

# The deconfining phase transition of $SO(N)$ gauge theories in 2+1 dimensions

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**ABSTRACT:** We calculate the deconfining temperature of  $SO(N)$  gauge theories in 2+1 dimensions, and determine the order of the phase transition as a function of  $N$ , for various values of  $N \in [4, 16]$ . We do so by extrapolating our lattice results to the infinite volume limit, and then to the continuum limit, for each value of  $N$ . We then extrapolate to the  $N = \infty$  limit and observe that the  $SO(N)$  and  $SU(N)$  deconfining temperatures agree in that limit. We find that the deconfining temperatures of all the  $SO(N)$  gauge theories appear to follow a single smooth function of  $N$ , despite the lack of a non-trivial centre for odd  $N$ . We also compare the deconfining temperatures of  $SO(6)$  with  $SU(4)$ , and of  $SO(4)$  with  $SU(2) \times SU(2)$ , motivated by the fact that these pairs of gauge theories share the same Lie algebras.

**KEYWORDS:** 1/ $N$  Expansion, Confinement, Lattice Quantum Field Theory, Quark-Gluon Plasma

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## 1 Introduction

While a great deal is known about the non-perturbative physics of  $SU(N)$  gauge theories from calculations on the lattice, much less is known about  $SO(N)$  gauge theories. In this paper we will show that  $SO(N)$  gauge theories in  $2 + 1$  dimensions possess a deconfining phase transition at a finite temperature  $T = T_c$ , just like the deconfining transition in  $SU(N)$  gauge theories. We will calculate its value and determine its nature for  $N = 4, 5, 6, 7, 8, 9, 12, 16$ . This will enable us to extrapolate to  $N = \infty$  where we can compare to the  $SU(\infty)$  extrapolated value [1]. This is interesting to do since  $SO(N)$  and  $SU(N)$  gauge theories have a common planar limit [2], and  $SO(2N)$  and  $SU(N)$  gauge theories are orbifold equivalent [3–9], so we expect that dimensionless ratios of common physical quantities, including the deconfining temperature, should be equal at  $N = \infty$  [10, 11]. We will perform further comparisons motivated by the fact that certain  $SO(N)$  and  $SU(N')$  gauge theories share the same Lie algebras, i.e.  $SO(3)$  and  $SU(2)$ ,  $SO(4)$  and  $SU(2) \times SU(2)$ ,  $SO(6)$  and  $SU(4)$ . To the extent that the difference in the global properties of the groups (such as the centre) is not important, we would expect the deconfining transition and temperature to be identical within each of these pairs of gauge theories, and this is something we shall attempt to check. Moreover assuming this identity, the known value of  $T_c$  in  $SU(2)$  provides us with a value for  $SO(3)$ , which we do not calculate directly (for reasons given below). In addition all these calculations will allow us to compare  $SO(2N)$  and  $SO(2N + 1)$  theories, which is interesting because  $SO(2N + 1)$  gauge theories have a trivial center in contrast to the non-trivial  $\mathbb{Z}_2$  center of  $SO(2N)$  theories.

While the calculations in this paper are primarily intended to establish the presence of the finite  $T$  transition and to investigate its properties, we shall choose to call it a deconfining transition, for both odd and even  $N$ , just like the one in  $SU(N)$  gauge theories. Of course that assumes that these theories are linearly confining at low  $T$ . While we shall provide some evidence for confinement at low  $T$  in this paper (see in particular the discussion in section 3.3), the explicit evidence for the confinement being linear is given in our companion paper on the glueball spectra and string tensions [12], where we show that the energy of closed flux tubes is (roughly) proportional to their length for

both odd and even  $N$ . Of course such numerical evidence possesses intrinsic limitations: we cannot distinguish between confinement that is exact and confinement to a very good approximation. However the quality of our numerical evidence is comparable to that which establishes linear confinement in  $D = 2 + 1$   $SU(N)$  gauge theories.

The paper is structured as follows. In section 2, we briefly review some well-known relations between  $SO(N)$  and  $SU(N)$  gauge theories, both at small and at large  $N$ . In section 3 we briefly describe the lattice setup, how to differentiate confining from non-confining phases, and we comment on what we know about confinement in  $SO(N)$  gauge theories. In section 4, we describe how we identify the location of the finite temperature transition and how we determine whether the transition is first or second order. Then in section 5, we describe how to calculate on a lattice the physical quantities that we shall use in order to express the transition temperature in physical units. The next few sections contain our results. First, in section 6, we calculate the infinite volume limit for each of the  $SO(N)$  gauge theories we consider, and hence the value of  $T_c$  at various lattice spacings. Then, in section 7, we use these values to calculate the continuum limit of the deconfining temperature for each group, briefly discussing the issues caused by the strong to weak coupling ‘bulk’ transition in  $D = 2 + 1$ . We then proceed in section 8 to calculate the large- $N$  limit of  $T_c$  for  $SO(2N)$  and  $SO(2N + 1)$  separately and together and in section 9, we compare the  $SO(N)$  and  $SU(N)$  deconfining temperatures both at  $N = \infty$ , and for pairs of  $SO(N)$  and  $SU(N')$  groups that share the same Lie algebra. Section 10, contains a summary of our conclusions. Appendix A contains our detailed tabulated results.

There are companion papers, both published [13] and in progress [12], that contain our results for the mass spectrum and string tension of  $SO(N)$  gauge theories. The latter paper describes the Monte Carlo algorithm in more detail, as well as providing more discussion of the ‘bulk’ transition. An earlier paper [14] contained our first, exploratory estimates of  $T_c$ . The values in the present paper are much more accurate and supersede those earlier values, although they are in fact consistent within errors.

## 2 Relations between $SO(N)$ and $SU(N)$

### 2.1 Lie algebra equivalences

The implications of the Lie algebra equivalence between certain  $SO(N)$  and  $SU(N')$  groups are discussed in more detail in [12, 13]. Here we merely summarise some points that are relevant to our present calculations. If we assume that the global structure of the groups is irrelevant to the physics (an assumption which needs to be tested) then we expect that colour singlet quantities are the same within each pair of theories. For example the mass gap or the deconfining temperature. String tensions on the other hand are associated with a flux that is in a certain representation, and this needs to be matched between the theories. In the following we summarise some points that are relevant to the calculations in this paper.

The first equivalence is between the Lie algebras of  $SO(3)$  and  $SU(2)$ . The  $SO(3)$  fundamental representation is equivalent to the  $SU(2)$  adjoint representation, so that the associated string tension should satisfy

$$\sigma_f|_{so3} = \sigma_{adj}|_{su2}. \quad (2.1)$$

Adjoint flux tubes in SU(2) are not expected to be stable and can, for example, decay into glueballs. So one expects the same to be true for SO(3) fundamental flux tubes. Of course, if the decay width is small enough, then just as for the mass of a narrow resonance, we can estimate a string tension. More importantly, glueball masses and  $T_c$  should be the same within SU(2) and SO(3), and the coupling  $g^2$  should satisfy [12, 14]

$$g^2|_{\text{so3}} = 4g^2|_{\text{su2}}. \tag{2.2}$$

In this paper we do not calculate  $T_c$  for SO(3) because the strong-to-weak coupling transition in our lattice theory occurs at such a small value of the lattice spacing that we would need to use very large lattices (in lattice units) and this would be computationally quite expensive. Of course, if one assumes that the physics of SU(2) and SO(3) is the same, as described above, then one can infer the value of  $T_c$  in SO(3) from the known value in SU(2), and compare it to the values obtained in SO( $N > 3$ ). We shall do this later in this paper.

As is also well known, SO(4) and SU(2)  $\times$  SU(2) share the same Lie algebra, with the latter forming a double cover of the former. In an SU(2)  $\times$  SU(2) theory the two SU(2) groups do not interact with each other and so the physics is directly related to that of SU(2). So if we assume that the physics of the SO(4) and SU(2)  $\times$  SU(2) gauge theories is the same, then the single particle spectrum and the value of  $T_c$  should be just as in SU(2). Because the fundamental SO(4) flux involves fundamental flux from both SU(2) groups, we expect

$$\sigma_f|_{\text{so4}} = 2\sigma_f|_{\text{su2}}. \tag{2.3}$$

We also expect the couplings to be related by [12, 14]

$$g^2|_{\text{so4}} = 2g^2|_{\text{su2}}. \tag{2.4}$$

Finally, we recall that SO(6) and SU(4) also share the same Lie algebra. We further recall that in SU(4)

$$\underline{4} \otimes \underline{4} = \underline{6} \oplus \underline{10} \tag{2.5}$$

where the  $\underline{6}$  corresponds to the  $k = 2$  antisymmetric representation and this maps to the fundamental  $\underline{6}$  of SO(6). To convert quantities in terms of the SU(4) fundamental string tension to the SU(4)  $k = 2A$  string tension, we shall use the known ratio of the SU(4)  $k = 2A$  and fundamental string tensions in  $D = 2 + 1$  [15]

$$\left. \frac{\sigma_{2A}}{\sigma_f} \right|_{\text{su4}} = 1.355(9). \tag{2.6}$$

In addition the couplings are related by [12, 14]

$$g^2|_{\text{so6}} = 2g^2|_{\text{su4}}. \tag{2.7}$$

Glueball masses and  $T_c$  should be the same for SO(6) and SU(4), in their common positive charge conjugation sector.

All the above relations assume that the differing global properties of the pairs of gauge groups do not affect the dynamics. It is not obvious that this is the case and one of our aims in this paper is to see if it is indeed the case for the properties of the deconfining transition.

## 2.2 Large- $N$

Just as with  $SU(N)$  gauge theories [16],  $SO(N)$  gauge theories at the diagrammatic level possess a smooth  $N \rightarrow \infty$  limit if one keeps  $g^2 N$  fixed [2]. Moreover the surviving planar diagrams are identical to those of  $SU(N)$  if one chooses [2]

$$g^2|_{SO(N)} = 2g^2|_{SU(N)}. \tag{2.8}$$

However there is a difference in the approach to the planar limit. The  $SO(N)$  gauge field propagator takes the form

$$\langle [A_\mu(x)]^i_j [A_\nu(y)]^k_l \rangle \propto \delta^i_l \delta^k_j - \delta^{ik} \delta_{lj}. \tag{2.9}$$

The first term on the right is the leading order double line description of an  $SU(N)$  gauge propagator. However, the second term is special to  $SO(N)$  gauge theories and corresponds to a ‘twisted’ propagator [17]. This leads to new non-oriented surfaces in double line graphs, which in turn means that corrections to the planar limit are  $O(1/N)$  rather than the  $O(1/N^2)$  one finds for  $SU(N)$ .

While the above diagrammatic analysis suggests that the large- $N$  physics of  $SU(N)$  and  $SO(N)$  gauge theories should be the same in their common positive charge conjugation sector of states, it does not guarantee that non-perturbative effects will not disrupt this expectation. However there exists a more general argument based on a large- $N$  orbifold equivalence [3–9]. One can apply an orbifold projection on a parent  $SO(2N)$  QCD-like theory to obtain a child  $SU(N)$  QCD theory [3–9] with the couplings related as in eq. (2.8). Since it has been shown that the large- $N$  physics of orbifold equivalent theories is indeed the same [10, 11], this tells us that the physics of large- $N$   $SO(2N)$  and  $SU(N)$  gauge theories should be identical within their common sector. In particular this should apply to the  $N \rightarrow \infty$  limit of the calculations of  $T_c$  in this paper.

## 3 Preliminaries

### 3.1 Lattice variables

Our variables are  $N \times N$   $SO(N)$  matrices  $U_l$  assigned to links  $l$ . We will often write  $U_l$  as  $U_\mu(x)$  where the link  $l$  emanates from the site  $x$  in the  $\mu$  direction. Our periodic lattice has dimensions  $L_s^2 L_t$ , with lattice spacing  $a$ . The partition function is

$$Z = \int \prod_l dU_l \exp\{-\beta S[U_l]\} \tag{3.1}$$

and we use a standard plaquette action  $\beta S$  where

$$\begin{aligned} \beta S &= \beta \sum_p \left( 1 - \frac{1}{N} \text{Tr}(U_p) \right) \\ \beta &= \frac{2N}{ag^2} \end{aligned} \tag{3.2}$$

with  $U_p$  the ordered product of  $U_l$  around the boundary of the plaquette  $p$ . The relation between  $\beta$  and  $g^2$  holds in the continuum limit; on the lattice it defines a lattice coupling that will become the standard continuum coupling when  $a \rightarrow 0$ . Note that different choices of action will lead to definitions of  $g^2$  that differ by  $O(a)$  corrections (which of course vanish in the continuum limit).

### 3.2 Finite temperature on the lattice

To calculate expectation values at a non-zero temperature  $T$ , we consider the Euclidean field theory on a periodic  $l_s^2 l_t$  space-time volume and take the thermodynamic limit  $l_s \rightarrow \infty$  so that we have a well-defined temperature,  $l_t = 1/T$ .

For convenience we shall use  $T = 1/l_t$  to define the ‘temperature’ of our system even in a finite volume.

On a  $L_s^2 L_t$  lattice with spacing  $a$ , we have  $l_s = aL_s$  and  $l_t = aL_t$ . The value of  $a$  is determined by the value of the inverse bare coupling,  $\beta = 2N/ag^2$ , that appears in the lattice action. So for  $L_s \rightarrow \infty$  a lattice field theory will have temperature  $T = 1/a(\beta)L_t$ . We can vary  $T$  at fixed  $L_t$  by varying  $\beta$  and hence  $a(\beta)$ . If we find that a deconfinement transition occurs at  $\beta = \beta_c$ , then the deconfining temperature is

$$T_c(a) = \frac{1}{a(\beta_c)L_t}. \tag{3.3}$$

If we increase  $L_t$ , the transition will occur at a smaller value of  $a$ . So by producing a sequence of such calculations we can extrapolate to the  $a = 0$  continuum limit.

### 3.3 The ‘temporal’ Polyakov loop, the center, and confinement

A useful order parameter for identifying the deconfining transition is the ‘temporal’ Polyakov loop,  $l_P$ . If the spatial starting point of the loop is  $\mathbf{x}$ , then the loop is defined by

$$l_P(\mathbf{x}) = \text{Tr} (U_t(\mathbf{x}, t = a)U_t(\mathbf{x}, t = 2a) \cdots U_t(\mathbf{x}, t = aL_t)). \tag{3.4}$$

This operator represents the world line of a static charge in the fundamental representation located at spatial site  $\mathbf{x}$ . So we can obtain the free energy  $F_{f\bar{f}}$  of a pair of such charges located at  $\mathbf{x}$  and  $\mathbf{y}$  respectively from the correlation function of two Polyakov loops at  $\mathbf{x}$  and  $\mathbf{y}$  with opposite orientations

$$e^{-\frac{1}{T}F_{f\bar{f}}(x,y)} = \langle l_P(\mathbf{x})l_P^T(\mathbf{y}) \rangle. \tag{3.5}$$

Assuming that the correlation function satisfies clustering, the correlation function decorrelates at large spatial distances

$$\langle l_P(\mathbf{x})l_P^T(\mathbf{y}) \rangle \xrightarrow{|\mathbf{x}-\mathbf{y}| \rightarrow \infty} |\langle l_P \rangle|^2. \tag{3.6}$$

Hence, if  $\langle l_P \rangle = 0$  then  $F_{f\bar{f}}(x,y) \rightarrow \infty$  as the separation  $|\mathbf{x} - \mathbf{y}| \rightarrow \infty$  which corresponds to confinement, although not necessarily to a linearly rising potential. (Recall that in  $D = 2+1$  the Coulomb interaction is already, by itself, logarithmically confining.) Similarly,

if  $\langle l_P \rangle \neq 0$ , then the free energy approaches a finite value at large spatial separation, and this will normally imply that the charges are not confined. A counterexample is when there are particles in the fundamental representation in the theory, which can then bind with the static charge to produce a colour singlet. This is the case in *QCD* where we have  $\langle l_P \rangle \neq 0$ , but the theory is confining (all physical states are colour singlet) even though the potential flattens out at large distances.

$SO(2N)$  gauge theories have a  $\mathbb{Z}_2$  centre symmetry under which the action and measure are invariant. We can generate a centre symmetry transformation by taking a non-trivial element  $z$  of the centre and multiplying all temporal links between two neighbouring time-slices by  $z$ . Unlike a contractible loop, the temporal Polyakov loop is not invariant under this symmetry,

$$l_P \rightarrow z l_P \tag{3.7}$$

so that its expectation value  $\langle l_P \rangle = 0$ , and the theory is confining, unless the centre symmetry is spontaneously broken, in which case we generically expect  $\langle l_P \rangle \neq 0$  and the theory is deconfining. So we expect that the deconfinement phase transition coincides with the spontaneous breakdown of the centre symmetry. This, of course, just parallels the well-known argument for  $SU(N)$  gauge theories. In addition the Lie algebra equivalences discussed in section 2.1 strongly suggest that both  $SO(4)$  and  $SO(6)$  must be confining at low  $T$ , just like  $SU(2)$  and  $SU(4)$  respectively. Moreover the large- $N$  equivalences discussed in section 2.2 strongly suggest that the  $SO(N \rightarrow \infty)$  theory is linearly confining, just like  $SU(N \rightarrow \infty)$ . All this (together with the numerical evidence for linear confinement in [12]), makes a convincing case that  $SO(2N)$  gauge theories are linearly confining at low  $T$ .

By contrast  $SO(2N+1)$  gauge theories have a trivial centre and so in general we would expect  $\langle l_P \rangle \neq 0$  at all  $T$ . Even so, this does not of itself preclude confinement. As a well-known example, recall that in *QCD* with a heavy enough but finite quark mass one has  $\langle l_P \rangle \neq 0$ , albeit very small, because of the explicit breaking of the centre symmetry by the fermion action. Nonetheless the theory still possesses a first order deconfining transition, which is continuously linked to that of the pure gauge theory (which is why we confidently label it as being deconfining). In *QCD* in this limit a long confining flux tube is in fact unstable, but with an extremely small decay width — the breaking is essentially a tunnelling phenomenon. So strictly speaking the theory is not linearly confining, although it is still believed to be physically confining in the sense that all finite-energy states are colour singlet. (And in practice the flux-tube breaking would not be visible in a direct numerical calculation of the potential.) Another well-known and more relevant example is provided by the  $SO(3)$  gauge theory. The Lie algebra equivalence with  $SU(2)$  (see section 2.1) strongly suggests that  $SO(3)$  is confining at low  $T$  with a second order deconfining transition at some non-zero  $T$ . However the fundamental flux tube of  $SO(3)$  is the adjoint flux tube of  $SU(2)$  which we expect to be unstable so that in  $SO(3)$  we are confident that  $\langle l_P \rangle \neq 0$  at any  $T$ . Indeed the direct physical interpretation of this is that the  $SO(3)$  fundamental source is screened by gluons which, in  $SO(3)$ , are in the same triplet as the fundamental. (Something that is not the case for  $SO(N \geq 4)$ .) A further directly relevant example is provided by  $SO(5)$ . This has the same Lie algebra as  $Sp(2)$ . (Note that there is another convention



where this is called  $\text{Sp}(4)$ .) There have been numerical investigations of  $D = 2 + 1 \text{ Sp}(2)$  demonstrating that it has a second order deconfining transition [18]. So we strongly expect  $\text{SO}(5)$  to also possess a deconfining transition. Yet another useful example is provided by  $G(2)$  which has a trivial center and yet has a deconfining transition [19] from a low  $T$  confining phase [20]. (For a discussion of the centre and confinement see e.g [20].) Finally, the diagrammatic (not orbifold) equivalence between  $\text{SO}(2N + 1 \rightarrow \infty)$  and  $\text{SU}(N \rightarrow \infty)$  (see section 2.2), strongly suggests that in  $\text{SO}(2N + 1 \rightarrow \infty)$  at  $N = \infty$  we have exact confinement at  $T = 0$  and we also have  $\langle l_P \rangle = 0$  at any  $T$ . Now if  $\text{SO}(3)$  (not to mention  $\text{SO}(5)$ ) and  $\text{SO}(2N + 1 \rightarrow \infty)$  are exactly confining at low  $T$ , then it appears very plausible that all  $\text{SO}(2N + 1)$  gauge theories are exactly confining at low  $T$ .

Even if  $\text{SO}(2N + 1)$  gauge theories are indeed confining, as we argued above, it is still interesting to ask if there is some exact order parameter based on the Polyakov loop. Since  $\langle l_P \rangle \neq 0$  in  $\text{SO}(3)$ , but  $\langle l_P \rangle = 0$  at  $N = \infty$ , it is plausible that  $\langle l_P \rangle \neq 0$  for any  $\text{SO}(2N + 1)$ , but  $\rightarrow 0$  as  $N \rightarrow \infty$ , and perhaps does this so rapidly that the non-zero value becomes invisible in a numerical calculation at moderate values of (odd)  $N$ . Returning to  $\text{SO}(3)$  we observe that it is the fundamental Polyakov loop of  $\text{SU}(2)$  that is exactly zero at low  $T$ , and since this corresponds to the spinorial of  $\text{SO}(3)$ , we expect that the corresponding spinorial Polyakov loop is exactly zero in  $\text{SO}(3)$ . This suggests the following speculation. In  $\text{SO}(2N + 1)$  gauge theories it is perhaps the spinorial Polyakov loop that is exactly zero (perhaps one can even locate a symmetry that ensures this) and this serves as the ‘ideal’ order parameter for (de)confinement. But since the dimension of the spinorial representation in  $\text{SO}(N)$  grows very rapidly with  $N$ , and one’s experience is that string tensions grow very roughly with the quadratic Casimir, it will presumably only be relevant to the low energy physics at small  $N$ . Simultaneously, we expect that the expectation value of the fundamental loop in  $\text{SO}(2N + 1)$  decreases very rapidly, perhaps exponentially in  $N$  if the tunnelling argument is correct, and it takes over as the ‘ideal’ order parameter at larger  $N$ . Assessing the plausibility of such a scenario is something that we will not do here, or in [12], since it would require explicit calculations with the spinorial representations of  $\text{SO}(N)$  gauge theories. But it is clearly something that would be interesting to do.

