Notes on SUSY gauge theories on three-sphere

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Abstract: We extend the formula for partition functions of $\mathcal{N} = 2$ superconformal gauge theories on S^3 obtained recently by Kapustin, Willett and Yaakov, to incorporate matter fields with arbitrary R-charge assignments. We use the result to check that the self-mirror property of $\mathcal{N} = 4$ SQED with two electron hypermultiplets is preserved under a certain mass deformation which breaks the supersymmetry to $\mathcal{N} = 2$.

Contents

1 Introduction

A major progress in the area of supersymmetric gauge theories has been made in recent years based on exact computation of path integral on some deformed or compact manifolds. In four dimensions, it was shown in the pioneering work [1] that the exact $\mathcal{N} = 2$ prepotential can be extracted from the path integral on Omega-deformed spacetime. In a similar manner, based on the localization principle, the partition function and Wilson loop observables of Seiberg-Witten theories on S^4 have been computed in [2]. These results led to a discovery of a remarkable relation between 4D gauge theories and 2D Liouville or Toda CFTs [3, 4], called AGT relation.

For 3D $\mathcal{N} = 2$ superconformal gauge theories on S^3, exact partition functions and Wilson loop observables have been obtained in [5]. The techniques developed there have been applied to further studies of various topics, such as Wilson loops [6, 7], 3D dualities [8] and large-N duality of topological string [9]. It has also been applied to the study of the ABJM theory at strong coupling, in particular its conjectured $\mathcal{O}(N^{3/2})$ growth of the degrees of freedom [10–12]. Another application has recently been made to the study of domain walls in 4D $\mathcal{N} = 2$ gauge theories [13, 14] in connection with the AGT relation.

The path integration of fields was performed in [5] for gauge theories with manifestly superconformally invariant Lagrangian. In particular, all the matter scalars and fermions are assigned canonical dimensions 1/2 and 1, respectively. Using the same technique, the partition functions of various $\mathcal{N} = 4$ superconformal gauge theories was computed in [8] as functions of the relevant (FI and mass) deformation parameters. One subtle issue there was that an $\mathcal{N} = 4$ vectormultiplet contains an $\mathcal{N} = 2$ chiral multiplet with non-canonical dimension. In [8], the contribution from such chiral multiplet to the partition function was argued to be trivial, by pointing out the existence of a SUSY-exact F-term deformation which lifts all of its component fields. One should be able to check this by more direct

means. Also, it remained unclear whether this property continues to hold when a mass to this chiral matter is turned on to break supersymmetry to $\mathcal{N} = 2$.

In this paper we extend the result of [5, 8] so that the matter chiral multiplets with arbitrary R-charge assignment can be incorporated. After summarizing in section 2 our notations for various geometric quantities in S^3, we give a supersymmetry transformation law of $\mathcal{N} = 2$ vector and chiral multiplets in section 3. There we also construct various supersymmetric Lagrangians; among them are the super Yang-Mills Lagrangian for vector-multiplets and kinetic Lagrangian for chiral multiplets. Similar Lagrangians were studied in the context of 4D $\mathcal{N} = 1$ gauge theories on $S^3 \times \mathbb{R}$ in [15–18]. They are both shown to be total superderivatives, so it follows that the partition function does not depend on the Yang-Mills coupling. Then, in section 4 we compute the one-loop determinant of general chiral matters on the saddle points parametrized by the vev of vectormultiplet scalars. The prescription to compute partition function for general $\mathcal{N} = 2$ gauge theories on S^3 is summarized in section 5. Finally, in section 6 we apply our result to check the self-mirror property of a certain $\mathcal{N} = 2$ SQED which has recently been studied in [14].

2   Three-sphere

The three-sphere is parametrized by an element g of the Lie group SU(2), and two copies of SU(2) symmetry act on g from the left and the right. We introduce the left-invariant (LI) and right-invariant (RI) one-forms $\mu^a = \mu^a_\nu d\xi^\nu$ and $\tilde{\mu}^a = \tilde{\mu}^a_\nu d\xi^\nu$,

$$g^{-1}dg = i\mu^a\gamma^a, \quad dgg^{-1} = i\tilde{\mu}^a\gamma^a, \tag{2.1}$$

where γ^a are Pauli matrices. These one-forms satisfy

$$d\mu^a = \epsilon^{abc}\mu^b\mu^c, \quad d\tilde{\mu}^a = -\epsilon^{abc}\tilde{\mu}^b\tilde{\mu}^c. \tag{2.2}$$

