

Elliptic fibrations with rank three Mordell-Weil group: F-theory with $U(1) \times U(1) \times U(1)$ gauge symmetry

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ABSTRACT: We analyze general F-theory compactifications with $U(1) \times U(1) \times U(1)$ Abelian gauge symmetry by constructing the general elliptically fibered Calabi-Yau manifolds with a rank three Mordell-Weil group of rational sections. The general elliptic fiber is shown to be a complete intersection of two non-generic quadrics in \mathbb{P}^3 and resolved elliptic fibrations are obtained by embedding the fiber as the generic Calabi-Yau complete intersection into $Bl_3\mathbb{P}^3$, the blow-up of \mathbb{P}^3 at three points. For a fixed base B , there are finitely many Calabi-Yau elliptic fibrations. Thus, F-theory compactifications on these Calabi-Yau manifolds are shown to be labeled by integral points in reflexive polytopes constructed from the nef-partition of $Bl_3\mathbb{P}^3$. We determine all 14 massless matter representations to six and four dimensions by an explicit study of the codimension two singularities of the elliptic fibration. We obtain three matter representations charged under all three $U(1)$ -factors, most notably a tri-fundamental representation. The existence of these representations, which are not present in generic perturbative Type II compactifications, signifies an intriguing universal structure of codimension two singularities of the elliptic fibrations with higher rank Mordell-Weil groups. We also compute explicitly the corresponding 14 multiplicities of massless hypermultiplets of a six-dimensional F-theory compactification for a general base B .

KEYWORDS: F-Theory, Superstring Vacua, D-branes

ARXIV EPRINT: [1310.0463](https://arxiv.org/abs/1310.0463)

Contents

1	Introduction and summary of results	1
2	Three ways to the elliptic curve with three rational points	4
2.1	The elliptic curve as intersection of two quadrics in \mathbb{P}^3	4
2.2	Resolved elliptic curve as complete intersection in $\text{Bl}_3\mathbb{P}^3$	6
2.3	Connection to the cubic in dP_2	10
2.3.1	Mapping the intersection of two quadrics in \mathbb{P}^3 to the cubic in \mathbb{P}^2	10
2.3.2	Mapping the intersection in $\text{Bl}_3\mathbb{P}^3$ to the Calabi-Yau onefold in dP_2	11
2.4	Weierstrass form with three rational points	14
3	Elliptic fibrations with three rational sections	15
3.1	Constructing Calabi-Yau elliptic fibrations	16
3.2	Basic geometry of Calabi-Yau manifolds with $\text{Bl}_3\mathbb{P}^3$ -elliptic fiber	18
3.3	All Calabi-Yau manifolds \hat{X} with $\text{Bl}_3\mathbb{P}^3$ -elliptic fiber over B	19
4	Matter in F-theory compactifications with a rank three Mordell-Weil group	22
4.1	Matter at the singularity loci of rational sections	24
4.1.1	The vanishing of two minors: special singularities of \hat{s}_S	24
4.1.2	The vanishing of three minors: singularities of all sections	26
4.2	Matter from singularities in the Weierstrass model	28
4.3	6D matter multiplicities and anomaly cancellation	31
5	Conclusions	35
A	The Weierstrass form of the elliptic curve with three rational points	36
B	Nef-partitions	40

1 Introduction and summary of results

Compactifications of F-theory [1–3] are a very interesting and broad class of string vacua, because they are on the one hand non-perturbative, but still controllable, and on the other hand realize promising particle physics. In particular, F-theory GUTs have drawn a lot of attention in the recent years, first in the context of local models following [4–7] and later also in compact Calabi-Yau manifolds [8–12], see e.g. [13–15] for reviews. Both of these approaches rely on the well-understood realization of non-Abelian gauge symmetries that are engineered by constructing codimension one singularities of elliptic fibrations [1–3, 16]

that have been classified in [17, 18].¹ In addition, the structure of these codimension one singularities governs the pattern of matter that is localized at codimension two singularities of the fibration [22], with some subtleties of higher codimension singularities uncovered recently in [23–25].²

Abelian gauge symmetries are crucial ingredients for extensions both of the standard model as well as of GUTs. However, the concrete construction of Abelian gauge symmetries as well as their matter content has only recently been addressed systematically in global F-theory compactifications. This is due to the fact that U(1) gauge symmetries in F-theory are not related to local codimension one singularities but to the global properties of the elliptic fibration of the Calabi-Yau manifold. Concretely, the number of U(1)-factors in an F-theory compactification is given by the rank of the Mordell-Weil group of the elliptic fibration³ [2, 3], see [30–34] for a mathematical background. The explicit compact Calabi-Yau manifolds with rank one [35] and the most general rank two [36, 37] Abelian sector have been constructed recently. In the rank two case, the general elliptic fiber is the generic elliptic curve in dP_2 and its Mordell-Weil group is rank two with the two generators induced from the ambient space dP_2 . The full six-dimensional spectrum of the Calabi-Yau elliptic fibrations with elliptic fiber in dP_2 has been determined in [37, 38] and chiral compactifications to four dimensions on Calabi-Yau fourfolds with G_4 -flux were constructed in [39, 40]. We note, that certain aspects of Abelian sectors in F-theory could be addressed in local models [9, 41–48]. In addition, special Calabi-Yau geometries realizing one U(1)-factor have been studied in [49–54].⁴

In this work we follow the systematic approach initiated in [35, 37] to construct elliptic curves with higher rank Mordell-Weil groups and their resolved elliptic fibrations, that aims at a complete classification of all possible Abelian sectors in F-theory. We construct the most general F-theory compactifications with $U(1) \times U(1) \times U(1)$ gauge symmetry by building elliptically fibered Calabi-Yau manifolds with rank three Mordell-Weil group. Most notably, we show that this forces us to leave the regime of hypersurfaces to represent these Calabi-Yau manifolds explicitly. In fact, the general elliptic fiber in the fully resolved elliptic fibration is naturally embedded as the generic Calabi-Yau complete intersection into $\text{Bl}_3\mathbb{P}^3$, the blow-up of \mathbb{P}^3 at three generic points. We show that this is the general elliptic curve \mathcal{E} with three rational points and a zero point. We determine the birational map to its Tate and Weierstrass form. All generic Calabi-Yau elliptic fibrations of \mathcal{E} over a given base B are completely fixed by the choice of three divisors in the base B . Furthermore, we show that every such F-theory vacuum corresponds to an integral in certain reflexive polytopes,⁵ that we construct explicitly.

¹A toolbox to construct examples of compact Calabi-Yau manifolds with a certain non-Abelian gauge group is provided by toric geometry, see [19–21].

²For a recent approach based on deformations, cf. [26]. See also [27] for a determination of BPS-states, including matter states, of (p,q)-strings using the refined topological string.

³See also [28, 29] for the interpretation of the torsion subgroup of the Mordell-Weil group as inducing non-simply connected non-Abelian group in F-theory.

⁴For a systematic study of rational sections on toric K3-surfaces we refer to [55].

⁵The correspondence between F-theory compactifications and (integral) points in a polytope has been noted in the toric case [56] and in elliptic fibrations with a general rank two Mordell-Weil group [39].

$U(1) \times U(1) \times U(1)$ -charged matter
$(1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1),$ $(1, 1, -1), (-1, -1, -2), (0, 1, 2), (1, 0, 2), (-1, 0, 1), (0, -1, 1), (0, 0, 2)$

Table 1. Matter representation for F-theory compactifications with a general rank-three Mordell-Weil group, labeled by their $U(1)$ -charges (q_1, q_2, q_3) .

As a next step, we determine the representations of massless matter in four- and six-dimensional F-theory compactifications by thoroughly analyzing the generic codimension two singularities of these elliptic Calabi-Yau manifolds. We find 14 different matter representations, cf. table 1, with various $U(1)^3$ -charges. Note, that the construction leads to representations that are symmetric under permutations of the first two $U(1)$ factors, but not the third one. Interestingly, we obtain three representations charged under all three $U(1)$ -factors, most notably a tri-fundamental representation. Matter in these representations is unexpected in perturbative Type II compactifications and might have interesting phenomenological implications. These results, in particular the appearance of a tri-fundamental representation, indicate an intriguing structure of the codimension two singularities of elliptic fibration with rank three Mordell Weil group.

Furthermore, we geometrically derive closed formulas for all matter multiplicities of charged hypermultiplets in six dimensions for F-theory compactifications on elliptically fibered Calabi-Yau threefolds over a general base B . As a consistency check, we show that the spectrum is anomaly-free. Technically, the analysis of codimension two singularities requires the study of degenerations of the complete intersection \mathcal{E} in $\text{Bl}_3\mathbb{P}^3$ and the computation of the homology classes of the determinantal varieties describing certain matter loci.

Along the course of this work we have encountered and advanced a number of technical issues. Specifically, we discovered three birational maps of the generic elliptic curve \mathcal{E} in $\text{Bl}_3\mathbb{P}^3$ to a non-generic form of the elliptic curve of [36, 37] in dP_2 . These maps are isomorphisms if the elliptic curve \mathcal{E} does not degenerate in a particular way. The dP_2 -elliptic curves we obtain are non-generic since one of the generators of the Mordell-Weil group of \mathcal{E} , with all its rational points being toric, i.e. induced from the ambient space $\text{Bl}_3\mathbb{P}^3$, maps to a non-toric rational point. It would be interesting to investigate, whether any non-toric rational point on dP_2 can be mapped to a toric point of \mathcal{E} in $\text{Bl}_3\mathbb{P}^3$. In addition, we see directly from this map that the elliptic curve in dP_3 can be obtained as a special case of the curve \mathcal{E} in $\text{Bl}_3\mathbb{P}^3$.

This work is organized as follows. In section 2 we construct the general elliptic curve \mathcal{E} . From the existence of the three rational points alone, we derive that \mathcal{E} is naturally represented as the complete intersection of two non-generic quadrics in \mathbb{P}^3 , see section 2.1. The resolved elliptic curve \mathcal{E} is obtained in section 2.2 as the generic Calabi-Yau complete intersection in $\text{Bl}_3\mathbb{P}^3$, where all its rational points are toric, i.e. induced from the ambient space. In section 2.3 we construct three canonical maps of this elliptic curve to the non-generic elliptic curves in dP_2 . In section 2.4 we find the Weierstrass form of the curve

\mathcal{E} along with the Weierstrass coordinates of all its rational points. We proceed with the construction of elliptically fibered Calabi-Yau manifolds \hat{X} with general elliptic fiber in $\text{Bl}_3\mathbb{P}^3$ over a general base B in section 3. First, we determine the ambient space and all bundles on B relevant for the construction of \hat{X} in section 3.1. We discuss the basic general intersections of \hat{X} in section 3.2 and classify all Calabi-Yau fibrations for a given base B in section 3.3. In section 4 we analyze explicitly the codimension two singularities of \hat{X} , which determine the matter representations of F-theory compactifications to six and four dimensions. We follow a two-step strategy to obtain the charges and codimension two loci of the 14 different matter representations of \hat{X} in sections 4.1 and 4.2, respectively. We also determine the explicit expressions for the corresponding matter multiplicities of charged hypermultiplets of a six-dimensional F-theory compactification on a threefold \hat{X}_3 with general base B . Our conclusions and a brief outlook can be found in 5. This work contains two appendices: in appendix A we present explicit formulae for the Weierstrass form of \mathcal{E} , and in appendix B we give a short account on nef-partitions, that have been omitted in the main text.

2 Three ways to the elliptic curve with three rational points

In this section we construct explicitly the general elliptic curve \mathcal{E} with a rank three Mordell-Weil group of rational points, denoted Q , R and S .

We find three different, but equivalent representations of \mathcal{E} . First, in section 2.1 we find that \mathcal{E} is naturally embedded into \mathbb{P}^3 as the complete intersection of two non-generic quadrics, i.e. two homogeneous equations of degree two. Equivalently, we embed \mathcal{E} in section 2.2 as the generic complete intersection Calabi-Yau into the blow-up $\text{Bl}_3\mathbb{P}^3$ of \mathbb{P}^3 at three generic points, which is effectively described via a nef-partition of the corresponding 3D toric polytope. In this representation the three rational points of \mathcal{E} and the zero point P descend from the four inequivalent divisors of the ambient space $\text{Bl}_3\mathbb{P}^3$. Thus, the Mordell-Weil group of \mathcal{E} is *toric*. Finally, we show in section 2.3 that \mathcal{E} can also be represented as a non-generic Calabi-Yau hypersurface in dP_2 . In contrast to the generic elliptic curve in dP_2 that has a rank two Mordell-Weil group [36, 37] which is toric, the onefold in dP_2 we find here exhibits a third rational point, say S , and has a rank three Mordell-Weil group. This third rational point, however, is *non-toric* in the presentation of \mathcal{E} in dP_2 . We note that there are three different maps of the quadric intersection in $\text{Bl}_3\mathbb{P}^3$ to an elliptic curve in dP_2 corresponding to the different morphisms from $\text{Bl}_3\mathbb{P}^3$ to dP_2 .

We emphasize that in the presentation of \mathcal{E} as a complete intersection in $\text{Bl}_3\mathbb{P}^3$ the rank four Mordell-Weil group is toric. Thus, as we will demonstrate in section 3 this representation is appropriate for the construction of resolved elliptic fibrations of \mathcal{E} over a base B .

2.1 The elliptic curve as intersection of two quadrics in \mathbb{P}^3

In this section we derive the embedding of \mathcal{E} with a zero point P and the rational points Q , R and S into \mathbb{P}^3 as the intersection of two non-generic quadrics. We follow the methods

described in [35, 37] used for the derivation of the general elliptic curves with rank one and two Mordell-Weil groups.

We note that the presence of the four points on \mathcal{E} defines a degree four line bundle $\mathcal{O}(P + Q + R + S)$ over \mathcal{E} . Let us first consider a general degree four line bundle \mathcal{M} over \mathcal{E} . Then the following holds, as we see by employing the Riemann-Roch theorem:

1. $H^0(\mathcal{E}, \mathcal{M})$ is generated by four sections, that we denote by u', v', w', t' .
2. $H^0(\mathcal{E}, \mathcal{M}^2)$ is generated by eight sections. However we know ten sections of M^2 , the quadratic monomials in $[u' : v' : w' : t']$, i.e. $u'^2, v'^2, w'^2, t'^2, u'v', u'w', u't', v'w', v't', w't'$.

The above first bullet point shows that $[u' : v' : w' : t']$ are of equal weight one and can be viewed as homogeneous coordinates on \mathbb{P}^3 . The second bullet point implies that $H^0(2\mathcal{M})$ is generated by sections we already know and that there have to be two relations between the ten quadratic monomials in $[u' : v' : w' : t']$, that we write as

$$\begin{aligned} s_1 t'^2 + s_2 u'^2 + s_3 v'^2 + s_4 w'^2 + s_5 t' u' + s_6 u' v' + s_7 u' w' + s_8 v' w' &= s_9 v' t' + s_{10} w' t', \\ s_{11} t'^2 + s_{12} u'^2 + s_{13} v'^2 + s_{14} w'^2 + s_{15} u' t' + s_{16} u' v' + s_{17} u' w' + s_{18} v' w' &= s_{19} v' t' + s_{20} w' t', \end{aligned} \quad (2.1)$$

Now specialize to $\mathcal{M} = \mathcal{O}(P+Q+R+S)$ and assume u' to vanish at all points P, Q, R, S . By inserting $u' = 0$ into (2.1) we should then get four rational solutions corresponding to the four points, i.e. other words (2.1) should factorize accordingly. However, this is not true for generic s_i taking values e.g. in the ring of functions of the base B of an elliptic fibration.⁶ Thus, we have to set the following coefficients s_i to zero,

$$s_1 = s_3 = s_4 = s_{11} = s_{13} = s_{14} = 0. \quad (2.3)$$

As we see below in section 2.2, this can be achieved globally, by blowing up \mathbb{P}^3 at three generic points.

For the moment, let us assume that (2.3) holds and determine P, Q, R, S . First we note that the presentation (2.1) for the elliptic curve \mathcal{E} now reads

$$\begin{aligned} s_2 u'^2 + s_5 u' t' + s_6 u' v' + s_7 u' w' &= s_9 v' t' + s_{10} w' t' - s_8 v' w', \\ s_{12} u'^2 + s_{15} u' t' + s_{16} u' v' + s_{17} u' w' &= s_{19} v' t' + s_{20} w' t' - s_{18} v' w', \end{aligned} \quad (2.4)$$

which is an intersection of two *non-generic* quadrics in \mathbb{P}^3 . Setting $u' = 0$ we obtain

$$0 = s_9 v' t' + s_{10} w' t' - s_8 v' w', \quad 0 = s_{19} v' t' + s_{20} w' t' - s_{18} v' w', \quad (2.5)$$

⁶In contrast, if we were considering an elliptic curve over an algebraically closed field, we could set some $s_i = 0$ by using the $\mathbb{P}GL(4)$ symmetries of \mathbb{P}^3 to eliminate some coefficients s_i . For example, $s_3 = 0$ can be achieved by making the transformation

$$u' \mapsto u' + kv', \quad \text{with } k \text{ obeying } (s_2 k^2 + s_6 k + s_3) = 0. \quad (2.2)$$

Solving this quadratic equation in k will, however, involve the square roots of s_i , which is only defined in an algebraically closed field. In particular, when considering elliptic fibrations the coefficients s_i will be represented by polynomials, of which a square root is not defined globally.

which has in the coordinates $[u' : v' : w' : t']$ the four solutions

$$\begin{aligned} P &= [0 : 0 : 0 : 1], & Q &= [0 : 1 : 0 : 0], & R &= [0 : 0 : 1 : 0], \\ S &= [0 : |M_1^S||M_3^S| : -|M_1^S||M_2^S| : -|M_3^S||M_2^S|]. \end{aligned} \tag{2.6}$$

Here we introduced the determinants $|M_i^S|$ of all three 2×2 -minors M_i^S reading

$$|M_1^S| = s_9 s_{20} - s_{10} s_{19}, \quad |M_2^S| = s_8 s_{19} - s_9 s_{18}, \quad |M_3^S| = s_8 s_{20} - s_{10} s_{18}, \tag{2.7}$$

that are obtained by deleting the $(4 - i)$ -th column in the matrix

$$M^S = \begin{pmatrix} s_9 & s_{10} & -s_8 \\ s_{19} & s_{20} & -s_{18} \end{pmatrix}, \tag{2.8}$$

where M^S is the matrix of coefficients in (2.5).

It is important to realize that the coordinates of the rational point S are products of determinants in (2.7), in particular when studying elliptic fibrations at higher codimension in the base B , cf. section 4. On the one hand, the vanishing loci of the determinant of a single determinant $|M_i^S|$ with $i = 1, 2, 3$ indicates the collisions of S with P , Q and R , respectively, i.e.

$$|M_1^S| = 0 : S = P, \quad |M_2^S| = 0 : S = Q, \quad |M_3^S| = 0 : S = R. \tag{2.9}$$

On the other hand the simultaneous vanishing of all $|M_i^S|$ is equivalent to the two constraints in (2.4) getting linearly dependent. Then, the elliptic curve \mathcal{E} degenerates to an I_2 -curve, i.e. two \mathbb{P}^1 's intersecting at two points, see the discussion around (2.27), with the point S becoming the entire $\mathbb{P}^1 = \{u = s_9 v' t' + s_{10} w' t' - s_8 v' w' = 0\}$.⁷ We note that this behavior of S indicates that in an elliptic fibration the point S will only give rise to a rational, not a holomorphic section of the fibration.