A final practical comment. Later on in this paper we shall take  $\text{SO}(7)$  as our typical example of  $\text{SO}(2N + 1)$  gauge theories, and we shall show that the value of  $\langle l_P \rangle$  at low  $T$  is extremely small, and indeed consistent with zero within our very small errors. So we can assert that, at the very least, we have a direct numerical demonstration of something close to exact confinement. And in [12] we shall show that, again within very small errors, this apparent confinement is in fact linear. Together with the above arguments this provides a justification for labelling the finite  $T$  transition that we study in this paper as being a ‘deconfining’ one.

## 4 Deconfining phase transitions

In an infinite spatial volume, a phase transition occurs when the free energy becomes a non-analytic function in one of its parameters. We will see that the  $\text{SO}(N)$  deconfining phase transition is second order for small  $N$  and first order for larger  $N$ . First order

phase transitions occur when there is a discontinuity in the first derivative of the free energy such that the second derivative is typically a delta function singularity. Second order phase transitions occur when there is a divergence in the second derivative in the free energy although the first derivative is continuous. This corresponds to a divergent correlation length.

On a finite volume, the partition function is finite so all derivatives are well-defined and analytic, so that there are no apparent non-analyticities. Finite size scaling tells us how the results at finite volumes should converge towards the expected non-analyticity as we increase the spatial volume size, allowing us to classify the transition.

### 4.1 First order transitions

Let  $O$  be an order parameter, such as the temporal Polyakov loop or plaquette averaged over the spatial volume. Suppose that it takes a value  $\langle O \rangle = O_c$  in the confined phase and  $\langle O \rangle = O_d$  in the deconfined phase. (For simplicity we shall assume here a single deconfined phase.) We can define a susceptibility  $\chi_O(V, T)$  for a volume  $V$  and temperature  $T$  by

$$\chi_O(V, T) = \mathcal{N}V \left( \langle O(T)^2 \rangle - \langle O(T) \rangle^2 \right) \tag{4.1}$$

for some constant  $\mathcal{N}$ . If we are in a single phase then the spatial average ensures that  $\left( \langle O(T)^2 \rangle - \langle O(T) \rangle^2 \right) \sim O(1/V)$  so that  $\chi_O(V, T) \sim O(V^0)$ , as long as the correlation length is finite, i.e. the mass gap is non-zero.

At the phase transition,  $T = T_c$ , in an infinite volume the free energies are equal. On a finite volume the phase transition is smeared out and there is no unique way to say at which value of  $T$  it occurs, but a sensible and standard choice is to choose  $T_c$  where the free energy densities are equal

$$f_c(T = T_c) = f_d(T = T_c) \tag{4.2}$$

where  $F_{c/d}(T) = f_{c/d}(T)V$  are the free energies for the confined and deconfined phases respectively. At  $T = T_c$  the system is equally likely to be in the confined and deconfined phases and so the order parameter takes values  $O_c$  and  $O_d$  with equal probability. Hence,

$$\chi_O(V, T_c) = \mathcal{N}V \left( \frac{(O_c^2 + O_d^2)}{2} - \frac{(O_c + O_d)^2}{4} \right) = \mathcal{N}V \left( \frac{(O_c - O_d)^2}{4} \right) \tag{4.3}$$

and so the peak height of the susceptibility should grow as  $\chi_{\max} = \mathcal{O}(V)$ . Note that the susceptibility peaks when the probability of being in the confining phase is 1/2 and that this is independent of the number of identical deconfined phases. Note also that here we neglect the  $O(1/\sqrt{V})$  fluctuations of  $O$  around its mean value in each phase.

So we conclude that a first order transition on finite volumes  $V$  is characterised by a susceptibility that forms a peak with height  $\chi_{\max} = \mathcal{O}(V)$  and that the whole peak is confined to a range  $\Delta\beta = \mathcal{O}(1/V)$ . So as  $V \rightarrow \infty$  the peak tends towards a  $\delta$ -function and in extrapolating  $T_c(V)$  to  $V = \infty$  one should use a leading  $\mathcal{O}(1/V)$  correction term.

## 4.2 Second order transitions

For a second order phase transition, the correlation length  $\xi \rightarrow \infty$  as  $T \rightarrow T_c$  if we are on an infinite volume. On a finite volume it will (effectively) approach the spatial lattice length  $L_s$  [21–23]. Let us define the reduced temperature by

$$t = (T - T_c)/T_c = (\beta - \beta_c)/\beta_c \equiv \Delta\beta \quad (4.4)$$

using  $T = 1/(aL_t) = \beta g^2/(2NL_t)$ , and the critical exponents  $\nu$  and  $\gamma$  by the standard relations

$$\begin{aligned} \xi &\sim |t|^{-\nu} \sim |\Delta\beta|^{-\nu} \\ \chi(T, L_s \rightarrow \infty) &\sim |t|^{-\gamma} \sim |\Delta\beta|^{-\gamma}. \end{aligned} \quad (4.5)$$

The standard finite size scaling analysis [21–23] then tells us that at the transition the susceptibility has a height  $\chi_{\max} = \mathcal{O}(L_s^{\frac{\gamma}{\nu}})$  over a half-width of  $\Delta\beta = \mathcal{O}(1/L_s^{\frac{1}{\nu}})$ . Note that the  $L_s \rightarrow \infty$  peak provides an envelope for the peaks at finite  $L_s$ , leading to a structure quite different from the  $\delta$ -function peak in a first order transition.

## 4.3 Scaling laws

From the above we infer that we can distinguish between first and second order transitions by examining the structure of the susceptibility peaks over a range of different spatial volumes. We summarise the scaling laws by the following relations. In  $D = 2 + 1$ , the phase transition occurs at

$$\begin{aligned} \frac{T_c(\infty) - T_c(V)}{T_c(\infty)} \sim \frac{1}{V} &\Rightarrow \beta_c(V) = \beta_c(\infty) \left[ 1 - h \left( \frac{L_t}{L_s} \right)^2 \right] && \text{1st order} \\ \frac{T_c(\infty) - T_c(V)}{T_c(\infty)} \sim \frac{1}{V^{\frac{1}{2\nu}}} &\Rightarrow \beta_c(V) = \beta_c(\infty) \left[ 1 - k \left( \frac{L_t}{L_s} \right)^{\frac{1}{\nu}} \right] && \text{2nd order} \end{aligned} \quad (4.6)$$

where  $h, k$  are constants and we use  $T = 1/(aL_t) = \beta g^2/(2NL_t)$ . In 2 spatial dimensions, the maximum of the susceptibility peak  $\chi_{\max}(V)$  depends on the spatial volume  $V$  as

$$\begin{aligned} \chi_{\max}(V) &= c_0 V + c_1 && \text{1st order} \\ \chi_{\max}(V) &= c_0 V^{\frac{\gamma}{2\nu}} + c_1 && \text{2nd order} \end{aligned} \quad (4.7)$$

for constants  $c_0$  and  $c_1$ . Hence, finite size scaling shows us how  $\beta_c(V)$  and  $\chi_{\max}(V)$  vary with the spatial volume  $V$ , and how to extrapolate  $\beta_c(V)$  to the infinite volume limit.

## 4.4 Useful order parameters

An order parameter for a phase transition is a quantity that distinguishes between the different phases and exhibits a non-analyticity at the transition, and it is this behaviour that allows us to determine if and where the deconfinement phase transition occurs. As remarked above, phase transitions correspond to non-analyticities in the derivatives of

the partition function  $Z$  with respect to  $\beta$ . So consider the first two derivatives for our lattice action

$$\begin{aligned} \left(\frac{1}{N_p} \frac{\partial}{\partial \beta}\right) \ln Z &\sim \langle \bar{U}_p \rangle \\ \left(\frac{1}{N_p} \frac{\partial}{\partial \beta}\right) \langle \bar{U}_p \rangle &= \langle \bar{U}_p^2 \rangle - \langle \bar{U}_p \rangle^2 \equiv \chi_{\bar{U}_p} / V \end{aligned} \tag{4.8}$$

where  $N_p$  is the number of plaquettes,  $\bar{U}_p = \frac{1}{N_p} \sum_p \left(\frac{1}{N} \text{tr}(U_p)\right)$  is the plaquette averaged over the lattice volume, and  $\chi_O = V \left(\langle O^2 \rangle - \langle O \rangle^2\right)$  is the susceptibility of the operator  $O$ . In the case of a first order transition we expect  $\langle \bar{U}_p \rangle$  to exhibit a finite discontinuity at  $T = T_c$ , and  $\chi_{\bar{U}_p}$  to be a  $\delta$ -function when  $V \rightarrow \infty$ . For a second order transition  $\langle \bar{U}_p \rangle$  will be continuous, but will have a divergent first derivative at  $T_c$  when  $V \rightarrow \infty$ , so that  $\chi_{\bar{U}_p}$  will display a divergence as described above. Thus  $\bar{U}_p$  appears to be the obvious order parameter for locating the phase transition.

Unfortunately, our calculations indicate that the plaquette susceptibility has a weakly varying signal over the phase transition — too weak in fact to be useful on the lattice volumes that we are able to contemplate using. To show what happens it is convenient to partition the plaquettes into those that are only spatial  $\bar{U}_s$  and those that have links in a temporal direction  $\bar{U}_t$ . Figure 1 shows the spatial plaquette susceptibility  $\chi_{\bar{U}_s}$  and the temporal plaquette susceptibility  $\chi_{\bar{U}_t}$  in the region of the phase transition for an SO(4)  $32^2_3$  volume (renormalised for purposes of comparison). We need a clear peak in the susceptibility to identify the location of the phase transition but we see instead that  $\chi_{\bar{U}_s}$  has no obvious peak structure while  $\chi_{\bar{U}_t}$  has only a very weak peak structure. For other SO( $N$ ) groups, we also typically find that  $\chi_{\bar{U}_{s,t}}$  have no useful peak structures on the volumes we use. Of course when  $L_s \rightarrow \infty$  the peaks should eventually appear and grow, but it does mean that for our purposes the plaquette susceptibility is not a useful order parameter.

An alternative order parameter is provided by the temporal Polyakov loop  $l_P$ . As described earlier, its expectation value has a direct relation to the free energy of an isolated charge, and it is therefore a natural order parameter for the deconfining transition. We shall shortly see that the Polyakov loop operator  $\bar{l}_P$  has a much clearer signal in the region of the phase transition, compared to the plaquette operators. Around the transition it tunnels between confined and deconfined phases so that  $\bar{l}_P$  takes discrete values with very small fluctuations around these. There is however a problem at finite  $V$ . If there is a non-trivial centre symmetry then tunnelling between the corresponding deconfined phases will cause  $\bar{l}_P$  to average to zero for  $T > T_c$ . This is not an issue for SO( $2N + 1$ ) gauge theories since these have a trivial center symmetry, but it is a problem for SO( $2N$ ) with its  $\mathbb{Z}_2$  center symmetry. The same problem arises, of course, for SU( $N$ ) gauge theories. The standard (if theoretically ugly) fix is to take the absolute value of the Polyakov loop after averaging it over the spatial volume

$$|\bar{l}_P| = \left| \frac{1}{L_s^2} \sum_{\mathbf{x}} l_P(\mathbf{x}) \right| \tag{4.9}$$

and to use  $|\overline{l_P}|$  as an order parameter and to construct an associated susceptibility from that,

$$\frac{\chi_{|\overline{l_P}|}}{L_s^2 L_t} = \langle |\overline{l_P}|^2 \rangle - \langle |\overline{l_P}| \rangle^2. \quad (4.10)$$

This has the disadvantage that  $\langle |\overline{l_P}| \rangle \neq 0$  in the confined phase as well as in the deconfined phase, but the values are very different and it has a very good signal in the region of the phase transition. We return to our SO(4)  $32^2 3$  lattice in figure 1, and plot the Polyakov loop susceptibility. We see that  $\chi_{|\overline{l_P}|}$  has a much clearer peak structure than the plaquette susceptibilities shown in the same figure.

So, in a plot of  $\langle |\overline{l_P}| \rangle$  against  $\beta$  in the neighbourhood of  $\beta_c$ , we would expect to see the value of  $\langle |\overline{l_P}| \rangle$  increase from near-zero to some non-zero value over a narrow range of  $\beta$ . For a first order transition this range shrinks to zero as the volume increases, becoming a discontinuity at  $V = \infty$ , while for a second order transition this range remains finite and there is no discontinuity, but the slope at  $\beta_c$  tends to  $\infty$ . We show an example, obtained on a  $20^2 3$  lattice in SO(6), in figure 2. We expect to see a corresponding peak in  $\chi_{|\overline{l_P}|}$  at  $\beta_c$ , as in figure 1. For a first order transition, we expect the susceptibility  $\chi_{|\overline{l_P}|}$  to approach a delta function singularity as  $V \rightarrow \infty$ . For a second order phase transition, we expect that the susceptibility  $\chi_{|\overline{l_P}|}$  has a peak over a finite range of  $\beta$  around  $\beta_c$ , with a cusp-like divergence at  $\beta_c$ .

For odd  $N$  there is no  $Z_2$  symmetry to be spontaneously broken, so we can use our cleaner original variable,  $\langle \overline{l_P} \rangle$ , to characterise the transition. In figure 3 we plot this quantity against  $\beta$  for a  $48^2 4$  lattice in SO(7) with a sharp transition visible near the middle of the range. (Since the lattice spacing varies roughly as  $1/\beta$ , the range  $\beta \in [20, 40]$  corresponds roughly to the range  $T/T_c \in [0.66, 1.5]$ .) Despite the lack of a centre symmetry, we find that for  $\beta \leq 26.0$  our values are all consistent with  $\langle \overline{l_P} \rangle$  being zero within errors, with values  $\sim \pm 10^{-5}$ . This behaviour motivates describing the transition as being ‘deconfining’ even if the low- $T$  vacuum eventually turns out not to be exactly confining.

#### 4.5 Tunnelling

We can represent the values of  $\overline{l_P}$  obtained from the sequence of field configurations generated at a given  $\beta$  in a Monte Carlo run as either a histogram over the entire run, or as a history plot along the run. For  $\beta < \beta_c$ , we expect the theory to be confining so that  $\langle \overline{l_P} \rangle \approx 0$ . On the histogram, we would expect that the values of  $\overline{l_P}$  form a narrow peak around zero while, on the history plot, we would expect the values to fluctuate around zero. For  $\beta > \beta_c$ , the system would be in a deconfined phase so that  $\langle \overline{l_P} \rangle \neq 0$ , and we would expect to see deconfined peaks at non-zero values on the histogram. For SO( $2N$ ) gauge theories, we would expect to see two deconfined peaks at non-zero values, reflecting the spontaneous breaking of the  $Z_2$  center symmetry, while, for SO( $2N + 1$ ) gauge theories, where the center symmetry is trivial, we would only expect one deconfined peak at a non-zero value.

For a first order transition we would expect that as we increase  $\beta$  towards  $\beta \approx \beta_c$ , and beyond, we should see deconfined peaks appear at non-zero values while the confined peak

at zero decreases. And in a history plot we would see jumps that reflect tunnelling between the confined and deconfined phases. Beyond  $\beta \approx \beta_c$  any tunnelling should be only between the two deconfined phases for even  $N$ , and no tunnelling for odd  $N$ . The behaviour for even  $N$  is illustrated for  $\text{SO}(6)$  on a  $20^2 3$  lattice in the histograms in figure 4 and the history plots in figure 5. For odd  $N$  we illustrate the expected behaviour in  $\text{SO}(7)$  on a  $48^2 4$  lattice in the history plot in figure 6 and the histograms in figure 7. The coexistence of both confining and deconfining peaks at a given  $\beta$  establishes that we have a first order transition in both  $\text{SO}(6)$  and  $\text{SO}(7)$ .

For a second order transition, there is no phase coexistence. As we increase  $\beta$ , we would expect the confined peak around zero to spread out and, once it disappears, the deconfined peaks emerge at  $\beta = \beta_c$ . On the history plot, we would expect to see significant fluctuations around zero for  $\beta < \beta_c$  before the onset of tunnelling between the deconfined phases for  $\beta > \beta_c$ . This is illustrated for the case of a  $28^2 2$  lattice in  $\text{SO}(4)$  in figure 8.

Hence, we can use both the histograms and history plots of  $\overline{l_P}$  to distinguish between first and second order transitions.

#### 4.6 Identifying $\beta_c$

To calculate  $\beta_c$  on a given volume  $V$  we need to locate the maximum of the susceptibility. We do so by first performing separate runs at different  $\beta$  values, and then doing more runs at values of  $\beta$  near the peak. We use the standard density of states reweighting method [24–26] to construct a smooth interpolating function through the measured values, whose maximum provides our estimate of  $\beta_c$  on the given volume  $V$ . For some very large spatial volumes, the values that arise in the reweighting algorithm exceed the machine precision. In principle this obstacle should be surmountable by some judicious alteration of the algorithm, but in these cases we choose instead to use curve fitting to find  $\beta_c$ , based on a logistic function for the Polyakov loop, which in practice turns out to have a comparable performance to that of our reweighting algorithm, as we see from figure 9 and table 1.

### 5 $\text{SO}(N)$ lattice calculations in $D = 2 + 1$

We generate sequences of lattice field configurations using an  $\text{SO}(N)$  adaptation of the  $\text{SU}(N)$  Cabbibo-Marinari heat bath algorithm [27], which we describe in our companion paper on the  $\text{SO}(N)$  spectrum, [12]. We use the plaquette action in eq. (3.2).

We express the deconfining temperature in physical units by calculating suitable mass scales  $\mu$  of the gauge theories at  $T = 0$  and then taking ratios  $aT_c/a\mu = T_c/\mu$ , which we can then extrapolate to the continuum limit in a standard way. Three such quantities are the string tension, coupling, and lightest scalar glueball mass (the mass gap). We now briefly describe how we calculate these on the lattice. We provide fuller details in [12].

#### 5.1 String tensions

To obtain the string tension, we calculate the energy  $E(l)$  of the lightest flux tube that winds around the spatial torus of size  $l$  on a lattice that corresponds to  $T \sim 0$ . To do this, we use correlators of zero-momentum sums of Polyakov loop operators, that have been

‘blocked’ to obtain a very good overlap onto the ground state [28, 29] supplemented by a standard variational calculation [30]. We expect that  $E(l) \rightarrow \sigma l$  for  $l$  large [12]. For finite  $l$ , we expect  $E(l)$  to be well-approximated by [31–34]

$$E(l) = \sigma l \left(1 - \frac{\pi}{3\sigma l^2}\right)^{\frac{1}{2}}. \tag{5.1}$$

By evaluating the string tension at  $\beta_c$ , we can then express the deconfining temperature in the dimensionless ratio  $T_c/\sqrt{\sigma}$ .

## 5.2 Couplings

In  $D = 2 + 1$  the coupling  $g^2$  provides a mass scale for the theory. In the continuum limit

$$\lim_{\beta \rightarrow \infty} \frac{\beta}{2N^2} = \frac{1}{ag^2N} \tag{5.2}$$

where  $g^2N$  is the ’t Hooft coupling which one keeps constant as  $N$  increases in order to have a smooth large- $N$  limit. At finite lattice spacing the coupling is scheme dependent, and in that sense not a physical quantity, but different choices of coupling differ at  $O(ag^2)$  and so converge to the same continuum limit. It makes sense to try and choose a coupling scheme within which that convergence is rapid. Previous calculations in  $D = 2 + 1$  SU( $N$ ) [30] have found it useful to employ the mean field improved coupling [35, 36]

$$\beta_I = \beta \left\langle \frac{1}{N} \text{tr}(U_p) \right\rangle. \tag{5.3}$$

We will choose to use this improved coupling to calculate the continuum value of  $T_c/g^2N$ .

## 5.3 Scalar glueball masses

SO( $N$ ) gauge theories have a glueball mass spectrum similar to that in SU( $N$ ) gauge theories, except that all glueballs have charge conjugation  $C = +$ . The lightest glueball has spin  $J = 0$  and parity  $P = +$  and it is the glueball mass that we can calculate most accurately. We evaluate the continuum glueball masses  $M_{0+}/\sqrt{\sigma}$  in [12, 13] and use these values as another way of expressing the deconfining temperature in physical units  $T_c/M_{0+} = T_c/\sqrt{\sigma} \times \sqrt{\sigma}/M_{0+}$ .

