

AdS₄ solutions of massive IIA from dyonic ISO(7) supergravity

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ABSTRACT: Explicit formulae are given for the consistent truncation of massive type IIA supergravity on the six-sphere to the SU(3)-invariant sector of $D = 4$ $\mathcal{N} = 8$ supergravity with dyonic ISO(7) gauging. These formulae are then used to construct AdS₄ solutions of massive type IIA via uplift on S^6 of the critical points of the $D = 4$ supergravity with at least SU(3) symmetry. We find a new $\mathcal{N} = 1$ solution with SU(3) symmetry, a new non-supersymmetric solution with SO(6) symmetry, and recover previously known solutions. We quantise the fluxes, calculate the gravitational free energies of the solutions and discuss the stability of the non-supersymmetric ones. Among these, a (previously known) G₂-invariant solution is found to be stable.

KEYWORDS: AdS-CFT Correspondence, Flux compactifications, Supergravity Models, Superstring Vacua

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1 Introduction

Supersymmetric solutions of string or M-theory containing four- or five-dimensional, for definiteness, anti-de-sitter (AdS) factors provide consistent backgrounds where the anti-de-Sitter/conformal field theory (AdS/CFT) correspondence [1–3] can be formulated precisely. Well known early examples include the maximally supersymmetric Freund-Rubin direct product solutions AdS₄ × S⁷ [4] and AdS₅ × S⁵ [5] of $D = 11$ and type IIB supergravity, respectively, where the spheres are equipped with their usual homogeneous round metrics with SO(8) and SO(6) symmetry. Less supersymmetric solutions in the Freund-Rubin class can be obtained by either quotienting S⁷ and S⁵, or by altogether replacing them with more general classes of manifolds. For instance, Sasaki-Einstein seven- and five-manifolds (of which S⁷ and S⁵ are special examples) generically admit two Killing spinors and thus give rise to $\mathcal{N} = 2$ AdS₄ and AdS₅ solutions. Besides the finite list of homogeneous Freund-Rubin solutions (see [6] for AdS₄ in $D = 11$), infinite families of inhomogeneous spaces, for example in the Sasaki-Einstein class [7, 8], give rise to infinite families of supersymmetric inhomogeneous Freund-Rubin solutions in $D = 11$ and IIB.

Warped product solutions, both supersymmetric and non-supersymmetric, involving AdS₄ and AdS₅ in $D = 11$ and type IIB, respectively, are also known. Smooth solutions of this type have been obtained by exploiting the consistent truncation of $D = 11$ on S⁷ [9, 10]

and type IIB on S^5 (see [11–13]) down to $\mathcal{N} = 8$ supergravity in $D = 4$ with $\text{SO}(8)$ [14] and $D = 5$ with $\text{SO}(6)$ [15] gauging. Any solution of these gauged supergravities uplifts on the corresponding sphere to $D = 11$ or type IIB. In particular, the (AdS) critical points of their scalar potentials respectively give rise to AdS_4 and AdS_5 solutions in $D = 11$ and IIB. Some examples include [16–18] in $D = 11$ and [19] in type IIB. Together with the warp factor, these solutions are typically supported by internal values of the supergravity forms. The metrics on the internal S^7 and S^5 are usually inhomogeneous and display isometry groups smaller than $\text{SO}(8)$ and $\text{SO}(6)$. A two-step variant of this uplifting method was introduced in [20], where a spherical truncation to some intermediate dimension, e.g. $D = 7$ [21, 22], followed by a further reduction on a suitable (usually hyperbolic) space was performed to obtain supersymmetric AdS_4 or AdS_5 solutions.

All of the above AdS_4 and AdS_5 direct product solutions and most (but not all, see e.g. [23]) of the warped products are partially supported by a non-vanishing (‘electric’) Freund-Rubin-like term $\hat{F}_{(4)} \sim \text{vol}(\text{AdS})_4$ in $D = 11$ and $\hat{F}_{(5)} \sim \text{vol}(\text{AdS})_5$ in type IIB. This reflects that the corresponding solutions describe conformal phases of the M2, D3 and, for [23], M5 brane field theories. Since type IIA supergravity also contains a four-form field strength $\hat{F}_{(4)}$, it would be natural to expect that the above plethora of AdS_4 solutions in $D = 11$ had a counterpart in type IIA. However, this is not the case: excluding the massless IIA solutions obtained by circle reduction from $D = 11$, only a handful of AdS_4 solutions in type IIA are known, either smooth and sourceless or singular and with sources. And most of them are only known numerically.

Massive type IIA supergravity [24] does admit a direct product Freund-Rubin solution $\text{AdS}_4 \times S^6$ [24], where S^6 is equipped with the usual, round, homogeneous, $\text{SO}(7)$ -symmetric metric: see equation (4.8). However, unlike its maximally supersymmetric counterparts in $D = 11$ [4] and type IIB [5], it breaks all supersymmetries. Furthermore, as it will be argued below, this solution is unstable. G -structure methods have been used to classify supersymmetric AdS_4 solutions in type IIA (as well as in other contexts) and, in some cases, these results have led to explicit classes of solutions. $\mathcal{N} = 1$ AdS_4 solutions with $\text{SU}(3)$ -structure were classified in [25] and [26]. An explicit class of $\mathcal{N} = 1$ solutions of massive type IIA involving the direct product of AdS_4 with a nearly-Kähler six-dimensional manifold was discovered in [25]. This class includes, in particular, the homogeneous $\mathcal{N} = 1$ direct product $\text{AdS}_4 \times S^6$, where the six-sphere is now regarded as $S^6 = \text{G}_2/\text{SU}(3)$ and the internal supergravity forms take values along the G_2 -invariant nearly-Kähler forms. In stark contrast with the Sasaki-Einstein situation [7, 8], the only analytically known nearly-Kähler six-manifolds are the four (and only four [27]) homogeneous cases,¹ including $S^6 = \text{G}_2/\text{SU}(3)$. A ‘half-flat’ generalisation of the nearly-Kähler condition [25] for $\mathcal{N} = 1$ AdS_4 solutions with $\text{SU}(3)$ -structure is known [26]. Also in that larger class, however, the only known solutions are homogeneous [31] (see also [32]).

Classifications of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ AdS_4 solutions of type IIA with local $\text{SU}(2)$ (that is, global $\text{SU}(3) \times \text{SU}(3)$) structure have also been carried out [33–36]. However,

¹Only very recently, existence results of inhomogeneous nearly-Kähler metrics on S^6 and other manifolds have been given [28]. See also [29, 30]. Analytic, closed form expressions for such metrics are not known.

\mathcal{N}	G_0	Stable?	F	ref.
$\mathcal{N} = 2$	$SU(3) \times U(1)$	yes	$2^{1/3} 3^{1/6} 5^{-1} \pi N^{5/3} k^{1/3}$	[43]
$\mathcal{N} = 1$	G_2	yes	$2^{-7} 3^{7/6} 5^{3/2} \pi N^{5/3} k^{1/3}$	[25]
$\mathcal{N} = 1$	$SU(3)$	yes	$2^{-17/3} 3^{1/6} 5^{3/2} \pi N^{5/3} k^{1/3}$	(4.4)
$\mathcal{N} = 0$	$SO(7)_+$	no	$2^{7/3} 3^{2/3} 5^{-13/6} \pi N^{5/3} k^{1/3}$	[24]
$\mathcal{N} = 0$	$SO(6)_+$	no	$2^{-1/2} 3^{2/3} 5^{-1} \pi N^{5/3} k^{1/3}$	(4.7)
$\mathcal{N} = 0$	G_2	yes	$2^{-3} 3^{13/6} 5^{-1} \pi N^{5/3} k^{1/3}$	[36]

Table 1. Summary of AdS₄ solutions of massive type IIA supergravity with supersymmetry \mathcal{N} and bosonic symmetry G_0 that are obtained from uplift of the analytical critical points of $D = 4$ $\mathcal{N} = 8$ dyonic ISO(7) supergravity with at least SU(3) symmetry. The stability against perturbations contained in the $D = 4$ supergravity, the gravitational free energy F of the solutions and the reference where they were first found are also shown.

these analyses lead to non-linear systems of first-order coupled differential equations for the supergravity fields that usually are very difficult to solve analytically. Some cohomogeneity-one $\mathcal{N} = 2$ AdS₄ solutions in this class are known numerically [37, 38]. Only very recently, a combination of G -structure methods [33] with a two-step reduction *à la* Maldacena-Núñez [20] has produced explicit, analytic, $\mathcal{N} = 1$ warped product AdS₄ solutions in massive IIA with (typically, but not necessarily, hyperbolic) internal spaces with $SU(3) \times SU(3)$ -structure [39] (see also [40]). Duality methods have been also used recently to obtain massless type IIA AdS₄ solutions with no Freund-Rubin term [41], similarly to [23]. See [42] for further recent classifications of supersymmetric AdS backgrounds of type IIA.

In this paper, we obtain new supersymmetric and non-supersymmetric AdS₄ solutions of massive type IIA and recover some previously known ones. We do this by-passing altogether the integration of difficult non-linear systems of first [33–36] or second order (see appendix B) differential equations. Instead, we use consistent truncation methods. In [43, 44] it was found that massive type IIA supergravity admits a consistent truncation on S^6 down to $D = 4$ $\mathcal{N} = 8$ supergravity with gauge group ISO(7) [45]. The gauging is of the dyonic type discussed in [46, 47]. By the consistency of the truncation, all $D = 4$ solutions uplift to massive IIA. In particular, the (AdS) critical points of the scalar potential give rise to AdS₄ solutions of massive type IIA, preserving supersymmetry in the process if present. This is exactly as in the uplifts to $D = 11$ and type IIB mentioned above.

Specifically, we uplift the critical points of the $D = 4$ supergravity with at least SU(3) symmetry that were classified in [45] (see table 3 of that reference for a summary) to obtain smooth, supersymmetric and non-supersymmetric, direct or warped product solutions of AdS₄ and S^6 . All the solutions are supported by a Freund-Rubin electric flux $\hat{F}_{(4)} \sim \text{vol}(\text{AdS})_4$ (or equivalently, by \hat{F}_6 flux on S^6) and some of them by a warp factor and internal supergravity forms as well. For each solution, the metric on S^6 displays an isometry related to the symmetry of the $D = 4$ critical point it uplifts from. In particular, we obtain new solutions with $\mathcal{N} = 1$ and SU(3) symmetry and $\mathcal{N} = 0$ and SO(6)₊ symmetry, the

subscript indicating that the corresponding $D = 4$ point is supported by $D = 4$ scalars, rather than pseudoscalars. We comment on generalisations and on the stability of the non-supersymmetric solutions. We also discuss flux quantisation and compute the gravitational free energy of the solutions. See table 1 for a summary.

We obtain these solutions by first working out explicit consistent truncation formulae of massive type IIA on S^6 to the full, dynamical, $SU(3)$ -invariant sector of dyonic $ISO(7)$ supergravity. We do this by particularising the general $\mathcal{N} = 8$ formulae given in [43, 44]. In other words, we explicitly obtain the full non-linear embedding of the entire, dynamical $SU(3)$ -invariant sector of (dyonic) $ISO(7)$ supergravity into (massive) type IIA at the level of the metric, dilaton and IIA form potentials (see equation (2.6)). These formulae allow for the uplift of any solution of the $D = 4$ theory, not only critical points of the potential. As emphasised in [44], these formulae do not depend on the Romans mass, which only enters the IIA embedding of the $D = 4$ bosons through the field strengths and scalar covariant derivatives. These formulae are thus also valid to uplift solutions of the $SU(3)$ -invariant sector of the purely electric $ISO(7)$ gauging [48] to massless type IIA. These should necessarily have non-constant scalars or non-trivial profiles for the vector fields, given the absence of critical points for the electric $ISO(7)$ gauging.

Section 2 discusses the embedding of the entire $SU(3)$ sector of $ISO(7)$ supergravity into type IIA. We discuss the regularity, symmetry and supersymmetry of the embedding and, in section 3, particularise it to some subsectors. Section 2 also contains the massive IIA field strengths evaluated on constant scalar configurations. Particularising further to the individual critical points in the $SU(3)$ sector, we obtain the solutions contained² in section 4. The paper concludes with two technical appendices. The foliation of S^6 with S^5 leaves, where S^5 is equipped with its usual Sasaki-Einstein structure, is described in appendix A. This is the structure that naturally emerges in the embedding studied in section 2 and the solutions of section 4. Appendix B specifies the system of equations, derived from the massive type IIA field equations, that the solutions of section 4 must obey.