In summary, we have found that the general elliptic curve \mathcal{E} with three rational points Q , R , S and a zero point P is embedded into \mathbb{P}^3 as the intersection of the two non-generic quadrics (2.4).

2.2 Resolved elliptic curve as complete intersection in $\text{Bl}_3\mathbb{P}^3$

In this section we represent the elliptic curve \mathcal{E} with a rank three Mordell-Weil group as a *generic* complete intersection Calabi-Yau in the ambient space $\text{Bl}_3\mathbb{P}^3$. As we demonstrate here, the three blow-ups in $\text{Bl}_3\mathbb{P}^3$ remove globally the coefficients in (2.3). In addition, the three blow-ups resolve all singularities of \mathcal{E} , that can appear in elliptic fibrations. Finally, we emphasize that the elliptic curve \mathcal{E} is a complete intersection associated to the nef-partition of the polytope of $\text{Bl}_3\mathbb{P}^3$, where we refer to appendix B for more details on nef-partitions.

⁷This curve can be seen to define a \mathbb{P}^1 either using adjunction or employing the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into \mathbb{P}^3 .

First, we recall the polytope of \mathbb{P}^3 and its nef-partition describing a complete intersection of quadrics. The polytope $\nabla_{\mathbb{P}^3}$ of \mathbb{P}^3 is the convex hull $\nabla_{\mathbb{P}^3} = \langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ of the four vertices

$$\rho_1 = (-1, -1, -1), \quad \rho_2 = (1, 0, 0), \quad \rho_3 = (0, 1, 0), \quad \rho_4 = (0, 0, 1), \quad (2.10)$$

corresponding to the homogeneous coordinates u', v', w' and t' , respectively. The anti-canonical bundle of \mathbb{P}^3 is $K_{\mathbb{P}^3}^{-1} = \mathcal{O}(4H)$, where H denotes the hyperplane class of \mathbb{P}^3 . Two generic degree two polynomials in the class $\mathcal{O}(2H)$ are obtained via (B.2) from the nef-partition of the polytope of \mathbb{P}^3 into ∇_1, ∇_2 reading

$$\nabla_{\mathbb{P}^3} = \langle \nabla_1 \cup \nabla_2 \rangle, \quad \nabla_1 = \langle \rho_1, \rho_2 \rangle, \quad \nabla_2 = \langle \rho_3, \rho_4 \rangle, \quad (2.11)$$

where \cup denotes the union of sets of a vector space. This complete intersection defines the elliptic curve in (2.1) with only the origin P .

Next, we describe the elliptic curve \mathcal{E} as a generic complete intersection associated to a nef-partition of $\text{Bl}_3\mathbb{P}^3$, the blow-up of \mathbb{P}^3 at three generic points, that we choose to be P, Q and R in (2.6). We first perform these blow-ups and determine the proper transform of \mathcal{E} by hand, before we employ toric techniques and nef-partitions.

The blow-up from \mathbb{P}^3 to $\text{Bl}_3\mathbb{P}^3$ is characterized by the blow-down map

$$u' = e_1 e_2 e_3 u, \quad v' = e_2 e_3 v, \quad w' = e_1 e_3 w, \quad t' = e_1 e_2 t. \quad (2.12)$$

It maps the coordinates $[u : v : w : t : e_1 : e_2 : e_3]$ on $\text{Bl}_3\mathbb{P}^3$ to the coordinates on $[u : v : w : t]$ on \mathbb{P}^3 . Here the $e_i = 0, i = 1, 2, 3$, are the exceptional divisors E_i of the the blow-ups at the points Q, R and P , respectively. We summarize the divisor classes of all homogeneous coordinates on $\text{Bl}_3\mathbb{P}^3$ together with the corresponding \mathbb{C}^* -actions that follow immediately from (2.12) as

	divisor class	\mathbb{C}^* -actions
u	$H - E_1 - E_2 - E_3$	1 1 1 1
v	$H - E_2 - E_3$	1 0 1 1
w	$H - E_1 - E_3$	1 1 0 1
t	$H - E_1 - E_2$	1 1 1 0
e_1	E_1	0 -1 0 0
e_2	E_2	0 0 -1 0
e_3	E_3	0 0 0 -1

(2.13)

Here H denotes the pullback of the hyperplane class H on \mathbb{P}^3 . The coordinates $[u : w : t], [u : v : t]$ and $[u : v : w]$ are the homogeneous coordinates on each $E_i \cong \mathbb{P}^2$, respectively, and can not vanish simultaneously. Together with the pullback of the Stanley-Reissner ideal of \mathbb{P}^3 this implies the following Stanley Reissner ideal on $\text{Bl}_3\mathbb{P}^3$,

$$SR = \{uvt, uwt, uvw, e_1v, e_2w, e_3t, e_1e_2, e_2e_3, e_1e_3\}. \quad (2.14)$$

This implies the following intersections of the four independent divisors on $\text{Bl}_3\mathbb{P}^3$,

$$H^3 = E_i^3 = 1, \quad E_i \cdot H = E_i \cdot E_j = 0, \quad i \neq j. \quad (2.15)$$

The proper transform under the map (2.12) of the constraints (2.4) describing \mathcal{E} read

$$\begin{aligned} p_1 &:= s_2 e_1 e_2 e_3 u^2 + s_5 e_1 e_2 u t + s_6 e_2 e_3 u v + s_7 e_1 e_3 u w - s_9 e_2 v t - s_{10} e_1 w t + s_8 e_3 v w, \\ p_2 &:= s_{12} e_1 e_2 e_3 u^2 + s_{15} e_1 e_2 u t + s_{16} e_2 e_3 u v + s_{17} e_1 e_3 u w - s_{19} e_2 v t - s_{20} e_1 w t + s_{18} e_3 v w. \end{aligned} \quad (2.16)$$

We immediately see that this complete intersection defines a Calabi-Yau onefold in $\text{Bl}_3\mathbb{P}^3$ employing (2.13), adjunction and noting that the anti-canonical bundle of $\text{Bl}_3\mathbb{P}^3$ reads

$$K_{\text{Bl}_3\mathbb{P}^3} = \mathcal{O}(4H - 2E_1 - 2E_2 - 2E_3). \quad (2.17)$$

From (2.6), (2.12) and (2.16) we readily obtain the points in P , Q , R and S on $\text{Bl}_3\mathbb{P}^3$. They are given by the intersection of (2.16) with the four inequivalent toric divisors on $\text{Bl}_3\mathbb{P}^3$, the divisor $D_u := \{u = 0\}$ and the exceptional divisors E_i . Their coordinates read

$$\begin{aligned} E_3 \cap \mathcal{E}: \quad P &= [s_{10}s_{19} - s_{20}s_9 : s_{10}s_{15} - s_{20}s_5 : s_{19}s_5 - s_{15}s_9 : 1 : 1 : 1 : 0], \\ E_1 \cap \mathcal{E}: \quad Q &= [s_{19}s_8 - s_{18}s_9 : 1 : -s_{19}s_6 + s_{16}s_9 : -s_{18}s_6 + s_{16}s_8 : 0 : 1 : 1], \\ E_2 \cap \mathcal{E}: \quad R &= [s_{10}s_{18} - s_{20}s_8 : -s_{10}s_{17} + s_{20}s_7 : 1 : s_{18}s_7 - s_{17}s_8 : 1 : 0 : 1], \\ D_u \cap \mathcal{E}: \quad S &= [0 : 1 : 1 : 1 : s_{19}s_8 - s_{18}s_9 : s_{10}s_{18} - s_{20}s_8 : s_{10}s_{19} - s_{20}s_9]. \end{aligned} \quad (2.18)$$

Here we made use of the Stanley-Reissner ideal (2.14) to set the coordinates to one that can not vanish simultaneously with $u = 0$, respectively, $e_i = 0$.

We emphasize that the coordinates (2.18) are again given by determinants of 2×2 -minors. Indeed, we can write (2.18) as

$$\begin{aligned} P &= [-|M_3^P| : |M_2^P| : -|M_1^P| : 1 : 1 : 1 : 0], \quad Q = [-|M_3^Q| : 1 : |M_2^Q| : -|M_1^Q| : 0 : 1 : 1], \\ R &= [|M_3^R| : -|M_2^R| : 1 : |M_1^R| : 1 : 0 : 1], \quad S = [0 : 1 : 1 : 1 : -|M_3^Q| : |M_3^R| : -|M_3^P|]. \end{aligned} \quad (2.19)$$

Here we defined the matrices

$$M^P = \begin{pmatrix} -s_5 & s_9 & s_{10} \\ -s_{15} & s_{19} & s_{20} \end{pmatrix}, \quad M^Q = \begin{pmatrix} -s_6 & -s_8 & s_9 \\ -s_{16} & -s_{18} & s_{19} \end{pmatrix}, \quad M^R = \begin{pmatrix} -s_7 & -s_8 & s_{10} \\ -s_{17} & -s_{18} & s_{20} \end{pmatrix} \quad (2.20)$$

with their 2×2 -minors $M_i^{P,Q,R}$ defined by deleting the $(4 - i)$ -th column. We emphasize that the minors of the matrix M^S in (2.7) can be expressed by the minors of the matrices in (2.20) and, thus, M^S does not appear in (2.19). The matrices $M^{P,Q,R}$ describe the two linear equations that we obtain by setting $e_3 = 0$, $e_2 = 0$ and $e_1 = 0$ in (2.16), respectively.

It is important to realize that the points P , Q and R are always distinct, as can be seen from (2.19) and the Stanley-Reissner ideal (2.14) since the exceptional divisors do not mutually intersect. However, the point S can agree with all other points, if the appropriate minors in (2.19) vanish. In fact, we see the following pattern,

$$|M_3^P| = 0 : S = P, \quad |M_3^Q| = 0 : S = Q, \quad |M_3^R| = 0 : S = R, \quad (2.21)$$

which will be relevant to keep in mind for the study of elliptic fibrations.

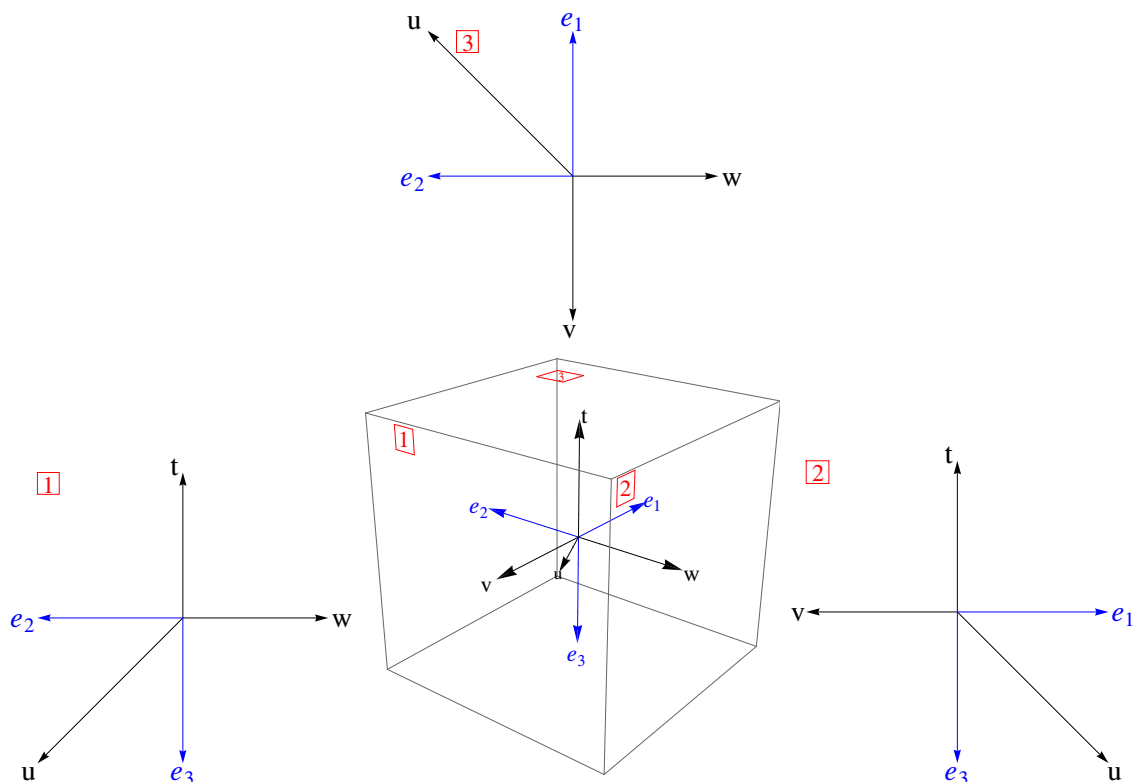


Figure 1. Toric fan of $\text{Bl}_3\mathbb{P}^3$ and the 2D projections to the three coordinate planes, each of which yielding the polytope of dP_2 .

We note that the elliptic curve \mathcal{E} degenerates into an I_2 -curve if, as explained before below (2.8), the rank of one of the matrices in (2.8) and (2.20) is one.⁸ In addition, one particular intersection in (2.18) no longer yields a point in \mathcal{E} , but an entire \mathbb{P}^1 . As discussed below in section 4 the points on \mathcal{E} , thus, will only lift to rational sections of an elliptic fibration of \mathcal{E} .

Finally, we show that the presentation of \mathcal{E} as the complete intersection (2.16) can be obtained torically from a nef-partition of the $\text{Bl}_3\mathbb{P}^3$. For this purpose we only have to realize that the blow-ups (2.12) can be realized torically by adding the following rays to the polytope of \mathbb{P}^3 in (2.10),

$$\rho_{e_1} = (-1, 0, 0), \quad \rho_{e_2} = (0, -1, 0), \quad \rho_{e_3} = (0, 0, -1). \quad (2.22)$$

The rays of the polytope of $\text{Bl}_3\mathbb{P}^3$ are illustrated in the center of figure 1.

Here the ray ρ_{e_i} precisely corresponds to the exceptional divisor $E_i = \{e_i = 0\}$. Then we determine the nef-partitions of this polytope $\nabla_{\text{Bl}_3\mathbb{P}^3}$ of $\text{Bl}_3\mathbb{P}^3$. We find that it admits a single nef-partition into ∇_1, ∇_2 reading

$$\nabla_{\text{Bl}_3\mathbb{P}^3} = \langle \nabla_1 \cup \nabla_2 \rangle, \quad \nabla_1 = \langle \rho_1, \rho_4, \rho_{e_1}, \rho_{e_2} \rangle \quad \nabla_2 = \langle \rho_2, \rho_3, \rho_{e_3} \rangle. \quad (2.23)$$

⁸We emphasize that the complete intersection (2.4) in \mathbb{P}^3 degenerates into only one \mathbb{P}^1 and becomes singular if one matrices in (2.20) has rank one, in contrast to the smooth I_2 -curve obtained from (2.16).

It is straightforward to check that the general formula (B.2) for the nef-partition at hand reproduces precisely the constraints (2.16).

2.3 Connection to the cubic in dP_2

In this section we construct three equivalent maps of the elliptic curve \mathcal{E} given as the intersection (2.16) in $\text{Bl}_3\mathbb{P}^3$ to the Calabi-Yau onefold in dP_2 . The elliptic curve we obtain will not be the generic elliptic curve in dP_2 found in [36, 37] with rank two Mordell-Weil group, but non-generic with a rank three Mordell-Weil group with one *non-toric generator*. The map of the toric generator of the Mordell-Weil group in $\text{Bl}_3\mathbb{P}^3$ to a non-toric generator in dP_2 will be manifest.

The presentation of \mathcal{E} as a non-generic hypersurface in dP_2 with a non-toric Mordell-Weil group allows us to use the results of [37] from the analysis of the generic dP_2 -curve. On the one hand, we can immediately obtain the birational map of \mathcal{E} in (2.16) to the Weierstrass model by first using the map to dP_2 and then by the map from dP_2 to the Weierstrass form. We present this map separately in section 2.4. On the other hand, the study of codimension two singularities in section 4 will essentially reduce to the analysis of codimension two singularities in fibrations with elliptic fiber in dP_2 . However, the additional non-toric Mordell-Weil generator as well as the non-generic hypersurface equation in dP_2 will give rise to a richer structure of codimension two singularities.

2.3.1 Mapping the intersection of two quadrics in \mathbb{P}^3 to the cubic in \mathbb{P}^2

As a preparation, we begin with a brief digression on the map of an elliptic curve with a single point P_0 given as a complete intersection of two quadrics in \mathbb{P}^3 to the cubic in \mathbb{P}^2 , where we closely follow [57, 58].

Let us assume that there is a rational point P_0 on the complete intersection of two quadrics with coordinates $[x_0 : x_1 : x_2 : x_3] = [0 : 0 : 0 : 1]$ in \mathbb{P}^3 .⁹ This implies the quadrics must have the form

$$Ax_3 + B = 0, \quad Cx_3 + D = 0, \tag{2.24}$$

where A, C are linear and B, D are quadratic polynomials in the variables x_0, x_1, x_2 . Assuming that A, C are generic, we obtain a cubic equation in \mathbb{P}^2 with coordinates $[x_0 : x_1 : x_2]$ ¹⁰ by solving (2.24) for x_3 ,

$$AD - BC = 0, \tag{2.25}$$

Here we have to require that $[x_0 : x_1 : x_2] \neq [0, 0, 0]$, because $x_3 = -\frac{B}{A} = -\frac{D}{C}$ has to be well-defined. Then, the inverse map from the cubic in \mathbb{P}^2 to the complete intersection (2.24) reads

$$[x_0 : x_1 : x_2] \mapsto \left[x_0 : x_1 : x_2 : x_3 = -\frac{B}{A} = -\frac{D}{C} \right]. \tag{2.26}$$

⁹We choose coordinates $[x_0 : x_1 : x_2 : x_3]$ on \mathbb{P}^3 in order to keep our discussion here general. We will identify the x_i with the coordinates used in sections 2.1 and (2.2) in section 2.3.2.

¹⁰We can think of this \mathbb{P}^2 as being obtained from \mathbb{P}^3 via a toric morphism defined by projection along one toric ray. In the case at hand this is the ray corresponding to $x_3 = 0$.

The original point $P_0 = [0 : 0 : 0 : 1]$ is mapped to the rational point given by the intersection of the two lines $A = 0, C = 0$. This can be seen by noting that $A = C = 0$ in (2.24) implies also $B = D = 0$ which is only solved if $[x_0 : x_1 : x_2] = [0 : 0 : 0]$.