The left-right invariant round metric with radius ℓ is

$$ds^2 = \frac{1}{2}\ell^2 \mathrm{tr}(dgdg^{-1}) = \ell^2\mu^a\mu^a = \ell^2\tilde{\mu}^a\tilde{\mu}^a. \tag{2.3}$$

We define the vielbein in the "LI frame" as $e^a = e^a_\mu d\xi^\mu = \ell\mu^a$. The spin connection in this frame is $\omega^{ab} = \varepsilon^{abc}\mu^c$ and satisfies $de^a + \omega^{ab}e^b = 0$. If we define the vielbein from $\tilde{\mu}^a$ ("RI frame"), the spin connection is $\tilde{\omega}^{ab} = -\varepsilon^{abc}\tilde{\mu}^c$.

Killing spinors. Killing spinor ϵ satisfies the following equation

$$D\epsilon \equiv d\epsilon + \frac{1}{4}\gamma^{ab}\omega^{ab}\epsilon = e^a\gamma^a\tilde{\epsilon}, \tag{2.4}$$

for a certain $\tilde{\epsilon}$. Here we used the notation $\gamma^{ab} \equiv \frac{1}{2}[\gamma^a, \gamma^b] = i\varepsilon^{abc}\gamma^c$. There are two types of Killing spinors. The first one is constant in the LI frame,

$$\epsilon = \epsilon_0 \text{ (constant)}, \quad \tilde{\epsilon} = +\frac{i}{2\ell}\epsilon. \tag{2.5}$$

The second one reads

$$\epsilon = g^{-1}\epsilon_0, \quad \tilde{\epsilon} = -\frac{i}{2\ell}\epsilon, \tag{2.6}$$

and is constant in the RI frame.

Killing vectors. Let us next introduce the vector fields $\mathcal{L}^a = \mathcal{L}^{a\mu}\frac{\partial}{\partial\xi^\mu}$ and $\mathcal{R}^a = \mathcal{R}^{a\mu}\frac{\partial}{\partial\xi^\mu}$ which generate the left and the right actions of SU(2). They can be determined from

$$\mathcal{L}^a g = i\gamma^a g, \quad \mathcal{R}^a g = ig\gamma^a. \tag{2.7}$$

The vector fields $\frac{i}{2}\mathcal{L}^a$ and $-\frac{i}{2}\mathcal{R}^a$ satisfy the standard commutation relations of SU(2) Lie algebra. It is also easy to find $\mathcal{R}^{a\nu}\mu_\nu^b = \mathcal{L}^{a\nu}\tilde{\mu}_\nu^b = \delta^{ab}$, in other words $\mathcal{R}^{a\nu}$ and $\mathcal{L}^{a\nu}$ are proportional to the inverse-vielbeins in LI or RI frames. The action of these Killing vector fields on the LI and RI one-forms is given by

$$\mathcal{L}^a\tilde{\mu}^b = 2\varepsilon^{abc}\tilde{\mu}^c, \quad \mathcal{R}^a\mu^b = -2\varepsilon^{abc}\mu^c, \quad \mathcal{L}^a\mu^b = \mathcal{R}^a\tilde{\mu}^b = 0. \tag{2.8}$$

It therefore follows that $\mu^1\mu^2\mu^3 = d^3\xi\det(\mu_\nu^a)$ can be used to define the invariant volume form.

3 SUSY theories on three-sphere

Here we review the construction of Euclidean 3D $\mathcal{N} = 2$ superconformal gauge theories on manifolds with Killing spinors [5], and extend it to non-conformal theories. We begin by summarizing our conventions for bilinear products of spinors.

$$\bar{\epsilon}\lambda = \bar{\epsilon}^\alpha C_{\alpha\beta}\lambda^\beta, \quad \bar{\epsilon}\gamma^\mu\lambda = \bar{\epsilon}^\alpha(C\gamma^\mu)_{\alpha\beta}\lambda^\beta, \quad \text{etc.} \tag{3.1}$$

Here C is the charge conjugation matrix. Noticing that C is antisymmetric and $C\gamma_\mu$ are symmetric, one finds

$$\bar{\epsilon}\lambda = \lambda\bar{\epsilon}, \quad \bar{\epsilon}\gamma^\mu\lambda = -\lambda\gamma^\mu\bar{\epsilon} \tag{3.2}$$

for all spinors $\bar{\epsilon}, \lambda$ which we assume to be Grassmann odd.

