

Constraining conformal field theories with a higher spin symmetry in $d > 3$ dimensions

Vasyl Alba and Kenan Diab

*Department of Physics, Princeton University,
Jadwin Hall, Princeton, NJ 08544 U.S.A.*

E-mail: valba@princeton.edu, kdiab@princeton.edu

ABSTRACT: We study unitary conformal field theories with a unique stress tensor and at least one higher-spin conserved current in $d > 3$ dimensions. We prove that every such theory contains an infinite number of higher-spin conserved currents of arbitrarily high spin, and that Ward identities generated by the conserved charges of these currents imply that the correlators of the stress tensor and the conserved currents of the theory must coincide with one of the following three possibilities: a) a theory of n free bosons (for some integer n), b) a theory of n free fermions, or c) a theory of n $\frac{d-2}{2}$ -forms. For d even, all three structures exist, but for d odd, it may be the case that the third structure (c) does not; if it does exist, it is unclear what theory, if any, realizes it. This is a generalization of the result proved in three dimensions by Maldacena and Zhiboedov [1]. This paper supersedes the previous paper by the authors [2].

KEYWORDS: Conformal and W Symmetry, Field Theories in Higher Dimensions, Higher Spin Symmetry

ARXIV EPRINT: [1510.02535](https://arxiv.org/abs/1510.02535)

Contents

1	Introduction	1
2	Definition of the lightcone limits	4
3	Charge conservation identities	7
4	Quasi-bilocal fields: basic properties	10
5	Quasi-bilocal fields: correlation functions	14
5.1	Symmetries of the quasi-bilocal operators	14
5.2	Correlation functions of the bosonic quasi-bilocal	15
5.3	Correlation functions of the fermionic and tensor quasi-bilocal	16
5.4	Normalization of the quasi-bilocal correlation functions	20
6	Constraining all the correlation functions	21
7	Discussion and conclusions	24
A	Form factors as Fourier transforms of correlation functions	26
B	Uniqueness of three-point functions in the tensor lightcone limit	27
C	Uniqueness of $\langle s22 \rangle$ for $s \geq 4$	28
D	Transformation properties of bilocal operators under K_-	29
D.1	Fermionic case	30
D.2	Tensor case	30
E	Proof that \mathcal{O}_q exists	31
F	The free Maxwell field in five dimensions	32

1 Introduction

Characterizing the theories dual to Vasiliev’s higher-spin gauge theories in anti de-Sitter space [3–5] under the AdS/CFT correspondence [6–8] has been a topic of active research for over ten years, starting from the conjecture of Klebanov and Polyakov that Vasiliev’s theory in four dimensions is dual to the critical $O(N)$ vector model in three dimensions [9, 10]. Under general principles of AdS/CFT, we expect that the conformal field theory duals to Vasiliev’s theories (when given appropriate boundary conditions) should also have higher-spin symmetry, so it is natural to try to classify all higher-spin conformal field theories.

In the case of CFT's in three dimensions, this task has already been accomplished by Maldacena and Zhiboedov [1], who showed that unitary conformal field theories with a unique stress tensor and a higher-spin current are essentially free in three dimensions. This can be viewed as an analogue of the Coleman-Mandula theorem [11, 12], which states that the maximum spacetime symmetry of theories with a nontrivial S-matrix is the super-Poincare group, along with any internal symmetries whose charges are Lorentz-invariant quantum numbers (i.e. are scalars with respect to the spacetime symmetry group).

In this paper, we will prove an analogue of the Coleman-Mandula theorem for generic conformal field theories in all dimensions greater than three. We will show that in any conformal field theory that (a) satisfies the unitary bound for operator dimensions, (b) satisfies the cluster decomposition axiom, (c) contains a symmetric conserved current of spin larger than 2, and (d) has a unique stress tensor in $d > 3$ dimensions, all correlation functions of symmetric currents of the theory are equal to the correlation functions of one of the following three theories — either the theory of n free bosons (for some integer n), a theory of n free fermions, or a theory of n free $\frac{d-2}{2}$ -forms.

Note that in odd dimensions, the free $\frac{d-2}{2}$ -form does not exist, and the status of our result is somewhat complicated. We do not show that there exists any solution to the conformal Ward identities that corresponds to this possibility in odd dimensions, although we do show that if one exists, it is unique. For every odd dimension $d \geq 7$, we know that an infinite tower of higher-spin currents must be present [13]. In $d = 5$, the unitarity forbids finite tower case, so only infinite tower is possible. Assuming that the solution exists and there are an infinite number of higher spin currents, we show that the correlation functions of the conserved currents of the theory may be understood as the analytic continuation of the correlation functions of the currents of the even-dimensional free $\frac{d-2}{2}$ -form theory to odd dimensions. Then, even under all these assumptions, we do not show that there exists any conformal field theory that realizes this solution. That is, it is possible that this structure may have no good microscopic interpretation for other reasons. For example, in odd dimensions it could be possible that some correlation function of some operator is not consistent with the operator product expansion in the sense that it cannot be decomposed in a sum over conformal blocks with non-negative coefficients (i.e. consistent with unitarity¹). Such questions are not explored in this work.

The paper by Boulanger, Ponomarev, Skvortsov, and Taronna [13] strongly indicates that all the algebras of higher-spin charges that are consistent with conformal symmetry are not only Lie algebras but associative. Hence, they are all reproduced by the universal enveloping construction of [14] with the conclusion that any such algebra must contain a symmetric higher-spin current. This implies that our result should be true even after relaxing our assumption that the higher-spin current is symmetric. The argument is structured as follows:

In section 2, we will present the main technical tool of the paper: we will define a particular limit of three-point functions of symmetric conserved currents called *lightcone*

¹There is an example of this phenomenon. If one considers a theory of N scalar fields ϕ_i and computes the four-point function of the operator $\phi^2 = \sum_i \phi_i \phi_i$, it turns out that N should be greater than 1, otherwise the theory is nonunitary.

limits. We will show that such correlation functions behave essentially like correlation functions of a free theory in these limits, enabling us to translate complicated Ward identities of the full theory into simpler ones involving only free field correlators. We will also compute the Fourier transformation of these correlation functions; this will ultimately allow us to simplify certain Ward identities into easily-analyzed polynomial equations.

The rest of the paper will then carry out proof of our main statement. The steps are as follows:

In section 3, we will solve the Ward identity arising from the action of the charge Q_s arising from a spin s current j_s on the correlator $\langle j_2 j_2 j_s \rangle$ in the lightcone limit, where j_2 is the stress tensor. We will show that the only possible solution is given by the free-field solution. This implies the existence of infinitely many conserved currents of arbitrarily high spin,² thereby giving rise to infinitely many charge conservation laws which powerfully constrain the theory.

In section 4, we will construct certain quasi-bilocal fields which roughly behave like products of free fields in the lightcone limit, yet are defined for any CFT. We will establish that all the higher-spin charges (whose existence was proven in the previous step) act on these quasi-bilocals in a particularly simple way.

In section 5, we will translate the action of the higher-spin charges on the quasi-bilocals into constraints on correlation functions of the quasi-bilocals. We will then show that these constraints are so powerful that they totally fix every correlation function of the quasi-bilocals to agree with the corresponding correlation function of a particular biprimary operator in free field theory on the lightcone.

In section 6, we show how the quasi-bilocal correlation functions can be used to prove that the three-point function of the stress tensor must be equal to the three-point function of either the free boson, the free fermion, or the free $\frac{d-2}{2}$ -form, even away from the lightcone limit. This is then used to recursively constrain every correlation function of the CFT to be equal to the corresponding correlation function in the free theory, finishing the proof.

This strategy is similar to the argument in the three-dimensional case given in [1]. There are two main differences between the three-dimensional case and the higher-dimensional cases that we must account for: first, the Lorentz group in $d > 3$ admits asymmetric representations, but the three-dimensional Lorentz group does not. By asymmetric, we

²The fact that the existence of a higher-spin current implies the existence of infinitely many other higher-spin currents has been proven before in the four-dimensional case [15] under the additional assumptions that the theory flows to a theory with a well defined S-matrix in the infrared, that the correlation function $\langle j_2 j_2 j_s \rangle \neq 0$, and that the scattering amplitudes of the theory have a certain scaling behavior. This statement was also proven for $d \neq 4, 5$ in [13] by classifying all the higher-spin algebras in all dimensions other than 4 and 5. We give a proof for the sake of completeness, and also because our techniques differ from those two papers.

mean that a current $J_{\mu_1 \dots \mu_n}$ is not invariant with respect to interchange of its indices. For example, in the standard (j_1, j_2) classification of representations of the four-dimensional Lorentz group induced from the isomorphism of Lie algebras $\mathfrak{so}(3, 1)_{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, these are the representations with $j_1 \neq j_2$. The existence of these representations means that many more structures are possible in $d > 3$ dimensions than in three dimensions (the asymmetric structures), and so many more coefficients have to be constrained in order to solve the Ward identities. We restrict our attention to Ward identities arising from the action of a symmetric charge to correlation functions of only symmetric currents; we will then show that asymmetric structures cannot appear in these Ward identities, making the exact solution of the identities possible.

Second, the space of possible correlation functions consistent with conformal symmetry is larger in $d > 3$ dimensions than in three dimensions. For example, consider the three-point function of the stress tensor $\langle j_2 j_2 j_2 \rangle$. It has long been known (see, e.g. [16–19]) that this correlation function factorizes into three structures in $d > 3$ dimensions, as opposed to only two structures in three dimensions (ignoring a parity-violating structure which is eliminated in three dimensions by the higher-spin symmetry). These three structures correspond to the correlation functions that appear in the theories of free bosons, free fermions, and free $\frac{d-2}{2}$ -forms. We will show that even though more structures are possible in four dimensions and higher, the Ward identities we need can still be solved.

We note that our work is related to a paper by Stanev [20], in which the four, five, and six-point correlation functions of the stress tensor were constrained in CFT's with a higher spin current in four dimensions. It was also shown that the pole structure of the general n -point function of the stress tensor coincides with that of a free field theory. Though this paper reaches the same conclusions, we do not make the rationality assumption [21] of that paper.

By investigating the Jacobi identity for the charges of the conformal algebra combined with at least one HS charge it was proved in [13] that in $d > 5$ there is a unique HS algebra that such charges can form. This algebra corresponds to the free boson and contains HS charges of all (at least even) spins. One of the assumptions was that only HS charges built out of the symmetric HS currents can contribute to the Lie bracket. In the present paper we relax this assumption, which results in two more solutions: free fermion and the $\frac{d-2}{2}$ -form field.

Finally, while this paper was being prepared, the paper [22] appeared in which they showed that unitary “Cauchy conformal fields”, which are fields that satisfy a certain first-order differential equation, are free in the sense that their correlation functions factorize on the 2-point function. Their result may be understood as establishing a similar result that applies even to certain fields which are not symmetric traceless, which we say nothing about.

2 Definition of the lightcone limits

The fundamental technical tool we need to extend into four dimensions and higher is the *lightcone limit*. In order to constrain the correlation functions of the theory to be equal to free field correlators, we will show that the three-point function of the $\langle j_2 j_2 j_2 \rangle$ must be equal

to $\langle j_2 j_2 j_2 \rangle$ for a free boson, a free fermion, or a free $\frac{d-2}{2}$ -form field - it cannot be some linear combination of these three structures. To this end, it will be helpful to split up the Ward identities of the theory into three different identities, each of which involves only one of the three structures separately. To do this, we will need to somehow project all the three-point functions of the theory into these three sectors. The lightcone limits accomplish this task.

Before defining the lightcone limits, we will set up some notation. As in [1], we are writing the flat space metric $ds^2 = dx^+ dx^- + d\vec{y}^2$ and contracting each current with lightline polarization vectors whose only nonzero component is in the minus direction: $j_s \equiv J_{\mu_1 \dots \mu_s} \epsilon^{\mu_1} \dots \epsilon^{\mu_s} = J_{- \dots -}$. We will also denote $\partial_1 \equiv \partial/\partial x_1^-$ and similarly for ∂_2 and ∂_3 . Thus, in all expressions where indices are suppressed, those indices are taken to be minus indices. There are two things we will establish:

1. We need to define an appropriate limit for each of the three cases, which, when applied to a three-point function of conserved currents $\langle j_{s_1} j_{s_2} j_{s_3} \rangle$, yields an expression proportional to an appropriate correlator of the free field theory. For example, in the bosonic case where all the currents are symmetric, we would like the lightcone limit to give us $\partial_1^{s_1} \partial_2^{s_2} \langle \phi \phi^* j_{s_3} \rangle_{\text{free}}$.
2. Second, we need to explicitly compute the free field correlator which we obtain from the lightcone limits. In the bosonic case where all currents are symmetric, this would mean that we need to compute the three-point function $\langle \phi \phi^* j_{s_3} \rangle$ in the free theory.

For the first task, we claim that the desired lightcone limits are:

$$\langle \underline{j_{s_1} j_{s_2} j_{s_3}} \rangle_b \equiv \lim_{|y_{12}| \rightarrow 0} |y_{12}|^{d-2} \lim_{x_{12}^+ \rightarrow 0} \langle j_{s_1} j_{s_2} j_{s_3} \rangle \propto \partial_1^{s_1} \partial_2^{s_2} \langle \phi \phi^* j_{s_3} \rangle_{\text{free}} \quad (2.1)$$

$$\langle \underline{j_{s_1} j_{s_2} j_{s_3}} \rangle_f \equiv \lim_{|y_{12}| \rightarrow 0} |y_{12}|^d \lim_{x_{12}^+ \rightarrow 0} \frac{1}{x_{12}^+} \langle j_{s_1} j_{s_2} j_{s_3} \rangle \propto \partial_1^{s_1-1} \partial_2^{s_2-1} \langle \psi \gamma_- \bar{\psi} j_{s_3} \rangle_{\text{free}} \quad (2.2)$$

$$\langle \underline{j_{s_1} j_{s_2} j_{s_3}} \rangle_t \equiv \lim_{|y_{12}| \rightarrow 0} |y_{12}|^{d+2} \lim_{x_{12}^+ \rightarrow 0} \frac{1}{(x_{12}^+)^2} \langle j_{s_1} j_{s_2} j_{s_3} \rangle \propto \partial_1^{s_1-2} \partial_2^{s_2-2} \langle F_{-\{\alpha\}} F_{-\{\alpha\}} j_{s_3} \rangle_{\text{free}} \quad (2.3)$$

Here, the subscript b , f , and t denote the bosonic, fermionic, and tensor lightcone limits. ϕ is a free boson, ψ is a free fermion, and F is the field tensor for a free $\frac{d-2}{2}$ -form field; the repeated $\{\alpha\}$ indices indicate Einstein summation over all other indices. For example, in four dimensions, the “tensor” structure is just the ordinary free Maxwell field. For conciseness, we will often refer to the free $\frac{d-2}{2}$ -form field as simply the “tensor field” or the “tensor structure”. Again, we emphasize that in odd dimensions, the free $\frac{d-2}{2}$ -form field does not exist. In odd dimensions, our claim is that the only possible structure with the scaling behavior captured by the tensor lightcone limit is the one which coincides with the naive analytic continuation of the correlation functions of the free $\frac{d-2}{2}$ form field to odd d .

