpha_a\frac{(1,1,-2)}{\sqrt{3}} + \beta_a(1,-1,0)$ and $n+1$ operators are fused to the origin and $m+1$ operators at infinity, one has the potentials in (3.2) with the parameters: $b_k \equiv \sum_{a=0}^{n}\hbar\beta_a z_a^k$ $(k=0,1,\cdots,n)$ and $b_{-k} \equiv -\sum_{a=0}^{n}\hbar\beta_a z_a^k$ $(k=1,\cdots,m)$. Similarly, $a_\ell = (\sqrt{3}c_\ell - b_\ell)/2$ and $c_k \equiv \sum_{a=0}^{m}\hbar\alpha_a z_a^k$ $(k=0,1,\cdots,n)$ and $c_{-k} \equiv -\sum_{a=0}^{m}\hbar\alpha_a z_a^k$ $(k=1,\cdots,m)$.

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The duality transform $x_i \to 1/x_i$ and $y_j \to 1/y_j$ induces the dual potential

$$\bar{V}_1(z) = \bar{b}_0 \log z - \sum_{k>0} \frac{\bar{b}_k}{kz^k} + \sum_{k>0} \frac{\bar{b}_{-k} z^k}{k}\,,$$
$$\bar{V}_2(z) = \bar{a}_0 \log z - \sum_{k>0} \frac{\bar{a}_k}{kz^k} + \sum_{k>0} \frac{\bar{a}_{-k} z^k}{k}\,, \tag{3.4}$$

where $\bar{a}_k = -a_{-k}$ and $\bar{b}_k = -b_{-k}$ when $k \neq 0$. $\bar{c}_0 = c_\infty$ and $\bar{a}_0 = a_\infty$ are fixed by the neutrality condition $b_0 + b_\infty + \epsilon(N - M/2) = \epsilon$ and $a_0 + a_\infty + \epsilon(M - N/2) = \epsilon$.

The loop equation at the classical/NS limit is summarized as the following. (More details can be found in appendix A). One is the quadratic equation related with the Virasoro symmetry:

$$X_1^2 + X_2^2 - X_1 X_2 + \epsilon(X_1' + X_2') + \xi_2 = 0\,, \tag{3.5}$$

where X_1 and X_2 are one-point resolvents (R_1 and R_2) shifted by potential whose integration variable is x_i and y_i, respectively: $X_1 = 2\left(R_1 + \frac{1}{3}\left(2V_1' + V_2'\right)\right)$ and $X_2 = 2(R_2 + \frac{1}{3}\left(V_1' + 2V_2'\right))$. $\xi_2(z)$ is the energy momentum tensor (Virasoro current) expectation value

$$\xi_2(z) = -2\epsilon(V_1'' + V_2'') - \frac{4}{3}(V_1'^2 + V_2'^2 + V_1'V_2') - F = \frac{\langle I_m | T(z) | I_n \rangle}{\langle I_m | I_n \rangle}\,. \tag{3.6}$$

where $F \equiv f_1 + f_2$ is defined in the appendix and has the mode expansion $f_1(z) + f_2(z) = \sum_{k=-m}^{n-1} d_k/z^{k+2}$. Therefore, $\xi_2(z)$ has the mode expansion

$$\xi_2(z) = \sum_{k=-2m}^{2n} \frac{A_k}{z^{k+2}} - \sum_{k=-m}^{n-1} \frac{d_k}{z^{k+2}}\,. \tag{3.7}$$

Here A_k is a constant

$$A_k = 2\epsilon(k+1)(a_k + b_k) - \frac{4}{3} \sum_{r+s=k} (a_r a_s + b_r b_s + a_r b_s)\,. \tag{3.8}$$

Since $T(z) = \sum_k L_k/z^{k+2}$, A_k $(n \le k \le 2n)$ is the eigenvalues of L_k of the irregular state $|I_n\rangle$, consistent with the definition of the irregular state of rank n. The mode d_k has an important role since it is related with the partition function as in the Virasoro case. When $0 \le k \le n-1$, one has

$$d_k = v_k\left(F_{(m;n)}\right)\,, \qquad v_k = \sum_{s>0} s\left(b_{s+k}\frac{\partial}{\partial b_s} + a_{s+k}\frac{\partial}{\partial a_s}\right)\,. \tag{3.9}$$

One may find its dual form if one replaces $b_k(a_k)$ and $k \neq 0$ with $\bar{b}_k$ and $\bar{a}_k$, respectively.

The other loop equation is cubic. One may put the cubic equation conveniently into two separate equations when combined with the previous quadratic one.

$$X_1^3 + \xi_2 X_1 + 3\epsilon X_1 X_1' + \epsilon^2 X_1'' = +\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\xi_2'\,, \tag{3.10}$$
$$X_2^3 + \xi_2 X_2 + 3\epsilon X_2 X_2' + \epsilon^2 X_2'' = -\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\xi_2'\,, \tag{3.11}$$

where ξ_3 is given in terms of the coefficients of the potential:

$$\xi_3(z) = \sum_{k=-3m}^{3n} \frac{B_k}{z^{k+3}} - \sum_{k=-2m}^{2n-1} \frac{e_k}{z^{k+3}}\,. \tag{3.12}$$

B_k is a constant

$$\begin{aligned} B_k =& \frac{4}{3\sqrt{3}} \sum_{r+s+t=k} \Big(2(b_r b_s b_t - a_r a_s a_t) + 3(b_r b_s a_t - a_r a_s b_t)\Big) \\ &- \frac{\sqrt{3}}{2}\epsilon \sum_{r+s=k} \Big(2(k+2)(b_r b_s - a_r a_s) + (r-s)(b_r a_s - a_r b_s)\Big) \\ &+ \frac{\sqrt{3}}{2}\epsilon^2 (k+1)(k+2)(b_k - a_k)\,, \end{aligned} \tag{3.13}$$

which comes from the terms

$$\left(\frac{4}{3\sqrt{3}}(2V_1'^3 + 3V_1'^2 V_2') + \sqrt{3}\epsilon(2V_1'V_1'' + V_2'V_1'') + \frac{\epsilon^2}{3}V_1'''\right) - \Big(1 \leftrightarrow 2\Big)\,. \tag{3.14}$$

It turns out that $\xi_3(z)$ is the expectation of the $\mathcal{W}_3$ current $W(z)$

$$\xi_3(z) = \frac{\langle I_m|W(z)|I_n\rangle}{\langle I_m|I_n\rangle} = \sum_k \frac{\langle W_k\rangle}{z^{k+3}}\,, \tag{3.15}$$

and B_k $(2n \le k \le 3n)$ is the W_k eigenvalue of the ket (right state). When $-3n \le k \le -2n$, B_k is the $-W_{-k}$ eigenvalue of the bra (left state) since $W_k^\dagger = -W_{-k}$ (this anti-hermiticity comes from our normalization. See appendix B.).

The moment e_k induces the flow equation. When $n \le k \le 2n-1$, e_k applies to the right state

$$e_k = \mu_k\left(F_{(m;n)}\right)\,, \qquad \mu_k = \sum_{\substack{k=r+s-t;\\ t>0}} \sqrt{3}t\left((a_r a_s + 2a_r b_s)\frac{\partial}{\partial a_t} - (b_r b_s + 2a_r b_s)\frac{\partial}{\partial b_t}\right)\,. \tag{3.16}$$

Its dual form applies to the left state.

It is worth to note that if one defines $\Psi_i(z) = \exp\left(\frac{1}{\epsilon}\int^z X_i(z')dz'\right)$ with $i = 1, 2$. Then, the loop equations (3.10) and (3.11) can be rewritten as a third order differential equation of $\Psi_i(z)$:

$$\left(\epsilon^3 \frac{\partial^3}{\partial z^3} + \xi_2 \epsilon \frac{\partial}{\partial z} + U_i(z)\right)\Psi_i(z) = 0\,, \tag{3.17}$$

where $U_1(z) = +\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\xi_2'$ and $U_2(z) = -\frac{2}{3\sqrt{3}}\xi_3 - \frac{\epsilon}{2}\xi_2'$.

3.2 Spectral curve and partition function

As shown in section 2, the symmetry present in the spectral curve will be used to find the partition function $\mathcal{Z}_{(m;n)}$. The loop equations (3.5), (3.10) and (3.11) are our starting point. Our first step is to introduce two monic polynomials of z with degree N and M: $P(z) = \left\langle \prod_{i=1}^N (z - x_i)\right\rangle = \prod_{\alpha=1}^N (z - t_\alpha)$ and $Q(z) = \left\langle \prod_j^M (z - y_j)\right\rangle = \prod_{\alpha=1}^M (z - w_\alpha)$. At the classical/NS limit, one has the resolvents as rational functions: $2R_1(z) = \epsilon P'(z)/P(z)$ and $2R_2(z) = \epsilon Q'(z)/Q(z)$. We rewrite the quadratic equation (3.5) and cubic equations in (3.10) and (3.11) in terms of the polynomials P and Q. The quadratic equation reduces to the second order differential equation:

$$\epsilon^2(P''Q - P'Q' + PQ'') + 2\epsilon(V_1'P'Q + V_2'PQ') = FPQ\,. \tag{3.18}$$

The cubic equations reduces to the third order differential equation:

$$\epsilon^3 P''' + 2\epsilon^2(2V_1' + V_2')P'' + \epsilon\left(4V_1'(V_1' + V_2') + 2\epsilon V_1'' - F\right)P' = G_1 P \tag{3.19}$$

$$\epsilon^3 Q''' + 2\epsilon^2(V_1' + 2V_2')Q'' + \epsilon\left(4V_2'(V_1' + V_2') + 2\epsilon V_2'' - F\right)Q' = G_2 Q\,. \tag{3.20}$$

where

$$G_1 = \sum_{k=-2m}^{2n-1} \frac{1}{z^{k+3}} \left\{ -\frac{2}{3\sqrt{3}} e_k + \frac{2}{3} \sum_{r+s=k} d_r(2b_s + a_s) \right\} - \frac{\epsilon}{2} \sum_{k=-m}^{n-1} \frac{(k+2)d_k}{z^{k+3}}\,, \tag{3.21}$$

$$G_2 = \sum_{k=-2m}^{2n-1} \frac{1}{z^{k+3}} \left\{ +\frac{2}{3\sqrt{3}} e_k + \frac{2}{3} \sum_{r+s=k} d_r(2a_s + b_s) \right\} - \frac{\epsilon}{2} \sum_{k=-m}^{n-1} \frac{(k+2)d_k}{z^{k+3}}\,. \tag{3.22}$$

Our next step is to find the mode d_a $(0 \le a \le n-1)$ and e_k $(n \le k \le 2n-1)$ and their duals if necessary. As noted in section 2, it is not easy to find the exact form of d_a and and e_k. We provide some examples of the partition function using the ϵ expansion method. Note that R_1 , R_2 and F are $O(\epsilon)$ whereas V_1 and V_2 are $O(1)$. Therefore, denoting $R_i = \sum_{k\ge1} R_i^{(k)}\epsilon^k$, $F = \sum_{k\ge1} F^{(k)}\epsilon^k$ and $G_i = \sum_{k\ge1} G_i^{(k)}\epsilon^k$, we have $R_1^{(1)}$ and $R_2^{(1)}$ at the leading order of the loop equations (3.19) and (3.20):

$$2R_1^{(1)} = \frac{G_1^{(1)}}{4V_1'(V_1' + V_2')}\,, \quad 2R_2^{(1)} = \frac{G_2^{(1)}}{4V_2'(V_1' + V_2')}\,. \tag{3.23}$$

The stationary point of the potentials, $V_1' = 0$, $V_2' = 0$ and $V_1' + V_2' = 0$ provide the pole structure of the resolvents (zeros of the polynomials P and Q). This is the reminiscence of the cut structure on the Riemann sheet which appears at large N limit. Let us denote the number of poles of the resolvent R_1 by N_k and R_2 by M_k so that $N = \sum_{k=1}^{2n} N_k$ and $M = \sum_{k=1}^{2n} M_k$. There are equal number of stationary points for V_1 and V_2. Therefore, we have identities from the filling fractions. When $1 \le k \le n$, we have

$$\oint_{\mathcal{A}_k} dz\, \frac{G_1^{(1)}}{4V_1'(V_1' + V_2')} = N_k\,, \quad \oint_{\mathcal{B}_k} dz\, \frac{G_2^{(1)}}{4V_2'(V_1' + V_2')} = M_k\,, \tag{3.24}$$

where the contours $\mathcal{A}_k$ and $\mathcal{B}_k$ encircle the stationary points of V_1 and V_2, respectively. When $n+1 \le k \le 2n$, we have

$$\oint_{\mathcal{C}_k} dz\, \frac{G_1^{(1)}}{4V_1'(V_1' + V_2')} = N_k\,, \quad \oint_{\mathcal{C}_k} dz\, \frac{G_2^{(1)}}{4V_2'(V_1' + V_2')} = M_k\,, \tag{3.25}$$

where $\mathcal{C}_k$ encircles the stationary point of $V_1 + V_2$. It turns out that $N_k = M_k$ since $R_1 - R_2$ has no poles inside $\mathcal{C}_k$. These identities provide $3n$-independent equations which solve d_a

and e_k in terms of the filling fraction at the lowest order in ϵ. One obtains the non-trivial contribution from the next order

$$\begin{aligned}2R_1^{(2)} &= \frac{G_1^{(2)}}{4V_1'(V_1'+V_2')} + \frac{(F^{(1)}-2V_1'')G_1^{(1)} - 2(2V_1'+V_2')(G_1^{(1)})'}{16(V_1'(V_1'+V_2'))^2} \\ &\quad + \frac{(2V_1'+V_2')V_1''G_1^{(1)}}{8(V_1')^3(V_1'+V_2')^2} + \frac{(2V_1'+V_2')(V_1''+V_2'')G_1^{(1)}}{8(V_1')^2(V_1'+V_2')^3} - \frac{(2V_1'+V_2')(G_1^{(1)})^2}{32(V_1'(V_1'+V_2'))^3}\,.\end{aligned} \tag{3.26}$$

$R_2^{(2)}$ is obtained if V_1 and V_2 are exchanged and $G_1^{(i)} \to G_2^{(i)}$.