# 6 Results: infinite volume limits

## 6.1 Methodology

We need to calculate  $\beta_c(V \rightarrow \infty)$  on our  $L_s^2 L_t$  lattices, to obtain the lattice deconfining temperature  $T_c = 1/a(\beta_c(V \rightarrow \infty))L_t$ . Using  $|\overline{l_P}|$  as our order parameter, for a given finite spatial volume  $V$ , we calculate  $\beta_c(V)$  by calculating the susceptibility  $\chi_{|\overline{l_P}|}$  for a range of  $\beta$  values, reweighting the data from those  $\beta$  values where we observe there to be tunnelling between the confined and deconfined phases, and then locating the maximum. If the lattice volume is too large for our reweighting algorithm, we follow the curve fitting procedure

mentioned above. Then  $\beta_c(V)$  is the  $\beta$  value that corresponds to a maximum in  $\chi_{|\bar{l}_P|}$ . This is illustrated in figure 10 on a  $20^2.3$  lattice in  $SO(6)$  where we see that the reweighted curve agrees well with our original data and that the estimates for  $\beta_c$  and  $\chi_{|\bar{l}_P|}(\beta_c)$  have very small errors.

Repeating this calculation for a range of  $V$  we can extrapolate to  $V = \infty$  using the finite size scaling formulae in eq. (4.6). In figure 11 we display such an infinite volume extrapolation for a second order transition in  $SO(4)$  with  $L_t = 2$ , and in figure 12 for a first order transition, in  $SO(16)$  with  $L_t = 3$ . In both cases we see that the extrapolation is precise and well-defined. As will be apparent when we list the results of our extrapolations, this is mostly the case, albeit with a significant number of exceptions where the fits are statistically poor.

Since the tunnelling in a first order transition is important to both identifying and locating the transition, it is useful to consider how this tunnelling varies with  $V$  and  $N$ . Using a standard argument, the tunnelling must proceed through an intermediate configuration where the two phases are separated by two spatial domain walls of length  $l_s = aL_s$ , with a probability of

$$P_W(T) \propto \exp\left(-\frac{2\sigma_W l_s}{T}\right) = \exp(-2a^2\sigma_W L_s L_t) \tag{6.1}$$

relative to the probability of a single phase at the same temperature. Here  $\sigma_W$  is the surface tension per unit length of the domain wall. (All this assumes that our Monte Carlo is a local process. If we have global updates, which are trivial to construct between the two deconfined phases, then this discussion will need changing.) Now just as in  $SU(N)$  we expect the surface tension to grow with  $N$  as  $\sigma_W \propto N^2$  [37, 38]. Hence, the probability of the domain walls and the probability of tunnelling decreases exponentially as either the volume  $V$  or as  $N$  increase. Thus transitions between the two states are increasingly rare at large  $V$ , especially at large  $N$ , and this provides an effective upper bound on the volumes we can consider at a given  $N$ . In addition to this, critical slowing down will also suppress the frequency of tunnelling as  $a(\beta_c)$  decreases.

Since the accuracy of our calculation of  $\beta_c(V)$  depends primarily on the number of tunnelling fluctuations, rather than the fluctuations within a given phase, we should, ideally, use errors in our reweighting procedure derived solely from the number of tunnellings. Since this is not straightforward to do, we instead used only data points from runs that clearly have tunnellings, but then used ‘naive’ errors, albeit based on large bin-sizes each of which would usually contain some tunnellings. While we believe this ‘fix’ is usually reliable, it nonetheless leaves a systematic error in our calculations which we only partially control, and this may be the reason for the very poor goodness of fit of a few of our  $V \rightarrow \infty$  extrapolations.

## 6.2 $SO(4)$ and $SO(5)$

The  $SO(4)$  and  $SO(5)$  deconfining phase transitions are second order. We can see this from the  $\bar{l}_P$  histograms, such as figure 8, which show a continuous transition from confined to deconfined phases as we increase  $\beta$ . We can also see this in susceptibility plots for



different spatial volumes at fixed  $L_t$ , such as figure 13, which show that, as the spatial volume increases, the susceptibility peak height increases, and the large volume susceptibility provides an envelope for the ones at smaller  $V$ .

For SO(4), we can use reweighting for  $2 \leq L_t \leq 4$  to calculate  $\beta_c$ . For  $L_t = 5$ , the susceptibility peak is at  $\beta \in [9.0, 10.0]$ . This is in the region of the ‘bulk’ transition which separates weak and strong coupling and which we will discuss later, and which affects the data so greatly that reweighting does not work. For  $L_t \geq 6$ , the spatial volumes become so large that we cannot reweight the data using our standard algorithm and so we curve fit instead. For smaller  $L_t$ , the values lie on a smooth curve with small errors and the reweighted values fit well with the original data. At larger  $L_t$ , the data is more scattered than at smaller  $L_t$ , although we can still estimate  $\beta_c$  with usefully small errors. We present the SO(4) values of  $\beta_c(V)$  for volumes  $V$  with  $L_t = 2, 3, 4, 6, 7, 8, 10, 12$  in tables 2 and 3.

To extrapolate  $\beta_c(V \rightarrow \infty)$  to the infinite volume limit using eq. (4.6), we need a value for the critical exponent  $\nu$ . We recall that the Svetitsky-Yaffe conjecture [39] puts the deconfining phase transition in the same universality class as the order/disorder transition of the spin system which is in the same spatial dimensions and which is invariant under the group that corresponds to the centre of the gauge group. For SO(2N) gauge groups, which have a  $\mathbb{Z}_2$  centre symmetry, this puts the deconfining phase transition in the same universality class as the  $D = 2$  Ising model. In the case of SO(4)  $\sim$  SU(2)  $\times$  SU(2) we would expect the deconfining phase transition to be in the universality class of two decoupled  $D = 2$  Ising models. In the case of SO(5), we know that Sp(2) forms the vector representation of SO(5), which also has a  $\mathbb{Z}_2$  centre symmetry so we would expect that its deconfining phase transition should also be in the universality class of the  $D = 2$  Ising model [18]. Since the order/disorder transition for the  $D = 2$  Ising model has critical exponents

$$\gamma = 1.75 \quad ; \quad \nu = 1 \tag{6.2}$$

we expect these to be the critical exponents of the SO(4) and SO(5) deconfining phase transitions. One can try to support this choice by fitting  $\nu$  to our actual data, but because the variation of  $\beta_c(V)$  is weak one needs a large lever arm in  $V$ , and very accurate data, to get a useful result. With our data the only useful fit for  $\nu$  is to the SO(4)  $L_t = 2$  data from which we obtain the estimate  $\nu = 0.88(19)$ , which provides some support for the universality based value, which we shall employ from now on.

We list the resulting SO(4)  $\beta_c(V = \infty)$  values in table 4 showing in each case the goodness of fit as measured by the value of  $\bar{\chi}_{\text{dof}}^2$  (chi-squared divided by the number of degrees of freedom). We see that the extrapolated values have small errors and most of the  $\bar{\chi}_{\text{dof}}^2$  values are reasonable. (One  $\bar{\chi}_{\text{dof}}^2$  value is very large, and this is due to a scatter among values with very small errors, which cannot be remedied by dropping values at the smallest  $V$ .)

For SO(5), we can use reweighting for  $2 \leq L_t \leq 6$  and curve fitting for  $L_t \geq 7$  to calculate  $\beta_c$ . Since the centre symmetry is trivial we cannot use that to argue that  $\langle \overline{l_P} \rangle \approx 0$  in the low  $T$  confined phase. However, our calculations show that this is indeed the case. We list the  $\beta_c(V)$  values for SO(5) with  $L_t = 2, 3, 4, 5, 6, 7, 8, 10$  in table 5 and for

the infinite volume limits in table 6. We see that the extrapolated values again have small errors and that the  $\bar{\chi}_{\text{dof}}^2$  values are mostly reasonable.

### 6.3 SO(6)

The SO(6) deconfining phase transition is (weakly) first order: the coexisting phases are apparent, as in figure 4, but are less well defined than for SO( $N \geq 7$ ). While susceptibility plots indicate that the transition has features from both first and second order transitions, the  $\bar{l}_P$  histograms (such as figure 4) show a clear first order phase coexistence. We extrapolate to the infinite volume limit using eq. (4.6). We list the  $\beta_c(V)$  values in table 7 and the infinite volume limits in table 8.

### 6.4 SO(7), SO(8), SO(9), SO(12), and SO(16)

The SO( $N \geq 7$ ) deconfining phase transitions are all first order, as is clear from the phase coexistence in the  $\langle \bar{l}_P \rangle$  histograms and from the susceptibility plots (such as figure 14) which show the whole peak shrinking and its height growing as  $V$  increases. For SO(7) and SO(9) our calculations show that, just as for SO(5), we have  $\langle \bar{l}_P \rangle \approx 0$  despite the absence of a non-trivial center symmetry.

We list the  $\beta_c(V)$  values in tables 9, 11, 13, 15, and 17 and the infinite volume limits, obtained using eq. (4.6), in tables 10, 12, 14, 16, and 18.

## 7 Results: continuum limits

### 7.1 Methodology

To extrapolate  $T_c$  to the continuum limit, i.e.  $a \rightarrow 0$  or equivalently  $\beta \rightarrow \infty$ , we express  $T_c$  in units of some other energy scale  $\mu$ , calculated at the same value of  $\beta$ , and extrapolate the resulting dimensionless ratio  $\lim_{\beta \rightarrow \infty} T_c/\mu$ . For the scale  $\mu$  we will use either the string tension,  $\mu = \sqrt{\sigma}$ , calculated at  $\beta_c$  and at  $T \approx 0$ , or the 't Hooft coupling,  $\mu = g^2 N$ .

Let us express the critical temperature in units of the string tension evaluated at the critical coupling  $\beta_c$  on a lattice corresponding to  $T \simeq 0$ ,

$$\frac{T_c}{\sqrt{\sigma}}(a) = \frac{1}{a(\beta_c)\sqrt{\sigma}L_t}. \tag{7.1}$$

Once we have  $T_c/\sqrt{\sigma}$  for each of our values of  $L_t$ , we take the continuum limit  $a \rightarrow 0$ . Since this is the ratio of two physical mass scales, we expect the leading correction to be  $\mathcal{O}(a^2)$  [40],

$$\frac{T_c}{\sqrt{\sigma}}(a) = \frac{T_c}{\sqrt{\sigma}}(a=0) + ca^2\sigma(a) + \dots \tag{7.2}$$

for some constant  $c$ .

We can similarly express the critical temperature in terms of the 't Hooft coupling,

$$\frac{T_c}{g^2 N} \equiv \frac{aT_c}{ag^2 N} = \frac{\beta_c}{2N^2} \frac{1}{L_t}. \tag{7.3}$$

As remarked earlier, at finite  $\beta$  the lattice coupling is scheme dependent, and we will choose to use the mean field improved coupling, replacing  $\beta$  by  $\beta_I = \beta \langle \frac{1}{N} \text{tr}(U_p) \rangle$  in the above. Once we have  $T_c/(g^2 N)$  for each of our  $L_t$  values, we can take the continuum limit

$$\frac{T_c}{g^2 N}(a) = \frac{T_c}{g^2 N}(a=0) + c a g^2 N + \dots \tag{7.4}$$

where the leading order correction is  $\mathcal{O}(a)$  rather than  $\mathcal{O}(a^2)$  since, unlike the string tension or glueball mass, the lattice coupling is not a physical quantity.

We note that the errors on the values of  $\beta_c$  and  $\beta_{I,c}$  are typically much smaller than on the  $a\sqrt{\sigma}$  lattice values. However this greater accuracy is offset by the fact that these values are ‘further away’ from the continuum limit in that the leading correction is  $\mathcal{O}(a)$  rather than  $\mathcal{O}(a^2)$ . Moreover one would naively expect  $T_c$  and  $\sigma$  to be more closely correlated than  $T_c$  and some lattice  $g^2$ , and so their ratio to be closer to its continuum value. For this reason we will place more stress on our continuum extrapolation of  $T_c/\sqrt{\sigma}$  than on  $T_c/g^2 N$ .

Finally we remark that we could equally well express the critical temperature in units of the lightest scalar glueball mass  $m_{0+}$ , by calculating this mass at each  $\beta_c$  in the  $T \simeq 0$  theory, and extrapolating the resulting dimensionless ratio to the continuum limit. However we do not do this here. Rather we simply obtain the continuum ratio  $T_c/m_{0+}$  from the continuum limit of  $m_{0+}/\sqrt{\sigma}$  calculated in [12] and our extrapolated value of  $T_c/\sqrt{\sigma}$ ,

$$\frac{T_c}{m_{0+}} = \frac{T_c/\sqrt{\sigma}}{m_{0+}/\sqrt{\sigma}}. \tag{7.5}$$

## 7.2 Bulk transition

Lattice gauge theories generally have some kind of ‘bulk’ transition between the regions of strong and weak coupling, where the coupling expansion changes from powers of  $\beta \propto 1/(a g^2)$  to powers of  $1/\beta \propto a g^2$ . Since an extrapolation to the continuum limit,  $\beta \rightarrow \infty$ , is only plausible, *a priori*, if made using values obtained in the weak coupling region, it is important to know where this bulk transition occurs.

With the  $\text{SO}(N)$  plaquette action, we find that the bulk transition seems to be characterised by the appearance of a very light excitation in the scalar glueball sector, with the rest of the glueball spectrum being essentially unaffected. Moreover we find that the visibility of this light excitation is sensitive to the lattice volume and that as  $N$  increases, we can use smaller volumes to identify the bulk transition in this way. This is an interesting and unusual transition, which we will describe in greater detail in our companion paper on the glueball spectrum [12]. For our present purposes, we only need to note that it provides an unambiguous way to identify the location of the bulk transition. We show the  $\beta$  values corresponding to this bulk transition in table 19 together with the range of  $L_t$  values for which the corresponding  $\beta_c$  lie in the weak coupling region. We note that the transition moves to weaker coupling as  $N$  decreases, making the weak coupling calculations more expensive at small  $N$ . This is why we have not performed  $\text{SO}(3)$  calculations, which one can estimate would necessitate using  $L_t > 10$  (and up to  $L_t \sim 20$  to have a useful lever arm for a continuum extrapolation).

To calculate the continuum limit of the deconfinement temperature, we shall use data corresponding to values in the weak coupling region, ignoring the data from  $L_t$  values that have  $\beta_c$  values in the strong coupling region. Occasionally, where the would-be susceptibility peak around  $\beta_c$  overlaps with this bulk transition, it may be grossly distorted by the very light scalar excitation (which can also affect the winding flux tube spectrum) and we are then unable to obtain a usefully precise value of  $\beta_c$ .

### 7.3 SO(4)

For SO(4), the  $\beta_c$  values for  $L_t < 5$  are in the strong coupling region whereas the  $\beta_c$  values for  $L_t > 5$  are in the weak coupling region. The deconfining transition for  $L_t = 5$  mixes with the bulk transition and we do not attempt to extract corresponding values of  $\beta_c$ . We give the corresponding values of  $T_c/\sqrt{\sigma}$  in table 20.

We extrapolate the values of  $T_c/\sqrt{\sigma}$  to the continuum limit using eq. (7.2). We display the data and the fits in figure 15. There are two separate fits on display. The first is to the weak-coupling data, obtained on lattices with  $L_t \geq 6$ . This data shows very little dependence on  $a$  and the fit with just the leading  $O(a^2)$  correction works well. This is no surprise because  $a^2\sigma \ll 1$  for all the weak coupling data. We obtain a continuum limit

$$\frac{T_c}{\sqrt{\sigma}}(a = 0) = 0.7702(88) \quad \bar{\chi}_{\text{dof}}^2 = 0.12 \quad \text{SO(4) (weak coupling)}. \quad (7.6)$$

The second fit is motivated by the fact that the three strong coupling values obtained on lattices with  $L_t \leq 4$ , appear to lie on a straight line. A linear fit as in eq. (7.2) works well and provides us with what we dub a ‘strong coupling’ continuum limit

$$\frac{T_c}{\sqrt{\sigma}}(a = 0) = 0.8638(21) \quad \bar{\chi}_{\text{dof}}^2 = 0.09 \quad \text{SO(4) (strong coupling)}. \quad (7.7)$$

This linearity of the strong coupling data is unexpected and indeed bizarre. It may just be an accident, in which case our exercise is meaningless. However it may be that  $a^2$  is small enough that the operator expansion of the lattice action in powers of  $a^2$  is viable even if the coupling expansion in powers of  $1/\beta$  is not. If so one might speculate that this provides some kind of strong coupling continuum limit. In any case, the true continuum limit of the SO(4) theory is the one extracted from the weak coupling values in eq. (7.6).

Similarly, we can calculate the critical temperatures in units of the 't Hooft coupling. The values of  $T_c/(g^2N)$  are listed in table 20. We can plot  $T_c/(g^2N)$  against  $ag^2N$  and extrapolate to the continuum limit using eq. (7.4). The continuum limit is

$$\frac{T_c}{g^2N}(a = 0) = 0.04567(43) \quad \bar{\chi}_{\text{dof}}^2 = 2.17 \quad \text{SO(4) (weak coupling)} \quad (7.8)$$

where we need to drop the  $L_t = 6$  point from the fit in order to obtain a reasonable  $\bar{\chi}_{\text{dof}}^2$  with just a leading order weak coupling correction. (Note that the strong coupling values do not fit onto a linear extrapolation in  $1/\beta_I$ , which is of course as expected.)

Finally, we express the critical temperature in units of the lightest scalar glueball mass  $M_{0+}$ . We use the continuum value of  $M_{0+}/\sqrt{\sigma}$  calculated in [12] with our above continuum

value of  $T_c/\sqrt{\sigma}$  to obtain

$$\frac{T_c}{M_{0+}}(a=0) = 0.2293(30) \quad \text{SO}(4) \text{ (weak coupling)}. \quad (7.9)$$

#### 7.4 SO(5) and SO(6)

For both SO(5) and SO(6), the  $\beta_c$  values for  $L_t \geq 5$  are in the weak coupling region and so can be used for a continuum extrapolation. To obtain the critical temperature in string tension units  $T_c/\sqrt{\sigma}$ , we calculate the string tension at each  $\beta_c$  as in SO(4). We list the resulting values for SO(5) in table 21 and for SO(6) in table 22. We display the continuum extrapolation for SO(6) in figure 16.

In the case of SO(5), unlike SO(4), there were difficulties in using a linear extrapolation in the weak coupling region due to peculiar variation in the value of  $T_c/\sqrt{\sigma}$ . To obtain a good fit we had to drop the two smallest  $L_t$  points in the weak coupling region. (Context: for no other  $N$  did we need to drop any weak coupling points.) SO(5) is the largest SO( $N$ ) group for which the transition is second order and it might be that this is behind this atypical behaviour. The continuum limit from within the weak coupling region is

$$\frac{T_c}{\sqrt{\sigma}}(a=0) = 0.7963(114), \quad \bar{\chi}_{\text{dof}}^2 = 0.003 \quad \text{SO}(5). \quad (7.10)$$

We also note that the strong coupling values fit less well with a linear extrapolation than they did for SO(4), giving a strong coupling ‘continuum’ extrapolation of  $T_c/\sqrt{\sigma} = 0.783(4)$  with a mediocre  $\bar{\chi}_{\text{dof}}^2 = 2.78$ .

SO(6) is the smallest group for which the transition is first order. The continuum limit taken from data within the weak coupling region is

$$\frac{T_c}{\sqrt{\sigma}}(a=0) = 0.8105(42), \quad \bar{\chi}_{\text{dof}}^2 = 0.16 \quad \text{SO}(6). \quad (7.11)$$

There is also a good linear extrapolation using the strong coupling values that gives  $T_c/\sqrt{\sigma} = 0.8144(20)$  with  $\bar{\chi}_{\text{dof}}^2 = 0.59$ . We note that here the strong coupling extrapolation is consistent with the true weak-coupling continuum limit.

We can also calculate the critical temperatures in units of the coupling.  $T_c/(g^2N)$ . The values are listed in table 21 and table 22. We can then plot  $T_c/(g^2N)$  against  $ag^2N$  and extrapolate to the continuum limit using eq. (7.4). The continuum limits, from within the weak coupling regions, are

$$\left. \frac{T_c}{g^2N} \right|_{a=0} = \begin{cases} 0.05544(92) & \bar{\chi}_{\text{dof}}^2 = 0.05 & \text{SO}(5) \\ 0.05996(19) & \bar{\chi}_{\text{dof}}^2 = 0.53 & \text{SO}(6) \end{cases}. \quad (7.12)$$

In the case of the SO(5) data, the points seem to lie on a smooth curve and do not exhibit the peculiar variation seen in the corresponding  $T_c/\sqrt{\sigma}$  values.

Finally, we can calculate the critical temperatures in units of the lightest scalar glueball. Using the values of  $M_{0+}/\sqrt{\sigma}$  from [12] we obtain

$$\left. \frac{T_c}{M_{0+}} \right|_{a=0} = \begin{cases} 0.2244(33) & \text{SO}(5) \\ 0.2232(14) & \text{SO}(6) \end{cases}. \quad (7.13)$$

## 7.5 SO(7), SO(8), SO(9), SO(12), and SO(16)

For SO(7) and SO(8) the  $\beta_c$  values are in the weak coupling region for  $L_t \geq 4$ , and for SO(9), SO(12), and SO(16) they are in the weak coupling region for  $L_t \geq 3$ .