Vectormultiplets. The vectormultiplet fields obey the following transformation laws,

$$\begin{aligned}
\delta A_\mu &= -\frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon), \\
\delta\sigma &= \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon), \\
\delta\lambda &= \frac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma + \frac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon, \\
\delta\bar{\lambda} &= \frac{1}{2}\gamma^{\mu\nu}\bar{\epsilon}F_{\mu\nu} + D\bar{\epsilon} - i\gamma^\mu\bar{\epsilon}D_\mu\sigma - \frac{2i}{3}\sigma\gamma^\mu D_\mu\bar{\epsilon}, \\
\delta D &= -\frac{i}{2}\bar{\epsilon}\gamma^\mu D_\mu\lambda - \frac{i}{2}D_\mu\bar{\lambda}\gamma^\mu\epsilon + \frac{i}{2}[\bar{\epsilon}\lambda, \sigma] + \frac{i}{2}[\bar{\lambda}\epsilon, \sigma] - \frac{i}{6}(D_\mu\bar{\epsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu D_\mu\epsilon). \tag{3.3}
\end{aligned}$$

Here and throughout this paper, D_μ denotes the gauge, local Lorentz and general covariant derivative, and γ^μ is the Dirac matrix with curved index which satisfies

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad \gamma^{\mu\nu} = i\varepsilon^{\mu\nu\rho}\gamma_\rho/\sqrt{g}. \quad (\varepsilon^{123} = 1) \tag{3.4}$$

Note that D_μ commutes with the vielbein e_μ^a and the Dirac matrices γ^a or γ^μ. The spinors $\epsilon, \bar{\epsilon}$ are assumed to satisfy Killing spinor equation. Denoting δ as the sum of unbarred and

barred parts, $\delta = \delta_\epsilon + \delta_{\bar{\epsilon}}$, one can show that two unbarred or two barred supersymmetries commute. Also, on all the fields except D the commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ becomes a sum of translation, gauge transformation, Lorentz rotation, dilation and R-rotation.

$$
\begin{aligned}
[\delta_\epsilon, \delta_{\bar{\epsilon}}]A_\mu &= \xi^\nu \partial_\nu A_\mu + \partial_\mu \xi^\nu A_\nu + D_\mu \Lambda, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\sigma &= \xi^\mu \partial_\mu \sigma + i[\Lambda, \sigma] + \rho\sigma, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\lambda &= \xi^\mu \partial_\mu \lambda + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\lambda + i[\Lambda, \lambda] + \frac{3}{2}\rho\lambda + \alpha\lambda, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\lambda} &= \xi^\mu \partial_\mu \bar{\lambda} + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar{\lambda} + i[\Lambda, \bar{\lambda}] + \frac{3}{2}\rho\bar{\lambda} - \alpha\bar{\lambda}, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]D &= \xi^\mu \partial_\mu D + i[\Lambda, D] + 2\rho D + \frac{1}{3}\sigma(\bar{\epsilon}\gamma^\mu\gamma^\nu D_\mu D_\nu \epsilon - \epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu \bar{\epsilon}), \quad (3.5)
\end{aligned}
$$

where

$$
\begin{aligned}
\xi^\mu &= i\bar{\epsilon}\gamma^\mu\epsilon, \\
\Theta^{\mu\nu} &= D^{[\mu}\xi^{\nu]} + \xi^\lambda \omega_\lambda^{\mu\nu}, \\
\Lambda &= -iA_\mu \bar{\epsilon}\gamma^\mu\epsilon + \sigma\bar{\epsilon}\epsilon, \\
\rho &= \frac{i}{3}(\bar{\epsilon}\gamma^\mu D_\mu \epsilon + D_\mu \bar{\epsilon}\gamma^\mu\epsilon), \\
\alpha &= \frac{i}{3}(D_\mu \bar{\epsilon}\gamma^\mu\epsilon - \bar{\epsilon}\gamma^\mu D_\mu \epsilon). \quad (3.6)
\end{aligned}
$$

In order for the supersymmetry algebra to close, the last term in the right hand side of $[\delta_\epsilon, \delta_{\bar{\epsilon}}]D$ needs to vanish. The Killing spinors therefore have to satisfy, in addition to (2.4), the following condition

$$
\gamma^\mu\gamma^\nu D_\mu D_\nu \epsilon = h\epsilon, \quad (3.7)
$$

with some scalar function h. The barred spinor $\bar{\epsilon}$ also has to satisfy the same equation with the same h. By combining this with Killing spinor equation (2.4), one obtains

$$
D_\mu \epsilon = \gamma_\mu \tilde{\epsilon}, \quad 3\gamma^\mu D_\mu \tilde{\epsilon} = h\epsilon. \quad (3.8)
$$