3.3 Partition function $\mathcal{Z}_{(0;n)}$

The irregular partition function $\mathcal{Z}_{(0;n)}$ has the potential with logarithmic and inverse powers only:

$$V_1(z) = b_0 \log z - \sum_{k=1}^{n} \frac{b_k}{kz^k}\,, \quad V_2(z) = a_0 \log z - \sum_{k=1}^{n} \frac{a_k}{kz^k}\,. \tag{3.27}$$

This partition function is the two-point correlation of one regular vertex at infinity and one irregular vertex at origin and is therefore, considered as the inner product between a regular state and an irregular state.

The partition function is the function of $2n$-variables, $\{b_1, \cdots, b_n\}$ and $\{a_1, \cdots, a_n\}$ and d_a $(0 \le a \le n-1)$ and e_a $(n \le a \le 2n-1)$ provide $2n$-flow equations. In this case, d_0 is simple to find: $d_0 = 2\epsilon(b_0 N + a_0 M) + \epsilon^2(N(N-1) + M(M-1) - NM)$ if one uses the large z expansion of the quadratic loop equation. Other quantities need more elaborate evaluation.

Let us consider the partition function $\mathcal{Z}_{(0;1)}$, the rank 1 case. We need e_1. Using the results (3.24), (3.24) and (3.26), one finds e_1:

$$\begin{aligned} e_1 =& 2\sqrt{3}\epsilon\left[(a_1+2b_1)(a_0M+b_0N) + 3b_0(a_0+b_0)(N-N_1)\xi_1 + 3b_0(a_0+b_0)N_1\xi_3\right] \\ &+\sqrt{3}\epsilon^2\Bigg[-\frac{3a_0\xi_3\left(\left(N^2+2N_1(N_1+M+1)-N(4N_1+M+1)\right)\xi_1 + N_1(N_1-1)\xi_3\right)}{\xi_1-\xi_3} \\ &\quad +3b_0\left(\left(N^2+N_1(N_1+3)-N(2N_1+3)\right)\xi_1 + N_1(N_1-1)\xi_3\right) + (a_1+2b_1)d_0^{(2)}\Bigg] + \mathcal{O}(\epsilon^3)\,, \end{aligned} \tag{3.28}$$

where $d_0^{(2)} = N(N-1) + M(M-1) - NM$ is the ϵ^2-order coefficient of d_0. In addition, $\xi_1 = -\frac{b_1}{b_0}$ and $\xi_3 = -\frac{a_1+b_1}{a_0+b_0}$ are stationary points of V_1 and $V_1 + V_2$, respectively. Using the flow equations $d_0 = v_0\left(F_{(0:1)}\right)$ and $e_1 = \mu_1\left(F_{(0;1)}\right)$, one can find the free energy and the partition function up to $O(\epsilon^2)$

$$\begin{aligned}\mathcal{Z}_{(0;1)} =& a_1^{-(2\epsilon a_0 M_1 + \epsilon^2 M_1(M_1-1))/\hbar^2} b_1^{-(2\epsilon b_0 N_1 + \epsilon^2 a_0 N_1(N_1-1))/\hbar^2} (a_1+b_1)^{-(2\epsilon N_2(a_0+b_0)+\epsilon^2 N_2(N_2-3))/\hbar^2} \\ &\times (a_0 b_1 - a_1 b_0)^{-\epsilon^2(N_1N_2+N_2+M_1N_2-M_1N_1)/\hbar^2}\,.\end{aligned} \tag{3.29}$$

3.4 Partition function $\mathcal{Z}_{(m;n)}$ and ICB

As in the Virasoro case, $\mathcal{Z}_{(m;n)}$ is equivalent to the two-point correlation of irregular vertex operators. One can evaluate the irregular conformal block (ICB) using the relation with the partition function [36]

$$\mathcal{F}_\Delta^{(m;n)}(\{a_{-k}, b_{-k} : a_k, b_k\}) = \frac{e^{\zeta_{(m;n)}} \mathcal{Z}_{(m;n)}(a_0, b_0; \{a_\ell, b_\ell\})}{\mathcal{Z}_{(0;n)}(a_0, b_0; \{a_k, b_k\}) \mathcal{Z}_{(0;m)}(\bar{a}_0, \bar{b}_0; \{\bar{a}_k, \bar{b}_k\})} . \tag{3.30}$$

This time the extra factor $e^{\zeta_{(m;n)}}$ is the generalization of the Virasoro case: $\hbar^2 \zeta_{(m:n)} = -\sum_k^{\min(m,n)} \frac{4}{3k}(2a_k a_{-k} + a_k b_{-k} + b_k a_{-k} + 2b_k b_{-k})$.

We find the partition function $\mathcal{Z}_{(1;1)}$ using the ϵ expansion. By rescaling $x_i \to x_i/a_{-1}$ and $y_j \to y_j/a_{-1}$, one obtains a_{-1} dependence and has the partition function with three parameters, $\eta_0 \equiv a_1 a_{-1}$, $t_1 \equiv b_1/a_1$ and $t_{-1} \equiv b_{-1}/a_{-1}$. Then, the partition function is to be evaluated from three flow equations: $d_0 = v_0(F_{(1;1)})$, $e_1 = \mu_1(F_{(1;1)})$ and $e_{-1} = \mu_{-1}(F_{(1;1)}) + \nu_{-1}^c$ where ν_{-1}^c is a constant (see appendix A). We use notations for the filling fractions as $M_0 = M_1 + M_2$, $N_0 = N_1 + N_2$ $(M_2 = N_2)$ and $M_\infty = M_{-1} + M_{-2}$, $N_\infty = N_{-1} + N_{-2}$ $(M_{-2} = N_{-2})$ with $M = M_0 + M_\infty$, $N = N_0 + N_\infty$. We have up to $\mathcal{O}(\epsilon^1)$,

$$d_0 = 2\epsilon(a_0 M_0 + b_0 N_0) + \mathcal{O}(\eta_0) , \tag{3.31}$$

$$\begin{aligned} e_1 = 2\sqrt{3}\epsilon\big[&a_0\left(M_0 a_1 + (2M_0 - 3N_0 + 3N_1)b_1\right) \\ &+ b_0\left((N_0 - 3N_1)a_1 - N_0 b_1\right)\big] + \mathcal{O}(\eta_0) , \end{aligned} \tag{3.32}$$

$$\begin{aligned} e_{-1} - \nu_{-1}^c = -2\sqrt{3}\epsilon\big[&a_0\left(M_\infty a_{-1} + (2M_\infty - 3N_\infty + 3N_{-1})b_{-1}\right) \\ &+ b_0\left((N_\infty - 3N_{-1})a_{-1} - N_\infty b_{-1}\right)\big] + \mathcal{O}(\eta_0) , \end{aligned} \tag{3.33}$$

and the partition function

$$\begin{aligned} \mathcal{Z}_{(1;1)} = \, & a_1^{-(2\epsilon a_0 M_1)/\hbar^2} a_{-1}^{-(2\epsilon a_\infty M_{-1})/\hbar^2} b_1^{-(2\epsilon b_0 N_1)/\hbar^2} b_{-1}^{-(2\epsilon b_\infty N_{-1})/\hbar^2} \\ & \times (a_1 + b_1)^{-2\epsilon(a_0 + b_0)/\hbar^2} (a_{-1} + b_{-1})^{-2\epsilon(a_\infty + b_\infty)/\hbar^2} \\ & \times \exp\Bigg[\frac{2\epsilon\eta_0}{\hbar^2 a_0 b_0 (a_0 + b_0)} \Big\{ b_0^2(M_1 - M_{-1}) + a_0^2(N_1 - N_{-1})t_1 t_{-1} + a_0 b_0 \big(M_1 - M_{-1} \\ & \qquad + (N_2 - N_{-2})(1 + t_1 + t_{-1}) + (N_1 + N_2 - N_{-1} - N_{-2})t_1 t_{-1}\big) \Big\} + \mathcal{O}(\eta_0^2) \Bigg] . \end{aligned} \tag{3.34}$$

This provides ICB

$$\begin{aligned} \mathcal{F}_\Delta^{(1;1)} = & \left[1 - \frac{4\eta_0}{3\hbar^2}(2 + t_1 + t_{-1} + 2t_1 t_{-1}) + \mathcal{O}(\eta_0^2)\right] + \epsilon\Bigg[\frac{2\eta_0}{\hbar^2 a_0 b_0 (a_0 + b_0)} \Big\{ b_0^2(M_1 - M_{-1}) \\ & + a_0^2(N_1 - N_{-1})t_1 t_{-1} + a_0 b_0 \big(M_1 - M_{-1} + (N_2 - N_{-2})(1 + t_1 + t_{-1}) \\ & + (N_1 + N_2 - N_{-1} - N_{-2})t_1 t_{-1}\big) \Big\} + \mathcal{O}(\eta_0^2) \Bigg] + \mathcal{O}(\epsilon^2) . \end{aligned} \tag{3.35}$$

One may use the perturbative method to find ICB using IMM with the relation (3.30). One may put the reference potentials $V^{(0)}$ and its perturbations $\Delta V^{(0)}$ for N_0 and M_0 variables:

$$
\begin{aligned}
V^{(0)}(x_I, y_J) &= \sum_{I=1}^{N_0} \left(b_0 \log x_I - \sum_{k=1}^{n} \frac{b_k}{k x_I^k} \right) + \sum_{J=1}^{M_0} \left(a_0 \log y_J - \sum_{k=1}^{m} \frac{a_k}{k y_J^k} \right) , \\
\Delta V^{(0)}(x_I, y_J) &= \sum_{I=1}^{N_0} \left(\sum_{\ell=1}^{n} \frac{b_{-\ell}}{\ell} x_I^\ell \right) + \sum_{J=1}^{M_0} \left(\sum_{\ell=1}^{n} \frac{a_{-\ell}}{\ell} y_J^\ell \right) .
\end{aligned} \tag{3.36}
$$

For the rest variables, N_∞ and M_∞ variables, one has the reference potential $V^{(\infty)}(\mu_K, \nu_L)$ and perturbation $\Delta V^{(\infty)}(\mu_K, \nu_L)$ which can be put into the similar form $V^{(0)}$, $\Delta V^{(0)}$ with dual variables if one uses the dual transformation $\mu_K \to 1/\mu_K$ and $\nu_L \to 1/\nu_L$. After this, one has ICB in the following form

$$
\begin{aligned}
\mathcal{F}_\Delta^{(m;n)}(\{a_{-k}, b_{-k} : a_k, b_k\}) = e^{\zeta_{(m;n)}} \Big\langle & \prod_{I,K} (1 - x_I \mu_K)^{2\beta} \prod_{J,L} (1 - y_J \nu_L)^{2\beta} \prod_{I,L} (1 - x_I \nu_L)^{-\beta} \\
& \times \prod_{J,K} (1 - y_J \mu_K)^{2\beta} e^{\frac{\sqrt{\beta}}{g} \left(\Delta V^{(0)}(x_I, y_J) + \Delta V^{(\infty)}(\mu_K, \nu_L) \right)} \Big\rangle_0 ,
\end{aligned} \tag{3.37}
$$

where the bracket denotes the expectation value with respect to the reference potentials, $V^{(0)}$ and $V^{(\infty)}$. One may obtain the expectation values using the large z expansion of the resolvents in the loop equations (3.10) and (3.11) of the reference partition functions.