We list the critical temperature values in string tension units  $T_c/\sqrt{\sigma}$  for these groups in tables 23, 24, 25, 26, and 27. The continuum limits are

$$\frac{T_c}{\sqrt{\sigma}} \Big|_{a=0} = \begin{cases} 0.8351(38) & \bar{\chi}_{\text{dof}}^2 = 0.98 & \text{SO(7)} \\ 0.8418(39) & \bar{\chi}_{\text{dof}}^2 = 0.05 & \text{SO(8)} \\ 0.8515(14) & \bar{\chi}_{\text{dof}}^2 = 0.30 & \text{SO(9)} \\ 0.8642(38) & \bar{\chi}_{\text{dof}}^2 = 0.02 & \text{SO(12)} \\ 0.8780(38) & \bar{\chi}_{\text{dof}}^2 = 0.15 & \text{SO(16)} \end{cases} . \quad (7.14)$$

We note that all these fits are very good.

Similarly, we can calculate the critical temperature in units of the coupling, as listed in tables 23, 24, 25, 26, and 27. We can then plot  $T_c/(g^2N)$  against  $ag^2N$  and extrapolate to the continuum limit using eq. (7.4). These plots have very similar forms to the corresponding  $T_c/\sqrt{\sigma}$  plots. The continuum limits are

$$\frac{T_c}{g^2N} \Big|_{a=0} = \begin{cases} 0.06478(18) & \bar{\chi}_{\text{dof}}^2 = 4.01 & \text{SO(7)} \\ 0.06809(16) & \bar{\chi}_{\text{dof}}^2 = 0.00 & \text{SO(8)} \\ 0.07043(7) & \bar{\chi}_{\text{dof}}^2 = 0.10 & \text{SO(9)} \\ 0.07552(14) & \bar{\chi}_{\text{dof}}^2 = 0.63 & \text{SO(12)} \\ 0.07947(17) & \bar{\chi}_{\text{dof}}^2 = 0.85 & \text{SO(16)} \end{cases} . \quad (7.15)$$

We can see that these continuum extrapolations are mostly good. (Given there is only one degree of freedom, the SO(7) fit is not unacceptable.)

Finally, we can calculate the critical temperatures in units of the lightest scalar glueball mass  $M_{0+}$ . Using the values for  $N = 7, 8, 12, 16$  calculated in [12] (there is no calculation for  $N = 9$ ) we find

$$\frac{T_c}{M_{0+}} \Big|_{a=0} = \begin{cases} 0.2234(12) & \text{SO(7)} \\ 0.2224(15) & \text{SO(8)} \\ 0.2217(18) & \text{SO(12)} \\ 0.2220(22) & \text{SO(16)} \end{cases} . \quad (7.16)$$

## 8 Results: large- $N$ limits

### 8.1 Deconfining temperature

In contrast to SU( $N$ ), the leading large- $N$  correction for SO( $N$ ) gauge theories is expected to be  $\mathcal{O}(1/N)$ . So we expect

$$\frac{T_c}{\mu} \Big|_{\text{SO}(N)} \stackrel{N \rightarrow \infty}{\simeq} \frac{T_c}{\mu} \Big|_{\text{SO}(\infty)} + \frac{c}{N} + \dots \quad (8.1)$$

with  $\mu$  a physical mass scale such as  $\sqrt{\sigma}$ ,  $m_{0+}$  or  $g^2N$ .

Since one of our aims is to compare the values of  $T_c$  for  $\text{SO}(2N)$  and  $\text{SO}(2N + 1)$  gauge theories, it would be useful to have an estimate of  $T_c$  in  $\text{SO}(3)$ . Since the  $\text{SO}(3)$  and  $\text{SU}(2)$  groups have the same Lie algebra, it is plausible to assume that they share the same value of  $T_c/M_{0+}$ , where  $M_{0+}$  is the mass of the lightest scalar glueball. We have to be more careful with  $T_c/\sqrt{\sigma}$  because the fundamental string tension in  $\text{SO}(3)$  corresponds to the adjoint in  $\text{SU}(2)$ . Now, we know that in  $\text{SU}(2)$   $T_c/\sqrt{\sigma} = 1.1238(88)$  [1] We also know that in  $\text{SU}(2)$   $M_{0+}/\sqrt{\sigma} = 4.7367(55)$  [41] and in  $\text{SO}(3)$   $M_{0+}/\sqrt{\sigma} = 2.980(24)$  [12]. From the ratio of these two numbers we extract an estimate of the ratio of fundamental string tensions in  $\text{SU}(2)$  and  $\text{SO}(3)$ . All this implies that the  $\text{SO}(3)$  continuum deconfining temperature in units of the string tension is

$$\frac{T_c}{\sqrt{\sigma}} = 0.7072(80) \quad \text{SO}(3). \quad (8.2)$$

We also know the  $\text{SO}(3)$  string tension  $\sqrt{\sigma}/(g^2N) = 0.04576(36)$  [12], which tells us that

$$\frac{T_c}{g^2N} = 0.03236(45) \quad \text{SO}(3). \quad (8.3)$$

Finally, we can also infer from the above that the  $\text{SO}(3)$  continuum deconfining temperature in units of the lightest scalar glueball mass is

$$\frac{T_c}{M_{0+}} = 0.2373(33) \quad \text{SO}(3). \quad (8.4)$$

We list the  $\text{SO}(N)$  deconfining temperatures in string tension units in table 28. We begin by applying a linear fit in  $1/N$  to just the  $\text{SO}(2N)$  values. We do so for two reasons. Firstly, we intend to compare this limit to the  $\text{SU}(N)$  large- $N$  limit motivated by the large- $N$  orbifold equivalence. Secondly,  $\text{SO}(2N + 1)$  has a different centre to  $\text{SO}(2N)$ , so the deconfinement properties might differ between the two sets of gauge theories and it is interesting to see if this is the case. In figure 17 we plot all our values of  $T_c/\sqrt{\sigma}$ , including that inferred for  $\text{SO}(3)$ , against  $1/N$ , and we also show the best leading-order fit to just the  $\text{SO}(2N)$  values. We see that the linear fit is very good. We also see that values for the  $\text{SO}(2N + 1)$  groups are consistent with lying on this fit. Indeed if we take all the values, including  $\text{SO}(3)$ , we obtain a very similar best fit:

$$\left. \frac{T_c}{\sqrt{\sigma}} \right|_{N \rightarrow \infty} = \begin{cases} 0.9152(48) & \bar{\chi}_{\text{dof}}^2 = 0.58 & \text{SO}(2N \geq 4) \\ 0.9231(45) & \bar{\chi}_{\text{dof}}^2 = 0.52 & \text{SO}(2N + 1 \geq 3) \\ 0.9194(33) & \bar{\chi}_{\text{dof}}^2 = 0.82 & \text{SO}(N \geq 3) \end{cases} . \quad (8.5)$$

We conclude that at our level of accuracy there is no evidence for any difference in the way  $T_c/\sqrt{\sigma}$  varies with  $N$  in  $\text{SO}(2N)$  and  $\text{SO}(2N + 1)$  gauge theories: the lack of a center symmetry in the latter appears to play no role. We also observe that the groups with a second order transition,  $\text{SO}(N \leq 5)$ , fall nicely on the smooth curve that describes the  $N$ -dependence of the first-order transitions,  $\text{SO}(N \geq 6)$ .

We can repeat the above, replacing  $\sqrt{\sigma}$  by the 't Hooft coupling  $g^2N$ . We list the  $\text{SO}(N)$  deconfining temperatures in units of  $g^2N$  in table 28. To fit to all the values of  $N$ ,

we need to include an additional  $O(1/N^2)$  correction and, to avoid a systematic bias, we do the same for our separate fits to odd and even  $N$ . We find:

$$\frac{T_c}{g^2 N} \Big|_{N \rightarrow \infty} = \begin{cases} 0.09139(49) & \bar{\chi}_{\text{dof}}^2 = 2.53 & \text{SO}(2N \geq 4) \\ 0.09039(63) & \bar{\chi}_{\text{dof}}^2 = 0.43 & \text{SO}(2N + 1 \geq 3) \\ 0.09160(35) & \bar{\chi}_{\text{dof}}^2 = 1.45 & \text{SO}(N \geq 3) \end{cases} . \quad (8.6)$$

We see that values of  $T_c/(g^2 N)$  obtained for the  $\text{SO}(2N + 1)$  groups are consistent with those for  $\text{SO}(2N)$ .

Finally, we list the  $\text{SO}(N)$  deconfining temperatures in units of the lightest scalar glueball mass in table 28 and plot these values in figure 18. We show a leading-order fit to  $\text{SO}(2N)$  for  $2N \geq 6$  since the data (if one pays attention to the  $\text{SO}(3)$  value) indicates the need for a higher order correction at the lower values of  $N$ . We do not fit odd  $N$  separately because we do not have available a glueball mass for  $\text{SO}(9)$ , and so the number of odd  $N$  values is too small to fit. We also perform fits with an additional  $O(1/N^2)$  correction to both  $\text{SO}(2N \geq 4)$  and to all our values,  $\text{SO}(N \geq 3)$ . Altogether, these fits give:

$$\frac{T_c}{M_{0+}} \Big|_{N \rightarrow \infty} = \begin{cases} 0.2209(28) & \bar{\chi}_{\text{dof}}^2 = 0.03 & \text{SO}(2N \geq 6) \\ 0.2189(23) & \bar{\chi}_{\text{dof}}^2 = 0.51 & \text{SO}(2N \geq 4) \\ 0.2230(39) & \bar{\chi}_{\text{dof}}^2 = 0.10 & \text{SO}(N \geq 3) \end{cases} . \quad (8.7)$$

We note that the linear fit is particularly flat compared to the linear fit in figure 17 and the one to  $T_c/g^2 N$ . Finally, we again see evidence that the  $\text{SO}(2N)$  and  $\text{SO}(2N + 1)$  values form a single smooth series.

## 8.2 Large- $N$ scaling

We have assumed throughout that the large- $N$  limit requires keeping  $g^2 N$  fixed and that the leading correction is  $O(1/N)$ , guided by the all-orders analysis of diagrams [2]. Certainly, keeping  $g^2 N$  fixed is necessary if one wants to obtain an  $\text{SO}(\infty)$  theory that is perturbative (and asymptotically free) at short distances. Here we ask whether our non-perturbative calculations support these assumptions.

Without assuming  $g^2 N$  scaling, we can test for the power of the leading correction by fitting

$$\frac{T_c}{\sqrt{\sigma}} \Big|_{\text{SO}(N \rightarrow \infty)} = c_0 + \frac{c_1}{N^\alpha} \quad \text{SO}(N \geq 3). \quad (8.8)$$

We find that  $\alpha = 1.13 \pm 0.14$ . If we assume  $g^2 N$  scaling then a similar analysis for  $T_c/g^2 N$ , over the range  $N \geq 7$  where we can get a good fit, gives  $\alpha = 0.88 \pm 0.16$ . So if the power of the correction is an integer, then this confirms that it must be  $O(1/N)$ .

It is amusing to see if our results also demand that  $g^2 N$  should be kept constant. We can fit

$$\frac{T_c}{g^2 N^\gamma} \Big|_{\text{SO}(N \rightarrow \infty)} = c_0 + \frac{c_1}{N} \quad \text{SO}(N \geq 7) \quad (8.9)$$

and doing so we find a tight constraint  $\gamma = 1.020 \pm 0.024$ , just as expected.



## 9 Comparison of $SO(N)$ and $SU(N)$ deconfining temperatures

### 9.1 $SO(4) \sim SU(2) \times SU(2)$

We know that  $SO(4)$  and  $SU(2) \times SU(2)$  share a common Lie algebra so it is interesting to see if they have the same deconfining temperatures and transitions. We have seen that the  $SO(4)$  deconfining phase transition is second order, just like  $SU(2)$ , and (within large errors) they appear to share the same critical exponents. Now, we expect that the fundamental flux of  $SO(4)$  contains the fundamental flux of both  $SU(2)$  groups from the product group  $SU(2) \times SU(2)$ , so that

$$\sigma|_{su2 \times su2} = 2 \sigma|_{su2}. \tag{9.1}$$

Hence, we expect that

$$\frac{T_c}{\sqrt{\sigma}} \Big|_{so4} = \frac{T_c}{\sqrt{\sigma}} \Big|_{su2 \times su2} = \frac{1}{\sqrt{2}} \frac{T_c}{\sqrt{\sigma}} \Big|_{su2}. \tag{9.2}$$

We know that the  $SU(2)$  deconfining temperature is  $T_c/\sqrt{\sigma} = 1.1238(88)$  [1] so that we can compare this to our value for  $SO(4)$ :

$$\begin{aligned} \frac{T_c}{\sqrt{\sigma}} &= 0.7702(88) && SO(4) \\ \frac{1}{\sqrt{2}} \frac{T_c}{\sqrt{\sigma}} &= 0.7946(62) && SU(2). \end{aligned} \tag{9.3}$$

We see that these values are within about  $2.25\sigma$  of each other which we consider to be reasonable agreement.

### 9.2 $SO(6) \sim SU(4)$

$SO(6)$  and  $SU(4)$  also share a common Lie algebra so it is also interesting to compare their deconfining transitions. We have seen that  $SO(6)$  is first order, but weakly so, and this is also the case for  $SU(4)$  [1, 42]. As we discussed earlier, the  $SO(6)$  fundamental string tension is equivalent to the  $SU(4)$   $k = 2$  anti-symmetric string tension so what we may expect is

$$\frac{T_c}{\sqrt{\sigma_f}} \Big|_{so6} = \frac{T_c}{\sqrt{\sigma_{2A}}} \Big|_{su4}. \tag{9.4}$$

Hence, to compare between the  $SO(6)$  and  $SU(4)$  deconfining temperatures measured in units of the fundamental string tension, we need the ratio of the  $k = 2A$  and fundamental string tensions in  $SU(4)$ , and this has been calculated to be  $\sigma_{2A}/\sigma_f|_{su4} = 1.355(9)$  in [15]. We also know from [1], that the  $SU(4)$  deconfining temperature is  $T_c/\sqrt{\sigma_f} = 0.9572(39)$  so that

$$\begin{aligned} \frac{T_c}{\sqrt{\sigma_f}} &= 0.8105(42) && SO(6) \\ \frac{T_c}{\sqrt{\sigma_{2A}}} &= 0.8223(61) && SU(4). \end{aligned} \tag{9.5}$$

We see that these values are in agreement, being within  $1.5\sigma$  of each other.

### 9.3 Large- $N$ (orbifold) equivalence

As remarked earlier, the existence of an orbifold projection from  $\text{SO}(2N)$  to  $\text{SU}(N)$  gauge theories means that they should have the same large- $N$  limit and in particular the same deconfining temperature in that limit when expressed in physical units. In addition we have the diagrammatic planar equivalence of  $\text{SO}(N)$  and  $\text{SU}(N)$ .

We list the  $\text{SO}(2N)$  and  $\text{SU}(N)$  [1] continuum values of  $T_c/\sqrt{\sigma}$  in table 29. We display the corresponding large- $N$  extrapolations in figure 19. The two large- $N$  limits are

$$\frac{T_c}{\sqrt{\sigma}} = \begin{cases} 0.9152(48) & \text{SO}(2N \rightarrow \infty) \\ 0.9030(29) & \text{SU}(N \rightarrow \infty) \end{cases} . \quad (9.6)$$

We see that these two values are within  $2\sigma$  of each other, which is consistent with the hypothesis that they are equal. As for the planar equivalence, we recall that fitting  $\text{SO}(N \geq 3)$  with a fit that is linear in  $1/N$  gives  $T_c/\sqrt{\sigma} = 0.9194(33)$  at  $N = \infty$ , and this is a less comfortable  $\sim 4\sigma$  from the  $\text{SU}(\infty)$  value.

Similarly, we list the  $\text{SO}(2N)$  and  $\text{SU}(N)$  [43] continuum values of  $T_c/(g^2 N)$  in table 29. To compare the  $\text{SO}(2N)$  and  $\text{SU}(N)$  values, we need to rescale the  $\text{SO}(N)$  values to  $\text{SO}(2N)$  values by doubling them (as we did in some earlier figures). This is due to the large- $N$  coupling matching  $g_{\text{SU}(N)}^2 N = g_{\text{SO}(2N)}^2 N$ , so that the large- $N$  limit is

$$\lim_{N \rightarrow \infty} \frac{T_c}{g_{\text{SU}(N)}^2 N} = \lim_{N \rightarrow \infty} \frac{T_c}{g_{\text{SO}(2N)}^2 N} = \lim_{N \rightarrow \infty} 2 \frac{T_c}{g_{\text{SO}(2N)}^2 2N} . \quad (9.7)$$

The two large- $N$  limits are

$$\frac{T_c}{g^2 N} = \begin{cases} 0.1828(10) & \text{SO}(2N \rightarrow \infty) \\ 0.1852(8) & \text{SU}(N \rightarrow \infty) \end{cases} \quad (9.8)$$

while a fit to  $\text{SO}(N \geq 3)$  gives  $T_c/g^2 N = 0.1832(7)$ . We see that these two values are no more than  $\sim 2\sigma$  from each other.

Finally, we list the  $\text{SO}(2N)$  and  $\text{SU}(N)$  [1, 41, 44] values of  $T_c/M_{0+}$  table 29. We display the two large- $N$  extrapolations in figure 20. The two large- $N$  limits obtained from these leading order fits are

$$\frac{T_c}{M_{0+}} = \begin{cases} 0.2209(28) & \text{SO}(2N \rightarrow \infty) \\ 0.2207(6) & \text{SU}(N \rightarrow \infty) \end{cases} . \quad (9.9)$$

and a higher order fit to all  $\text{SO}(N \geq 3)$  gives a value 0.2230(39). We see that all these values agree very well.

## 10 Conclusions

In this paper we identified a finite temperature transition in  $D = 2 + 1$   $\text{SO}(N)$  gauge theories for  $N = 4, 5, 6, 7, 8, 9, 12, 16$ . For  $N = 4, 5$  the transition appears to be second order, while for  $N \geq 6$  it appears to be first order. We did not attempt to calculate  $T_c$  for  $\text{SO}(3)$  because the inconvenient location of the ‘bulk’ transition would have made it computationally expensive. However, the close connection between  $\text{SO}(3)$  and  $\text{SU}(2)$  makes one confident that there is a deconfining transition in  $\text{SO}(3)$ , and we have used the  $\text{SU}(2)$  value of  $T_c$  to provide an estimate for the  $\text{SO}(3)$  value.

This transition appears to have all the characteristics of a deconfining transition for both even and odd  $N$ , and appears to be a phase transition rather than a cross-over. We gave some arguments, and provided some evidence, that  $\text{SO}(N)$  gauge theories are indeed confining at low  $T$ , and that this is the case not just for even  $N$  but also for odd  $N$  where the centre is trivial.

Our calculations were performed on a lattice in a finite volume, but our final results for the deconfining temperature,  $T_c$ , are for the continuum theory in an infinite volume. (Achieved through extrapolation, of course.) We find that dimensionless ratios such as  $T_c/\sqrt{\sigma}$ , where  $\sigma$  is the zero-temperature confining string tension, fall on a single sequence that can be interpolated by a smooth function of  $N$  for  $N \geq 3$  (with the value for  $\text{SO}(3)$  being inferred from the value in  $\text{SU}(2)$ ). That is to say, we can think of the  $N$ -dependence of non-Abelian  $\text{SO}(N)$  gauge theories as a continuous function of  $N$  for all  $N$ . In particular there is no evidence that even and odd  $N$  fall on two separate (even if converging) branches.

Somewhat remarkably we find that a simple leading-order large- $N$  expression suffices to fit all our calculated values of  $T_c/\sqrt{\sigma}$ :

$$\frac{T_c}{\sqrt{\sigma}} = 0.9194(33) - \frac{0.620(28)}{N} \quad ; \quad N \geq 3. \quad (10.1)$$

Such a ‘precocious’ large- $N$  scaling seems the norm for physical mass ratios in both  $\text{SO}(N)$  [12, 14] and  $\text{SU}(N)$  [30, 41, 44] gauge theories. Even more striking is how weakly the ratio  $T_c/M_{0+}$  depends on  $N$ , as we see in figure 20 and as highlighted by the smallness of the leading correction in our higher order fit

$$\frac{T_c}{M_{0+}} = 0.2230(39) - \frac{0.033(45)}{N} + \frac{0.23(12)}{N^2} \quad ; \quad N \geq 3. \quad (10.2)$$

A possible explanation for this weak  $N$ -dependence is discussed in [13].

As an aside, we remark that our results, as described in section 8.2, provide a non-perturbative confirmation of the expected large- $N$  scaling:  $g^2 N$  fixed, and  $O(1/N)$  leading corrections as  $N \rightarrow \infty$ .

As another (much less expected) aside, we recall that our values of  $T_c/\sqrt{\sigma}$  on the strong coupling side of the ‘bulk’ transition also appeared to extrapolate to  $a = 0$  with a simple  $O(a^2)$  correction. At small  $N$  this ‘strong coupling continuum limit’ was very different from the true weak coupling continuum limit, but at larger  $N$  they became consistent. This unexpected and bizarre behaviour strongly suggests that our interpretation of the ‘bulk’ transition as a simple strong-to-weak coupling transition is too naive.

