

# Freudenthal Gauge Theory

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**ABSTRACT:** We present a novel gauge field theory, based on the *Freudenthal Triple System* (*FTS*), a ternary algebra with mixed symmetry (not completely symmetric) structure constants. The theory, named *Freudenthal Gauge Theory* (*FGT*), is invariant under two (off-shell) symmetries: the gauge Lie algebra constructed from the *FTS* triple product and a novel *global* non-polynomial symmetry, the so-called *Freudenthal duality*.

Interestingly, a broad class of *FGT* gauge algebras is provided by the Lie algebras “of type  $\mathfrak{e}_7$ ” which occur as conformal symmetries of Euclidean Jordan algebras of rank 3, and as *U*-duality algebras of the corresponding (super)gravity theories in  $D = 4$ .

We prove a *No-Go Theorem*, stating the incompatibility of the invariance under *Freudenthal duality* and the coupling to space-time vector *and/or* spinor fields, thus forbidding non-trivial supersymmetric extensions of *FGT*.

We also briefly discuss the relation between *FTS* and the triple systems occurring in BLG-type theories, in particular focusing on superconformal Chern-Simons-matter gauge theories in  $D = 3$ .

**KEYWORDS:** Gauge Symmetry, Global Symmetries, Chern-Simons Theories

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**1 Introduction**

The idea that a ternary algebra might be an essential structure of physical theories has a long history.

In the early 70's, Nambu [1] proposed a generalized Hamiltonian system based on a ternary product, the *Nambu-Poisson bracket*. Despite some partial results (see e.g. [2] for a comprehensive review), the quantization of the Nambu-Poisson bracket remains a long-term puzzle.

However, ternary algebras and their applications to theoretical physics have been object of intense study over the last four decades. The Jordan triple product was exploited by Günaydin and Gürsey in their quest for a formulation of quantum mechanics over different division algebras, including octonions; this investigation led to the quadratic Jordan

formulation of quantum mechanics in terms of Jordan triple product [3, 4] that extends to the octonionic quantum mechanics of [5], which has no formulation over an Hilbert space. Later on, a unified construction of Lie algebras and Lie superalgebras over triple systems was achieved by Bars and Günaydin in [6], and in [7] various composite models based on ternary algebras were investigated. Two-dimensional superconformal algebras over triple systems were then constructed in [8]; in particular, Freudenthal triple systems were applied to  $\mathcal{N} = 4$  superconformal algebras and gauged WZW models in [9].

Recently, ternary algebras re-appeared in the study of  $M$ -theory by Bagger and Lambert [10] and by Gustavsson [11], in which a ternary Lie-3 algebra is proposed as the underlying gauge symmetry structure on a stack of supersymmetric  $M2$ -branes; this is the famous BLG theory (for a recent review and list of refs., see e.g. [12]). When taking the Nambu-Poisson bracket as an infinite-dimensional generalization of the Lie-3 bracket, one gets from the BLG theory a novel six-dimensional field theory, which can be interpreted as a non-commutative version of the  $M5$ -brane theory [13].

In the present paper, we propose a novel gauge field theory, based on another ternary algebra: the *Freudenthal Triple System*<sup>1</sup> ( $FTS$ ). We call this theory “*Freudenthal Gauge Theory*” (FGT). In its simplest setup, FGT contains a bosonic scalar field  $\phi(x)$  valued in the  $FTS$   $\mathfrak{K}$  together with a gauge field  $A_\mu(x)$  taking values in the symmetric product  $\mathfrak{K} \otimes_s \mathfrak{K}$ . Similar to the BLG theory, the gauge transformation is constructed from a triple product defined over the  $FTS$   $\mathfrak{K}$ . However, unlike the totally anti-symmetric Lie-3 bracket used in the BLG theory, in general the  $FTS$  triple product does not have a simple symmetry structure with respect to the exchange of a pair of its arguments. Nevertheless, one can still prove that the gauge invariance of FGT is guaranteed by the algebraic properties of the  $FTS$ .

Besides the *off-shell* gauge symmetry, FGT also possesses a novel *global (off-shell) symmetry*, the so-called *Freudenthal duality* ( $F$ -*duality*). This is a non-linear, non-polynomial mapping from  $\mathfrak{K}$  to  $\mathfrak{K}$ , relying on non-linear identities which can be traced back to the early days of the mathematical investigation of  $FTS$ 's [22]. The name *Freudenthal duality* is much more recent, and it was introduced within physical literature in [23], in the study of Maxwell-Einstein supergravity theories (MESGT's) in  $D = 4$  space-time dimensions based on symmetric scalar manifolds and with *non-degenerate* groups of type  $E_7$  [22–28] as generalized electric-magnetic ( $U$ -)duality<sup>2</sup> symmetries. In such a framework,  $F$ -*duality* was observed as a non-polynomial, anti-involutive mapping on  $\mathfrak{K}$ -valued black hole charges (i.e.

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<sup>1</sup>Historically, there are several different notions of *Freudenthal Triple System*, which differ by the symmetry structure of their triple product. They were introduced in mathematics in order to address different algebraic properties of the triple system. Although simply related, different definitions of  $FTS$  have different properties, which of course can be translated from one to another. In the physics literature, the  $FTS$  we focus on in this paper is sometimes also called *generalized Freudenthal Triple System*, which makes the derivation property more transparent.

Since there is no general agreement on the definition, we will simply denote the triple system in this paper by *Freudenthal Triple System* ( $FTS$ ). The  $FTS$  introduced in  $\mathcal{N} = 2$  Maxwell-Einstein supergravity and its  $\mathcal{N} > 2$  generalizations [14, 15] (see also e.g. [19] and [20, 21] for recent reviews) can be regarded as some special cases of it.

<sup>2</sup>Here  $U$ -duality is referred to as the “continuous” symmetries of [29, 30]. Their discrete versions are the  $U$ -duality non-perturbative string theory symmetries introduced in [31].

fluxes of the Abelian 2-form field strengths) which keeps the Bekenstein-Hawking [32, 33] black hole entropy invariant [23]. Further generalization to a generic  $\mathcal{N} = 2$  special Kähler geometry, to its  $\mathcal{N} > 2$  generalization and to the so-called effective black hole potential governing the scalar flows has been discussed in [24].

At any rate, FGT, in its simplest setup presented in this paper, can be regarded as the simplest gauge theory admitting *F-duality* as *global* symmetry. Despite the *No-Go theorem* proved in section 4.2, a slight generalization of the FGT will be presented in a companion paper [34].

Intriguingly, as discussed in section 5, FGT shares the same symmetry structures as the “quaternionic level” of Faulkner’s construction [35], which relates triple systems to pairs  $(\mathfrak{g}, \mathbf{V})$  of a metric Lie algebra  $\mathfrak{g}$  and a suitable representation  $\mathbf{V}$ . After the treatment [36, 37], an interesting similarity between FGT and the bosonic sector of  $\mathcal{N} = 3$ ,  $D = 3$  superconformal (SC) Chern-Simons-matter (CSM) gauge theories can be envisaged. An important difference relies in supersymmetry, which in FGT, as discussed in section 4, is essentially spoiled by the enforcement of *global* invariance under *F-duality*; this affects also other terms in the Lagrangian, e.g. the scalar potential (*quartic* in FGT, *sextic* in BLG-type theories).

All in all, we can observe that, with some important differences pointed out along the present investigation, the same symmetry structures are shared (with different implementations and physical meanings) by three (a priori very different) classes of theories, namely: ( $D = 3$ ) FGT (*non-supersymmetric*),  $D = 4$  MESGT (with various amounts of *local* supersymmetry) and  $D = 3$  SC CSM gauge theory (with  $\mathcal{N} = 3$  *global* supersymmetry). Further details and results will be reported in a companion paper [34].

This paper is organized as follows.

We start by recalling the relation between *FTS*, rank-3 Euclidean Jordan algebras and exceptional Lie algebras (section 2.1); the treatment is then generalized in section 2.2. The axiomatic definition of a *FTS* and the general symmetry of its structure constants are then discussed in sections 2.3 and 2.4. The *Freudenthal duality* for a generic *FTS* is introduced in section 2.5, along with a discussion of its basic properties.

The global transformation constructed from the *FTS* triple product is introduced in section 3.1, and its gauging is discussed in section 3.2. Then, in section 3.3 we propose a bosonic Lagrangian density that exhibits both *FTS* gauge transformations and (global) *F-duality* as *off-shell* symmetries, and we provide a detailed proof of its invariance under such symmetries. The class of FGT gauge Lie algebras *of type*  $\mathfrak{e}_7$  is considered in section 3.4, and the intriguing relation between the corresponding FGT and  $D = 4$  MESGT’s with *U-duality* symmetry given by such Lie algebras *of type*  $\mathfrak{e}_7$  is discussed in section 3.5.

The possible generalization of the simplest FGT Lagrangian introduced in section 3.3 is discussed in section 4, in which the *FTS*  $\mathfrak{K}$  is coupled to the most general algebraic system, and the mathematical structure required for a consistent definition of *F-duality* is investigated (section 4.1); a *No-Go theorem* is proved in section 4.2.

The intriguing similarities (and important differences) between FGT and (the bosonic sector of)  $\mathcal{N} = 3$  SC CSM gauge theories in  $D = 3$  are discussed in section 5.

The concluding section 6 contains a summary, along with some remarks and an outlook of further developments.

Three appendices conclude the paper. Appendices A and B respectively contain details on the *F-duality* and on the FGT scalar kinetic term, whereas appendix C lists the induced axioms needed for the discussion of the generalization of FGT and in the proof of the *No-Go theorem* of section 4.2.

As mentioned above, further results and more detailed analysis of some topics mentioned along the paper will be reported in a companion work [34].

## 2 Freudenthal Triple Systems (FTS's)

### 2.1 Rank-3 Jordan algebras and Lie algebras

The *Freudenthal Triple System (FTS)*  $\mathfrak{K}$  was first introduced by Freudenthal in his study of exceptional Lie algebras [38–40] (see also [41]). In the original construction,  $\mathfrak{K}$  is defined to be the direct sum of two copies of a *Jordan Triple System (JTS)*  $\mathfrak{J}$  and two copies of real numbers<sup>3</sup>  $\mathbb{R}$ :

$$\mathfrak{K}(\mathfrak{J}) \equiv \mathfrak{J} \oplus \mathfrak{J} \oplus \mathbb{R} \oplus \mathbb{R}. \tag{2.1}$$

Over the vector space  $\mathfrak{K}(\mathfrak{J})$ , one can introduce a *symplectic* invariant 2-form, as well as a *triple product*. The latter is defined via the completely symmetric tri-linear form (also known as *cubic norm*) of the *JTS*  $\mathfrak{J}$ , and it can be re-interpreted as a linear map  $\mathcal{L}_{\phi_I \phi_J}$  over  $\mathfrak{K}$  parametrized by a pair of elements  $\phi_I, \phi_J \in \mathfrak{K}$  (cfr. definition (2.13)).

In Freudenthal's construction of exceptional Lie algebras, the *JTS*  $\mathfrak{J}$  is restricted to a rank-3 *simple* Euclidean *Jordan algebra*  $\widehat{\mathfrak{J}}$ , namely  $\widehat{\mathfrak{J}} = \mathbb{R}$  or  $\widehat{\mathfrak{J}} = J_3^{\mathfrak{A}} \equiv H_3(\mathfrak{A})$ , where  $H_3(\mathfrak{A})$  stands for the algebra of Hermitian  $3 \times 3$  matrices with entries taking values in one of the four *normed division algebras*  $\mathfrak{A} = \mathbb{R}$  (real numbers),  $\mathbb{C}$  (complex numbers),  $\mathbb{H}$  (quaternions),  $\mathbb{O}$  (octonions) (see e.g. [42]). Then, by introducing in  $\mathfrak{K}(\widehat{\mathfrak{J}})$  the submanifold

$$\mathfrak{M}_{\widehat{\mathfrak{J}}} \equiv \{ \phi_I \in \mathfrak{K}(\widehat{\mathfrak{J}}) \mid \mathcal{L}_{\phi_I \phi_I} \phi_J = 0, \forall \phi_J \in \mathfrak{K}(\widehat{\mathfrak{J}}) \}, \tag{2.2}$$

the five *exceptional* (finite-dimensional) Lie algebras  $\mathfrak{G} = \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$  arise as the the direct sum of the algebra  $\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})$  that keeps  $\mathfrak{M}_{\widehat{\mathfrak{J}}}$  invariant, together with a copy of  $\mathfrak{su}(2)$  and two copies (namely, an  $\mathfrak{su}(2)$ -doublet) of  $\mathfrak{K}(\widehat{\mathfrak{J}})$  [38, 43]:

$$\mathfrak{G} = \text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}}) \oplus \mathfrak{su}(2) \oplus \mathfrak{K}(\widehat{\mathfrak{J}}) \oplus \mathfrak{K}(\widehat{\mathfrak{J}}). \tag{2.3}$$

As a vector space,  $\mathfrak{K}(\widehat{\mathfrak{J}})$  may be regarded as the *representation space* of a non-trivial<sup>4</sup> *symplectic* representation  $\mathbf{R}$  of the algebra  $\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})$  itself, introduced in (2.3):

$$\mathfrak{K}(\widehat{\mathfrak{J}}) \sim \mathbf{R}(\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})) . \tag{2.4}$$

<sup>3</sup>Namely, the ground field was chosen to be  $\mathbb{R}$ . Other choices are of course possible (such as  $\mathbb{Z}$  or  $\mathbb{C}$ ), but we will not deal with them in the present investigation.

<sup>4</sup>Such a representation is not necessarily the smallest one. A counter-example is provided e.g. by  $\mathfrak{sp}(6) = \text{Inv}(\mathfrak{M}_{J_3^{\mathbb{R}}})$ , whose smallest non-trivial symplectic irrep. is the fundamental  $\mathbf{6}$ . However,  $\mathfrak{K}(J_3^{\mathbb{R}})$  has dimension 14, and it is based on the rank-3 completely antisymmetric irrep.  $\mathbf{14}'$ , which exhibits a *completely symmetric* rank-4 invariant structure.

However, a suitable *FTS*  $\mathfrak{K}$  on the  $\mathbf{6}$  can also be constructed; see point 2 in section 5.

At least for  $\mathbf{R}$  irreducible,  $\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}})$  is maximally (and non-symmetrically) embedded into the symplectic algebra  $\mathfrak{sp}(\mathfrak{K}(\hat{\mathfrak{J}}))$  through the *Gaillard-Zumino (GZ) embedding* [44] (see also e.g. [78] for a recent review)

$$\begin{aligned} \mathfrak{sp}(\mathfrak{K}(\hat{\mathfrak{J}})) &\supset \text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}}); \\ \mathbf{Fund}(\mathfrak{sp}(\mathfrak{K}(\hat{\mathfrak{J}}))) &= \mathbf{R}(\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}})) . \end{aligned} \tag{2.5}$$

This can be regarded as a consequence of the following *Theorem* by Dynkin (Th. 1.5 of [45], more recently discussed e.g. in [46]): every *irreducible* group of unimodular linear transformations of the  $N$ -dimensional complex space (namely, a group of transformations which does not leave invariant a proper subspace of such a space) is maximal either in  $\text{SL}(N)$  (if the group does not have a bilinear invariant), or in  $\text{Sp}(N)$  (if it has a skew-symmetric bilinear invariant), or in  $O(N)$  (if it has a symmetric bilinear invariant). Exceptions to this rule are listed in table VII of [46].

For later convenience, we introduce the number  $f$  as (cfr. (2.4))

$$\dim_{\mathbb{R}} \mathbf{Fund}(\mathfrak{sp}(\mathfrak{K}(\hat{\mathfrak{J}}))) = \dim_{\mathbb{R}} \mathbf{R}(\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}})) = \dim_{\mathbb{R}} \mathfrak{K}(\hat{\mathfrak{J}}) \equiv f, \tag{2.6}$$

which is even whenever the symplectic 2-form on  $\mathfrak{K}(\hat{\mathfrak{J}})$  is *non-degenerate* (as we will assume throughout).