2 Truncation to the $SU(3)$ sector of $ISO(7)$ supergravity

We will now particularise the general $\mathcal{N} = 8$ consistent embedding formulae of [43, 44] to the $SU(3)$ -invariant bosonic sector of dyonic $ISO(7)$ supergravity. The embedding formulae are naturally expressed, at the level of the IIA metric, dilaton and form potentials, in terms of $D = 4$ $SU(3)$ -singlet fields in the restricted duality hierarchy introduced in [45]. The relevant $D = 4$ field content includes

$$\begin{aligned}
 & 1 \text{ metric :} && ds_4^2, \\
 & 6 \text{ scalars :} && \varphi, \chi, \phi, a, \zeta, \tilde{\zeta}, \\
 & 2 \text{ electric vectors and their magnetic duals :} && A^0, A^1, \tilde{A}_0, \tilde{A}_1, \\
 & 3 \text{ two-forms :} && B^0, B_1, B_2, \\
 & 2 \text{ three-forms :} && C^0, C^1,
 \end{aligned} \tag{2.1}$$

²Two of the previously known solutions we recover have also been rederived in the recent [49] using the uplifting formulae of [43, 44].

see section 3 of [45]. All these fields are real. The scalars parametrise a submanifold

$$\frac{\text{SU}(1,1)}{\text{U}(1)} \times \frac{\text{SU}(2,1)}{\text{SU}(2) \times \text{U}(1)} \tag{2.2}$$

of $E_{7(7)}/\text{SU}(8)$. As discussed in [45] following [50], not all of the fields in (2.1) carry independent degrees of freedom. The four-form field strengths of the three-form potentials C^0, C^1 can be dualised into functions on the scalar manifold (2.2), etc. See (3.9), (3.13)–(3.15) of [45] for the definitions of the field strengths of the form potentials in (2.1) and (3.17)–(3.19) of [45] for the duality conditions. These dualisations can be used to write the consistent embedding into the type IIA field strengths in terms of independent degrees of freedom only.

In this section we give the full non-linear embedding in type IIA of the $\text{SU}(3)$ sector of $\text{ISO}(7)$ supergravity, at the level of the IIA metric, dilaton and form potentials, using the restricted tensor hierarchy (2.1). We also give the field strengths evaluated on constant scalar configurations, and employ the duality relations for the four-form field strengths to express the Freund-Rubin term as a function of the $D = 4$ scalars (in full generality, not only for constant scalars). We also discuss the regularity, symmetry and supersymmetry of the embedding. Further examples on the use of the dualisation conditions can be found in section 3. See appendix A of [44] for our type IIA conventions.

2.1 Consistent embedding formulae

The embedding of the $\text{SU}(3)$ -invariant $D = 4$ p -forms, $p = 1, 2, 3$, in (2.1) into the $\mathcal{N} = 8$ $\text{SL}(7)$ -covariant restricted tensor hierarchy [45] of the $\text{ISO}(7)$ gauging was given in equation (3.4) of that reference. We can bring those definitions to the uplifting formulae (3.12), (3.13) of [44], and use the S^6 conventions of appendix A. Then, some simple algebra allows us to obtain the ten-dimensional embedding of these $D = 4$ p -forms. In contrast, the IIA embedding of the $D = 4$ scalars is computationally more intense. This can be achieved by bringing the $\text{SU}(3)$ -invariant $D = 4$ scalar matrix, given in appendix D.1 of [45], to the uplifting formulae (8), (10) of [43] (or, equivalently, (3.14)–(3.18) of [44]). The non-vanishing components of the $D = 4$ scalar matrix occur along the invariant metric δ_{ij} , $i = 1, \dots, 6$, and tensors J, Ω that define the $\text{SU}(3)$ -holonomy of the \mathbb{R}^6 factor in the ambient $\mathbb{R}^7 = \mathbb{R}^6 \times \mathbb{R}$ where S^6 is embedded. The embedding coordinates μ^I , $I = 1, \dots, 7$, adapted to this setting are given in (A.1), and the Killing vectors and their derivatives in (A.4) and (A.5). Using these formulae, we find that all contractions of μ^I and $\partial_m \mu^I$ with the scalar matrix happen to occur through the combinations (A.10) that define the Sasaki-Einstein structure $\mathbf{J}, \mathbf{\Omega}, \mathbf{\eta}$ of the S^5 within S^6 .

In order to present the result, it is useful to define the following combinations of $D = 4$ scalars

$$X \equiv 1 + e^{2\varphi} \chi^2, \quad Y \equiv 1 + \frac{1}{4} e^{2\phi} (\zeta^2 + \tilde{\zeta}^2), \tag{2.3}$$

and $D = 4$ scalars together with an angle α on S^6 (see appendix A),

$$\Delta_1 = \left(e^\varphi + \frac{1}{4} e^{2\phi+\varphi} (\zeta^2 + \tilde{\zeta}^2) \right) \sin^2 \alpha + e^{2\phi-\varphi} (1 + e^{2\varphi} \chi^2) \cos^2 \alpha, \quad (2.4)$$

$$\Delta_2 = e^\varphi \sin^2 \alpha + e^{2\phi-\varphi} \cos^2 \alpha. \quad (2.5)$$

With these definitions, the complete nonlinear embedding of the SU(3)-invariant field content (2.1) of ISO(7) supergravity into type IIA reads, in ten-dimensional Einstein frame,

$$d\hat{s}_{10}^2 = e^{\frac{1}{8}(2\phi-\varphi)} X^{1/4} \Delta_1^{1/2} \Delta_2^{1/8} \left[ds_4^2 + g^{-2} e^{-2\phi+\varphi} X^{-1} d\alpha^2 + g^{-2} \sin^2 \alpha \left(\Delta_1^{-1} ds^2(\mathbb{CP}^2) + X^{-1} \Delta_2^{-1} \tilde{\eta}^2 \right) \right],$$

$$e^{\hat{\phi}} = e^{\frac{1}{4}(6\phi+5\varphi)} X^{-1/2} \Delta_1^{-1} \Delta_2^{3/4},$$

$$\begin{aligned} \hat{A}_{(3)} = & \cos^2 \alpha (C^0 + A^0 \wedge B^0) + \sin^2 \alpha C^1 \\ & - g^{-1} \sin \alpha \cos \alpha \left(B_1 + \frac{1}{2} A^0 \wedge \tilde{A}_0 + \frac{1}{6} A^1 \wedge \tilde{A}_1 \right) \wedge d\alpha - \frac{1}{3} g^{-1} \sin^2 \alpha B_2 \wedge \tilde{\eta} \\ & + \frac{1}{3} g^{-2} \tilde{A}_1 \wedge \left[\sin^2 \alpha \mathbf{J} + \sin \alpha \cos \alpha d\alpha \wedge \tilde{\eta} \right] \\ & - \cos \alpha A^0 \wedge \left[g^{-2} e^{2\varphi} \chi X^{-1} \sin \alpha d\alpha \wedge \tilde{\eta} \right. \\ & \quad \left. + g^{-2} e^{2\phi+\varphi} \chi \Delta_1^{-1} \sin^2 \alpha \cos \alpha \mathbf{J} + \frac{1}{2} g^{-2} e^{2\phi+\varphi} \Delta_1^{-1} \sin^3 \alpha (\tilde{\zeta} \operatorname{Re} \Omega - \zeta \operatorname{Im} \Omega) \right] \\ & - g^{-3} e^{2\phi+\varphi} a \chi \Delta_1^{-1} \sin^3 \alpha \cos \alpha \mathbf{J} \wedge d\alpha - g^{-3} e^\varphi \chi Y \Delta_1^{-1} \sin^4 \alpha \mathbf{J} \wedge \tilde{\eta} \\ & + \frac{1}{2} g^{-3} \sin^2 \alpha \left((\zeta \operatorname{Re} \Omega + \tilde{\zeta} \operatorname{Im} \Omega) - e^{2\phi+\varphi} a \Delta_1^{-1} \sin^2 \alpha (\tilde{\zeta} \operatorname{Re} \Omega - \zeta \operatorname{Im} \Omega) \right) \wedge d\alpha \\ & + \frac{1}{2} g^{-3} e^{2\phi-\varphi} X \Delta_1^{-1} \sin^3 \alpha \cos \alpha (\tilde{\zeta} \operatorname{Re} \Omega - \zeta \operatorname{Im} \Omega) \wedge \tilde{\eta}, \end{aligned}$$

$$\begin{aligned} \hat{B}_{(2)} = & -\cos \alpha B^0 + g^{-1} \sin \alpha \tilde{A}_0 \wedge d\alpha + g^{-2} e^{2\varphi} \chi X^{-1} \sin \alpha d\alpha \wedge \tilde{\eta} \\ & + g^{-2} e^{2\phi+\varphi} \chi \Delta_1^{-1} \sin^2 \alpha \cos \alpha \mathbf{J} + \frac{1}{2} g^{-2} e^{2\phi+\varphi} \Delta_1^{-1} \sin^3 \alpha (\tilde{\zeta} \operatorname{Re} \Omega - \zeta \operatorname{Im} \Omega), \end{aligned}$$

$$\hat{A}_{(1)} = -\cos \alpha A^0 - g^{-1} a \sin \alpha d\alpha + g^{-1} e^{-\varphi} (X - Y) \Delta_2^{-1} \sin^2 \alpha \cos \alpha \tilde{\eta}. \quad (2.6)$$

The metric $ds^2(\mathbb{CP}^2)$ is the usual Fubini-Study metric on the \mathbb{CP}^2 base of the S^5 within S^6 , normalised so that the Ricci tensor is six times the metric, and

$$\tilde{\eta} \equiv \eta + g A^1 \equiv d\psi + \sigma + g A^1. \quad (2.7)$$

Here, ψ is the angle on the Hopf fiber of S^5 and σ is a one-form potential for \mathbf{J} : $d\sigma = 2\mathbf{J}$. These embedding formulae depend on the (non-vanishing) $D = 4$ electric gauge coupling g , but not on the magnetic coupling m . Thus, they simultaneously describe the embedding

of the SU(3)-invariant sector of the purely electric, $m = 0$, and dyonic, $m \neq 0$, ISO(7) gauging into massless and massive, respectively, type IIA supergravity.

The field strengths corresponding to the form potentials in (2.6) can be computed using their definitions, see (A.4) of [44] for our conventions. For simplicity, here we present them for constant $D = 4$ scalars and vanishing $D = 4$ two- and three-form field strength contributions. The Romans mass is given by [43]

$$\hat{F}_{(0)} = m, \quad (2.8)$$

and $\hat{F}_{(4)}$, $\hat{H}_{(3)}$ and $\hat{F}_{(2)}$ by

$$\begin{aligned} \hat{F}_{(4)} = & U \text{vol}_4 + \left(mg^{-4} e^{4\phi+2\varphi} \Delta_1^{-2} \left[\frac{1}{4} (\zeta^2 + \tilde{\zeta}^2) \sin^2 \alpha + \chi^2 \cos^2 \alpha \right] \right. \\ & \left. - 4g^{-3} e^\varphi \chi Y \Delta_1^{-1} \right) \sin^4 \alpha \text{vol}(\mathbb{CP}_2) \\ & + \left[mg^{-4} e^{2\phi+3\varphi} \chi^2 X^{-1} \Delta_1^{-1} \right. \\ & \quad - g^{-3} e^\varphi \chi \Delta_1^{-2} \Delta_2^{-1} \left(e^\varphi X^{-1} Y (e^{2\phi-\varphi} X(X+Y) + 2e^\varphi Y^2) \sin^4 \alpha \right. \\ & \quad \quad \left. + (e^{4\phi-2\varphi} X(X+Y) + 6e^{2\phi} Y^2) \sin^2 \alpha \cos^2 \alpha \right. \\ & \quad \left. \left. + 4e^{4\phi-2\varphi} XY \cos^4 \alpha \right) \right] \sin^3 \alpha \cos \alpha \mathbf{J} \wedge d\alpha \wedge \boldsymbol{\eta} \\ & + \frac{1}{2} \left[mg^{-4} e^{2\phi+3\varphi} \chi X^{-1} \Delta_1^{-1} \right. \\ & \quad - g^{-3} \Delta_1^{-2} \Delta_2^{-1} \left(e^\varphi Y (e^{2\phi} X + 3e^{2\varphi} Y) \sin^4 \alpha \right. \\ & \quad \quad \left. + (e^{4\phi-\varphi} X(X-Y) + 2e^{2\phi+\varphi} Y(Y+3X)) \sin^2 \alpha \cos^2 \alpha \right. \\ & \quad \left. \left. + (e^{4\phi-\varphi} X(3X+2Y) - e^{6\phi-3\varphi} X^2) \cos^4 \alpha \right) \right] \\ & \quad \times \sin^4 \alpha (\tilde{\zeta} \text{Re } \boldsymbol{\Omega} - \zeta \text{Im } \boldsymbol{\Omega}) \wedge d\alpha \wedge \boldsymbol{\eta}, \\ \hat{H}_{(3)} = & -g^{-2} e^{2\phi+\varphi} \chi \Delta_1^{-2} \left(e^\varphi X^{-1} Y (X + 2e^{-2\phi+2\varphi} Y) \sin^2 \alpha + (4e^\varphi Y - e^{2\phi-\varphi} X) \cos^2 \alpha \right) \\ & \quad \times \sin^3 \alpha \mathbf{J} \wedge d\alpha \\ & + \frac{1}{2} g^{-2} e^{2\phi+\varphi} \Delta_1^{-2} \left((e^\varphi Y + 2e^{2\phi-\varphi} X) \sin^2 \alpha + 3e^{2\phi-\varphi} X \cos^2 \alpha \right) \sin^2 \alpha \cos \alpha \\ & \quad \times (\tilde{\zeta} \text{Re } \boldsymbol{\Omega} - \zeta \text{Im } \boldsymbol{\Omega}) \wedge d\alpha \\ & - \frac{3}{2} g^{-2} e^{2\phi+\varphi} \Delta_1^{-1} \sin^3 \alpha (\zeta \text{Re } \boldsymbol{\Omega} + \tilde{\zeta} \text{Im } \boldsymbol{\Omega}) \wedge \boldsymbol{\eta}, \\ \hat{F}_{(2)} = & \left(mg^{-2} e^{2\phi+\varphi} \chi \Delta_1^{-1} + 2g^{-1} e^{-\varphi} (X-Y) \Delta_2^{-1} \right) \sin^2 \alpha \cos \alpha \mathbf{J} \\ & + \frac{1}{2} mg^{-2} e^{2\phi+\varphi} \Delta_1^{-1} \sin^3 \alpha (\tilde{\zeta} \text{Re } \boldsymbol{\Omega} - \zeta \text{Im } \boldsymbol{\Omega}) \\ & + \left[mg^{-2} e^{2\varphi} \chi X^{-1} + g^{-1} e^{-\varphi} (X-Y) \Delta_2^{-2} \left(2e^{2\phi-\varphi} \cos^4 \alpha \right. \right. \\ & \quad \left. \left. + e^{2\phi-\varphi} \cos^2 \alpha \sin^2 \alpha - e^\varphi \sin^4 \alpha \right) \right] \sin \alpha d\alpha \wedge \boldsymbol{\eta}. \end{aligned} \quad (2.9)$$