Since  $\text{SU}(N)$  gauge theories can be orbifold projected from  $\text{SO}(2N)$ , we expect them to have the same physics at large  $N$  [10, 11], and indeed we find that the values of  $T_c$  extrapolated to  $N = \infty$  are consistent with being equal, at the  $\sim 1\%$  level. We obtain a similar confirmation of the large- $N$  planar equivalence between  $\text{SO}(N)$  and  $\text{SU}(N)$  gauge theories. In addition we find that the values of  $T_c$  in  $\text{SO}(4)$  and  $\text{SO}(6)$  are consistent with those in  $\text{SU}(2) \times \text{SU}(2)$  and  $\text{SU}(4)$  respectively, indicating that for this physics at least the differences in group global structure are not important: theories with the same Lie algebra appear to possess the same value of  $T_c$ .

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## A Tables

Data Fit	$\beta_c$	$\chi(\beta_c)$	$\bar{\chi}_{\text{dof}}^2$
Reweighting	8.493(10)	25.40(30)	n/a
Gaussian	8.500(11)	25.66(40)	0.62
Logistic	8.500(6)	25.71(41)	0.64

**Table 1.** Comparisons between reweighting, Gaussian fits, and logistic fits to obtain  $\beta_c$  and  $\chi(\beta_c)$  on a  $40^2 4$  volume in SO(4).

$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{l}_P }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{l}_P }$
$20^2 2$	6.4748(4)	10.77(4)	$32^2 3$	7.534(3)	19.60(14)
$24^2 2$	6.4771(4)	13.64(5)	$36^2 3$	7.538(3)	23.22(16)
$28^2 2$	6.4788(3)	16.80(9)	$40^2 3$	7.539(1)	26.37(25)
$32^2 2$	6.4797(3)	20.16(8)	$44^2 3$	7.545(2)	31.09(38)
$36^2 2$	6.4813(4)	23.65(12)	$48^2 3$	7.546(3)	35.56(38)
$40^2 2$	6.4819(3)	27.58(14)	$52^2 3$	7.552(2)	40.58(53)
$48^2 2$	6.4822(4)	35.15(43)	$66^2 3$	7.552(2)	58.75(131)
$56^2 2$	6.4840(4)	44.42(63)	$80^2 3$	7.555(3)	81.02(193)
$60^2 2$	6.4850(4)	49.92(89)	$90^2 3$	7.557(3)	95.80(171)
$80^2 2$	6.4853(4)	78.57(132)	$40^2 4$	8.493(10)	25.40(30)
			$48^2 4$	8.501(6)	33.88(56)
			$56^2 4$	8.509(8)	42.02(93)
			$64^2 4$	8.526(7)	51.67(120)
			$72^2 4$	8.520(3)	59.46(140)
			$80^2 4$	8.535(6)	66.44(173)
			$88^2 4$	8.545(8)	75.23(296)

**Table 2.**  $\beta_c$  and  $\chi_{|\bar{l}_P|}$  in SO(4) for  $L_t \leq 4$  on various volumes.

$L_s^2 L_t$	$\beta_c$	$\chi_{ \overline{L_P} }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \overline{L_P} }$
36 <sup>2</sup> 6	11.110(31)	20.68(18)	64 <sup>2</sup> 8	14.005(37)	55.93(88)
48 <sup>2</sup> 6	10.924(17)	28.87(36)	80 <sup>2</sup> 8	13.836(24)	71.23(136)
60 <sup>2</sup> 6	10.861(9)	38.76(53)	96 <sup>2</sup> 8	13.901(16)	93.98(248)
72 <sup>2</sup> 6	10.837(14)	47.33(83)	112 <sup>2</sup> 8	13.712(16)	106.33(214)
84 <sup>2</sup> 6	10.809(21)	57.82(125)	128 <sup>2</sup> 8	13.736(14)	131.84(390)
96 <sup>2</sup> 6	10.824(14)	70.28(199)	144 <sup>2</sup> 8	13.767(20)	158.18(629)
120 <sup>2</sup> 6	10.835(8)	95.57(321)	80 <sup>2</sup> 10	17.096(31)	95.93(174)
42 <sup>2</sup> 7	12.685(45)	29.41(38)	90 <sup>2</sup> 10	16.919(35)	107.06(234)
56 <sup>2</sup> 7	12.494(22)	41.46(64)	100 <sup>2</sup> 10	16.870(24)	127.64(305)
70 <sup>2</sup> 7	12.397(12)	54.24(94)	110 <sup>2</sup> 10	16.790(53)	137.15(367)
84 <sup>2</sup> 7	12.303(12)	68.11(111)	120 <sup>2</sup> 10	16.731(24)	154.83(462)
98 <sup>2</sup> 7	12.224(12)	81.74(172)	140 <sup>2</sup> 10	16.725(23)	184.45(659)
112 <sup>2</sup> 7	12.261(19)	100.03(247)	72 <sup>2</sup> 12	20.648(62)	107.12(158)
126 <sup>2</sup> 7	12.274(19)	117.10(302)	84 <sup>2</sup> 12	20.202(66)	122.92(199)
			96 <sup>2</sup> 12	20.098(52)	144.13(252)
			120 <sup>2</sup> 12	19.797(29)	190.28(414)
			144 <sup>2</sup> 12	19.757(27)	236.11(603)

**Table 3.**  $\beta_c$  and  $\chi_{|\overline{L_P}|}$  in SO(4) for  $L_t \geq 6$  on various volumes.

$L_t$	$\beta_c(V \rightarrow \infty)$	$L_s$ range	$\bar{\chi}_{\text{dof}}^2$
2	6.4891(3)	$L_s \geq 20$	1.19
3	7.573(3)	$L_s \geq 32$	1.44
4	8.573(10)	$L_s \geq 40$	1.24
6	10.781(16)	$L_s \geq 48$	2.77
7	11.980(24)	$L_s \geq 42$	6.49
8	13.504(34)	$L_s \geq 64$	13.18
10	16.321(80)	$L_s \geq 90$	1.19
12	19.090(99)	$L_s \geq 84$	3.13

**Table 4.** Infinite volume limit of  $\beta_c$ , range of volumes used and  $\bar{\chi}_{\text{dof}}^2$  of extrapolation, for SO(4).

$L_s^2 L_t$	$\beta_c$	$\chi_{ \overline{L_P} }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \overline{L_P} }$
16 <sup>2</sup> 2	10.380(1)	10.75(3)	42 <sup>2</sup> 6	19.017(34)	28.35(49)
18 <sup>2</sup> 2	10.378(1)	12.82(4)	48 <sup>2</sup> 6	18.939(61)	31.74(41)
20 <sup>2</sup> 2	10.378(1)	14.74(7)	54 <sup>2</sup> 6	18.965(19)	36.39(49)
22 <sup>2</sup> 2	10.376(1)	16.83(9)	60 <sup>2</sup> 6	18.930(18)	40.39(76)
24 <sup>2</sup> 2	10.377(2)	18.53(12)	56 <sup>2</sup> 7	21.715(17)	46.57(67)
26 <sup>2</sup> 3	12.058(3)	13.41(14)	60 <sup>2</sup> 7	21.663(16)	48.70(77)
28 <sup>2</sup> 3	12.054(3)	14.07(15)	64 <sup>2</sup> 7	21.592(15)	53.02(81)
30 <sup>2</sup> 3	12.049(3)	14.76(12)	68 <sup>2</sup> 7	21.595(15)	55.50(90)
32 <sup>2</sup> 3	12.053(4)	15.28(15)	72 <sup>2</sup> 7	21.554(11)	58.01(99)
34 <sup>2</sup> 3	12.049(4)	16.02(25)	84 <sup>2</sup> 7	21.564(17)	69.48(159)
32 <sup>2</sup> 4	13.964(10)	16.42(12)	64 <sup>2</sup> 8	24.329(24)	63.20(102)
36 <sup>2</sup> 4	13.964(9)	18.90(13)	72 <sup>2</sup> 8	24.294(18)	72.14(124)
40 <sup>2</sup> 4	13.955(7)	21.27(18)	80 <sup>2</sup> 8	24.255(14)	80.30(154)
48 <sup>2</sup> 4	13.962(12)	24.13(27)	88 <sup>2</sup> 8	24.255(17)	85.77(180)
40 <sup>2</sup> 5	16.316(14)	20.75(14)	96 <sup>2</sup> 8	24.182(18)	83.53(197)
44 <sup>2</sup> 5	16.342(8)	24.15(23)	80 <sup>2</sup> 10	29.730(19)	109.84(163)
50 <sup>2</sup> 5	16.300(13)	26.37(20)	90 <sup>2</sup> 10	29.621(26)	116.99(200)
54 <sup>2</sup> 5	16.295(26)	27.87(18)	100 <sup>2</sup> 10	29.521(19)	130.84(236)
60 <sup>2</sup> 5	16.265(8)	30.58(29)	110 <sup>2</sup> 10	29.519(19)	141.11(272)
70 <sup>2</sup> 5	16.290(6)	35.40(40)	120 <sup>2</sup> 10	29.517(22)	159.53(391)

**Table 5.**  $\beta_c(V)$  and  $\chi_{|\overline{L_P}|}(V)$  for SO(5).

$L_t$	$\beta_c(V \rightarrow \infty)$	$L_s$ range	$\bar{\chi}_{\text{dof}}^2$
2	10.368(3)	$L_s \geq 16$	0.74
3	12.021(16)	$L_s \geq 26$	0.49
4	13.944(36)	$L_s \geq 32$	0.32
5	16.180(13)	$L_s \geq 40$	4.56
6	18.603(55)	$L_s \geq 42$	1.29
7	21.192(52)	$L_s \geq 56$	4.63
8	23.938(63)	$L_s \geq 64$	1.39
10	29.500(157)	$L_s \geq 100$	0.00

**Table 6.** Infinite volume limit of  $\beta_c$ , range of volumes used and  $\bar{\chi}_{\text{dof}}^2$  of extrapolation, for SO(5).

$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{l}_P }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{l}_P }$
8 <sup>2</sup> 2	15.175(2)	3.191(8)	28 <sup>2</sup> 5	25.479(24)	14.25(8)
10 <sup>2</sup> 2	15.185(1)	4.972(8)	32 <sup>2</sup> 5	25.496(16)	17.77(12)
12 <sup>2</sup> 2	15.192(1)	7.151(12)	40 <sup>2</sup> 5	25.501(14)	25.25(18)
12 <sup>2</sup> 3	17.810(9)	3.94(1)	48 <sup>2</sup> 5	25.549(14)	33.99(38)
16 <sup>2</sup> 3	17.793(4)	6.22(2)	56 <sup>2</sup> 5	25.577(11)	44.40(41)
20 <sup>2</sup> 3	17.821(4)	9.18(3)	60 <sup>2</sup> 5	25.589(10)	48.77(68)
24 <sup>2</sup> 3	17.831(4)	12.34(5)	42 <sup>2</sup> 6	29.781(26)	31.45(18)
28 <sup>2</sup> 3	17.833(3)	15.79(8)	48 <sup>2</sup> 6	29.727(18)	38.65(30)
32 <sup>2</sup> 3	17.839(3)	19.53(13)	54 <sup>2</sup> 6	29.791(33)	47.09(75)
28 <sup>2</sup> 4	21.295(6)	12.07(5)	60 <sup>2</sup> 6	29.796(31)	55.66(75)
32 <sup>2</sup> 4	21.358(8)	15.56(12)	66 <sup>2</sup> 6	29.819(9)	65.52(89)
36 <sup>2</sup> 4	21.356(8)	18.58(9)	44 <sup>2</sup> 7	34.031(37)	38.73(30)
40 <sup>2</sup> 4	21.352(4)	22.23(13)	50 <sup>2</sup> 7	33.907(25)	46.75(48)
44 <sup>2</sup> 4	21.370(6)	25.85(15)	56 <sup>2</sup> 7	34.078(49)	57.24(96)
			60 <sup>2</sup> 7	34.042(36)	62.06(65)
			64 <sup>2</sup> 7	34.021(19)	69.71(85)
			68 <sup>2</sup> 7	34.009(25)	75.83(107)
			72 <sup>2</sup> 7	34.092(23)	83.87(143)

**Table 7.**  $\beta_c(V)$  and  $\chi_{|\bar{l}_P|}(V)$  for SO(6).

$L_t$	$\beta_c(V \rightarrow \infty)$	$L_s$ range	$\bar{\chi}_{\text{dof}}^2$
2	15.205(3)	$L_s \geq 8$	0.62
3	17.854(3)	$L_s \geq 12$	0.83
4	21.399(6)	$L_s \geq 28$	6.42
5	25.613(11)	$L_s \geq 28$	2.25
6	29.872(21)	$L_s \geq 42$	2.79
7	34.113(32)	$L_s \geq 44$	4.44

**Table 8.** Infinite volume limit of  $\beta_c$ , range of volumes used and  $\bar{\chi}_{\text{dof}}^2$  of extrapolation, for SO(6).

$L_s^2 L_t$	$\beta_c$	$\chi_{ \overline{l_P} }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \overline{l_P} }$
$8^2 2$	20.963(3)	3.347(3)	$32^2 5$	36.909(20)	23.75(21)
$10^2 2$	20.960(3)	5.381(10)	$40^2 5$	36.885(19)	35.40(43)
$12^2 2$	20.953(3)	7.916(14)	$48^2 5$	36.895(24)	50.05(80)
$12^2 3$	25.148(29)	4.00(3)	$56^2 5$	36.930(22)	66.59(110)
$16^2 3$	25.022(14)	6.60(4)	$64^2 5$	36.909(19)	83.37(151)
$20^2 3$	24.982(13)	9.96(8)	$52^2 6$	43.088(25)	62.57(104)
$24^2 3$	24.988(5)	14.27(7)	$56^2 6$	43.089(10)	72.92(93)
$28^2 3$	25.011(7)	19.40(14)	$60^2 6$	43.164(14)	83.65(127)
$24^2 4$	30.721(24)	12.65(12)	$64^2 6$	43.129(17)	94.02(115)
$32^2 4$	30.714(21)	21.04(30)			
$40^2 4$	30.721(7)	31.62(27)			
$48^2 4$	30.727(6)	44.05(43)			
$56^2 4$	30.726(8)	58.67(67)			

**Table 9.**  $\beta_c(V)$  and  $\chi_{|\overline{l_P}|}(V)$  for SO(7).

$L_t$	$\beta_c(V \rightarrow \infty)$	$L_s$ range	$\bar{\chi}_{\text{dof}}^2$
2	20.947(6)	$L_s \geq 8$	0.82
3	24.992(11)	$L_s \geq 12$	5.46
4	30.729(9)	$L_s \geq 24$	0.14
5	36.913(20)	$L_s \geq 32$	0.81
6	43.311(62)	$L_s \geq 52$	5.00

**Table 10.** Infinite volume limit of  $\beta_c$ , range of volumes used and  $\bar{\chi}_{\text{dof}}^2$  of extrapolation, for SO(7).

$L_s^2 L_t$	$\beta_c$	$\chi_{ \overline{l_P} }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \overline{l_P} }$
$8^2 2$	27.583(4)	3.241(4)	$32^2 5$	50.073(23)	25.70(27)
$10^2 2$	27.594(3)	5.279(6)	$40^2 5$	50.163(21)	40.21(40)
$12^2 2$	27.605(3)	7.851(7)	$48^2 5$	50.178(37)	58.72(72)
$16^2 3$	33.488(22)	6.29(6)	$56^2 5$	50.247(22)	81.93(90)
$18^2 3$	33.503(17)	8.05(7)	$42^2 6$	58.560(18)	46.27(34)
$20^2 3$	33.468(13)	9.68(5)	$48^2 6$	58.742(24)	63.38(78)
$24^2 3$	33.521(8)	14.54(7)	$54^2 6$	58.703(19)	79.59(60)
$24^2 4$	41.573(14)	13.20(6)	$60^2 6$	58.758(12)	101.15(99)
$28^2 4$	41.600(18)	17.88(16)			
$32^2 4$	41.631(14)	23.71(17)			
$40^2 4$	41.686(14)	37.98(28)			

**Table 11.**  $\beta_c(V)$  and  $\chi_{|\overline{l_P}|}(V)$  for SO(8).



$L_t$	$\beta_c(V \rightarrow \infty)$	$L_s$ range	$\bar{\chi}_{\text{dof}}^2$
2	27.622(5)	$L_s \geq 8$	0.66
3	33.547(21)	$L_s \geq 16$	4.06
4	41.769(32)	$L_s \geq 24$	0.22
5	50.319(32)	$L_s \geq 32$	0.46
6	58.935(29)	$L_s \geq 42$	6.65

**Table 12.** Infinite volume limit of  $\beta_c$ , range of volumes used and  $\bar{\chi}_{\text{dof}}^2$  of extrapolation, for SO(8).

$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{l}_P }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{l}_P }$
16 <sup>2</sup> 3	43.443(9)	6.79(2)	20 <sup>2</sup> 4	54.449(28)	10.96(10)
18 <sup>2</sup> 3	43.425(6)	8.64(2)	24 <sup>2</sup> 4	54.375(28)	15.93(14)
20 <sup>2</sup> 3	43.427(7)	10.80(4)	28 <sup>2</sup> 4	54.393(19)	22.01(16)
22 <sup>2</sup> 3	43.431(7)	13.26(5)	32 <sup>2</sup> 4	54.505(17)	30.22(18)
24 <sup>2</sup> 3	43.460(7)	16.16(5)	40 <sup>2</sup> 4	54.464(14)	48.16(26)
26 <sup>2</sup> 3	43.457(6)	19.17(6)	48 <sup>2</sup> 4	54.434(12)	70.28(55)
30 <sup>2</sup> 3	43.424(7)	25.59(9)	24 <sup>2</sup> 5	65.634(38)	16.87(9)
36 <sup>2</sup> 3	43.433(12)	37.59(26)	28 <sup>2</sup> 5	65.654(46)	23.34(16)
42 <sup>2</sup> 3	43.399(8)	51.23(30)	32 <sup>2</sup> 5	65.661(38)	31.17(22)
			36 <sup>2</sup> 5	65.674(15)	40.16(22)
			40 <sup>2</sup> 5	65.648(20)	50.10(36)
			48 <sup>2</sup> 5	65.760(17)	75.33(53)

**Table 13.**  $\beta_c(V)$  and  $\chi_{|\bar{l}_P|}(V)$  for SO(9).

$L_t$	$\beta_c(V \rightarrow \infty)$	$L_s$ range	$\bar{\chi}_{\text{dof}}^2$
3	43.450(7)	$L_s \geq 16$	6.10
4	54.457(13)	$L_s \geq 20$	6.78
5	65.678(32)	$L_s \geq 24$	0.47

**Table 14.** Infinite volume limit of  $\beta_c$ , range of volumes used and  $\bar{\chi}_{\text{dof}}^2$  of extrapolation, for SO(9).

$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{l}_P }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{l}_P }$
6 <sup>2</sup> 2	63.57(1)	1.593(1)	12 <sup>2</sup> 4	101.81(5)	3.995(25)
7 <sup>2</sup> 2	63.59(1)	2.248(1)	16 <sup>2</sup> 4	102.11(8)	7.619(84)
8 <sup>2</sup> 2	63.58(1)	3.014(3)	20 <sup>2</sup> 4	102.19(4)	12.739(71)
8 <sup>2</sup> 3	80.72(1)	1.620(3)	24 <sup>2</sup> 4	102.28(4)	19.393(88)
10 <sup>2</sup> 3	80.83(1)	2.640(8)	16 <sup>2</sup> 5	123.03(10)	7.709(60)
12 <sup>2</sup> 3	80.96(1)	4.043(8)	20 <sup>2</sup> 5	123.26(8)	12.988(104)
14 <sup>2</sup> 3	81.05(1)	5.800(16)	24 <sup>2</sup> 5	123.52(4)	19.769(137)
16 <sup>2</sup> 3	81.12(1)	7.929(17)	28 <sup>2</sup> 5	123.62(6)	27.835(284)

**Table 15.**  $\beta_c(V)$  and  $\chi_{|\bar{l}_P|}(V)$  for SO(12).

$L_t$	$\beta_c(V \rightarrow \infty)$	$L_s$ range	$\bar{\chi}_{\text{dof}}^2$
2	63.610(14)	$L_s \geq 6$	3.35
3	81.299(17)	$L_s \geq 8$	0.70
4	102.424(45)	$L_s \geq 12$	0.16
5	124.011(15)	$L_s \geq 16$	0.34

**Table 16.** Infinite volume limit of  $\beta_c$ , range of volumes used and  $\bar{\chi}_{\text{dof}}^2$  of extrapolation, for SO(12).