We note that the case when A and C are co-linear, i.e. $A \sim C$, is special because the curve (2.24) describes no longer a smooth elliptic curve, but a \mathbb{P}^1 . Indeed, if $A = aC$ for a number a we can rewrite (2.24) as

$$B - aD = 0, \quad Cx_3 + D = 0, \tag{2.27}$$

where we can solve the second constraint for x_3 , given $C \neq 0$, so that we are left with the quadratic constraint $B - aD = 0$ in \mathbb{P}^2 , which is a \mathbb{P}^1 . This type of degeneration of the complete intersection (2.24) will be the prototype for the degenerations of the elliptic curve (2.16), that we find in section 4.

2.3.2 Mapping the intersection in $\text{Bl}_3\mathbb{P}^3$ to the Calabi-Yau onefold in dP_2

Next we apply the map of section 2.3.1 to the elliptic curve \mathcal{E} with three rational points. Since (2.4) is linear in all three coordinates v', w' and t' we will obtain according to the discussion below (2.24) three canonical maps to a cubic in \mathbb{P}^2 . In fact, these maps lift to maps of the elliptic curve (2.16) in $\text{Bl}_3\mathbb{P}^3$ to elliptic curves presented as Calabi-Yau hypersurfaces in dP_2 , as we demonstrate in the following.

We construct the map from the complete intersection (2.16) to the elliptic curve in dP_2 explicitly for the point R in (2.6), i.e. we identify $P_0 \equiv R$ and $[x_0 : x_1 : x_2 : x_3] = [u' : v' : t' : w']$ in the coordinates on \mathbb{P}^3 before the blow-up for the discussion in section 2.3.1. Next, we compare (2.24) to the complete intersection (2.16). After the blow-up (2.12), the point R is mapped to $e_2 = 0$ as noted earlier in (2.18). This allows us to identify A, C in (2.24) as those terms in (2.16) that do not vanish, respectively, B, D as the terms that vanish for $e_2 = 0$. Thus we effectively rewrite (2.16) in the form (2.24) with $x_3 \equiv w$ after the blow-up, since $w = 1$ follows from (2.14) for $e_2 = 0$, and obtain

$$\begin{aligned} A &= s_7 e_1 e_3 u + s_8 e_3 v - s_{10} e_1 t, & C &= s_{17} e_1 e_3 u + s_{18} e_3 v - s_{20} e_1 t, \\ B &= e_2 (s_2 e_1 e_3 u^2 + s_5 e_1 u t + s_6 e_3 u w - s_9 v t), & D &= e_2 (s_{12} e_1 e_3 u^2 + s_{15} e_1 u t + s_{16} e_3 u w - s_{19} v t). \end{aligned} \tag{2.28}$$

In particular, this identification implies that $R = \{e_2 = 0\}$ is mapped to $A = C = 0$ on dP_2 as required. Then, we solve both equations for w and obtain the hypersurface equation of the form

$$u(\tilde{s}_1 u^2 e_1^2 e_3^2 + \tilde{s}_2 u v e_1 e_3^2 + \tilde{s}_3 v^2 e_3^2 + \tilde{s}_5 u t e_1^2 e_3 + \tilde{s}_6 v t e_1 e_3 + \tilde{s}_8 t^2 e_1^2) + \tilde{s}_7 v^2 t e_3 + \tilde{s}_9 v t^2 e_1 = 0, \tag{2.29}$$

where we have set $e_2 = 1$ using one \mathbb{C}^* -action on $\text{Bl}_3\mathbb{P}^3$ as $B, D \sim e_2$ and $e_2 = 0$ implies $w = -\frac{B}{A} = -\frac{D}{C} = 0$ which is inconsistent with the SR-ideal (2.14). The coefficients \tilde{s}_i

in (2.29) read

	coefficients in dP_2 -curve projected along $[w : e_2]$	
\tilde{s}_1	$-s_{17}s_2 + s_{12}s_7$	
\tilde{s}_2	$-s_{18}s_2 - s_{17}s_6 + s_{16}s_7 + s_{12}s_8$	
\tilde{s}_3	$- M_1^Q = s_{16}s_8 - s_{18}s_6$	
\tilde{s}_5	$-s_{10}s_{12} + s_2s_{20} - s_{17}s_5 + s_{15}s_7$	(2.30)
\tilde{s}_6	$-s_{10}s_{16} - s_{18}s_5 + s_{20}s_6 - s_{19}s_7 + s_{15}s_8 + s_{17}s_9$	
\tilde{s}_7	$ M_3^Q = s_{18}s_9 - s_{19}s_8$	
\tilde{s}_8	$- M_2^P = -s_{10}s_{15} + s_{20}s_5$	
\tilde{s}_9	$- M_3^P = s_{10}s_{19} - s_{20}s_9$	

Here we have used the minors introduced in (2.7) and in (2.19), (2.20).

We note that the ambient space of (2.29) is dP_2 with homogeneous coordinates $[u : v : w : t : e_1 : e_3]$. The relevant dP_2 is obtained from $\text{Bl}_3\mathbb{P}^3$ by a toric morphism that is defined by projecting the polytope of $\text{Bl}_3\mathbb{P}^3$ generated by (2.10), (2.22) onto the plane that is perpendicular to the line through the rays ρ_3 and ρ_{e_2} . The rays of the fan are shown in the figure on the right of 1 that is obtained by the projection of the rays on the face number two of the cube. This can also be seen from the unbroken \mathbb{C}^* -actions in (2.13) and the SR-ideal (2.14) for $e_2 = 1$ and $w = 0$, or $e_2 = 0$ and $w = 1$. Then, the cubic (2.29) is a section precisely of the anti-canonical bundle of this dP_2 surface.

The general elliptic curve in dP_2 was studied in [36, 37] and shown to have a rank two Mordell-Weil group. However, the elliptic curve (2.29) has by construction a rank three Mordell-Weil group. Indeed, we see that the coefficients \tilde{s}_i are non-generic and precisely allow for a fourth rational point. This fourth point, however, does not descend from a divisor of the ambient space dP_2 and is not toric. In fact, the mapping of the four rational points (2.18) in the coordinates on dP_2 reads

$$\begin{aligned}
 P &= [-|M_3^P| : |M_2^P| : -|M_1^P| : 1 : 1 : 1 : 0] \mapsto [|M_3^P| : -|M_2^P| : 1 : 1 : 0], \\
 Q &= [-|M_3^Q| : 1 : |M_2^Q| : -|M_1^Q| : 0 : 1 : 1] \mapsto [-|M_3^Q| : 1 : -|M_1^Q| : 0 : 1], \\
 R &= [|M_3^R| : -|M_2^R| : 1 : |M_1^R| : 1 : 0 : 1] \mapsto [|M_3^R| : -|M_2^R| : |M_1^R| : 1 : 1], \\
 S &= [0 : 1 : 1 : 1 : -|M_3^Q| : |M_3^R| : -|M_3^P|] \mapsto [0 : 1 : 1 : -|M_3^Q| : -|M_3^P|].
 \end{aligned}
 \tag{2.31}$$

We see, that the points P , Q and S are mapped to the three toric points on the elliptic curve in dP_2 studied in [37], whereas the points R is mapped to a non-toric point.

The map from the complete intersection in $\text{Bl}_3\mathbb{P}^3$ to the elliptic curve (2.29) in dP_2 implies that the results from the analysis of [37], where the generic elliptic curve in dP_2 was considered, immediately apply. More precisely, renaming the coordinates $[u : v : t : e_1 : e_3]$ in (2.29) as $[u : v : w : e_1 : e_2]$ we readily recover equation (3.4) of [37]. Furthermore, the points P , Q and S in (2.31) immediately map to the origin and the two rational points of the rank two elliptic curve in dP_2 , that we denote in the following as \tilde{P} , \tilde{Q} and \tilde{R} . In the notation of [37] we thus rewrite (2.31) using (2.30) as

$$\begin{aligned}
 P &\mapsto \tilde{P} := [-\tilde{s}_9 : \tilde{s}_8 : 1 : 1 : 0], & Q &\mapsto \tilde{Q} := [-\tilde{s}_7 : 1 : \tilde{s}_3 : 0 : 1], \\
 S &\mapsto \tilde{R} := [0 : 1 : 1 : -\tilde{s}_7 : \tilde{s}_9].
 \end{aligned}
 \tag{2.32}$$

We emphasize that the origin P in the complete intersection in (2.16) is mapped to the origin \tilde{P} , which implies that the Weierstrass form of the curve in dP_2 will agree with the Weierstrass form of the curve (2.16), cf. section 2.4.

As we mentioned before, the point R is mapped to a non-toric point in dP_2 . This complicates the determination of the Weierstrass coordinates for R , for example. Fortunately, there are two other maps of the elliptic curve (2.16) to a curve in dP_2 in which the point R is mapped to a toric point and another point, either Q or P , are realized non-torically. Thus, we construct in the following a second map to an elliptic curve in dP_2 , where R is toric. Since the logic is completely analogous to the previous construction, we will be as brief as possible.

We choose $P_0 \equiv Q$ for the map to dP_2 . We recall from (2.18) that Q is realized as $e_1 = 0$ on the elliptic curve in $\text{Bl}_3\mathbb{P}^3$. Thus, we write (2.16) as

$$Av + B = 0, \quad Cv + D = 0, \tag{2.33}$$

where, as before, A and C are obtained by setting $e_1 = 0$ and B, D are the terms proportional to e_1 ,

$$\begin{aligned} A &= -s_9e_2t + s_6e_2e_3u + s_8e_3w, & C &= -s_{19}e_2t + s_{16}e_2e_3u + s_{18}e_3w, \\ B &= e_1(s_2e_2e_3u^2 + s_5e_2ut + s_7e_3uw - s_{10}wt), & D &= e_1(s_{12}e_2e_3u^2 + s_{15}e_2ut + s_{17}e_3uw - s_{20}wt). \end{aligned} \tag{2.34}$$

Thus, we obtain an elliptic curve in dP_2 with homogeneous coordinates $[u : w : t : e_2 : e_3]$ by solving (2.33) for v and by setting $e_1 = 1$ as required by the SR-ideal (2.14). The hypersurface constraint (2.25) takes the form

$$u(\hat{s}_1u^2e_2^2e_3^2 + \hat{s}_2uwe_2e_3^2 + \hat{s}_3w^2e_3^2 + \hat{s}_5ute_2^2e_3 + \hat{s}_6wte_2e_3 + \hat{s}_8t^2e_2^2) + \hat{s}_7w^2te_3 + \hat{s}_9wt^2e_2 = 0, \tag{2.35}$$

with coefficients \hat{s}_i defined as

	coefficients in dP_2 -curve projected along $[v : e_1]$	
\hat{s}_1	$-s_{16}s_2 + s_{12}s_6$	
\hat{s}_2	$-s_{18}s_2 + s_{17}s_6 - s_{16}s_7 + s_{12}s_8$	
\hat{s}_3	$- M_1^R = -s_{18}s_7 + s_{17}s_8$	
\hat{s}_5	$s_{19}s_2 - s_{16}s_5 + s_{15}s_6 - s_{12}s_9$	(2.36)
\hat{s}_6	$s_{10}s_{16} - s_{18}s_5 - s_{20}s_6 + s_{19}s_7 + s_{15}s_8 - s_{17}s_9$	
\hat{s}_7	$ M_3^R = s_{10}s_{18} - s_{20}s_8$	
\hat{s}_8	$- M_1^P = s_{19}s_5 - s_{15}s_9$	
\hat{s}_9	$ M_3^P = -\tilde{s}_9 = -s_{10}s_{19} + s_{20}s_9$	

where we have used (2.30). Analogously to the previous map, the ambient space of the hypersurface (2.35) is the dP_2 with homogeneous coordinates $[u : w : t : e_2 : e_3]$ that is obtained from $\text{Bl}_3\mathbb{P}^3$ by the toric morphism induced by projecting along the line through the rays ρ_2 and ρ_{e_1} . The rays of the fan are shown in the left figure of 1 that corresponds to the projection of the rays on the face number one. Then, the three rational points on \mathcal{E}

and the origin get mapped, in the coordinates $[u : w : t : e_2 : e_3]$ of dP_2 , to

$$\begin{aligned}
 P &= [-|M_3^P| : |M_2^P| : -|M_1^P| : 1 : 1 : 1 : 0] \mapsto [-|M_3^P| : -|M_1^P| : 1 : 1 : 1 : 0], \\
 Q &= [-|M_3^Q| : 1 : |M_2^Q| : -|M_1^Q| : 0 : 1 : 1] \mapsto [-|M_3^Q| : |M_2^Q| : -|M_1^Q| : 1 : 1 : 1], \\
 R &= [|M_3^R| : -|M_2^R| : 1 : |M_1^R| : 1 : 0 : 1] \mapsto [|M_3^R| : 1 : |M_1^R| : 0 : 1 : 1], \\
 S &= [0 : 1 : 1 : 1 : -|M_3^Q| : |M_3^R| : -|M_3^P|] \mapsto [0 : 1 : 1 : |M_3^R| : -|M_3^P|].
 \end{aligned}
 \tag{2.37}$$

As before, it is convenient to make contact to the notation of [37]. After the renaming $[u : w : t : e_2 : e_3] \rightarrow [u : v : w : e_1 : e_2]$ we obtain the hypersurface constraint (2.35) takes the standard form of eq. (3.4) in [37]. In addition, we see that the points P , R and S get mapped to the toric points on dP_2 , whereas Q maps to a non-toric point. Denoting the origin of the dP_2 -curve by \hat{P} and the two rational points by \hat{Q} , \hat{R} in order to avoid confusion, we then write (2.37) as

$$\begin{aligned}
 P &\mapsto \hat{P} := [-\hat{s}_9 : \hat{s}_8 : 1 : 1 : 0], & R &\mapsto \hat{Q} = [-\hat{s}_7 : 1 : \hat{s}_3 : 0 : 1], \\
 S &\mapsto \hat{R} = [0 : 1 : 1 : \hat{s}_7 : -\hat{s}_9].
 \end{aligned}
 \tag{2.38}$$

We note that there is a third map from (2.16) to dP_2 by solving for the variable t , respectively, e_3 (its fan would correspond to the upper figure in figure 1 that shows the projection of the rays in the face number three). Although this map is formally completely analogous to the above the maps, it is not very illuminating for our purposes since the chosen zero point P on \mathcal{E} maps to a non-toric point in dP_2 . In particular, the Weierstrass model with respect to P can not be obtained from this elliptic curve in dP_2 by simply applying the results of [37], where P by assumption has to be a toric point.

2.4 Weierstrass form with three rational points

Finally, we are prepared to obtain the Weierstrass model for the elliptic curve \mathcal{E} in (2.16) with respect to the chosen origin P along with the coordinates in Weierstrass form for the three rational points Q , R and S . We present three maps to a Weierstrass model in this work, each of which yielding an identical Weierstrass form, i.e. identical f , g in $y^2 = x^3 + fxz^4 + gz^6$. The details of the relevant computations as well as the explicit results can be found in appendix A.

The simplest two ways to obtain this Weierstrass form is by first exploiting the two presentations of the elliptic curve \mathcal{E} as the hypersurfaces (2.29) and (2.35) in dP_2 constructed in section 2.3.2 and by then using the birational map of [37] of the general elliptic curve in dP_2 to the Weierstrass form in $\mathbb{P}^2(1, 2, 3)$. In summary, we find the following schematic coordinates for the coordinates in Weierstrass form of the rational points Q , R and S

$$Q = [g_2^Q : g_3^Q : 1], \quad R = [g_2^R : g_3^R : 1], \quad S = [g_2^S : g_3^S : (s_{10}s_{19} - s_9s_{20})]
 \tag{2.39}$$

with the explicit expressions for $g_2^{Q,R,S}$ and $g_3^{Q,R,S}$ given in (A.11)–(A.15) in appendix A. The explicit form for f and g , along with the discriminant follow from the formulas in [37] in combination with (2.30), respectively, (2.36). In fact, we obtain (2.39) for Q and S by using the presentation (2.29) along with the maps (2.32) of the rational points Q and S

onto the two toric points in the dP_2 -elliptic curve, denoted by \tilde{Q} and \tilde{R} in this context. Then, we apply eqs. (3.11) and (3.12) of [37] for the coordinates in Weierstrass form of the two toric rational points on the elliptic curve in dP_2 . For concreteness, for the curve (2.29) the coordinates in Weierstrass form of the two points read

$$[g_2^Q : g_3^Q : z_Q] = \left[\frac{1}{12}(\tilde{s}_6^2 - 4\tilde{s}_5\tilde{s}_7 + 8\tilde{s}_3\tilde{s}_8 - 4\tilde{s}_2\tilde{s}_9), \frac{1}{2}(\tilde{s}_3\tilde{s}_6\tilde{s}_8 - \tilde{s}_2\tilde{s}_7\tilde{s}_8 - \tilde{s}_3\tilde{s}_5\tilde{s}_9 + \tilde{s}_1\tilde{s}_7\tilde{s}_9) : 1 \right] \quad (2.40)$$

for the point $\tilde{Q} = [-\tilde{s}_7 : 1 : \tilde{s}_3 : 0 : 1]$ and

$$\begin{aligned} g_2^S &= \frac{1}{12}(12\tilde{s}_7^2\tilde{s}_8^2 + \tilde{s}_9^2(\tilde{s}_6^2 + 8\tilde{s}_3\tilde{s}_8 - 4\tilde{s}_2\tilde{s}_9) + 4\tilde{s}_7\tilde{s}_9(-3\tilde{s}_6\tilde{s}_8 + 2\tilde{s}_5\tilde{s}_9)), \\ g_3^S &= \frac{1}{2}(2\tilde{s}_7^3\tilde{s}_8^3 + \tilde{s}_3\tilde{s}_9^3(-\tilde{s}_6\tilde{s}_8 + \tilde{s}_5\tilde{s}_9) + \tilde{s}_7^2\tilde{s}_8\tilde{s}_9(-3\tilde{s}_6\tilde{s}_8 + 2\tilde{s}_5\tilde{s}_9) \\ &\quad + \tilde{s}_7\tilde{s}_9^2(\tilde{s}_6^2\tilde{s}_8 + 2\tilde{s}_3\tilde{s}_8^2 - \tilde{s}_5\tilde{s}_6\tilde{s}_9 - \tilde{s}_2\tilde{s}_8\tilde{s}_9 + \tilde{s}_1\tilde{s}_9^2), \\ z_S &= \tilde{s}_9 \end{aligned} \quad (2.41)$$

for the point $\tilde{R} = [0 : 1 : 1 - \tilde{s}_7 : \tilde{s}_9]$, where we apply (2.30). The explicit result in terms of the coefficients s_i for both Q, S can be found in (A.11), respectively, (A.15).

In order to obtain the Weierstrass coordinates for the point R in (2.39) we invoke the map $R \mapsto \hat{Q}$ in (2.38) for the elliptic curve (2.35) in dP_2 . Here, the coordinates of $R \mapsto \hat{Q}$ are again given by (2.40) after replacing $\tilde{s}_i \rightarrow \hat{s}_i$. The explicit form for these coordinates in terms of the s_i is obtained using (2.36) and can be found in (A.13). We emphasize that the coordinates in Weierstrass form for S can also be obtained from the map $S \mapsto \hat{R}$ in (2.38) in combination with (2.36). They precisely agree with those in (A.15) deduced from the map $S \mapsto \tilde{R}$ and (2.30).