In contrast to the gauge potentials in (2.6), the field strengths (2.9) now do depend on the $D = 4$ magnetic gauge coupling m . This is a generic feature: see [44] for further discussion. By the consistency of the truncation, the metric and dilaton in (2.6) and the field strengths (2.9) solve all the field equations of massless or massive, if $m = 0$ or $m \neq 0$, type IIA supergravity provided the $D = 4$ field equations on constant scalar configurations are imposed. In other words, the field strengths (2.9) solve the type IIA field equations when evaluated on critical points of the $D = 4$ scalar potential. Critical points in the ISO(7) gauging only exist for $m \neq 0$, i.e. in the dyonic case. See section 4 for the resulting AdS₄ solutions of massive IIA. Let us emphasise that the configuration (2.6) is valid, more generally, for dynamical scalars and p -forms in the $D = 4$ hierarchy (2.1). The corresponding field strengths (namely, the extension of (2.9) to include the derivatives of the scalars along with the field strengths of the p -forms in (2.1)) obey the type IIA field equations provided their $D = 4$ counterparts are imposed.

Following the strategy of [44], the Freund-Rubin term in $\hat{F}_{(4)}$ can be determined by using the dualisation conditions for the $D = 4$ four-form field strengths. In the case at hand, the Freund-Rubin term $U \text{vol}_4 = \mathcal{H}_{(4)}^{IJ} \mu_I \mu_J$ [44] reduces to

$$U \text{vol}_4 = H_{(4)}^0 \cos^2 \alpha + H_{(4)}^1 \sin^2 \alpha, \tag{2.10}$$

where $H_{(4)}^0$ and $H_{(4)}^1$ are the four-form field strengths of the three-form potentials in (2.1) above (see (3.15) of [45]). Using the dualisation relations given in (3.19) of [45], the function U in (2.10) and (2.9) becomes

$$\begin{aligned} U = & \left[\frac{1}{2} g (1 + e^{2\varphi} \chi^2) \left(12 e^{2\phi-\varphi} - 2 e^{4\phi-3\varphi} (1 + e^{2\varphi} \chi^2)^2 - 3 e^{4\phi+\varphi} \chi^2 (\zeta^2 + \tilde{\zeta}^2) \right) \right. \\ & \left. + m e^{4\phi+3\varphi} \chi^3 \right] \cos^2 \alpha \\ & + \left[\frac{1}{2} g \left(8 e^\varphi + 2 e^{2\phi-\varphi} (1 + e^{2\varphi} \chi^2) + e^{2\phi+\varphi} (\zeta^2 + \tilde{\zeta}^2) (1 - 3 e^{2\varphi} \chi^2) \right) \right. \\ & \left. - \frac{1}{2} e^{4\phi+\varphi} (\zeta^2 + \tilde{\zeta}^2) \chi^2 (1 + e^{2\varphi} \chi^2) - \frac{1}{4} e^{4\phi+\varphi} (\zeta^2 + \tilde{\zeta}^2)^2 (1 + 3 e^{2\varphi} \chi^2) \right. \\ & \left. + \frac{1}{4} m e^{4\phi+3\varphi} \chi (\zeta^2 + \tilde{\zeta}^2) \right] \sin^2 \alpha. \tag{2.11} \end{aligned}$$

This expression can be equivalently obtained from the master formula (3.43) of [44].

Although the field strengths (2.9) are evaluated for constant scalars, the Freund-Rubin term (2.11) is general: it is valid for both constant and dynamical scalars. In general, the Freund-Rubin term can be expressed in terms of the scalar potential, the covariant derivatives of the three-form field strengths in the restricted $D = 4$ tensor hierarchy, and the vector field strengths, see equation (3.36) of [44]. Upon dualisation, the last two contributions become related to scalar equations of motion. In particular, the dualised scalar contributions should be related to derivatives of the scalar potential. This can be exhibited with the explicit parametrisation of the SU(3)-invariant sector that we are using. Indeed, the function U in (2.11) that defines the Freund-Rubin term can be checked to be

related to the SU(3)-invariant $D = 4$ scalar potential V (given in (3.11) of [45]) and its derivatives:

$$gU = -\frac{1}{3}V - \frac{1}{6}\left(\partial_\phi V - 2\partial_\varphi V + 2\chi\partial_\chi V - \zeta\partial_\zeta V - \tilde{\zeta}\partial_{\tilde{\zeta}}V\right)\cos^2\alpha + \frac{1}{12}\left(\partial_\phi V - \zeta\partial_\zeta V - \tilde{\zeta}\partial_{\tilde{\zeta}}V\right)\sin^2\alpha. \quad (2.12)$$

At a critical point of the scalar potential, this expression reduces to

$$gU|_0 = -\frac{1}{3}V_0, \quad (2.13)$$

in agreement with the general $\mathcal{N} = 8$ discussion of [44]. In (2.13), $|_0$ and V_0 denote evaluation at a critical point. See [51, 52] for a related discussion in a $D = 11$ on S^7 context.

2.2 Regularity, symmetry and supersymmetry

The local geometry corresponding to the type IIA configuration (2.6) can be regarded as a foliation, parametrised by α , of S^6 with S^5 leaves. The S^5 , in turn, comes naturally equipped with its Sasaki-Einstein structure. Namely, the S^5 can be viewed as a U(1), parametrised by ψ , fibered over \mathbb{CP}^2 , with fibers stretched or squashed (depending on the values of the $D = 4$ scalars) as a function of α . It is easy to see that this geometry extends globally in a smooth fashion provided the angles α, ψ are given the ranges

$$0 \leq \alpha \leq \pi, \quad 0 \leq \psi \leq 2\pi. \quad (2.14)$$

Globally, (2.6) is defined on a smooth S^2 bundle, parametrised by α, ψ , over \mathbb{CP}^2 . These angles do indeed parametrise an S^2 since, for all values of the $D = 4$ scalars, the relevant metric functions behave as

$$e^{2\phi-\varphi}\Delta_2^{-1}\sin^2\alpha \xrightarrow{\alpha \rightarrow 0} \alpha^2 + \mathcal{O}(\alpha^4), \quad e^{2\phi-\varphi}\Delta_2^{-1}\sin^2\alpha \xrightarrow{\alpha \rightarrow \pi} (\pi - \alpha)^2 + \mathcal{O}((\pi - \alpha)^4), \quad (2.15)$$

at both endpoints of the chosen range (2.14) for α . Moreover, this S^2 is smooth if the period of ψ is chosen as in (2.14). Similarly, the bundle over \mathbb{CP}^2 is also smooth because

$$e^{2\phi-\varphi}X\Delta_1^{-1}\sin^2\alpha \xrightarrow{\alpha \rightarrow 0} \alpha^2 + \mathcal{O}(\alpha^4), \quad e^{2\phi-\varphi}X\Delta_1^{-1}\sin^2\alpha \xrightarrow{\alpha \rightarrow \pi} (\pi - \alpha)^2 + \mathcal{O}((\pi - \alpha)^4). \quad (2.16)$$

The easiest way to see that the total space of the bundle is indeed S^6 is by noting that the local family of internal metrics in (2.6) can be continuously deformed to the usual sine-cone metric (A.3) by restricting the $D = 4$ scalars to the line

$$\varphi = \phi, \quad \chi = a = \zeta = \tilde{\zeta} = 0 \quad (2.17)$$

within the scalar manifold (2.2). The warp factor, dilaton and form potentials also remain smooth for all values of the $D = 4$ scalars.

The metric in (2.6) displays an SU(3) \times U(1) isometry group, where the first factor corresponds to the isometry of the Fubini-Study metric on \mathbb{CP}^2 , and the U(1) is generated

by the vector ∂_ψ dual to $\boldsymbol{\eta}$. For $\zeta \neq 0$ or $\tilde{\zeta} \neq 0$, this $U(1)$ is broken by the dependence of the supergravity forms on the Sasaki-Einstein two-form $\boldsymbol{\Omega}$, which carries $U(1)$ charge. The generic symmetry of the complete IIA configuration (2.6) is thus $SU(3)$, in agreement with the symmetry of the sector of $D = 4$ $ISO(7)$ supergravity from which it uplifts. When the $D = 4$ scalars are restricted to certain loci of (2.2), symmetry enhancements occur. As we have already noted, configurations (2.6) with

$$\zeta = \tilde{\zeta} = 0 \tag{2.18}$$

preserve $SU(3) \times U(1)$. The latter acts with cohomogeneity-one. When

$$\chi = \zeta = \tilde{\zeta} = 0, \tag{2.19}$$

the symmetry is enhanced to $SO(6)_+$, which again acts with cohomogeneity-one. For

$$\phi = \varphi, \quad \tilde{\zeta} = 2\chi, \quad a = \zeta = 0, \tag{2.20}$$

the symmetry is enhanced from $SU(3)$ to G_2 , which now acts homogeneously (see subsection 3.1). Finally, the symmetry is enhanced to a homogeneous $SO(7)_+$ when the $D = 4$ scalars take values on the locus (2.17). The loci (2.17)–(2.20) also define the sectors of $ISO(7)$ supergravity with the same symmetry, see section 3.5 of [45]. Therefore, the evaluation of equation (2.6) on these loci together with appropriate restrictions of the $D = 4$ p -forms determines the truncation of massive IIA to specific $D = 4$ sectors. We will discuss some of these truncations in more detail in section 3.

Since supersymmetry is preserved in the uplifting process, the type IIA configuration (2.6) is $8/32$ -supersymmetric for generic values of the $D = 4$ fields, reflecting the $\mathcal{N} = 2$ supersymmetry of the $SU(3)$ -invariant sector of $ISO(7)$ supergravity. Supersymmetry in (2.6) is realised by the presence of a global $SU(3) \times SU(3)$ -structure, or local $SU(2)$ -structure, on the internal S^6 of the type discussed in [34] (see also [35, 37]). Since (2.6) only depends on the Sasaki-Einstein structure of S^5 , these truncation formulae are still locally valid if S^5 is replaced with any Sasaki-Einstein manifold. In other words, (massive) type IIA supergravity can be consistently truncated on a local geometry of the form (2.6), based on an arbitrary Sasaki-Einstein five-manifold, to the $D = 4$ $\mathcal{N} = 2$ supergravity theory of section 3 of [45]. In this more general setting, the internal geometry will still be that of an S^2 bundle over the local Kähler-Einstein base of the Sasaki-Einstein manifold. The total space of the bundle will no longer correspond to the six-sphere, and will typically display orbifold-type singularities.

3 Further truncations

It is instructive to particularise the above $SU(3)$ -invariant consistent truncation formulae to further subsectors of the $D = 4$ theory with smaller field content and larger symmetry.

3.1 Truncation to the G_2 sector

The G_2 -invariant sector of $D = 4$ ISO(7) supergravity was discussed in [45], and the explicit truncation from type IIA, in [44]. Here, we will recover the latter truncation from the SU(3)-invariant truncation.

The $D = 4$ G_2 sector is recovered from the SU(3) sector by setting to zero all vectors and two-form potentials in (2.1), identifying the scalars as in (2.20), and the three-form potentials as $C^0 = C^1 \equiv C$. Bringing these identifications to (2.6), we find that all the α dependence combines with the forms $\mathbf{J}, \mathbf{\Omega}, \mathbf{\eta}$ into the combinations (A.14) that determine the homogeneous nearly-Kähler structure \mathcal{J}, Ω on S^6 . Specifically, (2.6) reduces to

$$\begin{aligned} d\hat{s}_{10}^2 &= e^{\frac{3}{4}\varphi} (1 + e^{2\varphi} \chi^2)^{\frac{3}{4}} ds_4^2 + g^{-2} e^{-\frac{1}{4}\varphi} (1 + e^{2\varphi} \chi^2)^{-\frac{1}{4}} ds^2(S^6), \\ e^{\hat{\phi}} &= e^{\frac{5}{2}\varphi} (1 + e^{2\varphi} \chi^2)^{-\frac{3}{2}}, \\ \hat{A}_{(3)} &= C + g^{-3} \chi \text{Im } \Omega, \quad \hat{B}_{(2)} = g^{-2} e^{2\varphi} \chi (1 + e^{2\varphi} \chi^2)^{-1} \mathcal{J}, \quad \hat{A}_{(1)} = 0. \end{aligned} \quad (3.1)$$

These agree with the formulae for the consistent truncation to the G_2 -invariant sector given in (4.3) of [44]. Likewise, the constant-scalar field strengths (2.9) reduce to the field strengths given in (4.4) of [44], after dropping the terms in the latter containing derivatives of scalars.