$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{L}_P }$	$L_s^2 L_t$	$\beta_c$	$\chi_{ \bar{L}_P }$
$4^2 2$	114.85(2)	0.595(1)	$6^2 4$	192.07(26)	1.016(7)
$5^2 2$	114.82(2)	0.998(2)	$8^2 4$	189.04(32)	1.804(25)
$6^2 2$	114.84(2)	1.512(3)	$10^2 4$	188.91(17)	3.011(42)
$6^2 3$	149.17(2)	0.940(2)	$12^2 4$	189.13(11)	4.617(47)
$8^2 3$	149.32(2)	1.832(3)	$14^2 4$	189.33(11)	6.725(74)
$10^2 3$	149.58(3)	3.141(4)	$16^2 4$	189.55(6)	9.221(40)
$12^2 3$	149.76(2)	4.839(10)	$8^2 5$	230.62(44)	1.972(18)
$14^2 3$	149.89(3)	6.897(20)	$10^2 5$	229.42(56)	3.013(44)
			$12^2 5$	228.60(5)	4.591(14)
			$14^2 5$	228.77(6)	6.572(18)
			$16^2 5$	229.01(7)	9.036(31)
			$20^2 5$	229.51(12)	15.429(88)

**Table 17.**  $\beta_c(V)$  and  $\chi_{|\bar{L}_P|}(V)$  for SO(16).

$L_t$	$\beta_c(V \rightarrow \infty)$	$L_s$ range	$\bar{\chi}_{\text{dof}}^2$
2	114.824(35)	$L_s \geq 4$	0.66
3	150.128(33)	$L_s \geq 8$	0.83
4	189.975(14)	$L_s \geq 12$	0.19
5	230.096(212)	$L_s \geq 14$	1.06

**Table 18.** Infinite volume limit of  $\beta_c$ , range of volumes used and  $\bar{\chi}_{\text{dof}}^2$  of extrapolation, for SO(16).

$N$	$L_s^2 L_t$	Bulk transition	Weak coupling region
4	$20^2 24$	$\beta \in [9.1, 10.2]$	$L_t \geq 6$
5	$12^2 24$	$\beta \in [13.5, 15.4]$	$L_t \geq 5$
6	$12^2 24$	$\beta \in [18.0, 21.3]$	$L_t \geq 5$
7	$8^2 24$	$\beta \in [23.5, 28.0]$	$L_t \geq 4$
8	$8^2 24$	$\beta \in [31, 35]$	$L_t \geq 4$
9	$4^2 24$	$\beta \in [37, 42]$	$L_t \geq 3$
12	$4^2 24$	$\beta \in [65, 73]$	$L_t \geq 3$
16	$2^2 24$	$\beta \in [111, 124]$	$L_t \geq 3$

**Table 19.** Location of the bulk transition on volumes  $L_s^2 L_t$ , and consequent range of  $L_t$  for which  $\beta_c$  is on weak coupling side.

$L_t$	$\beta_c(V \rightarrow \infty)$	$a\sqrt{\sigma}$	$T_c/\sqrt{\sigma}$	$\beta_I(V \rightarrow \infty)$	$T_c/(g^2 N)$	Coupling
2	6.4891(3)	0.6208(3)	0.8054(3)	3.7234(5)	0.05818(1)	Strong
3	7.573(3)	0.3970(7)	0.8397(14)	5.169(3)	0.05384(3)	
4	8.573(10)	0.2936(10)	0.8515(31)	6.290(11)	0.04914(8)	
6	10.781(16)	0.2146(17)	0.7766(60)	8.590(16)	0.04474(8)	Weak
7	11.980(24)	0.1845(11)	0.7743(44)	9.816(24)	0.04382(11)	
8	13.504(34)	0.1609(12)	0.7770(60)	11.364(34)	0.04439(13)	
10	16.322(80)	0.1206(13)	0.7718(81)	14.213(81)	0.04442(25)	
12	19.090(99)	0.1083(15)	0.7692(106)	17.099(76)	0.04453(20)	

**Table 20.** SO(4) critical temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , and in units of the (mean field improved) 't Hooft coupling,  $T_c/(g^2 N)$ , evaluated at  $\beta_c(V \rightarrow \infty)$ .

$L_t$	$\beta_c(V \rightarrow \infty)$	$a\sqrt{\sigma}$	$T_c/\sqrt{\sigma}$	$\beta_I(V \rightarrow \infty)$	$T_c/(g^2 N)$	Coupling
2	10.368(3)	0.6447(14)	0.7756(17)	5.811(5)	0.05811(5)	Strong
3	12.021(16)	0.4248(19)	0.7847(36)	8.044(19)	0.05363(13)	
4	13.944(36)	0.3214(15)	0.7778(36)	10.162(39)	0.05081(19)	
5	16.180(13)	0.2593(15)	0.7714(44)	12.497(13)	0.04999(5)	Weak
6	18.603(55)	0.2190(12)	0.7611(43)	14.985(56)	0.04995(19)	
7	21.192(52)	0.1877(11)	0.7613(45)	17.620(53)	0.05034(15)	
8	23.938(63)	0.1624(9)	0.7698(41)	10.401(64)	0.05100(16)	
10	29.500(157)	0.1281(10)	0.7801(62)	26.010(158)	0.05202(32)	

**Table 21.** SO(5) critical temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , and in units of the (mean field improved) 't Hooft coupling,  $T_c/(g^2 N)$ , evaluated at  $\beta_c(V \rightarrow \infty)$ .

$L_t$	$\beta_c(V \rightarrow \infty)$	$a\sqrt{\sigma}$	$T_c/\sqrt{\sigma}$	$\beta_I(V \rightarrow \infty)$	$T_c/(g^2N)$	Coupling
2	15.205(3)	0.6535(13)	0.7651(15)	8.460(4)	0.05875(3)	Strong
3	17.854(3)	0.4201(7)	0.7935(13)	11.964(4)	0.05539(2)	
4	21.399(6)	0.3106(10)	0.8050(27)	15.784(6)	0.05480(2)	
5	25.613(11)	0.2501(7)	0.7996(22)	20.147(11)	0.05596(3)	Weak
6	29.872(21)	0.2077(3)	0.8024(14)	24.497(21)	0.05670(5)	
7	34.113(32)	0.1774(4)	0.8053(19)	28.798(32)	0.05714(6)	

**Table 22.** SO(6) critical temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , and in units of the (mean field improved) 't Hooft coupling,  $T_c/(g^2N)$ , evaluated at  $\beta_c(V \rightarrow \infty)$ .

$L_t$	$\beta_c(V \rightarrow \infty)$	$a\sqrt{\sigma}$	$T_c/\sqrt{\sigma}$	$\beta_I(V \rightarrow \infty)$	$T_c/(g^2N)$	Coupling
2	20.947(6)	0.6571(8)	0.7610(10)	11.597(9)	0.05917(4)	Strong
3	24.992(11)	0.4181(7)	0.7972(13)	16.816(12)	0.05720(4)	
4	30.729(9)	0.3104(5)	0.8053(12)	22.924(10)	0.05848(2)	
5	36.913(20)	0.2455(7)	0.8147(22)	29.302(20)	0.05980(4)	Weak
6	43.311(62)	0.2023(6)	0.8239(26)	35.821(63)	0.06092(11)	

**Table 23.** SO(7) critical temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , and in units of the (mean field improved) 't Hooft coupling,  $T_c/(g^2N)$ , evaluated at  $\beta_c(V \rightarrow \infty)$ .

$L_t$	$\beta_c(V \rightarrow \infty)$	$a\sqrt{\sigma}$	$T_c/\sqrt{\sigma}$	$\beta_I(V \rightarrow \infty)$	$T_c/(g^2N)$	Coupling
2	27.622(5)	0.6586(8)	0.7591(9)	15.266(8)	0.05963(3)	Strong
3	33.547(21)	0.4179(7)	0.7977(14)	22.715(23)	0.05915(6)	
4	41.769(32)	0.3051(11)	0.8193(28)	31.414(33)	0.06136(6)	
5	50.319(32)	0.2415(4)	0.8281(15)	40.210(32)	0.06283(5)	Weak
6	58.935(29)	0.2003(5)	0.8320(20)	48.975(29)	0.06377(4)	

**Table 24.** SO(8) critical temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , and in units of the (mean field improved) 't Hooft coupling,  $T_c/(g^2N)$ , evaluated at  $\beta_c(V \rightarrow \infty)$ .

$L_t$	$\beta_c(V \rightarrow \infty)$	$a\sqrt{\sigma}$	$T_c/\sqrt{\sigma}$	$\beta_I(V \rightarrow \infty)$	$T_c/(g^2N)$	Coupling
3	43.450(7)	0.4150(2)	0.8032(4)	29.586(8)	0.060877(16)	Weak
4	54.457(13)	0.3025(4)	0.8263(11)	41.190(13)	0.063564(21)	
5	65.678(32)	0.2395(3)	0.8351(12)	52.714(33)	0.065079(40)	

**Table 25.** SO(9) critical temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , and in units of the (mean field improved) 't Hooft coupling,  $T_c/(g^2N)$ , evaluated at  $\beta_c(V \rightarrow \infty)$ .

$L_t$	$\beta_c(V \rightarrow \infty)$	$a\sqrt{\sigma}$	$T_c/\sqrt{\sigma}$	$\beta_I(V \rightarrow \infty)$	$T_c/(g^2N)$	Coupling
2	63.610(14)	0.6600(24)	0.7576(27)	35.213(22)	0.06113(4)	Strong
3	81.299(17)	0.4070(6)	0.8191(12)	56.114(18)	0.064947(21)	
4	102.424(45)	0.2977(6)	0.8399(18)	78.245(46)	0.06792(4)	Weak
5	124.011(15)	0.2354(12)	0.8497(43)	100.359(180)	0.06969(12)	

**Table 26.** SO(12) critical temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , and in units of the (mean field improved) 't Hooft coupling,  $T_c/(g^2N)$ , evaluated at  $\beta_c(V \rightarrow \infty)$ .

$L_t$	$\beta_c(V \rightarrow \infty)$	$a\sqrt{\sigma}$	$T_c/\sqrt{\sigma}$	$\beta_I(V \rightarrow \infty)$	$T_c/(g^2N)$	Coupling
2	63.610(14)	0.6438(25)	0.7766(31)	64.219(47)	0.06271(5)	Strong
3	81.299(17)	0.4003(12)	0.8328(24)	104.617(35)	0.068110(23)	
4	102.424(45)	0.2925(9)	0.8547(26)	146.288(207)	0.07143(10)	Weak
5	124.011(15)	0.2320(7)	0.8622(28)	187.090(218)	0.07308(9)	

**Table 27.** SO(16) critical temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , and in units of the (mean field improved) 't Hooft coupling,  $T_c/(g^2N)$ , evaluated at  $\beta_c(V \rightarrow \infty)$ .

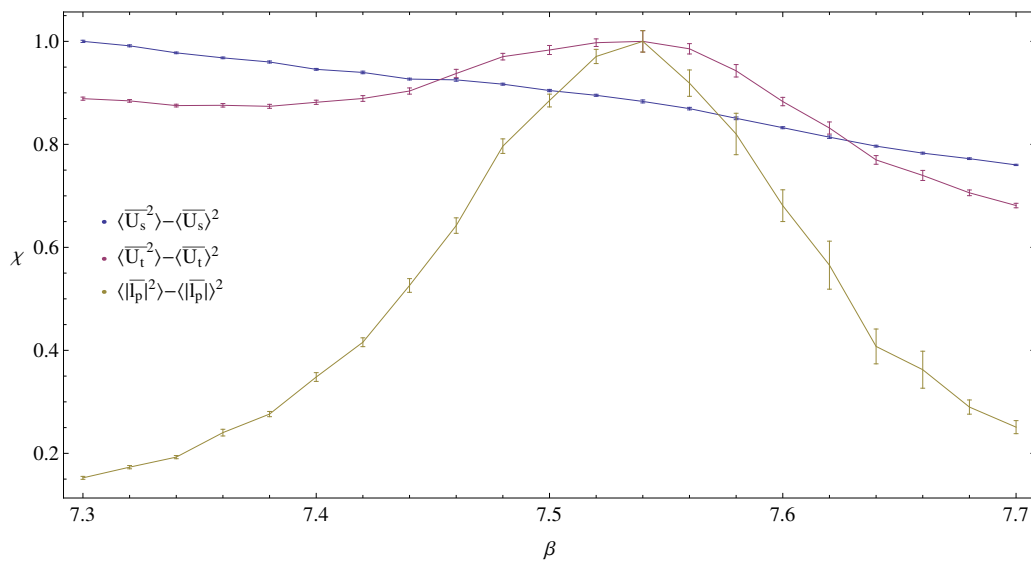
$N$	$T_c/\sqrt{\sigma}$	$\bar{\chi}_{\text{dof}}^2$	$T_c/(g^2N)$	$\bar{\chi}_{\text{dof}}^2$	$T_c/M_{0+}$
3*	0.7072(80)		0.03236(45)		0.2373(33)
4	0.7702(88)	0.12	0.04567(43)	2.17	0.2293(30)
5	0.7963(114)	0.00	0.05544(93)	0.05	0.2244(33)
6	0.8105(42)	0.16	0.05996(19)	0.53	0.2232(14)
7	0.8351(38)	0.98	0.06478(18)	4.01	0.2234(12)
8	0.8418(39)	0.05	0.06809(16)	0.00	0.2224(15)
9	0.8515(15)	0.30	0.07043(7)	0.10	
12	0.8642(38)	0.02	0.07552(14)	0.63	0.2217(18)
16	0.8780(38)	0.15	0.07947(17)	0.85	0.2220(22)

**Table 28.** SO( $N$ ) continuum limit of the deconfining temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , of the 't Hooft coupling,  $T_c/(g^2N)$ , and of the lightest scalar glueball mass,  $T_c/M_{0+}$ , with corresponding  $\bar{\chi}_{\text{dof}}^2$  of the fits. Note that we infer the SO(3) value (\*) from the SU(2) value.

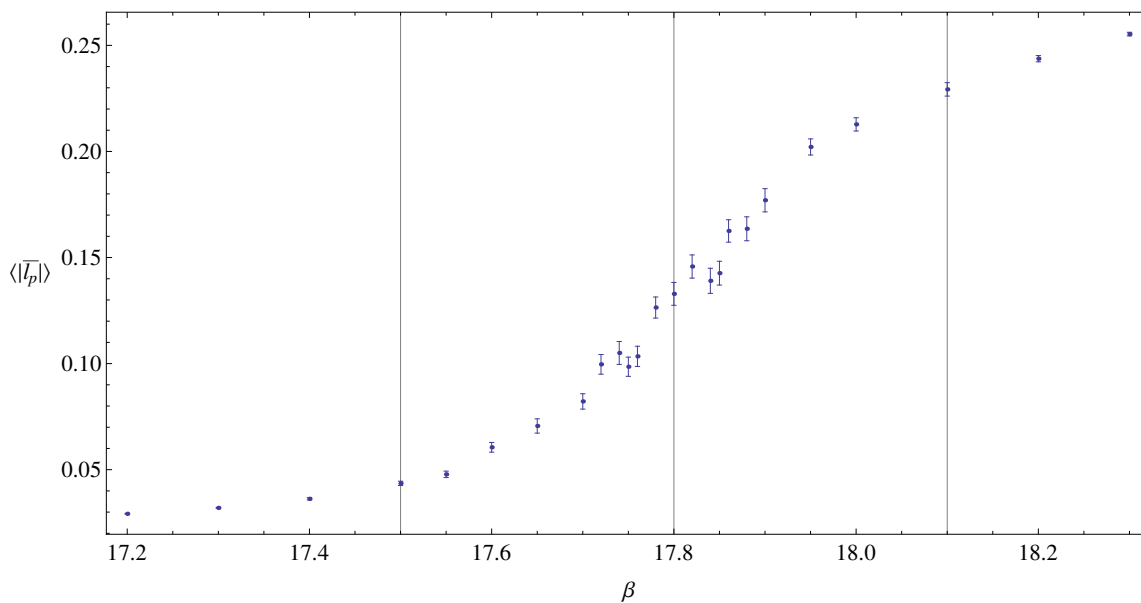
$N$	$T_c/\sqrt{\sigma}$		$T_c/g^2N$		$T_c/M_{0+}$	
	SO(2N)	SU( $N$ )	SO(2N)	SU( $N$ )	SO(2N)	SU( $N$ )
2	0.7702(88)	1.1238(88)	0.0913(9)	0.1998(34)	0.2293(30)	0.2373(19)
3	0.8105(42)	0.9994(40)	0.1199(4)	0.1904(12)	0.2232(14)	0.2288(10)
4	0.8418(39)	0.9572(39)	0.1362(3)	0.1884(12)	0.2224(15)	0.2259(11)
5		0.9380(19)		0.1874(10)		0.2233(7)
6	0.8642(38)	0.9300(48)	0.1510(3)	0.1873(8)	0.2217(18)	0.2232(12)
8	0.8780(38)	0.9144(41)	0.1589(3)	0.1849(10)	0.2220(22)	0.2207(11)

**Table 29.** SO(2N) and SU( $N$ ) [1] continuum limit of the deconfining temperature in units of the string tension,  $T_c/\sqrt{\sigma}$ , the 't Hooft coupling,  $T_c/(g^2N)$ , and the lightest scalar glueball mass,  $T_c/M_{0+}$ .

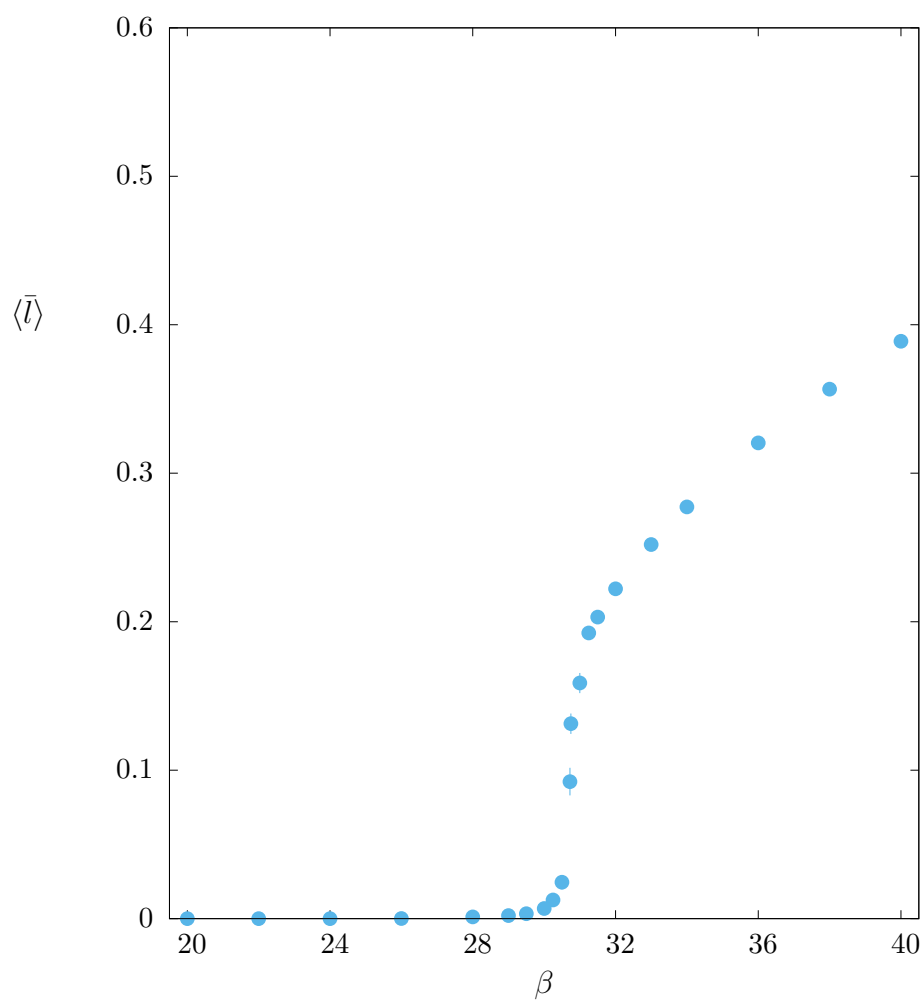
## B Figures



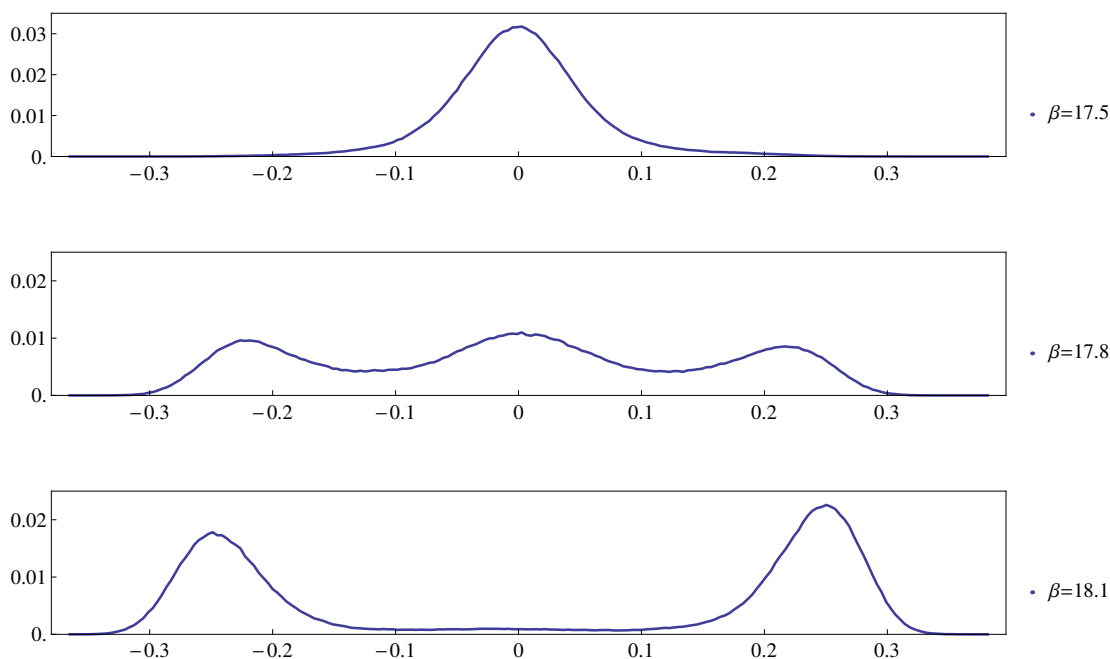
**Figure 1.** Susceptibility plots (renormalised) for the spatial plaquette  $\overline{U}_s$ , the temporal plaquette  $\overline{U}_t$ , and the modulus of the averaged Polyakov loop  $|\overline{l}_P|$  for SO(4) on a  $32^2 3$  lattice.