Alternatively, one can directly construct the birational map from (2.16) to the Weierstrass form by extension of the techniques of [35, 37], where x and y in $\mathbb{P}^2(1, 2, 3)$ are constructed as sections of appropriate line bundles that vanish with appropriate degrees at Q, R and S . However, the corresponding calculations are lengthy and the resulting Weierstrass model is identical to the one obtained from dP_2 . Thus, we have opted to relegate this analysis to appendix A.

3 Elliptic fibrations with three rational sections

In this section we construct resolved elliptically fibered Calabi-Yau manifolds $\mathcal{E} \rightarrow \hat{X} \xrightarrow{\pi} B$ over a base B with a rank three Mordell-Weil group. The map π denotes the projection to the base B and the general elliptic fiber $\mathcal{E} = \pi^{-1}(pt)$ over a generic point pt in B is the elliptic curve with rank three Mordell-Weil group of section 2. An elliptic Calabi-Yau manifold \hat{X} with all singularities at higher codimension resolved is obtained by fibering \mathcal{E} in the presentation (2.16). In addition, in this representation for \mathcal{E} the generators of the Mordell-Weil group are given by the restriction to \hat{X} of the toric divisors of the ambient space $\text{Bl}_3\mathbb{P}^3$ of the fiber, i.e. the Mordell-Weil group of the generic \hat{X} is toric.

We begin in section 3.1 with the construction of Calabi-Yau elliptic fibrations \hat{X} with rank three Mordell-Weil group over a general base B with the elliptic curve (2.16) as

the general elliptic fiber. We see that all these fibrations are classified by three divisors in the base B . Then in section 3.2 we compute the universal intersections on \hat{X} , that hold generically and are valid for any base B . Finally, in section 3.3 we classify all generic Calabi-Yau manifolds \hat{X} with elliptic fiber \mathcal{E} in $\text{Bl}_3\mathbb{P}^3$ over any base B . Each such F-theory vacua \hat{X} is labeled by one point in a particular polytope, that we determine.

The techniques and results in the following analysis are a direct extension to the ones used in [37–39] for the case of a rank two Mordell-Weil group.

3.1 Constructing Calabi-Yau elliptic fibrations

Let us begin with the explicit construction of the Calabi-Yau manifold \hat{X} . Abstractly, a general elliptic fibration of the given elliptic curve \mathcal{E} over a base B is given by defining the complete intersection (2.16) over the function field of B . In other words, we lift all coefficients s_i as well as the coordinates in (2.16) to sections of appropriate line bundles over B .

To each of the homogeneous coordinates on $\text{Bl}_3\mathbb{P}^3$ we assign a different line bundle on the base B . However, we can use the $(\mathbb{C}^*)^4$ -action in (2.13) to assign without loss of generality the following non-trivial line bundles

$$u \in \mathcal{O}_B(D_u), \quad v \in \mathcal{O}_B(D_v), \quad w \in \mathcal{O}_B(D_w), \quad (3.1)$$

with all other coordinates $[t : e_1 : e_2 : e_3]$ transforming in the trivial bundle on B . Here K_B denotes the canonical bundle on B , $[K_B]$ the associated divisor and D_u , D_v and D_w are three, at the moment, arbitrary divisors on B . They will be fixed later in this section by the Calabi-Yau condition on the elliptic fibration. The assignment (3.1) can be described globally by constructing the fiber bundle

$$\begin{array}{ccc} \text{Bl}_3\mathbb{P}^3 & \longrightarrow & \text{Bl}_3\mathbb{P}_B^3(D_u, D_v, D_w) \\ & & \downarrow \\ & & B \end{array} \quad (3.2)$$

The total space of this fibration is the ambient space of the complete intersection (2.16), that defines the elliptic fibration of \mathcal{E} over B .

Next, we require the complete intersection (2.16) to define a Calabi-Yau manifold in the ambient space (3.2). To this end, we first calculate the anti-canonical bundle of $\text{Bl}_3\mathbb{P}_B^3(D_u, D_v, D_w)$ via adjunction. We obtain

$$K_{\text{Bl}_3\mathbb{P}_B^3}^{-1} = 4H - 2E_1 - 2E_2 - 2E_3 + [K_B^{-1}] + D_u + D_v + D_w, \quad (3.3)$$

where we suppressed the dependence on the vertical divisors D_u , D_v and D_w for brevity of our notation and H as well as the E_i are the classes introduced in (2.13). For the complete intersection (2.16) to define a Calabi-Yau manifold \hat{X} in (3.2) we infer again from adjunction that the sum of the classes of the two constraints p_1 , p_2 has to be agree with $[K_{\text{Bl}_3\mathbb{P}_B^3}^{-1}]$. Thus, the Calabi-Yau condition reads

$$[p_1] + [p_2] \stackrel{!}{=} 4H - 2E_1 - 2E_2 - 2E_3 + [K_B^{-1}] + D_u + D_v + D_w. \quad (3.4)$$

We see from (2.13) that both constraints in (2.16) are automatically in the divisor class $2H - E_1 - E_2 - E_3$ w.r.t. the classes on the fiber $\text{Bl}_3\mathbb{P}^3$. Thus, (3.4) effectively reduces to a condition on the class of (2.16) in the homology of the base B . Denoting the part of the homology classes of the $[p_i]$ in the base B by $[p_1]^b$ and $[p_2]^b + D_v + D_w$, we obtain

$$[p_1]^b + [p_2]^b \stackrel{!}{=} [K_B^{-1}] + D_u. \quad (3.5)$$

Here we shifted the class $[p_2]^b \rightarrow D_v + D_w + [p_2]^b$ for reasons that will become clear in section 3.3.

Using this information we fix the line bundles on B in which the coefficients s_i take values. We infer from (2.16), (3.1) and the Calabi-Yau condition (3.5) the following assignments of line bundles,

section	line-bundle	section	line-bundle
s_2	$\mathcal{O}([K_B^{-1}] - D_u - [p_2]^b)$	s_{12}	$\mathcal{O}(-2D_u + D_v + D_w + [p_2]^b)$
s_5	$\mathcal{O}([K_B^{-1}] - [p_2]^b)$	s_{15}	$\mathcal{O}(-D_u + D_v + D_w + [p_2]^b)$
s_6	$\mathcal{O}([K_B^{-1}] - [p_2]^b - D_v)$	s_{16}	$\mathcal{O}(-D_u + D_w + [p_2]^b)$
s_7	$\mathcal{O}([K_B^{-1}] - [p_2]^b - D_w)$	s_{17}	$\mathcal{O}(-D_u + D_v + [p_2]^b)$
s_8	$\mathcal{O}([K_B^{-1}] - [p_2]^b + D_u - D_v - D_w)$	s_{18}	$\mathcal{O}([p_2]^b)$
s_9	$\mathcal{O}([K_B^{-1}] - [p_2]^b + D_u - D_v)$	s_{19}	$\mathcal{O}(D_w + [p_2]^b)$
s_{10}	$\mathcal{O}([K_B^{-1}] - [p_2]^b + D_u - D_w)$	s_{20}	$\mathcal{O}(D_v + [p_2]^b)$

(3.6)

We also summarize the complete line bundles of the homogeneous coordinates on $\text{Bl}_3\mathbb{P}^3$ by combining the classes in (2.13) and (3.1),

section	bundle
u	$\mathcal{O}(H - E_1 - E_2 - E_3 + D_u)$
v	$\mathcal{O}(H - E_2 - E_3 + D_v)$
w	$\mathcal{O}(H - E_1 - E_3 + D_w)$
t	$\mathcal{O}(H - E_1 - E_2)$
e_1	$\mathcal{O}(E_1)$
e_2	$\mathcal{O}(E_2)$
e_3	$\mathcal{O}(E_3)$

(3.7)

For later reference, we point out that the divisors associated to the vanishing of the coefficients \tilde{s}_7 , \hat{s}_7 and $\tilde{s}_9 = -\hat{s}_9$, denoted as $\tilde{\mathcal{S}}_7$, $\hat{\mathcal{S}}_7$ respectively \mathcal{S}_9 , in the two presentations (2.29) and (2.35) in dP_2 of the elliptic curves \mathcal{E} are given by

$$\begin{aligned} \tilde{\mathcal{S}}_7 &:= [-s_{19}s_8 + s_{18}s_9] = [K_B^{-1}] + D_u - D_v, & \hat{\mathcal{S}}_7 &:= [s_{10}s_{18} - s_{20}s_8] = [K_B^{-1}] + D_u - D_w, \\ \mathcal{S}_9 &:= [\tilde{s}_9] = [\hat{s}_9] = [-s_{10}s_{19} + s_{20}s_9] = D_u + [K_B^{-1}]. \end{aligned} \quad (3.8)$$

Here we have used the definitions in (2.30), respectively, (2.36) together with (3.6) and denoted the divisor classes of a section s_i by $[\cdot]$.

It is important to notice that the line bundles of the s_i admit an additional degree of freedom due to the choice of the class $[p_2]^b$, the divisor class of the second constraint p_2 in the homology of B . This is due to the fact that the Calabi-Yau condition (3.5) is a partition problem, that only fixes the sum of the classes $[p_1]^b, [p_2]^b$ but leaves the individual classes undetermined. For example, in complete intersections in a toric ambient space (3.2) the freedom of the class $[p_2]^b$ is fixed by finding all nef-partitions of the toric polytope associated to (3.2) that are consistent with the nef-partition (2.23) of the $\text{Bl}_3\mathbb{P}^3$ -fiber. We discuss the freedom in $[p_2]^b$ further in section 3.3.

3.2 Basic geometry of Calabi-Yau manifolds with $\text{Bl}_3\mathbb{P}^3$ -elliptic fiber

Let us next discuss the basic topological properties of the Calabi-Yau manifold \hat{X} .

We begin by constructing a basis D_A of the group of divisors $H^{(1,1)}(\hat{X})$ on \hat{X} that is convenient for the study of F-theory on \hat{X} . A basis of divisors on the generic complete intersection \hat{X} is induced from the basis of divisors of the ambient space $\text{Bl}_3\mathbb{P}^3(\tilde{\mathcal{S}}_7, \hat{\mathcal{S}}_7, \mathcal{S}_9)$ by restriction to \hat{X} . There are the vertical divisors D_α that are obtained by pulling back divisors D_α^b on the base B as $D_\alpha = \pi^*(D_\alpha^b)$ under the projection map $\pi : \hat{X} \rightarrow B$. In addition, each point P, Q, R and S on the elliptic fiber \mathcal{E} in (2.16) lifts to an in general rational section of the fibration $\pi : \hat{X} \rightarrow B$, that we denote by $\hat{s}_P, \hat{s}_Q, \hat{s}_R$ and \hat{s}_S , with \hat{s}_P the zero section. The corresponding divisor classes, denoted S_P, S_Q, S_R and S_S , then follow from (2.18) and (3.7) as

$$S_P = E_3, \quad S_Q = E_1, \quad S_R = E_2, \quad S_S = H - E_1 - E_2 - E_3 + \mathcal{S}_9 + [K_B], \quad (3.9)$$

where we denote, by abuse of notation, the lift of the classes H, E_1, E_2, E_3 of the fiber $\text{Bl}_3\mathbb{P}^3$ in (2.13) to classes in \hat{X} by the same symbol. For convenience, we collectively denote the generators of the Mordell-Weil group and their divisor classes as

$$\hat{s}_m = (\hat{s}_Q, \hat{s}_R, \hat{s}_S), \quad S_m = (S_Q, S_R, S_S) \quad m = 1, 2, 3. \quad (3.10)$$

The vertical divisors D_α together with the classes (3.9) of the rational points form a basis of $H^{(1,1)}(\hat{X})$. A basis that is better suited for applications to F-theory, however, is given by

$$D_A = (\tilde{S}_P, D_\alpha, \sigma(\hat{s}_m)), \quad A = 0, 1, \dots, h^{(1,1)}(B) + 4, \quad (3.11)$$

where the Hodge number $h^{(1,1)}(B)$ of the base B counts the number of vertical divisors D_α in \hat{X} . Here we have introduced the class [59, 60]

$$\tilde{S}_P = S_P + \frac{1}{2}[K_B^{-1}], \quad (3.12)$$

and have applied the Shioda map σ that maps the Mordell-Weil group of \hat{X} to a certain subspace of $H^{(1,1)}(\hat{X})$. The map σ is defined as

$$\sigma(\hat{s}_m) := S_m - \tilde{S}_P - \pi(S_m \cdot \tilde{S}_P), \quad (3.13)$$

where π , by abuse of notation, denotes the projection of $H^{(2,2)}(\hat{X})$ to the vertical homology $\pi^*H^{(1,1)}(B)$ of the base B . For every \mathcal{C} in $H^{(2,2)}(\hat{X})$ the map π is defined as

$$\pi(\mathcal{C}) = (\mathcal{C} \cdot \Sigma^\alpha)D_\alpha, \quad (3.14)$$

where we obtain the elements $\Sigma^\alpha = \pi^*(\Sigma_b^\alpha)$ in $H_4(\hat{X})$ as pullbacks from a dual basis Σ_b^α to the divisors D_α^b in B , i.e. $\Sigma_b^\alpha \cdot D_\beta^b = \delta_\beta^\alpha$.

Next, we list the fundamental intersections involving the divisors S_P , S_Q and S_R in (3.9), that will be relevant throughout this work:

Universal intersection:	$S_P \cdot F = S_m \cdot F = 1$ with general fiber $F \cong \mathcal{E}$, (3.15)
Rational sections:	$\pi(S_P^2 + [K_B^{-1}] \cdot S_P) = \pi(S_m^2 + [K_B^{-1}] \cdot S_m) = 0$, (3.16) $\tilde{\mathcal{S}}_7 = \pi(S_Q \cdot S_S)$, $\hat{\mathcal{S}}_7 = \pi(S_R \cdot S_S)$, $\mathcal{S}_9 = \pi(S_P \cdot S_S)$,
Holomorphic sections:	$S_P^2 + [K_B^{-1}] \cdot S_P = S_m^2 + [K_B^{-1}] \cdot S_m = 0$, (3.17)
Shioda maps:	$\sigma(\hat{s}_Q) = S_Q - S_P - [K_B^{-1}]$, $\sigma(\hat{s}_R) = S_R - S_P - [K_B^{-1}]$, (3.18) $\sigma(\hat{s}_R) = S_S - S_P - [K_B^{-1}] - \mathcal{S}_9$,

The first line (3.15) and the second line (3.16) are the defining property of a section of a fibration, whereas the fourth line only holds for a holomorphic section. The third line holds because the collision pattern of the points in (2.21) directly translates into intersections of their divisor classes S_m , where we made use of (2.30) and (2.36). In other words, (3.16) states that divisors $\tilde{\mathcal{S}}_7$, $\hat{\mathcal{S}}_7$, \mathcal{S}_9 are the codimension one loci where the sections collide with each other in the fiber \mathcal{E} . Finally, the result for the Shioda maps of the sections follows from their definitions in (3.13) and the intersections in (3.16).

For later reference, we also compute the intersection matrix of the Shioda maps $\sigma(\hat{s}_m)$, i.e. the height pairing, as

$$\pi(\sigma(\hat{s}_m) \cdot \sigma(\hat{s}_n)) = \begin{pmatrix} 2[K_B] & [K_B] & -\mathcal{S}_9 + \tilde{\mathcal{S}}_7 + [K_B] \\ [K_B] & 2[K_B] & -\mathcal{S}_9 + \hat{\mathcal{S}}_7 + [K_B] \\ -\mathcal{S}_9 + \hat{\mathcal{S}}_7 + [K_B] & -\mathcal{S}_9 + \tilde{\mathcal{S}}_7 + [K_B] & 2(-\mathcal{S}_9 + [K_B]) \end{pmatrix}_{mn}. \quad (3.19)$$

which readily follows from (3.18) and (3.16).

We note that all the above intersections (3.15), (3.16), (3.17), (3.18) and (3.19) are in completely analogous to the ones found in [37, 39, 53] for the case of an elliptic Calabi-Yau manifold with rank two Mordell-Weil group, see also [35, 54, 61, 62] for a discussion of intersections in the rank one case.

3.3 All Calabi-Yau manifolds \hat{X} with $\text{Bl}_3\mathbb{P}^3$ -elliptic fiber over B

Finally, we are equipped to classify the generic Calabi-Yau manifolds \hat{X} with elliptic fiber in $\text{Bl}_3\mathbb{P}^3$ and base B . This task reduces to a classification of all possible assignments of line bundles to the sections s_i in (3.6) so that the Calabi-Yau manifold \hat{X} is given by the generic complete intersection (2.16). Otherwise we expect additional singularities in \hat{X} ,

potentially corresponding to a minimal gauge symmetry in F-theory, either from non-toric non-Abelian singularities or from non-toric sections. We prove in the following that a generic Calabi-Yau manifold \hat{X} over a base B corresponds to a point in a certain polytope, that is related to the single nef-partition of the polytope of $\text{Bl}_3\mathbb{P}^3$ as explained below. The following discussion is similar in spirit to the one in [36, 39], that can agree with the toric classification of [56].

We begin with the basis expansion

$$D_u = n_u^\alpha D_\alpha, \quad D_v = n_v^\alpha D_\alpha, \quad D_w = n_w^\alpha D_\alpha, \quad (3.20)$$

into vertical divisors D_α , where the n_u^α , n_v^α and n_w^α are integer coefficients. For \hat{X} to be generic these coefficients are bounded by the requirement that all the sections s_i in (3.6) are generic, i.e. that the line bundles of which the s_i are holomorphic sections admit holomorphic sections. This is equivalent to all divisors in (3.6) being effective.

First, we notice that effectiveness of the sum $[s_i] + [s_{i+10}] \geq 0$ in (3.6) is guaranteed if the vector of integers $\mathbf{n}^\alpha = (n_u^\alpha, n_v^\alpha, n_w^\alpha)$ is an integral point in the rescaled polytope of $\text{Bl}_3\mathbb{P}^3$. Indeed, we can express the conditions of effectiveness of the divisors $[s_i] + [s_{i+10}]$ as the following set of inequalities in \mathbb{R}^3 ,

$$\frac{1}{-K^\alpha} \mathbf{n}^\alpha \cdot \nu_i \geq -1, \quad i = 1, \dots, 7, \quad (3.21)$$

where we also expand the canonical bundle K_B of the base B in terms of the vertical divisors D_α as

$$[K_B] = K^\alpha D_\alpha \quad (3.22)$$

with integer coefficients K^α . The entries of the vectors ν_i are extracted by first summing the rows of the two tables in (3.6), requiring the sum to be effective and then taking the coefficients of the the divisors D_u, D_v, D_w . The ν_i span the following polytope

$$\Delta_3 := \langle \nu_i \rangle = \left\langle \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle. \quad (3.23)$$

This is precisely the dual of the polytope $\nabla_{\text{Bl}_3\mathbb{P}^3}$ of $\text{Bl}_3\mathbb{P}^3$, where the latter polytope is the convex hull of the following vertices,

$$\nabla_{\text{Bl}_3\mathbb{P}^3} = \left\langle -\rho_1, \rho_{e_1}, \rho_4, \rho_3, \rho_{e_2}, \rho_{e_3}, \rho_1 \right\rangle. \quad (3.24)$$

We note that these vertices are related to the vertices in (2.10) and (2.22) by an $\text{SL}(3, \mathbb{Z})$ transformation. Thus, we confirm that the solutions to (3.21), for which all divisors $[s_i] + [s_{i+10}]$ are effective, are precisely given by vectors \mathbf{n}^α that take values for all α in the polytope of $\text{Bl}_3\mathbb{P}^3$ rescaled by the factor $-K^\alpha$.