We find $\mathcal{F}_\Delta^{(1;1)}$ as the simplest example. Up to the first order of a_1 and b_1 (also their duals $\bar{a}_1$ and $\bar{b}_1$), we have

$$
\begin{aligned}
\mathcal{F}_\Delta^{(1;1)} = 1 + \frac{1}{\hbar^2} \Big[& 2\epsilon^2 \left(\langle x_I \rangle \langle \mu_K \rangle + \langle y_J \rangle \langle \nu_L \rangle \right) - \epsilon^2 \left(\langle x_I \rangle \langle \nu_L \rangle + \langle y_J \rangle \langle \mu_K \rangle \right) \\
& + 2\epsilon \left(\bar{b}_1 \langle x_I \rangle + \bar{a}_1 \langle y_J \rangle + b_1 \langle \mu_K \rangle + a_1 \langle \nu_L \rangle \right) + \frac{4}{3} (2 a_1 \bar{a}_1 + a_1 \bar{b}_1 + b_1 \bar{a}_1 + 2 b_1 \bar{b}_1) \Big] .
\end{aligned} \tag{3.38}
$$

Here we omitted summation symbols inside the expectation value bracket for simplicity. Each expectation values can be read off from the order of z^{-4} of the loop equations (3.10) and (3.11) for the reference partition function $\mathcal{Z}_{(0;1)}$. Finally, we obtain ICB at the first order of a_1 and b_1

$$
\begin{aligned}
\mathcal{F}_\Delta^{(1;1)} = 1 - \frac{1}{9\hbar^2 \left(4\omega_0^2 + \Delta^2 (4\Delta - 3\epsilon^2) \right)} \Big[& 8\Delta \omega_{-1} \omega_1 \\
& + 12 \omega_0 (\omega_{-1} \ell_1 + \omega_1 \ell_{-1}) - \frac{9}{2} \Delta \ell_{-1} \ell_1 (4\Delta - 3\epsilon^2) \Big] ,
\end{aligned} \tag{3.39}
$$

where $\Delta = -\frac{4}{3}(\alpha^2 + \alpha\beta + \beta^2) + 2\epsilon(\alpha + \beta)$ with $\alpha = a_0 + \epsilon(M_0 - N_0/2)$, $\beta = b_0 + \epsilon(N_0 - M_0/2)$, $\ell_1 = -\frac{4}{3}(a_0(2a_1 + b_1) + b_0(a_1 + 2b_1)) + 4\epsilon(a_1 + b_1)$ and its dual ℓ_{-1} are the constant modes of $\xi_2(z)$. As in the Virasoro case, we may identify the expectation values of the Virasoro generators: $\ell_1 = \frac{\langle\Delta|L_1|I^{(1)}\rangle}{\langle\Delta|I^{(1)}\rangle}$, $\ell_{-1} = \frac{\langle I^{(1)}|L_{-1}|\Delta\rangle}{\langle I^{(1)}|\Delta\rangle}$ and $\Delta = \frac{\langle\Delta|L_0|I^{(1)}\rangle}{\langle\Delta|I^{(1)}\rangle}$ where $\langle\Delta|I^{(1)}\rangle$ is $\mathcal{Z}_{(0;1)}$ and $\langle I^{(1)}|\Delta\rangle$ is $\bar{\mathcal{Z}}_{(0;1)}$. In addition, ω_k is the mode appearing in $\xi_3(z)$. The constant mode $\omega_0 = \frac{1}{3\sqrt{3}}(\alpha-\beta)(4\alpha+2\beta-3\epsilon)(2\alpha+4\beta-3\epsilon)$ is identified as $\frac{\langle\Delta|W_0|I^{(1)}\rangle}{\langle\Delta|I^{(1)}\rangle}$ and the other modes are expectation values: $\omega_1 = \frac{\langle\Delta|W_1|I^{(1)}\rangle}{\langle\Delta|I^{(1)}\rangle} = B_1 - e_1$ and ω_{-1} is its dual. We check that the ϵ expansion of the above $\mathcal{F}_\Delta^{(1;1)}$ in (3.39) is in complete agreement with (3.35). It is also noted that ICB is manifestly dual invariant.

ICB of (3.39), obtained from the perturbation of the irregular matrix model is convenient to find the irregular state in terms of descendants. The irregular state of the rank 1 has the form

$$\begin{aligned}\frac{|I^{(1)}\rangle}{\langle\Delta|I^{(1)}\rangle} = 1 - \frac{1}{9\hbar\left(\omega_0^2+\Delta^2\left(\Delta-\frac{c-2}{32}\right)\right)}\Bigg[&(2\Delta\omega_1-3\omega_0\ell_1)W_{-1}|\Delta\rangle \\ &-\left(3\omega_0\omega_1+\frac{9}{2}\Delta\left(\Delta-\frac{c-2}{32}\right)\ell_1\right)L_{-1}|\Delta\rangle\Bigg]+\cdots\end{aligned} \tag{3.40}$$

where $c = 2+24\epsilon^2$ is the central charge and $\cdots$ refers to the higher descendant. The irregular state (3.40) has no semi-degenerate condition at the first level in contrast to the state constructed in [30, 41, 42] where L_{-1} is related to W_{-1} descendant. Instead, the coefficient ω_1 is not a simple constant and is given in terms of the flow equation with respect to the proper normalization $\mathcal{Z}_{(0:1)}$. This feature also appeared in Virasoro irregular state with rank 2 and higher [20]. However, here in Toda irregular state, the non-trivial feature appears even for the rank 1 and at the first descendant level.

4 Relation with gauge theories

The Nekrasov partition function for $\mathcal{N}=2$ is given as the product of the perturbative part and instanton part:

$$Z_{\text{full}}(\vec{q};\vec{a},m;\epsilon) = Z_{\text{perturb}}Z_{\text{inst}} = Z_{\text{tree}}Z_{\text{1loop}}Z_{\text{inst}}\,. \tag{4.1}$$

According to AGT, the Nekrasov partition function of U(N) gauge theory with $N_f = 2N$ is related with the correlation function of Toda field theory. Explicitly, the correlation function on a sphere with $n+3$ punctures corresponds to a linear quiver n of SU(N) gauge groups:

$$\langle V_n(\infty)V_{n-1}(1)V_{n-2}(q_1)\cdots V_2(q_1\cdots q_{n-3})V_1(0)\rangle \propto \int |Z_{\text{full}}(\vec{q};\vec{a},m;\epsilon)|^2 \tag{4.2}$$

Especially the instanton part of Nekrasov partition function Z_{inst} corresponds to the conformal blocks $\mathcal{F}_\Delta$ in the CFT side; $Z_{\text{inst}} = \mathcal{F}_\Delta$. One may try to prove this conjecture by evaluating the conformal block directly [19, 31, 32, 43].

One may equally check the AGT conjecture by comparing the Seiberg-Witten curve of the gauge theory with the corresponding spectral curve from CFT side. Seiberg-Witten curve of SU(N) gauge theory with $N_f = 2N$ is given in [44–46],

$$x^n = \sum_{k=2}^{n} \frac{P_{2k}^{(k)}(z)}{(z(z-1)(z-q)))^k} x^{n-k}\,, \tag{4.3}$$

where $P_{2k}(z)$ is a $2k$ polynomial in z and $q = e^{i\pi\tau}$ with τ the gauge coupling constant $\tau = \theta/\pi + 8\pi i/g^2$.

On the other hand, from the matrix model with A_{n-1} quiver one may have the spectral curve of the type

$$x^n = \sum_{k=2}^{n} (-1)^{k-1} \phi_k(z)\; x^{n-k}, \tag{4.4}$$

where $\phi_k(z)$ is the expectation value of the conformal current with dimension k. This was obtained from the the expectation value of the Miura transform of $\det(x - i\hbar\partial\varphi(z)) = 0$ with Toda field φ in the matrix model context. Even though this spectral curve holds at $\epsilon \to 0$ limit (Note that this limit corresponds to the large N limit where N is the number of the eigenvalues of the matrix), one may identify parameters between two theories so that the pole structure of the curves matches each other as done in [44]. One may have more accurate spectral curve if one uses the loop equation of the matrix model.

This conjecture still works for irregular conformal blocks. Gaiotto [15] obtained the irregular singularity for the simplest example of $\mathcal{W}_2$ and identify the result with the asymptotically free theories of SU(2) gauge theory with $N_f < 4$. Explicitly, SU(2) gauge theory with $N_f = 4$ reduces to SU(2) pure Yang-Mills theory if the masses μ_i of hyper-multiplets are to infinity $\mu_1, \ldots, \mu_4 \to \infty$ and $q \to 0$, while keeping $q \prod_{I=1}^{4} \mu_I = \Lambda^4$. Then the Seiberg-Witten curve is given as $x^2 = \phi_2(z)$ where

$$\phi_2 = \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}\,, \tag{4.5}$$

which contains the strongest irregular singularity with odd power on the Riemann sphere. u parameterizes the Coulomb branch. This irregular singularity of odd power is not properly represented in terms of matrix model as yet unless one considers the special limit from the even power singularity case so that the highest even power term vanishes.

For $N_f = 2$, one may have the SW curve with the irregular singularity with degree 4

$$\phi_2 = \frac{\Lambda^2}{z^4} + \frac{2m\Lambda}{z^3} + \frac{2u}{z^2} + \frac{2\tilde{m}\Lambda}{z} + \Lambda^2 \tag{4.6}$$

where $q\mu_2\mu_4 = 4\Lambda^2$, $m = \mu_1 - \frac{\epsilon}{2}$ and $\tilde{m} = \mu_3 - \frac{\epsilon}{2}$. In this case ϕ_2 is ξ_2 in (2.5) with $n = m = 1$, and the Nekrasov instanton partition function is checked to be the same as the partition function $Z_{(1;1)}$ of the irregular matrix model of the rank (1;1) [47, 48]. In addition, $Z_{(1;1)}$ corresponds to the inner product of two irregular states whose eigenvalues of L_2 and L_1 are given from ϕ_2; Λ^2 and $2m\Lambda$ for the ket (irregular module) and Λ^2 and $2\tilde{m}\Lambda$ for the bra, respectively. The Coulomb branch parameter u is related to d_0 and is fixed by the filling fraction of the matrix model, the contour integral of the resolvent, which is identified with the contour integral of the Seiberg-Witten one-form xdz. In addition, from the fixed

d_0 the partition function is obtained from the corresponding Virasoro flow equation as seen in section 2. Considering this result, one may naturally assume that the AGT conjecture works for the colliding limit as well with higher ranks $(m;n)$. For the special case $(0;n)$, one has the irregular singularity of type D_{2n} considered in [12, 37] which is associated with a regular puncture of degree 2 and an irregular puncture of degree $2n+2$ on the sphere. Specifically, $Z_{(0;1)}$ corresponds to D_2 and $Z_{(0;2)}$ corresponds to D_4.

One may extend the analysis to the case with $\mathcal{W}_N$ symmetry. For $\mathcal{W}_3$, one has the SW curve with the cubic equation. One can have the same spectral curve from the Penner-type matrix model with the multi-log potential

$$V_1(z) = \sum_{k=1}^{p} m_k \log(q_k - z)\,, \qquad V_2(z) = \sum_{k=1}^{p} \tilde{m}_k \log(q_k - z)\,, \tag{4.7}$$

which corresponds to the linear quiver $p-2$ SU(3) gauge theories [46]. For instance, in [44], the potential (4.7) for $p=3$ was considered, which corresponds to SU(3) gauge theory with six massive flavors. The relation between six mass parameters and parameters in the potential was made by using the residue relation of the one-form xdz.

At the colliding limit, we used the potential (3.2) and obtained the two cubic equations (3.10) and (3.11). In fact, the two cubic equations reduce to one in the large N limit ($\epsilon \to 0$),

$$x^3 + \frac{\xi_2}{4} x - \frac{\xi_3}{12\sqrt{3}} = 0\,, \tag{4.8}$$

where $x = 2X_1$ or $-2X_2$. Note that the spectral curve (4.8) has the same form even before the colliding limit, where the explicit form of ξ_2, ξ_3 is different. At the colliding limit, ξ_2 and ξ_3 have irregular singularities which indicate positive modes of the Virasoro and $\mathcal{W}_3$ currents as we have shown in (3.6) and (3.15). In addition, the spectral curve provides further information about the gauge theory with the same SW curve. For example, if one considers the potential (3.27) of the rank $(0;n)$, then the spectral curve (at $\epsilon \to 0$ limit) is identified with the SW curve of the type IV Argyres-Douglas theory [49]:

$$\begin{aligned} x^3 &+ \left(\frac{v_1}{z^{2n+2}} + \frac{v_2}{z^{2n+1}} + \cdots + \frac{v_{2n}}{z^3} + \frac{v_{2n+1}}{z^2}\right) x \\ &+ \left(\frac{1}{z^{3n+3}} + \frac{\omega}{z^{3n+2}} + \frac{u_1}{z^{3n+1}} + \cdots + \frac{u_{3n-2}}{z^4} + \frac{u_{3n-1}}{z^3}\right) = 0\,, \end{aligned} \tag{4.9}$$

where the dominant singular part in ξ_3 is normalized as 1. Comparing this SW curve with the explicit expression ξ_2 in (3.7) and ξ_3 in (3.12), one notes that $(2n+1)$ parameters $(v_1, \cdots, v_{n+1}$ and $\omega, u_1, \cdots, u_{n-1})$ are fixed by the $(2n+2)$ parameters of the potential (3.27) with the proper normalization. The mass parameters v_{2n+1} and u_{3n-1} are determined by the additional constants d_0 and e_0.

Finally, the Coulomb branch parameters $v_{n+2}, \cdots, v_{2n}$ and $u_n, \cdots, u_{3n-2}$ are related with the contour integral of the SW one-form in the gauge theory side. From the matrix model side, the Coulomb branch parameters are given in terms of d_i $(i = 1, \cdots, n-1)$ and e_j $(j = 1, \cdots, 2n-1)$ and can be fixed by $(3n-2)$ independent filling fractions which is related with the cut structure of the spectral curve. Once the Coulomb branch parameter are known, one can find the partition function since the values d_k and e_k are directly related with the flow equations, which are the strong merit of the matrix model approach.