The justification for the first two equations comes from the generating functions obtained in [18, 19]; in those references, the three-point functions for correlation functions of conserved currents with y_{12} and x_{12}^+ dependence of those types was uniquely characterized, and so taking the limit of those expressions as indicated gives us the claimed result. In the tensor case, [19] did not find a unique structure, but rather, a one-parameter family of

possible structures. Nevertheless, all possible structures actually coincide in the lightcone limit, as is proven in appendix B.

We note that parity-violating structures cannot appear after taking these lightcone limits. This is because the all-minus component of every parity violating structure allowed by conformal invariance in $d > 3$ dimensions is identically zero. To see this, observe that all parity-violating structures for three-point functions consistent with conformal symmetry must have exactly one $\epsilon_{\mu_1\mu_2\dots\mu_d}$ tensor contracted with polarization vectors and differences in coordinates. Only two of these differences are independent of each other, and all polarization vectors in the all-minus components are set to be equal. Thus, there are only three unique objects that can be contracted with the ϵ tensor, but we need d unique objects to obtain a nonzero contraction. Thus, all parity-violating structures have all-minus components equal to zero.

Later in our argument, we will need expressions for the Fourier transformation of the lightcone-limit three point function of two free fields and a spin s current with respect to the variables x_1^- and x_2^- in the theories of a free boson, a free fermion, and a free $\frac{d-2}{2}$ -form field. The computation for each of the three cases is straightforward and is given explicitly in appendix A. The results are as follows:

$$F_s^b \equiv \langle \phi \phi^* j_s \rangle \propto (p_2^+)^s {}_2F_1 \left(2 - \frac{d}{2} - s, -s, \frac{d}{2} - 1, p_1^+/p_2^+ \right) \quad (2.4)$$

$$F_s^f \equiv \langle \psi \gamma_- \bar{\psi} j_s \rangle \propto (p_2^+)^{s-1} {}_2F_1 \left(1 - \frac{d}{2} - s, -s, \frac{d}{2}, p_1^+/p_2^+ \right) \quad (2.5)$$

$$F_s^b \equiv \langle \underline{F_{-\{\alpha\}}} \underline{F_{-\{\alpha\}}} j_s \rangle \propto (p_2^+)^{s-2} {}_2F_1 \left(-\frac{d}{2} - s, -s, \frac{d}{2} + 1, p_1^+/p_2^+ \right) \quad (2.6)$$

Here, ${}_2F_1$ is the hypergeometric function, and the proportionality sign in each formula indicates that we have omitted an overall nonsingular function which we are not interested in. That they are indeed nonsingular is also proven in appendix A.

Before continuing, we emphasize that the three lightcone limits we have defined do not cover all possible lightcone behaviors which can be realized in a conformal field theory. We define only these three limits because one crucial step in our proof is to constrain the three-point function of the stress tensor $\langle j_2 j_2 j_2 \rangle$, which has only these three scaling behaviors.

Furthermore, though we have discussed only symmetric currents, one could hope that similar expressions could be generated for asymmetric currents — that is, lightcone limits of correlation functions of asymmetric currents are generated by one of the three free field theories discussed here. Unfortunately, running the same argument in [19] fails in the case of asymmetric currents in multiple ways. Consider the current $\langle j_2 j_s \bar{j}_s \rangle$, where j_s is some asymmetric current and \bar{j}_s is its conjugate. To determine how such a correlator could behave the lightcone limit, one could write out all the allowed conformally invariant structures consistent with the spin of the fields, and seeing how each one behaves in the lightcone limits. Unlike the symmetric cases, one finds that in the lightcone limit many independent structures exist, and these structures behave differently depending on which pair of coordinates we take the lightcone limit. To put it another way, for a symmetric

current s , one has the decomposition:

$$\langle j_2 j_s j_s \rangle = \sum_{j \in \{b, f, t\}} \langle j_2 j_s j_s \rangle_j \tag{2.7}$$

where the superscript j denotes the result after taking corresponding lightcone limit in any of the three pairs of coordinates (all of which yield the same result), and the corresponding structures can be understood as arising from some free theory. In the case of asymmetric j_s , this instead becomes a triple sum

$$\langle j_2 j_s \bar{j}_s \rangle = \sum_{j, k, l \in \{b, f, t\}} \langle j_2 j_s \bar{j}_s \rangle_{(j, k, l)} \tag{2.8}$$

where each sum corresponds to taking a lightcone limit in each of the three different pairs of coordinates, and we do not know how to interpret the independent structures in terms of a free field theory. This tells us that for asymmetric currents, the lightcone limit no longer achieves its original goal of helping us split up the Ward identities into three identities which can be analyzed independently; each independent structure could affect multiple different Ward identities. Again, we emphasize that this does not exclude the possibility of a different lightcone limit reducing the correlators of asymmetric currents to those of some other free theory. It simply means that our techniques are not sufficient to constrain correlation functions involving asymmetric currents, so we will restrict our attention to correlation functions that involve only symmetric currents.

3 Charge conservation identities

We will now use the results of the previous section to prove that every CFT with a higher-spin current contains infinitely many higher-spin currents of arbitrarily high (even) spin. We note that this result was proven in a different way in [13] for all dimensions other than $d = 4$ and $d = 5$, wherein they showed that there is a unique higher-spin algebra in $d \neq 4, 5$ and showed that there are infinitely many higher-spin currents. The discussion below is a different proof of this statement based on analysis of the constraints that conservation of the higher-spin charge imposes, and the techniques we develop here will be used later. As before, we treat the bosonic, fermionic, and tensor cases separately.

Before beginning, we will tabulate a few results about commutation relations that we will use freely throughout from this section onwards. Their proofs are identical to those in [1], and are therefore omitted:

1. If a current j' appears (possibly with some number of derivatives) in the commutator $[Q_s, j]$, then j appears in $[Q_s, j']$.
2. Three-point functions of a current with odd spin with two identical currents of even spin are zero: $\langle j_s j_s j_{s'} \rangle = 0$ if s is even and s' is odd.

3. The commutator of a symmetric current with a charge built from another symmetric current contains only symmetric currents and their derivatives:

$$[Q_s, j_{s'}] = \sum_{s''=\max[s'-s+1,0]}^{s'+s-1} \alpha_{s,s',s''} \partial^{s'+s-1-s''} j_{s''} \quad (3.1)$$

The proof of this statement requires an additional step since one needs to exclude asymmetric currents contracted with invariant symbols like the ϵ tensor. For example, consider what structures could appear in $[Q_2, j_2]$ in four dimensions. In $SU(2)$ indices, this object has three dotted and three undotted spinor indices, so one could imagine that a structure like $\epsilon_{abj}{}^{abcd\dot{c}\dot{d}\dot{e}}$ could appear in $[Q_2, j_2]$. However, $[Q_2, j_2]$ has conformal dimension 5, and the unitarity bound constrains the current j , which transforms in the $(5/2, 3/2)$ representation, to have conformal dimension at least $d - 2 + s = 6$, which is impossible. The proof for a general commutator $[Q_s, j_{s'}]$ follows in an identical manner.

4. $[Q_s, j_2]$ contains ∂j_s . This was actually proven for all dimensions in appendix A of [1]. Item 1 then implies that $[Q_s, j_s]$ contains $\partial^{2s-3} j_2$.

In these statements, we are implicitly ignoring the possibility of parity violating structures. For example, the three-point function $\langle 221 \rangle$, which is related to the $U(1)$ gravitational anomaly, may not be zero in a parity violating theory. As mentioned in section 2, however, the all-minus components of every parity-violating structure consistent with conformal symmetry is identically zero, so they will not appear in any of our identities here.

Let's start with the bosonic case. Consider the charge conservation identity arising from the action of Q_s on $\langle \underline{22}_b s \rangle$:

$$0 = \langle [Q_s, 2] \underline{2}_b s \rangle + \langle 2 [Q_s, 2]_b s \rangle + \langle \underline{22}_b [Q_s, s] \rangle \quad (3.2)$$

If s is symmetric, we may use the general commutation relation (3.1) and the lightcone limit (2.1) to expand this equation out in terms of free field correlators:

$$0 = \partial_1^2 \partial_2^2 \left(\gamma (\partial_1^{s-1} + (-1)^s \partial_2^{s-1}) \langle \underline{\phi\phi^*} s \rangle_{\text{free}} + \sum_{2 \leq k < 2s-1 \text{ even}} \tilde{\alpha}_k \partial_3^{2s-1-k} \langle \underline{\phi\phi^*} k \rangle_{\text{free}} \right) \quad (3.3)$$

Note that the sum over k is restricted to even currents since $\langle 22k \rangle = 0$ for odd k . In addition, the fact that the coefficient in front of the ∂_2^{s-1} term is constrained to be $(-1)^s$ times the coefficient for the ∂_1^{s-1} term arises from the symmetry of $\langle \phi(x_1) \phi^*(x_2) j_s(x_3) \rangle$ under interchange of x_1 and x_2 .

Now, we apply our Fourier space expressions for the three-point functions given in section 2. In the Fourier transformed variables, derivatives along the minus direction turn into multiplication by the momenta in the plus direction. After ‘‘cancelling out’’ the overall derivatives, which just yields an overall factor of $(p_1^+)^2 (p_2^+)^2$, the relevant equation is:

$$0 = \gamma ((p_1^+)^{s-1} + (-1)^s (p_2^+)^{s-1}) F_s(p_1^+, p_2^+) + \sum_{2 \leq k < 2s-1 \text{ even}} \tilde{\alpha}_k (p_1^+ + p_2^+)^{2s-1-k} F_k(p_1^+, p_2^+) \quad (3.4)$$

The solution of (3.4) is not easy to obtain by direct calculation. We can make two helpful observations, however. First, not all coefficients can be zero. This is because we know 2 appears in $[Q_s, s]$, so at least $\tilde{\alpha}_2$ is not zero. Second, we know that the free boson exists (there is a CFT with higher spin symmetry), and therefore, the coefficients one obtains from that theory would exactly solve this equation. We will show that this solution is unique.

Suppose we have two sets of coefficients $(\gamma, \{\tilde{\alpha}_k\})$ and $(\gamma', \{\tilde{\beta}_k\})$ that solve this equation. First, suppose $\gamma \neq 0$ and $\gamma' \neq 0$. Then, we can normalize the coefficients so that $\gamma = \gamma'$ are equal for the two solutions. Then, subtract the two solutions from each other so that the γ terms vanish. If we evaluate the result at some arbitrary nonzero value of p_2^+ , we may absorb all overall p_2^+ factors into the coefficients and re-express the equation as a polynomial identity in a single variable $z \equiv p_1^+/p_2^+$:

$$0 = \sum_{2 \leq k < 2s-1 \text{ even}} \tilde{\delta}_k (1+z)^{2s-1-k} {}_2F_1\left(2 - \frac{d}{2} - k, -k, \frac{d}{2} - 1, -z\right) \quad (3.5)$$

Then, the entire right hand side is divisible by $1+z$ since s is even, so we may divide both sides by $1+z$. Setting $z = -1$, since ${}_2F_1(a, a, 1, 1) \neq 0$ for all negative half-integers a , we conclude that $\tilde{\delta}_{2s-2} = 0$. Then, the entire right hand side is proportional to $(1+z)^2$, so we may divide it out. Then, setting $z = -1$ again, we find $\tilde{\delta}_{2s-4} = 0$. Repeating this procedure, we conclude that all coefficients are zero, and therefore, that the two solutions are identical. On the other hand, suppose one of the solutions has $\gamma = 0$. Then, the same argument establishes that all the coefficients $\tilde{\alpha}_k$ are zero. As noted earlier, however, the trivial solution is disallowed. Therefore, the solution is unique and coincide with one for free boson. Thus, we have infinitely many even conserved currents, as desired.

In the fermionic case, precisely the same analysis works. The action of Q_s on $\langle 22_f s \rangle$ for symmetric s leads to

$$0 = \partial_1^2 \partial_2^2 \left(\gamma (\partial_1^{s-2} + (-1)^{s-1} \partial_2^{s-2}) \langle \psi \bar{\psi} s \rangle + \sum_{2 \leq k < 2s-2 \text{ even}} \tilde{\alpha}^k \partial_3^{2s-2-k} \langle \psi \bar{\psi} k \rangle \right), \quad (3.6)$$

Then, converting this expression to form factors and running the same analysis from the bosonic case verbatim establishes that the unique solution to this equation is the one arising in the theory of a free fermion.

In the tensor case, the argument again passes through exactly as before, except for two subtleties: first, unlike in the bosonic and fermionic case, we do not have unique expressions for the three-point functions of currents with the tensor-type coordinate dependence, so this only demonstrates that the free-field solution is an admissible solution, but not necessarily the unique solution. Nevertheless, in the lightcone limit, all possible structures for three-point functions coincide with the free-field answer.³ This was proven in appendix B.

Second, there may not exist a solution to the Ward identities in odd dimensions, because the free $\frac{d-2}{2}$ -form does not exist in odd dimensions. However, if any solution exists, our argument shows that it is unique. In $d \geq 7$, it is known that there is a unique

³Actually, we proved that correlators of the form $\langle 22s \rangle$ have a unique tensor structure even away from the lightcone limit. The proof, however, is very technical, and it is given in appendix C.

higher-spin algebra containing the tower of higher-spin currents described in the bosonic case [13]. In $d = 5$, our technique shows that if there is a solution for the Ward identity in the tensor lightcone limit, then it is unique. We do not prove, however, that there is an infinite tower of higher spin currents or that there is exactly one current of every spin. Finite dimensional representations would be inconsistent with unitarity. We do not explore this question further in this work. Henceforth, we assume that our theory does indeed contain the infinite tower of higher-spin currents necessary for our analysis.