Next we determine the conditions inferred from each individual class $[s_i]$ in (3.6) being effective. We obtain the following *two sets of conditions*, whose solutions, given also below, yield the set of all generic elliptic fibrations \hat{X} with a general rank three Mordell-Weil group over a given base B :

$$\begin{aligned}
 &1) \quad 0 \leq ([p_2]^b)^\alpha \leq -K_B^\alpha, \quad (3.25) \\
 &2) \quad \mathbf{n}^\alpha \cdot \nu_i \geq K^\alpha + ([p_2]^b)^\alpha, \quad \nu_i \in \nabla_1, \quad \mathbf{n}^\alpha \cdot \nu_i \geq -([p_2]^b)^\alpha, \quad \nu_i \in \nabla_2.
 \end{aligned}$$

These conditions are solved by any \mathbf{n}^α being integral points in the following Minkowski sum of the polyhedra ∇_1, ∇_2 defined in (3.29),

$$\mathbf{n}^\alpha \in -(K^\alpha + ([p_2]^b)^\alpha)\nabla_1 + ([p_2]^b)^\alpha\nabla_2, \quad \forall \alpha = 1, \dots, h^{(1,1)}(B). \quad (3.26)$$

Here the two conditions for $[p_2]^b$ in the first line of (3.25) follow from $[s_5], [s_{18}] \geq 0$ and the first, respectively, second set of conditions in the second line follow from the first, respectively, second table in (3.6). In addition, we have expanded the class $[p_2]^b$ into a basis D_α as

$$[p_2]^b = ([p_2]^b)^\alpha D_\alpha \quad (3.27)$$

and have introduced the points ν_i that define two polytopes

$$\begin{aligned}
 \Delta_1 := \langle \nu_i \rangle_{0 \leq i \leq 6} &= \left\langle \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle, \\
 \Delta_2 := \langle \nu_i \rangle_{7 \leq i \leq 12} &= \left\langle \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle. \quad (3.28)
 \end{aligned}$$

Next, we show how we have constructed the solutions (3.26) to (3.25). To this end, it we only have to notice that the two polytopes Δ_1, Δ_2 are the duals in the sense of (B.1) of the following two polytopes ∇_1, ∇_2 ,

$$\nabla_1 = \left\langle -\rho_1, \rho_{e_1}, \rho_4, \rho_3 \right\rangle, \quad \nabla_2 = \left\langle \rho_{e_2}, \rho_{e_3}, \rho_1 \right\rangle, \quad (3.29)$$

where the vectors ρ_i, ρ_{e_i} were defined in (2.10), (2.22). These two polytopes correspond to the unique nef-partition of (3.24). Now, we first fix the class $[p_2]^b$ such that the first conditions in (3.25) are met. Second, for each allowed class for $[p_2]^b$ we solve the second set of conditions in (3.25) for the vectors \mathbf{n}^α . However, these are just the duality relations between the Δ_i and ∇_j , rescaled by appropriate factors. Consequently, the solutions are precisely given by the integral points in the Minkowski sum of the polyhedra in (3.26). Here we emphasize again that both coefficients in (3.26) are positive integers by means of the first condition in (3.25).

In summary, we have shown that for a given base B a generic elliptically fibered Calabi-Yau manifold \hat{X} with general elliptic fiber \mathcal{E} given by (2.16) in $\text{Bl}_3\mathbb{P}^3$ corresponds to an integral point \mathbf{n}^α in the polyhedron (3.26) for every α and for every class $[p_2]^b$ obeying $0 \leq [p_2]^b \leq [K_B^{-1}]$. The coordinates of the point \mathbf{n}^α are the coefficients of the divisors D_u, D_v, D_w in the expansion (3.20) into vertical divisors D_α .

4 Matter in F-theory compactifications with a rank three Mordell-Weil group

In this section we analyze the codimension two singularities of the elliptic fibration of \hat{X} to determine the matter representations of corresponding F-theory compactifications to six and four dimensions. We find 14 different singlet representations in sections 4.1 and 4.2. Then, we determine the explicit matter multiplicities of these 14 matter fields in six-dimensional F-theory compactification on a Calabi-Yau threefold \hat{X}_3 with a general two-dimensional base B in section 4.3. The following discussion is based on techniques developed in [37–39] for the case of a rank two Mordell-Weil group, to which we refer for more background on some technical details.

We begin with an outline of the general strategy to determine matter in an F-theory compactification on a Calabi-Yau manifold with a higher rank Mordell-Weil group. First, we recall that in general rational curves c_{mat} obtained from resolving a singularity of the elliptic fibration at codimension two in the base B give rise to matter in F-theory due to the presence of light M2-brane states in the F-theory limit. In elliptically fibered Calabi-Yau manifolds with a non-Abelian gauge symmetry in F-theory, these codimension two singularities are located on the divisor in the base B , which supports the 7-branes giving rise to the non-Abelian gauge group. Technically, the discriminant of the elliptic fibration takes the form $\Delta = z^n(k + \mathcal{O}(z))$, where z vanishes along the 7-brane divisor and k is a polynomial independent of z . Then, the codimension two singularities are precisely given by the intersections of $z = 0$ and $k = 0$.

This is in contrast to elliptic fibrations with only a non-trivial Mordell-Weil group, i.e. only an Abelian gauge group, since the elliptic fibration over codimension one has only I_1 -singularities and the discriminant does not factorize in an obvious way. Thus, the codimension two codimension singularities are not contained in a simple divisor in B and have to be studied directly. In fact, the existence of a rational section, denoted by say \hat{s}_Q , means that there is a solution to the Weierstrass form (WSF) of the form $[x^Q : y^Q : z^Q] = [g_2^Q : g_3^Q : 1]$.¹¹ Here g_2^Q and g_3^Q are sections of K_B^{-2} and K_B^{-3} , respectively.¹² Thus, the presence of \hat{s}_Q implies the factorization

$$(y - g_3^Q z^3)(y + g_3^Q z^3) = (x - g_2^Q z^2)(x^2 + g_2^Q x z^2 + g_4^Q z^4) \tag{4.1}$$

for appropriate g_4^Q . Parametrizing the discriminant Δ in terms of the polynomials in (4.1), we see that it vanishes of order two at the codimension two loci in B reading

$$g_3^Q = 0, \quad \hat{g}_4^Q := g_4^Q + 2(g_2^Q)^2 = 0. \tag{4.2}$$

These two conditions lead to a factorization of both sides of (4.1), so that a conifold singularity is developed at $y = (x - g_2^Q z^2) = 0$.

¹¹Sections with $z^Q = b$ for a section b of a line bundle $\mathcal{O}([b])$ on the base B and with g_2^Q, g_3^Q sections of $K_B^{-2} \otimes \mathcal{O}(2[b])$, respectively, $K_B^{-3} \otimes \mathcal{O}(3[b])$, can be studied similarly. We only have to assume that we are at a locus with $b \neq 0$. Then we can employ the \mathbb{C}^* -action to set $z^Q = 1, x^Q = \frac{g_2^Q}{b^2}, y^Q = \frac{g_3^Q}{b^3}$.

¹²For concreteness and for comparison to [35, 37], in the special case of the base $B = \mathbb{P}^2$, the sections $g_2^Q = g_6, g_3^Q = g_9$ are polynomials of degree 6, respectively, 9.

It is evident that the section \hat{s}_Q passes automatically through the singular point of the elliptic curve. Thus, in the resolved elliptic curve \mathcal{E} where the singular point $y = (x - g_2^Q z^2) = 0$ is replaced by a Hirzebruch-Jung sphere tree of intersecting \mathbb{P}^1 's,¹³ the section \hat{s}_Q automatically intersects at least one \mathbb{P}^1 . This implies that the loci (4.2) in the base contain matter charged under $U(1)_Q$ associated to \hat{s}_Q , as can be seen from the charge formula

$$q_Q = c_{\text{mat}} \cdot (S_Q - S_P). \tag{4.3}$$

Here S_Q, S_P denote the divisor classes of \hat{s}_Q and the zero section \hat{s}_P , respectively. In fact, the locus (4.2) contains the codimension two loci supporting *all* matter charged under $U(1)_Q$, without distinguishing between matter with different $U(1)_Q$ -charges. The loci of the different matter representations correspond to the irreducible components of (4.2), that can in principle be obtained by finding all associated prime ideals of (4.2) of codimension two in B . Unfortunately, in many concrete setups this is computationally unfeasible and we have to pursue a different strategy to obtain the individual matter representations that has already been successful in the rank two case in [35, 37].

For the following analysis of codimension two singularities of \hat{X} we identify the irreducible components of (4.2) corresponding to different matter representations in two qualitatively different ways:

- 1) One type of codimension two singularities corresponds to singularities of the sections \hat{s}_m and \hat{s}_P . This analysis, see section 4.1, is performed in the presentation of \mathcal{E} as the complete intersection (2.16) in $\text{Bl}_3\mathbb{P}^3$, where the rational sections are given by (2.19). In fact, when a rational section \hat{s}_m or the zero section \hat{s}_P is ill-defined, the resolved elliptic curve splits into an I_2 -curve with one \mathbb{P}^1 representing the original singular fiber and the other \mathbb{P}^1 representing the singular section.
- 2) The second type of codimension two singularities has to be found directly in the Weierstrass model. The basic idea is isolate special solutions to (4.2) by supplementing the two equations (4.2) by further constraints that have to vanish in addition in order for a certain matter representation to be present. We refer to section 4.2 for concrete examples. It is then possible to find the codimension two locus along which all these constraints vanish simultaneously. We note that for the geometry \hat{X} there are three rational sections, thus, three factorizations of the form (4.1) and loci (4.2), that have to be analyzed separately.

A complete analysis of codimension two singularities following the above two-step strategy should achieve a complete decomposition of (4.2) for all sections of \hat{X} into irreducible components. It would be interesting to prove this mathematical for the codimension two singularities of \hat{X} we find in this section. As a consistency check of our analysis of codimension two singularities we find, we determine the full spectrum, including multiplicities, of charged hypermultiplets of a six-dimensional F-theory compactification and check that six-dimensional anomalies are cancelled, cf. section 4.3.

¹³In F-theory compactifications with only Abelian groups the resolved elliptic fibers are expected to be I_2 -curves, i.e. two \mathbb{P}^1 's intersecting at two points.

4.1 Matter at the singularity loci of rational sections

Now that the strategy is clear, we will look for the first type of singularities in this subsection. These are the codimension two loci in the base where the rational sections are singular in $\text{Bl}_3\mathbb{P}^3$. This precisely happens when the coordinates (2.18), (2.19) of any of the rational sections take values in the Stanley-Reisner ideal (2.14) of $\text{Bl}_3\mathbb{P}^3$.

There are two reasons why codimension two loci with singular rational sections are good candidates for I_2 -fibers. First, the elliptic fibration of \hat{X} is smooth,¹⁴ thus, the indeterminacy of the coordinates of the sections in the fiber may imply that the section is not a point, but an entire \mathbb{P}^1 . Second, as was remarked in [35] and [37], if we approach the codimension two singularity of the section along a line in the base B the section has a well defined coordinate given by the slope of the line. Thus, approaching the singularity along lines of all possible slopes the section at the singular point is identified with the \mathbb{P}^1 formed by all slopes. In fact, specializing the elliptic curve to each locus yielding a singularity of a rational section we observe a splitting of the elliptic curve into an I_2 -curve. We note that it is crucial to work in $\text{Bl}_3\mathbb{P}^3$, because only in this space the fiber is fully resolved space by the exceptional divisors E_i , in contrast to the curve (2.4) in \mathbb{P}^3 .

4.1.1 The vanishing of two minors: special singularities of \hat{s}_S

In order to identify singularities of rational sections, let us take a close look at the Stanley-Reisner ideal (2.14). It contains monomials with two variables of the type $e_i e_j$ and monomials with three variables of the type uXY , where X and Y are two variables out of the set $\{v, w, t\}$. In this subsection we look for singular sections whose coordinates are forbidden by the elements $e_i e_j$.

From the coordinates (2.19) of the rational sections we infer that this type of singular behavior can only occur for the section \hat{s}_S , whose coordinates in the fiber \mathcal{E} are

$$S = [0 : 1 : 1 : 1 : s_{19}s_8 - s_{18}s_9 : s_{10}s_{18} - s_{20}s_8 : s_{10}s_{19} - s_{20}s_9]. \quad (4.4)$$

There are three codimension two loci where S is singular, reading

$$\{s_8 = s_{18} = 0\}, \quad \{s_9 = s_{19} = 0\}, \quad \{s_{10} = s_{20} = 0\}. \quad (4.5)$$

It is important to note that the matrices (2.8), (2.20) retain rank two at these loci, since only two of their 2×2 -minors, being identified with the coordinates (2.19), have vanishing determinant. Next, we inspect the constraint (2.16) of the elliptic curve at these loci.

At all these three codimension two loci, we see that the elliptic curve in (2.16) takes the common form

$$Au + BY = 0, \quad Cu + DY = 0. \quad (4.6)$$

Here Y is one of the variables $\{v, w, t\}$ and the polynomials B, D are chosen to be independent of u and Y , which fixes the polynomials A, C uniquely. This complete intersection describes a reducible curve. This can be seen by rewriting it as

$$(AD - BC)u = 0, \quad Au + BY = Cu + DY = 0, \quad (4.7)$$

¹⁴This is clear for toric bases B .

which we obtained by solving for the variable Y in the first equation of (4.6) and requiring consistency with the second equation.

Now, we directly see that one solution to (4.7) is given by $\{u = 0, Y = 0\}$. This is a \mathbb{P}^1 as is clear from the remaining generators of the SR-ideal after setting the coordinates that are not allowed to vanish to one using the \mathbb{C}^* -actions. The second solution, which also describes a \mathbb{P}^1 , is given by the vanishing of the determinant in the first equation in (4.7), which implies that the two constraint in the second equation become dependent. Thus, the two \mathbb{P}^1 's of the I_2 -curve are given by

$$c_1 = \{u = 0, Y = 0\}, \quad c_2 = \{AD - BC = 0, Cu + DY = 0\}. \quad (4.8)$$

As an example, let us look at the loci $\{s_8 = s_{18} = 0\}$ in (4.5) in detail. In this case the elliptic curve \mathcal{E} given in (2.16) takes the form

$$\begin{aligned} u(s_2e_1e_2e_3u + s_5e_1e_2t + s_6e_2e_3v + s_7e_1e_3w) &= t(s_9e_2v + s_{10}e_1w), \\ u(s_{12}e_1e_2e_3u + s_{15}e_1e_2t + s_{16}e_2e_3v + s_{17}e_1e_3w) &= t(s_{19}e_2v + s_{20}e_1w). \end{aligned} \quad (4.9)$$

This complete intersection is in the form (4.6) by identifying $Y = t$ and setting

$$\begin{aligned} A &= (s_2e_1e_2e_3u + s_5e_1e_2t + s_6e_2e_3v + s_7e_1e_3w), & B &= -(s_9e_2v + s_{10}e_1w), \\ C &= (s_{12}e_1e_2e_3u + s_{15}e_1e_2t + s_{16}e_2e_3v + s_{17}e_1e_3w), & D &= -(s_{19}e_2v + s_{20}e_1w). \end{aligned} \quad (4.10)$$

Then the two \mathbb{P}^1 's of the I_2 -curve are given by c_1, c_2 in (4.8).

Equipped with the equations for the individual curves c_1, c_2 we can now calculate the intersections with the sections and the charge of the hypermultiplet that is supported there. The intersections of the curve defined c_1 can be readily obtained from the toric intersections of $\text{Bl}_3\mathbb{P}^3$. It has intersection -1 with the section S_S , intersection one with the sections S_Q, S_R and zero with S_P , where the last intersection is clear from the existence of the term e_3t in the Stanley-Reisner ideal (2.14). The intersections with c_2 can be calculated either directly from (4.8) or from the fact, that the intersections of a section with the total class $F = c_1 + c_2$ have to be one.

We summarize our findings as:

Loci	Curve	$\cdot S_P$	$\cdot S_Q$	$\cdot S_R$	$\cdot S_S$
$s_8 = s_{18} = 0$	$c_1 = \{u = t = 0\}$	0	1	1	-1
	c_2	1	0	0	2
$s_9 = s_{19} = 0$	$c_1 = \{u = w = 0\}$	1	1	0	-1
	c_2	0	0	1	2
$s_{10} = s_{20} = 0$	$c_1 = \{u = v = 0\}$	1	0	1	-1
	c_2	0	1	0	2

(4.11)

Here we denoted the intersection pairing by ‘ \cdot ’ and we also computed the intersections of the sections with the I_2 -curves at the other two codimension two loci in (4.5). In these cases, we identified $Y = w$, respectively, $Y = v$.

We proceed with the calculation of the charges in each case employing the charge formula (4.3). We note that the isolated curve c_{mat} is always the curve in the I_2 -fiber that does not intersect the zero section S_P . We obtain the charges:

Loci	q_Q	q_R	q_S
$s_8 = s_{18} = 0$	1	1	-1
$s_9 = s_{19} = 0$	0	1	2
$s_{10} = s_{20} = 0$	1	0	2

(4.12)

4.1.2 The vanishing of three minors: singularities of all sections

The remaining singularities of the rational sections occur if the three of the determinants of the minors of the matrices (2.8), (2.20) vanish. This implies that three coordinates (2.19) of a section are forbidden by the SR-ideal (2.14), which happens also for the sections \hat{s}_P , \hat{s}_Q , \hat{s}_R , in addition to \hat{s}_S , due to the elements uXY with X, Y in $\{v, w, t\}$.

Before analyzing these loci, we emphasize that the three vanishing conditions are a codimension two phenomenon because the vanishing of the determinants of three minors of the same matrix is not independent. In fact, these codimension two loci can be viewed as determinantal varieties describing the loci where the rank of each of the matrices in (2.8), (2.20) jump from two to one, which is clearly a codimension two phenomenon.