5 Conclusion and outlook

We develop a new mechanism to evaluate the irregular conformal block using the Virasoro and W symmetry. We use the loop equation of the irregular matrix model which encodes all the details of the conformal symmetry. At the classical/NS limit, the loop equation does not contain the multi-point resolvent terms and reduces to the simple spectral curve which contains the first derivative of the resolvent. The special feature of the spectral curve is that it contains not only constants of motion but also flow equations corresponding to the conformal symmetry. The flow equations are defined on the parameter space of the potential of the irregular matrix model, and its generators represent the Virasoro and W symmetry. We present the details of the flow equations and how to obtain the partition function and irregular conformal block. The irregular conformal block is related with the partition function of the Argyres-Douglas theory according to AGT conjecture, if one uses the parameter relations between these two theories whose details can be found, for example, in [30].

It is noted that the spectral curve and flow equation are not restricted to the irregular conformal block. The method can be applied to the regular conformal block at the classical/NS limit. Using the similar flow equation, one can find the partition function [22, 40]. Even though the partition function is simply obtained, the relation of the positions of the primary operators is not. For example, 5-point Liouville conformal block with one degenerate operator reduces to Painlevé VI as presented in [50]. It seems to be worthwhile to investigate the connection between the positions of the multi-point regular conformal block.

Nekrasov partition function and its counter part, regular conformal block are represented in terms of Young diagrams [3, 4, 43]. Irregular conformal block should also be represented in the same way, which is not well understood yet. On the other hand, conformal symmetry is reinstated in the *degenerate double affine Hecke algebra*(DDAHA) and Nekrasov partition function was studied in terms of DDAHA [51]. In the same way, the irregular conformal block can be better understood using DDAHA. There was a few attempts to investigate this connection [33, 34] and it should be worth finding DDAHA representation of the irregular conformal block.

Finally, the mixture of bulk and micro Coulomb charges in two dimensions is an interesting system whose interaction is represented in terms of the logarithmic potential. If the system is fine-tuned so that the system shows the conformal symmetry, then the matrix model should play the role. In addition, if the bulk charges are localized so that they are idealized in terms of finite number of multi-poles, then the free energy of the irregular matrix model can be useful.

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A Loop equation of A_2 irregular matrix model

The A_2 irregular matrix model (3.1) has the Virasoro and W_3 symmetry which is represented in terms of loop equations [36, 52–54]. We put the multi-point resolvent as

$$R_{K_1;\cdots;K_s}(z_1,\cdots,z_s)=\beta\left(\frac{g}{\sqrt{\beta}}\right)^{2-s}\left\langle\sum_{i_1=1}^{N_{K_1}}\frac{1}{z_1-\lambda_{i_1;K_1}}\cdots\sum_{i_s=1}^{N_{K_s}}\frac{1}{z_s-\lambda_{i_s;K_s}}\right\rangle_{\text{connected}}. \tag{A.1}$$

where denoting $\lambda_{i;1}=x_i$, $\lambda_{j;2}=y_j$. One obtains the quadratic loop equation if one performs the conformal transformation of the integration variables $x_i\to x_i+\varepsilon/(x_i-z)$ and $y_j\to y_j+\varepsilon/(y_j-z)$ which provides the Virasoro symmetry:

$$\begin{aligned}&R_1(z)^2+R_2(z)^2-R_1(z)R_2(z)+V_1'(z)R_1(z)+V_2'(z)R_2(z)\\&\quad+\frac{\hbar Q}{2}\left(R_1'(z)+R_2'(z)\right)-\frac{\hbar^2}{4}\left(R_{1;1}(z,z)-R_{1;2}(z,z)+R_{2;2}(z,z)\right)=\frac{f_1(z)+f_2(z)}{4},\end{aligned} \tag{A.2}$$

where $f_1(z):=4g\sqrt{\beta}\sum_i^{N_1}\left\langle\frac{V_1'(z)-V_1'(x_i)}{z-x_i}\right\rangle$ and $f_2(z):=4g\sqrt{\beta}\sum_i^{N_2}\left\langle\frac{V_2'(z)-V_2'(y_j)}{z-y_j}\right\rangle$. Here $\langle\cdots\rangle$ denotes the expectation value with respect to the A_2 matrix model.

W_3 symmetry is given in terms of cubic loop equation [36]

$$\begin{aligned}0=&-R_1^2R_2+R_1R_2^2-V_1'\left(R_1^2+V_1'R_1-\frac{f_1}{4}\right)+V_2'\left(R_2^2+V_2'R_2-\frac{f_2}{4}\right)+\frac{g_1-g_2}{4}\\&+\frac{\hbar Q}{4}\left[3(V_2'R_2'-V_1'R_1')+R_1R_2'-R_1'R_2+2(R_2R_2'-R_1R_1')+V_2''R_2-V_1''R_1+\frac{f_1'-f_2'}{4}\right]\\&+\frac{\hbar^2Q^2}{8}(R_2''-R_1'')+\frac{\hbar^2}{4}\left[V_1'R_{1;1}-V_2'R_{2;2}+R_{1;1}R_2-R_{2;2}R_1-2R_{1;2}(R_2-R_1)\right]\\&+\frac{\hbar^3Q}{16}\left[R_{1;1}'-R_{2;2}'+\lim_{\bar z\to z}\left(\frac{\partial}{\partial z}R_{1;2}(z,\bar z)-\frac{\partial}{\partial\bar z}R_{1;2}(z,\bar z)\right)\right]+\frac{\hbar^4}{16}(R_{1;2;2}-R_{1;1;2}),\end{aligned} \tag{A.3}$$

where $g_1(z):=4g^2\beta\sum_{i,j}\left\langle\frac{V_1'(z)-V_1'(x_i)}{(z-x_i)(x_i-y_j)}\right\rangle$ and $g_2(z):=4g^2\beta\sum_{i,j}\left\langle\frac{V_1'(z)-V_2'(y_j)}{(z-y_j)(y_j-x_i)}\right\rangle$. This is obtained after varying the integration variables $x_i\to x_i+\sum_{j=1}^{N_2}\frac{\epsilon}{(x_i-z)(x_i-y_j)}$ and $y_j\to y_j+\sum_{i=1}^{N_1}\frac{\epsilon}{(y_j-z)(x_i-y_j)}$.

At the classical/NS limit ($\hbar\to 0$, $b\to\infty$ while $\hbar b=\epsilon$ finite), each multi-point resolvent is finite but due to the factor $\hbar$, the multi-point resolvent terms drop out and the loop equations are given in the simple form:

$$X_1^2+X_2^2-X_1X_2+2\epsilon(X_1'+X_2')=-\xi_2\,, \tag{A.4}$$

$$X_1^2X_2-X_1X_2^2+\frac{\epsilon}{2}\left[(2X_1+X_2)X_1'-(X_1+2X_2)X_2'\right]+\frac{\epsilon^2}{2}(X_1''-X_2'')=\frac{2}{3\sqrt{3}}\xi_3\,, \tag{A.5}$$

where $X_1/2 = R_1 + (2V_1' + V_2')/3$, $X_2/2 = R_2 + (V_1' + 2V_2')/3$ and

$$\xi_2 = -2\epsilon(V_1'' + V_2'') - \frac{4}{3}\left[(V_1')^2 + V_1' V_2' + (V_2')^2\right] - (f_1 + f_2)\,, \tag{A.6}$$

$$\begin{aligned}\xi_3 =& \frac{4\sqrt{3}}{9}\left(2(V_1')^3 + 3(V_1')^2 V_2' - 3V_1'(V_2')^2 - 2(V_2')^3\right) \\ &+ \sqrt{3}\epsilon(2V_1'V_1'' + V_2'V_1'' - 2V_2'V_2'' - V_1'V_2'') + \frac{\sqrt{3}}{2}\epsilon^2(V_1''' - V_2''') \\ &+ \sqrt{3}\left((f_1 - 2f_2)V_1' + (2f_1 - f_2)V_2'\right) + 3\sqrt{3}(g_1 - g_2) + \frac{3\sqrt{3}}{4}\epsilon(f_1' - f_2')\,,\end{aligned} \tag{A.7}$$

ξ_2 and ξ_3 look complicated but can be written in a compact form if one uses the mode expansion,

$$\xi_2 = \sum_{k=-2m}^{2n} \frac{A_k}{z^{k+2}} - \sum_{k=-m}^{n-1} \frac{d_k}{z^{k+2}}\,, \qquad \xi_3 = \sum_{k=-3m}^{3n} \frac{M_k}{z^{k+3}} - \sum_{k=-2m}^{2n-1} \frac{e_k}{z^{k+3}}\,, \tag{A.8}$$

where A_k and B_k are constants (here we use the notation c_ℓ which is related with $a_\ell = (\sqrt{3}c_\ell - b_\ell)/2$)

$$\begin{aligned}A_k =& \epsilon(k+1)(b_k + \sqrt{3}c_k) - \sum_{r+s=k}(b_r b_s + c_r c_s)\,, \\ B_k =& \sum_{r+s+t=k}(3c_r\, b_s\, b_t - c_r\, c_s\, c_t) + \frac{3}{2}\epsilon \sum_{r+s=k}(s+1)\left(\sqrt{3}c_r c_s - 2c_r b_s - \sqrt{3}b_r b_s\right) \\ &+ \frac{3}{4}\epsilon^2(k+1)(k+2)(\sqrt{3}b_k - c_k)\,.\end{aligned}$$

d_k is the mode of $f_1 + f_2 = \sum_{k=-m}^{n-1} \frac{d_k}{z^{k+2}}$ and induces the flow equation when $-(m-1) \le k \le n-1$;

$$d_k = \begin{cases} v_k^\partial\left(-\hbar^2 \log \mathcal{Z}_{(m;n)}\right)\,, & 0 \le k \le n-1 \\ 2\epsilon(b_k N_b + c_k N_c) + u_k^\partial\left(-\hbar^2 \log \mathcal{Z}_{(m;n)}\right)\,, & -(m-1) \le k \le -1 \end{cases} \tag{A.9}$$

$$v_k^\partial = \sum_{\substack{r-s=k \\ 0<s}} s\left(b_r \frac{\partial}{\partial b_s} + c_r \frac{\partial}{\partial c_s}\right)\,, \qquad u_k^\partial = \sum_{\substack{r-s=k \\ s<0}} (-s)\left(b_r \frac{\partial}{\partial b_s} + c_r \frac{\partial}{\partial c_s}\right)$$

where $N_b = N - M/2$, $N_c = \sqrt{3}M/2$. On the other hand, d_{-m} is constant, $d_{-m} = 2\epsilon(b_{-m}N_b + c_{-m}N_c)$.

The mode e_k is defined as

$$-\sum_{k=-2m}^{2n-1} \frac{e_k}{z^{k+3}} = \sqrt{3}\left((f_1 - 2f_2)V_1' + (2f_1 - f_2)V_2'\right) + 3\sqrt{3}(g_1 - g_2) + \frac{3\sqrt{3}}{4}\epsilon(f_1' - f_2')\,,$$

and also induces the flow equation. When $-(2m-1) \leq k \leq 2n-1$, we have

$$e_k = \begin{cases} -\frac{1}{\mathcal{Z}_{(m;n)}} \hbar^2 \mu_k^{\partial}\, \mathcal{Z}_{(m;n)}\,, & n \leq k \leq 2n-1 \\ -\frac{1}{\mathcal{Z}_{(m;2n-k)}} \left(\hbar^2 \mu_k^{\partial} + \hbar^4 \mu_k^{\partial^2}\right) \mathcal{Z}_{(m;2n-k)}\Big|_{\{b_{k>n}, c_{k>n}\}\to 0}\,, & 0 \leq k \leq n-1 \\ \nu_k^c - \frac{1}{\mathcal{Z}_{(2m+k;n)}} \left(\hbar^2 \nu_k^{\partial} + \hbar^4 \nu_k^{\partial^2}\right) \mathcal{Z}_{(2m+k;n)}\Big|_{\{b_{-k>m}, c_{-k>m}\}\to 0}\,, & -(m-1) \leq k \leq -1 \\ \nu_k^c - \frac{1}{\mathcal{Z}_{(m;n)}} \hbar^2 \nu_k^{\partial}\, \mathcal{Z}_{(m;n)}\,, & -(2m-1) \leq k \leq -m\,. \end{cases} \tag{A.10}$$

where

$$\begin{aligned} \mu_k^{\partial} = & \sum_{\substack{r+s-t=k \\ t>0}} \frac{t}{2} \left(3c_r c_s \frac{\partial}{\partial c_t} - 6c_r b_s \frac{\partial}{\partial b_t} - 3b_r b_s \frac{\partial}{\partial c_t}\right) \\ & - \frac{3}{2}\epsilon \sum_{\substack{r-s=k \\ s>0}} \frac{s}{2} \left[(1+r)\left(\sqrt{3}c_r \frac{\partial}{\partial c_s} - 2b_r \frac{\partial}{\partial c_s} - \sqrt{3}b_r \frac{\partial}{\partial b_s}\right)\right. \\ & \qquad \left. +(1-s)\left(\sqrt{3}c_r \frac{\partial}{\partial c_s} - 2c_r \frac{\partial}{\partial b_s} - \sqrt{3}b_r \frac{\partial}{\partial b_s}\right)\right], \end{aligned} \tag{A.11}$$