4 Quasi-bilocal fields: basic properties

In this section, we will define a set of quasi-bilocal operators, one for each of the three lightcone limits, and characterize the charge conservation identities arising from the action of the higher-spin currents. As we will explain in section 5, these charge conservation identities will turn out to be so constraining that the correlation functions of the quasi-bilocal operators are totally fixed. This will then enable us to recursively generate all the correlation functions of the theory and prove that the three-point function of the stress tensor can exhibit only one of the three possible structures allowed by conformal symmetry. As in the three-dimensional case, we define the quasi-bilocal operators on the lightcone as operator product expansions of the stress tensor with derivatives “integrated out”:

$$\underline{\underline{22}}_b = \partial_1^2 \partial_2^2 B(x_1, x_2) \tag{4.1}$$

$$\underline{\underline{22}}_f = \partial_1 \partial_2 F_-(x_1, x_2) \tag{4.2}$$

$$\underline{\underline{22}}_t = V_{--}(x_1, x_2) \tag{4.3}$$

The motivation behind these definitions can be understood by appealing to what these expressions look like in free field theory. There, they will be given by simple products of free fields:

$$B(x_1, x_2) \sim: \phi(x_1)\phi^*(x_2) : + : \phi(x_2)\phi^*(x_1) : \tag{4.4}$$

$$F_-(x_1, x_2) \sim: \bar{\psi}(x_1)\gamma_- \psi(x_2) : - : \bar{\psi}(x_2)\gamma_- \psi(x_1) : \tag{4.5}$$

$$V_{--} \sim: F_{-\{\alpha\}}(x_1)F_{-\{\alpha\}}(x_2) : \tag{4.6}$$

It is clear from the basic properties of our lightcone limits that when they are inserted into correlation functions with another conserved current j_s , they will be proportional to an appropriate free field correlator. Since $\langle \underline{\underline{22}}_s \rangle = 0$ for odd s , only the correlation functions with even s will be nonzero:

$$\langle B(\underline{x_1, x_2})j_s \rangle \propto \langle \phi(x_1)\phi^*(x_2)j_s(x_3) \rangle_{\text{free}} \tag{4.7}$$

$$\langle F_-(\underline{x_1, x_2})j_s \rangle \propto \langle \psi(x_1)\gamma_- \bar{\psi}(x_2)j_s(x_3) \rangle_{\text{free}} \tag{4.8}$$

$$\langle V_{--}(\underline{x_1, x_2})j_s \rangle \propto \langle F_{-\{\alpha\}}(x_1)F_{-\{\alpha\}}(x_2)j_s(x_3) \rangle_{\text{free}} \tag{4.9}$$

Of course, away from the lightcone, things will not be so simple: we have not even defined the quasi-bilocal operators there, and their behavior there is the reason why they are not true bilocals. In fact, even on the lightcone, these expressions are not fully conformally

invariant: the contractions of indices performed in equations (4.8) and (4.9) are only invariant under the action of the collinear subgroup of the conformal group defined by the line connecting x_1 and x_2 . For now, however, the lightcone properties enumerated above are enough to establish the commutator of Q_s with the bilocals. As usual, we begin with the bosonic case: assume that $\langle \underline{22}_b \underline{2} \rangle \neq 0$. Our goal is to show that

$$[Q_s, B(\underline{x}_1, \underline{x}_2)] = (\partial_1^{s-1} + \partial_2^{s-1})B(\underline{x}_1, \underline{x}_2). \quad (4.10)$$

This can be shown using the same arguments as [1]. To begin, notice that the action of Q_s commutes with the lightcone limit. Thus,

$$\langle [Q_s, B]j_k \rangle = \langle [Q_s, j_2]j_2 j_k \rangle + \langle j_2 [Q, j_2]j_k \rangle = -\langle j_2 j_2 [Q_s, j_k] \rangle = \langle [Q_s, j_2 j_2]j_k \rangle \quad (4.11)$$

This immediately leads to:

$$[Q_s, B(\underline{x}_1, \underline{x}_2)] = (\partial_1^{s-1} + \partial_2^{s-1})\tilde{B}(\underline{x}_1, \underline{x}_2) + (\partial_1^{s-1} - \partial_2^{s-1})B'(\underline{x}_1, \underline{x}_2), \quad (4.12)$$

Here, \tilde{B} is built from even currents, while B' is built from odd currents. This makes the whole expression symmetric. We would like to show that $B' = 0$. Therefore, suppose otherwise so that some current $j_{s'}$ has nontrivial overlap with B' . Then, the charge conservation identity $0 = \langle [Q_{s'}, B' j_2] \rangle$ yields

$$0 = \langle [Q_{s'}, B'(\underline{x}_1, \underline{x}_2)] j_2 \rangle + \langle B'(\underline{x}_1, \underline{x}_2) [Q_{s'}, j_2] \rangle, \quad (4.13)$$

$$\Rightarrow 0 = \gamma \left(\partial_1^{s'-1} - \partial_2^{s'-1} \right) \langle \phi \bar{\phi} j_2 \rangle + \sum_{k=0}^{s'+1} \tilde{\alpha}_k \partial^{s'+1-k} \langle \phi \bar{\phi} j_k \rangle. \quad (4.14)$$

Using the same techniques as the previous section, we obtain

$$0 = \gamma \left((p_1^+)^{s'-1} - (p_2^+)^{s'-1} \right) F_2(p_1^+, p_2^+) + \sum_{k=0}^{s'+1} \tilde{\alpha}_k (p_1^+ + p_2^+)^{s'+1-k} F_k(p_1^+, p_2^+). \quad (4.15)$$

In this sum, $\tilde{\alpha}_{s'} \neq 0$ because $j_{s'} \subset [Q_{s'}, 2]$. Therefore, we can use the same procedure as before to show that all $\tilde{\alpha}_k$ are nonzero if they are nonzero for the free field theory. In particular, since $\tilde{\alpha}_1$ is not zero for the complex free boson, the overlap between j_1 and B' is not zero. Now, let's consider

$$0 = \langle [Q_s, B j_1] \rangle = (\partial_1^{s-1} - \partial_2^{s-1}) \langle B' j_1 \rangle + \langle B [Q_s, j_1] \rangle, \quad (4.16)$$

where Q_s is a charge corresponding to any even higher-spin current appearing in the operator product expansion of $j_2 j_{2b}$. We have shown the first term is not zero. We will prove that the second term must be equal to zero to get a contradiction. Specifically, we will show that there are no even currents in $[Q_s, j_1]$. Since B is proportional to $\underline{22}$, and since $\langle \underline{22s} \rangle = 0$ for all odd s , this yields the desired conclusion.

Consider the action of Q_s on $\langle \underline{221} \rangle$. We obtain the now-familiar form:

$$0 = \gamma \left((p_1^+)^{s-1} - (p_2^+)^{s-1} \right) F_1(p_1^+, p_2^+) + \sum_{k=0}^s \tilde{\alpha}_k (p_1^+ + p_2^+)^{s-k} F_k(p_1^+, p_2^+) \quad (4.17)$$

We want to show that $\alpha_k = 0$ for even k . Recall the definition of F_k :

$$F_k = (p_2^+)^k {}_2F_1 \left(2 - \frac{d}{2} - k, -k, \frac{d}{2} - 1, -\frac{p_1^+}{p_2^+} \right) \quad (4.18)$$

$$= \sum_{i=0}^k c_i^k (p_1^+)^i (p_2^+)^{s-i} \quad (4.19)$$

The hypergeometric coefficients c_i^k have the property that $c_i^k = (-1)^k c_{k-i}^k$. Now, we collect terms in equation (4.17) proportional to $(p_1^+)^s$ and $(p_2^+)^s$ — each sum must vanish separately for the entire polynomial to vanish. We obtain

$$\gamma + \sum_{0 \leq k \leq s \text{ odd}} \alpha_k u_k + \sum_{0 \leq k \leq s \text{ even}} \alpha_k v_k = 0 \quad (4.20)$$

$$-\gamma - \sum_{0 \leq k \leq s \text{ odd}} \alpha_k u_k + \sum_{0 \leq k \leq s \text{ even}} \alpha_k v_k = 0 \quad (4.21)$$

Here, u_k and v_k are sums of products of coefficients of the hypergeometric function and the binomial expansion of $(p_1^+ + p_2^+)^{s-k}$; we do not care about their properties except that, with the signs indicated above, they are strictly positive, as can be verified by direct calculation. By adding and subtracting these equations, we obtain two separate equations that must be satisfied by the odd and even coefficients separately

$$\gamma + \sum_{0 \leq k \leq s \text{ odd}} \alpha_k u_k = 0 \quad (4.22)$$

$$\sum_{0 \leq k \leq s \text{ even}} \alpha_k v_k = 0 \quad (4.23)$$

Exactly analogously, we may do the same procedure to every other pair of monomials $(p_1^+ + 1^+)^a (p_2^+)^{s-a}$ and $(p_1^+)^{s-a} (p_2^+)^a$ to turn the constraints for the two monomials into constraints for the even and odd coefficients (where we're considering γ as an odd coefficient) separately. Hence, by multiplying each term by the monomial from which it was computed and then resumming, we find that the original identity (4.17) actually splits into two separate identities that must be satisfied. For the even terms, this identity is:

$$0 = \sum_{0 \leq k \leq s \text{ even}} \alpha_k (p_1^+ + p_2^+)^{s-k} (p_2^+)^k {}_2F_1 \left(2 - \frac{d}{2} - k, -k, \frac{d}{2} - 1, -\frac{p_1^+}{p_2^+} \right) \quad (4.24)$$

Then, we may again use the argument from section 3 to conclude that all $\alpha_k = 0$ for even k , which is what we wanted. Thus, $B' = 0$.

Now we would like to show that $B = \tilde{B}$. First of all we will show that \tilde{B} is nonzero. Consider the charge conservation identity

$$0 = \langle [Q_s, B j_2] \rangle = (\partial_1^{s-1} + \partial_2^{s-1}) \langle \tilde{B} \rangle + \langle B, [Q_s, 2] \rangle \quad (4.25)$$

Since $[Q_s, j_2] \supset \partial j_s$, and since $\langle B s \rangle \neq 0$, the second term in that identity is nonzero, and so \tilde{B} must be nonzero. Now we can normalize the currents in such a way that j_2 has the same

overlap with \tilde{B} and B . After normalization, we know that $B - \tilde{B}$ does not contain any spin 2 current because the stress tensor is unique, by hypothesis. Now, we will show that $B - \tilde{B}$ is zero by contradiction. Suppose $B - \tilde{B}$ is nonzero. Then, there is a current j_s whose overlap with $B - \tilde{B}$ is nonzero. Then, the charge conservation identity for the case $s > 2$ is

$$0 = \langle [Q_s, (B - \tilde{B}) j_2] \rangle, \quad (4.26)$$

$$0 = \gamma \left((p_1^+)^{s-1} + (p_2^+)^{s-1} \right) F_2(p_1^+, p_2^+) + \sum_{k=0}^{s+1} \tilde{\alpha}_k (p_1^+ + p_2^+)^{s+1-k} F_k(p_1^+, p_2^+), \quad (4.27)$$

where we assume that $\tilde{\alpha}_s \neq 0$. Then, we can again run the same analysis as section 3 to conclude that since $\tilde{\alpha}_s \neq 0$, we must have $\tilde{\alpha}_2 \neq 0$ — that is, j_2 has nonzero overlap with $B - \tilde{B}$, which is a contradiction. It means that $B - \tilde{B}$ has no overlap with any currents j_s for $s > 2$. The only possibility is to overlap only with spin zero currents. Suppose that there is a current j'_0 that overlaps with $B - \tilde{B}$, where the prime distinguishes it from a spin 0 current j_0 that could appear in B . We first show that $\langle j_0 j'_0 \rangle = 0$. Consider the charge conservation identity the action Q_4 on $\langle (B - \tilde{B}) j_0 \rangle$. The action of the charge is $[Q_4, 0] = \partial^3 j_0 + \partial j_2 + \dots$, where the \dots represent terms that cannot overlap with $\underline{22}$ (from which B is constructed) or the even currents that appear in \tilde{B} . By hypothesis, $B - \tilde{B}$ has no overlap with j_2 , so the identity simplifies to $\langle j_0 j'_0 \rangle = 0$. Then, since j'_0 is nonzero, it should have nontrivial overlap with some Q_s . Now, recall the fact that if a current j' appears (possibly with some number of derivatives) in the commutator of $[Q_s, j]$, then j appears in $[Q_s, j']$. Thus, there should be a current current of spin $s'' < s$ such that $[Q_s, j_{s''}] = j'_0 + \dots$. The action Q_s on $\langle (B - \tilde{B}) j_{s''} \rangle$ is

$$\langle [Q_s, (B - \tilde{B}) j_{s''}] \rangle = \partial_3^3 \langle (B - \tilde{B}) j'_0 \rangle + \partial \langle (B - \tilde{B}) j_2 \rangle, \quad (4.28)$$

Here, we have used that the action of Q_s on B and \tilde{B} is identical because $B' = 0$. Then, since the second term is zero, thus the first term is equal to zero as well. Thus, $B - \tilde{B}$ has no overlap with any currents and is equal to zero, as desired.

In the fermionic case, we can run almost the same argument as in the bosonic case, except there is no discussion of a possible j_0 , since there is no conserved spin zero current in the free fermion theory. We obtain the action of the charge on the fermionic quasi-bilocal is

$$[Q_s, F_-(\underline{x}_1, \underline{x}_2)] = (\partial_1^{s-1} + \partial_2^{s-1}) F_-(\underline{x}_1, \underline{x}_2). \quad (4.29)$$

In the tensor case, we again can repeat the argument to obtain

$$[Q_s, V_{--}(\underline{x}_1, \underline{x}_2)] = (\partial_1^{s-1} + \partial_2^{s-1}) V_{--}(\underline{x}_1, \underline{x}_2) \quad (4.30)$$

In this case, there is neither a conserved spin 0 or spin 1 current in the free tensor theory. The argument works the same way, however, if we consider j_3 instead of j_1 in the steps of the argument that require it.

5 Quasi-bilocal fields: correlation functions

In this section, we will discuss how to precisely define the quasi-bilocal operators in a way that makes their symmetries manifest. In particular, each of the three bilocals will be bi-primary operators in some sense. This will allow us to argue that the correlation functions of the bilocals should agree with an appropriate corresponding free-field result. We will then explore what this implies for the full theory in section 6.