From (2.3) and (2.5), it thus follows that the invariance subalgebra  $\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}})$  can be equivalently defined as the intersection of two Lie algebras: the *symplectic* one  $\mathfrak{sp}(\mathfrak{K}(\hat{\mathfrak{J}}))$  in (2.5) and the *exceptional* one  $\mathfrak{G} (= \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8)$  in (2.3):

$$\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}}) = \mathfrak{sp}(\mathfrak{K}(\hat{\mathfrak{J}})) \cap \mathfrak{G}. \tag{2.7}$$

## 2.2 General case

Within Freudenthal's formulation, the above construction can be repeated for a *generic* FTS  $\mathfrak{K}$ , by generalizing (2.2) to the submanifold

$$\mathfrak{M}_{\hat{\mathfrak{J}}} \equiv \{ \phi_I \in \mathfrak{K}(\hat{\mathfrak{J}}) \mid \mathcal{L}_{\phi_I} \phi_J = 0, \forall \phi_J \in \mathfrak{K}(\hat{\mathfrak{J}}) \}, \tag{2.8}$$

and thus introducing its invariance algebra  $\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}})$ .

It is however worth remarking that, in this general case, neither  $\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}})$  nor

$$\mathfrak{G} = \text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}}) \oplus \mathfrak{su}(2) \oplus \mathfrak{K}(\hat{\mathfrak{J}}) \oplus \mathfrak{K}(\hat{\mathfrak{J}}) \tag{2.9}$$

(this latter generalizing (2.3) to a generic *JTS*  $\hat{\mathfrak{J}}$ ), along with their possible non-compact real forms, are necessarily simple.

Nonetheless, it still holds that, as a vector space,  $\mathfrak{K}(\hat{\mathfrak{J}})$  may be regarded as the *representation space* of the relevant *symplectic* representation  $\mathbf{R}$  of the invariance subalgebra  $\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}})$  of  $\mathfrak{M}_{\hat{\mathfrak{J}}}$  (2.8):

$$\mathfrak{K}(\hat{\mathfrak{J}}) \sim \mathbf{R}(\text{Inv}(\mathfrak{M}_{\hat{\mathfrak{J}}})) . \tag{2.10}$$

Before proceeding to analyze the axiomatic definition of *FTS*, we remark that, as mentioned in footnote 1, in the mathematics literature there are several different notions of *FTS*, which differ by the symmetry structure of the corresponding triple product (see for instance [22, 41, 47]). All of these “*FTS*”s are closely inter-related by simple redefinitions; however, because they exhibit different symmetry properties, some algebraic properties of the *FTS* are manifest only within a specific formulation.

### 2.3 Axiomatic definition

We define an *FTS* to be a particular *Symplectic Triple System* [48, 49], which is a symplectic vector space  $\mathfrak{K}$  equipped with a (not necessarily completely symmetric) *triple product*

$$T : \begin{cases} \mathfrak{K} \otimes \mathfrak{K} \otimes \mathfrak{K} \rightarrow \mathfrak{K}; \\ \phi_I, \phi_J, \phi_K \mapsto T(\phi_I, \phi_J, \phi_K). \end{cases} \quad (2.11)$$

In the following, for brevity’s sake, we will denote  $T(\phi_I, \phi_J, \phi_K) \equiv \phi_I \phi_J \phi_K$ .

By introducing the symplectic form as<sup>5</sup>

$$\langle \cdot, \cdot \rangle : \begin{cases} \mathfrak{K} \otimes_a \mathfrak{K} \rightarrow \mathbb{R}; \\ \phi_I, \phi_J \mapsto \langle \phi_I, \phi_J \rangle, \end{cases} \quad (2.12)$$

in an *FTS* the triple product (2.11) satisfies the following *axioms*:

- (i)  $\phi_I \phi_J \phi_K = \phi_J \phi_I \phi_K$ ;
- (ii)  $\phi_I \phi_J \phi_K = \phi_I \phi_K \phi_J + 2\lambda \langle \phi_J, \phi_K \rangle \phi_I + \lambda \langle \phi_I, \phi_K \rangle \phi_J - \lambda \langle \phi_I, \phi_J \rangle \phi_K$ ;
- (iii)  $\phi_L \phi_M (\phi_I \phi_J \phi_K) = (\phi_L \phi_M \phi_I) \phi_J \phi_K + \phi_I (\phi_L \phi_M \phi_J) \phi_K + \phi_I \phi_J (\phi_L \phi_M \phi_K)$ ;
- (iv)  $\langle \phi_L \phi_M \phi_I, \phi_J \rangle + \langle \phi_I, \phi_L \phi_M \phi_J \rangle = 0$ ,

where  $\lambda$  is an arbitrary (real) constant.<sup>6</sup>

By introducing, for any pair  $\phi_L, \phi_M \in \mathfrak{K}$ , a linear operator  $\mathcal{L}_{\phi_L \phi_M} \in \mathfrak{gl}(\mathfrak{K})$  acting on  $\phi_K \in \mathfrak{K}$  as

$$\mathcal{L}_{\phi_I \phi_J} : \begin{cases} \mathfrak{K} \otimes_s \mathfrak{K} \rightarrow \mathfrak{K}; \\ \phi_I, \phi_J \mapsto \mathcal{L}_{\phi_I \phi_J} \phi_K \equiv \phi_I \phi_J \phi_K, \end{cases} \quad (2.13)$$

axiom (iii) yields that  $\mathcal{L}_{\phi_I \phi_J}$  is a *derivation* with respect to the *FTS* triple product  $T$  (2.11).

<sup>5</sup>Subscripts “s” and “a” respectively stand for *symmetric* and *antisymmetric*.

<sup>6</sup>Axioms (i)-(iv) define the most general *FTS*  $\mathfrak{K}$ , which does not necessarily enjoys the decomposition (2.1) in terms of an underlying *JTS*  $\mathfrak{J}$  (as in the original Freudenthal’s construction).

A counterexample is provided by Example 1 of [41], in which  $\mathfrak{g} = \mathfrak{sp}(2l)$ . In  $\mathcal{N} = 1, D = 4$  supergravity, this corresponds to a theory in which the scalar fields parametrize the *upper Siegel half-plane*; see e.g. a recent treatment in [53].

On the other hand, axiom (i) implies

$$\mathcal{L}_{\phi_I\phi_J} = \mathcal{L}_{\phi_J\phi_I}, \quad (2.14)$$

which justifies the *symmetric* tensor product of  $\mathfrak{K}$ 's in the definition (2.13) itself.

By virtue of the definition (2.13), one can reformulate axioms (iii) and (iv) as follows:

$$(iii') \quad \mathcal{L}_{\phi_L\phi_M}(\phi_I\phi_J\phi_K) = (\mathcal{L}_{\phi_L\phi_M}\phi_I)\phi_J\phi_K + \phi_I(\mathcal{L}_{\phi_L\phi_M}\phi_J)\phi_K + \phi_I\phi_J(\mathcal{L}_{\phi_L\phi_M}\phi_K);$$

$$(iv') \quad \mathcal{L}_{\phi_L\phi_M}\langle\phi_I, \phi_J\rangle = \langle\mathcal{L}_{\phi_L\phi_M}\phi_I, \phi_J\rangle + \langle\phi_I, \mathcal{L}_{\phi_L\phi_M}\phi_J\rangle = 0.$$

In particular, the reformulation (iv') of axiom (iv) makes manifest the fact the symplectic form  $\langle\cdot, \cdot\rangle$  (2.12) is *invariant* under  $\mathcal{L}_{\phi_I\phi_J}$ . Thus,  $\mathcal{L}_{\phi_I\phi_J}$  is valued in a certain Lie algebra  $\mathfrak{g}$ , which exhibits a symplectic bilinear invariant structure in the relevant representation  $\mathbf{R}$  to which  $\phi_I$  belongs. *At least* when such a representation space is *irreducible*, through the GZ embedding [44], or equivalently through the abovementioned Dynkin Theorem [45]

$$\mathfrak{g} \stackrel{GZ}{\subset} \mathfrak{sp}(\mathfrak{K}) \subset \mathfrak{gl}(\mathfrak{K}) : \mathbf{R}(\mathfrak{g}) = \mathbf{Fund}(\mathfrak{sp}) = \mathbf{Fund}(\mathfrak{gl}), \quad (2.15)$$

one has

$$\mathcal{L}_{\phi_I\phi_J} \in \mathfrak{g} \stackrel{GZ}{\subset} \mathfrak{sp}(\mathfrak{K}) \subset \mathfrak{gl}(\mathfrak{K}). \quad (2.16)$$

Within Freudenthal's construction, an important class of algebras is given by  $\mathfrak{g} = \text{Inv}(\mathfrak{M}_3)$  introduced above. The Lie algebra  $\mathfrak{g}$  will be identified below as the gauge Lie algebra of the *Freudenthal gauge theory*.

It is worth remarking here that for  $\lambda \neq 0$  axiom (iv) can actually be derived from axioms (i)-(iii). Mathematically, whenever  $\lambda \neq 0$  axiom (ii) yields a compatibility condition that constrains the structure of the triple product (2.11) and the symplectic form (2.12), and hence the non-trivial algebraic structure of the *FTS* itself. We anticipate that axiom (iii) can be regarded as the "*FTS counterpart*" of the so-called "*fundamental identity*" of Lie-3 algebras (see section 5). On the other hand, for  $\lambda = 0$  axioms (i)-(iii) reduce to the defining properties of a Lie-3 algebra over Grassmannian numbers, which in general is *not* a *FTS*. And hence, in order to restore the algebraic structure of the *FTS*  $\mathfrak{K}$ , one has to further impose axiom (iv) as a compatibility condition between the (now totally symmetric) triple product (2.11) and the symplectic form (2.12).

At any rate, in the present investigation we regard an *FTS*  $\mathfrak{K}$  as a *Symplectic Triple System* [48, 49] with  $\lambda \neq 0$ , and we include (iv) (or equivalently (iv')) as part of the defining axioms, so that the most generic situation will be considered.

## 2.4 *FTS* structure constants and their invariance

In order to make our treatment more explicit yet basis-dependent, it is convenient to introduce a basis  $\{e_a\}$  of  $\mathfrak{K}$ , such that  $\phi = \phi^a e_a$  ( $a = 1, \dots, f$ ;  $f = \dim_{\mathbb{R}}(\mathfrak{K})$ , (2.6)). Thus, one can define the *symplectic metric*  $\omega_{ab}$  and the *FTS (triple product) structure constants*  $f_{abc}^d$  respectively as

$$\langle e_a, e_b \rangle \equiv \omega_{ab} = -\omega_{ba};$$



$$e_a e_b e_c \equiv f_{abc}{}^d e_d. \tag{2.17}$$

As mentioned above,  $\omega_{ab}$  is *invariant* under  $\mathfrak{g}$  (recall (2.15) and (2.16)). Furthermore, when  $\omega_{ab}$  is *non-degenerate* (which we will always assume to hold true in this paper), an isomorphism is defined between the vector space  $\mathfrak{K}$  and its dual space, and hence one can lower<sup>7</sup> the last index of the *FTS* structure constants as follows:

$$f_{abcd} \equiv f_{abc}{}^e \omega_{ed}. \tag{2.18}$$

By virtue of definitions (2.17), the defining axioms (i)-(iv) of the *FTS*  $\mathfrak{K}$  can be rewritten as follows:

- (i)  $f_{abcd} = f_{bacd}$ ;
- (ii)  $f_{abcd} = f_{acbd} + 2\lambda\omega_{ad}\omega_{bc} - \lambda\omega_{ca}\omega_{bd} - \lambda\omega_{ab}\omega_{cd}$ ;
- (iii)  $f_{abc}{}^d f_{efd}{}^g = f_{efc}{}^d f_{abd}{}^g + f_{ecf}{}^d f_{adb}{}^g + f_{fce}{}^d f_{bda}{}^g$ ;
- (iv)  $f_{abcd} = f_{abdc}$ .

It is worth stressing here that the non-complete symmetry of the *FTS* triple product  $T$  (2.11) (as yielded by axioms (i) and (ii)) implies the non-complete symmetry of the rank-4 tensor of *FTS* structure constants  $f_{abcd}$  (2.18). However, note that axioms (i), (ii), and (iv) imply the *structure constants* to be symmetric also under exchange of the first and last pair of its indices:

$$f_{abcd} = f_{cdab}, \tag{2.19}$$

a property which will be important in the construction of a Chern-Simons action for the gauge fields of the “*Freudenthal gauge theory*” (see next sections).

Summarizing, the general symmetry properties of  $f_{abcd}$ , as implied by axioms (i), (ii) and (iv), are given by

$$f_{abcd} = f_{((ab),(cd))}. \tag{2.20}$$

$f_{abc}{}^d$  and  $f_{abcd}$  are rank-4 invariant tensors of the Lie algebra  $\mathfrak{g}$  (2.15)–(2.16). Under certain further restrictions (see point 2 in section 5), the symmetry can be extended to  $\mathfrak{sp}(\mathfrak{K})$  itself. It is here worth recalling that Kantor gave a complete classification of the finite dimensional triple systems that can arise in Lie algebras [50] (see also [51]); in particular, Kantor and Skopets showed that there is a one-to-one correspondence between simple Lie algebras and simple *FTS*’s with a non-degenerate bilinear form [52].

## 2.5 Freudenthal duality

Whenever the completely symmetric part of  $f_{abcd}$  is non-vanishing, from the definition of the *FTS* triple product (2.11) and of the symplectic form (2.12) one can define a *quartic*

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<sup>7</sup>We adopt the NE-WS convention when raising or lowering the indices using the symplectic metric.

$\mathfrak{g}$ -invariant structure  $\Delta(\phi)$  for any  $\phi \in \mathfrak{K}$ , as follows<sup>8</sup> (cfr. (25c) of [23];  $T(\phi) \equiv \phi\phi\phi$ ):

$$\Delta : \begin{cases} \mathfrak{K} \rightarrow \mathbb{R}; \\ \phi \mapsto \Delta(\phi) \equiv \frac{1}{2} \langle \phi\phi\phi, \phi \rangle = \frac{1}{2} f_{abcd} \phi^a \phi^b \phi^c \phi^d. \end{cases} \quad (2.21)$$

Such a quartic form has appeared in physical literature e.g. in the formula for the Bekenstein-Hawking [32, 33] entropy of spherically symmetric, asymptotically flat, static, extremal black hole solutions of  $D = 4$  supergravity theories whose  $U$ -duality Lie algebra is a particular non-compact, real form of  $\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})$ , namely the *conformal* Lie algebra  $\mathfrak{g} = \text{conf}(\widehat{\mathfrak{J}})$  of  $\widehat{\mathfrak{J}}$  itself (see e.g. [19] and [54] for a review, and a list of refs.).

Interestingly,  $\Delta$  also occurs in the duality-invariant expression of the cosmological constant of some  $AdS_4$  vacua (and of the corresponding central charge of the dual CFT's) of general  $\mathcal{N} = 2$  *gauged* supergravities underlying flux compactifications of type *II* theories [79].

The fact that  $f_{(abcd)} \neq 0$  which allows for the existence of (primitive) *quartic*  $\mathfrak{g}$ -invariant structure  $\Delta(\phi)$  characterizes the pair  $(\mathfrak{g} = \text{conf}(\widehat{\mathfrak{J}}), \mathbf{R})$  as a (*non-degenerate*) Lie algebra of type  $\mathfrak{e}_7$ , defined axiomatically by the axioms (a)-(c) of [22]:  $\mathbf{R}$  is a representation space of  $\mathfrak{g}$  such that

- (a)  $\mathbf{R}$  possesses a *non-degenerate*, skew-symmetric bilinear  $\mathfrak{g}$ -invariant form (cfr. (2.12) and (2.17));
- (b)  $\mathbf{R}$  possesses a *completely symmetric*, rank-4  $\mathfrak{g}$ -invariant structure  $f_{(abcd)}$  ( given by the completely symmetric part of (2.18)), which allows to define

$$q(x, y, z, w) \equiv f_{(abcd)} x^a y^b z^c w^d = 2\Delta(x, y, z, w); \quad (2.22)$$

- (c) by defining a ternary product  $\mathbf{T}(x, y, z)$  on  $\mathbf{R}$  as

$$\langle \mathbf{T}(x, y, z), w \rangle \equiv q(x, y, z, w), \quad (2.23)$$

then one has

$$3 \langle \mathbf{T}(x, x, y), \mathbf{T}(y, y, y) \rangle = \langle x, y \rangle q(x, y, y, y). \quad (2.24)$$

Note that, from (2.22) and (2.23),  $\mathbf{T}(x, y, z)$  is the the completely symmetric part of the triple product  $T$  (2.11) on  $\mathfrak{K} \sim \mathbf{R}$ .