$$\mu_k^{\partial^2} = - \sum_{\substack{r-s-t=k\,, \\ s,t>0}} \frac{s\,t}{4} \left(3c_r \frac{\partial}{\partial c_s} \frac{\partial}{\partial c_t} - 6b_r \frac{\partial}{\partial b_s} \frac{\partial}{\partial c_t} - 3c_r \frac{\partial}{\partial b_s} \frac{\partial}{\partial b_t}\right), \tag{A.12}$$

$$\begin{aligned} \nu_k^{\partial} = & \sum_{\substack{r+s-t=k \\ t<0}} \frac{(-t)}{2} \left(3c_r c_s \frac{\partial}{\partial c_t} - 6b_r c_s \frac{\partial}{\partial b_t} - 3b_r b_s \frac{\partial}{\partial c_t}\right) \\ & + \epsilon \sum_{\substack{r-s=k \\ s<0}} \frac{(-s)}{2} \left(6c_r N_c \frac{\partial}{\partial c_s} - 6b_r N_b \frac{\partial}{\partial c_s} - 6b_r N_c \frac{\partial}{\partial b_s} - 6c_r N_b \frac{\partial}{\partial b_s}\right) \\ & - \frac{3}{2}\epsilon \sum_{\substack{r-s=k \\ s<0}} \frac{(-s)}{2} \left[(1+r)\left(\sqrt{3}c_r \frac{\partial}{\partial c_s} - 2b_r \frac{\partial}{\partial c_s} - \sqrt{3}b_r \frac{\partial}{\partial b_s}\right)\right. \\ & \qquad \left. + (1-s)\left(\sqrt{3}c_r \frac{\partial}{\partial c_s} - 2c_r \frac{\partial}{\partial b_s} - \sqrt{3}b_r \frac{\partial}{\partial b_s}\right)\right], \end{aligned} \tag{A.13}$$

$$\nu_k^{\partial^2} = - \sum_{\substack{r-s-t=k \\ s,t<0}} \frac{s\,t}{4} \left(3c_r \frac{\partial}{\partial c_s} \frac{\partial}{\partial c_t} - 6b_r \frac{\partial}{\partial b_s} \frac{\partial}{\partial c_t} - 3c_r \frac{\partial}{\partial b_s} \frac{\partial}{\partial b_t}\right), \tag{A.14}$$

$$\begin{aligned} \nu_k^c = & + \epsilon \sum_{r+s=k} \left(3N_c\, c_r c_s - 6N_b\, c_r b_s - 3N_c\, b_r b_s\right) + \epsilon^2 \left(3c_k N_c^2 - 6b_k N_b N_c - 3c_k N_b^2\right) \\ & - \frac{3}{2}\epsilon^2 \left[(k+1)\left(\sqrt{3}c_k N_c - 2b_k N_c - \sqrt{3}b_k N_b\right) + \left(\sqrt{3}c_k N_c - 2c_k N_b - \sqrt{3}b_k N_b\right)\right]. \end{aligned} \tag{A.15}$$

Note that the ϵ terms in μ_k^{∂} vanished identically when $k \geq n$ and same for ν_k^{∂} when $k \leq -m$. And $e_{-2m} = \nu_{-2m}^c$ is a constant.

Note that we introduced the extended partition function in (A.10); $\mathcal{Z}_{(m;2n-k)}$ for $0 \le k \le n-1$ and $\mathcal{Z}_{(2m+k;n)}$ for $-(m-1) \le k \le -1$. This is because g_1 and g_2 can have the expectation values $\langle 1/x_i^r \rangle$, $\left\langle 1/y_j^r \right\rangle$ with $-(2m+k) \le r < -m$ and $n < r \le 2n-k$. To represent these expectation values in terms of derivatives of the partition function, we need to extend the parameter space up to b_{2n-k}, c_{2n-k} when $0 \le k \le n-1$, and up to $b_{-(2m+k)}$ and $c_{-(2m+k)}$ when $-(m-1) \le k \le -1$. After evaluation of the derivatives, we put the parameters zero [36].

B Representation of W3 currents

ξ_2 and ξ_3 are the expectation values of the Virasoro and $\mathcal{W}_3$ current:

$$\xi_2 = \frac{\langle I_m|T(z)|I_n\rangle}{\langle I_m|I_n\rangle}\,, \qquad \xi_3 = \frac{\langle I_m|W(z)|I_n\rangle}{\langle I_m|I_n\rangle}\,. \tag{B.1}$$

One can check that the modes of ξ_2 and ξ_3 in (A.8) are compatible with the $\mathcal{W}_3$ algebraic commutation relation:

$$[L_p, L_q] = (p-q)L_{p+q} + \frac{c}{12}(p^3-p)\delta_{p,-q}\,, \tag{B.2}$$

$$[L_p, W_q] = (2p-q)W_{p+q}\,, \tag{B.3}$$

$$\begin{aligned}-\frac{2}{9}\left(\frac{32}{22+5c}\right)[W_p, W_q] &= \frac{c}{3\cdot 5!}(p^2-1)(p^2-4)p\delta_{p,-q} + \frac{16}{22+5c}(p-q)\Lambda_{p+q} \\ &\quad + (p-q)\left(\frac{1}{15}(p+q+2)(p+q+3) - \frac{1}{6}(p+2)(q+2)\right)L_{p+q}\,,\end{aligned} \tag{B.4}$$

where[2]

$$\Lambda_p = \sum_{k=-\infty}^{\infty} :L_k L_{p-k}: + \frac{1}{5}x_p L_p\,,$$

$$x_{2\ell} = (\ell+1)(\ell-1)\,, \qquad x_{2\ell+1} = (2+\ell)(1-\ell)\,,$$

and the central charge $c = 2 + 24\epsilon^2$.

Note that the negative generators L_{-k} and W_{-k} ($k > 0$) obtained in (A.9) and (A.10) are left representation in the sense that negative generators should act on the bra $\langle I_m|$. However, to check the commutation relation (B.4) we need to find right representations of negative generators acting on ket $|I_n\rangle$. To find the right representation we follow the trick used in [36]: use the transformation of the integration variables $x_i \to x_i + \varepsilon/x_i^r$ and $y_j \to y_j + \varepsilon/y_j^r$ to obtain two identities which can be used to find the relation for d_k ($k \le -1$):

$$d_k = -\left.\frac{\left(\hbar^2 v_k^{\partial} + \hbar^4 v_k^{\partial^2}\right)\mathcal{Z}_{(m;n-k)}}{\mathcal{Z}_{(m;n-k)}}\right|_{\{b_{k>n}, c_{k>n}\}\to 0}\,, \tag{B.5}$$

[2] If one rescales W_p as $i\frac{3}{\sqrt{2}}\sqrt{\frac{22+5c}{32}}W_p$, then the algebra reduces to the original Fateev and Zamolodchikov convention [55]. Note that because of the factor i, the modes of $W(z)$ are anti-hermitian: $W_k^\dagger = -W_k$.

where

$$v_k^{\partial^2} = -\sum_{-(r+s)=k} \frac{r\,s}{4}\left(\frac{\partial}{\partial b_r}\frac{\partial}{\partial b_s} + \frac{\partial}{\partial c_r}\frac{\partial}{\partial c_s}\right) + \frac{\epsilon}{2}k(k+1)\left(\frac{\partial}{\partial b_{-k}} + \sqrt{3}\frac{\partial}{\partial c_{-k}}\right), \tag{B.6}$$

When $k < -m$, one realizes that d_k vanishes identically and $v_{-1}^{\partial^2} = 0$ by definition.

Likewise, after the change of variables $x_i \to x_i + \sum_{j=1}^{N_2} \frac{\varepsilon}{(x_i - y_j)x_i^r}$ and $y_j \to y_j + \sum_{i=1}^{N_1} \frac{\varepsilon}{(x_i - y_j)y_j^r}$, one finds the right representation of the negative mode $k \leq -1$ $\mathcal{W}_3$ current:

$$e_k = -\left.\frac{(\hbar^2 \mu_k^{\partial} + \hbar^4\, \mu_k^{\partial^2} + \hbar^6\, \mu_k^{\partial^3})\, \mathcal{Z}_{(m;2n-k)}}{\mathcal{Z}_{(m;2n-k)}}\right|_{\{b_{k>n}, c_{k>n}\}\to 0}, \tag{B.7}$$

where

$$\begin{aligned}\mu_k^{\partial^3} = &-\sum_{-(r+s+t)=k} \frac{r\,s\,t}{8}\left(3\frac{\partial}{\partial b_r}\frac{\partial}{\partial b_s}\frac{\partial}{\partial b_t} - \frac{\partial}{\partial c_r}\frac{\partial}{\partial c_s}\frac{\partial}{\partial c_t}\right)\\ &+ \frac{3}{2}\epsilon \sum_{-(r+s)=k} \frac{r\,s\,(1-s)}{4}\left(\sqrt{3}\frac{\partial}{\partial c_r}\frac{\partial}{\partial c_s} - 2\frac{\partial}{\partial c_r}\frac{\partial}{\partial b_s} - \sqrt{3}\frac{\partial}{\partial b_r}\frac{\partial}{\partial b_s}\right)\\ &+ \frac{3}{8}\epsilon^2 k(k+1)(k+2)\left(\sqrt{3}\frac{\partial}{\partial b_{-k}} - \frac{\partial}{\partial c_{-k}}\right).\end{aligned} \tag{B.8}$$

We have $e_{k<-2m} = 0$ and $\mu_k^{\partial^3} = 0$ for $k = -1, -2$.

If we define the differential operator v_k and μ_k by

$$v_k = \begin{cases} v_k^{\partial}\,, & -1 \leq k \leq n-1 \\ v_k^{\partial} + v_k^{\partial^2}\,, & k \leq -2 \end{cases}, \qquad \mu_k = \begin{cases} \mu_k^{\partial}\,, & n \leq k \leq 2n-1 \\ \mu_k^{\partial} + \mu_k^{\partial^2}\,, & -2 \leq k \leq n-1 \\ \mu_k^{\partial} + \mu_k^{\partial^2} + \mu_k^{\partial^3}\,, & k \leq -3\,. \end{cases} \tag{B.9}$$

then, the right representation of the Virasoro and $\mathcal{W}_3$ currents has the expression

$$\mathcal{L}_k = \begin{cases} 0\,, & 2n < k \\ A_k\,, & n \leq k \leq 2n \\ A_k + v_k\,, & -2m \leq k \leq n-1 \\ v_k\,, & k < -2m \end{cases}, \qquad \Omega_k = \begin{cases} 0\,, & 3n < k \\ M_k\,, & 2n \leq k \leq 3n \\ M_k + \mu_k\,, & -3m \leq k \leq 2n-1 \\ \mu_k\,, & k < -3m\,. \end{cases} \tag{B.10}$$

where $\langle I_m|L_k|I_n\rangle := \mathcal{L}_k \langle I_m|I_n\rangle$ and $\langle I_m|W_k|I_n\rangle := \Omega_k \langle I_m|I_n\rangle$. One can check that the right representation satisfies the commutation relations (B.4).

C Perturbation method to find flow equations in A_2 model

In this section, we apply another method to find flow equations, with no need to assume ϵ to be small. Instead, we will suppose hierarchy in the Toda momentum a_k and b_k.