5.1 Symmetries of the quasi-bilocal operators

Let us first consider the case of the bosonic bilocal operator $B(x_1, x_2)$. Recall that, on the lightcone, the bilocals should imitate products of the appropriate free fields. In the bosonic free-field theory, the operator product expansion of $\phi(x_1)\phi^*(x_2)$ is composed of all of the even-spin currents of the theory with appropriate numbers of derivatives and factors of $(x_1 - x_2)$ so that the expression has the correct conformal dimension. More explicitly, we may write:

$$\phi(x_1)\phi^*(x_2) = \sum_{\text{even } s \geq 0} b_s^{\text{free}}(x_1, x_2) \quad (5.1)$$

$$b_s^{\text{free}}(x_1, x_2) = \sum_{(k,l)|s+l-k=0} c_{skl}(x_1 - x_2)^k \partial^l j_s \left(\frac{x_1 + x_2}{2} \right) \quad (5.2)$$

All the coefficients c_{skl} may be computed exactly in the free theory just by Taylor expansion. We have shown that all the currents j_s exist in our theory for all even s . So we may define an analogous quantity in our theory as follows:

$$B(x_1, x_2) = \sum_{\text{even } s \geq 0} b_s(x_1, x_2) \quad (5.3)$$

$$b_s(x_1, x_2) = \sum_{(k,l)|s+l-k=0} c'_{skl}(x_1 - x_2)^k \partial^l j_s \left(\frac{x_1 + x_2}{2} \right) \quad (5.4)$$

Here, the c' are some other coefficients which are to be determined by demanding that this definition of B coincide with the definition given on the lightcone in the previous section, i.e. that $\partial_1^2 \partial_2^2 B(x_1, x_2) = \underline{\underline{22}}_b$. We claim that this can be accomplished by choosing the c' coefficients such that $\langle B(\underline{x}_1, \underline{x}_2) j_s \rangle \propto \langle \underline{\phi}(\underline{x}_1) \underline{\phi}^*(\underline{x}_2) j_s \rangle_{\text{free}}$. To see that there exists such a choice of c' which can achieve this condition, we explicitly compare $\langle B j_s \rangle$ and $\langle \phi \phi^* j_s \rangle_{\text{free}}$ term by term using (5.2) and (5.4). Each term in both of these correlation functions has the structure $(x_1 - x_2)^k \partial^l \langle j_{s'} j_s \rangle$ with coefficient $c_{s'kl}$ and $c'_{s'kl}$, respectively. Two-point functions of primary operators in CFT's are determined up to a constant, so each term is identical up to a possible scaling, which can be eliminated by choosing the c' coefficient appropriately. Then, by applying $\partial_1^2 \partial_2^2$ to both sides of $\langle B(\underline{x}_1, \underline{x}_2) j_s \rangle \propto \langle \underline{\phi}(\underline{x}_1) \underline{\phi}^*(\underline{x}_2) j_s \rangle_{\text{free}}$, we find that our definition coincides on the lightcone, as desired. This construction works the same way for the fermionic and tensor quasi-bilocals with analogous results, except that the quasi-bilocals in those cases carry the appropriate spin structure.

Since the conformal transformation properties of a conserved current j_s is theory-independent in the sense that it is completely fixed by its spin and conformal dimension, it

is manifest from this definition that the bosonic quasi-bilocal $B(x_1, x_2)$ has the same transformation properties under the full conformal group as a product of free bosons. That is, it is a scalar bi-primary field with a conformal dimension of 1 with respect to each argument.

On the other hand, consider the fermionic and tensor quasi-bilocals F_- and V_{--} . The same line of reasoning tells us that they will transform like products of free fields contracted in a particular way: F_- will transform like $:\psi\gamma_-\bar{\psi}:$ does in the free fermionic theory, and V_{--} will transform like $:F_{-\{\alpha\}}F_-^{\{\alpha\}}:$ does in the theory of a free $\frac{d-2}{2}$ -form.⁴ These contractions, however, are not preserved by the full conformal group — the special conformal transformations orthogonal to the — direction will ruin the structure of the Lorentz contraction. Thus, even in the free theory, these objects are not preserved by the full conformal group. They are only preserved by the so-called collinear conformal group generated by K_-, P_+, J_{+-} , and D , where K, P, J , and D are the generators of special conformal transformations, translations, boosts, and dilatations, respectively. It is clear from the structure of the conformal algebra that the commutation relations of this subset of conformal generators closes, so it forms a proper sub-algebra. Thus, what we are allowed to conclude is that F_- and V_{--} are bi-primary operators with respect to this collinear subgroup, not the conformal group. Nevertheless, this will still be enough symmetry for our purposes.

The key fact which is still true for this more restricted set of symmetries is that under K_- , the special conformal transformation in the — direction, the n -point function of fermionic and tensor quasi bi-primaries should scale separately in each variable. That is, under K_- , if $x \rightarrow x'$ and $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$, we have

$$\begin{aligned} \langle F_-(\underline{x}'_1, \underline{x}'_2), \dots, F_-(\underline{x}'_{2n-1}, \underline{x}'_{2n}) \rangle \\ = \Omega(x_1)^{d/2-1} \dots \Omega(x_{2n})^{d/2-1} \langle F_-(\underline{x}_1, \underline{x}_2), \dots, F_-(\underline{x}_{2n-1}, \underline{x}_{2n}) \rangle \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} \langle V_{--}(\underline{x}'_1, \underline{x}'_2), \dots, V_{--}(\underline{x}'_{2n-1}, \underline{x}'_{2n}) \rangle \\ = \Omega(x_1)^{d/2-1} \dots \Omega(x_n)^{d/2-1} \langle V_{--}(\underline{x}_1, \underline{x}_2), \dots, V_{--}(\underline{x}_{2n-1}, \underline{x}_{2n}) \rangle \end{aligned} \quad (5.6)$$

The proof of these two statements is given in appendix D.

5.2 Correlation functions of the bosonic quasi-bilocal

Now we will discuss the structure of the n -point functions of the quasi-bilocals. Again, let's begin with the bosonic case. We wish to constrain $\langle B(\underline{x}_1, \underline{x}_2) \dots B(\underline{x}_{2n-1}, \underline{x}_{2n}) \rangle$. We established that $B(x_1, x_2)$ has the transformation properties of a product of two free fields under the full conformal group — i.e. it is a bi-primary field with dimension $\frac{d-2}{2}$ in each variable. That means that the n -point function can only depend on distances between

⁴Technically, the argument given above for the symmetries of the bosonic quasi-bilocal only works for even dimensions in the tensorial case since the free $\frac{d-2}{2}$ -form exists only in even dimensions, so the matching procedure can't be carried out naively in odd dimensions. On the other hand, it is evident from the definition (4.3) that it has at least the collinear conformal symmetry since there are no derivatives to be “integrated out”.

coordinates d_{ij} and have conformal dimension $\frac{d-2}{2}$ with respect to each variable. Since x_1 and x_2 are lightlike separated, d_{12} cannot appear, and similarly for every pair of arguments of the same bilocal. There is also a permutation symmetry: B is symmetric in its two arguments, and the n point function must be symmetric under interchange of any pair of the identical B 's. Finally, there is the higher-spin symmetry. In the bosonic case, the charge conservation identity (4.10) imposes the simple relation

$$\sum_{i=1}^{2n} \partial_i^{s-1} \langle B(\underline{x}_1, \underline{x}_2) \dots B(\underline{x}_{2n-1}, \underline{x}_{2n}) \rangle, \quad \text{for all even } s \tag{5.7}$$

As shown in appendix E of [1], this fixes the x^- dependence of the n -point function to have the particular form:

$$\sum_{\sigma \in S^{2n}} g_{\sigma} \left(x_{\sigma(1)}^- - x_{\sigma(2)}^-, x_{\sigma(3)}^- - x_{\sigma(4)}^-, \dots, x_{\sigma(2n-1)}^- - x_{\sigma(2n)}^- \right) \tag{5.8}$$

where S^{2n} is the set of permutations of $2n$ elements. The point is that the x_i^- dependence of the n -point function is constrained such that, for each g_{σ} , x_i^- can only appear in a difference with one and only one other coordinate. This is a very strong constraint. The conformal symmetry tells us that each g_{σ} in the above series can be written as a product of a dimensionful function of distances with the correct dimension in each variable times a smooth, dimensionless function of conformal cross-ratios. The constraint on the functional form of g_{σ} , however, forbids all such functions except the trivial function 1, because each cross ratio separately violates the constraint.

Putting it all together, we conclude that the n -point function has to be proportional to a sum of terms with equal coefficients, each of which is a product $\prod d_{ij}^{-(d-2)}$, where the product has n terms corresponding to some partition of the $2n$ points into pairs where no pair contains two arguments of the same bilocal. For example, the two-point function is:

$$\langle B(\underline{x}_1, \underline{x}_2) B(\underline{x}_3, \underline{x}_4) \rangle = \tilde{N}_b \left(\frac{1}{d_{13}^{d-2} d_{24}^{d-2}} + \frac{1}{d_{14}^{d-2} d_{23}^{d-2}} \right), \tag{5.9}$$

where \tilde{N}_b is a constant of proportionality. One immediately notes that the expressions one obtains this way for all n -point functions of the quasi-bilocals are proportional to the n -point function of $:\phi(x_1)\phi(x_2):$ in a theory of free bosons.

5.3 Correlation functions of the fermionic and tensor quasi-bilocal

In the fermionic and tensor cases, we claim that the correlation functions of the quasi-bilocals still coincide with the correlation functions of the corresponding free field theories, despite the fact that the fermionic and tensor quasi-bilocals have less symmetry than the bosonic quasi-bilocal. The argument, however, is somewhat more complicated due to the reduced amount of symmetry. The proof is essentially the same for both the fermionic and tensor cases, so we will only present the argument for the tensor case. Our general strategy will be to compare the constraints that one obtains from the definition of V_{--} as the lightcone limit $\underline{22}_t$ with the constraints one obtains from the symmetries of V_{--} as

established by its definition away from the lightcone given at the beginning of this section. In the bosonic case, we only used the latter, but in the fermionic case and tensor case, we will need the former as well.

First, we consider what the $2n$ -point function of T_{--} is away from the lightcone. We know from the definition of $V_{--} = \underline{\underline{22}}_t$ that if we take n lightcone limits of this $2n$ point function in each pair of adjacent arguments $(x_1, x_2), (x_3, x_4), \dots (x_{2n-1}, x_{2n})$, we will obtain the n point function of $V_{--}(x_1, x_2)$. It may be the case that the definition of V_{--} given earlier as a sum of currents and descendants (with appropriate derivatives and powers of x) will yield a different result away from the lightcone, but nevertheless, it must agree with the $2n$ -point function of T_{--} in the lightcone limit.

Generically, the $2n$ point function of T_{--} with arguments in arbitrary locations can be decomposed as a polynomial in some basis of conformally invariant structures. One convenient basis is the $\{H_{ij}, V_i\}$ space defined in [23]. In this basis, we may write

$$\langle T_{--}(x_1) \cdots T_{--}(x_{2n}) \rangle = \frac{\langle \langle T_{--}(x_1) \cdots T_{--}(x_{2n}) \rangle \rangle}{d_{12}^{d-2} d_{23}^{d-2} \cdots d_{2n-1,2n}^{d-2} d_{2n,1}^{d-2}} \quad (5.10)$$

where

$$\langle \langle T_{--}(x_1) \cdots T_{--}(x_{2n}) \rangle \rangle = \sum_i f_i(\{u_j\}) \left(\prod_{k<l} H_{kl}^{h_i^{(k,l)}} \right) \left(\prod_{k<l<m} V_{k,lm}^{v_i^{(k,lm)}} \right) \quad (5.11)$$

where $f_i(\{u_j\})$ is an arbitrary function of cross-ratios $\{u_j\}$, the h_{kl} and v_i coefficients satisfy

$$\sum_{l,m|k<l<m} v_i^{(k,lm)} + \sum_{n|k<n} h_i^{(k,n)} = 2 \quad \text{for all } i, k \quad (5.12)$$

and the conformal invariants are

$$V_{k,lm} = \frac{x_{kl}^+}{d_{kl}^2} + \frac{x_{km}^+}{d_{km}^2} \quad (5.13)$$

$$H_{kl} = \frac{-2(x_{kl}^+)^2}{d_{kl}^4} \quad (5.14)$$

Note that this decomposition omits structures which contain the epsilon tensor, which all vanish in our formalism because we contract all free indices with the same polarization vector in the $-$ direction.

We would like to understand the properties of this decomposition under the tensor lightcone limit (2.3). First, note that the universal dimensionful factor of distances that is factored out of $\langle \langle T \dots T \rangle \rangle$ in (5.10) is conventional. In principle, one could choose it to be something different and compensate by appropriate redefinitions of f_i . We have chosen it as shown in order to simplify the structure of this function under the lightcone limit. More precisely, the distances corresponding to pairs of points that become lightlike separated $d_{12}, d_{34}, \dots, d_{2n-1,2n}$ vanish in the lightcone limit, so they cannot explicitly appear in the correlation function, and we have chosen the universal factor so that this property is manifest. To see this, note that when we take the lightcone limit (2.3) of this general

structure, the part of this universal factor corresponding to the distances between points that become lightlike separated — i.e. $d_{12}^{-d+2} d_{34}^{-d+2} \dots d_{2n-1,2n}^{-d+2}$ - becomes $d_{12}^4 d_{34}^4 \dots d_{2n-1,2n}^4$. This residual factor is exactly cancelled out by the V and H terms corresponding to the x^+ factors stripped away in (2.3). To see this, recall that the light-cone limits of correlation functions are well-defined and non-divergent,⁵ so any structure consistent with conformal symmetry needs to appear with enough V 's and H 's with appropriate indices to cancel out the factors of $(x_{12}^+)^{-2}, (x_{34}^+)^{-2}, \dots, (x_{2n-1,2n}^+)^{-2}$ that appear in the lightcone limit. As noted earlier, these factors of V 's and H 's come with exactly two powers each of $d_{12}^2, \dots, d_{2n-1,2n}^2$, which is exactly what is needed to cancel out the residual term.

Thus, after the lightcone limit, the most general structure that can appear in the n -point function of V_{--} is:

$$\langle V_{--}(x_1, x_2) \dots V_{--}(x_{2n-1}, x_{2n}) \rangle = \langle T_{--}(x_1) T_{--}(x_2)_t \dots T_{--}(x_{2n-1}) T_{--}(x_{2n})_t \rangle \quad (5.15)$$

$$= \sum_i \frac{f_i(\{u_j\})}{d_{23}^{d-2} \dots d_{2n,1}^{d-2}} \prod_{k,l} \left(\frac{x_{kl}^+}{d_{kl}^2} \right)^{c_{kl}} \quad (5.16)$$

where the product over k and l is understood to be restricted to pairs (k, l) not corresponding to x_k, x_l lightlike separated, and $\sum c_{kl} = 2n$.

We can determine which terms of this form are consistent with the symmetries of V_{--} . Consider the n -point correlation function of V_{--} . Its transformation properties under Lorentz transformations and dilatations tell us that we must have $2n +$ indices in the numerator of the correlation function, and that the overall scaling dimension of the n -point function should be $2n \times d/2 = dn$. Then, as mentioned before, since V_{--} is a bi-primary under the collinear conformal group, the n -point function should scale appropriately in each variable separately after acting with K_- according to (5.6). In order to satisfy this constraint, for each independent structure appearing in the correlation function and each index i , we must have 2 factors of x_{ij}^+ in the numerator (not necessarily the same j for each of the 2 factors) and $d + 2$ powers of d_{ik} in the denominator for some k (again, not necessarily the same k for each of the $d + 2$ factors). Once we have picked such a denominator, there is still some ambiguity since conformally invariant functions f_i can still appear after imposing this constraint (since they are fixed by K_-), and such functions can change the denominator. What is tightly constrained here is the numerator — i.e. the spin structure. “Imbalanced” structures with that would otherwise be allowed by Lorentz symmetry, scaling symmetry, and permutation symmetry cannot appear. For example, for the two-point function $\langle V_{--} V_{--} \rangle$ in four dimensions, structures such as $\frac{(x_{13}^+)^4}{d_{13}^2} + \frac{(x_{24}^+)^4}{d_{24}^2}$ do not satisfy (5.6). Note that the numerators which are allowed by this constraint are precisely the ones that appear in free-field correlation functions (i.e. the ones arising from Wick contractions of free fields) and no others.