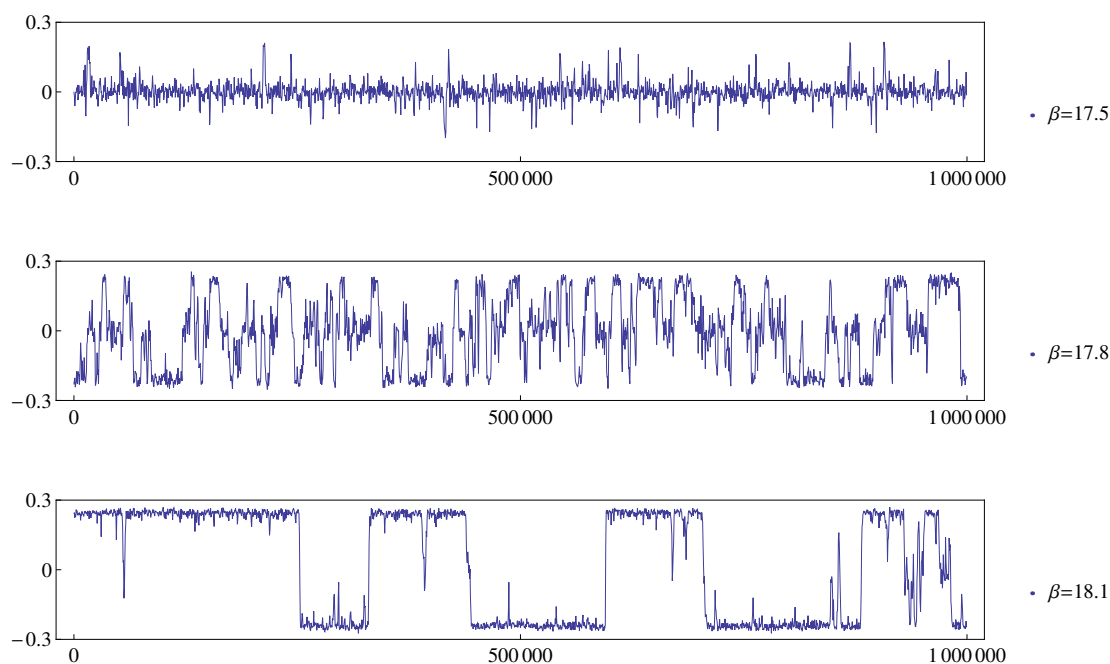
**Figure 2.** The Polyakov loop  $\langle |\overline{l}_P| \rangle$  for SO(6) on a  $20^2 3$  lattice. The vertical lines, spanning the transition, correspond to  $\beta = (\beta_-, \beta_0, \beta_+) = (17.5, 17.8, 18.1)$ .



**Figure 3.** Average value of Polyakov loop on a  $48^2 4$  lattice in  $SO(7)$  versus  $\beta$ .

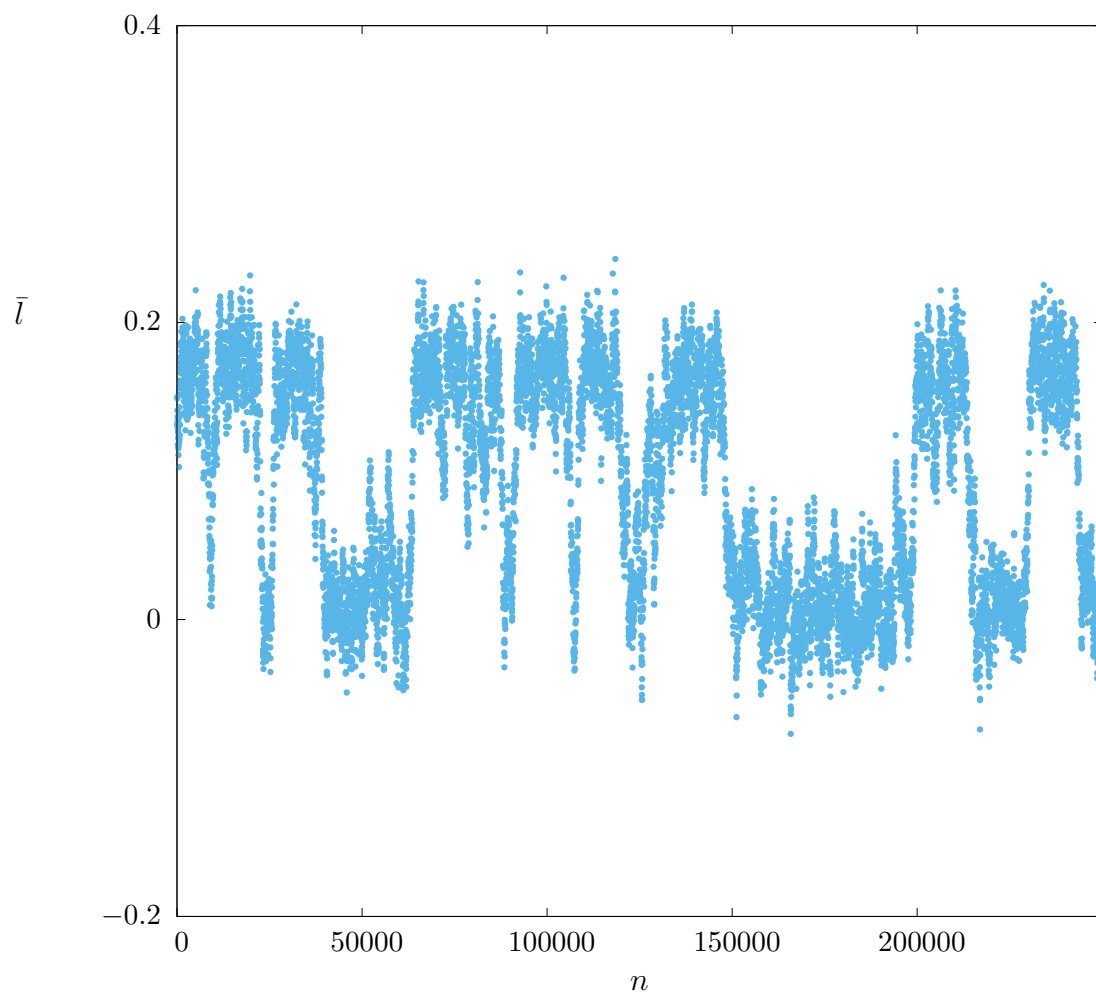


**Figure 4.** Polyakov loop  $\overline{l_P}$  histograms at  $\beta = \beta_-, \beta_0, \beta_+$  (see figure 2) for SO(6) on a  $20^{23}$  lattice.

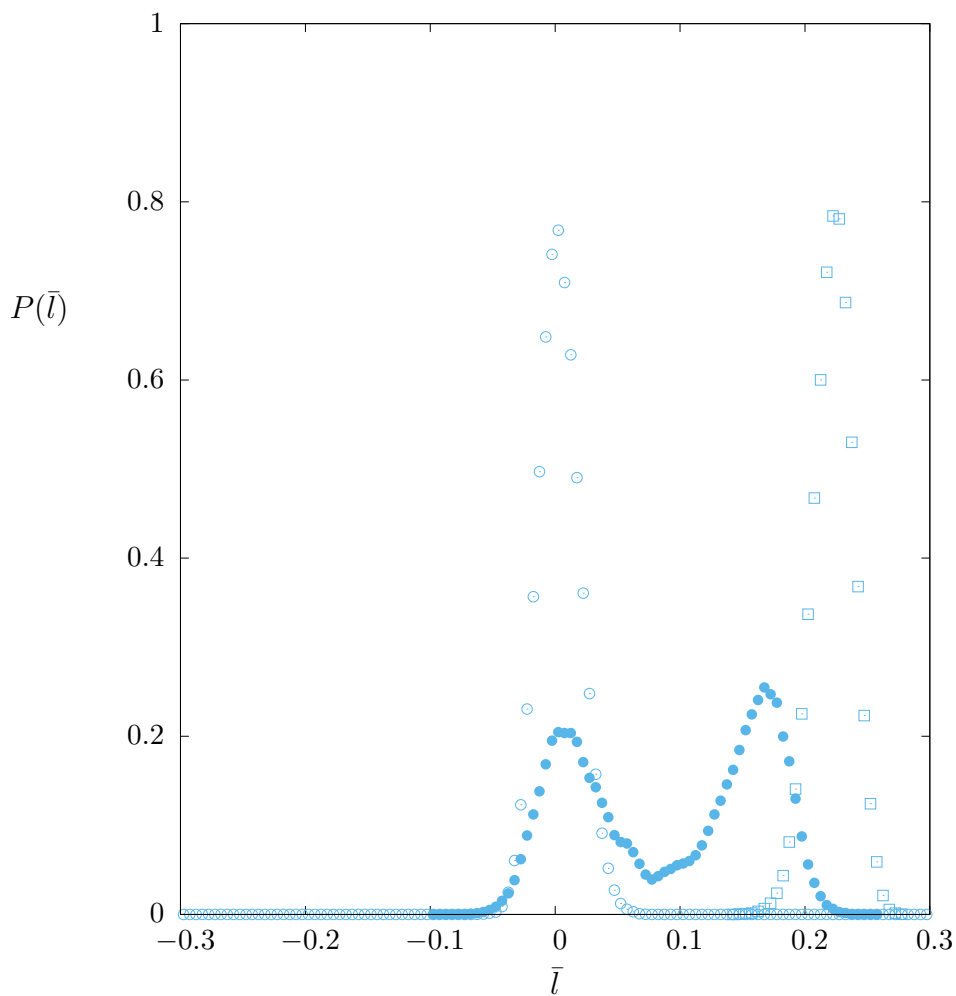


**Figure 5.** Polyakov loop  $\overline{l_P}$  history plots at  $\beta = \beta_-, \beta_0, \beta_+$  (see figure 2) for SO(6) on a  $20^{23}$  lattice from a run of  $10^6$  configurations.

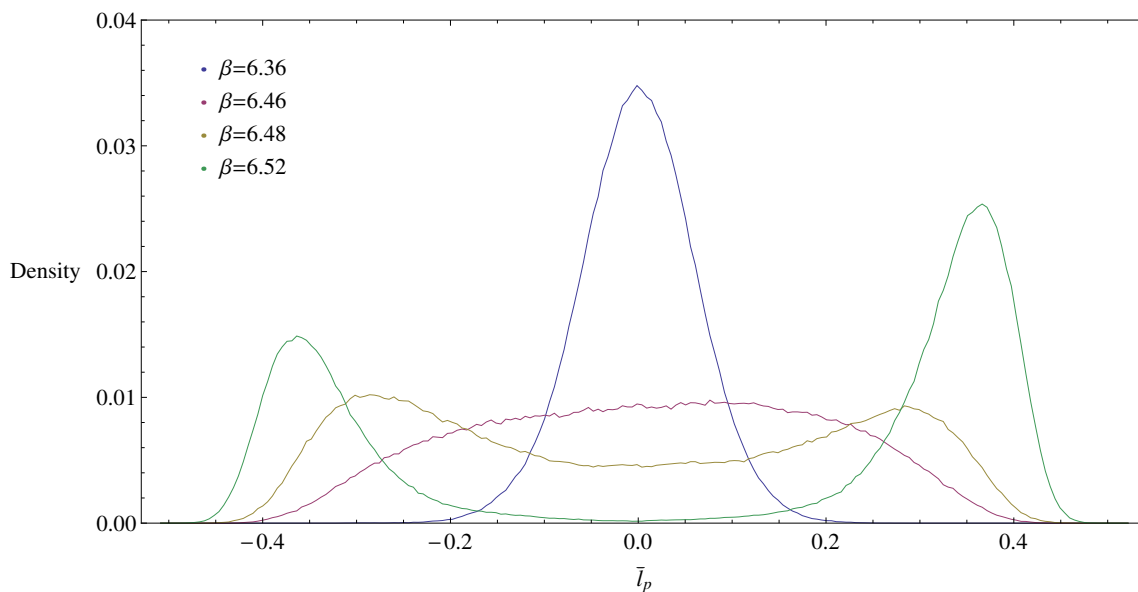




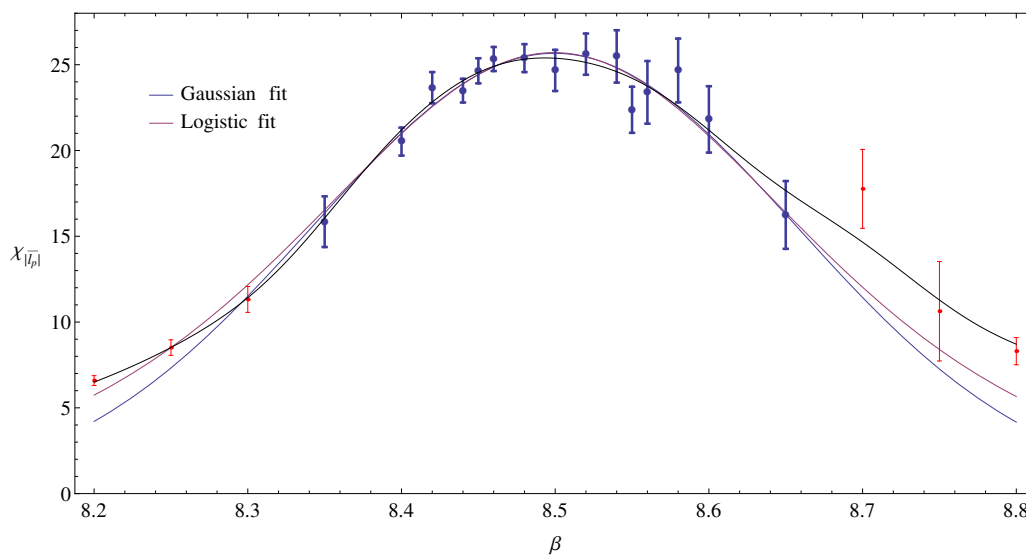
**Figure 6.** The value of the averaged Polyakov loop, taken every 25 sweeps, on a  $48^2 4$  lattice in  $SO(7)$  versus the number of sweeps  $n$ .



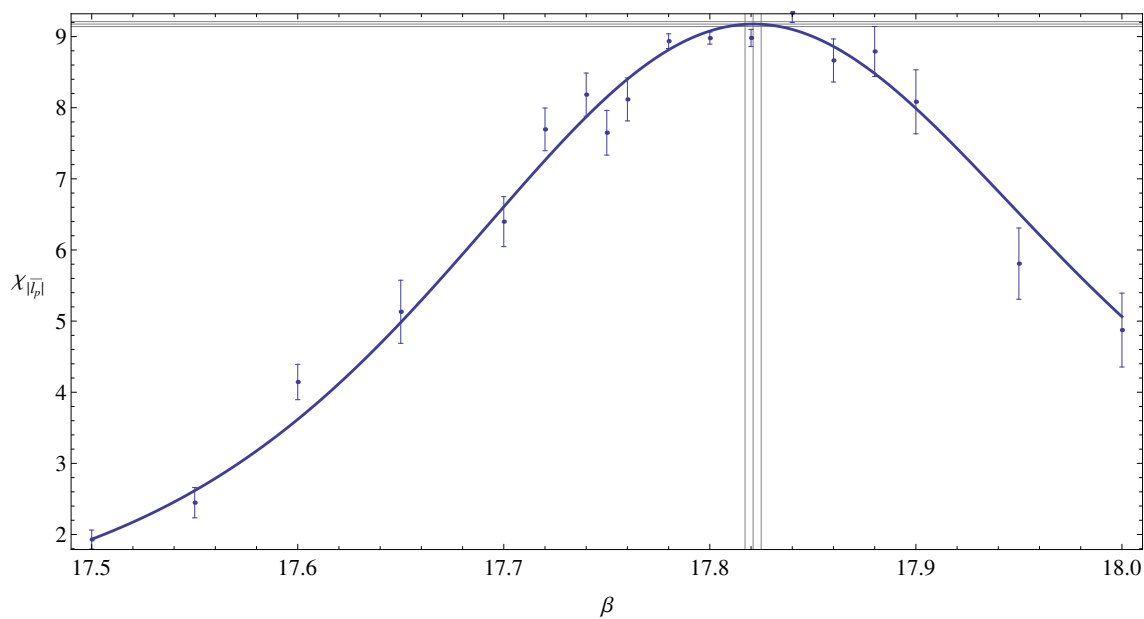
**Figure 7.** Probability (unnormalised) of the Polyakov loop averaged over the spatial volume of a  $48^2 \times 4$  lattice in  $SO(7)$  for  $T \simeq 0.96T_c$ ,  $\circ$ ,  $T \simeq T_c$ ,  $\bullet$ , and  $T \simeq 1.04T_c$ ,  $\square$ .



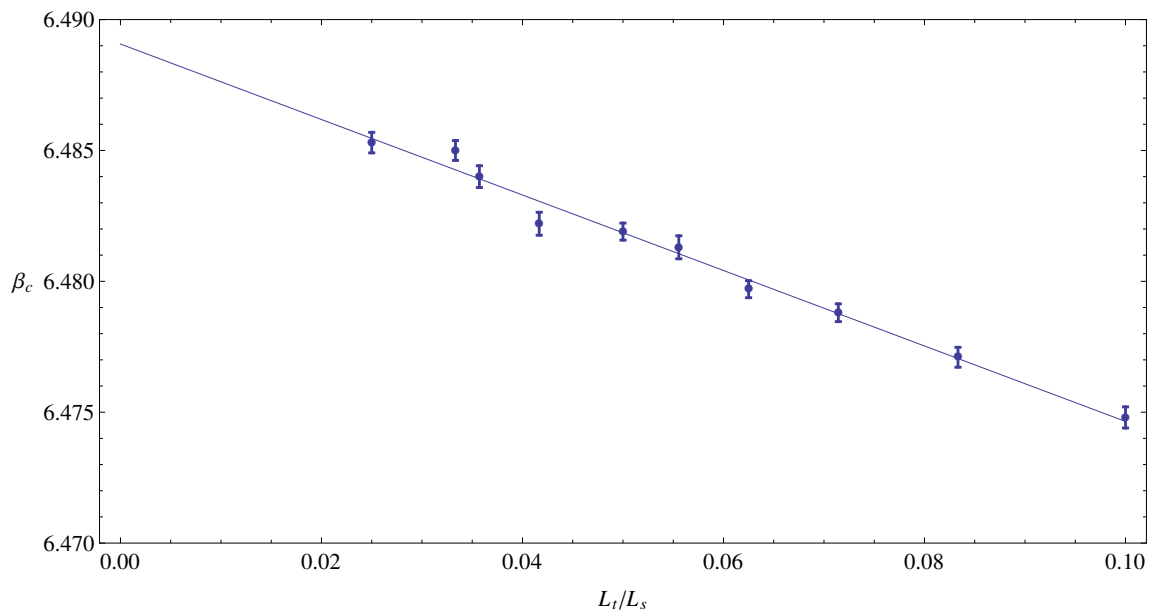
**Figure 8.** Polyakov loop  $\overline{l_P}$  histograms for SO(4) on a  $28^2$  lattice.