3.2 Dilatons

Here we give an illustrative example of how the dualisation conditions of [45] can be used to simplify not only the Freund-Rubin term, but also the three-form field strength contributions. For simplicity, we focus on the consistent subsector of (2.1) containing only the dilatons φ, ϕ , the two form B_1 and the three-forms C^0, C^1 , with all other fields set to zero. This is a consistent subsector both of the $SO(6)_+$ invariant sector and of the neutral scalar model of [43]. From (3.14), (3.15) of [45], the field strengths in this subsector are

$$H_{(3)1} = dB_1 + 2g(C^1 - C^0), \quad H_{(4)}^0 = dC^0, \quad H_{(4)}^1 = dC^1. \quad (3.2)$$

These are subject to the dualisation conditions

$$H_{(3)1} = 2 * (d\varphi - d\phi), \quad H_{(4)}^0 = g (6 e^{2\phi-\varphi} - e^{4\phi-3\varphi}) \text{vol}_4, \quad H_{(4)}^1 = g (4 e^\varphi + e^{2\phi-\varphi}) \text{vol}_4, \quad (3.3)$$

which arise as a particular case of (3.18), (3.19) of [45].

In this sector, the functions (2.4), (2.5) simplify to

$$\Delta_1 = \Delta_2 \equiv \Delta = e^\varphi \sin^2 \alpha + e^{2\phi-\varphi} \cos^2 \alpha, \quad (3.4)$$

and the type IIA embedding (2.6) to

$$\begin{aligned} d\hat{s}_{10}^2 &= e^{\frac{1}{8}(2\phi-\varphi)} \Delta^{5/8} \left[ds_4^2 + g^{-2} e^{-2\phi+\varphi} d\alpha^2 + g^{-2} \Delta^{-1} \sin^2 \alpha d\hat{s}^2(S^5) \right], \\ e^{\hat{\phi}} &= e^{\frac{1}{4}(6\phi+5\varphi)} \Delta^{-1/4}, \\ \hat{A}_{(3)} &= \cos^2 \alpha C^0 + \sin^2 \alpha C^1 - g^{-1} \sin \alpha \cos \alpha B_1 \wedge d\alpha, \quad \hat{B}_{(2)} = 0, \quad \hat{A}_{(1)} = 0. \end{aligned} \quad (3.5)$$

Here, $d\tilde{s}^2(S^5)$ is the round, $SO(6)$ -invariant metric on the foliating S^5 , normalised so that the Ricci tensor equals 4 times the metric. Now, with the help of the definitions (3.2), the four-form field strength corresponding to the configuration (3.5) can be written as

$$\hat{F}_{(4)} = H_{(4)}^0 \cos^2 \alpha + H_{(4)}^1 \sin^2 \alpha - g^{-1} \sin \alpha \cos \alpha H_{(3)1} \wedge d\alpha . \quad (3.6)$$

Finally, the dualisation conditions (3.3) can be used to rewrite $\hat{F}_{(4)}$ in terms of independent $D = 4$ fields only (ϕ , φ and the metric, through the Hodge dual), as

$$\begin{aligned} \hat{F}_{(4)} = g \left[(6 e^{2\phi-\varphi} - e^{4\phi-3\varphi}) \cos^2 \alpha + (4 e^\varphi + e^{2\phi-\varphi}) \sin^2 \alpha \right] \text{vol}_4 \\ + 2g^{-1} \sin \alpha \cos \alpha d\alpha \wedge *(d\varphi - d\phi) . \end{aligned} \quad (3.7)$$

The Freund-Rubin term here is a particularisation of (2.11). Terms like $*(d\varphi - d\phi)$ in $\hat{F}_{(4)}$ have long been known to enter similar spherical truncation formulae, see e.g. (3.1) of [53] or the formulae in [54]. Such term has been obtained here using the systematics of the duality hierarchy.

It is further consistent to set $\varphi = \phi$ in the above formulae. The resulting expressions correspond to the truncation of IIA to the $SO(7)_+$ -invariant sector of $ISO(7)$ supergravity.

4 New AdS_4 solutions of massive type IIA

We will now use the explicit formulae (2.6), (2.9) for the ten-dimensional uplift of the $SU(3)$ -invariant sector of $D = 4$ (dyonic) $ISO(7)$ supergravity, to obtain sourceless solutions of massive type IIA supergravity. All $D = 4$ critical points with at least $SU(3)$ symmetry are AdS , and their location in the scalar manifold (2.2) can be found in table 3 of [45]. Evaluating (2.6), (2.9) at the scalar locations given in that table, we find classical solutions of massive type IIA of the form $AdS_4 \times S^6$, with the product possibly warped and with various metrics on S^6 . The supersymmetry (or lack thereof) and bosonic symmetry of the $D = 4$ critical points is carried over into the ten-dimensional solutions. The former, if present, endows the S^6 with a particular G -structure. The latter manifests itself as the subgroup of the isometry group of the relevant metric on S^6 that also preserves the IIA form fields. The uplift of critical points with symmetry group G_0 such that $G_2 \subset G_0$ leads to homogeneous solutions, in agreement with the discussion in section 2.2. On the other hand, critical points with $SU(3) \subset G_0 \subset G_2$, lead to cohomogeneity-one solutions in ten dimensions. By the analysis of section 2.2, all these solutions are smooth.

The $\mathcal{N} = 1$ $SU(3)$ solution (4.4) and the $\mathcal{N} = 0$ $SO(6)_+$ solution (4.7) are new. The uplift of the $D = 4$ $\mathcal{N} = 2$ point with $SU(3) \times U(1)$ symmetry is omitted here, as it was already given in [43]. All other solutions have already appeared in the literature, as they have been previously obtained by other methods. In any case, we have checked that all these configurations solve the massive type IIA field equations, see appendix B. This provides a double-check on our uplifting formulae and on the consistency of the truncation.

The solutions below are presented in ten-dimensional Einstein frame. We have rescaled the external four-dimensional metric so that AdS_4 is always unit radius. The metrics thus

have the local form

$$d\hat{s}_{10}^2 = e^{2A} \left(ds^2(\text{AdS}_4) + ds_6^2 \right). \quad (4.1)$$

From (2.6) and (2.13), the warp factor and Freund-Rubin term become

$$e^{2A} \equiv -6 e^{\frac{1}{8}(2\phi-\varphi)} X^{1/4} \Delta_1^{1/2} \Delta_2^{1/8} V_0^{-1} \quad , \quad U_{\text{rescaled}} = -12 g^{-1} V_0^{-1} \quad , \quad (4.2)$$

where $V_0 < 0$ is the value of the SU(3)-invariant scalar potential, (3.11) of [45], at each critical point. All the solutions depend on the $D = 4$ couplings g and m through combinations L and e^{ϕ_0} that we define for each solution on an individual basis. Although the uplifting formulae (2.6) for the metric, dilaton and form potentials only depend on g , and m only enters the embedding through the field strengths (2.8), (2.9), all the fields in the solutions below develop a dependence on both g and m (via L and e^{ϕ_0}). The reason for this is that the position in the salar space (2.2) of the $D = 4$ critical points depends on both couplings (as $\sim m^{\frac{1}{3}} g^{-\frac{1}{3}}$), see table 3 of [45].

We comment on generalisations of these solutions and, for the non-supersymmetric ones, on their stability against fluctuations contained in the $D = 4$ $\mathcal{N} = 8$ dyonic ISO(7) supergravity. We conclude with a flux quantisation analysis. The latter shows that all stable solutions are also good massive type IIA string theory backgrounds. See table 1 in the introduction for a summary.

4.1 Supersymmetric solutions

Recall that dyonic ISO(7) supergravity contains supersymmetric points with ($\mathcal{N} = 2$, SU(3) \times U(1)), ($\mathcal{N} = 1$, SU(3)) and ($\mathcal{N} = 1$, G₂) residual supersymmetry and bosonic symmetry. The massive IIA solution corresponding to the $\mathcal{N} = 2$ point has already been given in [43]. For that reason, we omit the full solution here and merely give a set of potentials for the internal field strengths:³

$$\begin{aligned} L^{-3} e^{\frac{1}{4}\phi_0} \hat{A}_{(3)} &= 6\sqrt{3} \frac{\sin^4 \alpha}{3 + \cos 2\alpha} \mathbf{J} \wedge \boldsymbol{\eta} \quad , \\ L^{-2} e^{-\frac{1}{2}\phi_0} \hat{B}_{(2)} &= -6\sqrt{2} \frac{\sin^2 \alpha \cos \alpha}{3 + \cos 2\alpha} \mathbf{J} - \frac{3}{\sqrt{2}} \sin \alpha d\alpha \wedge \boldsymbol{\eta} \quad , \\ L^{-1} e^{\frac{3}{4}\phi_0} \hat{A}_{(1)} &= \sqrt{6} \frac{\sin^2 \alpha \cos \alpha}{5 + \cos 2\alpha} \boldsymbol{\eta} \quad . \end{aligned} \quad (4.3)$$

These follow from particularising (2.6) to the $\mathcal{N} = 2$, SU(3) \times U(1) critical point, and then performing a gauge transformation to remove the constant $D = 4$ Stückelberg scalar a . The contributions from the external three-forms C^0 and C^1 have been excluded here. Following [43], we have defined the constants $L^2 \equiv 2^{-\frac{5}{8}} 3^{-1} g^{-\frac{25}{12}} m^{\frac{1}{12}}$ and $e^{\phi_0} \equiv 2^{\frac{1}{4}} g^{\frac{5}{6}} m^{-\frac{5}{6}}$, so that, for this solution, $g = 2^{-\frac{3}{10}} 3^{-\frac{1}{2}} L^{-1} e^{-\frac{1}{20}\phi_0}$ and $\hat{F}_{(0)} \equiv m = 3^{-\frac{1}{2}} L^{-1} e^{-\frac{5}{4}\phi_0}$.

³The field strengths of the solution in [43] were in fact computed from this set of potentials, and then double-checked with (2.9). A set of potentials has also been recently given in [55].

The $\mathcal{N} = 1$ SU(3) extremum gives rise to a new $\mathcal{N} = 1$ solution of massive type IIA supergravity. It reads

$$\begin{aligned}
 ds_{10}^2 &= L^2 (3 + \cos 2\alpha)^{1/2} (2 + \cos 2\alpha)^{1/8} \left[ds^2(\text{AdS}_4) \right. \\
 &\quad \left. + \frac{6}{5} d\alpha^2 + \frac{24 \sin^2 \alpha}{5(3 + \cos 2\alpha)} ds^2(\mathbb{CP}^2) + \frac{18 \sin^2 \alpha}{5(2 + \cos 2\alpha)} \boldsymbol{\eta}^2 \right], \\
 e^{\hat{\phi}} &= e^{\phi_0} \frac{(2 + \cos 2\alpha)^{3/4}}{3 + \cos 2\alpha}, \\
 L^{-3} e^{\frac{1}{4}\phi_0} \hat{F}_{(4)} &= \frac{3\sqrt{5}}{\sqrt{2}} \text{vol}(\text{AdS}_4) - \frac{96\sqrt{6}}{25} \frac{4 + 3 \cos 2\alpha}{(3 + \cos 2\alpha)^2} \sin^4 \alpha \text{vol}(\mathbb{CP}^2) \\
 &\quad - \frac{72\sqrt{6}}{25} \frac{6 + \cos 2\alpha}{(2 + \cos 2\alpha)(3 + \cos 2\alpha)} \sin^3 \alpha \cos \alpha \mathbf{J} \wedge d\alpha \wedge \boldsymbol{\eta} \\
 &\quad + \frac{648\sqrt{2}}{25} \frac{\sin^4 \alpha}{(2 + \cos 2\alpha)(3 + \cos 2\alpha)} \text{Im } \boldsymbol{\Omega} \wedge d\alpha \wedge \boldsymbol{\eta}, \\
 L^{-2} e^{-\frac{1}{2}\phi_0} \hat{H}_{(3)} &= -\frac{96}{5\sqrt{5}} \frac{\sin^3 \alpha}{(3 + \cos 2\alpha)^2} \mathbf{J} \wedge d\alpha - \frac{24\sqrt{3}}{5\sqrt{5}} \frac{11 + \cos 2\alpha}{(3 + \cos 2\alpha)^2} \sin^2 \alpha \cos \alpha \text{Im } \boldsymbol{\Omega} \wedge d\alpha \\
 &\quad - \frac{72\sqrt{3}}{5\sqrt{5}} \frac{\sin^3 \alpha}{3 + \cos 2\alpha} \text{Re } \boldsymbol{\Omega} \wedge \boldsymbol{\eta}, \\
 L^{-1} e^{\frac{3}{4}\phi_0} \hat{F}_{(2)} &= -\frac{4\sqrt{6}}{\sqrt{5}} \frac{\sin^2 \alpha \cos \alpha}{(2 + \cos 2\alpha)(3 + \cos 2\alpha)} \mathbf{J} - \frac{3\sqrt{6}}{\sqrt{5}} \frac{\sin \alpha \cos 2\alpha}{(2 + \cos 2\alpha)^2} d\alpha \wedge \boldsymbol{\eta} \\
 &\quad - \frac{12\sqrt{2}}{\sqrt{5}} \frac{\sin^3 \alpha}{3 + \cos 2\alpha} \text{Im } \boldsymbol{\Omega}. \tag{4.4}
 \end{aligned}$$