Now, let's impose the higher-spin constraint, which stipulates that the correlation function must be a sum of terms g_σ which have the functional form given by (5.8). Since

⁵As we remarked before, this is only true a priori if we subtract off the bosonic and fermionic pieces, but we will show in section 6 that if any one of the three lightcone limits are nonzero, it follows that the other two are zero, so this subtraction procedure is not actually necessary.

that constraint only involves the dependence in the x^- direction, it does not constrain the numerator, which involves only terms involving the x_i^+ variables. However, it does restrict the denominator to only have each index i involved in a power d_{ik} for one specific k since d_{ik} does depend nontrivially on x_{ik}^- . That is, the denominator is built out of terms like d_{ik}^{d+2} . This constrains the f_i powerfully. Since each cross ratio separately violates the higher spin constraint, the only f_i that can appear are the ones whose product with a denominator satisfying the higher-spin constraint is another denominator satisfying that constraint. That is, once we have picked a denominator, the f_i can only be very specific kinds of rational functions. We can still generate terms that don't appear in the free-field result, however, because the spin structure in the numerator doesn't have to match the index structure of the denominator. For example, the following structure could in principle appear in the four-point function of V_{--} , but obviously this structure is not generated in the free theory:

$$\frac{(x_{14}^+)^2(x_{27}^+)^2(x_{36}^+)^2(x_{58}^+)^2}{(d_{13}d_{24}d_{57}d_{68})^{d+2}} \quad (5.17)$$

This structure has a numerator which is consistent with free field theory but a denominator that does not match the result one would obtain from the free field propagator. Another possibility is to write a structure where the numerator corresponds to the connected part of the free-field correlator — i.e. the two factors of x_{ij}^+ appear with different j for some i .

$$\frac{x_{13}^+x_{32}^+x_{28}^+x_{86}^+x_{67}^+x_{75}^+x_{54}^+x_{41}^+}{(d_{13}d_{24}d_{57}d_{68})^{d+2}} \quad (5.18)$$

Purely on symmetry considerations, these terms are consistent with the general structure (5.16). Indeed, one can set (5.17) and (5.18) equal to (5.16) to explicitly solve for the function $f_i(\{u_j\})$ that generates it, and one can check that this f_i is indeed conformally invariant, as required. These structures are inconsistent, however, with cluster decomposition. To see this, we examine the tensor analogue of (5.4):

$$V_{--}(x_1, x_2) = \sum_{\text{even } s \geq 2} v_{--}^s(x_1, x_2) \quad (5.19)$$

$$v_{--}^s(x_1, x_2) = \sum_{k,l} c_{kl} (x_1 - x_2)^k \partial^l j_s \left(\frac{x_1 + x_2}{2} \right) \quad (5.20)$$

Comparing the conformal dimension of the left and right hand side yields the constraint that $s + l - k = 2$. Hence, by setting $x_1 = x_2$, we extract the $k = 0$ piece, forcing $l = 0$ and $s = 2$ (since $s = 1$ is not realized in the tensor sector). That is, $V_{--}(x, x) = T_{--}(x)$. By performing this projection on each factor of V_{--} in the correlation function (i.e. setting $x_1 = x_2$, $x_3 = x_4$, etc.), we obtain an expression for the n -point function of T_{--} , which we know must satisfy cluster decomposition since T is a local operator. Then, by taking the points to be separated very far apart from each other, we obtain constraints on how the structures must simplify. For example, we know that if we take x_1 and x_3 to be very far from all the other points, we must have that

$$\langle T_{--}(x_1)T_{--}(x_3) \dots T_{--}(x_{2n-1}) \rangle \implies \langle T_{--}(x_1)T_{--}(x_3) \rangle \langle T_{--}(x_5) \dots T_{--}(x_{2n-1}) \rangle \quad (5.21)$$

This factorization property is not satisfied by the structure (5.17), for example. Indeed, the only way to satisfy all such constraints arising from cluster decomposition is to have all powers of x_{ij}^+ appear with the corresponding factor of d_{ij}^{d-2} in the denominator, modulo trivial equalities such as $x_{13}^+ = x_{14}^+$ (which arise since points which are taken to be — separated in the lightcone limit have the same difference in the + direction). If it appears with the wrong d_{ij} factor in the denominator (again, modulo the trivial relabelings of the spin structure), it cannot satisfy the cluster decomposition identity arising from taking the two points appearing in that factor to be very far from all the other points. The spin structure required by the factorization will simply not be present.

Hence, the only allowed terms are the ones that are built from free-field propagators $(x_{ij}^+)^2/d_{ij}^{d+2}$. Permutation symmetry implies that the coefficients of all the structures that can appear are the same up to disconnected terms which are fixed, as before, by cluster decomposition. This implies that the n -point function of bilocals V_{--} are exactly the same as the n -point function of stress tensors in free field theory up to a possible overall constant.

Clearly, this entire argument works for the fermionic case as well with only minor modifications - the projection procedure that isolates the contribution from the stress tensor is slightly more complicated since it appears at first order, not zeroth order, in x_{12} in the fermionic analogue of (5.20), and the correlation function is permutation anti-symmetric instead of symmetric because fermions anticommute. All other steps are the same, and we conclude that in the fermionic case, the n -point functions of bilocals are also given by the free field result. For example, the two-point functions of fermionic and tensor quasi-bilocals are given by

$$\langle F_-(x_1, x_2) F_-(x_3, x_4) \rangle = \tilde{N}_f \left(\frac{x_{13}^+ x_{24}^+}{d_{13}^d d_{24}^d} - \frac{x_{14}^+ x_{23}^+}{d_{14}^d d_{23}^d} \right) \quad (5.22)$$

$$\langle V_{--}(x_1, x_2) V_{--}(x_3, x_4) \rangle = \tilde{N}_t \left(\frac{(x_{13}^+)^2 (x_{24}^+)^2}{d_{13}^{d+2} d_{24}^{d+2}} + \frac{(x_{14}^+)^2 (x_{23}^+)^2}{d_{14}^{d+2} d_{23}^{d+2}} \right) \quad (5.23)$$

where \tilde{N}_f and \tilde{N}_t are overall constants that we will presently analyze.

5.4 Normalization of the quasi-bilocal correlation functions

Now, let's fix the the overall constants \tilde{N}_b , \tilde{N}_f , and \tilde{N}_t in front of each n -point function. We claim that they all are fixed by the normalization of the two-point function of the bilocals. This can be seen by considering how one can obtain the n -point function of quasi-bilocals from the $n - 1$ point function. We know the n -point function of some quasi-bilocal \mathcal{A} is:

$$\underbrace{\langle \mathcal{A} \dots \mathcal{A} \rangle}_{n \text{ copies of } \mathcal{A}} = \tilde{N}_n g(d_{ij}) \quad (5.24)$$

where g is some known function which agrees with the result for the n -point function of the corresponding free theory bilocal. Each bilocal contains the stress tensor j_2 in its OPE, so we can consider acting on both sides with the projector P which isolates the contribution of j_2 from the first bilocal. We have already seen, for example, that for the tensor bilocal, this projector just sets $x_1 = x_2$. Then, we can integrate over the coordinate x_1 . This

yields the action of the dilatation operator on the $n - 1$ point function, whose eigenvalue will be some multiple of the conformal dimension of the appropriate free field. So by this procedure, we can fix the coefficient in front of the n -point function in terms of the $n - 1$ point function. So by recursion, all the coefficients of the correlation functions are fixed by the coefficient \tilde{N} appearing in front of the two-point function.

6 Constraining all the correlation functions

We have shown now that the n -point functions of all the quasi-bilocal fields exactly coincide with the corresponding free-field result for a theory of N free fields of appropriate spin for some N (which we will show later must be an integer). Now, we will explain how to use these facts to constrain all the other correlation functions of the theory. We will start by proving that the three point function $\langle 222 \rangle$ must be either equal to the result for a free boson, a free fermion, or a free $\frac{d-2}{2}$ form. That is, if we write the most general possible form:

$$\langle 222 \rangle = c_b \langle 222 \rangle_{\text{free boson}} + c_f \langle 222 \rangle_{\text{free fermion}} + c_t \langle 222 \rangle_{\text{free tensor}}, \quad (6.1)$$

then the result will be consistent with higher-spin symmetry only if $(c_b, c_f, c_t) \propto (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$.

We first show that if $\langle \underline{22}_b 2 \rangle \neq 0$ then $\langle \underline{22}_f 2 \rangle = 0 = \langle \underline{22}_t 2 \rangle$. Consider the action of Q_4 on $\langle \underline{22}_b 2 \rangle$. By exactly the same analysis as the charge conservation identities of section 3, we obtain exactly the same expression as equation (3.3), except the summation starts from $j = 0$. Thus, the existence of the spin 4 current implies the existence of a spin 0 current with $\langle \underline{22}_b 0 \rangle \neq 0$. The action of charge Q_4 on j_0 is

$$[Q_4, j_0] = \partial^3 j_0 + \partial j_2 + \text{no overlap with } \underline{22}_b \quad (6.2)$$

Now consider the charge conservation identities arising from the action of Q_4 on $\langle \underline{22}_f 0 \rangle$ and $\langle \underline{22}_t 0 \rangle$. Since $\langle \underline{22}_f 0 \rangle = 0 = \langle \underline{22}_t 0 \rangle$, we conclude $\langle \underline{22}_f 2 \rangle = 0 = \langle \underline{22}_t 2 \rangle$, as desired.

Now, assume that $\langle \underline{22}_b 2 \rangle = 0$. It suffices to show that if $\langle \underline{22}_t 2 \rangle \neq 0$, then $\langle \underline{22}_f 2 \rangle = 0$. In this case, by hypothesis, the quasi-bilocal V_{--} is nonzero. The results of the previous section tell us that the three point function of the tensor quasi-bilocal is proportional to:

$$\langle V_{--}(\underline{x}_1, \underline{x}_2) V_{--}(\underline{x}_3, \underline{x}_4) V_{--}(\underline{x}_5, \underline{x}_6) \rangle \propto \frac{(x_{13}^+)^2 (x_{25}^+)^2 (x_{46}^+)^2}{d_{13}^{\frac{d}{2}+2} d_{25}^{\frac{d}{2}+2} d_{46}^{\frac{d}{2}+2}} + \text{perm.} \quad (6.3)$$

and this precisely coincides with the three-point function of the free field operator $v_{--}(x_1, x_2) =: F_{-\{\alpha\}}(x_1) F_{-\{\alpha\}}(x_2) :$

$$\langle V_{--}(x_1, x_2) V_{--}(x_3, x_4) V_{--}(x_5, x_6) \rangle \propto \langle v_{--}(x_1, x_2) v_{--}(x_3, x_4) v_{--}(x_5, x_6) \rangle \quad (6.4)$$

Now, take x_1 and x_2 very close together and expand both sides of this equation in powers of $(x_1 - x_2)$. The zeroth order term of v is clearly the normal ordered product $: F_{-\alpha}(\frac{x_1+x_2}{2}) F_{-\alpha}(\frac{x_1+x_2}{2}) :$ — this is precisely the free field stress tensor. On the other hand, we know from the previous section that the term in V_{--} which is zeroth order in

$(x_1 - x_2)$ — i.e. the term that arises from setting $x_1 = x_2$, is just the stress tensor of the theory T_{--} . Repeating the same procedure for the pairs of coordinates (x_3, x_4) and (x_5, x_6) , we obtain the desired result:

$$\langle 222 \rangle = \langle 222 \rangle_{\text{free tensor}} \tag{6.5}$$

$$\Rightarrow \langle \underline{22}_f 2 \rangle = \langle \underline{22}_b 2 \rangle = 0 \tag{6.6}$$

as required. Therefore, since the stress-energy tensor is unique,

$$\langle 222 \rangle_b \neq 0 \Rightarrow \langle 222 \rangle_f = 0, \quad \langle 222 \rangle_t = 0, \quad \underline{j_2 j_2}_b = \sum_{k=0}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_f = 0, \quad \underline{j_2 j_2}_t = 0, \tag{6.7}$$

$$\langle 222 \rangle_f \neq 0 \Rightarrow \langle 222 \rangle_b = 0, \quad \langle 222 \rangle_t = 0, \quad \underline{j_2 j_2}_f = \sum_{k=1}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_b = 0, \quad \underline{j_2 j_2}_t = 0, \tag{6.8}$$

$$\langle 222 \rangle_t \neq 0 \Rightarrow \langle 222 \rangle_b = 0, \quad \langle 222 \rangle_f = 0, \quad \underline{j_2 j_2}_t = \sum_{k=1}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_b = 0, \quad \underline{j_2 j_2}_f = 0, \tag{6.9}$$

where square brackets denotes currents and their descendants. This establishes the claim that the three-point function of the stress tensor coincides with the answer for some free theory.

At this point, we would like to stress that the factorization property we have proven here holds only for conformal field theories that satisfy the unitarity bound for the dimensions of operators. Clearly, all unitary CFT's have this property, but it is possible to conceive of non-unitary CFT's which also satisfy it. Without the unitarity bound's constraint on operator dimensions, however, various operators we have not considered could appear in all the charge conservation identities we have written. These operators make it possible to construct theories where the three-point function of the stress tensor decomposes as a nontrivial superpositions of the bosonic, fermionic, and tensor sectors. For example, we show in appendix F that the free five-dimensional Maxwell field is a non-unitary conformal field theory whose stress tensor decomposes into a superposition of all three sectors.

Returning to the main argument, we may now obtain all the other correlation functions, we may expand equation (6.4) to higher orders in $x_1 - x_2$, and use the correlation functions obtained at lower orders to fix the ones that appear at higher orders. For example, at second order in $x_1 - x_2$, we have:

$$v_{--} = (x_1 - x_2)^2 \left(: \partial^2 F_{-\alpha} \left(\frac{x_1 + x_2}{2} \right) F_{-\alpha} \left(\frac{x_1 + x_2}{2} \right) : \right. \\ \left. + : \partial F_{-\alpha} \left(\frac{x_1 + x_2}{2} \right) \partial F_{-\alpha} \left(\frac{x_1 + x_2}{2} \right) : \right), \tag{6.10}$$

and V_{--} contains terms involving only the spin 2, 3, and 4 currents. Using our answers for $\langle 222 \rangle$ and our knowledge that $\langle 223 \rangle = 0$, we can then fix $\langle 224 \rangle$ to agree with the free field theory. This procedure recursively fixes all the correlators in the free tensor sector. The argument flows identically for the free bosonic and free fermionic sectors, except that

the zeroth order term will not fix $\langle 222 \rangle$, but some lower-order current. For example, in the bosonic theory, the zeroth order term will fix $\langle 000 \rangle$, and one will need to carry out the power series expansion to higher orders in order to fix the correlators of the higher-spin conserved currents.