Recently, the role of Lie algebras of type  $\mathfrak{e}_7$  was investigated in supergravity in some detail (see section 3.5). In section 5 Brown's definition of Lie algebras of type  $\mathfrak{e}_7$  [22] will be discussed in relation to *F<sub>T</sub>S* and *Freudenthal gauge theory*.

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<sup>8</sup>Even if here  $f_{abcd}$  is not (necessarily) completely symmetric in the present framework, we adopt the same normalization of [23] and [24].

From the *FTS* axioms discussed in subsections 2.3 and 2.4, one can show that  $\Delta(\phi)$  is *invariant* under the following transformation:

$$\mathcal{F} : \begin{cases} \mathfrak{K} \rightarrow \mathfrak{K}; \\ \phi \mapsto \mathcal{F}(\phi) \equiv \text{sgn}(\Delta(\phi)) \frac{T(\phi)}{\sqrt{6|\lambda\Delta(\phi)|}} \equiv \tilde{\phi}, \end{cases} \quad (2.25)$$

namely that

$$\Delta(\phi) = \Delta(\tilde{\phi}), \quad (2.26)$$

The proof can be found in appendix A (which generalizes the treatment of [23], in turn referring to [22], to *FTS* defined by axioms (i)-(iv); see also [24]). In the physics literature, the map  $\mathcal{F}$  (2.25) has been called “*Freudenthal Duality*” (or *F-duality* for short); it was first observed in [23] as a symmetry of the Bekenstein-Hawking [32, 33] entropy-area formula for black holes, and then further generalized<sup>9</sup> in [24].

In the rest of this subsection, we list some brief remarks; further details will be reported in a forthcoming paper [34].

**(I) Anti-involutivity.** The *F-duality*  $\mathcal{F}$  (2.25) is an *anti-involution* in  $\mathfrak{K}$  [22–24]:

$$\begin{aligned} \mathcal{F} \circ \mathcal{F} &= -Id; \\ \tilde{\tilde{\phi}} &= -\phi. \end{aligned} \quad (2.27)$$

This holds whenever  $\phi$  is an element in  $\mathfrak{M}_{\mathfrak{J}}^c$ , which is the complement in  $\mathfrak{K}$  of the submanifold (recall (2.8))

$$\mathfrak{M}_{\mathfrak{J}}|_{I=J} \equiv \{\phi \in \mathfrak{K} \mid \mathcal{L}_{\phi}\phi \equiv T(\phi) = 0\} \subset \mathfrak{K}. \quad (2.28)$$

In addition to this, for  $\lambda \neq 0$  and for any  $\phi \in \mathfrak{K}$ , the *F-duality* map and its image  $\tilde{\phi}$  (namely, the “*F-dual*” scalar field) are defined *iff*  $\Delta(\phi) \neq 0$ . Whenever  $\text{Inv}(\mathfrak{M}_{\mathfrak{J}})$  is non-empty and thus its corresponding action determines a *stratification* of the symplectic vector space  $\mathfrak{K}(\mathfrak{J}) \sim \mathbf{R}(\text{Inv}(\mathfrak{M}_{\mathfrak{J}}))$  (cfr. (2.10)), this can also be equivalently stated as the requirement that  $\phi$  belongs to the rank-4 orbit of  $\mathfrak{K}$  under the action of  $\text{Inv}(\mathfrak{M}_{\mathfrak{J}})$  itself.

**(II)  $\mathbb{Z}_4$ -grading.** The *anti-involutivity* (2.27) of  $\mathcal{F}$  yields a  $\mathbb{Z}_4$ -grading of the symplectic vector space  $\mathfrak{K}$ . This interesting property will be investigated in [34].

**(III) *F-duality* is not an *FTS* derivation.** The non-linear map over  $\mathfrak{K}$  provided by *F-duality* (2.25) is *not* a *derivation* with respect to the triple product (2.11) over  $\mathfrak{K}$ . Thus, such a mathematical structure cannot be consistently used to define an infinitesimal transformation. This means that the invariance (2.26) is rather a *global* symmetry (“*duality*”) of  $\mathfrak{K}$ , and thus a *global (off-shell)* symmetry of the corresponding gauge theory; see next sections.

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<sup>9</sup>In the nomenclature introduced in [24], (2.25) (which preserves the homogeneity in  $\phi$ ) defines the *non-polynomial “on-shell”* version of *F-duality*; other possible versions and generalizations are discussed therein.

### 3 Freudenthal Gauge Theory (FGT)

In the present section, we will introduce the gauge theory based on the *FTS* discussed in section 2. As anticipated, this theory, whose consistent (bosonic) Lagrangian density is proposed in subsection 3.3, will be named “*Freudenthal Gauge Theory*” (*FGT*).

As it will become clear, our construction resembles very much the one of BLG theory [10, 11]. However, we present here a detailed analysis, also in order to make several remarks addressing the differences between *FGT* (and thus *FTS*) and the triple systems-related gauge theories, especially in  $D = 3$  (see the discussion in section 5).

#### 3.1 From global symmetry...

We consider a real scalar field  $\phi(x)$  valued in a *FTS*  $\mathfrak{K}$  over  $\mathbb{R}$ , and we aim at constructing a Lagrangian density functional  $\mathbf{L}[\phi(x)]$  with the desired symmetry.

Clearly,  $\mathbf{L}[\phi(x)]$  must be a  $\mathfrak{K}$ -scalar, and thus all its terms must be of the form

$$\mathbf{L}[\phi(x)] \sim \alpha(\phi) \langle f(\phi), g(\phi) \rangle, \tag{3.1}$$

$$\alpha : \begin{cases} \mathfrak{K} \rightarrow \mathbb{R}; \\ \phi(x) \mapsto \alpha(\phi(x)); \end{cases} \tag{3.2}$$

$$f, g : \begin{cases} \mathfrak{K} \rightarrow \mathfrak{K}; \\ \phi(x) \mapsto f(\phi(x)); \quad \phi(x) \mapsto g(\phi(x)). \end{cases} \tag{3.3}$$

At each point  $x$  in space-time,  $f(\phi(x))$  and  $g(\phi(x))$  are elements of the subalgebra  $\mathfrak{K}_{\phi(x)} \subset \mathfrak{K}$  generated by the element  $\phi(x) \in \mathfrak{K}$ . More precisely, elements of  $\mathfrak{K}_{\phi(x)}$  are homogeneous polynomials of odd degree in  $\phi(x)$ , with the multiplication defined by the *non-associative* (cfr. axiom (iii)) triple product  $T$  (2.11) over  $\mathfrak{K}$ .

The *FTS* axiom (iii) (or equivalently (iii')), along with the definition (2.13), allow for a consistent definition of an *infinitesimal* transformation  $\mathcal{L}_\Lambda \in \mathfrak{sp}(\mathfrak{K})$  (recall (2.16)), such that

$$[f((Id + \mathcal{L}_\Lambda)\phi(x)) - f(\phi(x))]_{\text{linear order}} = \mathcal{L}_\Lambda f(\phi(x)), \tag{3.4}$$

where the parameters of the transformation are denoted by

$$\Lambda \in \mathfrak{K} \otimes_s \mathfrak{K}. \tag{3.5}$$

Note that only elements in the *symmetric* tensor product  $\mathfrak{K} \otimes_s \mathfrak{K}$  can generate a transformation  $\mathcal{L}_\Lambda$ , because the antisymmetric part  $\mathfrak{K} \otimes_a \mathfrak{K}$  is projected out by the symmetry property under the exchange of the first two entries of the triple product  $T$  (cfr. axiom (i)).

Crucially, axiom (iv) (or equivalently (iv')) states that for any  $f(\phi), g(\phi) \in \mathfrak{K}$ , the symplectic product  $\langle f(\phi), g(\phi) \rangle$  (defined in (2.12) and in (2.17)) is *invariant* under  $\mathcal{L}_\Lambda$ :

$$\mathcal{L}_\Lambda \langle f(\phi), g(\phi) \rangle = \langle \mathcal{L}_\Lambda f(\phi), g(\phi) \rangle + \langle f(\phi), \mathcal{L}_\Lambda g(\phi) \rangle = 0. \tag{3.6}$$

By the same argument, all  $\mathfrak{K}$ -scalar real functions  $\alpha(\phi)$  (3.2) are necessarily of this form, namely

$$\alpha(\phi) \sim \langle h(\phi), l(\phi) \rangle \tag{3.7}$$

for some functions  $h(\phi)$  and  $l(\phi)$  of the same kind as  $f(\phi)$  and  $g(\phi)$  defined in (3.3).

Thus, one can conclude that any Lagrangian density functional  $\mathbf{L}$  of the form (3.1) is *invariant*<sup>10</sup> under the infinitesimal transformation (3.4). In other words, by the four axioms (i)-(iv) of *FTS*, any Lagrangian  $\mathbf{L}$  of the form (3.1) is guaranteed to be *invariant* under the *global* symmetry generated by  $\mathcal{L}_\Lambda$  (3.4).

It should also be remarked here that the definitions (2.21) and (2.25) imply that the *F-dual* field  $\tilde{\phi}(x)$  is also an element of  $\mathfrak{K}_{\phi(x)}$ . Therefore,  $\tilde{\phi}(x)$  transforms in the very same way as  $\phi(x)$  under the global symmetry  $\mathcal{L}_\Lambda$  (3.4).

As already pointed out above, the invariance (3.6) of the symplectic product  $\langle \cdot, \cdot \rangle$  (2.12) in  $\mathfrak{K}$  under the action of the infinitesimal transformation  $\mathcal{L}_\Lambda$  implies that the latter is not simply an element in  $\mathfrak{gl}(\mathfrak{K})$ , but rather it generally belongs to the Lie algebra  $\mathfrak{g}$  (2.15)–(2.16).

### 3.2 ...to gauge symmetry

We will now proceed to gauge the *global* symmetry introduced in subsection 3.1, by promoting the infinitesimal generator  $\Lambda$  (3.5) to be a function  $\Lambda(x)$  over space-time. Correspondingly, this will identify  $\mathfrak{g}$  (2.15)–(2.16) as the *gauge* algebra.

As done in subsection 2.3, by adopting a basis  $\{e_a\}$  for  $\mathfrak{K}$ , one can generally write down the gauge transformation of a  $\mathfrak{K}$ -valued scalar field  $\phi(x) = \phi^a(x)e_a$  in the following form (recall (2.17)):

$$\mathcal{L}_\Lambda \phi(x) = \Lambda^{ab}(x) \mathcal{L}_{e_a e_b} \phi(x) = f_{abc}{}^d \Lambda^{ab}(x) \phi^c(x) e_d, \quad (3.8)$$

where  $\Lambda^{ab}(x)$  denotes the rank-2 tensor generating the gauge transformation itself. Note that axiom (i) of *FTS* implies that such a tensor is *symmetric* (cfr. (2.14)):

$$\mathcal{L}_{e_a e_b} = \mathcal{L}_{e_b e_a} \Leftrightarrow \Lambda^{ab}(x) = \Lambda^{ba}(x), \quad (3.9)$$

which is consistent with (3.5). When  $\Lambda^{ab}$  is *constant* over space-time, one consistently re-obtains the *global* symmetry considered in subsection 3.1.

By recalling (2.16), one can define the linear operator  $\hat{\Lambda} \in \mathfrak{g}$  as<sup>11</sup>

$$\hat{\Lambda}_b{}^a \equiv f_{cdb}{}^a \Lambda^{cd}, \quad (3.10)$$

such that the gauge symmetry transformation (3.8) of a field  $\phi(x)$  is nothing but a matrix multiplication by the linear operator  $\hat{\Lambda}$ :

$$\mathcal{L}_\Lambda \phi^a = \hat{\Lambda}_b{}^a \phi^b. \quad (3.11)$$

As discussed at the end of subsection 3.1, the gauge transformation of the *F-dual* field  $\tilde{\phi}(x)$  (2.25) is by construction the following one:

$$\mathcal{L}_\Lambda \tilde{\phi}^a = \hat{\Lambda}_b{}^a \tilde{\phi}^b. \quad (3.12)$$

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<sup>10</sup>Note that no mentioning of invariance under (*global*; cfr. point (IV) of subsection 2.5) *Freudenthal duality*  $\mathcal{F}$  (2.25) (which will be a crucial ingredient of FGT; see subsection 3.3) has been made so far; indeed, it is immediate to check that the Lagrangian density functional  $\mathbf{L}$  (3.1) is *not invariant* under  $\mathcal{F}$  (2.25).

<sup>11</sup>In the following treatment, we will often drop the explicit  $x$ -dependence in order to simplify the notation, whenever confusion is unlikely to occur.

Next, we introduce a *gauge field*

$$A_\mu(x) \equiv A_\mu^{ab}(x) e_a \otimes_s e_b, \quad (3.13)$$

which is a 1-form valued in<sup>12</sup>  $\mathfrak{K} \otimes_s \mathfrak{K}$ . Correspondingly, a  $\mathfrak{g}$ -valued *gauge covariant derivative*  $D_\mu$  acting on the scalar field  $\phi^a(x)$  can be defined as:

$$D_\mu \phi^a(x) \equiv \partial_\mu \phi^a(x) - (\widehat{A}_\mu)_b^a(x) \phi^b(x), \quad (3.14)$$

where

$$(\widehat{A}_\mu)_b^a(x) \equiv f_{cdb}^a A_\mu^{cd}(x) \quad (3.15)$$

is the corresponding 1-form linear operator in  $\mathfrak{g}$ .

It is worth remarking that both definitions (3.10) and (3.15) can respectively be regarded as images of the rank-2 symmetric tensor  $\Lambda^{ab}(x)$  (3.5) of infinitesimal gauge parameters and of the corresponding rank-2 symmetric tensor  $A_\mu^{ab}(x)$  (3.13) of 1-form gauge potentials, under a map (dubbed “*hat*” map), defined through the *FTS structure constants*  $f_{abc}^d$  (2.17) as follows:

$$\widehat{\cdot} : \begin{cases} \mathfrak{K} \otimes_s \mathfrak{K} \rightarrow \mathfrak{g}; \\ \Psi^{ab}(x) e_a \otimes_s e_b \mapsto f_{cdb}^a \Psi^{cd}(x) \equiv \widehat{\Psi}_b^a. \end{cases} \quad (3.16)$$

The “*hat*” map (3.16) allows one to implement (generally  $\mathfrak{g}$ -valued) infinitesimal gauge transformation  $\mathcal{L}_\Lambda$  defined via the *FTS* triple product in terms of standard matrix multiplication (in  $\mathfrak{gl}(\mathfrak{K})$ ). As such, this map provides an explicit matrix realization of the *gauge Lie algebra*  $\mathfrak{g}$  of the FGT, by means of an embedding (*local* in space-time) analogous to the *local* embedding  $\mathfrak{K}_{\phi(x)} \subset \mathfrak{K}$  mentioned below (3.3).

Then, the requirement of  $D_\mu \phi(x)$  to transform under the gauge symmetry  $\mathcal{L}_\Lambda$  in the same way as  $\phi(x)$ , i.e.

$$\mathcal{L}_\Lambda (D_\mu \phi^a(x)) = (\mathcal{L}_\Lambda D_\mu) \phi^a(x) + D_\mu (\mathcal{L}_\Lambda \phi)^a(x) \equiv \widehat{\Lambda}_b^a(x) (D_\mu \phi)^b(x) \quad (3.17)$$

consistently fixes the gauge transformation  $\widehat{A}_\mu(x)$  as follows:

$$\mathcal{L}_\Lambda \widehat{A}_\mu(x) = \partial_\mu \widehat{\Lambda}(x) - [\widehat{A}_\mu(x), \widehat{\Lambda}(x)] \equiv D_\mu \widehat{\Lambda}(x), \quad (3.18)$$

namely  $\widehat{A}_\mu(x)$  transforms as a  $\mathfrak{g}$ -valued 1-form.