C.1 $\mathcal{Z}_{(0;1)}$

Expanding in terms of z, to the highest power of (3.18), z^{N+M-2} shows that

$$d_0 = \epsilon^2\big(N(N-1) - NM + M(M-1)\big) + 2\epsilon[Nb_0 + Ma_0]\,, \tag{C.1}$$

Expanding (3.19) in terms of z, for z^{N-3-k} we have

$$\begin{aligned} 0 =\; & \epsilon^3 P_{N-k}(N-k)(N-k-1)(N-k-2) \\ & + 2\epsilon^2 \sum_{t=-m}^{n} P_{N-k+t}(N-k+t)(N-k-2)(2b_t + a_t) \\ & + \epsilon\Bigg(\sum_{t=-2m}^{2n} P_{N-k+t}(N-k+t)A_t - \sum_{t=-m}^{n-1} P_{N-k+t}(N-k+t)d_t \\ & \quad + \frac{4}{3}\sum_{t=-2m}^{2n} P_{N-k+t}(N-k+t)\left[\sum_{s=-m}^{n}(2b_s+a_s)(2b_{t-s}+a_{t-s})\right]\Bigg) \\ & - \sum_{t=-2m}^{2n-1} P_{N-k+t}\left[-\frac{2}{3\sqrt{3}}e_t + \frac{2}{3}\sum_{s=-m}^{n-1} d_s(2b_{t-s}+a_{t-s})\right] + \frac{\epsilon}{2}\sum_{t=-m}^{n-1} P_{N-k+t}(t+2)d_t\,. \end{aligned} \tag{C.2}$$

The next power z^{N+M-3} of (3.18) gives

$$\begin{aligned} & P_{N-1}\Big\{\epsilon^2\big[M - 2(N-1)\big] - 2\epsilon b_0\Big\} + Q_{M-1}\Big\{\epsilon^2\big[N - 2(M-1)\big] - 2\epsilon a_0\Big\} \\ & \quad = -2\epsilon\big[Nb_1 + Ma_1\big]\,, \end{aligned} \tag{C.3}$$

Then let's turn back to eq. (3.19). For z^{N-3-k},

$$\begin{aligned} 0 = P_{N-k}\Bigg\{ & -\left[-\frac{2}{3\sqrt{3}}e_0 + \frac{2}{3}d_0(2b_0+a_0) - \epsilon d_0\right] + \epsilon^3(N-k)(N-k-1)(N-k-2) \\ & + 2\epsilon^2(N-k)(N-k-2)(2b_0+a_0) + \epsilon(N-k)\left[-d_0 + 2\epsilon(b_0+a_0) + 4(b_0^2 + a_0b_0)\right]\Bigg\} \\ + P_{N-k+1}\Bigg\{ & -\left[-\frac{2}{3\sqrt{3}}e_1 + \frac{2}{3}d_0(2b_1+a_1)\right] \\ & + 2\epsilon^2(N-k+1)(N-k-2)(2b_1+a_1) + 4\epsilon(N-k+1)[\epsilon(b_1+a_1) + (2b_0b_1 + a_1b_0 + a_0b_1)]\Bigg\} \\ + P_{N-k+2}\Big\{ & \epsilon(N-k+2)[4(b_1^2 + a_1b_1)]\Big\}\,, \end{aligned} \tag{C.4}$$

where we have used the definition of A_k in (3.8). The corresponding equations of Q_{M-k} can be obtained by setting $P_{N-k} \to Q_{M-k}$, $e_k \to -e_k$ and $b_k \to a_k$, $a_k \to b_k$.

At each power of z, we have identities,

$$\begin{aligned} z^{N-3}: \quad & \left[-\frac{2}{3\sqrt{3}}e_0 + \frac{2}{3}d_0(2b_0+a_0) - \epsilon d_0\right] = \\ & \epsilon^3 N(N-1)(N-2) + 2\epsilon^2 N(N-2)(2b_0+a_0) \\ & \quad + \epsilon N[-d_0 + 2\epsilon(b_0+a_0) + 4(b_0^2 + a_0b_0)]\,, \end{aligned} \tag{C.5}$$

$$z^{N-4}: \quad 0 = P_{N-1}\Big\{ -[-\frac{2}{3\sqrt{3}}e_0 + \frac{2}{3}d_0(2b_0+a_0) - \epsilon d_0] + \epsilon^3(N-1)(N-2)(N-3)$$
$$+ 2\epsilon^2(N-1)(N-3)(2b_0+a_0) + \epsilon(N-1)[-d_0 + 2\epsilon(b_0+a_0) + 4(b_0^2+a_0b_0)]\Big\}$$
$$+ \Big\{ - \Big[-\frac{2}{3\sqrt{3}}e_1 + \frac{2}{3}d_0(2b_1+a_1)\Big] \tag{C.6}$$
$$+ 2\epsilon^2 N(N-3)(2b_1+a_1) + 4\epsilon N[\epsilon(b_1+a_1) + (2b_0b_1 + a_1b_0 + a_0b_1)]\Big\},$$

$$z^{N-5}: \quad 0 = P_{N-2}\Big\{ - \Big[-\frac{2}{3\sqrt{3}}e_0 + \frac{2}{3}d_0(2b_0+a_0) - \epsilon d_0\Big] + \epsilon^3(N-2)(N-3)(N-4)$$
$$+ 2\epsilon^2(N-2)(N-4)(2b_0+a_0) + \epsilon(N-2)[-d_0 + 2\epsilon(b_0+a_0) + 4(b_0^2+a_0b_0)]\Big\}$$
$$+ P_{N-1}\Big\{ - \Big[-\frac{2}{3\sqrt{3}}e_1 + \frac{2}{3}d_0(2b_1+a_1)\Big] \tag{C.7}$$
$$+ 2\epsilon^2(N-1)(N-4)(2b_1+a_1) + 4\epsilon(N-1)[\epsilon(b_1+a_1) + (2b_0b_1+a_1b_0+a_0b_1)]\Big\}$$
$$+ \Big\{\epsilon N[4(b_1^2 + a_1b_1)]\Big\},$$

To find e_1, we use perturbation assuming $|b_1| \ll |a_1|$ so that $|P_{N-k}| \sim |b_1|^k$. Then at the first order we have from the above equations (C.6) and (C.7):

$$z^{N-4}: \quad \frac{2}{3\sqrt{3}}e_1^{(1)} = \frac{2}{3}d_0a_1 - 2\epsilon^2 N(N-1)a_1 - 4\epsilon N b_0 a_1 \equiv B_1(b_0,a_0)a_1\,, \tag{C.8}$$
$$z^{N-5}: \quad 0 = P_{N-1}^{(1)}\Big\{ - \Big[-\frac{2}{3\sqrt{3}}e_1^{(1)} + \frac{2}{3}d_0a_1\Big] \tag{C.9}$$
$$+ 2\epsilon^2(N-1)(N-4)a_1 + 4\epsilon(N-1)[\epsilon a_1 + a_1b_0]\Big\} + \Big\{\epsilon N[4a_1b_1]\Big\},$$
$$P_{N-1}^{(1)} = \frac{Nb_1}{\epsilon(N-1)+b_0}\,. \tag{C.10}$$

At the second order, we have

$$z^{N-4}: \quad 0 = P_{N-1}^{(1)}\Big\{ - \Big[-\frac{2}{3\sqrt{3}}e_0 + \frac{2}{3}d_0(2b_0+a_0) - \epsilon d_0\Big] + \epsilon^3(N-1)(N-2)(N-3)$$
$$+ 2\epsilon^2(N-1)(N-3)(2b_0+a_0) + \epsilon(N-1)[-d_0 + 2\epsilon(b_0+a_0) + 4(b_0^2+a_0b_0)]\Big\}$$
$$+ \Big\{ - \Big[-\frac{2}{3\sqrt{3}}e_1^{(2)} + \frac{4}{3}d_0b_1\Big] + 4\epsilon^2 N(N-3)b_1 + 4\epsilon N[\epsilon b_1 + (2b_0b_1 + a_0b_1)]\Big\}, \tag{C.11}$$

$$\frac{2}{3\sqrt{3}}e_1^{(2)} = \frac{4}{3}d_0b_1 - 4\epsilon N(2b_0+a_0)b_1 - 4\epsilon^2 N(N-2)b_1 + \frac{Nb_1}{\epsilon(N-1)+b_0}\Big\{4\epsilon(b_0+a_0)b_0$$
$$+ 2\epsilon^2(b_0+a_0) + 2\epsilon^2(2N-3)(2b_0+a_0) + 3\epsilon^3(N-1)(N-2) - \epsilon d_0\Big\} \tag{C.12}$$
$$\equiv B_2(b_0,a_0)b_1\,.$$

Up to $\mathcal{O}(b_1)$ we find $e_1 = e_1^{(1)} + e_1^{(2)}$. If one expands e_1 in terms of ϵ, it reads

$$\begin{aligned}\frac{2}{3\sqrt{3}}e_1 = &\frac{4}{3}\epsilon\left[(a_1+2b_1)(a_0M+b_0N) - 3b_0N(a_1+b_1)\right] \\ &+\epsilon^2\left[\frac{2}{3}(a_1+2b_1)(N(N-1)+M(M-1)-NM)-2N(N-1)(a_1+b_1)-2NM\frac{a_0}{b_0}b_1\right] \\ &+\mathcal{O}(\epsilon^3).\end{aligned} \tag{C.13}$$

This result is in perfect agreement with (3.28), expanding up to $\mathcal{O}(b_1)$. In fact, the perturbative condition $|b_1| \ll |a_1|$ is equivalent to choosing the filling fraction $N_1 = N$, and $N_2 = 0$.

In this way we can find that

$$\begin{aligned} e_1 &= e_1^{(1)} + e_1^{(2)} + e_1^{(3)} + \ldots + e_1^{(k+2)} + \ldots \\ &= B_1(b_0,a_0)a_1 + B_2(b_0,a_0)b_1 + B_3(b_0,a_0)b_1\frac{b_1}{a_1} + \ldots + B_{k+2}(b_0,a_0)b_1\left(\frac{b_1}{a_1}\right)^k + \ldots\end{aligned} \tag{C.14}$$

The flow equations for rank 1 case are

$$-\hbar^2 v_0 \log \mathcal{Z}_1 = d_0\,, \qquad -\hbar^2\mu_1 \log\mathcal{Z}_1 = e_1\,, \tag{C.15}$$

where $v_0 = b_1\frac{\partial}{\partial b_1} + a_1\frac{\partial}{\partial a_1}$ and $\mu_1 = \sqrt{3}(a_1^2+2a_1b_1)\frac{\partial}{\partial a_1} - \sqrt{3}(2a_1b_1+b_1^2)\frac{\partial}{\partial b_1}$. From the first equation of (C.15) we find

$$\hbar^2 \log \mathcal{Z}_1 = -d_0\log a_1 + H(t)\,, \tag{C.16}$$

where $t := b_1/a_1$ and $H(t)$ is a homogeneous solution to v_0. Put $H(t)$ into the second equation of (C.15), we get

$$3(t+1)t\frac{\partial H(t)}{\partial t} = \frac{1}{\sqrt{3}}\frac{e_1}{a_1} - (1+2t)d_0\,. \tag{C.17}$$

From (C.14), it is clear that

$$\frac{e_1}{a_1} = B_1(b_0,a_0) + B_2(b_0,a_0)t + B_3(b_0,a_0)t^2 + \ldots \tag{C.18}$$

Therefore, we have

$$H(t) = \frac{1}{3}\left(\frac{B_1}{\sqrt{3}} - d_0\right)\log t - \frac{1}{3}\left(\frac{B_1-B_2+B_3}{\sqrt{3}} + d_0\right)\log(t+1) + \frac{B_3}{3\sqrt{3}}t + \ldots\,, \tag{C.19}$$

and partition function

$$\mathcal{Z}_{(0;1)} = \mathcal{N} a_1^{-d_0/\hbar^2} t^{\left(\frac{B_1}{\sqrt{3}}-d_0\right)/3\hbar^2}(t-1)^{-\left(\frac{B_1-B_2+B_3}{\sqrt{3}}+d_0\right)/3\hbar^2} e^{B_3t/(3\sqrt{3})}\,, \tag{C.20}$$

where $\mathcal{N}$ is a function of a_0, b_0 and B_k with $k \geq 4$.

C.2 $\mathcal{Z}_{(0;2)}$

We need d_0, d_1, e_2 and e_3 to obtain the partition function. From the highest power z^{N+M-2} of the quadratic equation (3.16) we obtain the expression of d_0 for any rank n. Now for rank 2 case, from the second highest power z^{N+M-3} we have

$$\begin{aligned} P_{N-1}&\Big\{-d_0+\epsilon^2\big[(N-1)(N-2)-(N-1)M+(M-1)M\big]+2\epsilon\big[(N-1)b_0+Ma_0\big]\Big\}\\ &+Q_{M-1}\Big\{-d_0+\epsilon^2\big[(M-1)(M-2)-(M-1)N+(N-1)N\big]+2\epsilon\big[Nb_0+(M-1)a_0\big]\Big\}\\ &\quad=d_1-2\epsilon\big[Nb_1+Ma_1\big]\,, \end{aligned} \tag{C.21}$$