Then, one could consider correlation functions that have indices set to values other than minus. This works in exactly the same way, since the operator product expansion of two currents with minus indices will contain currents with other indices. This has the effect of doubling the number of bilocals required to build a correlation function, since we need to take an extra OPE to fix the index structure. Thus, an n -point function with non-minus indices can be fixed from $2n$ bilocals. Thus, we have fixed every correlation function from currents at appear in successive OPE's of two stress tensors, including those of every higher-spin current.

The last thing we will argue is that the normalization of the correlation functions matches the normalization for some free theory. For example, in the theory of N free bosons, the two-point function of $\sum_{i=1}^N : \phi_i \phi_i^* :$ will have overall coefficient N . The same is true for the fermionic and tensor cases. One might wonder if the overall coefficient \tilde{N} of the quasi-bilocal could be non-integer, which would imply that it could not coincide with any theory of N free bosons. We will now argue that this is not possible. We start with the bosonic case, which works similarly to the argument presented in [1]:

in a theory of N free bosons, consider the operator

$$\mathcal{O}_{q,\text{free}} = \delta_{[j_1, \dots, j_q]}^{[i_1, \dots, i_q]} (\phi^{i_1} \partial \phi^{i_2} \dots \partial^{q-1} \phi^{i_q}) (\phi^{j_1} \partial \phi^{j_2} \dots \partial^{q-1} \phi^{j_q}) \quad (6.11)$$

Here, δ is the totally antisymmetric delta function that arises from a partial contraction of ϵ symbols:

$$\delta_{[j_1, \dots, j_q]}^{[i_1, \dots, i_q]} \propto \epsilon^{i_1, \dots, i_q, i_{q+1}, \dots, i_N} \epsilon_{j_1, \dots, j_q, i_{q+1}, \dots, i_N} \quad (6.12)$$

We claim that in the full theory, there exists an operator \mathcal{O}_q whose correlation functions coincide with the correlation functions of $\mathcal{O}_{q,\text{free}}$ in the free theory. The proof of this is given in appendix E.

Consider the norm of the state that \mathcal{O}_q generates. This is computed by the two point function $\langle \mathcal{O}_q \mathcal{O}_q \rangle$. It is obvious from the definition of \mathcal{O}_q that it arises from the contraction of q bilocal fields, so this correlator is a polynomial in N of order q . The antisymmetry of the totally antisymmetric function in the definition of $\mathcal{O}_{q,\text{free}}$ enforces that the correlation function vanishes at $q > N$. So we know all the roots of the polynomial, and hence the correlation function is proportional to $N(N-1) \dots (N-(q-1))$. Now, consider an analytic continuation of this correlator to non-integer \tilde{N} . By taking $q = \lfloor N \rfloor + 2$, we find that this product is negative, which is impossible for the norm of a state. Since the correlators of \mathcal{O}_q are forced to agree with the correlators of some operator in the full CFT, we conclude that the normalization \tilde{N} of the scalar quasi-bilocals must be an integer.

The same argument can be ran in the tensor case for an operator defined similarly:

$$\mathcal{O}_q = \delta_{[j_1, \dots, j_q]}^{[i_1, \dots, i_q]} (F_{-\{\alpha_1\}}^{i_1} \partial F_{-\{\alpha_2\}}^{i_2} \dots \partial^{q-1} F_{-\{\alpha_q\}}^{i_q}) (F_{-\{\alpha_1\}}^{j_1} \partial F_{-\{\alpha_2\}}^{j_2} \dots \partial^{q-1} F_{-\{\alpha_q\}}^{j_q}) \quad (6.13)$$

We again conclude that the normalization constant \tilde{N} must be an integer.

The construction in the fermionic case is somewhat simpler. We know j_2 appears in F_- , and we can define an operator $\mathcal{O}_q = (j_2)^q$ by extracting the term in the operator product expansion of q copies of j_2 whose correlation functions coincide with the free fermion operator $(j_2)_{\text{free}}^q$. In the theory of N free fermions, $j_2 = \sum_i (\partial\psi_i)\gamma_- \bar{\psi}_i - \psi_i\gamma_- (\partial\bar{\psi}_i)$, where here i is the flavor index for the N fermions. By antisymmetry of the fermions, we know that \mathcal{O}_q will be zero if $q \geq N$. Then, as in the bosonic case, we can consider the norm of the state that \mathcal{O}_q generates, which is computed by $\langle \mathcal{O}_q \mathcal{O}_q \rangle$, and the rest of the argument runs as before. Thus, the normalization \tilde{N} of the fermionic bilocals must be an integer.

It is worth noting the relationship between this result and one of the primary motivations for studying higher-spin CFT's — holographic dualities involving Vasiliev gravity in an anti-de Sitter space. As mentioned earlier, it has been conjectured that Vasiliev gravity is conjectured to be dual to a theory of N free scalar fields in the $O(N)$ singlet sector. This implies a relationship between the vacuum energy of Vasiliev gravity at tree-level and the free energy of a scalar field, namely, that $F_{\text{Vasiliev}}/G_N \sim NF_{\text{scalar}}$, where G_N is the Newton constant. Our result implies that this normalization constant N , and therefore, the Newton constant G_N is quantized in the Vasiliev theory in any dimension.

It must be noted, however, that we cannot claim that this quantization can be seen perturbatively in N . Recent work of Giombi and Klebanov [24] have shown that the one-loop correction to the vacuum energy of minimally coupled type A Vasiliev gravity in anti-de Sitter background does not vanish as expected. This was interpreted as a shift of $N \rightarrow N-1$ in the tree-level calculation of the vacuum free energy. Our result cannot predict such a shift or any other $1/N$ corrections that appear in higher orders in perturbation theory. We claim only that the exact result, after summing all loop corrections, must be quantized.

7 Discussion and conclusions

In this paper, we have shown that in a unitary conformal field theory in $d > 3$ dimensions with a unique stress tensor and a symmetric conserved current of spin higher than 2, the three-point function of the stress tensor must coincide with the three-point function of the stress tensor in either a theory of free bosons, a theory of free fermions, or a theory of free $\frac{d-2}{2}$ -forms. This implies that all the correlation functions of symmetric currents of the theory coincide with the those in the corresponding free field theory.

Our technique was to use a set of appropriate lightcone limits to transform the data of certain key Ward identities into simple polynomial equations. Even though we could not directly solve for the coefficients in these identities like in three dimensions, we were nevertheless able to show that the only solution these Ward identities admit is the one furnished by the appropriate free field theory. This was the key step that allowed us to defined bilocal operators which were used to show that the three-point function of the stress tensor must agree with a free field theory. This in turn fixed all the other correlators of the theory to agree with those in the same free field theory. These results can be understood as an extension of the techniques and conclusions of [1] from three dimensions to all dimensions higher than three.

We stress that our classification into the bosonic, fermionic, and tensor free field theories depends somewhat sharply on our assumption that a unique stress tensor exists. Other

free field theories with higher spin symmetry exist in $d > 3$ dimensions, such as a theory of free gravitons. This theory, however, does not have a stress tensor, and we make no statement about how the correlation functions of such theories are constrained, and analogously for theories with many stress tensors. On the other hand, we may consider the possibility of multiple stress tensors. It was argued in [1] that the result holds if there are two stress tensors $\{j_2, \hat{j}_2\}$ instead of just one. The idea is that one can define a basis $\{J_2, \hat{J}_2\}$ of vectors $\{j_2, \hat{j}_2\}$ such that each vector generates action along itself. As a result correlation function of n currents J_2 and n currents \hat{J}_2 factorizes into product of correlation function of n currents J_2 and correlation function of n currents \hat{J}_2 . So one can argue that in some sense theories are dynamically decoupled. This argument carries over to our result totally unchanged, and so our result also holds in the case of two stress tensors. In the case of more than two stress tensors one have to find analogous basis. We do not comment on the possibility of more than two stress tensors.

In [25], it was shown that for $3d$ conformal theories with slightly broken higher-spin symmetry it is possible to constrain the three point functions in the leading order in large parameter N . Similar analysis for D dimensional case is out of the scope of the present paper. But we expect that similar conclusion should be valid.

Moreover, we have not computed correlation functions or commutators for asymmetric currents and charges. In [14], it was shown that if one considers the possible algebras of charges in theories that contain asymmetric currents in four dimensions, a one-parameter family of algebras exists. This may suggest the existence of nontrivial higher-spin theories, though our result indicates that at least the subalgebra generated by the symmetric currents must agree with free field theory.

We also stress that the tensor structure is not well understood in all dimensions. In even dimensions, it corresponds to the theory of a free $\frac{d-2}{2}$ -form field, which does not exist in odd dimensions. In odd dimensions, the structure may not exist, and even it does, there may not exist a conformal field theory which realizes it. Our argument only tells us that if there is a solution of the conformal and higher-spin Ward identities corresponding to this structure, then it is unique. If the structure exists, we only know for a fact that it contains an infinite tower of higher-spin currents for $d \geq 7$ and in this case, the theory, if it exists, has the correlation functions we claimed. In $d = 5$, it is not known if all the higher-spin currents must be present. Assuming they are present, our results also flow through in $d = 5$. Even then, the tensor structure in odd dimensions could fail to have a good microscopic interpretation for many other reasons. For example, the four-point function of the stress tensor in this sector may not be consistent with the operator product expansion in the sense that it may not be decomposable as a sum over conformal blocks — i.e. it may be possible to continue all the correlation functions to odd dimensions, but not the blocks. We have not explored this question.

Acknowledgments

We would like to thank J. Maldacena for countless helpful discussions and invaluable guidance on this project, A. Zhiboedov for many useful discussions and collaboration on parts

of this paper, and J. Penedones for his help understanding the role of the collinear conformal group in the fermionic and tensor cases. We would also like to thank E. Skvortsov and D. Ponomarev for their help in understanding their recent paper.

The work of VA was supported by the National Science Foundation under Grant No. PHY-1314198. The work of KD was supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE 1148900.

A Form factors as Fourier transforms of correlation functions

In this appendix, we will explicitly calculate the Fourier-transformed, lightcone-limit three-point functions F_s^b , F_s^f , and F_s^v cited in section 2. Let's start with the bosonic case. We want to compute the relevant Fourier transformation of the three-point function $\langle \phi(x_1)\phi^*(x_2)j_s(x_3) \rangle$. The explicit form of $j_s(x_3)$ is given in [26] as:

$$j_s = \sum_{k=0}^s c_k \partial^k \phi \partial^{s-k} \phi^*, \quad c_k = \frac{(-1)^k}{2} \frac{\binom{s}{k} \binom{s+d-4}{k+\frac{d}{2}-2}}{\binom{s+d-4}{\frac{d}{2}-2}} \quad (\text{A.1})$$

Wick's theorem and translation invariance of the correlators yields that::

$$\langle \phi(x_1)\phi^*(x_2)j_s(x_3) \rangle = \sum c_i (\partial_3^i \langle \phi(x_1)\phi^*(x_3) \rangle) (\partial_3^{s-i} \langle \phi(x_3)\phi^*(x_2) \rangle) \quad (\text{A.2})$$

$$= \sum c_i (\partial_1^i \langle \phi(x_1)\phi^*(x_3) \rangle) (\partial_2^{s-i} \langle \phi(x_3)\phi^*(x_2) \rangle) \quad (\text{A.3})$$

Then, we may Fourier transform term by term with respect to x_1^- and x_2^- . Recalling that the propagator of a scalar field is $(x^2)^{\frac{2-d}{2}}$ and that in the lightcone limit, $x_1^+ = x_2^+$ and $\vec{y}_1 = \vec{y}_2$, we obtain:

$$\begin{aligned} & \partial_1^{s-i} \partial_2^i \langle \phi(x_1)\phi^*(x_3) \rangle \langle \phi(x_3)\phi^*(x_2) \rangle \\ & \longrightarrow i^s (p_1^+)^{s-i} (p_2^+)^i \int dx_1^- dx_2^- e^{ip_1^+ x_1^-} e^{ip_2^+ x_2^-} \frac{1}{(x_{13}^+ x_{13}^- + \vec{y}_{13}^2)^{\frac{d-2}{2}}} \frac{1}{(x_{23}^+ x_{23}^- + \vec{y}_{23}^2)^{\frac{d-2}{2}}} \end{aligned} \quad (\text{A.4})$$

$$= \frac{i^s (p_1^+)^{s-i} (p_2^+)^i}{(x_{13}^+)^{d-2}} \int dx_1^- dx_2^- e^{ip_1^+ x_1^-} e^{ip_2^+ x_2^-} \frac{1}{(x_{13}^- + \frac{\vec{y}_{13}^2}{x_{13}^+})^{\frac{d-2}{2}}} \frac{1}{(x_{23}^- + \frac{\vec{y}_{13}^2}{x_{13}^+})^{\frac{d-2}{2}}} \quad (\text{A.5})$$

$$= \frac{i^s (p_1^+)^{s-i} (p_2^+)^i}{(x_{13}^+)^{d-2}} \left(\int dx_1^- e^{ip_1^+ x_1^-} \frac{1}{(x_1^- - \bar{x})^{\frac{d-2}{2}}} \right) \left(\int dx_2^- e^{ip_2^+ x_2^-} \frac{1}{(x_2^- - \bar{x})^{\frac{d-2}{2}}} \right) \quad (\text{A.6})$$

Here, we have defined $\bar{x} = x_3^- - \frac{\vec{y}_{13}^2}{x_{13}^+}$. Depending on the parity of d , each integral has either a pole of order $\frac{d-2}{2}$ at \bar{x} or a branch point at \bar{x} . Our prescription for evaluating this integral is as follows: first, we shift x_1^- and x_2^- by \bar{x} so that the singularity is at 0, and then we will move the singularity from 0 to $\text{sign}(p)\epsilon$. Then, the integral can be evaluated by Schwinger parameterization. For example, suppose p_1^+ and p_2^+ are positive. Following our procedure, the x_1 integral becomes:

$$\int_{-\infty}^{\infty} dx_1^- e^{ip_1^+ x_1^-} \frac{1}{(x_1^- - \bar{x})^{\frac{d-2}{2}}} = e^{ip_1^+ \bar{x} + p_1^+ \epsilon} \int_{-\infty}^{\infty} dy e^{ip_1^+ y} \frac{1}{(y - i\epsilon)^{\frac{d-2}{2}}} \quad (\text{A.7})$$

$$= e^{ip_1^+ \bar{x} + p_1^+ \epsilon} \int_{-\infty}^{\infty} dy \int_0^{\infty} ds \frac{i}{\Gamma(\frac{d-2}{2})} e^{ip_1^+ y} s^{\frac{d-4}{2}} e^{-is(y-i\epsilon)} \quad (\text{A.8})$$

$$= \frac{ie^{ip_1^+ \bar{x} + p_1^+ \epsilon}}{\Gamma(\frac{d-2}{2})} \int_0^{\infty} ds 2\pi \delta(s - p_1^+) e^{ip_1^+ y} s^{\frac{d-4}{2}} e^{-s\epsilon} \quad (\text{A.9})$$

$$= \frac{2\pi i e^{ip_1^+ \bar{x}}}{\Gamma(\frac{d-2}{2})} (p_1^+)^{\frac{d-4}{2}} \quad (\text{A.10})$$

This function is indeed nonsingular, as required. The x_2 integral has exactly the same form, and so gives the same answer. Hence, we obtain that the Fourier transform of $\langle \phi \phi^* j_s \rangle$ is indeed proportional to $\sum c_i (p_1^+)^i (p_2^+)^{s-i}$, where the proportionality factor is a nonsingular function. The, noting that the coefficients c_i are the coefficients for the hypergeometric function with appropriate arguments, we obtain the answer cited in the text:

$$F_s^b \equiv \langle \phi \phi^* j_s \rangle \propto (p_2^+)^s {}_2F_1 \left(2 - \frac{d}{2} - s, -s, \frac{d}{2} - 1, p_1^+/p_2^+ \right) \quad (\text{A.11})$$

The fermionic and tensor cases can be tackled in exactly the same way. There are only two differences. First, the propagator in the free fermion and free tensor theories are $(x^2)^{\frac{1-d}{2}}$ and $(x^2)^{\frac{-d}{2}}$, respectively, as compared with the free scalar propagator $(x^2)^{\frac{2-d}{2}}$. Second, the coefficients in the expression for j_s are different, as can be checked from the expressions in [27, 28] or in [26]. The end result is that the arguments of the hypergeometric function are different in the way claimed in the text.