Concretely, for the section \hat{s}_P to be singular, the three minors that have to vanish are $|M_3^P| = |M_2^P| = |M_1^P| = 0$, which implies the conditions

$$\frac{s_5}{s_{15}} = \frac{s_{10}}{s_{20}} = \frac{s_9}{s_{19}}. \tag{4.13}$$

Similarly, for \hat{s}_Q to be singular, we impose $|M_3^Q| = |M_2^Q| = |M_1^Q| = 0$, which yields

$$\frac{s_6}{s_{16}} = \frac{s_8}{s_{18}} = \frac{s_9}{s_{19}}. \tag{4.14}$$

For a singular section \hat{s}_R , we require $|M_3^R| = |M_2^R| = |M_1^R| = 0$, which is equivalent to

$$\frac{s_{10}}{s_{20}} = \frac{s_8}{s_{18}} = \frac{s_7}{s_{17}}. \tag{4.15}$$

Finally, the section \hat{s}_S is singular at $|M_3^Q| = |M_3^R| = |M_3^P| = 0$, or equivalently at

$$\frac{s_{10}}{s_{20}} = \frac{s_8}{s_{18}} = \frac{s_9}{s_{19}}. \tag{4.16}$$

We remark that the vanishing of the three minors in all these cases excludes the loci (4.5) of the previous subsection.

All these singularities imply a reducible curve of a form similar to (2.27), however, adapted to the ambient space $\text{Bl}_3\mathbb{P}^3$. In fact, at each of the loci (4.13)–(4.16) the complete intersection (2.16) takes the form

$$AX + BY = 0, \quad CX + DY = 0, \tag{4.17}$$

for appropriate polynomials A, B, C, D with A and C collinear, that is $A = aC$, and the pair of coordinates $[X : Y]$ forming a \mathbb{P}^1 .¹⁵ Then, we can multiply the second equation by a and subtract from the first equation, to obtain

$$(B - aD)Y = 0, \quad AX + BY = 0. \tag{4.18}$$

From this we see that the two solutions are given by

$$c_1 = \{Y = A = 0\}, \quad c_2 = \{B - aD = AX + BY = 0\}, \tag{4.19}$$

that describe two \mathbb{P}^1 's intersecting at two points. Thus the complete intersection (4.18) is an I_2 -curve.

One example in detail. Let us focus on the locus in (4.14) where the section \hat{s}_Q is singular. The complete intersection (2.16) then takes the form

$$\begin{aligned} v(-e_2s_9t + e_2e_3s_6u + e_3s_8w) + e_1(e_2s_5tu + e_2e_3s_2u^2 - s_{10}tw + s_7e_3uw) &= 0, \\ v(-e_2s_{19}t + e_2e_3s_{16}u + e_3s_{18}w) + e_1(e_2s_{15}tu + e_2e_3s_{12}u^2 - s_{20}tw + e_3s_{17}uw) &= 0. \end{aligned}$$

This is of the form (4.17) as we see by identifying $X = v$ and $Y = e_1$ and by setting

$$\begin{aligned} A &= -e_2s_9t + e_2e_3s_6u + e_3s_8w, & B &= e_2s_5tu + e_2e_3s_2u^2 - s_{10}tw + s_7e_3uw, \\ C &= -e_2s_{19}t + e_2e_3s_{16}u + e_3s_{18}w, & D &= e_2s_{15}tu + e_2e_3s_{12}u^2 - s_{20}tw + e_3s_{17}uw \end{aligned} \tag{4.20}$$

with $A = (s_8/s_{18})C$ collinear at the locus (4.14). Then, the two \mathbb{P}^1 's in this I_2 -curve are given by (4.19) with the identifications (4.20).

Next, we obtain the intersections of the curves c_1, c_2 with the rational sections, that follow directly from the toric intersections of $\text{Bl}_3\mathbb{P}^3$. We find the intersections

Loci	Curve	· S_P	· S_Q	· S_R	· S_S
$ M_3^Q = M_2^Q = M_1^Q = 0$	c_1	0	-1	0	1
	c_2	1	2	1	0

(4.21)

As expected, the total fiber $F = c_1 + c_2$ has intersections $S_m \cdot F = 1$ with all sections.

Repeating the procedure with the other codimension two loci (4.13), (4.15) and (4.16), we obtain the intersections of the split elliptic curve with the sections as

Loci	Curve	· S_P	· S_Q	· S_R	· S_S
$ M_3^R = M_2^R = M_1^R = 0$	c_1	0	0	-1	1
	c_2	1	1	2	0
$ M_3^P = M_2^P = M_1^P = 0$	c_1	-1	0	0	1
	c_2	2	1	1	0
$ M_3^Q = M_3^R = M_3^P = 0$	c_1	1	1	1	-1
	c_2	0	0	0	2

(4.22)

¹⁵When \hat{s}_S becomes singular, we identify $Y = u$ and $X = 1$. However, A, C still become collinear and the argument applies.

With these intersection numbers and the charge formula (4.3) we obtain the charges

Loci	q_Q	q_R	q_S
$ M_3^Q = M_2^Q = M_1^Q = 0$	-1	0	1
$ M_3^R = M_2^R = M_1^R = 0$	0	-1	1
$ M_3^P = M_2^P = M_1^P = 0$	-1	-1	-2
$ M_3^Q = M_3^R = M_3^P = 0$	0	0	2

(4.23)

Relation to dP_2 . In section 2.3.2 we saw that the elliptic curve \mathcal{E} can be mapped to two¹⁶ non-generic anti-canonical hypersurfaces in dP_2 . It is expected that some of the singularities we just found map to the singularities in the dP_2 -elliptic curve. We recall from [36, 37], that the Calabi-Yau hypersurfaces (2.29), (2.35) in dP_2 have singular sections at the codimension two loci given by $\tilde{s}_3 = \tilde{s}_7 = 0$ ($\hat{s}_3 = \hat{s}_7 = 0$), $\tilde{s}_8 = \tilde{s}_9 = 0$ ($\hat{s}_8 = \hat{s}_9 = 0$) and $\tilde{s}_7 = \tilde{s}_9 = 0$ ($\hat{s}_7 = \hat{s}_9 = 0$), respectively.

In tables (2.30) and (2.36) we readily identified the minors of the matrices in (2.20) with the some of the coefficients \tilde{s}_i and \hat{s}_j . This implies a relationship between the singular codimension two loci of the elliptic curves in $\text{Bl}_3\mathbb{P}^3$ and in the two dP_2 -varieties, that we summarize in the following table:

$\text{Bl}_3\mathbb{P}^3$ -singularity	Singularity of curve in (2.29)	Singularity of curve in (2.35)
$ M_3^Q = M_2^Q = M_1^Q = 0$	$\tilde{s}_3 = \tilde{s}_7 = 0$	Q non-toric
$ M_3^R = M_2^R = M_1^R = 0$	R non-toric	$\hat{s}_3 = \hat{s}_7 = 0$
$ M_3^P = M_2^P = M_1^P = 0$	$\tilde{s}_8 = \tilde{s}_9 = 0$	$\hat{s}_8 = \hat{s}_9 = 0$
$ M_3^Q = M_3^R = M_3^P = 0$	$\tilde{s}_7 = \tilde{s}_9 = 0$	$\hat{s}_7 = \hat{s}_9 = 0$

(4.24)

In each case, three out of the four singular loci (4.23) yield singularities of the toric sections in the dP_2 -elliptic curve. The other singular locus in the curve in $\text{Bl}_3\mathbb{P}^3$ is not simply given by the vanishing of two coefficients \tilde{s}_i , respectively \hat{s}_j , because the non-toric rational sections becomes singular. Nevertheless, the elliptic curve in dP_2 admits a factorization at the singular locus of the non-toric section, i.e. it splits into an I_2 -curve, due to the non-genericity of the corresponding coefficients \tilde{s}_i or \hat{s}_j .

4.2 Matter from singularities in the Weierstrass model

As mentioned in the introduction of this subsection, *all* the loci of matter charged under a section \hat{s}_m satisfy the equations $g_3^m = 0$ and $\hat{g}_4^m = 0$. Since we have three rational sections \hat{s}_m , the WSF admits three possible factorizations of the form (4.1), each of which implying a singular elliptic fiber at the loci $g_3^{Q,R,S} = \hat{g}_4^{Q,R,S} = 0$ with $\hat{g}_4^{R,S}$ defined analogous to (4.2). In this subsection we separate solutions to these equations by requiring additional constraints to vanish.

¹⁶There are actually three dP_2 maps if we are willing to give up the zero point as a toric point. See section 2.3.2 for more details.

We can isolate matter with simultaneous U(1)-charges. The idea is the following. If the matter is charged under two sections, both sections have to pass through the singularity in the WSF. This requires the x -coordinates $g_2^{m_1}, g_2^{m_2}$ of the sections to agree,¹⁷

$$\delta g_2^{m_1, m_2} := g_2^{m_1} - g_2^{m_2} \stackrel{!}{=} 0, \tag{4.25}$$

for any two sections \hat{s}_{m_1} and \hat{s}_{m_2} . The polynomial (4.25) has a smaller degree than the other two conditions (4.2) and in fact it will be one of the two polynomials of the complete intersection describing the codimension two locus. The other constraint will be $g_3^m = 0$ for m either m_1 or m_2 .

If we solve for two coefficients in these two polynomials and insert the solution back into the elliptic curve (2.16) we observe a reducible curve of the form (4.18). In this I_2 -curve, one \mathbb{P}^1 is automatically intersected once by both sections \hat{s}_{m_1} and \hat{s}_{m_2} . This means that a generic solution of equations (4.2), (4.25) support matter with charges one under $U(1)_{m_1} \times U(1)_{m_2}$.

Let us be more specific for matter charged under the sections \hat{s}_Q and \hat{s}_R , that is matter transforming under $U(1)_Q \times U(1)_R$. The conditions (4.2) and (4.25) read

$$\delta g_2^{QR} := g_2^Q - g_2^R \stackrel{!}{=} 0, \quad g_3^Q = 0, \quad \hat{g}_4^Q = 0, \tag{4.26}$$

and the codimension to locus is given by the complete intersection $\delta g_2^{QR} = g_3^Q = 0$. In fact the constraint \hat{g}_4^Q, \hat{g}_4^R are in the ideal generate by $\langle \delta g_2^{QR}, g_3^Q \rangle$.

We proceed to look for matter charged under $U(1)_Q \times U(1)_S$. In this case, because of the section \hat{s}_S having a non-trivial z -component, the right patch of the WSF is $z \equiv \tilde{z}^S = s_{10}s_{19} - s_{20}s_9$, cf. (2.39). Thus, the constrains (4.2) and (4.25) take the form

$$\delta g_2^{QS} := g_2^S - (\tilde{z}^S)^2 g_2^Q \stackrel{!}{=} 0, \quad g_3^S = 0, \quad \hat{g}_4^S = 0. \tag{4.27}$$

Instead of using these polynomials, we will use two slightly modified polynomials that generate the same ideal. They were defined in [37] where they were denoted by $\delta g'_6$ and g'_9 and defined as

$$\delta(g_2^{QS})' := \tilde{s}_7 \tilde{s}_8^2 + \tilde{s}_9(-\tilde{s}_6 \tilde{s}_8 + \tilde{s}_5 \tilde{s}_9) = 0, \quad (g_3^{QS})' := \tilde{s}_3 \tilde{s}_8^2 - \tilde{s}_2 \tilde{s}_8 \tilde{s}_9 + \tilde{s}_1 \tilde{s}_9^2 = 0, \tag{4.28}$$

Here we have to use the map (2.30) to obtain these polynomials in terms of the coefficients s_i . We will see in section 4.3 that these polynomials are crucial to obtain the matter multiplicities of this type of charged matter fields.

Similarly, for matter charged under $U(1)_R \times U(1)_S$ we demand

$$\delta g_2^{RS} := g_2^S - (\tilde{z}^S)^2 g_2^R \stackrel{!}{=} 0, \quad g_3^S = 0, \quad \hat{g}_4^S = 0. \tag{4.29}$$

For this type of locus we will also use the modified polynomials $\delta(g_2^{RS})'$ and $\delta(g_3^{RS})'$ that can be obtained from (4.28) by replacing all the coefficients $\tilde{s}_i \rightarrow \hat{s}_i$ and by using (2.36).

¹⁷Here we assume that the z -coordinates of both sections are $z = 1$, for simplicity.

Next, we look for matter charged under all $U(1)$ factors $U(1)_Q \times U(1)_R \times U(1)_S$. This requires the three sections to collide and pass through the singular point $y = 0$ in the WSF, at codimension two. The four polynomials that are required to vanish simultaneously are

$$\delta g_2^{QS} = 0, \quad (z^S)^2 \delta g_2^{RS} = 0, \quad g_3^S = 0, \quad \hat{g}_4^S = 0, \quad (4.30)$$

where the first two conditions enforce a collision of the three sections in the elliptic fiber. In order for a codimension two locus to satisfy all these constraints simultaneously, all the polynomials (4.30) should factor as

$$p = h_1 p_1 + h_2 p_2, \quad (4.31)$$

where h_1 and h_2 are the polynomials whose zero-locus defines the codimension two locus in question. To obtain the polynomials we use the Euclidean algorithm twice. We first divide all polynomials in (4.30) by the lowest order polynomial available, which is δg_2^{QR} and take the biggest common factor from all residues. This is the polynomial h_1 and it reads

$$\begin{aligned} h_1 = & (s_{10}^2 s_{15} s_{16} s_{19} + s_{10}^2 s_{12} s_{19}^2 + s_{10} s_{15} s_{18} s_{19} s_5 + s_{10} s_{17} s_{19}^2 s_5 - s_{10} s_{16} s_{19} s_{20} s_5 \\ & - s_{18} s_{19} s_{20} s_5^2 - s_{10} s_{15}^2 s_{18} s_9 - s_{10} s_{15} s_{17} s_{19} s_9 - s_{10} s_{15} s_{16} s_{20} s_9 - 2 s_{10} s_{12} s_{19} s_{20} s_9 \\ & + s_{15} s_{18} s_{20} s_5 s_9 - s_{17} s_{19} s_{20} s_5 s_9 + s_{16} s_{20}^2 s_5 s_9 + s_{15} s_{17} s_{20} s_9^2 + s_{12} s_{20}^2 s_9^2). \end{aligned} \quad (4.32)$$

The knowledge of h_1 allows us to repeat the Euclidean algorithm. We reduce the polynomials (4.30) by (4.32) and again obtain the second common factor from the residues of all polynomials reading

$$\begin{aligned} h_2 = & s_{10}^2 s_{19} (s_{15} s_{16} + s_{12} s_{19}) - s_{10} [s_{15}^2 s_{18} s_9 + s_{19} (-s_{17} s_{19} s_5 + s_{16} s_{20} s_5 + 2 s_{12} s_{20} s_9) \\ & + s_{15} (-s_{18} s_{19} s_5 + s_{17} s_{19} s_9 + s_{16} s_{20} s_9)] + s_{20} [s_{18} s_5 (-s_{19} s_5 + s_{15} s_9) \\ & + s_9 (-s_{17} s_{19} s_5 + s_{16} s_{20} s_5 + s_{15} s_{17} s_9 + s_{12} s_{20} s_9)]. \end{aligned} \quad (4.33)$$

To confirm that these polynomials define the codimension two locus we were looking for, we check that all the constraints (4.30) are in the ideal generated by $\langle h_1, h_2 \rangle$.

Finally, if there are no more smaller ideals, i.e. special solutions, of $g_3^m = \hat{g}_4^m = 0$ we expect its remaining solutions to be generic and to support matter charged under only the section \hat{s}_m , i.e. matter with charges $q_m = 1$, and $q_n = 0$ for $n \neq m$. In summary, we find that matter at a generic point of the following loci has the following charges,

Generic point in locus	q_Q	q_R	q_S
$g_2^{QR} = g_3^Q = 0$	1	1	0
$(g_2^{QS})' = (g_3^S)' = 0$	1	0	1
$(g_2^{RS})' = (g_3^S)' = 0$	1	0	1
$h_1 = h_2 = 0$	1	1	1
$g_3^Q = \hat{g}_4^Q = 0$	1	0	0
$g_3^R = \hat{g}_4^R = 0$	0	1	0
$g_3^S = \hat{g}_4^S = 0$	0	0	1

(4.34)

In each of these six cases we checked explicitly the factorization of the complete intersection (2.27) for \mathcal{E} into an I_2 -curve, then computed the intersections of the sections \hat{s}_P, \hat{s}_m , $m = Q, R, S$ and obtained the charges by applying the charge formula (4.3).

4.3 6D matter multiplicities and anomaly cancellation

In this section we specialize to six-dimensional F-theory compactifications on an elliptically fibered Calabi-Yau threefolds \hat{X}_3 over a general two-dimensional base B with generic elliptic fiber given by (2.16). We work out the spectrum of charged hypermultiplets, that transform in the 14 different singlet representations found in sections 4.1 and 4.2. To this end, we compute the explicit expressions for the multiplicities of these 14 hypermultiplets. We show consistency of this charged spectrum by checking anomaly-freedom.

The matter multiplicities are given by the homology class of the irreducible locus that supports a given matter representation. As discussed above, some of these irreducible matter loci can only be expressed as prime ideals, of which we can not directly compute the homology classes. Thus, we have to compute matter multiplicities successively, starting from the complete intersections Loc_{CI} in (4.34) that support multiple matter fields of different type. We found, that at the generic point of the complete intersection Loc_{CI} one type of matter is supported, but at special points Loc_s^i different matter fields are located. We summarize this as

$$\cup_i \text{Loc}_s^i \subset \text{Loc}_{\text{CI}}. \tag{4.35}$$

Thus, first we calculate all multiplicities of matter located at all these special loci Loc_s^i and then subtract them from the complete intersection Loc_{CI} in which they are contained with a certain degree. This degree is given by the order of vanishing of resultant, that has already been used in a similar context in [37]. It is defined as follows. Given two polynomials (r, s) in the variables (x, y) , if $(0, 0)$ is a zero of both polynomials, its degree is given by the order of vanishing of the resultant $h(y) := \text{Res}_x(r, s)$ at $y = 0$.

This is a straightforward calculation when the variables (x, y) are pairs of the coefficients s_i . However, for more complicated loci we will need to treat full polynomials (p_1, p_2) as these variables, for example $x = \tilde{s}_7, y = \tilde{s}_9$ or $x = \delta g_6, y = g_9$. In this case we have to solve for two coefficients s_i, s_j from $\{p_1 = x, p_2 = y\}$, then replace them in (r, s) and finally proceed to take the resultant in x and y .

There is one technical caveat, when we are considering polynomials (p_1, p_2) that contain multiple different matter multiplets. We choose the coefficients s_i, s_j in such a way that the variables (x, y) only parametrize the locus of the hypermultiplets we are interested in. This is achieved by choosing s_i, s_j we are solving for so that the polynomials of the locus we are *not* interested in appear as denominators and are, thus, forbidden. For example, let us look at the loci $|M_3^Q| = |M_3^P| = 0$. This complete intersection contains the loci of the hypermultiplets with charges $(0, 0, 2)$ at the generic point and with charges $(0, 1, 2)$ at the special locus $s_9 = s_{19} = 0$, cf. (4.12), respectively, (4.23). Let us focus on the former hypermultiplets. We set

$$|M_3^Q| = s_{18}s_9 - s_{19}s_8 \equiv x, \quad |M_3^P| = s_{10}s_{19} - s_{20}s_9 \equiv y, \tag{4.36}$$

and solve for s_8 and s_{20} to obtain

$$s_8 = \frac{(s_{18}s_9 - x)}{s_{19}}, \quad s_{20} = \frac{(s_{10}s_{19} + y)}{s_9}. \quad (4.37)$$

From this, it is clear the locus $s_9 = s_{19} = 0$ corresponding to hypermultiplets with charges $(0, 1, 2)$ is excluded because of the denominators. Thus, (x, y) indeed parametrize the locus of the hypermultiplets of charges $(0, 0, 2)$.