To proceed further, we introduce the gauge field strength 2-form

$$\widehat{F}_{\mu\nu} \equiv -[D_\mu, D_\nu] = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - [\widehat{A}_\mu, \widehat{A}_\nu] \in \mathfrak{g}, \quad (3.19)$$

whose infinitesimal gauge transformation can consistently be computed to be

$$\mathcal{L}_\Lambda \widehat{F}_{\mu\nu} = [\widehat{F}_{\mu\nu}, \widehat{\Lambda}]. \quad (3.20)$$

The matrix embedding of  $\mathcal{L}_\Lambda$  into  $\mathfrak{g}$  provided by the “*hat*” map (3.16) also ensures that the “trace” of the field strength  $\widehat{F}_{\mu\nu}(x)$  (3.19) is  *$\mathfrak{g}$ -gauge invariant*; in the next subsection, this fact will be used to work out a bosonic Lagrangian for FGT.

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<sup>12</sup>Note that the *symmetric* nature of the tensor product in (3.13) does not imply any loss of generality, due to the axiom (i) of *FTS* (yielding  $f_{cdb}^a = f_{(cd)b}^a$ ).

### 3.3 The Lagrangian

We are now going to propose a consistent bosonic Lagrangian for the FGT.

By recalling definitions (2.21) and (2.25) and considering the lowest possible order in the scalar field  $\phi(x)$ , one can introduce the following (generally non-polynomial) term

$$\langle \phi, \tilde{\phi} \rangle = \text{sgn}(\Delta(\phi)) \frac{\langle \phi, T(\phi) \rangle}{\sqrt{6} |\lambda \Delta(\phi)|} = -\sqrt{\frac{2}{3|\lambda|}} \sqrt{|\Delta(\phi)|}, \quad (3.21)$$

which is homogeneous of degree 2 in  $\phi(x)$ . As discussed in subsection 3.2, the gauge covariant derivatives of both  $\phi(x)$  and its  $F$ -dual field  $\tilde{\phi}(x)$  transform as vectors under the gauge transformation  $\mathcal{L}_\Lambda$ ; therefore, a consistent kinetic term for scalar fields reads

$$-\frac{1}{2} \langle D_\mu \phi, D^\mu \tilde{\phi} \rangle, \quad (3.22)$$

whose *gauge invariance* is guaranteed by the *FTS* axioms (i)-(iv), (3.6), and by the very treatment of subsection 3.2.

From axiom (iv) (or equivalently (3.6)) and (3.21), it follows that for any sufficiently smooth function  $V : \mathbb{R} \rightarrow \mathbb{R}$ , then<sup>13</sup>

$$V(\Delta(\phi)) \quad (3.23)$$

is a *gauge invariant* real function of  $\phi$ :

$$\mathcal{L}_\Lambda (V(\Delta(\phi))) = 0, \quad (3.24)$$

which therefore can be taken as a *gauge invariant* potential in the bosonic FGT action.

By exploiting the matrix embedding of  $\mathfrak{g}$ -valued *Freudenthal gauge transformations*  $\mathcal{L}_\Lambda$  (realized by the “hat” map (3.16)), one can construct a Maxwell term for the *gauge invariant* kinetic term for the gauge field  $\hat{A}_\mu(x)$ .

By introducing the Minkowski metric  $\eta_{\mu\nu} = \eta^{\mu\nu}$  and a function  $\mathcal{N}(\Delta(\phi))$  coupling vector and scalar fields, for  $D \geq 4$  the following kinetic Maxwell term can be constructed:

$$\begin{aligned} \frac{1}{4} \mathcal{N}(\Delta(\phi)) \text{Tr}(\hat{F}^2) &\equiv \frac{1}{4} \mathcal{N}(\Delta(\phi)) (\hat{F}_{\mu\nu})_a^b (\hat{F}^{\mu\nu})_b^a \\ &= \frac{1}{4} \mathcal{N}(\Delta(\phi)) \eta^{\mu\lambda} \eta^{\nu\rho} f_{cda}^b f_{efb}^a F_{\mu\nu}^{cd} F_{\lambda\rho}^{ef} \\ &= -\frac{1}{4} \mathcal{N}(\Delta(\phi)) \eta^{\mu\lambda} \eta^{\nu\rho} f_{cdag} f_{efbh} \omega^{ah} \omega^{gb} F_{\mu\nu}^{cd} F_{\lambda\rho}^{ef}. \end{aligned} \quad (3.25)$$

The *gauge invariance* of (3.25) results from the simple computation

$$\mathcal{L}_\Lambda \left( \frac{1}{4} \mathcal{N}(\Delta(\phi)) \text{Tr}(\hat{F}^2) \right) = \frac{1}{4} \mathcal{L}_\Lambda (\mathcal{N}(\Delta(\phi))) \text{Tr}(\hat{F}^2) + \frac{1}{4} \mathcal{N}(\Delta(\phi)) \mathcal{L}_\Lambda (\text{Tr}(\hat{F}^2)) \quad (3.26)$$

$$= \frac{1}{2} \mathcal{N}(\Delta(\phi)) \text{Tr}([\hat{F}, \hat{\Lambda}] \hat{F}) = 0, \quad (3.27)$$

<sup>13</sup>Actually, by recalling definitions (3.2) and (3.3), one could have chosen  $V(\alpha(\phi)(f(\phi), g(\phi)))$  as the most general *gauge invariant* potential term. However, the invariance also under *F-duality*  $\mathcal{F}$  (2.25), as we do impose in FGT (see further below), further restricts the choice to  $V(\Delta(\phi))$ , as given by (3.23).

where (3.24) has been used for the function  $\mathcal{N}$ , the field strength gauge transformation property (3.20) has been recalled, and the cyclicity of the trace has been exploited.

Thus, by merging (3.22), (3.23) and (3.25), the following (bosonic) Lagrangian for the “Freudenthal gauge theory”(FGT) can be written down:

$$\mathbf{L}[\phi(x), F_{\mu\nu}(x)]_{D \geq 4} = -\frac{1}{2} \langle D_\mu \phi, D^\mu \tilde{\phi} \rangle + \frac{1}{4} \mathcal{N}(\Delta(\phi)) \text{Tr}(\widehat{F}^2) - V(\Delta(\phi)), \quad (3.28)$$

whose simplest (“minimal”) version corresponds to setting  $V(\Delta(\phi)) = \Delta(\phi)$  (quartic scalar potential) and  $\mathcal{N}(\Delta(\phi)) = 1$ :

$$\mathbf{L}_{\text{minimal}}[\phi(x), F_{\mu\nu}(x)]_{D \geq 4} = -\frac{1}{2} \langle D_\mu \phi, D^\mu \tilde{\phi} \rangle + \frac{1}{4} \text{Tr}(\widehat{F}^2) - \Delta(\phi). \quad (3.29)$$

Remarkably, the FGT Lagrangian density functional  $\mathbf{L}[\phi(x), F_{\mu\nu}(x)]_{D \geq 4}$  (3.28) is not only *invariant* under the *off-shell* gauge Lie algebra  $\mathfrak{g}$  introduced in subsections 3.1–3.2, but also under the *F-duality*  $\mathcal{F}$  (2.25), which acts as a *global (off-shell) symmetry*.<sup>14</sup> In order to check this, one should simply recall (2.26), as well as the *anti-involutivity* (2.27) of  $\mathcal{F}$  (2.25) itself and the anti-symmetry of the symplectic product used to construct the scalar kinetic term (3.22). In particular, the  $\mathcal{F}$ -invariance of the latter reads (recall point (IV) of subsection 2.5):

$$\begin{aligned} \mathcal{F}\left(\eta^{\mu\nu} \langle D_\mu \phi, D_\nu \tilde{\phi} \rangle\right) &= \eta^{\mu\nu} \langle D_\mu \tilde{\phi}, D_\nu(-\phi) \rangle = \eta^{\mu\nu} \langle D_\nu \phi, D_\mu \tilde{\phi} \rangle \\ &= \eta^{\mu\nu} \langle D_\mu \phi, D_\nu \tilde{\phi} \rangle, \end{aligned} \quad (3.30)$$

where in the second line one does not necessarily have to use the the symmetry of the Minkowski space-time metric  $\eta^{\mu\nu}$ , because, the scalar kinetic term is symmetric under the exchange of its space-time indices:

$$\langle D_\mu \phi, D_\nu \tilde{\phi} \rangle = \langle D_\nu \phi, D_\mu \tilde{\phi} \rangle, \quad (3.31)$$

as shown in appendix B.

It should be remarked here that in the above construction the dimension  $D$  of space-time does not necessarily need to be specified. As mentioned, the ( $\phi$ -coupled) Maxwell kinetic vector term (3.25) is well defined in  $D \geq 4$ . Moreover, in  $D = 4$  a *topological* (theta) term can also be introduced, along with its vector-scalar coupling function  $\mathcal{M}(\Delta(\phi))$ :

$$\frac{1}{4} \mathcal{M}(\Delta(\phi)) \text{Tr}(\widehat{F} \wedge \widehat{F}), \quad (3.32)$$

and its *gauge invariance* and  $\mathcal{F}$ -*invariance* once again follow from (3.24), (3.20), (2.26) and the the cyclicity of the trace.

Thus, in  $D = 4$ , the bosonic Lagrangian density (3.28) can be completed as follows:

$$\begin{aligned} \mathbf{L}[\phi(x), F_{\mu\nu}(x)]_{D=4} &= -\frac{1}{2} \langle D_\mu \phi, D^\mu \tilde{\phi} \rangle - V(\Delta(\phi)) \\ &\quad + \frac{1}{4} \mathcal{N}(\Delta(\phi)) \text{Tr}(\widehat{F}^2) + \frac{1}{4} \mathcal{M}(\Delta(\phi)) \text{Tr}(\widehat{F} \wedge \widehat{F}). \end{aligned} \quad (3.33)$$

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<sup>14</sup>From point (IV) of subsection 2.5, the *Freudenthal duality*  $\mathcal{F}$  (2.25) is not a *derivation* with respect to the *FTS* triple product (2.11) over  $\mathfrak{K}$ , and thus with respect to the *FTS*-based *gauge* transformation introduced above.



Even if in the above construction the dimension  $D$  of space-time does not necessarily need to be specified, it should be stressed that in  $D \geq 4$  the FGT is *non-unitary* whenever the gauge Lie algebra  $\mathfrak{g}$  is *non-compact* (and thus with a Cartan-Killing metric which is *not* positive-definite). Indeed, we recall that in the present investigation we consider the FTS to be defined on the ground field  $\mathbb{R}$  (cfr. footnote 1); this constrains the pair  $(\mathfrak{g}, \mathbf{R})$  such that  $\mathbf{R}$  is a real representation space of the real algebra  $\mathfrak{g}$ . The latter, *at least* in the examples related to conformal symmetries of  $JTS$   $\widehat{\mathfrak{J}} = \widehat{\mathfrak{J}}$  (treated in section 3.4 and reported in table 1), is *non-compact*.

On the other hand, in  $D = 3$  space-time dimensions this does not hold any more, and the non-compactness of the (real) gauge Lie algebra  $\mathfrak{g}$  is not inconsistent with unitarity of the theory. Indeed,  $\mathbf{R}$  is always assumed to possess a positive-definite inner product (for unitarity of the corresponding gauge theory), but the gauge fields are not propagating (and they are in  $\mathbf{Adj}(\mathfrak{g})$ ), and therefore  $\mathfrak{g}$  does not necessarily have to be endowed with a positive-definite product, thus allowing for non-compact (real) forms of  $\mathfrak{g}$  itself. As we discuss in section 5, this is particularly relevant for the connection between  $D = 3$  FGT and (the bosonic sector of) superconformal Chern-Simons-matter gauge theories in  $D = 3$ .

Moreover, in  $D = 3$  a Chern-Simons (CS) term for the gauge sector can be considered, with the same form as in the BLG theory (cfr. (45) of [10]):

$$\frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f_{abcd}A_{\mu}^{ab}\partial_{\nu}A_{\lambda}^{cd} + \frac{2}{3}f_{cda}{}^gf_{efgb}A_{\mu}^{ab}A_{\nu}^{cd}A_{\lambda}^{ef}\right), \quad (3.34)$$

whose consistence in FGT follows from  $FTS$  axioms (i) and (iv). The  $\mathcal{F}$ -invariance of the CS term (3.34) is trivial (it does not depend on  $\phi$  at all), while its *gauge invariance* can be easily proved by exploiting the symmetry property (2.19) of  $FTS$  structure constants  $f_{abcd}$ .

Thus, in  $D = 3$  one can propose the following bosonic FGT Lagrangian density:

$$\begin{aligned} \mathbf{L}[\phi(x), F_{\mu\nu}(x)]_{D=3} &= -\frac{1}{2}\langle D_{\mu}\phi, D^{\mu}\tilde{\phi}\rangle - V(\Delta(\phi)) \\ &\quad + \frac{1}{2}\varepsilon^{\mu\nu\lambda}\left(f_{abcd}A_{\mu}^{ab}\partial_{\nu}A_{\lambda}^{cd} + \frac{2}{3}f_{cda}{}^gf_{efgb}A_{\mu}^{ab}A_{\nu}^{cd}A_{\lambda}^{ef}\right). \end{aligned} \quad (3.35)$$

### 3.4 Gauge algebras of type $\mathfrak{e}_7$

An interesting class of gauge algebras  $\mathfrak{g}$  (2.15)–(2.16) for the FGT can be obtained by considering symmetry algebras of Jordan algebras  $\widehat{\mathfrak{J}}$  themselves. Indeed, a particular non-compact, real form of the decomposition (2.3) reads

$$\mathfrak{qconf}(\widehat{\mathfrak{J}}) = \mathfrak{conf}(\widehat{\mathfrak{J}}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{K}(\widehat{\mathfrak{J}}) \oplus \mathfrak{K}(\widehat{\mathfrak{J}}), \quad (3.36)$$

where  $\mathfrak{conf}(\widehat{\mathfrak{J}})$  and  $\mathfrak{qconf}(\widehat{\mathfrak{J}})$  respectively denote the *conformal* and *quasi-conformal*<sup>15</sup> Lie algebras of rank-3 *simple* Euclidean *Jordan algebras*  $\widehat{\mathfrak{J}}$ . Note that  $\mathfrak{conf}(\widehat{\mathfrak{J}})$  is nothing but

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<sup>15</sup>The novel, non-linear geometric quasi-conformal realizations of groups were first discovered by Günaydin, Koepsell and Nicolai in [16], by exploiting the underlying  $FTS$ , and showing that they extend to the complex forms and hence to different real forms of the corresponding groups. In the subsequent papers [17] and [18], the quasi-conformal realizations of  $D = 3$   $U$ -duality groups of Maxwell-Einstein supergravity theories, respectively with with 8 and at least 16 supersymmetries, have been determined. See e.g. [19], for a review and a list of refs..

a particular non-compact, real form of  $\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})$ ; this is also consistent with the fact that  $\text{conf}(\widehat{\mathfrak{J}})$  is nothing but the *automorphism* Lie algebra of  $\mathfrak{K}(\widehat{\mathfrak{J}})$  itself:

$$\text{conf}(\widehat{\mathfrak{J}}) \sim \text{aut}(\mathfrak{K}(\widehat{\mathfrak{J}})). \tag{3.37}$$

Analogously, also formulæ (2.4)–(2.7) hold at the suitable non-compact real level, by respectively replacing  $\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})$  and  $\mathfrak{sp}(\mathfrak{K}(\widehat{\mathfrak{J}}))$  with  $\text{conf}(\widehat{\mathfrak{J}})$  and<sup>16</sup>  $\mathfrak{sp}(f, \mathbb{R})$ . In particular, (2.7) can be recast as

$$\text{conf}(\widehat{\mathfrak{J}}) = \mathfrak{sp}(f, \mathbb{R}) \cap \mathfrak{qconf}(\widehat{\mathfrak{J}}). \tag{3.38}$$

The decompositions (2.3) and (3.36), as well as the whole treatment above, also hold for rank-3 *semi-simple* Euclidean Jordan algebras of the type

$$\widehat{\mathfrak{J}} = \mathbb{R} \oplus \mathbf{\Gamma}_{m,n}, \tag{3.39}$$

where  $\mathbf{\Gamma}_{m,n}$  is a rank-2 Jordan algebra with a quadratic form of pseudo-Euclidean signature  $(m, n)$ , i.e. the Clifford algebra of  $O(m, n)$  [80]. However, in this case the corresponding Lie algebra  $\mathfrak{G}$  in (2.3) (or  $\mathfrak{qconf}(\widehat{\mathfrak{J}})$  in (3.36)) is a *classical* Lie algebra, namely a (pseudo-)orthogonal algebra.