The solution depends on the $D = 4$ gauge coupling constants g , m through the combinations L , e^{ϕ_0} , which are now defined as $L^2 \equiv 2^{-\frac{7}{2}} 3^{-1} 5^{\frac{11}{8}} g^{-\frac{25}{12}} m^{\frac{1}{12}}$ and $e^{\phi_0} \equiv 2 \cdot 5^{\frac{1}{4}} g^{\frac{5}{6}} m^{-\frac{5}{6}}$, so that $g = 2^{-\frac{17}{10}} 3^{-\frac{1}{2}} 5^{\frac{7}{10}} L^{-1} e^{-\frac{1}{20}\phi_0}$ and $\hat{F}_{(0)} \equiv m = 2^{-\frac{1}{2}} 3^{-\frac{1}{2}} 5 L^{-1} e^{-\frac{5}{4}\phi_0}$. A set of internal potentials for this solution follows from (2.6):

$$\begin{aligned}
 L^{-3} e^{\frac{1}{4}\phi_0} \hat{A}_{(3)} &= -\frac{48\sqrt{6}}{25} \frac{\sin^4 \alpha}{3 + \cos 2\alpha} \mathbf{J} \wedge \boldsymbol{\eta} + \frac{72\sqrt{2}}{25} \sin^2 \alpha \text{Re } \boldsymbol{\Omega} \wedge d\alpha \\
 &\quad - \frac{288\sqrt{2}}{25} \frac{\sin^3 \alpha \cos \alpha}{3 + \cos 2\alpha} \text{Im } \boldsymbol{\Omega} \wedge \boldsymbol{\eta}, \\
 L^{-2} e^{-\frac{1}{2}\phi_0} \hat{B}_{(2)} &= \frac{24}{5\sqrt{5}} \frac{\sin^2 \alpha \cos \alpha}{3 + \cos 2\alpha} \mathbf{J} + \frac{6}{5\sqrt{5}} \sin \alpha d\alpha \wedge \boldsymbol{\eta} - \frac{24\sqrt{3}}{5\sqrt{5}} \frac{\sin^3 \alpha}{3 + \cos 2\alpha} \text{Im } \boldsymbol{\Omega}, \\
 L^{-1} e^{\frac{3}{4}\phi_0} \hat{A}_{(1)} &= -\frac{2\sqrt{6}}{\sqrt{5}} \frac{\sin^2 \alpha \cos \alpha}{2 + \cos 2\alpha} \boldsymbol{\eta}. \tag{4.5}
 \end{aligned}$$

In fact, we first computed the metric and dilaton in (4.4) and the form potentials in (4.5) using the formulae (2.6). The field strengths in (4.4) were then computed from (4.5) using their type IIA definitions (see (A.4) of [44] for our conventions), and then double-checked with (2.9). In (4.5), as in (4.3), the constant $D = 4$ scalar a has been gauged away.

Recall from section 2.1 that the Fubini-Study metric $ds^2(\mathbb{CP}^2)$ that appears in (4.4) is normalised so that the Ricci tensor equals 6 times the metric, and $\mathbf{J}, \mathbf{\Omega}, \mathbf{\eta}$ are the forms that characterise the Sasaki-Einstein structure of a foliating S^5 inside S^6 . In particular, \mathbf{J} can be regarded as the Kähler form of \mathbb{CP}^2 . The Hopf fiber of the S^5 is stretched,⁴ and the vector ∂_ψ , dual to the one-form $\mathbf{\eta} = d\psi + \sigma$ along the fiber, is a U(1) Killing vector of the metric. The isometry is therefore $SU(3) \times U(1)$, and acts with cohomogeneity-one. However, the dependence of the supergravity forms on $\mathbf{\Omega}$ breaks the symmetry of the complete IIA solution to $SU(3)$, in agreement with the symmetry of the $D = 4$ critical point. The $\mathcal{N} = 1$ supersymmetry of the solution (4.4) is reflected by the presence of a global $SU(3) \times SU(3)$ -structure, or local $SU(2)$ -structure, of the type discussed in [34] (see also [35]). Since the solution only depends on the Sasaki-Einstein structure of S^5 , a more general class of $\mathcal{N} = 1$ solutions of massive type IIA can be obtained from (4.4) by replacing S^5 with an arbitrary Sasaki-Einstein five-manifold. The resulting solutions will have other topologies than S^6 and will typically become singular.

The $\mathcal{N} = 1$ G₂ critical point uplifts via (2.6), (2.9), or equivalently (3.1), to the solution

$$\begin{aligned}
 d\hat{s}_{10}^2 &= L^2 \left(ds^2(\text{AdS}_4) + \frac{12}{5} ds^2(S^6) \right), \quad e^{\hat{\phi}} = e^{\phi_0}, \quad L^{-1} e^{\frac{3}{4}\phi_0} \hat{F}_{(2)} = -\frac{\sqrt{3}}{\sqrt{5}} \mathcal{J}, \\
 L^{-3} e^{\frac{1}{4}\phi_0} \hat{F}_{(4)} &= \frac{3\sqrt{15}}{4} \text{vol}(\text{AdS}_4) + \frac{54}{25} \mathcal{J} \wedge \mathcal{J}, \quad L^{-2} e^{-\frac{1}{2}\phi_0} \hat{H}_{(3)} = -\frac{12\sqrt{3}}{5\sqrt{5}} \text{Re } \Omega, \quad (4.6)
 \end{aligned}$$

first found by Behrndt and Cvetic in [25] using G -structure techniques. We have defined the constants $L^2 \equiv 2^{-\frac{43}{12}} 3^{-\frac{5}{8}} 5^{\frac{11}{8}} g^{-\frac{25}{12}} m^{\frac{1}{12}}$ and $e^{\phi_0} \equiv 2^{-\frac{1}{6}} 15^{\frac{1}{4}} g^{\frac{5}{6}} m^{-\frac{5}{6}}$, so that $g = 2^{-\frac{9}{5}} 3^{-\frac{3}{10}} 5^{\frac{7}{10}} L^{-1} e^{-\frac{1}{20}\phi_0}$ and $\hat{F}_{(0)} \equiv m = \frac{5}{4} L^{-1} e^{-\frac{5}{4}\phi_0}$. The metric $ds^2(S^6)$ is the Einstein metric on the unit radius S^6 (normalised so that the Ricci tensor equals 5 times the metric), now regarded as the homogeneous space $G_2/SU(3)$. The real two-form \mathcal{J} and holomorphic (3,0)-form Ω are G₂-invariant, and endow S^6 with its natural homogeneous nearly-Kähler structure, see appendix A.1. As noted in [25], (4.1) can be promoted to a class of $\mathcal{N} = 1$ solutions by replacing S^6 with an arbitrary nearly-Kähler manifold.

4.2 Non-supersymmetric solutions

We now turn to discuss the uplift of the non-supersymmetric critical points of dyonic ISO(7) supergravity with at least $SU(3)$ symmetry. In this sector, $\mathcal{N} = 0$ points with residual $SO(7)_+, G_2$ and $SO(6)_+$ symmetry are known analytically. In addition, two points with $SU(3)$ symmetry are known numerically. We will focus on the analytic points –only a few comments about the numerical ones will be made at the end of this section.

⁴The fiber is stretched in the sense that, at the $\mathcal{N} = 1, SU(3)$ critical point, $X^{-1}\Delta_1\Delta_2^{-1} \geq 1$ for all α . In contrast, the fiber of the solution (12) of [43] is squashed since, at the $\mathcal{N} = 2, SU(3) \times U(1)$ critical point, $X^{-1}\Delta_1\Delta_2^{-1} \leq 1$ for all α .

The $SO(6)_+$ critical point gives rise to a new non-supersymmetric solution of massive type IIA:

$$d\hat{s}_{10}^2 = L^2(5 + 3 \cos 2\alpha)^{\frac{5}{8}} \left[ds^2(\text{AdS}_4) + d\alpha^2 + \frac{8 \sin^2 \alpha}{5 + 3 \cos 2\alpha} d\tilde{s}^2(S^5) \right], \quad (4.7)$$

$$e^{\hat{\phi}} = e^{\phi_0} (5 + 3 \cos 2\alpha)^{-\frac{1}{4}}, \quad \hat{F}_{(4)} = 8L^3 e^{-\frac{1}{4}\phi_0} \text{vol}(\text{AdS}_4), \quad \hat{H}_{(3)} = 0, \quad \hat{F}_{(2)} = 0.$$

Here, $d\tilde{s}^2(S^5)$ is the round, Einstein metric on the unit radius $S^5 = SO(6)/SO(5)$, normalised so that the Ricci tensor equals 4 times the metric. We have defined the constants $L^2 \equiv 2^{-\frac{7}{3}} g^{-\frac{25}{12}} m^{\frac{1}{12}}$ and $e^{\phi_0} \equiv 2^{\frac{4}{3}} g^{\frac{5}{6}} m^{-\frac{5}{6}}$, so that $g = 2^{-\frac{11}{10}} L^{-1} e^{-\frac{1}{20}\phi_0}$ and $\hat{F}_{(0)} \equiv m = \sqrt{2} L^{-1} e^{-\frac{5}{4}\phi_0}$. The group $SO(6)$ acts by isometries with cohomogeneity-one.

This solution is unstable, given that the spectrum [56] of the $SO(6)_+$ point of dyonic ISO(7) supergravity contains BF-bound-violating modes. This spectrum coincides, within their $\mathcal{N} = 8$ truncations, with that [57] of the $SU(4)_-$ point [58] of $D = 4$ $SO(8)$ -gauged supergravity [14]. The instability of the solution (4.7) is thus formally analogous to that [57] of the Pope-Warner solution [59] of $D = 11$ supergravity that arises from uplift on S^7 of the $SU(4)_-$ point. The actual details on how the instability affects the massive IIA $SO(6)_+$ solution (4.7) and the $D = 11$ Pope-Warner $SU(4)_-$ solution will differ, given the very different structure of the solutions. We also note that a wider class of non-supersymmetric solutions with topologies different from S^6 , and possibly singular, may be obtained from (4.7) simply by replacing the round S^5 with any Einstein five-manifold M_5 . The resulting symmetry will then be that of M_5 , and the stability issue would need to be readdressed on a case-by-case basis. Little hope should be harboured about the solution (4.7) becoming stable for other Einstein manifolds M_5 : the instabilities were found to persist [60] for similar generalisations of the Pope-Warner solution.

The solutions corresponding to the uplift of the $\mathcal{N} = 0$ $SO(7)_+$ and G_2 points were previously known, although the stability comments we present here are new. The $SO(7)_+$ critical point gives rise to a homogeneous, Freund-Rubin type solution first found by Romans [24]. In our conventions, this solution reads

$$d\hat{s}_{10}^2 = L^2 \left(ds^2(\text{AdS}_4) + \frac{5}{2} ds^2(S^6) \right), \quad e^{\hat{\phi}} = e^{\phi_0},$$

$$\hat{F}_{(4)} = \sqrt{10} L^3 e^{-\frac{1}{4}\phi_0} \text{vol}(\text{AdS}_4), \quad \hat{H}_{(3)} = 0, \quad \hat{F}_{(2)} = 0. \quad (4.8)$$

Here, $ds^2(S^6)$ is the round, Einstein metric on the unit radius $S^6 = SO(7)/SO(6)$, with normalisation such that the Ricci tensor equals 5 times the metric. We have defined the constants $L^2 \equiv 2 \cdot 5^{-\frac{25}{24}} g^{-\frac{25}{12}} m^{\frac{1}{12}}$ and $e^{\phi_0} \equiv 5^{\frac{5}{12}} g^{\frac{5}{6}} m^{-\frac{5}{6}}$, so that $g = \sqrt{\frac{2}{5}} L^{-1} e^{-\frac{1}{20}\phi_0}$ and $\hat{F}_{(0)} \equiv m = \sqrt{2} L^{-1} e^{-\frac{5}{4}\phi_0}$.