From the cubic equation (3.19), we have

$$\begin{aligned} z^{N-3}:\quad &\left[-\frac{2}{3\sqrt{3}}e_0+\frac{2}{3}d_0(2b_0+a_0)-\epsilon d_0\right]\\ &\quad=\epsilon^3N(N-1)(N-2)+2\epsilon^2N(N-2)(2b_0+a_0)+\epsilon N(A_0-d_0)+\frac{4}{3}\epsilon N(2b_0+a_0)^2\,, \end{aligned} \tag{C.22}$$

$$\begin{aligned} z^{N-4}:\quad &P_{N-1}\left[-\frac{2}{3\sqrt{3}}e_0+\frac{2}{3}d_0(2b_0+a_0)-\epsilon d_0\right]\\ &+\left[-\frac{2}{3\sqrt{3}}e_1+\frac{2}{3}d_0(2b_1+a_1)+\frac{2}{3}d_1(2b_0+a_0)-\frac{3}{2}\epsilon d_1\right]\\ &\quad=\epsilon^3P_{N-1}(N-1)(N-2)(N-3)\\ &\quad+2\epsilon^2\Big(P_{N-1}(N-1)(N-3)(2b_0+a_0)+N(N-3)(2b_1+a_1)\Big)\\ &\quad+\epsilon\Big(P_{N-1}(N-1)(A_0-d_0)+N(A_1-d_1)\Big)\\ &\quad+\frac{4}{3}\epsilon\Big(P_{N-1}(N-1)(2b_0+a_0)^2+2N(2b_0+a_0)(2b_1+a_1)\Big)\,, \end{aligned} \tag{C.23}$$

$$\begin{aligned} z^{N-5}:\quad &P_{N-2}\left[-\frac{2}{3\sqrt{3}}e_0+\frac{2}{3}d_0(2b_0+a_0)-\epsilon d_0\right]\\ &+P_{N-1}\left[-\frac{2}{3\sqrt{3}}e_1+\frac{2}{3}d_0(2b_1+a_1)+\frac{2}{3}d_1(2b_0+a_0)-\frac{3}{2}\epsilon d_1\right]\\ &+\left[-\frac{2}{3\sqrt{3}}e_2+\frac{2}{3}d_0(2b_2+a_2)+\frac{2}{3}d_1(2b_1+a_1)\right]\\ &\quad=\epsilon^3P_{N-2}(N-2)(N-3)(N-4)\\ &\quad+2\epsilon^2\Big(P_{N-2}(N-2)(N-4)(2b_0+a_0)+P_{N-1}(N-1)(N-4)(2b_1+a_1)\\ &\quad+N(N-4)(2b_2+a_2)\Big)\\ &\quad+\epsilon\Big(P_{N-2}(N-2)(A_0-d_0)+P_{N-1}(N-1)(A_1-d_1)+NA_2\Big)\\ &\quad+\frac{4}{3}\epsilon\Big(P_{N-2}(N-2)(2b_0+a_0)^2+2P_{N-1}(N-1)(2b_0+a_0)(2b_1+a_1)\\ &\quad+N[2(2b_0+a_0)(2b_2+a_2)+(2b_1+a_1)^2]\Big)\,, \end{aligned} \tag{C.24}$$

$$
\begin{aligned}
z^{N-6}:\quad & P_{N-3}\left[-\frac{2}{3\sqrt{3}}e_0+\frac{2}{3}d_0(2b_0+a_0)-\epsilon d_0\right] \\
&+P_{N-2}\left[-\frac{2}{3\sqrt{3}}e_1+\frac{2}{3}d_0(2b_1+a_1)+\frac{2}{3}d_1(2b_0+a_0)-\frac{3}{2}\epsilon d_1\right] \\
&+P_{N-1}\left[-\frac{2}{3\sqrt{3}}e_2+\frac{2}{3}d_0(2b_2+a_2)+\frac{2}{3}d_1(2b_1+a_1)\right] \\
&+\left[-\frac{2}{3\sqrt{3}}e_3+\frac{2}{3}d_1(2b_2+a_2)\right] \\
&\quad=\epsilon^3P_{N-3}(N-3)(N-4)(N-5) \qquad\qquad (\text{C.25})\\
&\quad+2\epsilon^2\Big(P_{N-3}(N-3)(N-5)(2b_0+a_0)+P_{N-2}(N-2)(N-5)(2b_1+a_1) \\
&\quad+P_{N-1}(N-1)(N-5)(2b_2+a_2)\Big) \\
&\quad+\epsilon\Big(P_{N-3}(N-3)(A_0-d_0)+P_{N-2}(N-2)(A_1-d_1)+P_{N-1}(N-1)A_2 \\
&\quad+4N(2b_1b_2+a_1b_2+a_2b_1)\Big) \\
&\quad+\frac{4}{3}\epsilon\Big(P_{N-3}(N-3)(2b_0+a_0)^2+2P_{N-2}(N-2)(2b_0+a_0)(2b_1+a_1) \\
&\quad+P_{N-1}(N-1)[2(2b_0+a_0)(2b_2+a_2)+(2b_1+a_1)^2]\Big)\,, \\
z^{N-7}:\quad & P_{N-4}\left[-\frac{2}{3\sqrt{3}}e_0+\frac{2}{3}d_0(2b_0+a_0)-\epsilon d_0\right] \\
&+P_{N-3}\left[-\frac{2}{3\sqrt{3}}e_1+\frac{2}{3}d_0(2b_1+a_1)+\frac{2}{3}d_1(2b_0+a_0)-\frac{3}{2}\epsilon d_1\right] \\
&+P_{N-2}\left[-\frac{2}{3\sqrt{3}}e_2+\frac{2}{3}d_0(2b_2+a_2)+\frac{2}{3}d_1(2b_1+a_1)\right] \\
&+P_{N-1}\left[-\frac{2}{3\sqrt{3}}e_3+\frac{2}{3}d_1(2b_2+a_2)\right] \\
&\quad=\epsilon^3P_{N-4}(N-4)(N-5)(N-6) \qquad\qquad (\text{C.26})\\
&\quad+2\epsilon^2\Big(P_{N-4}(N-4)(N-6)(2b_0+a_0)+P_{N-3}(N-3)(N-6)(2b_1+a_1) \\
&\quad+P_{N-2}(N-2)(N-6)(2b_2+a_2)\Big) \\
&\quad+\epsilon\Big(P_{N-4}(N-4)(A_0-d_0)+P_{N-3}(N-3)(A_1-d_1)+P_{N-2}(N-2)A_2 \\
&\quad+P_{N-1}(N-1)A_3+4Nb_2(b_2+a_2)\Big) \\
&\quad+\frac{4}{3}\epsilon\Big(P_{N-4}(N-4)(2b_0+a_0)^2+2P_{N-3}(N-3)(2b_0+a_0)(2b_1+a_1) \\
&\quad+P_{N-2}(N-2)[2(2b_0+a_0)(2b_2+a_2)+(2b_1+a_1)^2] \\
&\quad+2P_{N-1}(N-1)(2b_1+a_1)(2b_2+a_2)\Big)\,.
\end{aligned}
$$

Perturbation holds if we require $|b_2/b_1| \ll |b_1| \ll 1$, $|b_2| \ll |a_2|$, and $|b_1| \sim |a_1|$, so that

$|P_{N-k}| \sim |b_2/b_1|^k$ is ensured. Then at the first order of the perturbation, we have: from the quadratic equation:

$$d_0 = \epsilon^2\big(N(N-1) - NM + M(M-1)\big) + 2\epsilon[Nb_0 + Ma_0]\,, \tag{C.27}$$

$$z^{N+M-3}: \qquad d_1^{(1)} = 2\epsilon\big[Nb_1 + Ma_1\big]\,. \tag{C.28}$$

From the equations (C.23) to (C.26):

$$z^{N-4}: \quad \left[-\frac{2}{3\sqrt{3}}e_1^{(1)} + \frac{2}{3}d_0(2b_1+a_1) + \frac{2}{3}d_1^{(1)}(2b_0+a_0) - \frac{3}{2}\epsilon d_1^{(1)}\right]$$
$$= 2\epsilon^2 N(N-3)(2b_1+a_1) + \epsilon N(A_1 - d_1^{(1)}) + \frac{8}{3}\epsilon N(2b_0+a_0)(2b_1+a_1)\,, \tag{C.29}$$

$$z^{N-5}: \quad \left[-\frac{2}{3\sqrt{3}}e_2^{(1)} + \frac{2}{3}d_0 a_2 + \frac{2}{3}d_1^{(1)}(2b_1+a_1)\right] \tag{C.30}$$
$$= 2\epsilon^2 N(N-4)a_2 + \epsilon N A_2 + \frac{4}{3}\epsilon N[2(2b_0+a_0)a_2 + (2b_1+a_1)^2]\,,$$

$$z^{N-6}: \quad \left[-\frac{2}{3\sqrt{3}}e_3^{(1)} + \frac{2}{3}d_1^{(1)}a_2\right] = 4\epsilon N a_2 b_1\,, \tag{C.31}$$

$$z^{N-7}: \quad P_{N-1}^{(1)}\left[-\frac{2}{3\sqrt{3}}e_3^{(1)} + \frac{2}{3}d_1^{(1)}a_2\right] \tag{C.32}$$
$$= \epsilon\left(P_{N-1}^{(1)}(N-1)A_3 + 4Nb_2a_2\right) + \frac{8}{3}\epsilon P_{N-1}(N-1)(2b_1+a_1)a_2\,.$$

Thus d_0, d_1, e_2 and e_3 are obtained.

C.3 $\mathcal{Z}_{(1;1)}$

We need d_0, e_{-1} and e_1 to evaluate. From the power expansion of the quadratic equation, we know

$$z^{N+M-1}: \quad d_{-1} = 2\epsilon\big[Nb_{-1} + Ma_{-1}\big]\,, \tag{C.33}$$

$$z^{N+M-2}: \quad d_{-1}(P_{N-1} + Q_{M-1}) + d_0 \tag{C.34}$$
$$= \epsilon^2\big(N(N-1) - NM + M(M-1)\big) + 2\epsilon[Nb_0 + Ma_0]$$
$$+ 2\epsilon[b_{-1}\big((N-1)P_{N-1} + NQ_{M-1}\big) + a_{-1}\big((M-1)Q_{M-1} + MP_{N-1}\big)]\,,$$

$$z^{N+M-3}: \quad d_{-1}(P_{N-2} + Q_{M-2} + P_{N-1}Q_{M-1}) + d_0(P_{N-1} + Q_{M-1}) \tag{C.35}$$
$$= P_{N-1}\Big\{\epsilon^2\big[(N-1)(N-2) - (N-1)M + (M-1)M\big] + 2\epsilon\big[(N-1)b_0 + Ma_0\big]\Big\}$$
$$+ Q_{M-1}\Big\{\epsilon^2\big[(M-1)(M-2) - (M-1)N + (N-1)N\big] + 2\epsilon\big[Nb_0 + (M-1)a_0\big]\Big\}$$
$$+ 2\epsilon[b_{-1}\big((N-2)P_{N-2} + NQ_{M-2} + (N-1)P_{N-1}Q_{M-1}\big)$$
$$+ a_{-1}\big((M-2)Q_{M-2} + MP_{N-2} + (M-1)Q_{M-1}P_{N-1}\big)]\,,$$

From the cubic equation

$$z^{N-1}: \quad \left[-\frac{2}{3\sqrt{3}}e_{-2}+\frac{2}{3}d_{-1}(2b_{-1}+a_{-1})\right]=\frac{4}{3}\epsilon N(2b_{-1}+a_{-1})^2+\epsilon N A_{-2}\,, \tag{C.36}$$

$$\begin{aligned}
z^{N-2}: \quad & P_{N-1}\left[-\frac{2}{3\sqrt{3}}e_{-2}+\frac{2}{3}d_{-1}(2b_{-1}+a_{-1})\right] \\
&+\left[-\frac{2}{3\sqrt{3}}e_{-1}+\frac{2}{3}d_0(2b_{-1}+a_{-1})+\frac{2}{3}d_{-1}(2b_0+a_0)-\frac{\epsilon}{2}d_{-1}\right] \\
&\quad =2\epsilon^2N(N-1)(2b_{-1}+a_{-1}) \\
&\quad +\frac{4}{3}\epsilon\Big(2N(2b_{-1}+a_{-1})(2b_0+a_0)+P_{N-1}(N-1)(2b_{-1}+a_{-1})^2\Big) \\
&\quad +\epsilon\Big(P_{N-1}(N-1)A_{-2}+N(A_{-1}-d_{-1})\Big)\,,
\end{aligned}$$

$$\begin{aligned}
z^{N-3}: \quad & P_{N-2}\left[-\frac{2}{3\sqrt{3}}e_{-2}+\frac{2}{3}d_{-1}(2b_{-1}+a_{-1})\right] \\
&+P_{N-1}\left[-\frac{2}{3\sqrt{3}}e_{-1}+\frac{2}{3}d_0(2b_{-1}+a_{-1})+\frac{2}{3}d_{-1}(2b_0+a_0)-\frac{\epsilon}{2}d_{-1}\right] \\
&+\left[-\frac{2}{3\sqrt{3}}e_0+\frac{2}{3}d_0(2b_0+a_0)+\frac{2}{3}d_{-1}(2b_1+a_1)-\epsilon d_0\right] \\
&\quad =\epsilon^3N(N-1)(N-2) \qquad\qquad (\text{C.37}) \\
&\quad +2\epsilon^2\Big(N(N-2)(2b_0+a_0)+P_{N-1}(N-1)(N-2)(2b_{-1}+a_{-1})\Big) \\
&\quad +\frac{4}{3}\epsilon\Big(N(2b_0+a_0)^2+2P_{N-1}(N-1)(2b_{-1}+a_{-1})(2b_0+a_0) \\
&\quad +P_{N-2}(N-2)(2b_{-1}+a_{-1})^2\Big) \\
&\quad +\epsilon\Big(P_{N-2}(N-2)A_{-2}+P_{N-1}(N-1)(A_{-1}-d_{-1})+N(A_0-d_0)\Big)\,,
\end{aligned}$$