Table 1 lists the entries of (3.36) for rank-3 Euclidean Jordan algebras, also including the cases  $\widehat{\mathfrak{J}} = J_3^{\mathfrak{A}_s} \equiv H_3(\mathfrak{A}_s)$ , where  $\mathfrak{A}_s = \mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s$  are the *split* version of  $\mathbb{C}, \mathbb{H}, \mathbb{O}$ , respectively (see e.g. [19] for further elucidation and list of refs.). The role of  $\mathfrak{K}(\widehat{\mathfrak{J}})$ 's and their symmetries in supergravity is discussed in the next subsection 3.5.

It is also worth recalling here that the Lie algebra  $\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})$  (or equivalently  $\text{conf}(\widehat{\mathfrak{J}})$ ) is “of type  $\mathfrak{e}_7$ ” [22], as recalled in section 2.5, and in the mathematical literature its *symplectic* (real) representation  $\mathbf{R}$  is sometimes called *minuscule* irrep. (see e.g. [55]).

### 3.5 FGT and supergravity

Summarizing, a class of gauge algebras (and representations) for FGT is provided by the conformal Lie algebras  $\text{conf}$  of (simple and semi-simple) Euclidean, rank-3 algebras  $\widehat{\mathfrak{J}}$ , listed in table 1, along with their (real) symplectic representation  $\mathbf{R}$ . The pair  $(\text{conf}(\widehat{\mathfrak{J}}), \mathbf{R})$  characterizes  $\text{conf}(\widehat{\mathfrak{J}})$  as a Lie algebra of type  $\mathfrak{e}_7$  [22].

Interestingly,  $\text{conf}(\widehat{\mathfrak{J}})$  is the  $U$ -duality<sup>17</sup> Lie algebra of  $D = 4$  Maxwell-Einstein supergravity theories (MESGT's) related to the FTS  $\mathfrak{K}(\widehat{\mathfrak{J}})$  [14, 15] (see also e.g. [19] and [20, 21] for recent reviews, and list of refs.).

Indeed, within such a class of theories, the decomposition (3.36) can be further interpreted as the Cartan decomposition of the  $\mathfrak{qconf}(\widehat{\mathfrak{J}})$  ( $U$ -duality algebra in  $D = 3$ ) with respect to  $\text{conf}(\widehat{\mathfrak{J}})$  ( $U$ -duality algebra in  $D = 4$ ). In particular,  $\mathbf{R}(\text{conf}(\widehat{\mathfrak{J}}))$  listed in table 1 is the representation in which the 2-form field strengths of the  $D = 4$  Abelian vector

<sup>16</sup>Note that  $\mathfrak{sp}(f, \mathbb{R})$  is the *maximally non-compact* (*split*) real form of  $\mathfrak{sp}(\mathfrak{K}(\widehat{\mathfrak{J}}))$ .

<sup>17</sup>Here  $U$ -duality is referred to as the “continuous” symmetries of [29, 30]. Their discrete versions are the  $U$ -duality non-perturbative string theory symmetries introduced by Hull and Townsend [31].

$\widehat{\mathfrak{J}}$	$\mathfrak{conf}(\widehat{\mathfrak{J}})$	$\mathfrak{qconf}(\widehat{\mathfrak{J}})$	$\mathbf{R}(\mathfrak{conf}(\widehat{\mathfrak{J}}))$	$\mathcal{N}$
$\mathbb{R}$	$\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{g}_{2(2)}$	<b>4</b>	2
$\mathbb{R} \oplus \mathbb{R}$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{so}(3, 4)$	<b>(2, 3)</b>	2
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{so}(4, 4)$	<b>(2, 2, 2)</b>	2
$\mathbb{R} \oplus \Gamma_{m,n}$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(m+1, n+1)$	$\mathfrak{so}(m+2, n+2)$	<b>(2, m+n+2)</b>	2 ( $m=1$ ) 4 ( $m=5$ )
$J_3^{\mathbb{R}}$	$\mathfrak{sp}(6, \mathbb{R})$	$\mathfrak{f}_{4(4)}$	<b>14'</b>	2
$J_3^{\mathbb{C}}$	$\mathfrak{su}(3, 3)$	$\mathfrak{e}_{6(2)}$	<b>20</b>	2
$J_3^{\mathbb{C}^s}$	$\mathfrak{sl}(6, \mathbb{R})$	$\mathfrak{e}_{6(6)}$	<b>20</b>	0
$M_{1,2}(\mathbb{O})$	$\mathfrak{su}(1, 5)$	$\mathfrak{e}_{6(-14)}$	<b>20</b>	5
$J_3^{\mathbb{H}}$	$\mathfrak{so}^*(12)$	$\mathfrak{e}_{7(-5)}$	<b>32<sup>(l)</sup></b>	2, 6
$J_3^{\mathbb{H}^s}$	$\mathfrak{so}(6, 6)$	$\mathfrak{e}_{7(7)}$	<b>32<sup>(l)</sup></b>	0
$J_3^{\mathbb{O}}$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{8(-24)}$	<b>56</b>	2
$J_3^{\mathbb{O}^s}$	$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{8(8)}$	<b>56</b>	8

**Table 1.** Conformal  $\mathfrak{conf}(\widehat{\mathfrak{J}})$  and quasi-conformal  $\mathfrak{qconf}(\widehat{\mathfrak{J}})$  Lie algebras associated to rank-3 Euclidean Jordan algebras. The relevant symplectic irrep.  $\mathbf{R}$  of  $\mathfrak{conf}(\widehat{\mathfrak{J}})$  is also reported. In particular,  $14'$  denotes the rank-3 antisymmetric irrep. of  $\mathfrak{sp}(6, \mathbb{R})$ , whereas  $32$  and  $32'$  are the two chiral spinor irreps. of  $\mathfrak{so}^*(12)$ . Note that  $\mathfrak{conf}(J_3^{\mathbb{R}})$  and  $\mathfrak{qconf}(J_3^{\mathbb{R}})$  are the *maximally non-compact (split)* real forms of the corresponding compact Lie algebra.  $M_{1,2}(\mathbb{O})$  is the *JTS* generated by  $2 \times 1$  vectors over  $\mathbb{O}$  [14, 15]. Note the Jordan algebraic isomorphisms  $\Gamma_{1,1} \sim \mathbb{R} \oplus \mathbb{R}$ , and  $\Gamma_{1,0} \sim \mathbb{R}$ . The number of spinor supercharges  $\mathcal{N}$  of the corresponding supergravity theory in  $D = 4$  (cfr. subsection 3.5) is also listed.

potentials sit, along with their duals. As mentioned above,  $\mathfrak{conf}(\widehat{\mathfrak{J}})$  is nothing but  $\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})$ , possibly specified as a suitable *non-compact* real algebra.<sup>18</sup>

At least in  $D = 3, 4, 5, 6$ , the theories of this class all exhibit (Abelian vector multiplets') scalar manifolds which are *symmetric* cosets.<sup>19</sup> In particular, the coset Lie generators in  $D = 4$  and  $D = 3$  Lorentzian space-time dimensions are respectively given by  $\mathfrak{conf}(\widehat{\mathfrak{J}})$  and  $\mathfrak{qconf}(\widehat{\mathfrak{J}})$  modded out by their maximal compact subalgebra (*mcs*).

The number of spinor supercharges  $\mathcal{N}$  of the  $D = 4$  supergravity theory is reported in table 1. In particular, the theories associated to  $\widehat{\mathfrak{J}} = J_3^{\mathbb{R}} \equiv H_3(\mathfrak{A})$  are usually dubbed

<sup>18</sup>In fact, as a maximal subalgebra of  $\mathfrak{qconf}(\widehat{\mathfrak{J}})$ , in this framework the Lie algebra  $\text{Inv}(\mathfrak{M}_{\widehat{\mathfrak{J}}})$  can be compact (with commuting subalgebra  $\mathfrak{su}(2)$ ) or non-compact (with commuting subalgebra  $\mathfrak{sl}(2, \mathbb{R})$ ), depending on whether the Kaluza-Klein reduction from  $D = 4 \rightarrow 3$  is performed along a space-like or time-like direction, respectively; in turn, this mathematically corresponds to perform a *c-map* [81] or a *c\*-map* (see e.g. [82]) on the  $D = 4$  (vector multiplets') scalar manifold.

<sup>19</sup>A particular case is given by  $M_{1,2}(\mathbb{O})$ , which (cfr. caption of table 1) is a *JTS* generated by  $2 \times 1$  vectors over  $\mathbb{O}$  [14, 15]. It is related to supergravity with 20 local supersymmetries, which exists only in  $D = 4$  ( $\mathcal{N} = 5$  [56]) and in  $D = 3$  ( $\mathcal{N} = 10$ ; see e.g. [57] and refs. therein).

“magical” MESGT’s [14, 15], whereas the  $\mathcal{N} = 2$ ,  $D = 4$  theories corresponding to  $\widehat{\mathfrak{J}} = \mathbb{R}$ ,  $\mathbb{R} \oplus \mathbb{R}$  and  $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$  are the so-called  $T^3$ ,  $ST^2$  and  $STU$  models [58, 59]. It should also be remarked that  $\widehat{\mathfrak{J}} = J_3^{\mathbb{H}}$  is related to both  $\mathcal{N} = 2$  and  $\mathcal{N} = 6$  theories, which in fact share the very same bosonic sector [14, 15, 60–62].

As discussed in subsection 2.1,  $F$  $T$  $S$ ’s  $\mathfrak{K}(\widehat{\mathfrak{J}})$  (with  $\widehat{\mathfrak{J}}$  simple) exhibit a close relationships with exceptional Lie algebras, as given by (2.3). As listed in table 1, when considering suitable non-compact, real forms, (2.3) enjoys the reinterpretation (3.36): in other words, exceptional Lie algebras occur as quasi-conformal Lie algebras of the corresponding simple Jordan algebras  $\widehat{\mathfrak{J}}$  [38, 43]. In this respect, it is worth adding that classical (namely, pseudo-orthogonal) Lie algebras also occur as quasi-conformal Lie algebras of rank-3 semi-simple Euclidean Jordan algebras of the type<sup>20</sup> (3.39) [18].

These facts provide indication of possible links between  $F$  $G$  $T$  and Yang-Mills (exceptional) gauge theories.

At bosonic level, differences and similarities between the  $F$  $G$  $T$  and the class of MESGT’s under consideration can be observed by comparing e.g. the  $D = 3$   $F$  $G$  $T$  Lagrangian density (3.35) with the bosonic sector of the (ungauged) MESGT ( $D = 4$ ) Lagrangian density (cfr. e.g. the treatment in [63], and refs. therein)

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{4}\text{Im}(\mathcal{N}_{\Lambda\Sigma})F_{\mu\nu}^\Lambda F^{\Sigma|\mu\nu} - \frac{e^{-1}}{8}\epsilon^{\mu\nu\rho\sigma}\text{Re}(\mathcal{N}_{\Lambda\Sigma})F_{\mu\nu}^\Lambda F_{\rho\sigma}^\Sigma. \quad (3.40)$$

Besides the presence of the Einstein-Hilbert term, there are crucial differences: in the  $F$  $G$  $T$  the scalar fields  $\phi$  fit into  $\mathbf{R}(\mathfrak{g})$  and the vectors arise from the gauging of the  $F$  $T$  $S$  triple product symmetry algebra  $\mathfrak{g}$ ; as a consequence, the derivatives acting on  $\phi$  are covariantized, as discussed in sections 3.2 and 3.3. On the other hand, in the corresponding ( $D = 4$ ) supergravity framework, the Abelian two-form field strengths fit into  $\mathbf{R}(\mathfrak{g} = \mathbf{conf}(\widehat{\mathfrak{J}}))$ , while the scalar fields are in a suitable representation of the maximal compact subalgebra  $mcs(\mathfrak{g})$ . Furthermore, as discussed above, in  $F$  $G$  $T$  the gauge algebra  $\mathfrak{g} = \mathbf{conf}(\widehat{\mathfrak{J}})$  and the corresponding global Freudenthal duality are off-shell symmetries of the theory, whereas in the MESGT’s under consideration  $\mathfrak{g} = \mathbf{conf}(\widehat{\mathfrak{J}})$  is only an on-shell symmetry.<sup>21</sup> It is also worth pointing out that on the gravity side supersymmetry seems to be an accidental feature; indeed, we recall that for  $\widehat{\mathfrak{J}} = J_3^{\mathbb{C}s}$  and  $J_3^{\mathbb{H}s}$ , the corresponding theories of gravity coupled to Maxwell and scalar fields are not supersymmetric; possible supersymmetrization of  $F$  $G$  $T$  will be discussed in section 4.

It will be interesting to investigate these relations in future studies; see also the discussion in section 5.

<sup>20</sup>The quasi-conformal realizations constructed in [16–18] correspond to non-linear geometric constructions that leave invariant a generalized lightcone with respect to a quartic distance function. As such, they are different from the algebraic constructions of Lie algebras over triple systems given in the mathematics literature (see e.g. [38, 43]).

<sup>21</sup>One can construct manifestly  $U$ -invariant Lagrangians, but at the price of a non-manifest Lorentz-invariance [64–66] or of doubling the field strengths’ degrees of freedom (doubled formalism [67]; for recent advances in relation to Freudenthal duality, see e.g. [68]).

## 4 Generalization?

In the previous section, we have constructed a consistent Lagrangian for the *Freudenthal gauge theory* (FGT), based on the FTS  $\mathfrak{K}(\mathfrak{J})$ , with  $\mathfrak{K}$ -valued scalar field  $\phi(x)$ , admitting both (*off-shell*) FTS *gauge* symmetry and (*off-shell*) *global Freudenthal-duality* symmetry  $\mathcal{F}$ .

The most important kind of generalization would concern an FGT-type Lagrangian involving some *vector* fields *and/or spinor* fields, which is again invariant under both *FTS gauge* and *Freudenthal duality* symmetries; indeed, this would be a necessary condition for a supersymmetric (non-trivial) extension of FGT. Moreover, such a generalization is of interest to the physicists, since it potentially might define a *sigma-model* type theory if the space-time considered in this paper is regarded as the *world-volume* of some extended objects (for instance, *M2-branes*), and correspondingly the *vector* fields conceived as the image of the *world-volume* in some target space.

However, in subsections 4.1–4.2 we shall prove that, within some minimal reasonable assumptions, such a generalization is not possible.

### 4.1 Coupling to a vector space

Let us start the analysis by coupling a generic *FTS*  $\mathfrak{K}$  to a generic vector space  $\mathfrak{V}$ , over which one can introduce suitable algebraic structures and make it into an algebra; for instance, spinors can be regarded as vectors with an anti-symmetric binary product that yields the Fermi statistics. In this way, our discussion for the formal algebraic system  $\mathfrak{V}$  will cover the most generic space that couples to  $\mathfrak{K}$ .