The spectrum [56] of the $SO(7)_+$ point of dyonic ISO(7) supergravity contains modes below the BF bound. The solution (4.8) is, thus, unstable. The $SO(7)_+$ point has the same spectrum, within their respective $\mathcal{N} = 8$ theories, as the $SO(7)_\pm$ points of $SO(8)$ supergravity [14]. In this sense, the instability of the $SO(7)_+$ massive IIA solution (4.8) is qualitatively similar to the instability of the de Wit-Nicolai [16] and Englert [61] solutions

of $D = 11$ supergravity, that respectively arise from uplift on S^7 of the $\text{SO}(7)_+$ and $\text{SO}(7)_-$ critical points of the $D = 4$ $\text{SO}(8)$ gauging. The precise details about how the instability affects these $\text{SO}(7)$ -invariant solutions in $D = 10$ and $D = 11$ may again differ. The unstable modes lie in the **27** of $\text{SO}(7)$. Since this representation is irreducible under G_2 (in particular, there is no branching into singlets), the unstable modes are truncated out from the G_2 -invariant, universal nearly-Kähler truncation of [62, 63]. This is analogous to the observation [57] that the universal M-theory truncation on (skew-whiffed) Sasaki-Einstein manifolds [64] is also blind to the instability of the Pope-Warner solution [59]. See [65] for the spectrum of this solution within the Sasaki-Einstein truncation of [64]. In this $D = 11$ case, the instability is already visible in larger left-invariant truncations of M-theory on S^7 [66, 67]. The latter truncations, however, do not have a massive type IIA on S^6 analog. A wider class of non-supersymmetric solutions can be obtained from (4.8) if S^6 is replaced with any six-dimensional Einstein space M_6 , as noted in [24]. The symmetry will now be that of M_6 , and the stability issue should in principle be readdressed on a case-by-case basis.

The $\mathcal{N} = 0$ G_2 critical point gives rise to a homogeneous solution first found in [36] using G -structure techniques. In our conventions, it reads

$$d\hat{s}_{10}^2 = L^2 \left(ds^2(\text{AdS}_4) + 2 ds^2(S^6) \right), \quad e^{\hat{\phi}} = e^{\phi_0}, \quad L^{-1} e^{\frac{3}{4}\phi_0} \hat{F}_{(2)} = \sqrt{2} \mathcal{J},$$

$$L^{-3} e^{\frac{1}{4}\phi_0} \hat{F}_{(4)} = \frac{3}{\sqrt{2}} \text{vol}(\text{AdS}_4) - \sqrt{6} \mathcal{J} \wedge \mathcal{J}, \quad L^{-2} e^{-\frac{1}{2}\phi_0} \hat{H}_{(3)} = 2\sqrt{3} \text{Re} \Omega, \quad (4.9)$$

where $L^2 \equiv 2^{-\frac{11}{6}} 3^{\frac{3}{8}} g^{-\frac{25}{12}} m^{\frac{1}{12}}$ and $e^{\phi_0} \equiv 2^{\frac{1}{3}} 3^{\frac{1}{4}} g^{\frac{5}{6}} m^{-\frac{5}{6}}$, so that $g = 2^{-\frac{9}{10}} 3^{\frac{1}{5}} L^{-1} e^{-\frac{1}{20}\phi_0}$ and $\hat{F}_{(0)} \equiv m = \sqrt{\frac{3}{2}} L^{-1} e^{-\frac{5}{4}\phi_0}$. The rest of the symbols are like in the supersymmetric solution (4.1). A new observation about this solution is that it is perturbatively stable, at least against $D = 4$ perturbations contained either in the $\mathcal{N} = 8$ $\text{ISO}(7)$ dyonic theory or in the G_2 -invariant, nearly-Kähler truncation [62, 63]. See [68] for the spectrum of the $\mathcal{N} = 0$, G_2 -invariant point within (the ‘massless mode’) $D = 4$ $\mathcal{N} = 8$ dyonic $\text{ISO}(7)$ supergravity and [45] for the spectrum within the ‘massive mode’ nearly-Kähler truncation [62, 63]. The solution (4.2) can be generalised by replacing S^6 with an arbitrary nearly-Kähler manifold. The stability of this more general class of solutions should be readdressed. The $\mathcal{N} = 0$ G_2 critical point of dyonic $\text{ISO}(7)$ supergravity does not have a counterpart in the purely electric $\text{SO}(8)$ gauging [14]. It does, however, in the dyonic $\text{SO}(8)$ gauging [46], but the latter does not uplift to $D = 11$ [10, 69]. For this reason, the G_2 -invariant massive IIA solution (4.2) does not have a $D = 11$ companion solution with the same symmetry, unlike (4.7), (4.8).

We conclude by noting that the formulae (2.6), (2.9) can be also used to uplift the two non-supersymmetric critical points with $\text{SU}(3)$ symmetry, using the numerical values given in table 3 of [45]. The resulting massive IIA configurations are qualitatively similar, though $\mathcal{N} = 0$, to the solution (4.4). By the spectrum analysis of [45], these two solutions can be declared to be stable under perturbations contained in $D = 4$ $\mathcal{N} = 8$ dyonic $\text{ISO}(7)$ supergravity.

4.3 Flux quantisation and free energies

As we will now show, the classical solutions of the previous subsections survive flux quantisation. Thus, the perturbatively stable ones extend to well defined massive type IIA string theory backgrounds. We will restrict our attention to the topologically S^6 solutions, and will not consider the quantisation of the more general classes of solutions that we discussed. Flux quantisation for the Behrndt-Cvetic [25] solution (4.1) has already been studied in [63]. See also [70] for general considerations about flux quantisation of AdS_4 solutions in massive type IIA.

The classical solutions presented in sections 4.1 and 4.2 depend on the gauge coupling constants g and m of the $D = 4$ supergravity. We have given this dependence in terms of two other constants, L and ϕ_0 ,

$$L^2 \propto g^{-\frac{25}{12}} m^{\frac{1}{12}}, \quad e^{\phi_0} \propto g^{\frac{5}{6}} m^{-\frac{5}{6}}, \quad (4.10)$$

more amenable to $D = 10$ interpretation: L sets the scale of the AdS_4 and S^6 radii and e^{ϕ_0} is proportional to the string coupling constant g_s . The two proportionality coefficients in (4.10) are dimensionless, order one and solution-dependent –we have given them below each solution. Classically, L^2 and e^{ϕ_0} can be rescaled to one without loss of generality using global symmetries of type IIA supergravity, see e.g. (A.7), (A.9) of [44]. At the quantum level, however, L and e^{ϕ_0} become fixed in terms of the integral Page charges k and N associated to $\hat{F}_{(0)}$ and $\hat{F}_{(6)}$,

$$k = 2\pi\ell_s \hat{F}_{(0)} \equiv 2\pi\ell_s m, \quad N = -\frac{1}{(2\pi\ell_s)^5} \int_{S^6} e^{\frac{1}{2}\hat{\phi}} \hat{*}\hat{F}_{(4)} + \hat{B}_{(2)} \wedge d\hat{A}_{(3)} + \frac{1}{6}m \hat{B}_{(2)} \wedge \hat{B}_{(2)} \wedge \hat{B}_{(2)}, \quad (4.11)$$

where $\ell_s = \sqrt{\alpha'}$ is the string length. Note that d of the integrand in the $\hat{F}_{(6)}$ quantisation condition gives the left-hand-side of the $\hat{F}_{(4)}$ equation of motion, see (A.5) of [44]. From a ten-dimensional perspective, the integers k and N respectively count the quanta of Romans mass and the number of D2-branes.

Working out the integral in (4.11) for each individual solution, we can solve for L and e^{ϕ_0} in terms of k and N on a case-by-case basis as

$$L \propto N^{\frac{5}{24}} k^{\frac{1}{24}} \ell_s, \quad e^{\phi_0} \propto N^{-\frac{1}{6}} k^{-\frac{5}{6}}, \quad (4.12)$$

where the proportionality coefficients are again dimensionless, solution-dependent and order one. More concretely:

$$\begin{aligned} \mathcal{N} = 2, \text{U}(3) & : L = 2^{-\frac{7}{48}} 3^{-\frac{7}{24}} \pi^{\frac{3}{8}} N^{\frac{5}{24}} k^{\frac{1}{24}} \ell_s, & e^{\phi_0} = 2^{\frac{11}{12}} 3^{-\frac{1}{6}} \pi^{\frac{1}{2}} N^{-\frac{1}{6}} k^{-\frac{5}{6}}, \\ \mathcal{N} = 1, \text{G}_2 & : L = 2^{-\frac{13}{8}} 3^{-\frac{5}{48}} 5^{\frac{11}{16}} \pi^{\frac{3}{8}} N^{\frac{5}{24}} k^{\frac{1}{24}} \ell_s, & e^{\phi_0} = 2^{\frac{1}{2}} 3^{\frac{1}{12}} 5^{\frac{1}{4}} \pi^{\frac{1}{2}} N^{-\frac{1}{6}} k^{-\frac{5}{6}}, \\ \mathcal{N} = 1, \text{SU}(3) & : L = 2^{-\frac{19}{12}} 3^{-\frac{7}{24}} 5^{\frac{11}{16}} \pi^{\frac{3}{8}} N^{\frac{5}{24}} k^{\frac{1}{24}} \ell_s, & e^{\phi_0} = 2^{\frac{5}{3}} 3^{-\frac{1}{6}} 5^{\frac{1}{4}} \pi^{\frac{1}{2}} N^{-\frac{1}{6}} k^{-\frac{5}{6}}, \\ \mathcal{N} = 0, \text{SO}(7)_+ & : L = 2^{\frac{2}{3}} 3^{\frac{5}{24}} 5^{-\frac{25}{48}} \pi^{\frac{3}{8}} N^{\frac{5}{24}} k^{\frac{1}{24}} \ell_s, & e^{\phi_0} = 2^{\frac{2}{3}} 3^{-\frac{1}{6}} 5^{\frac{5}{12}} \pi^{\frac{1}{2}} N^{-\frac{1}{6}} k^{-\frac{5}{6}}, \\ \mathcal{N} = 0, \text{SO}(6)_+ & : L = 2^{-1} 3^{\frac{5}{24}} \pi^{\frac{3}{8}} N^{\frac{5}{24}} k^{\frac{1}{24}} \ell_s, & e^{\phi_0} = 2^2 3^{-\frac{1}{6}} \pi^{\frac{1}{2}} N^{-\frac{1}{6}} k^{-\frac{5}{6}}, \\ \mathcal{N} = 0, \text{G}_2 & : L = 2^{-\frac{3}{4}} 3^{\frac{19}{48}} \pi^{\frac{3}{8}} N^{\frac{5}{24}} k^{\frac{1}{24}} \ell_s, & e^{\phi_0} = 2 \cdot 3^{\frac{1}{12}} \pi^{\frac{1}{2}} N^{-\frac{1}{6}} k^{-\frac{5}{6}}. \end{aligned} \quad (4.13)$$

In our topologically S^6 solutions, the only flux quantisation conditions that can be imposed are (4.11) and thus the classical constants L and e^{ϕ_0} in each solution become univocally fixed in terms of the quantum fluxes through (4.13). Although we have carried out this calculation for all solutions, stable or unstable, the flux quantisation analysis is only sensible in the former case. Thus, the perturbatively stable solutions (in particular, the supersymmetric solutions) can be promoted to solutions at the quantum level. They provide good massive type IIA string theory backgrounds.

This case-by-case analysis can also be used to solve for the classical constants g and m of the $D = 4$ supergravity in terms of the quantum numbers N and k . We find⁵

$$g = 2^{\frac{4}{5}} 3^{-\frac{1}{5}} \pi^{\frac{3}{5}} (2\pi\ell_s)^{-1} N^{-\frac{1}{5}}, \quad m = (2\pi\ell_s)^{-1} k, \quad (4.14)$$

with the same proportionality coefficients for all solutions, as it of course must be given that g , m and N , k characterise the theory, not only the individual solutions. The second relation in (4.14) is a mere rearrangement of the first equation in (4.11). However, the fact that we do get the same expression for g in terms of N for all solutions provides a non-trivial crosscheck on our calculations. One might have naively thought that the dimensionless electric and magnetic couplings, $g' = 2\pi\ell_s g$ and $m' \equiv k = 2\pi\ell_s m$, of the $D = 4$ supergravity should obey the Dirac quantisation condition, $g'm' = 2\pi n$, for some integer n . It is evident from (4.14) that this is not true. In retrospect, the IIA embedding of the $D = 4$ supergravity makes it clear why this relation will not hold: Dirac quantisation conditions must hold for Poincaré-dual Ramond-Ramond fluxes $\hat{F}_{(p+2)}$ and $\hat{F}_{(8-p)}$ [71]; but $\hat{F}_{(0)}$ and $\hat{F}_{(6)}$, to which m and g are respectively related, are not dual to each other. A purely four-dimensional argument can be also put forward: the $D = 4$ theory can be always formulated in a symplectic duality frame where all charges are electric and there are no magnetic charges.

It was argued in [38] that, in contrast to the celebrated massless case, massive type IIA string theory cannot reach a strong coupling regime. This phenomenon was illustrated with some of the (analytically or numerically) known $\text{AdS}_4 \times \mathbb{CP}^3$ backgrounds. It was found that the string frame curvature radius in string units, L_{string} , and the string coupling $g_s \sim e^{\phi_0}$ both stay bounded for all values of the fluxes on those backgrounds. The string coupling g_s was shown to initially increase with N , as in the massless IIA $\mathcal{N} = 6$ solution (corresponding to the IIA reduction of the $D = 11$ Freund-Rubin solution [4]), but then reach a second phase where g_s decreases as L_{string} becomes large for N large compared to k : $L_{\text{string}} \propto N^{\frac{1}{6}} k^{-\frac{1}{6}} \ell_s$ and $e^{\phi_0} \propto N^{-\frac{1}{6}} k^{-\frac{5}{6}}$. All our $\text{AdS}_4 \times S^6$ solutions behave as in this second phase for all values of the fluxes, as can be seen by translating the individual values of the Einstein frame L in (4.13) into the string frame via $L_{\text{string}} = L e^{\frac{1}{4}\phi_0}$. Note also that k here corresponds to n_0 in [38].