B Uniqueness of three-point functions in the tensor lightcone limit

Our goal in this section is to show that the free tensor solution for the lightcone limit of three-point functions explained in section 2 is indeed unique, at least in the lightcone limit.

Note that Lorentz symmetry constrains the propagator of spin j field to be of the form

$$\langle \psi_{-j}(x) \bar{\psi}_{-j}(0) \rangle \propto (x^+)^{2j}. \quad (\text{B.1})$$

Generically, according to [19], the most generic conformally invariant expression one can write down for a three-point function of symmetric conserved currents with tensor-type coordinate dependence is:

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \frac{1}{x_{12}^{d-2} x_{23}^{d-2} x_{13}^{d-2}} \sum_{a,b,c} \left((\Lambda_1^2 \alpha_{a,b,c} + \Lambda_2 \beta_{a,b,c}) (P_{12} P_{21})^a Q_1^b \right. \\ \left. \times (P_{23} P_{32})^c (P_{13} P_{31})^{-a-b+s_1} Q_2^{-a-c+s_2} Q_3^{a+b-c-s_1+s_3} \right) \quad (\text{B.2})$$

where the $\alpha_{a,b,c}$ and $\beta_{a,b,c}$ are free coefficients, and the Λ_i are defined as:

$$\Lambda_1 = Q_1 Q_2 Q_3 + [Q_1 P_{23} P_{32} + Q_2 P_{13} P_{31} + Q_3 P_{12} P_{21}], \quad (\text{B.3})$$

$$\Lambda_2 = 8 P_{12} P_{21} P_{23} P_{32} P_{13} P_{31}. \quad (\text{B.4})$$

Here, the P and Q invariants are defined as in [29] and [30]. However, for the choice of polarization vector $\epsilon^\mu = \epsilon^-$ there is a nontrivial relation:

$$\Lambda_2|_{\epsilon_i^\mu = \epsilon^-} = -2\Lambda_1^2|_{\epsilon_i^\mu = \epsilon^-}, \quad \Lambda_1|_{\epsilon_i^\mu = \epsilon^-} = \frac{1}{4} \frac{x_{12}^+ x_{23}^+ x_{13}^+}{x_{12}^2 x_{23}^2 x_{13}^2} (\epsilon^-)^3. \quad (\text{B.5})$$

Therefore, in the case $\epsilon^\mu = \epsilon^-$ the expression for this three-point function greatly simplifies. Instead of having two sets of undetermined coefficients c_a and d_a , one can combine the Λ_i 's into a single prefactor $\alpha_1 \Lambda_1^2 + \alpha_2 \Lambda_2$, where the α_i are arbitrary and can be chosen to be convenient; to produce exact agreement with the canonically normalized free-tensor theory, we will choose $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2(d-2)}$. Now, we take the lightcone limit, which corresponds to the point where

$$P_{23}P_{32} = 0, \quad Q_1 = - \left(\frac{P_{13}P_{31}}{Q_3} + \frac{P_{12}P_{21}}{Q_2} \right) \quad (\text{B.6})$$

in P_{ij}, Q_i space. Then, the three-point function reduces to

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \frac{\Lambda_1^2 + \Lambda_2 / (2(d-2))}{x_{12}^{d-2} x_{23}^{d-2} x_{13}^{d-2}} \sum_{a=0}^{s_1-2} c_a (P_{12}P_{21})^a (P_{13}P_{31})^{s_1-2-a} Q_2^{s_2-a} Q_3^{s_3-s_1+a}, \quad (\text{B.7})$$

Now, the c_a can be fixed demanding that all currents are conserved. The result is given by the following recurrence relation, with $c_0 = 1$:

$$\frac{c(a+1)}{c(a)} = \frac{(s_1 - 2 - a)(s_1 + \frac{d-4}{2} - a)(s_2 + a + \frac{d-2}{2})}{(a+1)(a + \frac{d-2}{2} + 2)(s_1 + s_3 + \frac{d-4}{2} - 2 - a)}$$

This solution exactly coincides with the free tensor solution, as required.

C Uniqueness of $\langle s22 \rangle$ for $s \geq 4$

Define

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \frac{\langle \langle j_{s_1} j_{s_2} j_{s_3} \rangle \rangle}{x_{12}^{d-2} x_{23}^{d-2} x_{13}^{d-2}}. \quad (\text{C.1})$$

Using the previous defined V and H conformal invariants, we can write the most general expression for a conformally invariant correlation function as follows:

$$\begin{aligned} \langle \langle j_s j_2 j_2 \rangle \rangle = & V_1^{s-4} \left[a_1 H_{1,2}^2 H_{1,3}^2 + a_2 (V_1 V_2 H_{1,2} H_{1,3}^2 + V_1 V_3 H_{1,2}^2 H_{1,3}) + a_3 V_1^2 H_{1,2} H_{1,3} H_{2,3} + \right. \\ & + a_4 (V_1^2 V_3^2 H_{1,2}^2 + V_1^2 V_2^2 H_{1,3}^2) + a_5 V_1^2 V_2 V_3 H_{1,2} H_{1,3} + \\ & + a_6 (V_1^3 V_2 H_{1,3} H_{2,3} + V_1^3 V_3 H_{1,2} H_{2,3}) + a_7 (V_1^3 V_2 V_3^2 H_{1,2} + V_1^3 V_2^2 V_3 H_{1,3}) + \\ & \left. + a_8 V_1^4 H_{2,3}^2 + a_9 V_1^4 V_2 V_3 H_{2,3} + a_{10} V_1^4 V_2^2 V_3^2 \right]. \quad (\text{C.2}) \end{aligned}$$

The coefficients can be solved by imposing charge conservation. For example, in $d = 4$ we obtain:

$$a_1 = - \frac{a_7 (s-3)(s-1)(s-2)^2}{32(s+1)(s+4)} + \frac{a_4 (s-5)(s-3)s(s-2)}{8(s+1)(s+4)} + \frac{a_5 (s-3)(s-2)}{8(s+4)}, \quad (\text{C.3})$$

$$a_2 = -\frac{a_4(s-2)^2}{s+4} + \frac{a_7(s-1)(s-2)}{4(s+4)} - \frac{a_5(s-2)}{2(s+4)}, \quad (\text{C.4})$$

$$a_3 = -\frac{8a_4(s^2-3s-1)}{(s+1)(s+4)} + \frac{a_5(s-8)}{2(s+4)} + \frac{a_7(s-1)(2s-1)}{(s+1)(s+4)}, \quad (\text{C.5})$$

$$a_6 = \frac{12a_4(s-2)}{(s-1)(s+4)} + \frac{6a_5}{(s-1)(s+4)} + \frac{a_7(s-2)}{2(s+4)}, \quad (\text{C.6})$$

$$a_8 = \frac{a_7(s-2)(s^2+11s-2)}{4s(s+1)(s+4)} - \frac{6a_4(s-5)}{(s+1)(s+4)} + \frac{a_5(s-2)}{s(s+4)}, \quad (\text{C.7})$$

$$a_9 = \frac{a_7(s^2+8s-8)}{s(s+4)} - \frac{24a_4(s-2)}{(s-1)(s+4)} + \frac{4a_5(s-2)(s+2)}{(s-1)s(s+4)}, \quad (\text{C.8})$$

$$a_{10} = \frac{a_7(s^2+8s+4)}{s(s+4)} - \frac{24a_4(s+1)}{(s-1)(s+4)} + \frac{4a_5(s+1)(s+2)}{(s-1)s(s+4)}. \quad (\text{C.9})$$

Therefore, $\langle\langle j_s j_2 j_2 \rangle\rangle_t$ depends only on three parameters. The bosonic light-cone limit of this function is zero if

$$a_5 = \frac{a_7(s-2)(s-1)}{4(s+1)} - \frac{a_4(s-5)s}{s+1}. \quad (\text{C.10})$$

The fermionic light-cone limit of this function is also zero if

$$a_4 = \frac{a_7}{4}. \quad (\text{C.11})$$

Therefore, $\langle\langle s22 \rangle\rangle_t$ depends only on one parameter or in other words it is unique up to a rescaling⁶

$$\langle\langle j_s j_2 j_2 \rangle\rangle_t \propto V_1^{s-2} \left[H_{12}^2 V_3^2 + (H_{23} V_1 + V_2 (H_{13} + 2V_1 V_3))^2 + H_{12} (H_{13} + 2V_1 V_3) (H_{23} + 2V_2 V_3) \right], \quad (\text{C.12})$$

In arbitrary dimension $d > 3$, the full expression is:

$$\begin{aligned} \langle\langle j_s j_2 j_2 \rangle\rangle_t &= V_1^{s-2} \left[(H_{23} V_1 + H_{13} V_2 + H_{12} V_3 + 2V_2 V_3 V_1)^2 + \frac{2}{(d-2)} H_{12} H_{13} H_{23} \right] \\ &= V_1^{s-2} \left[\Lambda_1^2 + \frac{1}{2(d-2)} \Lambda_2 \right]. \end{aligned} \quad (\text{C.13})$$

This formula coincides with the expression that was proposed in [19], and we have proven that this structure is unique.

D Transformation properties of bilocal operators under K_-

In this appendix, we will prove (5.5) and (5.6) by computing the action of a finite conformal transformation on them. The same results can be proven using the infinitesimal transformations, e.g. by using equation (3) of [31] and supplying the correct representation matrices for the Lie algebra of the Lorentz group. One can then check that the two computations agree by expanding our results to first order in b (remembering that only b^- is nonzero for K_-).

⁶In [18] it was proven that there are only three structures for $\langle\langle 22s \rangle\rangle$ in $d=4$.

D.1 Fermionic case

Consider a special conformal transformation

$$x^\mu \rightarrow y^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2} \quad (\text{D.1})$$

Under K_- , the parameter $b^\mu = b^- \delta_-^\mu$. We know that F_- has the same transformation properties as the contraction of free fields $\bar{\psi} \gamma_- \psi$ on the lightcone. Since K_- sends the lightcone into the lightcone, V_- transforms the same way as $\bar{\psi} \gamma_- \psi$ under K_- . Using the well-known expression for the finite conformal transformation of a Dirac spinor (e.g. [32])

$$\psi(y) = \left| \frac{\partial y}{\partial x} \right|^{\Delta-1/2} (1 - b_\mu x_\nu \gamma^\mu \gamma^\nu) \psi(x) \quad (\text{D.2})$$

$$\bar{\psi}(y) = \left| \frac{\partial y}{\partial x} \right|^{\Delta-1/2} \bar{\psi}(x) (1 - b_\mu x_\nu \gamma^\nu \gamma^\mu) \quad (\text{D.3})$$

we may therefore compute:

$$F_-(y_1, y_2) \sim \bar{\psi}(y_1) \gamma^+ \psi(y_2) \quad (\text{D.4})$$

$$= \left| \frac{\partial y_1}{\partial x_1} \right|^{\Delta-1/2} \left| \frac{\partial y_2}{\partial x_2} \right|^{\Delta-1/2} \bar{\psi}(x_1) (1 - b_\mu(x_1)_\nu \gamma^\nu \gamma^\mu) \gamma^+ (1 - b_\mu(x_2)_\nu \gamma^\mu \gamma^\nu) \psi(x_2) \quad (\text{D.5})$$

$$= \left| \frac{\partial y_1}{\partial x_1} \right|^{\Delta-1/2} \left| \frac{\partial y_2}{\partial x_2} \right|^{\Delta-1/2} \bar{\psi}(x_1) \times [\gamma^+ - b_+(x_1)_\nu \gamma^\nu \gamma^+ \gamma^+ - \gamma^+ b_+(x_2)_\nu \gamma^+ \gamma^\nu + b_+(x_1)_\nu \gamma^\nu \gamma^+ \gamma^+ b_+(x_2)_\mu \gamma^+ \gamma^\mu] \psi(x_2) \quad (\text{D.6})$$

$$= \left| \frac{\partial y_1}{\partial x_1} \right|^{\Delta-1/2} \left| \frac{\partial y_2}{\partial x_2} \right|^{\Delta-1/2} \bar{\psi}(x_1) \gamma^+ \psi(x_2) \quad (\text{D.7})$$

$$= \Omega^{d/2-1}(x_1) \Omega^{d/2-1}(x_2) F_-(x_1, x_2) \quad (\text{D.8})$$

The cancellations occur because $\gamma^+ \gamma^+ = \eta^{++} = 0$. This is exactly equation (5.5).