By inserting this into $\gamma^{\mu\nu}D_\mu D_\nu \epsilon = -\frac{1}{4}R\epsilon$ one finds $h = -\frac{3R}{8}$, where R is the scalar curvature of the 3D manifold. For S^3 of radius ℓ one has $R = \frac{6}{\ell^2}$ and therefore

$$
h = -\frac{9}{4\ell^2}. \quad (3.9)
$$

Note that all the Killing spinors on S^3 satisfy $D_\mu \epsilon = \pm\frac{i}{2\ell}\gamma_\mu\epsilon$, so they automatically satisfy the additional condition (3.7). So the additional condition does not reduce the number of supersymmetry on S^3.

The parameters ρ, α are associated to dilation and R-rotation, respectively. The above result shows that the fields $(A_\mu, \sigma, \lambda, D)$ have dimensions $(1, 1, 3/2, 2)$, and $(\lambda, \bar{\lambda})$ are assigned the R-charge $(1, -1)$.

Matter multiplets. The fields in a chiral multiplet coupled to a gauge symmetry transform as follows,

$$
\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi, \\
\delta\bar{\phi} &= \epsilon\bar{\psi}, \\
\delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \frac{i}{3}\gamma^\mu D_\mu\epsilon\phi + \bar{\epsilon}F, \\
\delta\bar{\psi} &= i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + i\bar{\phi}\sigma\bar{\epsilon} + \frac{i}{3}\bar{\phi}\gamma^\mu D_\mu\bar{\epsilon} + \bar{F}\epsilon, \\
\delta F &= \epsilon(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi), \\
\delta\bar{F} &= \bar{\epsilon}(i\gamma^\mu D_\mu\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\bar{\lambda}).
\end{aligned}
\tag{3.10}
$$

Here we assumed the fields ϕ, ψ, F ($\bar{\phi}, \bar{\psi}, \bar{F}$) to be column vectors (resp. row vectors) on which the vectormultiplet fields act as matrices from the left (right). The supersymmetry algebra closes off-shell. The commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ on matter fields reads

$$
\begin{aligned}
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\phi &= \xi^\mu\partial_\mu\phi + i\Lambda\phi + \frac{\rho}{2}\phi - \frac{\alpha}{2}\phi, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\phi} &= \xi^\mu\partial_\mu\bar{\phi} - i\bar{\phi}\Lambda + \frac{\rho}{2}\bar{\phi} + \frac{\alpha}{2}\bar{\phi}, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\psi &= \xi^\mu\partial_\mu\psi + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\psi + i\Lambda\psi + \rho\psi + \frac{\alpha}{2}\psi, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\psi} &= \xi^\mu\partial_\mu\bar{\psi} + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar{\psi} - i\bar{\psi}\Lambda + \rho\bar{\psi} - \frac{\alpha}{2}\bar{\psi}, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]F &= \xi^\mu\partial_\mu F + i\Lambda F + \frac{3\rho}{2}F + \frac{3\alpha}{2}F, \\
[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{F} &= \xi^\mu\partial_\mu\bar{F} - i\bar{F}\Lambda + \frac{3\rho}{2}\bar{F} - \frac{3\alpha}{2}\bar{F}.
\end{aligned}
\tag{3.11}
$$

This shows that the fields (ϕ, ψ, F) are assigned the canonical dimensions $(1/2, 1, 3/2)$. Two unbarred or two barred supersymmetries can be easily shown to commute, except on the auxiliary fields. On F one finds

$$
[\delta_\epsilon, \delta_{\epsilon'}]F = \frac{1}{3}\phi \cdot (\epsilon\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon' - \epsilon'\gamma^\mu\gamma^\nu D_\mu D_\nu\epsilon),
\tag{3.12}
$$

which vanishes if ϵ, ϵ' satisfy the constraint (3.7). Similarly, the commutator of two barred supersymmetries vanish on $\bar{F}$ only if the two barred Killing spinors satisfy the same constraint.