Thus, we are considering an *extended* vector space

$$\mathfrak{N} \equiv \mathfrak{K} \otimes \mathfrak{V}, \tag{4.1}$$

whose element, denoted by  $\Phi$ , is the tensor product of an element  $\phi \in \mathfrak{K}$  and an element  $v \in \mathfrak{V}$ , i.e.

$$\Phi \equiv \phi \otimes v \in \mathfrak{N}. \tag{4.2}$$

In order to be able to construct a Lagrangian density functional  $\mathbf{L}[\Phi(x)]$  for the fields  $\Phi(x) \in \mathfrak{N}$  obtained from promoting an element  $\Phi \in \mathfrak{N}$  to a  $\mathfrak{N}$ -valued space-time field  $\Phi(x)$ , one starts by introducing a bilinear form (namely, the *metric*)

$$\langle \cdot, \cdot \rangle : \begin{cases} \mathfrak{N} \otimes \mathfrak{N} \rightarrow \mathbb{R}; \\ \Phi_I, \Phi_J \mapsto \langle \Phi_I, \Phi_J \rangle, \end{cases} \tag{4.3}$$

defined for any two  $\Phi_{I,J} = \phi_{I,J} \otimes v_{I,J}$  in  $\mathfrak{N}$ . *Via* direct evaluation, (4.3) induces a *metric* on  $\mathfrak{V}$  itself:

$$\langle \Phi_I, \Phi_J \rangle = \langle \phi_I \otimes v_I, \phi_J \otimes v_J \rangle = \langle \phi_I, \phi_J \rangle \times (v_I, v_J)_{\mathfrak{V}}, \quad \forall \Phi_I, \Phi_J \in \mathfrak{N}, \tag{4.4}$$

where “ $\times$ ” is here multiplication by a scalar (real) factor, and

$$(\cdot, \cdot)_{\mathfrak{V}} : \begin{cases} \mathfrak{V} \otimes \mathfrak{V} \rightarrow \mathbb{R}; \\ v_I, v_J \mapsto (v_I, v_J)_{\mathfrak{V}}, \end{cases} \quad (4.5)$$

is the *induced metric* over  $\mathfrak{V}$ . Note that the symmetry property of  $(\cdot, \cdot)_{\mathfrak{V}}$  (4.5) is to be determined by the required symmetry property of the metric  $\langle \cdot, \cdot \rangle$  (4.3) over  $\mathfrak{N}$  (by also recalling the anti-symmetry of the symplectic form (2.12) over  $\mathfrak{K}$ ).

Furthermore, in order to consistently define the *Freudenthal duality*  $\mathcal{F}$  of this *extended* theory, one needs to introduce a triple product

$$\mathcal{T} : \begin{cases} \mathfrak{N} \otimes \mathfrak{N} \otimes \mathfrak{N} \rightarrow \mathfrak{N}; \\ \Phi_I, \Phi_J, \Phi_K \mapsto \mathcal{T}(\Phi_I, \Phi_J, \Phi_K) \equiv \Phi_I \Phi_J \Phi_K, \end{cases} \quad (4.6)$$

defined for any three elements  $\Phi_I, \Phi_J, \Phi_K \in \mathfrak{N}$ , which would then induce a tri-linear *triple product* on  $\mathfrak{V}$  itself:

$$[\cdot, \cdot, \cdot]_{\mathfrak{V}} : \begin{cases} \mathfrak{V} \otimes \mathfrak{V} \otimes \mathfrak{V} \rightarrow \mathfrak{V}; \\ v_I, v_J, v_K \mapsto [v_I, v_J, v_K]_{\mathfrak{V}}. \end{cases} \quad (4.7)$$

In order to proceed further, we make here a plausible conjecture that *Freudenthal duality*  $\mathcal{F}$  can be defined only for algebraic systems satisfying the axioms (i)-(iv) of an *FTS*, introduced in subsection 2.3. As a consequence, we require the metric (4.3) to be an *anti-symmetric* bilinear form (and append this as axiom (o)), thus obtaining the following five axioms for the algebra  $\mathfrak{N}$ :

- (o)  $\langle \Phi_I, \Phi_J \rangle = -\langle \Phi_J, \Phi_I \rangle$
- (i)  $\Phi_I \Phi_J \Phi_K = \Phi_J \Phi_I \Phi_K$
- (ii)  $\Phi_I \Phi_J \Phi_K = \Phi_I \Phi_K \Phi_J + 2\mu \langle \Phi_J, \Phi_K \rangle \Phi_I + \mu \langle \Phi_I, \Phi_K \rangle \Phi_J - \mu \langle \Phi_I, \Phi_J \rangle \Phi_K$
- (iii)  $\Phi_L \Phi_M (\Phi_I \Phi_J \Phi_K) = (\Phi_L \Phi_M \Phi_I) \Phi_J \Phi_K + \Phi_I (\Phi_L \Phi_M \Phi_J) \Phi_K + \Phi_I \Phi_J (\Phi_L \Phi_M \Phi_K)$
- (iv)  $\langle \Phi_L \Phi_M \Phi_I, \Phi_J \rangle + \langle \Phi_I, \Phi_L \Phi_M \Phi_J \rangle = 0,$

where  $\mu$  plays the role of the real parameter  $\lambda$  introduced above for the *FTS*  $\mathfrak{K}$ .

Then, by repeating for the algebra  $\mathfrak{N}$  the very same construction discussed in section 3 for the *FTS*  $\mathfrak{K}$ , one gets the most general Lagrangian density functional  $\mathbf{L}[\Phi(x)]$  invariant under the two desired symmetries, namely under both (*off-shell*) *FTS gauge* symmetry and (*off-shell*) *global Freudenthal-duality* symmetry  $\mathcal{F}$ .

## 4.2 A no-go theorem

However, this seemingly smooth construction of an *extended* FGT coupled to vector *and/or* spinor fields suffers from some severe constraints, which actually spoils the above generalization.

Indeed, axioms (o)-(iv) of  $\mathfrak{N}$  induce a set of corresponding axioms for the metric (4.5) and the triple product (4.7) induced on  $\mathfrak{V}$  (in addition to the ones already introduced for other physical reasons, such as the ones yielded by the Bose *and/or* Fermi statistics for the fields  $v_I \in \mathfrak{V}$ ); the reader can find the full set of such axioms for  $\mathfrak{V}$  in appendix C.

Among them, axiom (B.iii) induced from the *derivation* property of  $\mathfrak{N}$  leads to a particularly strong constraint. In order to realize this, let us restrict to a subalgebra

$$\mathfrak{N}_\phi \equiv \mathfrak{K}_\phi \otimes \mathfrak{V} \subset \mathfrak{N}, \quad (4.8)$$

where  $\mathfrak{K}_\phi$  is the subalgebra in  $\mathfrak{K}$  generated by a single generator  $\phi \in \mathfrak{K}$  (see also subsection 3.1). Then, by taking five elements of the form

$$\Phi_{L,M,I,J,K} \equiv \phi \otimes v_{L,M,I,J,K} \in \mathfrak{N}_\phi \quad (4.9)$$

and inserting them into axiom (B.iii) of appendix C, the following simplified (weaker) condition on the algebraic structure of  $\mathfrak{V}$  is achieved:

$$\phi \phi T(\phi) \otimes \left( [v_L, v_M, [v_I, v_J, v_K]_{\mathfrak{V}}]_{\mathfrak{V}} - [v_I, v_J, [v_L, v_M, v_K]_{\mathfrak{V}}]_{\mathfrak{V}} \right) = 0, \quad (4.10)$$

where the simplification comes from the fact that over the subalgebra  $\mathfrak{K}_\phi$ ,  $\mathcal{L}_{\phi T(\phi)}$  and  $\mathcal{L}_{T(\phi)\phi}$  act as *annihilation operators*, whose proof can be found in appendix A.

Moreover, we observe that, as holding for  $\mathfrak{K}$  (cfr. definition (2.13)) for any two elements  $v_L, v_M \in \mathfrak{V}$  one gets a linear operator (generally  $\mathfrak{gl}(\mathfrak{V})$ -valued, whenever it is non-zero)  $\mathcal{L}_{v_L v_M}$ , whose action is evaluated by the triple product (4.7) as:

$$\mathcal{L}_{v_L v_M} : \begin{cases} \mathfrak{V} \otimes \mathfrak{V} \rightarrow \mathfrak{V}; \\ v_L, v_M \mapsto \mathcal{L}_{v_L v_M} v_I \equiv [v_L, v_M, v_I]_{\mathfrak{V}}, \end{cases} \quad (4.11)$$

Then, by using definition (4.11), the weaker form (4.10) of the axiom (B.iii) can be recast as a condition on the matrix commutator in  $\mathfrak{gl}(\mathfrak{V})$ :

$$[\mathcal{L}_{v_L v_M}, \mathcal{L}_{v_I v_J}] = 0, \quad \forall v_{I,J,L,M} \in \mathfrak{V}. \quad (4.12)$$

Under the assumption that the metric (4.3) in  $\mathfrak{N}$  is *non-degenerate* (which we understand throughout<sup>22</sup>), the condition (4.12) can be satisfied in only two instances:

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<sup>22</sup>For instance, if the metric (4.3) of the algebra  $\mathfrak{N}$  is *degenerate*, it can be proved that a coupling to a Grassmannian number degree of freedom is possible. However, since there is no fermion bilinear for a one-dimensional spinor, this is a rather trivial theory, in which the fermionic degree of freedom never plays any role, and it cannot mix up with the bosonic degree of freedom. In such a theory, the structure is essentially the same as the one pertaining to a single  $\mathfrak{K}$ -valued (real) scalar field, and hence a consistent implementation of invariance under (*global, off-shell*) *Freudenthal duality* is possible. We plan to investigate further this issue in future work.

[ I ] when  $\dim_{(\mathbb{R})}\mathfrak{V} = 1$ , i.e.

$$\mathfrak{N} = \mathfrak{K} \otimes \mathbb{R}, \tag{4.13}$$

which is the case of a single  $\mathfrak{K}$ -valued (real) scalar field discussed in sections 2–3;

[ III ] when the set

$$\{\mathcal{L}_{v_I v_J} \in \mathfrak{gl}(\mathfrak{V}) \mid v_I, v_J \in \mathfrak{V}\} \subset \mathfrak{gl}(\mathfrak{V}) \tag{4.14}$$

is a subset of the *Cartan subalgebra* of  $\mathfrak{gl}(\mathfrak{V})$ , namely<sup>23</sup> (recall definitions (4.5) and (4.11)):

$$\mathcal{L}_{v_I v_J} v_K = [v_I, v_J, v_K]_{\mathfrak{V}} = (v_I, v_J)_{\mathfrak{V}} \times v_K. \tag{4.15}$$

The triple product  $[\cdot, \cdot, \cdot]_{\mathfrak{V}}$  (4.7) defined by (4.15) satisfies the strong form of axiom (B.iii) and most of other axioms of appendix C. However, *at least* within the assumption of *non-degeneracy* of the metric of the algebra  $\mathfrak{N}$  (cfr. footnote 19), it is refuted by axiom (B.ii) whenever  $\mathfrak{K}$  is larger than a single-generator algebra  $\mathfrak{K}_{\phi}$ . ■

This completes the proof of the following

**No-go theorem.** Assuming the metric of the algebraic system  $\mathfrak{N}$  (4.1) to be *non-degenerate* and the *Freudenthal duality*  $\mathcal{F}$  to be defined only for  $\mathfrak{N}$  satisfying all the four *FTS* axioms introduced in subsection 2.3, then it is *not* possible to construct a Lagrangian density functional  $\mathbf{L}[\Phi(x)]$  for a  $\mathfrak{K}$ -valued vector/spinor field  $\Phi(x)$  which admits both (*off-shell*) *FTS gauge* symmetry and (*off-shell*) *global F-duality* symmetry  $\mathcal{F}$ .

## 5 FGT and ( $\mathcal{N} = 3, D = 3$ ) SC CSM gauge theories

We will now briefly make some observations on the relation between *Freudenthal gauge theory* (FGT) (based on *Freudenthal triple systems* (FTS's)) and the intense research on triple systems and gauge theories, in which remarkable advances were achieved after the seminal papers of Bagger and Lambert [10] and Gustavsson [11]. A more detailed analysis will be reported in [34].

Here, we will focus on the relation to superconformal (SC) Chern-Simons-matter (CSM) gauge theories in  $D = 3$  (in which the  $\mathcal{R}$ -symmetry structure is richer); we will mainly refer to the mathematical treatment of [36] and [37] (see also [69]); for an extensive list of refs. on BLG theories and related developments, besides [36, 37, 69], we address the reader e.g. to the recent comprehensive review [12].

We anticipate that the symmetry properties (2.20) of the FTS structure constants on which the FGT is based are generally different from the ones pertaining to the structure

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<sup>23</sup>In general, instead of (4.15) one may propose

$$\mathcal{L}_{v_I v_J} v_K = [v_I, v_J, v_K]_{\mathfrak{V}} = h((v_I, v_J)_{\mathfrak{V}}) \times v_K,$$

for any function  $h : \mathbb{R} \rightarrow \mathbb{R}$ , as the most generic possibility [ III ]. However, the tri-linearity of the triple product  $[\cdot, \cdot, \cdot]_{\mathfrak{V}}$  (4.7) in  $\mathfrak{V}$  requires the function  $h$  to be linear. Since the constant term of the linear function  $h$  leads to a trivial triple product and is easily refuted by the other axioms of appendix C, one can conclude that, up to an overall (real) factor, (4.15) is the most generic possibility [ III ].



constants on which the BLG-type theories (such as the ones investigated e.g. in [70, 71], among others) rely. Among SC CSM  $D = 3$  gauge theories, the symmetry (2.20) is indeed consistent *only* with  $\mathcal{N} = 3$  (see e.g. [37], and refs. therein). Disregarding the global (*off-shell*) *Freudenthal duality*, ( $D = 3$ ) FGT could be viewed as an alternative, purely *bosonic sector* of the corresponding  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theory. In fact, as analyzed in section 3.3, in FGT the non-vanishing of  $f_{(abcd)}$  allows for terms in the Lagrangian which differ from the usual ones in BLG theories; for instance, the simplest FGT scalar potential is *quartic* in the scalar fields (essentially given by  $\Delta$  (2.21); see (3.29)), whereas in BLG theories it is of order six (see e.g. (19) of [10]).

We start by observing that the set of axioms (*i*), (*iii*) and (*iv*) defining an FTS (as given in section 2.4) match the set of axioms (*a*), (*b*) and (*c*) defining the triple systems based on *quaternionic unitary representations*  $\mathbf{W}$  of a metric Lie algebra  $\mathfrak{g}$ , as discussed in [36] and [37] (see e.g. appendix A.2.4 of [37], and axioms (125)-(127) therein); in particular, the FTS axiom (*iii*) is nothing but the so-called *fundamental identity* of the triple system (see e.g. (127) of [37]). In turn, the treatment of [36] and [37] is based on a construction due to Faulkner [35, 72], which essentially constructs triple systems from pairs  $(\mathfrak{g}, \mathbf{V})$ , where  $\mathbf{V}$  is a suitable representation<sup>24</sup> of  $\mathfrak{g}$  [36].

The  $\mathfrak{g}$ -irreducible decomposition of the rank-4  $\mathfrak{g}$ -invariant structure in  $\mathbf{W}$  is given by (124) of [37] (also, cfr. table 2 therein):

$$S^2 S^2 \mathbf{W} \cong S^4 \mathbf{W} \oplus \mathbf{W}^{(2,2)}. \tag{5.1}$$

In tensor notation, a reformulation<sup>25</sup> of (5.1) reads as follows ( $a, b \in \mathbb{R}$ ):

$$f_{abcd} = a f_{(abcd)} + b \omega_{a(c} \omega_{d)b}. \tag{5.2}$$

(5.2) is consistent with the general symmetry of the FTS structure constants' tensor  $f_{abcd}$  given by (2.20); furthermore, *Freudenthal duality*  $\mathcal{F}$  (2.25) can be consistently introduced whenever  $f_{(abcd)} \neq 0$ .

It is worth remarking that Brown's definition of Lie algebra  $(\mathfrak{g}, \mathbf{R})$  of type  $\mathfrak{e}_7$  [22] (cfr. (a)–(c) in section 2.5) can be extended to include also the not completely symmetric part  $\omega_{a(c} \omega_{d)b}$  of (5.2) as follows:  $\mathbf{R}$  is a representation space of  $\mathfrak{g}$  such that

( $\hat{a}$ )  $\mathbf{R}$  possesses a *non-degenerate*, skew-symmetric bilinear  $\mathfrak{g}$ -invariant form  $\omega$  (cfr. (2.12) and (2.17));

( $\hat{b}$ )  $\mathbf{R}$  possesses a rank-4  $\mathfrak{g}$ -invariant structure  $f_{abcd}$  (5.2), which allows to define

$$\hat{q}(x, y, z, w) \equiv f_{abcd} x^a y^b z^c w^d = 2 \hat{\Delta}(x, y, z, w); \tag{5.3}$$

( $\hat{c}$ ) by defining a ternary product  $\hat{\mathbf{T}}(x, y, z)$  on  $\mathbf{R}$  as

$$\langle \hat{\mathbf{T}}(x, y, z), w \rangle \equiv \hat{q}(x, y, z, w), \tag{5.4}$$

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<sup>24</sup>The fourth axiom (*quaternionic condition*; see e.g. (128) of [37]) defining Faulkner's triple systems based on  $(\mathfrak{g}, \mathbf{W})$  is essentially related to the existence of a skew-symmetric symplectic invariant bilinear form  $\omega$  which raises and lowers indices.