$$\begin{aligned}
z^{N-4}: \quad & P_{N-3}\left[-\frac{2}{3\sqrt{3}}e_{-2}+\frac{2}{3}d_{-1}(2b_{-1}+a_{-1})\right] \\
&+P_{N-2}\left[-\frac{2}{3\sqrt{3}}e_{-1}+\frac{2}{3}d_0(2b_{-1}+a_{-1})+\frac{2}{3}d_{-1}(2b_0+a_0)-\frac{\epsilon}{2}d_{-1}\right] \\
&+P_{N-1}\left[-\frac{2}{3\sqrt{3}}e_0+\frac{2}{3}d_0(2b_0+a_0)+\frac{2}{3}d_{-1}(2b_1+a_1)-\epsilon d_0\right] \\
&+\left[-\frac{2}{3\sqrt{3}}e_1+\frac{2}{3}d_0(2b_1+a_1)\right] \\
&\quad =\epsilon^3P_{N-1}(N-1)(N-2)(N-3) \qquad\qquad (\text{C.38}) \\
&\quad +2\epsilon^2\Big(N(N-3)(2b_1+a_1)+P_{N-1}(N-1)(N-3)(2b_0+a_0) \\
&\quad +P_{N-2}(N-2)(N-3)(2b_{-1}+a_{-1})\Big) \\
&\quad +\frac{4}{3}\epsilon\Big(2N(2b_0+a_0)(2b_1+a_1)+P_{N-1}(N-1)(2b_0+a_0)^2 \\
&\quad +2P_{N-2}(N-2)(2b_{-1}+a_{-1})(2b_0+a_0)+P_{N-3}(N-3)(2b_{-1}+a_{-1})^2\Big) \\
&\quad +\epsilon\Big(P_{N-3}(N-3)A_{-2}+P_{N-2}(N-2)(A_{-1}-d_{-1}) \\
&\quad +P_{N-1}(N-1)(A_0-d_0)+NA_1\Big)\,,
\end{aligned}$$

$$
\begin{aligned}
z^{N-5}: \quad & P_{N-4}\left[-\frac{2}{3\sqrt{3}}e_{-2}+\frac{2}{3}d_{-1}(2b_{-1}+a_{-1})\right] \\
&+P_{N-3}\left[-\frac{2}{3\sqrt{3}}e_{-1}+\frac{2}{3}d_0(2b_{-1}+a_{-1})+\frac{2}{3}d_{-1}(2b_0+a_0)-\frac{\epsilon}{2}d_{-1}\right] \\
&+P_{N-2}\left[-\frac{2}{3\sqrt{3}}e_0+\frac{2}{3}d_0(2b_0+a_0)+\frac{2}{3}d_{-1}(2b_1+a_1)-\epsilon d_0\right] \\
&+P_{N-1}\left[-\frac{2}{3\sqrt{3}}e_1+\frac{2}{3}d_0(2b_1+a_1)\right] \\
&\quad=\epsilon^3 P_{N-2}(N-2)(N-3)(N-4) \\
&\quad+2\epsilon^2\Big(P_{N-1}(N-1)(N-4)(2b_1+a_1)+P_{N-2}(N-2)(N-4)(2b_0+a_0) \\
&\quad+P_{N-3}(N-3)(N-4)(2b_{-1}+a_{-1})\Big) \\
&\quad+\frac{4}{3}\epsilon\Big(2P_{N-1}(N-1)(2b_0+a_0)(2b_1+a_1)+P_{N-2}(N-2)(2b_0+a_0)^2 \\
&\quad+2P_{N-3}(N-3)(2b_{-1}+a_{-1})(2b_0+a_0)+P_{N-4}(N-4)(2b_{-1}+a_{-1})^2\Big) \\
&\quad+\epsilon\Big(P_{N-4}(N-4)A_{-2}+P_{N-3}(N-3)(A_{-1}-d_{-1}) \\
&\quad+P_{N-2}(N-2)(A_0-d_0)+P_{N-1}(N-1)A_1\Big)+4\epsilon N(b_1^2+a_1b_1)\,,
\end{aligned} \tag{C.39}
$$

The corresponding equations of Q_{M-k} can be obtained by setting $P_{N-k}\to Q_{M-k}$, $e_k\to -e_k$ and $b_k\to a_k$. Again we can apply perturbation method.
To apply perturbation, we assume $|b_1|\ll|a_1|$, $|b_1b_{-1}|\ll 1$ and $|a_1a_{-1}|\ll 1$ so that $|P_{N-k}|\sim|b_1|^k$ is ensured.[3] Then at the first order, we have from the quadratic equation:

$$
z^{N+M-2}: \qquad d_0^{(1)}=\epsilon^2\big(N(N-1)-NM+M(M-1)\big)+2\epsilon[Nb_0+Ma_0]\,, \tag{C.40}
$$

From the cubic equation:

$$
\begin{aligned}
z^{N-2}: \qquad & \left[-\frac{2}{3\sqrt{3}}e_{-1}^{(1)}+\frac{2}{3}d_0^{(1)}(2b_{-1}+a_{-1})+\frac{2}{3}d_{-1}(2b_0+a_0)-\frac{\epsilon}{2}d_{-1}\right] \\
&=2\epsilon^2N(N-1)(2b_{-1}+a_{-1})+\frac{8}{3}\epsilon N(2b_{-1}+a_{-1})(2b_0+a_0)+\epsilon N(A_{-1}-d_{-1})\,,
\end{aligned} \tag{C.41}
$$

$$
\begin{aligned}
e_{-1}^{(1)}&=4\sqrt{3}\epsilon\left[M\left(a_0a_{-1}+a_0b_{-1}+b_0a_{-1}\right)-N\left(b_0b_{-1}+a_0b_{-1}+b_0a_{-1}\right)\right] \\
&+\sqrt{3}\epsilon^2\bigg[a_{-1}\left(M^2-2N^2+2NM+2N-\frac{5}{2}M\right) \\
&+b_{-1}\left(2M^2-N^2-2NM-2M+\frac{5}{2}N\right)\bigg]\,,
\end{aligned} \tag{C.42}
$$

[3]Although under this condition Q_{M-k} cannot be found by perturbation, d_k and e_k can be totally fixed by symmetry. Notice that d_k is invariant under the transformation $N\to M$ and $a_l\to b_l$, while e_k is anti-invariant when $N\to M$ and $a_l\to b_l$. Thus the explicit dependence of P_{N-k} is enough to determine d_k and e_k.

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$$z^{N-3}: \quad \left[-\frac{2}{3\sqrt{3}}e_0^{(1)}+\frac{2}{3}d_0^{(1)}(2b_0+a_0)-\epsilon d_0^{(1)}\right] \tag{C.43}$$
$$=\epsilon^3 N(N-1)(N-2)+2\epsilon^2 N(N-2)(2b_0+a_0)+\frac{4}{3}\epsilon N(2b_0+a_0)^2+\epsilon N(A_0-d_0)\,,$$

$$z^{N-4}: \quad \left[-\frac{2}{3\sqrt{3}}e_1^{(1)}+\frac{2}{3}d_0^{(1)}a_1\right]=2\epsilon^2 N(N-3)a_1+4\epsilon^2 Na_1+4\epsilon Nb_0a_1\,, \tag{C.44}$$

$$e_1^{(1)}=a_1\Big(2\sqrt{3}\epsilon[Ma_0-2Nb_0]+\sqrt{3}\epsilon^2[-2N(N-1)-NM+M(M-1)]\Big)\,, \tag{C.45}$$

$$z^{N-5}: \quad P_{N-1}^{(1)}\left[-\frac{2}{3\sqrt{3}}e_1^{(1)}+\frac{2}{3}d_0^{(1)}a_1\right]-4\epsilon Na_1b_1 \tag{C.46}$$
$$=P_{N-1}^{(1)}\Bigg(2\epsilon^2(N-1)(N-4)a_1+\frac{8}{3}\epsilon(N-1)(2b_0+a_0)a_1$$
$$+\epsilon(N-1)\left(4\epsilon a_1-\frac{4}{3}(2a_0a_1+b_0a_1)\right)\Bigg)\,,$$

$$P_{N-1}^{(1)}=\frac{Nb_1}{3\epsilon(N-1)+b_0}\,. \tag{C.47}$$

Second order contribution is given as follows:

$$z^{N+M-2}: \quad d_0^{(2)}=-2\epsilon[b_{-1}P_{N-1}^{(1)}+a_{-1}Q_{M-1}^{(1)}]\,, \tag{C.48}$$

$$d_0^{(2)}=-2\epsilon\left(\frac{Nb_1b_{-1}}{3\epsilon(N-1)+b_0}+\frac{Ma_1a_{-1}}{3\epsilon(M-1)+a_0}\right)\,. \tag{C.49}$$

$$z^{N-2}: \quad P_{N-1}\left[\frac{4}{3}\epsilon(2b_{-1}+a_{-1})^2+\epsilon NA_{-2}\right]+\left[-\frac{2}{3\sqrt{3}}e_{-1}^{(2)}+\frac{2}{3}d_0^{(2)}(2b_{-1}+a_{-1})\right]=0\,, \tag{C.50}$$

$$e_{-1}^{(2)}=2\sqrt{3}\epsilon\left(\frac{(2a_{-1}+b_{-1})N}{3\epsilon(N-1)+b_0}b_1b_{-1}-\frac{(a_{-1}+2b_{-1})M}{3\epsilon(M-1)+a_0}a_1a_{-1}\right)\,. \tag{C.51}$$

$$z^{N-4}: \quad P_{N-1}^{(1)}\left[-\frac{2}{3\sqrt{3}}e_0^{(1)}+\frac{2}{3}d_0^{(1)}(2b_0+a_0)-\epsilon d_0^{(1)}\right]+\left[-\frac{2}{3\sqrt{3}}e_1^{(2)}+\frac{4}{3}d_0^{(1)}b_1\right]$$
$$=\epsilon^3 P_{N-1}^{(1)}(N-1)(N-2)(N-3) \tag{C.52}$$
$$+2\epsilon^2\Big(N(N-3)2b_1+P_{N-1}^{(1)}(N-1)(N-3)(2b_0+a_0)\Big)$$
$$+\frac{4}{3}\epsilon\Big(4N(2b_0+a_0)b_1+P_{N-1}^{(1)}(N-1)(2b_0+a_0)^2\Big)$$
$$+\epsilon\left(P_{N-1}^{(1)}(N-1)(A_0-d_0)+N\left[4\epsilon b_1-\frac{4}{3}(2b_0+a_0)b_1\right]\right)\,,$$

$$e_1^{(2)}=2\sqrt{3}b_1\Big(\epsilon[2Ma_0-3Na_0-4Nb_0]+\epsilon^2[-2N^2+5N-NM+M(M-1)]\Big)$$
$$+\frac{3\sqrt{3}}{2}\frac{Nb_1}{3\epsilon(N-1)+b_0}\Big(4\epsilon(a_0+b_0)b_0+2\epsilon^2[b_0(3N-5)+a_0(2N-2-M)]$$
$$+\epsilon^3[2(N-1)(N-3)-NM+M(M-1)]\Big)\,. \tag{C.53}$$

From this consideration we have

$$d_0 = d_0^{(1)} + d_0^{(2)} + \ldots = D_0 + D_1 b_{-1} b_1 + D_2 a_{-1} a_1 + \text{higher order}\,, \tag{C.54}$$

$$e_1 = e_1^{(1)} + e_1^{(2)} + \ldots = D_3 a_1 + D_4 b_1 + \text{higher order}\,, \tag{C.55}$$

$$\begin{aligned} e_{-1} &= e_{-1}^{(1)} + e_{-1}^{(2)} + \ldots \\ &= D_5 b_{-1} + D_6 a_{-1} + D_7 (2a_{-1} + b_{-1}) b_1 b_{-1} + D_8 (a_{-1} + 2b_{-1}) a_1 a_{-1} + \text{higher order}\,, \end{aligned} \tag{C.56}$$

where all D_k are functions of b_0, a_0, which can be read off from the above equations. We compare these results with the ϵ expansion (3.31)–(3.33) and find they agree with each other, by calculating (3.31)–(3.33) further to order $\mathcal{O}(\eta_0^2)$ with $M = M_1$ and $N = N_1$:

$$d_0^{(1;1)} = 2\epsilon (b_0 N + a_0 M) - 2\epsilon \left(\frac{M}{a_0} a_1 a_{-1} + \frac{N}{b_0} b_1 b_{-1} \right) + \mathcal{O}(\eta_0^2)\,, \tag{C.57}$$

$$\begin{aligned} e_1^{(1;1)} &= 2\sqrt{3}\epsilon \left[a_0 (a_1 + 2b_1) M - b_0 (2a_1 + b_1) N \right] \\ &\quad - 2\sqrt{3}\epsilon \left(\frac{(a_1 + 2b_1) M}{a_0} a_1 a_{-1} - \frac{(2a_1 + b_1) N}{b_0} b_1 b_{-1} \right) + \mathcal{O}(\eta_0^2)\,, \end{aligned} \tag{C.58}$$

$$\begin{aligned} e_{-1}^{(1;1)} &= 4\sqrt{3}\epsilon \left[a_0 \left(a_{-1} M + b_{-1} (M - N) \right) + b_0 \left(a_{-1} (M - N) - b_{-1} N \right) \right] \\ &\quad - 2\sqrt{3}\epsilon \left(\frac{(a_{-1} + 2b_{-1}) M}{a_0} a_1 a_{-1} - \frac{(2a_{-1} + b_{-1}) N}{b_0} b_1 b_{-1} \right) + \mathcal{O}(\eta_0^2)\,. \end{aligned} \tag{C.59}$$

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