D.2 Tensor case

We'll start with the four-dimensional case for ease of notation and then at the end, we'll describe how one can generalize the computation to all dimensions. Consider a special conformal transformation

$$x^\mu \rightarrow y^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2} \quad (\text{D.9})$$

Under K_- , the parameter $b^\mu = b^- \delta_-^\mu$. We know that V_- has the same transformation properties as the contraction of free fields $F_{-\mu} F_-^\mu$ on the lightcone. Since K_- sends the lightcone into the lightcone, V_- transforms the same way as $F_{-\mu} F_-^\mu$ under K_- . We therefore compute:

$$V_-(y_1, y_2) = \left| \frac{\partial y_1}{\partial x_1} \right|^{-\tau_F/d} \left| \frac{\partial y_2}{\partial x_2} \right|^{-\tau_F/d} \frac{\partial x_1^\mu}{\partial y_1^\alpha} \frac{\partial x_1^\nu}{\partial y_1^\beta} \frac{\partial x_2^\lambda}{\partial y_2^\gamma} \frac{\partial x_2^\rho}{\partial y_2^\delta} \eta^{\alpha\beta} F_{\mu\nu}(x_1) F_{\lambda\rho}(x_2) \quad (\text{D.10})$$

$$= (1 - b^- x_1^+)^{\tau_F} (1 - b^- x_2^+)^{\tau_F} (1 - b^- x_1^+)^2 \eta^{\alpha\beta} F_{-\alpha}(x_1) F_{-\beta}(x_2) \quad (\text{D.11})$$

$$= (1 - b^- x_1^+) (1 - b^- x_2^+) V_{--}(x_1, x_2) \quad (\text{D.12})$$

$$= \Omega(x_1) \Omega(x_2) V_{--}(x_1, x_2) \quad (\text{D.13})$$

In the above manipulations, $\tau_F = \Delta - s = 0$ is the twist of F , and in the second to last line, we used that $x_1^+ = x_2^+$ (because the points x_1 and x_2 are — separated by hypothesis). This immediately implies (5.6) in the four-dimensional case. In general dimensions, the twist of F will not be 0, but rather $\Delta - s = d/2 - s$, and we will have a corresponding number of extra factors of $\partial x/\partial y$ to contract with the additional indices of F . This will make the exponent of the Ω factors equal to $\frac{d}{2} - 1$ instead of 1.

E Proof that \mathcal{O}_q exists

In this appendix, we will prove that an operator \mathcal{O}_q whose correlation functions agree with the corresponding free field operator $\mathcal{O}_{q,\text{free}}$ defined in (6.11) exists in the operator spectrum of every conformal field theory with higher-spin symmetry. As usual, we will consider the bosonic case, since the tensor case works almost in precisely the same way. To prove our statement, we will show that in the free theory, for any $q \leq N$

$$A_{q,N}(x_1, x_2, \dots, x_{q+1}) \equiv \underbrace{\langle \phi^2 \phi^2 \dots \phi^2 \mathcal{O}_{q,\text{free}} \rangle}_{q \text{ copies}} \neq 0 \quad (\text{E.1})$$

Here, $\phi^2 = \sum_i \phi_i^2$, which is known to appear in the OPE of two stress tensors. Thus, if we prove (E.1), then we would know that $\mathcal{O}_{q,\text{free}}$ appears in the operator product expansion of $2q$ copies of the free field stress tensor j_2 . Then, just as knowing the OPE structure of products of free field stress tensors allowed us to obtain conserved currents from products of the quasi-bilocal fields, we can obtain \mathcal{O}_q in the full theory by defining it to be the operator appearing in the operator product expansion of $2q$ copies of j_2 in the full theory whose correlation functions coincide with the correlation functions of $\mathcal{O}_{q,\text{free}}$ in the free theory. Thus, it suffices to prove (E.1).

First, note that we can immediately reduce to the $q = N$ case. This follows from the structure of the Wick contractions in $A_{q,N}$. To see this, note that every term in $\mathcal{O}_{q,\text{free}}$ involves exactly q of the N bosons, each of which appears twice for a total of $2q$ fields. Since ϕ^2 is bilinear in the the fields, the product of q copies of ϕ^2 will also contain $2q$ fields. Hence, we will need all the ϕ^2 fields to be contracted with the $\mathcal{O}_{q,\text{free}}$ fields in order to obtain a nonzero answer. Thus, for each term in $\mathcal{O}_{q,\text{free}}$, none of the $N - q$ flavors not appearing in that term will contribute, and so we can partition the terms in $A_{q,N}$ according to which of the q flavors appear. Since the correlation function is manifestly symmetric under relabelings of the N ϕ_i fields, this implies that each group of terms in this partition will equally contribute to the total correlation function an amount exactly equal to $A_{q,q}$. Hence, $A_{q,N} = \binom{N}{q} A_{q,q}$, so it suffices to show $A_{q,q}$ is nonzero.

Then, note that since $\mathcal{O}_{q,\text{free}}$ contains exactly two copies of each of the q ϕ_i fields, each of the q factors of ϕ^2 must contribute a different ϕ_i field for the contraction to be nonzero. Since $\mathcal{O}_{q,\text{free}}$ is manifestly invariant under arbitrary relabelings of the ϕ_i fields,

we may relabel each term so that the first copy of ϕ^2 contributes ϕ_1^2 , the second copy of ϕ^2 contributes ϕ_2^2 and so on. That is, we have

$$A_{q,q} = q! \langle \phi_1^2(x_1) \phi_2^2(x_2) \dots \phi_q^2(x_q) \mathcal{O}_{q,\text{free}}(x_{q+1}) \rangle \quad (\text{E.2})$$

The correlator on the right-hand side can be easily computed by direct evaluation of the Wick contractions. To illustrate, consider the result given by the term in $\mathcal{O}_{q,\text{free}}$ corresponding to setting the internal indices $i_k = j_k = k$ for all $k \in \{1, 2, \dots, q\}$. The contribution of this term is, up to a sign, given by:

$$\prod_{k=1}^q \partial_{q+1}^{k-1} x_{k,q+1}^{2-d} \quad (\text{E.3})$$

This is a rational function whose numerator is an integer. All other terms in the correlation function will be generated by permuting the powers of the partial derivatives that appear. Hence, each term in the overall sum will depend differently only each x_i , and the overall sum cannot cancel because the numerators have no x_i dependence. Thus, the correlation we wanted to show is nonzero is indeed nonzero, completing the proof.

F The free Maxwell field in five dimensions

Consider the theory of a free Maxwell field in d dimensions. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial A)^2 \quad (\text{F.1})$$

where $\xi = \frac{d}{d-4}$. As was noted in [33], this theory is a conformal field theory with higher spin symmetry, but it is non-unitary in dimension $d > 4$. We claim that this theory is an example of a conformal, non-unitary theory where the three-point function of the stress tensor does not coincide with one of the three free structures described in the body of the paper. This can be checked by explicit calculation. The canonical stress energy tensor is not trace-free, and it may be improved using the procedure of [34]. The result is

$$\begin{aligned} T^{--} = & 4\partial_+ A^\rho \partial_+ A_\rho + \partial^\rho A^- \partial_\rho A^- - 4\partial_+ A^\rho \partial_\rho A^- + 4\frac{(d-4)}{d} A^- \partial_+ (\partial A)_+ \\ & + \frac{1}{(d-2)} \left[4a(\partial A) \partial_+ A^- + 4a A^- \partial_+ (\partial A)_+ + 4a \partial_+ A^\rho \partial_\rho A^- + 4a A^\rho \partial_\rho \partial_+ A^- + \right. \\ & \quad \left. + 16b A_\rho \partial_+^2 A^\rho + 16b \partial_+ A_\rho \partial_+ A^\rho - 2a A^- \partial^2 A^- - 2a \partial^\rho A^- \partial_\rho A^- \right] - \\ & - 2\frac{(d-4)}{(d-1)} \left[\partial_+ A_\rho \partial_+ A^\rho + A_\rho \partial_+^2 A^\rho \right], \end{aligned} \quad (\text{F.2})$$

where $a = 2 - d/2, b = d/4 - 1$. Now, the three point function $\langle T_{--} T_{--} T_{--} \rangle$ can be evaluated by Wick contraction, and the result can be decomposed as follows:

$$\langle T_{--} T_{--} T_{--} \rangle = c_s \langle T_{--} T_{--} T_{--} \rangle_s + c_f \langle T_{--} T_{--} T_{--} \rangle_f + c_t \langle T_{--} T_{--} T_{--} \rangle_t, \quad (\text{F.3})$$

where $c_s = \frac{12125}{576}, c_f = -\frac{1000}{9}, c_t = \frac{54179}{576}$. This demonstrates that unitarity is a necessary assumption for our result; the three-point function of the stress tensor is not the same as the result for an appropriate free field theory. It is a superposition of the three possible structures.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a higher spin symmetry*, *J. Phys. A* **46** (2013) 214011 [[arXiv:1112.1016](https://arxiv.org/abs/1112.1016)] [[INSPIRE](#)].
- [2] V. Alba and K. Diab, *Constraining conformal field theories with a higher spin symmetry in $D = 4$* , [arXiv:1307.8092](https://arxiv.org/abs/1307.8092) [[INSPIRE](#)].
- [3] S.E. Konstein, M.A. Vasiliev and V.N. Zaikin, *Conformal higher spin currents in any dimension and AdS/CFT correspondence*, *JHEP* **12** (2000) 018 [[hep-th/0010239](https://arxiv.org/abs/hep-th/0010239)] [[INSPIRE](#)].
- [4] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in $(A)dS_d$* , *Phys. Lett. B* **567** (2003) 139 [[hep-th/0304049](https://arxiv.org/abs/hep-th/0304049)] [[INSPIRE](#)].
- [5] M.A. Vasiliev, *Higher spin gauge theories in various dimensions*, *Fortsch. Phys.* **52** (2004) 702 [[hep-th/0401177](https://arxiv.org/abs/hep-th/0401177)] [[INSPIRE](#)].
- [6] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](https://arxiv.org/abs/hep-th/9711200)] [[INSPIRE](#)].
- [7] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](https://arxiv.org/abs/hep-th/9802109)] [[INSPIRE](#)].
- [8] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](https://arxiv.org/abs/hep-th/9802150)] [[INSPIRE](#)].
- [9] I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical $O(N)$ vector model*, *Phys. Lett. B* **550** (2002) 213 [[hep-th/0210114](https://arxiv.org/abs/hep-th/0210114)] [[INSPIRE](#)].
- [10] E. Sezgin and P. Sundell, *Holography in 4D (super) higher spin theories and a test via cubic scalar couplings*, *JHEP* **07** (2005) 044 [[hep-th/0305040](https://arxiv.org/abs/hep-th/0305040)] [[INSPIRE](#)].
- [11] S.R. Coleman and J. Mandula, *All possible symmetries of the S matrix*, *Phys. Rev.* **159** (1967) 1251 [[INSPIRE](#)].
- [12] R. Haag, J.T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S matrix*, *Nucl. Phys. B* **88** (1975) 257 [[INSPIRE](#)].
- [13] N. Boulanger, D. Ponomarev, E.D. Skvortsov and M. Taronna, *On the uniqueness of higher-spin symmetries in AdS and CFT*, *Int. J. Mod. Phys. A* **28** (2013) 1350162 [[arXiv:1305.5180](https://arxiv.org/abs/1305.5180)] [[INSPIRE](#)].
- [14] N. Boulanger and E.D. Skvortsov, *Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime*, *JHEP* **09** (2011) 063 [[arXiv:1107.5028](https://arxiv.org/abs/1107.5028)] [[INSPIRE](#)].
- [15] Z. Komargodski and A. Zhiboedov, *Convexity and liberation at large spin*, *JHEP* **11** (2013) 140 [[arXiv:1212.4103](https://arxiv.org/abs/1212.4103)] [[INSPIRE](#)].
- [16] Y. Stanev, *Stress-energy tensor and $U(1)$ current operator product expansions in conformal QFT*, *Bulg. J. Phys.* **15** (1988) 93 [[INSPIRE](#)].
- [17] H. Osborn and A.C. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) 311 [[hep-th/9307010](https://arxiv.org/abs/hep-th/9307010)] [[INSPIRE](#)].

- [18] Y.S. Stanev, *Correlation functions of conserved currents in four dimensional conformal field theory*, *Nucl. Phys. B* **865** (2012) 200 [[arXiv:1206.5639](#)] [[INSPIRE](#)].
- [19] A. Zhiboedov, *A note on three-point functions of conserved currents*, [arXiv:1206.6370](#) [[INSPIRE](#)].
- [20] Y.S. Stanev, *Constraining conformal field theory with higher spin symmetry in four dimensions*, *Nucl. Phys. B* **876** (2013) 651 [[arXiv:1307.5209](#)] [[INSPIRE](#)].
- [21] N.M. Nikolov and I.T. Todorov, *Rationality of conformally invariant local correlation functions on compactified Minkowski space*, *Commun. Math. Phys.* **218** (2001) 417 [[hep-th/0009004](#)] [[INSPIRE](#)].
- [22] D. Friedan and C.A. Keller, *Cauchy conformal fields in dimensions $d > 2$* , [arXiv:1509.07475](#) [[INSPIRE](#)].
- [23] M.S. Costa, J. Penedones, D. Poland and S. Rychkov, *Spinning conformal correlators*, *JHEP* **11** (2011) 071 [[arXiv:1107.3554](#)] [[INSPIRE](#)].
- [24] S. Giombi and I.R. Klebanov, *One loop tests of higher spin AdS/CFT*, *JHEP* **12** (2013) 068 [[arXiv:1308.2337](#)] [[INSPIRE](#)].
- [25] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, *Class. Quant. Grav.* **30** (2013) 104003 [[arXiv:1204.3882](#)] [[INSPIRE](#)].
- [26] D. Anselmi, *Higher spin current multiplets in operator product expansions*, *Class. Quant. Grav.* **17** (2000) 1383 [[hep-th/9906167](#)] [[INSPIRE](#)].
- [27] V.K. Dobrev, A.K. Ganchev and O.I. Iordanov, *Conformal operators from spinor fields. 1. Symmetric tensor case*, *Phys. Lett. B* **119** (1982) 372 [[INSPIRE](#)].
- [28] V. Dobrev and A.K. Ganchev, *Conformal operators from spinor fields: antisymmetric tensor case*, submitted to *Teor. Mat. Fiz.*, (1982) [[INSPIRE](#)].
- [29] S. Giombi, S. Prakash and X. Yin, *A note on CFT correlators in three dimensions*, *JHEP* **07** (2013) 105 [[arXiv:1104.4317](#)] [[INSPIRE](#)].
- [30] I. Todorov, *Conformal field theories with infinitely many conservation laws*, *J. Math. Phys.* **54** (2013) 022303 [[arXiv:1207.3661](#)] [[INSPIRE](#)].
- [31] S. Weinberg, *Six-dimensional methods for four-dimensional conformal field theories*, *Phys. Rev. D* **82** (2010) 045031 [[arXiv:1006.3480](#)] [[INSPIRE](#)].
- [32] R. Jackiw and S.Y. Pi, *Tutorial on scale and conformal symmetries in diverse dimensions*, *J. Phys. A* **44** (2011) 223001 [[arXiv:1101.4886](#)] [[INSPIRE](#)].
- [33] S. El-Showk, Y. Nakayama and S. Rychkov, *What Maxwell theory in $d \neq 4$ teaches us about scale and conformal invariance*, *Nucl. Phys. B* **848** (2011) 578 [[arXiv:1101.5385](#)] [[INSPIRE](#)].
- [34] J. Polchinski, *Scale and conformal invariance in quantum field theory*, *Nucl. Phys. B* **303** (1988) 226 [[INSPIRE](#)].