We begin the computation of multiplicities with the simplest singularities in 4.1.1 located at the vanishing-loci of two coefficients $s_i = s_j = 0$. Their multiplicities are directly given by their homology classes, that are simply the product of the classes $[s_i]$, $[s_j]$. We obtain

Loci	q_Q	q_R	q_S	Multiplicity
$s_8 = s_{18} = 0$	1	1	-1	$[s_8] \cdot [s_{18}]$
$s_9 = s_{19} = 0$	0	1	2	$[s_9] \cdot [s_{19}]$
$s_{10} = s_{20} = 0$	1	0	2	$[s_{10}] \cdot [s_{20}]$

(4.38)

Next we proceed to calculate the multiplicities of the loci given by the vanishing of three minors given in (4.23). The most direct way of obtaining these multiplicities is by using the Porteous formula to obtain the first Chern class of a determinantal variety. However, we will use here a simpler approach that yields the same results.

It was noted in section 4.1.2, that the locus described by the vanishing of the three minors can be equivalently represented as the vanishing of only two minors, after excluding the zero locus from the vanishing of the two coefficients s_i, s_j that appear in both two minors. Thus, the multiplicities can be calculated by multiplying the homology classes of the two minors and subtracting the homology class $[s_i] \cdot [s_j]$ of the locus $s_i = s_j = 0$.

For example the multiplicity of the locus $|M_3^Q| = |M_2^Q| = |M_1^Q| = 0$ can be obtained from multiplying the classes of $|M_3^Q| = |M_1^Q| = 0$ and subtracting the multiplicity of the locus $s_8 = s_{18} = 0$ that satisfies these two equations, but not $M_2^Q = -s_6s_{19} + s_9s_{16}$:

$$\begin{aligned} x_{(-1,0,1)} &= [|M_3^Q|] \cdot [|M_1^Q|] - [s_8] \cdot [s_{18}] \\ &= ([p_2^b]^2 + [p_2]^b \cdot (\hat{\mathcal{S}}_7 + \cdot \tilde{\mathcal{S}}_7 - 3\tilde{\mathcal{S}}_9) + [K_B^{-1}] \cdot \tilde{\mathcal{S}}_7 + \tilde{\mathcal{S}}_7^2 - \hat{\mathcal{S}}_7 \cdot \mathcal{S}_9 - 2\tilde{\mathcal{S}}_7 \cdot \mathcal{S}_9 + 2\mathcal{S}_9^2), \end{aligned} \quad (4.39)$$

Here we denote the multiplicity of hypermultiplets with charge (q_Q, q_R, q_S) by $x_{(q_Q, q_R, q_S)}$, indicate homology classes of sections of line bundles by $[\cdot]$, as before, and employ (3.6), (2.30) and the divisors defined in (3.8) to obtain the second line. Calculating the other multiplicities in a similarly we obtain

Charges	Loci	Multiplicity
$(-1, 0, 1)$	$ M_3^Q = M_2^Q = M_1^Q = 0$	$x_{(-1,0,1)} = [M_1^Q] \cdot [M_3^Q] - [s_8] \cdot [s_{18}]$
$(0, -1, 1)$	$ M_3^R = M_2^R = M_1^R = 0$	$x_{(0,-1,1)} = [M_1^R] \cdot [M_3^R] - [s_8] \cdot [s_{18}]$
$(-1, -1, -2)$	$ M_3^P = M_2^P = M_1^P = 0$	$x_{(-1,-1,-2)} = [M_2^P] \cdot [M_3^P] - [s_{10}] \cdot [s_{20}]$
$(0, 0, 2)$	$ M_3^Q = M_3^Q = M_3^R = 0$	$x_{(0,0,2)} = [M_3^Q] \cdot [M_3^P] - [s_{19}][s_9]$

(4.40)

It is straightforward but a bit lengthy to use (3.6) in combination with (2.30), (2.36) to obtain, as demonstrated in (4.39), the expressions for the multiplicities of all these matter fields explicitly. We have shown one possible way of calculating the multiplicities in (4.40), i.e. choosing one particular pair of minors. We emphasize that the same results for the multiplicities can be obtained by picking any other the possible pairs of minors.

Finally we calculate the hypermultiplets of the matter found in the WSF, as discussed in section 4.2. In each case, in order to calculate the multiplicity of the matter located at a generic point of the polynomials (4.34) we need to first identify all the loci, which solve one particular constraint in (4.34), but support other charged hypermultiplets. Then, we have to find the respective orders of vanishing of the polynomial in (4.34) at these special loci using the resultant technique explained below (4.35). Finally, we compute the homology class of the complete intersection under consideration in (4.34) subtract the homology classes of the special loci with their appropriate orders.

We start with the matter with charges (1, 1, 1) in (4.34) which is located at a generic point of the locus $h_1 = h_2 = 0$. In this case, the degree of vanishing of the other loci are given by

Charge	$x_{(1,1,-1)}$	$x_{(0,1,2)}$	$x_{(1,0,2)}$	$x_{(-1,0,1)}$	$x_{(0,-1,1)}$	$x_{(-1,-1,-2)}$	$x_{(0,0,2)}$
(1, 1, 1)	0	1	1	0	0	4	0

(4.41)

Here we labeled the loci that are contained in $h_1 = h_2 = 0$ by the multiplicity of matter which supported on them. We note that the other six matter fields in (4.34) do not appear in this table, because the matter with charges (1, 1, 1) is contained in their loci, as we demonstrate next. This implies that the multiplicity of the hypermultiplets with charge (1, 1, 1) is given by

$$\begin{aligned}
 x_{(1,1,1)} &= [h_1] \cdot [h_2] - x_{(0,1,2)} - x_{(1,0,2)} - 4x_{(-1,-1,-2)}, \\
 &= 4[K_B^{-1}]^2 - 3([p_2]^b)^2 - 2[K_B^{-1}]\hat{\mathcal{S}}_7 - 3([p_2]^b) \cdot \hat{\mathcal{S}}_7 - 2[K_B^{-1}] \cdot \tilde{\mathcal{S}}_7 - 3([p_2]^b) \cdot \tilde{\mathcal{S}}_7 \\
 &\quad - 2\hat{\mathcal{S}}_7 \cdot \tilde{\mathcal{S}}_7 + 2[K_B^{-1}]\mathcal{S}_9 + 9([p_2]^b)\mathcal{S}_9 + 5\hat{\mathcal{S}}_7 \cdot \mathcal{S}_9 + 5\tilde{\mathcal{S}}_7 \cdot \mathcal{S}_9 - 8\mathcal{S}_9^2,
 \end{aligned}
 \tag{4.42}$$

where the first term is the class of the complete intersection $h_1 = h_2 = 0$ and the three following terms are the necessary subtractions that follow from (4.41). The homology classes of h_1, h_2 can be obtained by determining the class of one term in (4.32), respectively, (4.33) using (3.6).

Proceeding in a similar way for the hypermultiplets with charges (1, 0, 1), (0, 1, 1) and (1, 1, 0) we get the following orders of vanishing of the loci supporting the remaining matter fields:

Charges	$x_{(1,1,-1)}$	$x_{(0,1,2)}$	$x_{(1,0,2)}$	$x_{(-1,0,1)}$	$x_{(0,-1,1)}$	$x_{(-1,-1,-2)}$	$x_{(0,0,2)}$	$x_{(1,1,1)}$
(1, 0, 1)	0	0	4	0	0	4	0	1
(0, 1, 1)	0	4	0	0	0	4	0	1
(1, 1, 0)	1	0	0	0	0	1	0	1

(4.43)

We finally obtain the multiplicities of these matter fields by computing the homology class of the corresponding complete intersection in (4.34) and subtracting the multiplicities the matter fields contained in these complete intersections with the degrees determined in (4.43). We obtain

$$\begin{aligned}
 x_{(1,0,1)} &= 2[K_B^{-1}]^2 + 3([p_2]^b)^2 + 2[K_B^{-1}]\hat{\mathcal{S}}_7 + 3([p_2]^b)\hat{\mathcal{S}}_7 - 3[K_B^{-1}]\tilde{\mathcal{S}}_7 + 3([p_2]^b)\tilde{\mathcal{S}}_7 \\
 &\quad + 2\hat{\mathcal{S}}_7\tilde{\mathcal{S}}_7 + \tilde{\mathcal{S}}_7^2 + 2[K_B^{-1}]\mathcal{S}_9 - 9([p_2]^b)\mathcal{S}_9 - 5\hat{\mathcal{S}}_7\mathcal{S}_9 - 4\tilde{\mathcal{S}}_7\mathcal{S}_9 + 6\mathcal{S}_9^2, \\
 x_{(0,1,1)} &= 2[K_B^{-1}]^2 + 3([p_2]^b)^2 - 3[K_B^{-1}]\hat{\mathcal{S}}_7 + 3([p_2]^b)\hat{\mathcal{S}}_7 + \hat{\mathcal{S}}_7^2 + 2[K_B^{-1}]\tilde{\mathcal{S}}_7 \\
 &\quad + 3([p_2]^b)\tilde{\mathcal{S}}_7 + 2\hat{\mathcal{S}}_7\tilde{\mathcal{S}}_7 + 2[K_B^{-1}]\mathcal{S}_9 - 9([p_2]^b)\mathcal{S}_9 - 4\hat{\mathcal{S}}_7\mathcal{S}_9 - 5\tilde{\mathcal{S}}_7\mathcal{S}_9 + 6\mathcal{S}_9^2, \\
 x_{(1,1,0)} &= 2[K_B^{-1}]^2 + 3([p_2]^b)^2 + 2[K_B^{-1}]\hat{\mathcal{S}}_7 + 3([p_2]^b)\hat{\mathcal{S}}_7 + 2[K_B^{-1}]\tilde{\mathcal{S}}_7 + 3([p_2]^b)\tilde{\mathcal{S}}_7 \\
 &\quad + \hat{\mathcal{S}}_7\tilde{\mathcal{S}}_7 - 3[K_B^{-1}]\mathcal{S}_9 - 9([p_2]^b)\mathcal{S}_9 - 4\hat{\mathcal{S}}_7\mathcal{S}_9 - 4\tilde{\mathcal{S}}_7\mathcal{S}_9 + 7\mathcal{S}_9^2. \tag{4.44}
 \end{aligned}$$

Finally for the hypermultiplets of charges (1, 0, 0), (0, 1, 0) and (0, 0, 1) we obtain the following degrees of vanishing of the loci supporting the other matter fields:

Charges	$x_{(1,1,-1)}$	$x_{(0,1,2)}$	$x_{(1,0,2)}$	$x_{(-1,0,1)}$	$x_{(0,-1,1)}$	$x_{(-1,-1,-2)}$	$x_{(0,0,2)}$	$x_{(1,0,1)}$	$x_{(0,1,1)}$	$x_{(1,1,0)}$	$x_{(1,1,1)}$
(1, 0, 0)	1	0	1	1	0	1	0	1	0	1	1
(0, 1, 0)	1	1	0	0	1	1	0	0	1	1	1
(0, 0, 1)	1	16	16	1	1	16	16	1	1	0	1

(4.45)

Again we first computing the homology class of the complete intersection in (4.34) supporting the hypermultiplets with charges (1, 0, 0), (0, 1, 0), respectively, (0, 0, 1) and subtracting the multiplicities the matter fields contained in these complete intersections with the degrees determined in (4.45). We obtain

$$\begin{aligned}
 x_{(1,0,0)} &= 4[K_B^{-1}]^2 - 3([p_2]^b)^2 - 2[K_B^{-1}]\hat{\mathcal{S}}_7 - 3([p_2]^b)\hat{\mathcal{S}}_7 + 2[K_B^{-1}]\tilde{\mathcal{S}}_7 - 3([p_2]^b)\tilde{\mathcal{S}}_7 \\
 &\quad - \hat{\mathcal{S}}_7\tilde{\mathcal{S}}_7 - 2\tilde{\mathcal{S}}_7^2 - 2[K_B^{-1}]\mathcal{S}_9 + 9([p_2]^b)\mathcal{S}_9 + 4\hat{\mathcal{S}}_7\mathcal{S}_9 + 5\tilde{\mathcal{S}}_7\mathcal{S}_9 - 6\mathcal{S}_9^2, \\
 x_{(0,1,0)} &= 4[K_B^{-1}]^2 - 3([p_2]^b)^2 + 2[K_B^{-1}]\hat{\mathcal{S}}_7 - 3([p_2]^b)\hat{\mathcal{S}}_7 - 2\hat{\mathcal{S}}_7^2 - 2[K_B^{-1}]\tilde{\mathcal{S}}_7 \\
 &\quad - 3([p_2]^b)\tilde{\mathcal{S}}_7 - \hat{\mathcal{S}}_7\tilde{\mathcal{S}}_7 - 2[K_B^{-1}]\mathcal{S}_9 + 9([p_2]^b)\mathcal{S}_9 + 5\hat{\mathcal{S}}_7\mathcal{S}_9 + 4\tilde{\mathcal{S}}_7\mathcal{S}_9 - 6\mathcal{S}_9^2, \\
 x_{(0,0,1)} &= 4[K_B^{-1}]^2 - 4([p_2]^b)^2 + 2[K_B^{-1}]\hat{\mathcal{S}}_7 - 4([p_2]^b)\hat{\mathcal{S}}_7 - 2\hat{\mathcal{S}}_7^2 + 2[K_B^{-1}]\tilde{\mathcal{S}}_7 - 4([p_2]^b)\tilde{\mathcal{S}}_7 \\
 &\quad - 2\hat{\mathcal{S}}_7\tilde{\mathcal{S}}_7 - 2\tilde{\mathcal{S}}_7^2 + 2[K_B^{-1}]\mathcal{S}_9 + 12([p_2]^b)\mathcal{S}_9 + 6\hat{\mathcal{S}}_7\mathcal{S}_9 + 6\tilde{\mathcal{S}}_7\mathcal{S}_9 - 10\mathcal{S}_9^2.
 \end{aligned}$$

We conclude by showing that the spectrum of the theory we have calculated is anomaly-free, which serves also as a physically motivated consistency check for the completeness of analysis of codimension two singularities presented in sections 4.1 and 4.2. We refer to [63, 64] for a general account on anomaly cancellation and to [35, 37, 61] for the explicit form of the anomaly cancellation conditions adapted to the application to F-theory, cf. for example eq. (5.1) in [37]. Indeed, we readily check that the spectrum (4.38), (4.40), (4.42), (4.44) and (4.46) together with the height pairing matrix b_{mn} reading

$$b_{mn} = -\pi(\sigma(\hat{s}_m) \cdot \sigma(\hat{s}_n)) = \begin{pmatrix} -2[K_B] & -[K_B] & \mathcal{S}_9 - \tilde{\mathcal{S}}_7 - [K_B] \\ -[K_B] & -2[K_B] & \mathcal{S}_9 - \hat{\mathcal{S}}_7 - [K_B] \\ \mathcal{S}_9 - \hat{\mathcal{S}}_7 - [K_B] & \mathcal{S}_9 - \tilde{\mathcal{S}}_7 - [K_B] & 2(\mathcal{S}_9 - [K_B]) \end{pmatrix}_{mn} \tag{4.46}$$

with $m, n = 1, 2, 3$ all mixed gravitational-Abelian and purely-Abelian anomalies in eq. (5.1) of [37] are canceled.

5 Conclusions

In this work we have analyzed F-theory compactifications with $U(1) \times U(1) \times U(1)$ gauge symmetry that are obtained by compactification on the most general elliptically fibered Calabi-Yau manifolds with a rank three Mordell-Weil group. We have found that the natural presentation of the resolved elliptic fibration with three rational sections is given by a Calabi-Yau complete intersection \hat{X} with general elliptic fiber given by the unique Calabi-Yau complete intersection in $\text{Bl}_3\mathbb{P}^3$. We have shown that all F-theory vacua obtained by compactifying on a generic \hat{X} over a given general base B are classified by certain reflexive polytopes related to the nef-partition of $\text{Bl}_3\mathbb{P}^3$.

We have analyzed the geometry of these elliptically fibered Calabi-Yau manifolds \hat{X} in detail, in particular the singularities of the elliptic fibration at codimension two in the base B . This way we could identify the 14 different matter representations of F-theory compactifications on \hat{X} to four and six dimensions. We have found three matter representations that are simultaneously charged under all three $U(1)$ -factors, most notably a tri-fundamental representation. This unexpected representation is present because of the presence of a codimension two locus in B , along which all the four constraints in (4.30), δg_2^{QR} , δg_2^{QS} , g_3^Q and \hat{g}_4^Q , miraculously vanish simultaneously. We could explicitly identify the two polynomials describing this codimension two locus algebraically in (4.32), (4.33) by application of the Euclidean algorithm. These results point to an intriguing structure of codimension two singularities encoded in the elliptic fibrations with higher rank Mordell-Weil groups.

We also determined the multiplicities of the massless charged hypermultiplets in six-dimensional F-theory compactifications with general two-dimensional base B . The key to this analysis was the identification of the codimension two loci of all matter fields, which required a two-step strategy where first the singularities of the rational sections in the resolved fibration with $\text{Bl}_3\mathbb{P}^3$ -elliptic fiber have to be determined and then the remaining singularities that are visible in the singular Weierstrass form. We note that the loci of the former matter are determinantal varieties, whose homology classes we determine in general. The completeness of our strategy has been cross-checked by verifying 6D anomaly cancellation.

We would like to emphasize certain technical aspects in the analysis of the elliptic fibration. Specifically, we constructed three birational maps of the elliptic curve \mathcal{E} in $\text{Bl}_3\mathbb{P}^3$ to three different elliptic curves in dP_2 . On the level of the toric ambient spaces $\text{Bl}_3\mathbb{P}^3$ and dP_2 these maps are toric morphisms. The general elliptic curves in these toric varieties are isomorphic, whereas the map breaks down for the degenerations of \mathcal{E} in section 4.1.1. Besides loop-holes of this kind, we expect the degeneration of $\text{Bl}_3\mathbb{P}^3$ -elliptic fibrations to be largely captured by the degenerations of the non-generic dP_2 -fibrations.