We conclude with the calculation of the gravitational free energy of these solutions. The gravitational free energy F of a unit radius AdS_4 solution is given by $F = \pi/(2G_4)$ [72], where G_4 is the effective four-dimensional Newton's constant. Plugging the solutions into

⁵Perhaps a more suggestive rewrite for g is $g^5 = 5 v(S^6) (2\pi\ell_s)^{-5} N^{-1}$, with $v(S^6) = 16\pi^3/15$ the volume of the unit radius round six-sphere.

the ten-dimensional action (see (A.1) of [44] for our conventions), we obtain

$$F = \frac{16\pi^3}{(2\pi\ell_s)^8} \int_{S^6} e^{8A} \text{vol}_6, \tag{4.15}$$

where e^{2A} is the warp factor (4.2) and vol_6 is the volume form corresponding to the internal metric ds_6^2 in (4.1). We have evaluated this integral for each solution, and have traded the resulting L^8 dependence for the quantum numbers N and k via (4.13). The results appear in table 1 of the introduction. For completeness, the table also shows the free energy of the $\mathcal{N} = 2$ solution, which was already given in [43]. These free energies turn out to be inversely proportional to the four-dimensional cosmological constant at the corresponding $D = 4$ critical point. Indeed, for any two free energies F_1 and F_2 in table 1, and any $D = 4$ cosmological constants V_1, V_2 in table 3 of [45], it is straightforward to verify that

$$\frac{F_1}{F_2} = \frac{V_2}{V_1}. \tag{4.16}$$

The AdS_4 solutions contained in this paper should be dual to Chern-Simons-matter theories with a single gauge group $\text{SU}(N)$ at level k of the type considered in [73]. A match of gravitational and field theory free energies was given for the $\mathcal{N} = 2$ solution in [43]. It would be very interesting to understand better the holography of the $\mathcal{N} = 1$ solutions.

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A Foliation of S^6 with Sasaki-Einstein leaves

Here we describe how the natural embedding of S^6 in \mathbb{R}^7 determines the foliation of the former with S^5 leaves, the latter equipped with their natural Sasaki-Einstein structure. Let $\mu^I, I = 1, \dots, 7$, parametrise S^6 as the locus $\delta_{IJ}\mu^I\mu^J = 1$ in \mathbb{R}^7 , and let $y^m, m = 1, \dots, 6$, be the S^6 angles. It is convenient to split the index $I = (i, 7), i = 1, \dots, 6$, and the μ^I as

$$\mu^i = \sin \alpha \tilde{\mu}^i, \quad i = 1, \dots, 6, \quad \mu^7 = \cos \alpha, \tag{A.1}$$

where

$$0 \leq \alpha \leq \pi \tag{A.2}$$

is one of the S^6 angles y^m , and $\tilde{\mu}^i$ characterise the S^5 within S^6 as the locus $\delta_{ij}\tilde{\mu}^i\tilde{\mu}^j = 1$ in $\mathbb{R}^6 \subset \mathbb{R}^7$. In terms of (A.1), the round metric on S^6 with radius g , given in e.g. (E.2) of [44] with $n = 6$, acquires the familiar sine-cone form

$$d\tilde{s}^2(S^6) = g^{-2} \left(d\alpha^2 + \sin^2 \alpha d\tilde{s}^2(S^5) \right), \tag{A.3}$$

where $d\tilde{s}^2(S^5)$ is the round metric (E.2) of [44] with $n = 5$ and $g = 1$. Denoting by $\tilde{m} = 1, \dots, 5$ the directions along the S^5 , the Killing vectors $K^{mIJ} = 2g^{-2}\tilde{g}^{mn}\mu^{[I}\partial_n\mu^{J]}$ of the round metric (A.3) are

$$K_{ij}^\alpha = 0, \quad K_{ij}^{\tilde{m}} = \tilde{K}_{ij}^{\tilde{m}}, \quad K_{i7}^\alpha = -\tilde{\mu}_i, \quad K_{i7}^{\tilde{m}} = -\cot\alpha\tilde{g}^{mn}\partial_n\tilde{\mu}_i, \quad (\text{A.4})$$

and the tensors $K_{mn}^{IJ} = 4g^{-2}\partial_{[m}\mu^J\partial_n]\mu^I$ are

$$\begin{aligned} K_{\alpha\tilde{m}}^{ij} &= 4g^{-2}\sin\alpha\cos\alpha\tilde{\mu}^{[i}\partial_{\tilde{m}}\tilde{\mu}^{j]}, & K_{\tilde{m}\tilde{n}}^{ij} &= g^{-2}\sin^2\alpha\tilde{K}_{\tilde{m}\tilde{n}}^{ij}, \\ K_{\alpha\tilde{m}}^{i7} &= 2g^{-2}\sin^2\alpha\partial_{\tilde{m}}\tilde{\mu}^i, & K_{\tilde{m}\tilde{n}}^{i7} &= 0. \end{aligned} \quad (\text{A.5})$$

In the above expressions, $\partial_m\mu^I$ means derivative of $\mu^I = \mu^I(y^m)$ with respect to y^m .

Let us now consider the canonical Sasaki-Einstein structure on S^5 . Recall, more generally, that a Sasaki-Einstein structure on a five-manifold M_5 is an $SU(2)$ -structure, therefore determined by a real $(1,1)$ -form \mathbf{J} , a complex decomposable $(2,0)$ -form $\mathbf{\Omega}$, and a real one-form $\boldsymbol{\eta}$. These forms are subject to the following algebraic

$$\iota_{\partial_\psi}\mathbf{J} = \iota_{\partial_\psi}\mathbf{\Omega} = 0, \quad \mathbf{\Omega} \wedge \bar{\mathbf{\Omega}} = 2\mathbf{J} \wedge \mathbf{J} \neq 0, \quad \mathbf{J} \wedge \mathbf{\Omega} = 0, \quad (\text{A.6})$$

and differential relations

$$d\boldsymbol{\eta} = 2\mathbf{J}, \quad d\mathbf{J} = 0, \quad d\mathbf{\Omega} = 3i\mathbf{\Omega} \wedge \boldsymbol{\eta}. \quad (\text{A.7})$$

In (A.6), we have introduced a $U(1)$ angle ψ so that the vector ∂_ψ is dual to $\boldsymbol{\eta}$. We can locally write $\boldsymbol{\eta} = d\psi + \sigma$, where σ is a one-form potential for \mathbf{J} , with $d\sigma = 2\mathbf{J}$. This $SU(2)$ -structure specifies an Einstein metric $ds^2(M_5)$, normalised so that the Ricci tensor is 4 times the metric. Locally, this metric can be written as

$$ds^2(M_5) = ds^2(\text{KE}_4) + \boldsymbol{\eta}^2, \quad (\text{A.8})$$

where $ds^2(\text{KE}_4)$ is a metric defined, at least locally, on a positively curved Kähler-Einstein manifold or orbifold, normalised so that the Ricci tensor is 6 times the metric. The vector ∂_ψ is a Killing vector of (A.8). The cone $C(M_5) = \mathbb{R}^+ \times M_5$ over M_5 has $SU(3)$ holonomy. The $(1,1)$ -form J and $(3,0)$ -form Ω that the latter condition defines on $C(M_5)$ can be written as

$$J = r^2\mathbf{J} + r dr \wedge \boldsymbol{\eta}, \quad \Omega = r^2(dr + i r \boldsymbol{\eta}) \wedge \mathbf{\Omega}, \quad (\text{A.9})$$

where r is a coordinate on the \mathbb{R}^+ factor. The closure of J and Ω , required by $SU(3)$ -holonomy, is equivalent to the Sasaki-Einstein torsion conditions (A.7).

For the specific case of S^5 , we have that $\text{KE}_4 = \mathbb{C}\mathbb{P}^2$ and $C(S^5) = \mathbb{R}^6$. Following a similar strategy described in appendix E of [44], the Sasaki-Einstein forms $\boldsymbol{\eta}$, \mathbf{J} , $\mathbf{\Omega}$ on S^5 can be written in terms of the embedding coordinates $\tilde{\mu}^i$ and the components J_{ij} , Ω_{ijk} of the $SU(3)$ -holonomy forms on \mathbb{R}^6 as

$$\boldsymbol{\eta} = J_{ij}\tilde{\mu}^i d\tilde{\mu}^j, \quad \mathbf{J} = \frac{1}{2}J_{ij}d\tilde{\mu}^i \wedge d\tilde{\mu}^j, \quad \mathbf{\Omega} = \frac{1}{2}\Omega_{ijk}\tilde{\mu}^i d\tilde{\mu}^j \wedge d\tilde{\mu}^k. \quad (\text{A.10})$$

These expressions are crucial to bring the consistent truncation formulae of type IIA down to the SU(3)-invariant sector of ISO(7) supergravity to the form (2.6).

We conclude with the relation of the homogeneous nearly-Kähler structure \mathcal{J} , Ω on S^6 and the canonical Sasaki-Einstein structure of its foliating S^5 . Introducing a frame e^I , $I = 1, \dots, 7$ on \mathbb{R}^7 , and splitting $\mathbb{R}^7 = \mathbb{R}^6 \times \mathbb{R}$, the associative and coassociative forms ψ , $\tilde{\psi}$ on \mathbb{R}^7 can be written in terms of the SU(3)-holonomy forms J , Ω on the \mathbb{R}^6 factor as

$$\psi = J \wedge e^7 + \text{Re } \Omega, \quad \tilde{\psi} = \frac{1}{2} J \wedge J + \text{Im } \Omega \wedge e^7. \quad (\text{A.11})$$

From here, the following non-vanishing components can be read off:

$$\psi_{ijk} = (\text{Re } \Omega)_{ijk}, \quad \psi_{ij7} = J_{ij}, \quad \tilde{\psi}_{ijkl} = 3J_{[ij}J_{k]l}, \quad \tilde{\psi}_{ijk7} = (\text{Im } \Omega)_{ijk}. \quad (\text{A.12})$$

Finally, introducing (A.12) and (A.1) into

$$\mathcal{J} = \frac{1}{2} \psi_{IJK} \mu^I d\mu^J \wedge d\mu^K, \quad \Omega = \frac{1}{6} \left(\psi_{JKL} - i \tilde{\psi}_{IJKL} \mu^I \right) d\mu^J \wedge d\mu^K \wedge d\mu^L \quad (\text{A.13})$$

(see appendix E of [44]), the homogeneous nearly-Kähler forms on S^6 can be written in terms of the Sasaki-Einstein forms on the foliating S^5 as

$$\begin{aligned} \mathcal{J} &= \sin^2 \alpha \cos \alpha \mathbf{J} + \sin^3 \alpha \text{Re } \Omega + \sin \alpha d\alpha \wedge \boldsymbol{\eta}, \\ \text{Re } \Omega &= -\sin^3 \alpha \mathbf{J} \wedge d\alpha + \sin^2 \alpha \cos \alpha \text{Re } \Omega \wedge d\alpha - \sin^3 \alpha \text{Im } \Omega \wedge \boldsymbol{\eta}, \\ \text{Im } \Omega &= -\sin^4 \alpha \mathbf{J} \wedge \boldsymbol{\eta} + \sin^3 \alpha \cos \alpha \text{Re } \Omega \wedge \boldsymbol{\eta} + \sin^2 \alpha \text{Im } \Omega \wedge d\alpha. \end{aligned} \quad (\text{A.14})$$

Using the Sasaki-Einstein conditions (A.6), (A.7) for \mathbf{J} , Ω , $\boldsymbol{\eta}$, the forms \mathcal{J} , Ω in (A.14) can indeed be doublechecked to satisfy the nearly-Kähler relations

$$\Omega \wedge \bar{\Omega} = -\frac{4i}{3} \mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J} \neq 0, \quad \mathcal{J} \wedge \Omega = 0, \quad (\text{A.15})$$

and

$$d\mathcal{J} = 3 \text{Re } \Omega, \quad d \text{Im } \Omega = -2 \mathcal{J} \wedge \mathcal{J}. \quad (\text{A.16})$$

B SU(3)-invariant AdS₄ solutions of massive type IIA

The solutions we have presented in the main text are of the local form⁶

$$\begin{aligned} d\hat{s}_{10}^2 &= e^{2X(\alpha)} ds^2(\text{AdS}_4) + e^{2A(\alpha)} d\alpha^2 + e^{2B(\alpha)} ds^2(\text{KE}_4) + e^{2C(\alpha)} \boldsymbol{\eta}^2, \quad \hat{\phi} = \phi(\alpha), \\ \hat{F}_{(4)} &= \mu_0 \text{vol}_4 + A_4(\alpha) \text{vol}(\text{KE}_4) + B_4(\alpha) \mathbf{J} \wedge d\alpha \wedge \boldsymbol{\eta} \\ &\quad + C_4(\alpha) \text{Re } \Omega \wedge d\alpha \wedge \boldsymbol{\eta} + D_4(\alpha) \text{Im } \Omega \wedge d\alpha \wedge \boldsymbol{\eta}, \end{aligned}$$