For matter multiplets with non-canonical assignments of dimensions, we put the supersymmetry transformation rule as follows,

$$
\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi, \\
\delta\bar{\phi} &= \epsilon\bar{\psi}, \\
\delta\psi &= i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \frac{2qi}{3}\gamma^\mu D_\mu\epsilon\phi + \bar{\epsilon}F, \\
\delta\bar{\psi} &= i\gamma^\mu\bar{\epsilon}D_\mu\bar{\phi} + i\bar{\phi}\sigma\bar{\epsilon} + \frac{2qi}{3}\bar{\phi}\gamma^\mu D_\mu\bar{\epsilon} + \bar{F}\epsilon,
\end{aligned}
$$

$$\delta F = \epsilon(i\gamma^\mu D_\mu \psi - i\sigma\psi - i\lambda\phi) + \frac{i}{3}(2q-1)D_\mu\epsilon\gamma^\mu\psi,$$

$$\delta\bar{F} = \bar{\epsilon}(i\gamma^\mu D_\mu\bar{\psi} - i\bar{\psi}\sigma + i\bar{\phi}\bar{\lambda}) + \frac{i}{3}(2q-1)D_\mu\bar{\epsilon}\gamma^\mu\bar{\psi}. \tag{3.13}$$

The supersymmetry algebra then becomes

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}]\phi = \xi^\mu\partial_\mu\phi + i\Lambda\phi + q\rho\phi - q\alpha\phi,$$

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\phi} = \xi^\mu\partial_\mu\bar{\phi} - i\bar{\phi}\Lambda + q\rho\bar{\phi} + q\alpha\bar{\phi},$$

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}]\psi = \xi^\mu\partial_\mu\psi + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\psi + i\Lambda\psi + \left(q+\frac{1}{2}\right)\rho\psi + (1-q)\alpha\psi,$$

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{\psi} = \xi^\mu\partial_\mu\bar{\psi} + \frac{1}{4}\Theta_{\mu\nu}\gamma^{\mu\nu}\bar{\psi} - i\bar{\psi}\Lambda + \left(q+\frac{1}{2}\right)\rho\bar{\psi} + (q-1)\alpha\bar{\psi},$$

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}]F = \xi^\mu\partial_\mu F + i\Lambda F + (q+1)\rho F + (2-q)\alpha F,$$

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}]\bar{F} = \xi^\mu\partial_\mu\bar{F} - i\bar{F}\Lambda + (q+1)\rho\bar{F} + (q-2)\alpha\bar{F}. \tag{3.14}$$

The lowest components are now assigned the dimension q and R-charge $\mp q$. The supersymmetry algebra closes off-shell when the Killing spinors $\epsilon, \bar{\epsilon}$ satisfy (3.7), (3.9).

Supersymmetric Lagrangians. The Chern-Simons Lagrangian for $\mathcal{N}=2$ vectormultiplet is invariant under supersymmetry.

$$\mathcal{L}_{\text{CS}} = \text{Tr}\left[\frac{1}{\sqrt{g}}\varepsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda\right) - \bar{\lambda}\lambda + 2D\sigma\right]. \tag{3.15}$$

Given a gauge-invariant chiral multiplet of R-charge $q=2$ usually called *superpotential*, its F-term is invariant under supersymmetry up to total derivatives.

$$\delta F = iD_\mu(\epsilon\gamma^\mu\psi), \quad \delta\bar{F} = iD_\mu(\bar{\epsilon}\gamma^\mu\bar{\psi}). \tag{3.16}$$

These terms are invariant under δ for any Killing spinors $\epsilon, \bar{\epsilon}$. In addition, chiral matter multiplets with canonical dimensions have the kinetic Lagrangian,

$$\mathcal{L} = D_\mu\bar{\phi}D^\mu\phi - i\bar{\psi}\gamma^\mu D_\mu\psi + \frac{3}{4\ell^2}\bar{\phi}\phi + i\bar{\psi}\sigma\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi + i\bar{\phi}D\phi + \bar{\phi}\sigma^2\phi + \bar{F}F, \tag{3.17}$$

which is invariant under supersymmetry if the Killing spinors $\epsilon, \bar{\epsilon}$ satisfy (3.7), (3.9). If the Lagrangian is made of the above three types of terms, the theory is superconformal at the classical level.