<sup>25</sup>Here, we will not deal with issues of generality of the reformulation (5.2) of (5.1).

then one has

$$3 \langle \widehat{\mathbf{T}}(x, x, y), \widehat{\mathbf{T}}(y, y, y) \rangle = \langle x, y \rangle \widehat{q}(x, y, y, y). \quad (5.5)$$

By enhancing  $f_{abcd} = f_{(abcd)}$  to a not completely symmetric  $f_{abcd}$  given by (5.2), one can conclude that, by virtue of  $(\widehat{a})$ , the real parameters  $a$  and  $b$  can always be chosen such that the inclusion of  $\omega_{a(c\omega_d)b}$  in Brown's definition [22] yields nothing but an equivalent definition of a Lie algebra of type  $\mathfrak{e}_7$ ; however, as pointed out below, the presence or absence of the term  $\omega_{a(c\omega_d)b}$  matters in order to make contact with *FTS*'s.

Note that the  $\lambda$ -dependent *FTS*-defining axiom (ii) was not mentioned so far. However, at least for the class of pairs  $(\mathfrak{g}, \mathbf{R}) = (\mathbf{conf}(\widehat{\mathfrak{J}}), \mathbf{R})$  reported in table 1, the parameters  $a$  and  $b$  can be fixed consistently with axiom (ii), by further elaborating (5.2) as

$$f_{abcd} = 6\lambda f_{(abcd)} - 2\lambda \omega_{a(c\omega_d)b}. \quad (5.6)$$

For pairs  $(\mathfrak{g}, \mathbf{R}) = (\mathbf{conf}(\widehat{\mathfrak{J}}), \mathbf{R})$  with  $\mathfrak{g}$  simple, both (5.6) and the parameter  $\lambda$  acquires a very simple group-theoretical meaning. Indeed, exploiting the results of [73], (5.6) can be rewritten as

$$f_{abcd} = -3\tau f_{(abcd)} + \tau \omega_{a(c\omega_d)b} = t_{ab}^\alpha t_{cd}^\beta g_{\alpha\beta}, \quad (5.7)$$

where  $t_{ab}^\alpha = t_{(ab)}^\alpha$  is the ( $\mathfrak{g}$ -invariant) realization of the generators of  $\mathfrak{g}$  in  $\mathbf{R}$ ; the indices  $\alpha$  and  $a$  respectively are in **Adj** and  $\mathbf{R}$  of  $\mathfrak{g}$ , whose Cartan-Killing metric is  $g_{\alpha\beta}$ . Therefore,  $f_{abcd}$  can be defined as the adjoint-trace of the product of two realizations of generators of  $\mathfrak{g}$  in its representation  $\mathbf{R}$ . Moreover, the parameter [73]

$$\tau \equiv \frac{2\dim_{\mathbb{R}} \mathbf{Adj}(\mathfrak{g})}{\dim_{\mathbb{R}} \mathbf{R}(\mathfrak{g}) (\dim_{\mathbb{R}} \mathbf{R}(\mathfrak{g}) + 1)} = -2\lambda \quad (5.8)$$

expresses the ratio between the sets of indices  $\alpha$  and  $ab = (ab)$  of  $t_{ab}^\alpha$  (in the treatment above, we set  $\dim_{\mathbb{R}} \mathbf{R}(\mathfrak{g}) \equiv f$ ; cfr. (2.6)). By virtue of the *Gaillard-Zumino embedding* (2.5) [44] (or, equivalently of the aforementioned Theorem by Dynkin [45, 46]),  $\tau$  expresses the fraction of generators of  $\mathfrak{sp}(f, \mathbb{R})$  which generate its maximal (generally non-symmetric) sub-algebra  $\mathfrak{g}$ . Indeed, it holds that

$$0 < \tau \leq 1 \Leftrightarrow -\frac{1}{2} \leq \lambda < 0. \quad (5.9)$$

By a suitable generalization of the analysis of [83], explicitly worked out in [71], the choice of  $f_{abcd}$  given by (5.7) can be made also for the pairs  $(\mathfrak{g}, \mathbf{R}) = (\mathbf{conf}(\widehat{\mathfrak{J}}), \mathbf{R})$  with  $\mathfrak{g}$  semi-simple. However, in these cases the last step of (5.2) does not hold:

$$f_{abcd} = -3\tau f_{(abcd)} + \tau \omega_{a(c\omega_d)b} \neq t_{ab}^\alpha t_{cd}^\beta g_{\alpha\beta}; \quad (5.10)$$

in fact, the explicit expression of  $t_{ab}^\alpha t_{\alpha|cd}$  for these cases has been computed in [71], and it is such that [70]

$$g_{\alpha\beta} t_{(ab)}^\alpha t_{c)d}^\beta = 0.$$

Thus, the *FTS* (the triple system on which the FGT is based) turns out to be related to the *quaternionic level* of Faulkner’s construction [35] of triple systems from pairs  $(\mathfrak{g}, \mathbf{V})$ , which has been recently re-analyzed by [36, 37, 69] within  $D = 3$  SC CSM gauge theories.

An important difference with the latter framework is the fact that, in the treatment of the present paper, *FTS* is defined on the ground field  $\mathbb{R}$  (recall footnote 1); this constrains the pair  $(\mathfrak{g}, \mathbf{V}) = (\mathfrak{g}, \mathfrak{R})$  such that  $\mathbf{V}$  is a *real* representation space of the (*non-compact*) real algebra  $\mathfrak{g}$ ; some examples, related to conformal symmetries of *JTS*  $\mathfrak{J} = \widehat{\mathfrak{J}}$ , are reported in table 1. As mentioned in section 3.3, we point out that this is not inconsistent with the physical constraint on matter representations in  $D = 3$  SC CSM gauge theories; indeed,  $\mathbf{V} = \mathbf{W}$  is always assumed to possess a positive-definite inner product (for *unitarity* of the corresponding gauge theory), but CS gauge fields are not propagating (and they are in  $\mathbf{Adj}(\mathfrak{g})$ ), and therefore  $\mathfrak{g}$  does not necessarily have to be endowed with a positive-definite product, thus allowing for *non-compact* (real) forms of  $\mathfrak{g}$ .

The expression (5.2) of the *FTS* structure constants’ tensor  $f_{abcd}$  (or, equivalently, for the rank-4  $\mathfrak{g}$ -invariant structure in  $\mathbf{W}$  in  $(\mathfrak{g}, \mathbf{V} = \mathbf{W})$ -based Faulkner’s construction of triple systems [35]) entails two “extremal” cases:

1. The case in which  $f_{abcd}$  is *completely symmetric* (and therefore *Freudenthal duality*  $\mathcal{F}$  (2.25) can be consistently introduced). This corresponds to  $b = 0$  and (up to redefinition)  $a = 1$  in (5.2):

$$f_{abcd} = f_{(abcd)}, \tag{5.11}$$

which characterizes Brown’s definition [22] of  $(\mathfrak{g}, \mathbf{W})$  as a Lie algebra *of type*  $\mathfrak{e}_7$  (cfr. axiom (b) in section 2.5). The corresponding triple system has been called *quaternionic Lie triple system (qLTS)* in [37]. However, this triple system is *not* relevant for application to (BLG-type) gauge theories. Indeed, for *positive-definite*  $\mathbf{W}$  (as assumed for unitarity of the corresponding gauge theory),  $f_{abcd}$  is nothing but the Riemann tensor of a symmetric hyper-Kähler manifold, which is *Ricci-flat*; however, any homogeneous Ricci-flat Riemannian manifold is actually *Riemann-flat* [84, 85]. Thus, a *positive-definite*  $\mathbf{W}$  in *qLTS* (5.11) is necessarily the trivial representation (cfr. corollary 6 in [37]). Remarkably, this result has a consistent interpretation in the *FTS* framework. Indeed, it can be checked that (5.11), when plugged into the *FTS* axiom (iii) (*fundamental identity*) and contracted with  $x^a x^b y^c y^e y^f y^g$ , does *not* yield the axiom (c) which defines a Lie algebra *of type*  $\mathfrak{e}_7$  [22]. In other words,  $(\mathfrak{g}, \mathbf{W})$  *of type*  $\mathfrak{e}_7$  [22] is *not* consistent with the *FTS* introduced in sections 2.5–2.4; in particular, the *fundamental identity* (iii) is *not* consistent with axiom (c) of Lie algebras *of type*  $\mathfrak{e}_7$  [22]. As a consequence, the limit of the defining axioms (i)-(iv) in which  $f_{abcd}$  is taken to be *completely symmetric* (5.11) is ill defined; a non-trivial  $\lambda \rightarrow 0$  limit in (i)-(iv) can still be implemented, but it yields an *FTS* which does not fulfill the symmetry condition (5.11) [34].

2. The case in which  $f_{abcd}$  lacks its *completely symmetric* part. This corresponds to  $a = 0$  and (up to redefinition)  $b = 1$  in (5.2):

$$f_{abcd} = \omega_{a(c}\omega_{d)b}. \tag{5.12}$$

In this case the *Freudenthal duality*  $\mathcal{F}$  (2.25) cannot be consistently introduced. The corresponding triple system has been called *anti-Lie triple system* (*aLTS*) in [37]; it characterizes  $\mathcal{N} = 4$  and  $\mathcal{N} = 5$  SC CSM gauge theories in  $D = 3$ , as thoroughly analyzed in [37] (see also table 6 therein), by elaborating on previous literature (see refs. therein). A prototypical case (treated in Example 1 of [41]) is provided by a consistent limit of (5.6), given by<sup>26</sup> (recall (2.6))  $\mathfrak{g} = \mathfrak{sp}(f, \mathbb{R})$  and  $\mathbf{W} = \mathbf{f}$  (fundamental irrep.). Since

$$S^4\mathbf{f} \equiv (\mathbf{f} \times \mathbf{f} \times \mathbf{f} \times \mathbf{f})_s \tag{5.13}$$

is irreducible in  $\mathfrak{sp}(f, \mathbb{R})$  and contains no singlets, it follows that  $f_{(abcd)} = 0$ . On the other hand, since  $\mathbf{Adj}(\mathfrak{sp}(f, \mathbb{R})) = S^2\mathbf{f} \equiv (\mathbf{f} \times \mathbf{f})_s$ , the definition (5.8) also yields  $\tau = 1$ , and therefore (5.12) is recovered from (5.6). It is worth remarking that in this case the resulting *FTS* is not endowed with a manifestly *JTS*-covariant structure (2.1) as in the original Freudenthal's formulation [38–40]; the corresponding (super)gravity theory in  $D = 4$  can have *at most*<sup>27</sup>  $\mathcal{N} = 1$  local supersymmetry, and has a (non-special) Kähler scalar coset with algebra  $\mathfrak{sp}(f, \mathbb{R}) \oplus \mathfrak{u}(f/2)$  (*upper Siegel half-plane*).

The general triple system under consideration, which interpolates between *qLTS* (5.11) and *aLTS* (5.12), is endowed with an  $f_{abcd}$  given by (5.2) with *both*  $a$  and  $b$  *non-vanishing*. As anticipated, among SC CSM gauge theories in  $D = 3$ , this is consistent only with  $\mathcal{N} = 3$  (see e.g. [37], and refs. therein), which is thus the only amount of (global) supersymmetry for which *Freudenthal duality*  $\mathcal{F}$  (2.25) could a priori be implemented, even if its enforcement as a *global (off-shell)* symmetry is in contrast with supersymmetry itself, as implied by the *No-Go theorem* proved in section 4.2.

It is worth observing that this general case is also consistent with the “extension” of the definition of Lie algebras *of type*  $\mathfrak{e}_7$  (based on axioms  $(\widehat{a})$ - $(\widehat{c})$  above); indeed, up to some redefinitions, the real parameters  $a$  and  $b$  can always be chosen such that (5.2), when plugged into the *FTS* axiom (iii) and contracted with  $x^a x^b y^c y^e y^f y^g$ , does yield the axiom  $(\widehat{c})$  introduced above; the term  $\omega_{a(c\omega_d)b}$  plays a key role in this result.

The above treatment hints for the existence of a class of  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theories in which the gauge Lie algebra and its matter representation are given by

$$(\mathfrak{g}, \mathbf{V}) = \left( \mathbf{conf}(\widehat{\mathfrak{J}}), \mathbf{R} \right), \tag{5.14}$$

namely they are respectively given by the *conformal* symmetries of rank-3, Euclidean Jordan algebras, and by their relevant symplectic irreps.  $\mathbf{R}$ , as reported in table 1.

In this respect, by recalling section 3.5,  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theories based on (5.14) share the same symmetry (with different physical meanings) of two other distinct classes of theories:

- $D = 4$  Maxwell-Einstein (super)gravity theories (ME(S)GT) (with various amount  $\mathcal{N}$  of local supersymmetry) having *symmetric* scalar manifolds, as discussed in section 3.5 (and reported in table 1);

<sup>26</sup>Recall that, under the assumption that  $\omega$  is non-degenerate,  $f$  is even.

<sup>27</sup>In this theory, the consistency of  $\mathcal{N} = 1$  local supersymmetry with a symplectic structure of electric and magnetic fluxes has been studied e.g. in [53]; see also [86].

- $D = 3$  *Freudenthal gauge theories* (FGT's) based on an *FTS*  $\mathfrak{K} \sim \mathbf{R}(\text{conf}(\widehat{\mathfrak{J}}))$ . The consistency of FGT with (global) supersymmetry is an important difference with respect to  $\mathcal{N} = 3$  SC CSM gauge theories. Indeed, the *No-Go Theorem* proved in section 4.2 essentially states that global (*off-shell*) *Freudenthal duality* is *not* consistent with a non-trivial coupling to space-time vector/spinor fields, which in turn is a necessary condition for supersymmetry.

These relations among  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theories,  $D = 4$  ME(S)GT's and FGT's can actually be extended to the general case in which the pair  $(\mathfrak{g}, \mathbf{V} = \mathbf{W})$  defines a generic *FTS* (based on axioms (i)-(iv)) corresponding, in the sense outlined above, to the “*quaternionic level*” of Faulkner’s construction [35–37, 69, 72].

We plan to investigate this interesting interplay of symmetries in future work [34] (also in view of possible AdS/CFT applications). In particular, as anticipated above, when disregarding the global (*off-shell*) *Freudenthal duality*, it would be interesting to consider the consistency of ( $D = 3$ ) FGT as an alternative, purely *bosonic sector* of the corresponding  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theory. In fact, as analyzed in section 3.3, in FGT the non-vanishing of  $f_{(abcd)}$  allows for terms in the Lagrangian which differ from the usual ones in BLG theories; for instance, the simplest FGT scalar potential is *quartic* in the scalar fields (essentially given by  $\Delta$  (2.21); see (3.29)), whereas in BLG theories it is of order six (see e.g. (19) of [10]).

## 6 Concluding remarks

In this paper, we have introduced the *Freudenthal Gauge Theory* (FGT), a gauge theory invariant under two *off-shell* symmetries: a local, *gauge* symmetry constructed from a *Freudenthal Triple System* (FTS)  $\mathfrak{K}$ , and a *global* symmetry based on the so-called *Freudenthal Duality* (*F-duality*)  $\mathcal{F}$ .

We have presented the most general bosonic action invariant under these two symmetries, containing a single  $\mathfrak{K}$ -valued scalar field  $\phi(x)$  and a gauge field  $A_\mu^{ab}(x) \in \mathfrak{K} \otimes_S \mathfrak{K}$ . The algebraic structure of the *FTS* ensures that the FGT is well defined and has the required properties.