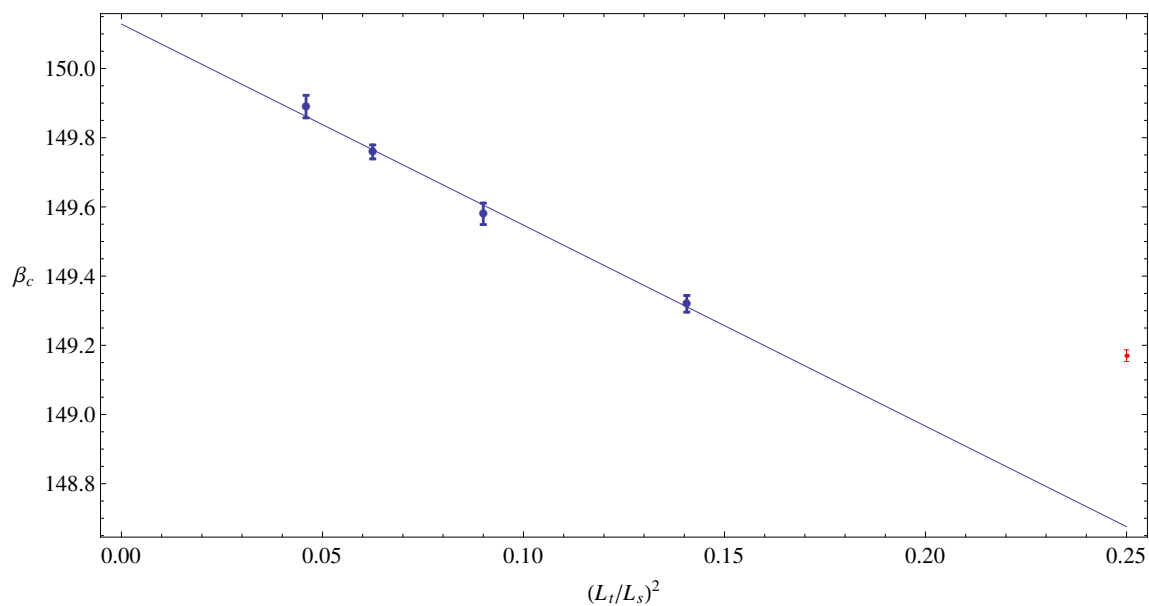
**Figure 9.** Fitting the Polyakov loop susceptibility in SO(4) on a  $40^2$  lattice. The points represent our calculations while the continuous black line represents reweighted values. Other lines are curve fits, using the data from the dark points rather than the light points to reduce the  $\bar{\chi}_{\text{dof}}^2$  of the fits.



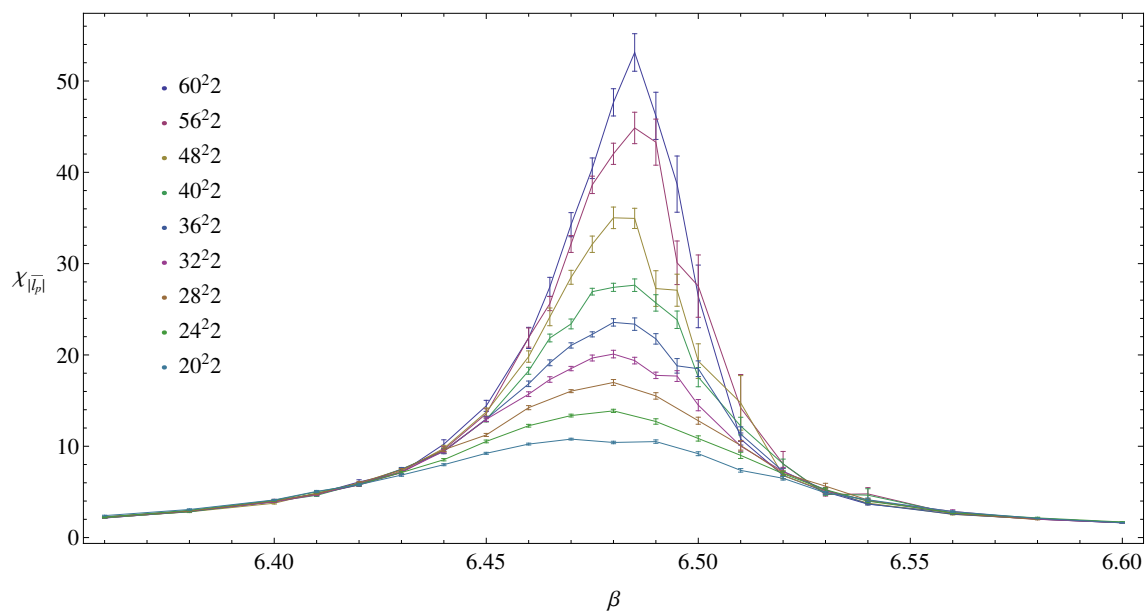
**Figure 10.** The reweighted susceptibility compared to our data, for a  $20^2 3$  lattice in  $SO(6)$ . The vertical and horizontal lines correspond to the values at the maximum of the susceptibility with its error:  $\beta_c = 17.821(4)$  and  $\chi_{|P|}(\beta_c) = 9.18(3)$ .



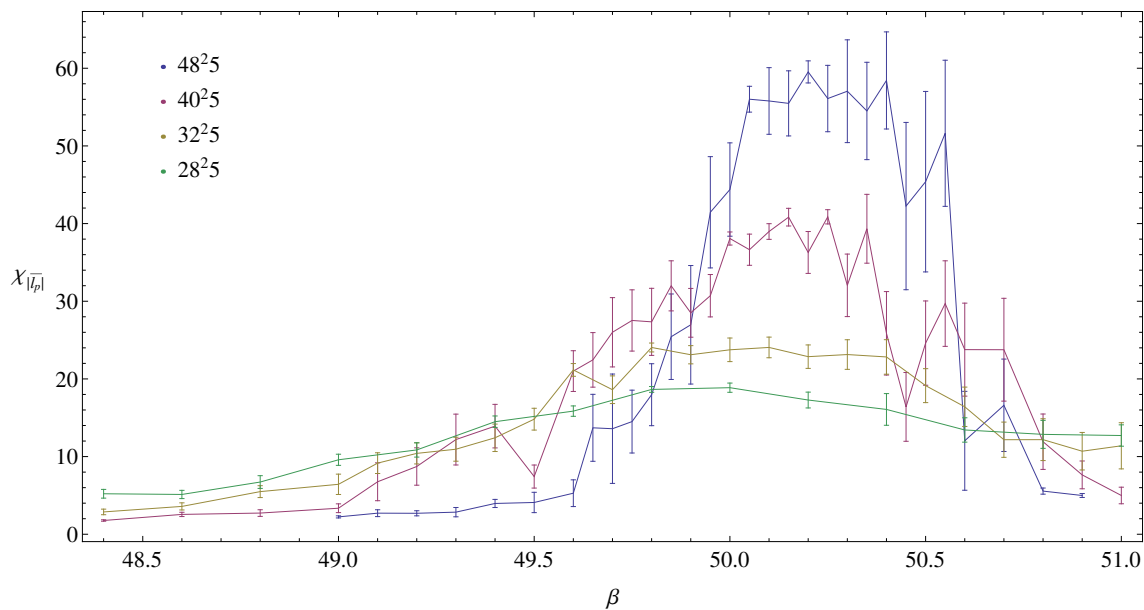
**Figure 11.** The infinite volume extrapolation for  $SO(4)$  with  $L_t = 2$ .



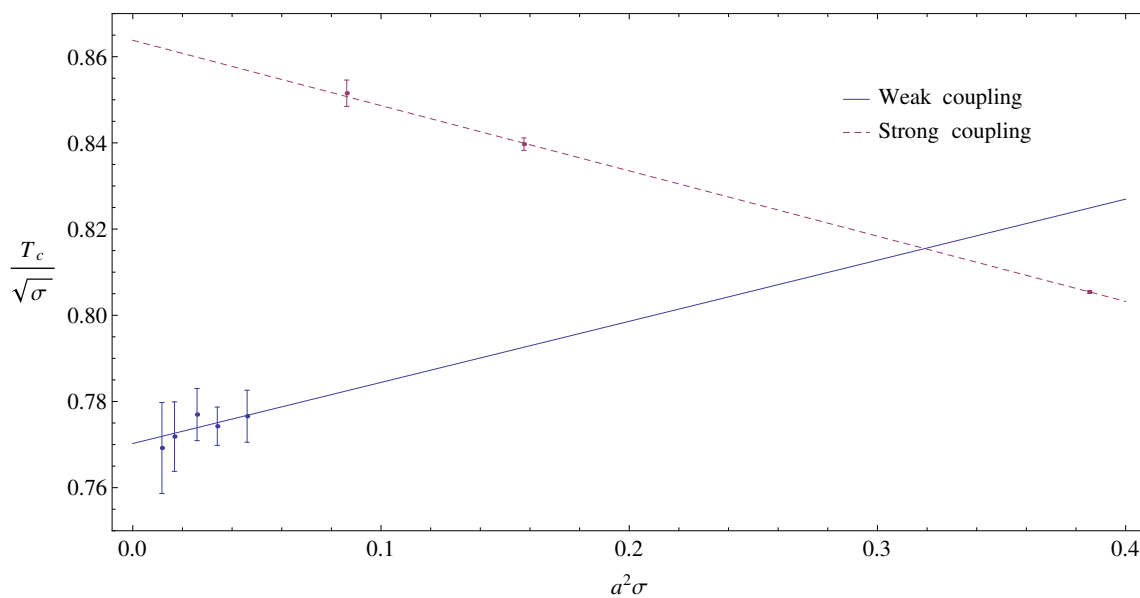
**Figure 12.** The infinite volume extrapolation (using dark points only) for SO(16) with  $L_t = 3$  and with a range of  $L_s$  values.



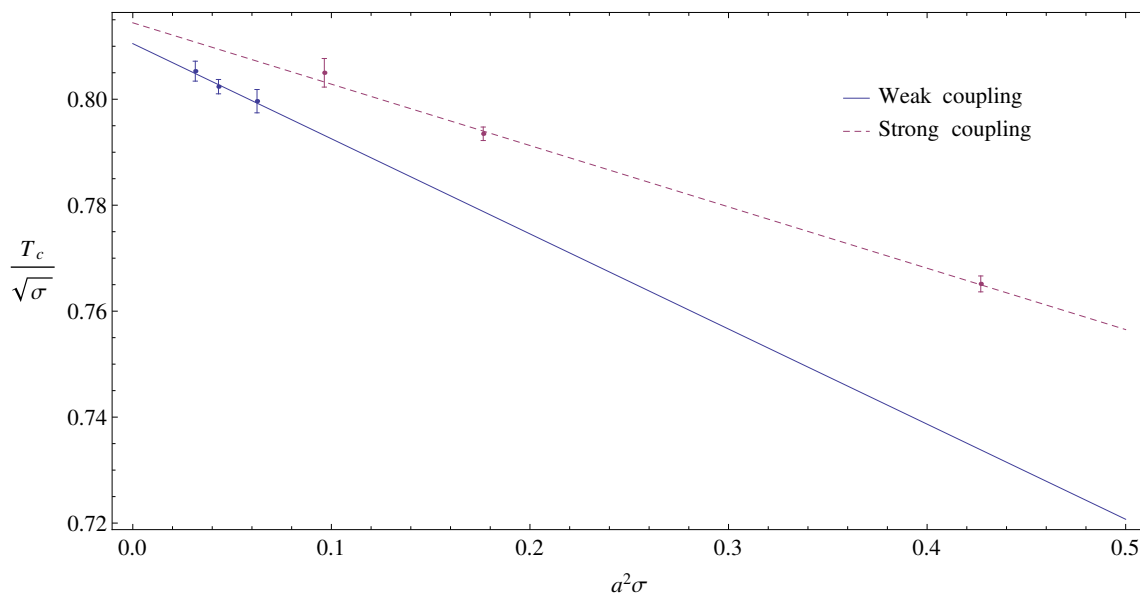
**Figure 13.** Susceptibility volume dependence for SO(4) and  $L_t = 2$ .



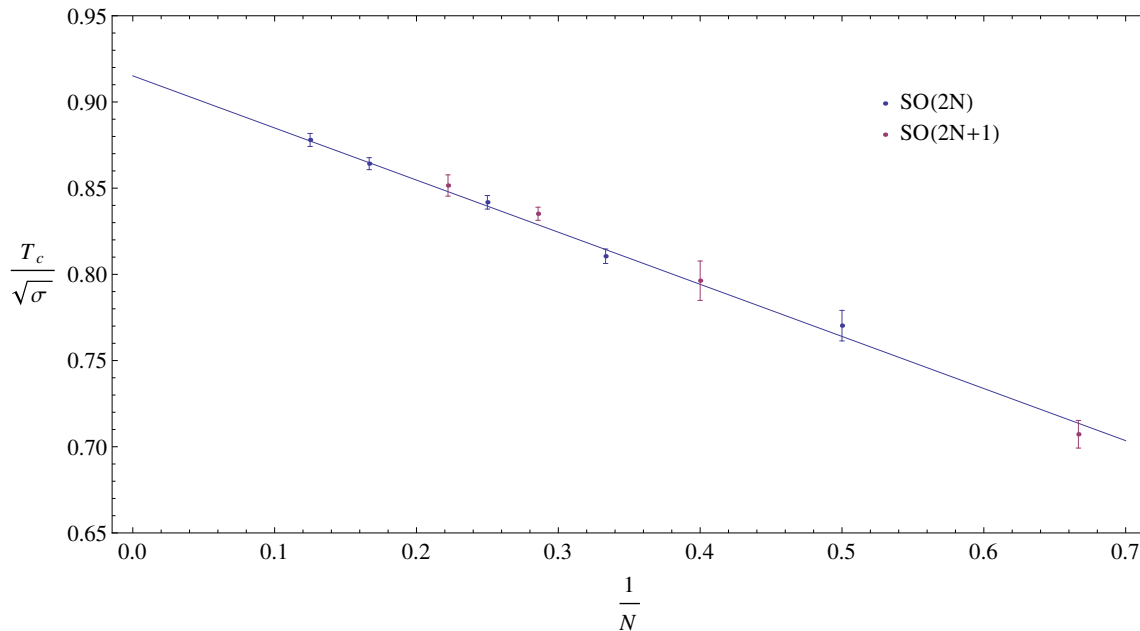
**Figure 14.** Susceptibility volume dependence for SO(8) and  $L_t = 5$ .



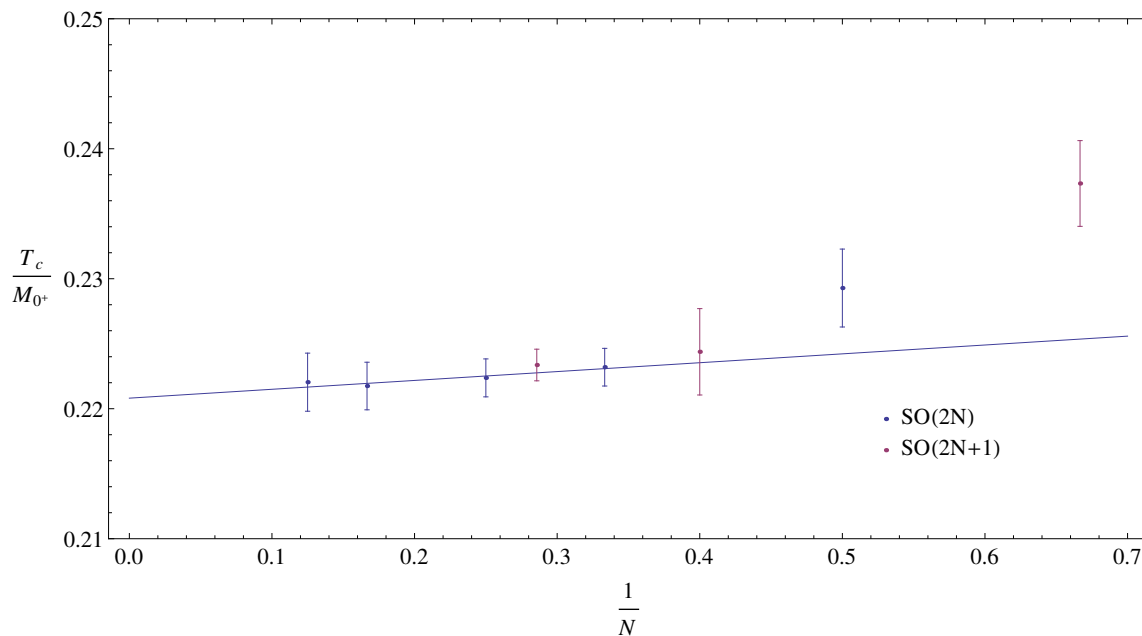
**Figure 15.** Continuum extrapolation of SO(4) deconfining temperature in units of the string tension.



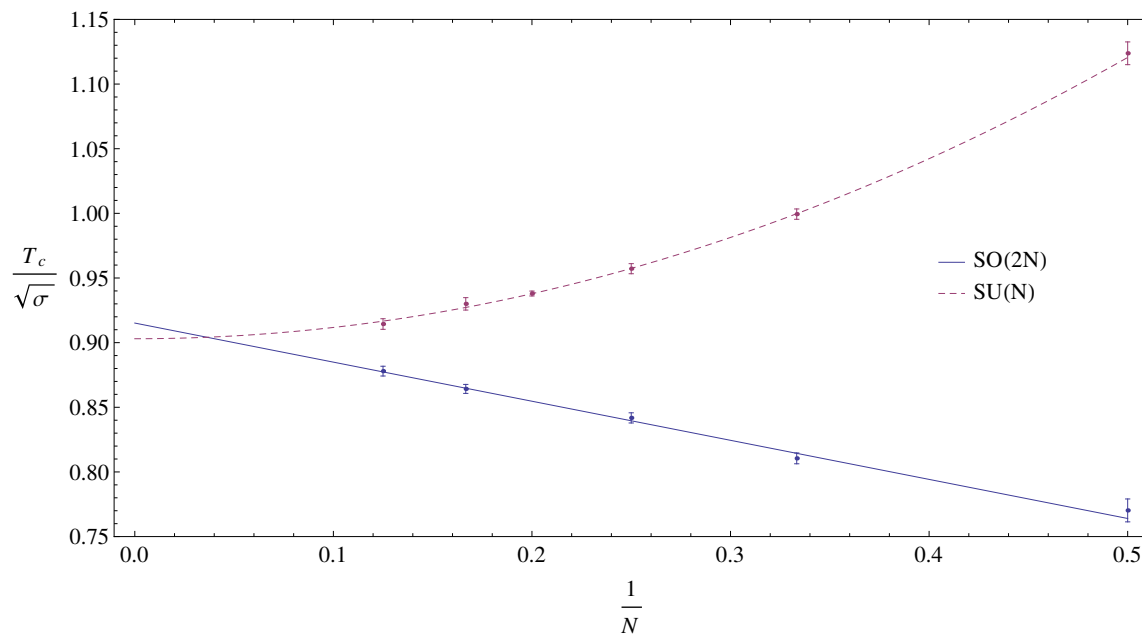
**Figure 16.** Continuum extrapolation of  $SO(6)$  deconfining temperature in units of the string tension.



**Figure 17.** Large- $N$  extrapolation of  $SO(2N)$  deconfining temperature in units of the string tension. Plotted points include both  $SO(2N)$  and  $SO(2N + 1)$ .

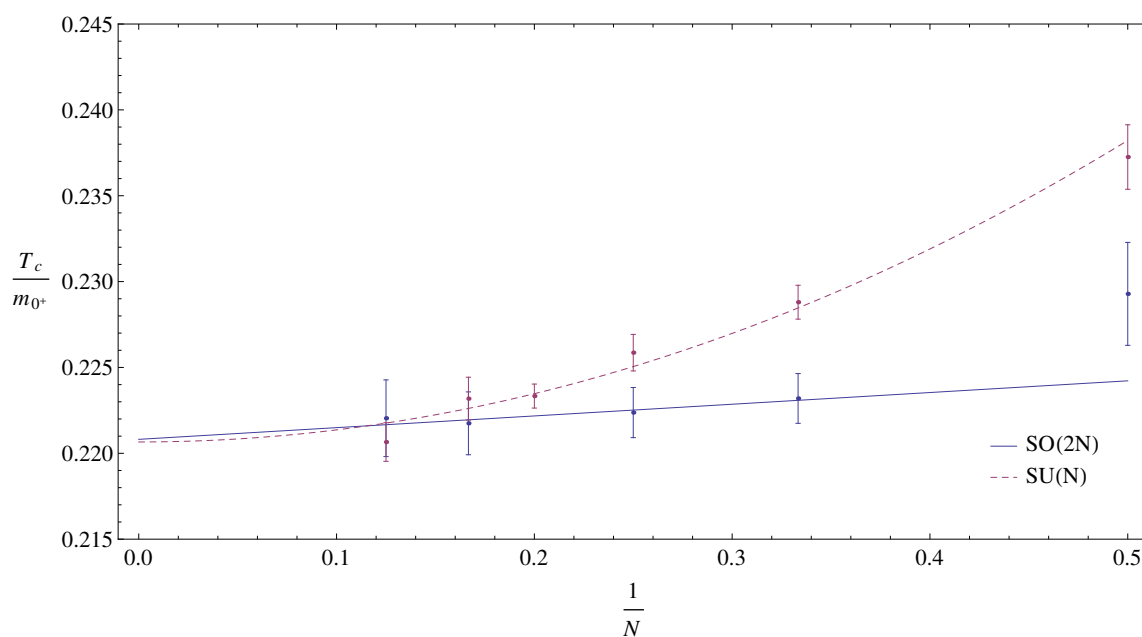


**Figure 18.** Large- $N$  extrapolation of  $SO(2N)$  deconfining temperature in units of the lightest scalar glueball mass. Plotted points include both  $SO(2N)$  and  $SO(2N + 1)$ .



**Figure 19.** Large- $N$  extrapolations of  $SO(2N)$  and  $SU(N)$  deconfining temperatures in units of the string tension.





**Figure 20.** Large- $N$  extrapolation of  $SO(2N)$  and  $SU(N)$  deconfining temperatures in units of the lightest scalar glueball mass.

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