It would be important for future works to systematically add non-Abelian gauge groups to the rank three Abelian sector of F-theory on \hat{X} . This requires to classify the possible

ways to engineer appropriate codimension one singularities of the elliptic fibration of \hat{X} . A straightforward way to obtain many explicit constructions of non-Abelian gauge groups is to employ the aforementioned birational maps to dP_2 , because every codimension one singularity of the dP_2 -elliptic fibration automatically induces an according singularity of the $\text{Bl}_3\mathbb{P}^3$ -elliptic fibration. In particular, many concrete I_4 -singularities, i.e. $\text{SU}(5)$ groups, can be obtained by application of the constructions of I_4 -singularities of dP_2 -elliptic fibrations in [36, 37, 54]. However, it would be important to analyze whether all codimension one singularities of \hat{X} are induced by singularities of the corresponding dP_2 -elliptic fibrations. For phenomenological applications, it would then be relevant to determine the matter representations for all possible $\text{SU}(5)$ -GUT sectors that can be realized in Calabi-Yau manifolds \hat{X} with $\text{Bl}_3\mathbb{P}^3$ -elliptic fiber. Compactifications with $\text{Bl}_3\mathbb{P}^3$ -elliptic fiber might lead to new implications for particle physics: e.g., the appearance of **10**-representations with different $\text{U}(1)$ -factors, which does not seem to appear in the rank-two Mordell-Weil constructions, and the intriguing possibility for the appearance of **5**-representations charged under all three $\text{U}(1)$ -factors, i.e. quadruple-fundamental representations, which are not present in perturbative Type II compactifications.

Furthermore, for explicit 4D GUT-model building, it would be necessary to combine the analysis of this work with the techniques of [39] to obtain chiral four-dimensional compactifications of F-theory. The determination of chiral indices of 4D matter requires the determination of all matter surfaces as well as the construction of the general G_4 -flux on Calabi-Yau fourfolds \hat{X} with general elliptic fiber in $\text{Bl}_3\mathbb{P}^3$, most desirable in the presence of an interesting GUT-sector. Furthermore the structure of Yukawa couplings should to be determined by an analysis of codimension three singularities of the fibration.

Acknowledgments

We gratefully acknowledge discussions and correspondence with Lasha Berezhiani, Yi-Zen Chu, Thomas Grimm and in particular Antonella Grassi and Albrecht Klemm. M.C. thanks CERN Theory Division and Aspen Center for Physics for hospitality. D.K. thanks the Bethe Center for Theoretical Physics Bonn and the Mitchell Institute at Texas A&M University for hospitality. The research is supported by the DOE grant DE-SC0007901 (M.C., H.P., D.K.), the NSF String Vacuum Project Grant No. NSF PHY05-51164 (H.P.), Dean's Funds for Faculty Working Group (M.C., D.K.), the Fay R. and Eugene L. Langberg Endowed Chair (M.C.) and the Slovenian Research Agency (ARRS) (M.C.).

A The Weierstrass form of the elliptic curve with three rational points

The main text made extensive use of the mapping of the elliptic curve \mathcal{E} with Mordell-Weil rank three to the Calabi-Yau hypersurface in dP_2 . Specifically, the calculation of the coordinates of the rational points, the Weierstrass form and the discriminant were all performed employing the results for the dP_2 -elliptic curve in [37]. Following [35, 37], that we refer when needed, in this appendix we calculate the Weierstrass form and the coordinates of the three rational points directly from the three elliptic curve \mathcal{E} .

In order to motivate the approach below, we briefly summarize how to obtain the Tate form of an elliptic curve with the zero point P . Given an elliptic curve with one marked point P , we can obtain the Tate equation with respect to this point by finding the sections of $\mathcal{O}(kP)$, $k = 1, \dots, 6$. The coordinate z will be the only section of $\mathcal{O}(P)$, the coordinate x is a section of $\mathcal{O}(2P)$ independent of z^2 , and y is a section of $\mathcal{O}(3P)$ independent of z^3 and xz . The Tate equation is obtained from the linear relation between the sections of $\mathcal{O}(6P)$.

Coordinates x , y and z . To obtain the birational map from the complete intersection (2.4) in \mathbb{P}^3 to the Tate form, we need to construct the Weierstrass coordinates x , y and z as sections of the line bundles $\mathcal{O}(kP)$ on \mathcal{E} with $k = 1, 2, 3$. In section 2.1 we found a basis for the bundle $\mathcal{M} = \mathcal{O}(P + Q + R + S)$, as well as a basis for \mathcal{M}^2 and a choice of basis for \mathcal{M}^3 . The sections of $\mathcal{O}(kP)$ are obtained from linear combinations of $\mathcal{O}(kM)$ that vanish with degree k at the points Q , R and S .

From the discussion in section 2.1, the section z can be taken to be $z := u'$. To find x , we take an eight-dimensional basis of $H^0(\mathcal{E}, \mathcal{M}^2)$ and construct the most general linear combination. The coefficient of u'^2 is set to zero in order for x to be independent of z^2 . Thus, the ansatz for the variable x reduces to

$$x := at'^2 + cv'^2 + dw'^2 + et'u' + fu'v' + gu'w' + hv'w'. \quad (\text{A.1})$$

Six out of the seven coefficients are fixed by imposing zeroes of order two at the three points Q , R and S . The last coefficient can be eliminated by an overall scaling. Solving the constraints but keeping h as the overall scaling coefficient, we obtain

$$\begin{aligned} a &= \frac{h(s_{10}s_{19} - s_{20}s_9)^2}{(s_{10}s_{18} - s_{20}s_8)(-s_{19}s_8 + s_{18}s_9)}, & c &= d = 0, & f &= h \frac{(s_{19}s_6 - s_{16}s_9)}{s_{19}s_8 - s_{18}s_9}, \\ g &= h \frac{(s_{10}s_{17} - s_{20}s_7)}{s_{10}s_{18} - s_{20}s_8}, \\ e &= -hs_{10} \frac{s_{18}s_{19}s_5 - s_{19}s_{20}s_6 + s_{19}^2s_7 + s_{15}s_{19}s_8 - 2s_{15}s_{18}s_9 - s_{17}s_{19}s_9 - s_{16}s_{20}s_9}{(s_{10}s_{18} - s_{20}s_8)(-s_{19}s_8 + s_{18}s_9)} \\ &\quad - h \frac{s_{10}^2s_{16}s_{19} + s_{20}[s_9(s_{18}s_5 + s_{20}s_6 + s_{15}s_8 + s_{17}s_9) - s_{19}(2s_5s_8 + s_7s_9)]}{(s_{10}s_{18} - s_{20}s_8)(-s_{19}s_8 + s_{18}s_9)}. \end{aligned}$$

Finally consider $y \in \mathcal{O}(3P)$ as a section linearly independent of u^3 and ux . We make the ansatz

$$y := \tilde{a}t'^3 + \tilde{c}v'^3 + \tilde{d}w'^3 + \tilde{f}t'u'^2 + \tilde{g}u'^2v' + \tilde{h}u'^2w' + \tilde{i}u'v'^2 + \tilde{j}u'w'^2 + \tilde{k}u'v'w' + \tilde{l}v'^2w', \quad (\text{A.2})$$

where again, all but one of the coefficients can be fixed by demanding y to have zeroes of degree three at Q , R and S and the free coefficient is an overall scaling. The solutions of these coefficients are long and not illuminating, thus we will not be presented here but can be provided on request.

Tate equations and Weierstrass form. Once the sections x , y and z are known, we impose the Tate form

$$y^2 + a_1 yxz + a_3 yz^3 = x^3 + a_4 x^2 z^4 + a_6 z^6 \tag{A.3}$$

to hold in the ideal generated by the complete intersection (2.4). First we exploit the free scalings of x and y to obtain coefficients equal to one in front of the monomials x^3 and y^2 in (A.1) and (A.2). Then we compute all the monomials in equation (A.3) after inserting $z = u'$, (A.1) and (A.2) and reduce by the ideal generated by the polynomials (2.4). Finally, from a comparison of coefficient, we obtain 23 equations that can be solved uniquely for the five Tate coefficients a_i . Unfortunately the results are long and not illuminating and are again provided on request.

From the Tate form (A.3), the Weierstrass form

$$y^2 = x^3 + fxz^4 + gz^6 \tag{A.4}$$

is obtained by the variable transformation

$$x \mapsto x + \frac{1}{12}b_2z^2, \quad y \mapsto y + \frac{1}{2}a_1xz + \frac{1}{2}a_3z^3 \tag{A.5}$$

with the following definitions

$$\begin{aligned} f &= -\frac{1}{48}(b_2^2 - 24b_4), & g &= -\frac{1}{864}(-b_2^3 + 36b_2b_4 - 216b_6), \\ b_2 &= a_1^2 + 4a_2, & b_4 &= a_1a_3 + 2a_4, & b_6 &= a_3^2 + 4a_6, \\ \Delta &= -16(4f^3 + 27g^2) = -8b_4^3 + \frac{1}{4}b_2^2b_4^2 + 9b_2b_4b_6 - \frac{1}{4}b_6b_2^3 - 27b_6. \end{aligned} \tag{A.6}$$

Rational points in the Weierstrass form. Equipped with the Weierstrass form (A.4) of the curve, we calculate the coordinates $[x^m : y^m : z^m] = [g_2^m : g_3^m : b^m]$ of all the rational points $m = P, Q, R, S$. By construction the point P is mapped to the zero section, that is the point $[\lambda^2 : \lambda^3 : 0]$.

The coordinates of the other points are all obtained through the following procedure: let us call the generic point N with Tate coordinates $[x^N : y^N : z^N]$. First, we find a section of degree two, denoted x' , that vanishes with degree three at the point N . In this case we need to make use of the full basis of $\mathcal{O}(2M)$ that includes u^2 . The vanishing at degree two already fixes most of the coefficients as in (A.1). The condition of vanishing at degree three fixes the new coefficient of u^2 . Restoring the variables x and z we obtain

$$x'|_N = x + \tilde{g}_m z^2. \tag{A.7}$$

Then, the coordinate x^N of N is given in terms of z^N by requiring $x'|_N = 0$. The coordinate y^N is determined by inserting the values for z^N , x^N into the Tate form (A.3). Finally, the coordinates in Weierstrass form are obtained by the transformations (A.5).

We summarize our results for the coordinates of the rational points Q , R and S in the following. We obtain the coordinates of the form

$$[x^Q, y^Q, z^Q] = [g_2^Q : g_3^Q : 1], \tag{A.8}$$

$$[x^R, y^R, z^R] = [g_2^R : g_3^R : 1], \tag{A.9}$$

$$[x^S, y^S, z^S] = [g_2^S : g_3^S : (s_{10}s_{19} - s_{20}s_9)], \tag{A.10}$$

where we have made the following definitions:

$$g_2^Q = \frac{1}{12} \left[8(s_{10}s_{15} - s_{20}s_5)(s_{18}s_6 - s_{16}s_8) + (s_{10}s_{16} + s_{18}s_5 - s_{20}s_6 + s_{19}s_7 - s_{15}s_8 - s_{17}s_9)^2 \right. \\ \left. - 4(s_{10}s_{12} - s_2s_{20} + s_{17}s_5 - s_{15}s_7)(s_{19}s_8 - s_{18}s_9) \right. \\ \left. + 4(s_{18}s_2 + s_{17}s_6 - s_{16}s_7 - s_{12}s_8)(s_{10}s_{19} - s_{20}s_9) \right], \quad (\text{A.11})$$

$$g_3^Q = \frac{1}{2} \left[(-s_{10}s_{15} + s_{20}s_5)(-s_{18}s_6 + s_{16}s_8)(-s_{10}s_{16} - s_{18}s_5 + s_{20}s_6 - s_{19}s_7 + s_{15}s_8 + s_{17}s_9) \right. \\ \left. - (s_{10}s_{15} - s_{20}s_5)(s_{18}s_2 + s_{17}s_6 - s_{16}s_7 - s_{12}s_8)(-s_{19}s_8 + s_{18}s_9) \right. \\ \left. - (s_{10}s_{12} - s_2s_{20} + s_{17}s_5 - s_{15}s_7)(s_{18}s_6 - s_{16}s_8)(s_{10}s_{19} - s_{20}s_9) \right. \\ \left. + (s_{17}s_2 - s_{12}s_7)(s_{19}s_8 - s_{18}s_9)(s_{10}s_{19} - s_{20}s_9) \right], \quad (\text{A.12})$$

$$g_2^R = \frac{1}{12} \left[-4(s_{10}s_{18} - s_{20}s_8)(s_{19}s_2 - s_{16}s_5 + s_{15}s_6 - s_{12}s_9) \right. \\ \left. + 8(-s_{18}s_7 + s_{17}s_8)(s_{19}s_5 - s_{15}s_9) + (s_{10}s_{16} - s_{18}s_5 - s_{20}s_6 + s_{19}s_7 + s_{15}s_8 - s_{17}s_9)^2 \right. \\ \left. - 4(s_{18}s_2 - s_{17}s_6 + s_{16}s_7 - s_{12}s_8)(s_{10}s_{19} - s_{20}s_9) \right], \quad (\text{A.13})$$

$$g_3^R = \frac{1}{2} \left[(s_{18}s_2 - s_{17}s_6 + s_{16}s_7 - s_{12}s_8)(s_{10}s_{18} - s_{20}s_8)(s_{19}s_5 - s_{15}s_9) \right. \\ \left. + (s_{18}s_7 - s_{17}s_8)(s_{19}s_5 - s_{15}s_9)(-s_{10}s_{16} + s_{18}s_5 + s_{20}s_6 - s_{19}s_7 - s_{15}s_8 + s_{17}s_9) \right. \\ \left. + (s_{16}s_2 - s_{12}s_6)(s_{10}s_{18} - s_{20}s_8)(s_{10}s_{19} - s_{20}s_9) \right. \\ \left. - (s_{18}s_7 - s_{17}s_8)(s_{19}s_2 - s_{16}s_5 + s_{15}s_6 - s_{12}s_9)(s_{10}s_{19} - s_{20}s_9) \right], \quad (\text{A.14})$$

$$g_2^S = \frac{1}{12} \left\{ 12(s_{10}s_{18} - s_{20}s_8)^2(s_{19}s_5 - s_{15}s_9)^2 \right. \\ \left. + (s_{10}s_{19} - s_{20}s_9)^2 [8(-s_{18}s_7 + s_{17}s_8)(s_{19}s_5 - s_{15}s_9) \right. \\ \left. + (s_{10}s_{16} - s_{18}s_5 - s_{20}s_6 + s_{19}s_7 + s_{15}s_8 - s_{17}s_9)^2 \right. \\ \left. - 4(s_{18}s_2 - s_{17}s_6 + s_{16}s_7 - s_{12}s_8)(s_{10}s_{19} - s_{20}s_9) \right] \\ \left. + 4(s_{10}s_{18} - s_{20}s_8)(-s_{10}s_{19} + s_{20}s_9) \times [\right. \\ \left. - 3(s_{19}s_5 - s_{15}s_9)(s_{10}s_{16} - s_{18}s_5 - s_{20}s_6 + s_{19}s_7 + s_{15}s_8 - s_{17}s_9) \right. \\ \left. + 2(s_{19}s_2 - s_{16}s_5 + s_{15}s_6 - s_{12}s_9)(-s_{10}s_{19} + s_{20}s_9) \right] \left. \right\}, \quad (\text{A.15})$$

$$g_3^S = \frac{1}{2} \left\{ 2(s_{10}s_{18} - s_{20}s_8)^3(s_{19}s_5 - s_{15}s_9)^3 \right. \\ \left. + (s_{10}s_{18} - s_{20}s_8)(s_{10}s_{19} - s_{20}s_9)^2 [2(-s_{18}s_7 + s_{17}s_8)(s_{19}s_5 - s_{15}s_9)^2 \right. \\ \left. + (s_{19}s_5 - s_{15}s_9)(s_{10}s_{16} - s_{18}s_5 - s_{20}s_6 + s_{19}s_7 + s_{15}s_8 - s_{17}s_9)^2 \right. \\ \left. - (s_{18}s_2 - s_{17}s_6 + s_{16}s_7 - s_{12}s_8)(s_{19}s_5 - s_{15}s_9)(s_{10}s_{19} - s_{20}s_9) \right. \\ \left. + (s_{19}s_2 - s_{16}s_5 + s_{15}s_6 - s_{12}s_9)(s_{10}s_{16} - s_{18}s_5 - s_{20}s_6 + s_{19}s_7 + s_{15}s_8 - s_{17}s_9)(s_{10}s_{19} - s_{20}s_9) \right. \\ \left. + (-s_{16}s_2 + s_{12}s_6)(s_{10}s_{19} - s_{20}s_9)^2 \right] \\ \left. + (-s_{18}s_7 + s_{17}s_8)(-s_{10}s_{19} + s_{20}s_9)^3 \times \right. \\ \left. [- (s_{19}s_5 - s_{15}s_9)(s_{10}s_{16} - s_{18}s_5 - s_{20}s_6 + s_{19}s_7 + s_{15}s_8 - s_{17}s_9) \right. \\ \left. + (s_{19}s_2 - s_{16}s_5 + s_{15}s_6 - s_{12}s_9)(-s_{10}s_{19} + s_{20}s_9) \right] \\ \left. + (s_{10}s_{18} - s_{20}s_8)^2(s_{19}s_5 - s_{15}s_9)(-s_{10}s_{19} + s_{20}s_9) \times \right. \\ \left. [- 3(s_{19}s_5 - s_{15}s_9)(s_{10}s_{16} - s_{18}s_5 - s_{20}s_6 + s_{19}s_7 + s_{15}s_8 - s_{17}s_9) \right. \\ \left. + 2(s_{19}s_2 - s_{16}s_5 + s_{15}s_6 - s_{12}s_9)(-s_{10}s_{19} + s_{20}s_9) \right] \left. \right\}. \quad (\text{A.17})$$

B Nef-partitions

Here we recall the very basic definitions and results about nef-Partitions. We refer for example to [65] for a detailed mathematical account.

Definition Let $X = \mathbb{P}_\nabla$ be a toric variety with a corresponding polytope ∇ , a normal fan of the polytope ∇ and rays $\rho \in \Sigma(1)$ with associated divisors D_ρ . Given a partition of $\Sigma(1) = I_1 \cup \dots \cup I_k$, into k disjoint subsets, there are divisors $E_j = \sum_{\rho \in I_j} D_\rho$ such that $-K_X = E_1 + \dots + E_k$. This decomposition is called a nef-partition if for each j , E_j is a Cartier divisor spanned by its global sections.

We denote the convex hull of the rays in I_j as ∇_j and their dual polytopes by Δ_j , which are defined as

$$\Delta_j = \{m \in \mathbb{Z}^3 | \langle m, \rho_i \rangle \geq -\delta_{ij} \text{ for } \rho_i \in \nabla_j\}. \quad (\text{B.1})$$

The generic global sections, h_j of D_j are computed according to the expression

$$h_j = \sum_{m \in \Delta_j \cap \mathbb{Z}^3} a_m \prod_{j=1}^k \prod_{\rho_i \in \nabla_j} x_i^{\langle m, \rho^i \rangle + \delta_{ij}}, \quad \delta_{ij} = \begin{cases} 1 & \text{for } \rho_i \in \nabla_j \\ 0 & \text{else.} \end{cases} \quad (\text{B.2})$$

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