One of the building blocks of FGT is the *F-duality*  $\mathcal{F}$ , which is a non-linear *anti-involutive* duality ( $\mathcal{F}^2 = -Id$ ) which gives, up to a sign, a one-to-one pairing of elements in  $\mathfrak{K}$ .

In section 4, we have also analyzed the possibility of generalizing the simple setup presented in section 3 by coupling to space-time vector *and/or* spinor fields, which is a necessary condition for supersymmetry and is usually a relatively simple step in the construction of gauge theories. Within the assumption<sup>28</sup> that *Freudenthal duality*  $\mathcal{F}$  can be defined only for algebraic systems satisfying the *FTS* axioms (i)-(iv) (see subsection 2.3) we have proved a *No-Go theorem* (which holds true if the metric of the system is *non-degenerate*), which essentially forbids the coupling to space-time vector *and/or* spinor fields.

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<sup>28</sup>We leave the possible relaxation of the assumptions on  $\mathcal{F}$  *and/or* on the metric of the algebraic system to further future investigation. Concerning the case of *degenerate* metric, see also footnote 19.

However, we point out that such a coupling is possible *at least* if one relaxes the requirement of invariance under *F-duality*. Despite the fact that in our treatment there is, a priori, no restriction on the space-time dimension  $D$ , *non-compact* gauge Lie algebras  $\mathfrak{g}$  generally yield non-unitary theories in  $D \geq 4$  (cfr. the remark below (3.33)). However, in  $D = 3$  this is no more a problem, and the resulting (*non-Freudenthal-invariant*) FGT can contain both bosonic and fermionic degrees of freedom together with the Chern-Simons term.

In  $D = 3$ , some intriguing similarities (and important differences) between FGT and superconformal (SC) Chern-Simons-matter (CSM) gauge theories with  $\mathcal{N} = 3$  global supersymmetry have been discussed in section 5. Indeed, among SC CSM gauge theories in  $D = 3$ , a generic *FTS* is *only* consistent for  $\mathcal{N} = 3$  (see e.g. [37], and refs. therein), which is thus the only amount of (global) supersymmetry for which *Freudenthal duality*  $\mathcal{F}$  (2.25) could a priori be implemented, even if its enforcement as a *global (off-shell)* symmetry is in contrast with supersymmetry itself, as implied by the *No-Go theorem* proved in section 4.2.

It is worth recalling here that our treatment hints for the existence of a class of  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theories in which the gauge Lie algebra is given by (5.14), namely by the *conformal* algebras  $\mathfrak{g} = \text{conf}(\widehat{\mathfrak{J}})$  of rank-3, Euclidean Jordan algebras, and by their relevant symplectic irreps.  $\mathbf{R}$ , as reported in table 1. In this respect, such  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theories share the same symmetry (with different physical meanings) of two other distinct classes of theories: **I**]  $D = 4$  Maxwell-Einstein (super)gravity theories (ME(S)GT) (with various amount  $\mathcal{N}$  of local supersymmetry) with *symmetric* scalar manifolds, as discussed in section 3.5 (and reported in table 1); **II**]  $D = 3$  FGT's based on an *FTS*  $\mathfrak{K} \sim \mathbf{R}(\text{conf}(\widehat{\mathfrak{J}}))$ .

These relations among  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theories,  $D = 4$  ME(S)GT's and  $D = 3$  FGT's can actually be extended to the general case in which the pair  $(\mathfrak{g}, \mathbf{V} = \mathbf{W})$  defines a generic *FTS* (based on axioms (i)-(iv)) corresponding, as discussed in section 5, to the “*quaternionic level*” of Faulkner’s construction [35–37, 69, 72].

We plan to investigate this interesting interplay of symmetries in future work [34] (also in view of possible AdS/CFT applications). In particular, when disregarding the global (*off-shell*) *Freudenthal duality*, it will be interesting to consider the consistency of  $D = 3$  FGT as an alternative, purely *bosonic sector* of the corresponding  $\mathcal{N} = 3$ ,  $D = 3$  SC CSM gauge theory. In fact, as analyzed in section 3.3, in FGT the non-vanishing of  $f_{(abcd)}$  allows for terms in the Lagrangian which differ from the usual ones in BLG theories; for instance, the simplest FGT scalar potential is *quartic* in the scalar fields (essentially given by  $\Delta$  (2.21); see (3.29)), whereas in BLG theories it is of order six (see e.g. (19) of [10]).

The close relation between the particular class  $\mathfrak{K}(\widehat{\mathfrak{J}})$  of *FTS*'s and exceptional Lie algebras  $\mathfrak{g}$  (discussed in sections 2.1 and 3.4) could also be used to investigate the possible relation (*duality?*) between FGT and Yang-Mills gauge theory with exceptional gauge Lie algebra  $\mathfrak{g}$ . This is certainly possible, but one should recall that exceptional Lie groups cannot be embedded into standard matrix groups, and thus the resulting Yang-Mills theory would not have the standard Maxwell term constructed from trace over matrices. Geometrically, a better way to understand this model is by noting that the exceptional Lie groups can be embedded as matrix groups over octonions  $\mathbb{O}$  [74]; thus, the  $\mathfrak{K}(\widehat{\mathfrak{J}})$ -based FGT would be *dual* to a standard Yang-Mills theory over<sup>29</sup>  $\mathbb{O}$ .

<sup>29</sup>For similar formulations, see e.g. [75–77], and refs. therein.

The present investigation proved the *quartic* polynomial  $\Delta$  (2.21) to be *invariant* not only under *Freudenthal duality*  $\mathcal{F}$  (2.25), but also under the (*global* or *gauged*) transformation based on the *FTS* triple product (2.11). It will be interesting to investigate the physical meaning of such an invariance of  $\Delta$  e.g. within black hole physics [23] and flux compactifications [79], in which  $\Delta$  occurs in relation respectively to the Bekenstein-Hawking [32, 33] black hole entropy and to the cosmological constant. Interesting recent advances on Freudenthal duality [68, 87] might also lead to further developments in understanding FGT.

Finally, we would like to point out that the *FTS* has another intriguing geometrical interpretation in terms of the so-called *metasymplectic geometry*, introduced decades ago by Freudenthal [38] [88]. In such a geometric framework, two points can define, instead of a line passing through them as in the standard geometry, two more relations, called *interwoven* and *hinged*. Furthermore, to each set of points there corresponds a set of *dual* geometrical objects called *symplecta*, satisfying relations which are *dual* to the aforementioned three ones among the points. In this bizarre geometrical setup, the *FTS* axioms acquire a natural geometrical interpretation, and the relation to the exceptional Lie algebras becomes more transparent. We leave the possible physical interpretation of such a fascinating geometry within FGT for further future investigation.

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### A Freudenthal duality

In this appendix, generalizing the treatment of [23] (in turn referring to [22]) to a generic *FTS*  $\mathfrak{K}$  (see also [24]), we present the proof that the *quartic* polynomial  $\Delta(\phi)$  (2.21) is *invariant* under the *Freudenthal duality*  $\mathcal{F}$  (2.25).

By recalling definition (2.13), we can restate the derivation property (*FTS* axiom (iii')) as follows:

$$[\mathcal{L}_{\phi_L\phi_M}, \mathcal{L}_{\phi_I\phi_J}]\phi_K = \mathcal{L}_{(\phi_L\phi_M\phi_I)\phi_J}\phi_K + \mathcal{L}_{\phi_I(\phi_L\phi_M\phi_J)}\phi_K. \tag{A.1}$$

Since this equation is true for any element  $\phi_K \in \mathfrak{K}$ , it is true as an operator equation for  $\mathcal{L}_{\phi_I\phi_J}$ . Setting  $I = J = L = M$ , we find that

$$[\mathcal{L}_{\phi\phi}, \mathcal{L}_{\phi\phi}] = \mathcal{L}_{T(\phi)\phi} + \mathcal{L}_{\phi T(\phi)} = 2\mathcal{L}_{\phi T(\phi)} \tag{A.2}$$

where the FTS axiom (i) of subsection 2.3 has been used. Since the commutator of an operator with itself must vanish, the above equation must be equal to zero:

$$\mathcal{L}_{\phi T(\phi)} = 0 \quad (\text{A.3})$$

This means, again by the derivation property of  $\mathcal{L}$ , that both  $\mathcal{L}_{T(\phi)\phi}$  and  $\mathcal{L}_{\phi T(\phi)}$  act like *annihilation operators* on any element  $\phi_K \in \mathfrak{K}$ .

Then, by recalling the definition (2.21), from the FTS axiom (ii) of subsection 2.3 one obtains:

$$\begin{aligned} \mathcal{L}_{T(\phi)T(\phi)}\phi &= T(\phi)T(\phi)\phi \\ &= T(\phi)\phi T(\phi) + 2\lambda \langle T(\phi), \phi \rangle T(\phi) + \lambda \langle T(\phi), \phi \rangle T(\phi) - \lambda \langle T(\phi), T(\phi) \rangle \phi \\ &= 6\lambda \Delta(\phi) T(\phi); \end{aligned} \quad (\text{A.4})$$

$$\mathcal{L}_{\phi\phi}T(\phi) = \phi\phi T(\phi) = -6\lambda \Delta(\phi)\phi. \quad (\text{A.5})$$

Consequently, the direct evaluation of  $T(T(\phi))$  reads:

$$\begin{aligned} T(T(\phi)) &= \mathcal{L}_{T(\phi)T(\phi)}T(\phi) = 6\lambda \Delta(\phi) \left( T(\phi)\phi\phi + \phi T(\phi)\phi + \phi\phi T(\phi) \right) \\ &= -(6\lambda \Delta(\phi))^2 \phi. \end{aligned} \quad (\text{A.6})$$

From result (A.6), by assuming  $6\lambda \Delta(\phi) \neq 0$  (see discussion in subsection 2.5, in particular point (III)), one can check that the following two statements hold true:

1. The *Freudenthal duality*  $\mathcal{F}$  (2.25) is an *anti-involution* in the FTS  $\mathfrak{K}$ , namely it squares to negative identity (cfr. (2.27) and point (I) of subsection 2.5):

$$\mathcal{F}^2 \equiv \mathcal{F} \circ \mathcal{F} = -Id. \quad (\text{A.7})$$

2. The *quartic* polynomial  $\Delta(\phi)$  (2.21) is *invariant* under the *Freudenthal duality*  $\mathcal{F}$  (2.25), namely (cfr. (2.26))

$$\Delta(\phi) = \Delta(\tilde{\phi}), \quad q.e.d. \quad \blacksquare$$

## B Space-time symmetry of scalar kinetic term

In order to prove the symmetry (3.31) of the FGT kinetic scalar term under the exchange of its space-time indices, one needs to re-write it only in terms of the  $\mathfrak{K}$ -valued scalar field  $\phi(x)$ , by recalling the definitions (2.21) and (2.25) of the *quartic* polynomial  $\Delta(\phi)$  and of *F-dual* field  $\tilde{\phi}(x)$ .

One starts by computing the FTS gauge covariant derivative of  $\tilde{\phi}(x)$ , as follows:

$$\begin{aligned} D_\mu \tilde{\phi}(x) &= \text{sgn}(\Delta(\phi)) \frac{1}{\sqrt{6}} D_\mu \left( \frac{T(\phi)}{\sqrt{|\lambda \Delta(\phi)|}} \right) \\ &= \frac{\text{sgn}(\Delta(\phi))}{\sqrt{6|\lambda \Delta(\phi)|}} \left[ 3\mathcal{L}_{\phi\phi} D_\mu \phi + 6\lambda \langle D_\mu \phi, \phi \rangle \phi + \frac{\langle D_\mu \phi, T(\phi) \rangle}{\Delta(\phi)} T(\phi) \right] \end{aligned} \quad (\text{B.1})$$



As an aside, notice that the  $\Delta(\phi)$  in the denominator of the last term does not have absolute signs attached to it. Plugging this expression into the kinetic term (prior to contraction with  $\eta^{\mu\nu}$ ) yields its following explicit re-writing only in terms of  $\phi(x)$ :

$$\frac{1}{2}\langle D_\mu\phi, D_\nu\tilde{\phi}\rangle = \frac{\text{sgn}(\Delta(\phi))}{2\sqrt{6|\lambda\Delta(\phi)|}} \left[ 3\langle D_\mu\phi, \mathcal{L}_{\phi\phi}D_\nu\phi\rangle + 6\lambda\langle D_\mu\phi, \phi\rangle\langle D_\nu\phi, \phi\rangle + \frac{1}{\Delta(\phi)}\langle D_\mu\phi, T(\phi)\rangle\langle D_\nu\phi, T(\phi)\rangle \right]. \quad (\text{B.2})$$

On the other hand, the second and third term of (B.2) are manifestly symmetric under  $\mu \leftrightarrow \nu$ , the symmetry of the first term can be proved as follows:

$$\langle D_\mu\phi, \mathcal{L}_{\phi\phi}D_\nu\phi\rangle = -\langle \mathcal{L}_{\phi\phi}D_\mu\phi, D_\nu\phi\rangle = \langle D_\nu\phi, \mathcal{L}_{\phi\phi}D_\mu\phi\rangle, \quad (\text{B.3})$$

thus implying the result (3.31). ■

## C Axioms of $\mathfrak{V}$

As discussed in subsection 4.2, we report here the five axioms induced on  $\mathfrak{V}$  by the five axioms (o)-(iv) of the algebra  $\mathfrak{N}$  (in addition to the ones already introduced on  $\mathfrak{V}$  for other physical reasons, such as the ones required by the Bose *and/or* Fermi statistics for the fields  $v_I \in \mathfrak{V}$ ). In particular, in the proof of the **no-go theorem** in subsection 4.2, a crucial role is played by axioms (B.iii) and (B.ii).

$$(B.o) \quad (v_I, v_J)_{\mathfrak{V}} = (v_J, v_I)_{\mathfrak{V}};$$

$$(B.i) \quad [v_I, v_J, v_K]_{\mathfrak{V}} = [v_J, v_I, v_K]_{\mathfrak{V}};$$

$$\begin{aligned} (B.ii) \quad & (\phi_I\phi_J\phi_K) \otimes \left( [v_I, v_J, v_K]_{\mathfrak{V}} - [v_I, v_K, v_J]_{\mathfrak{V}} \right) \\ & = \langle \phi_J, \phi_K \rangle \phi_I \otimes \left( 2\mu(v_J, v_K)_{\mathfrak{V}} \times v_I - 2\lambda[v_I, v_J, v_K]_{\mathfrak{V}} \right) \\ & + \langle \phi_I, \phi_K \rangle \phi_J \otimes \left( \mu(v_I, v_K)_{\mathfrak{V}} \times v_J - \lambda[v_I, v_J, v_K]_{\mathfrak{V}} \right) \\ & - \langle \phi_I, \phi_J \rangle \phi_K \otimes \left( \mu(v_I, v_J)_{\mathfrak{V}} \times v_K - \lambda[v_I, v_J, v_K]_{\mathfrak{V}} \right); \end{aligned}$$

$$\begin{aligned} (B.iii) \quad 0 = & (\phi_L\phi_M\phi_I)\phi_J\phi_K \otimes \left( [v_L, v_M, [v_I, v_J, v_K]_{\mathfrak{V}}]_{\mathfrak{V}} - [v_L, v_M, v_I]_{\mathfrak{V}}, v_J, v_K \right)_{\mathfrak{V}} \\ & + \phi_I(\phi_L\phi_M\phi_J)\phi_K \otimes \left( [v_L, v_M, [v_I, v_J, v_K]_{\mathfrak{V}}]_{\mathfrak{V}} - [v_I, [v_L, v_M, v_J]_{\mathfrak{V}}, v_K]_{\mathfrak{V}} \right) \\ & + \phi_I\phi_J(\phi_L\phi_M\phi_K) \otimes \left( [v_L, v_M, [v_I, v_J, v_K]_{\mathfrak{V}}]_{\mathfrak{V}} - [v_I, v_J, [v_L, v_M, v_K]_{\mathfrak{V}}]_{\mathfrak{V}} \right); \end{aligned}$$

$$(B.iv) \quad \left( [v_L, v_M, v_I]_{\mathfrak{V}}, v_J \right)_{\mathfrak{V}} + \left( v_I, [v_L, v_M, v_J]_{\mathfrak{V}} \right)_{\mathfrak{V}} = 0.$$

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