⁶The warp factor and the Freund-Rubin constant were denoted in the main text by e^{2A} and U_{rescaled} : see (4.2). In this appendix, these quantities are denoted by e^{2X} and μ_0 . The function X should not cause any confusion with X defined in the main text, in (2.3). Also in this appendix, e^{2A} is used to denote a metric function different than the warp factor. This should not cause any confusion either. Finally, note that the G₂-invariant solutions (4.1), (4.2) can also be brought to the local form (B.1) through the identifications (A.14).

$$\begin{aligned}
\hat{H}_{(3)} &= B_3(\alpha) \mathbf{J} \wedge d\alpha + C_3(\alpha) \operatorname{Re} \boldsymbol{\Omega} \wedge d\alpha + D_3(\alpha) \operatorname{Im} \boldsymbol{\Omega} \wedge d\alpha \\
&\quad + E_3(\alpha) \operatorname{Re} \boldsymbol{\Omega} \wedge \boldsymbol{\eta} + F_3(\alpha) \operatorname{Im} \boldsymbol{\Omega} \wedge \boldsymbol{\eta} , \\
\hat{F}_{(2)} &= A_2(\alpha) \mathbf{J} + B_2(\alpha) d\alpha \wedge \boldsymbol{\eta} + C_2(\alpha) \operatorname{Re} \boldsymbol{\Omega} + D_2(\alpha) \operatorname{Im} \boldsymbol{\Omega} ,
\end{aligned} \tag{B.1}$$

where AdS_4 is unit radius, μ_0 is a constant, $X(\alpha)$, etc., are functions of the angle α and \mathbf{J} , $\boldsymbol{\Omega}$, $\boldsymbol{\eta}$ are the usual forms on a Sasaki-Einstein five-manifold M_5 (see appendix A) with metric $ds^2(\operatorname{KE}_4)$ on the local Kähler-Einstein base normalised so that the Ricci tensor equals 6 times the metric. When $M_5 = S^5$ so that $\operatorname{KE}_4 = \mathbb{C}\mathbb{P}^2$, (B.1) is the most general AdS_4 configuration of type IIA supergravity with $\operatorname{SU}(3)$ symmetry. In this appendix we work out the differential and algebraic equations that the functions $X(\alpha)$, etc., must obey for (B.1) to solve the Bianchi identities and equations of motion of massive type IIA supergravity. We have used the equations in this appendix to verify that the solutions presented in the main text, that we obtained by using the uplifting formulae (2.6), (2.9), do indeed solve the type IIA field equations.

In order to write the Einstein equation below, we find it convenient to introduce the local internal metric

$$ds_6^2 = e^{2X(\alpha)} \left(e^{2A(\alpha)} d\alpha^2 + e^{2B(\alpha)} ds^2(\operatorname{KE}_4) + e^{2C(\alpha)} \boldsymbol{\eta}^2 \right) , \tag{B.2}$$

which is conformal to the internal metric that appears in (B.1). The Ricci tensor of the metric (B.2) with curved indices reads

$$\begin{aligned}
R_{\alpha\alpha} &= -(4X - A + 4B + C)'' - e^{-4X+A-4B-C} \left(e^{4X-A+4B+C} (X+A)' \right)' \\
&\quad + (X+A)'(9X - A + 8B + 2C)' - 4((X+B)')^2 - ((X+C)')^2 , \\
R_{mn} &= - \left[e^{-4X-A-2B-C} \left(e^{4X-A+4B+C} (X+B)' \right)' + 2e^{-2B+2C} - 6 \right] g_{mn} , \\
R_{\psi\psi} &= -e^{-4X-A-4B+C} \left(e^{4X-A+4B+C} (X+C)' \right)' + 4e^{-4B+4C} ,
\end{aligned} \tag{B.3}$$

with g_{mn} the components of $ds^2(\operatorname{KE}_4)$, and $R_{\alpha m} = R_{\alpha\psi} = R_{m\psi} = 0$. Here and in the following, we drop the explicit α dependence and denote with a prime the derivative with respect to it.

The IIA Bianchi identities (see (A.3) of [44]) impose the restrictions

$$\begin{aligned}
A_4' - 4B_4 - 2A_2B_3 - 2C_2C_3 - 3D_2D_3 &= 0 , \\
C_2E_3 + D_2F_3 &= 0 , \\
E_3' - 3D_3 &= 0 , \\
F_3' + 3C_3 &= 0 , \\
A_2' - 2B_2 - mB_3 &= 0 , \\
C_2' - mC_3 &= 0 , \\
D_2' - mD_3 &= 0 ,
\end{aligned}$$

$$\begin{aligned}
 3D_2 - mE_3 &= 0, \\
 3C_2 + mF_3 &= 0.
 \end{aligned}
 \tag{B.4}$$

Turning now to the equations of motion (see (A.5) of [44]), we find that the $\hat{F}_{(4)}$ equation of motion gives

$$\begin{aligned}
 (e^{\frac{1}{2}\phi+4X-A-C} B_4)' - 2e^{\frac{1}{2}\phi+4X+A-4B+C} A_4 + \mu_0 B_3 &= 0, \\
 (e^{\frac{1}{2}\phi+4X-A-C} C_4)' + \mu_0 C_3 &= 0, \\
 (e^{\frac{1}{2}\phi+4X-A-C} D_4)' + \mu_0 D_3 &= 0, \\
 3e^{\frac{1}{2}\phi+4X-A-C} D_4 + \mu_0 E_3 &= 0, \\
 3e^{\frac{1}{2}\phi+4X-A-C} C_4 - \mu_0 F_3 &= 0,
 \end{aligned}
 \tag{B.5}$$

the $\hat{H}_{(3)}$ equation of motion gives

$$\begin{aligned}
 (e^{-\phi+4X-A+C} B_3)' - e^{\frac{1}{2}\phi+4X+A-4B+C} A_2 A_4 - e^{\frac{1}{2}\phi+4X-A-C} B_2 B_4 \\
 - m e^{\frac{3}{2}\phi+4X+A+C} A_2 - \mu_0 B_4 &= 0, \\
 (e^{-\phi+4X-A+C} C_3)' - e^{\frac{1}{2}\phi+4X+A-4B+C} C_2 A_4 - e^{\frac{1}{2}\phi+4X-A-C} B_2 C_4 + 3e^{-\phi+4X+A-C} F_3 \\
 - m e^{\frac{3}{2}\phi+4X+A+C} C_2 - \mu_0 C_4 &= 0, \\
 (e^{-\phi+4X-A+C} D_3)' - e^{\frac{1}{2}\phi+4X+A-4B+C} D_2 A_4 - e^{\frac{1}{2}\phi+4X-A-C} B_2 D_4 - 3e^{-\phi+4X+A-C} E_3 \\
 - m e^{\frac{3}{2}\phi+4X+A+C} D_2 - \mu_0 D_4 &= 0, \\
 e^{-\phi+4X-A+C} B_3 - \frac{1}{2} e^{\frac{1}{2}\phi+4X-A-C} (A_2 B_4 + C_2 C_4 + D_2 D_4) \\
 - \frac{1}{4} m e^{\frac{3}{2}\phi+4X-A+4B-C} B_2 - \frac{1}{4} \mu_0 A_4 &= 0,
 \end{aligned}
 \tag{B.6}$$

the $\hat{F}_{(2)}$ equation of motion gives

$$\begin{aligned}
 (e^{\frac{3}{2}\phi+4X-A+4B-C} B_2)' - 4e^{\frac{3}{2}\phi+4X+A+C} A_2 \\
 + 2e^{\frac{1}{2}\phi+4X-A-C} (B_3 B_4 + C_3 C_4 + D_3 D_4) &= 0, \\
 E_3 C_4 + F_3 D_4 &= 0,
 \end{aligned}
 \tag{B.7}$$

and the dilaton equation of motion gives

$$\begin{aligned}
 (e^{4X-A+4B+C} \phi')' - \frac{3}{2} e^{\frac{3}{2}\phi+4X+A+C} (A_2^2 + C_2^2 + D_2^2) - \frac{3}{4} e^{\frac{3}{2}\phi+4X-A+4B-C} B_2^2 \\
 + e^{-\phi+4X-A+C} (B_3^2 + C_3^2 + D_3^2) + e^{-\phi+4X+A-C} (E_3^2 + F_3^2) \\
 - \frac{1}{4} e^{\frac{1}{2}\phi+4X+A-4B+C} A_4^2 - \frac{1}{2} e^{\frac{1}{2}\phi+4X-A-C} (B_4^2 + C_4^2 + D_4^2) \\
 - \frac{5}{4} m^2 e^{\frac{5}{2}\phi+4X+A+4B+C} + \frac{1}{4} \mu_0^2 e^{\frac{1}{2}\phi-4X+A+4B+C} &= 0.
 \end{aligned}
 \tag{B.8}$$

Finally, the external components of the $D = 10$ Einstein equation produce an equation for the warp factor,

$$\begin{aligned}
& (e^{4X-A+4B+C} X')' - \frac{1}{8} e^{\frac{3}{2}\phi+4X+A+C} (A_2^2 + C_2^2 + D_2^2) - \frac{1}{16} e^{\frac{3}{2}\phi+4X-A+4B-C} B_2^2 \\
& - \frac{1}{4} e^{-\phi+4X-A+C} (B_3^2 + C_3^2 + D_3^2) - \frac{1}{4} e^{-\phi+4X+A-C} (E_3^2 + F_3^2) \\
& - \frac{3}{16} e^{\frac{1}{2}\phi+4X+A-4B+C} A_4^2 - \frac{3}{8} e^{\frac{1}{2}\phi+4X-A-C} (B_4^2 + C_4^2 + D_4^2) \\
& + \frac{1}{16} m^2 e^{\frac{5}{2}\phi+4X+A+4B+C} - \frac{5}{16} \mu_0^2 e^{\frac{1}{2}\phi-4X+A+4B+C} + 3e^{2X+A+4B+C} = 0,
\end{aligned} \tag{B.9}$$

and the internal components give the equations

$$R_{\alpha\alpha} = T_{\alpha\alpha}, \quad R_{mn} = T_{mn}, \quad R_{\psi\psi} = T_{\psi\psi}, \quad C_3 E_3 + D_3 F_3 = 0, \tag{B.10}$$

where the left-hand-sides have been given in (B.3), and the right-hand-sides are

$$\begin{aligned}
T_{\alpha\alpha} & \equiv \frac{1}{2} (\phi')^2 + 8 (X')^2 - \frac{1}{4} e^{\frac{3}{2}\phi+2A-4B} (A_2^2 + C_2^2 + D_2^2) + \frac{3}{8} e^{\frac{3}{2}\phi-2C} B_2^2 \\
& + \frac{1}{2} e^{-\phi-4B} (B_3^2 + C_3^2 + D_3^2) - \frac{1}{2} e^{-\phi+2A-4B-2C} (E_3^2 + F_3^2) \\
& - \frac{3}{8} e^{\frac{1}{2}\phi+2A-8B} A_4^2 + \frac{1}{4} e^{\frac{1}{2}\phi-4B-2C} (B_4^2 + C_4^2 + D_4^2) \\
& + \frac{1}{8} m^2 e^{\frac{5}{2}\phi+2A} - \frac{1}{8} \mu_0^2 e^{\frac{1}{2}\phi-8X+2A} + 3e^{-2X+2A}, \\
T_{mn} & \equiv \left[\frac{1}{4} e^{\frac{3}{2}\phi-2B} (A_2^2 + C_2^2 + D_2^2) - \frac{1}{8} e^{\frac{3}{2}\phi-2A+2B-2C} B_2^2 \right. \\
& + \frac{1}{8} e^{\frac{1}{2}\phi-6B} A_4^2 - \frac{1}{4} e^{\frac{1}{2}\phi-2A-2B-2C} (B_4^2 + C_4^2 + D_4^2) \\
& \left. + \frac{1}{8} m^2 e^{\frac{5}{2}\phi+2B} - \frac{1}{8} \mu_0^2 e^{\frac{1}{2}\phi-8X+2B} + 3e^{-2X+2B} \right] g_{mn}, \\
T_{\psi\psi} & \equiv -\frac{1}{4} e^{\frac{3}{2}\phi-4B+2C} (A_2^2 + C_2^2 + D_2^2) + \frac{3}{8} e^{\frac{3}{2}\phi-2A} B_2^2 \\
& - \frac{1}{2} e^{-\phi-2A-4B+2C} (B_3^2 + C_3^2 + D_3^2) + \frac{1}{2} e^{-\phi-4B} (E_3^2 + F_3^2) \\
& - \frac{3}{8} e^{\frac{1}{2}\phi-8B+2C} A_4^2 + \frac{1}{4} e^{\frac{1}{2}\phi-2A-4B} (B_4^2 + C_4^2 + D_4^2) \\
& + \frac{1}{8} m^2 e^{\frac{5}{2}\phi+2C} - \frac{1}{8} \mu_0^2 e^{\frac{1}{2}\phi-8X+2C} + 3e^{-2X+2C}.
\end{aligned} \tag{B.11}$$

We have explicitly verified that the solutions given in the main text solve all the equations (B.4)–(B.10) and, hence, are indeed solutions of massive type IIA supergravity. It is apparent that without a solution generating technique like (2.6), (2.9), it would have been extremely difficult to find analytic solutions in closed form to these equations.

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