There are Lagrangians which are not superconformal but are still invariant under some supersymmetry. In the following we look for the quantities which are invariant if the parameters $\epsilon, \bar{\epsilon}$ satisfy

$$D_\mu\epsilon = \frac{i}{2\ell}\gamma_\mu\epsilon, \quad D_\mu\bar{\epsilon} = \frac{i}{2\ell}\gamma_\mu\bar{\epsilon}. \tag{3.18}$$

Under this additional condition, the commutator $[\delta_\epsilon, \delta_{\bar{\epsilon}}]$ does not give rise to dilation since ρ of (3.6) vanishes. The Killing vector $\bar{\epsilon}\gamma^a\epsilon$ is constant in the LI frame, so the commutator of supersymmetry is a linear sum of $\mathcal{R}^a$ and local Lorentz, gauge, R-transformations. This

restricted supersymmetry is therefore regarded as an analogue of Poincaré supersymmetry in flat space.

As an example, let us first look for a kinetic Lagrangian for matter fields with non-canonical dimensions. We take (3.17) as the trial Lagrangian for non-canonical matters. Its variation under the supersymmetry (3.13) is given by

$$
\begin{aligned}
\delta\mathcal{L} = {}& -\frac{i}{3}(2q-1)\bar{\phi}D_\mu\bar{\epsilon}\gamma^\mu(-i\gamma^\nu D_\nu\psi + i\sigma\psi + i\lambda\phi) \\
& +\frac{i}{3}(2q-1)(iD_\mu\bar{\psi}\gamma^\mu + i\bar{\psi}\sigma - i\bar{\phi}\bar{\lambda})\gamma^\nu D_\nu\epsilon\phi \\
& +\frac{i}{3}(2q-1)(\bar{F}D_\mu\epsilon\gamma^\mu\psi - \bar{\psi}\gamma^\mu D_\mu\bar{\epsilon}F).
\end{aligned} \tag{3.19}
$$

Using (3.18) one can rewrite this as $\delta\mathcal{L} = -\delta\mathcal{L}_{\mathrm{nc}}$, where

$$
\mathcal{L}_{\mathrm{nc}} = \frac{i(2q-1)}{\ell}\bar{\phi}\sigma\phi - \frac{(2q-1)}{2\ell}\bar{\psi}\psi - \frac{(2q-1)(2q-3)}{4\ell^2}\bar{\phi}\phi. \tag{3.20}
$$

Thus $\mathcal{L}_{\mathrm{mat}} = \mathcal{L} + \mathcal{L}_{\mathrm{nc}}$ is a supersymmetric kinetic Lagrangian.

$$
\begin{aligned}
\mathcal{L}_{\mathrm{mat}} = {}& D_\mu\bar{\phi}D^\mu\phi + \bar{\phi}\sigma^2\phi + \frac{i(2q-1)}{\ell}\bar{\phi}\sigma\phi + \frac{q(2-q)}{\ell^2}\bar{\phi}\phi + i\bar{\phi}D\phi + \bar{F}F \\
& -i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\psi}\sigma\psi - \frac{(2q-1)}{2\ell}\bar{\psi}\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi.
\end{aligned} \tag{3.21}
$$

Another example is the Yang-Mills Lagrangian for vectormultiplet. We start from the standard $\mathcal{N} = 2$ SYM Lagrangian and improve its supersymmetry variation by adding terms of order ℓ^{-1} and ℓ^{-2}. The supersymmetric Lagrangian finally becomes

$$
\mathcal{L}_{\mathrm{YM}} = \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}\left(D + \frac{\sigma}{\ell}\right)^2 + \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar{\lambda}[\sigma,\lambda] - \frac{1}{4\ell}\bar{\lambda}\lambda\right). \tag{3.22}
$$

Finally, there is an analogue of FI D-term for abelian vectormultiplet.

$$
\mathcal{L}_{\mathrm{FI}} \equiv D - \frac{\sigma}{\ell}, \quad \delta\mathcal{L}_{\mathrm{FI}} = -\frac{i}{2}D_\mu(\bar{\epsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu\epsilon). \tag{3.23}
$$

Note that $\mathcal{L}_{\mathrm{mat}}$ and $\mathcal{L}_{\mathrm{YM}}$ can be expressed as total-superderivatives,

$$
\begin{aligned}
\bar{\epsilon}\epsilon \cdot \mathcal{L}_{\mathrm{mat}} &= \delta_{\bar{\epsilon}}\delta_\epsilon\left(\bar{\psi}\psi - 2i\bar{\phi}\sigma\phi + \frac{2(q-1)}{\ell}\bar{\phi}\phi\right), \\
\bar{\epsilon}\epsilon \cdot \mathcal{L}_{\mathrm{YM}} &= \delta_{\bar{\epsilon}}\delta_\epsilon\mathrm{Tr}\left(\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma\right),
\end{aligned} \tag{3.24}
$$

but $\mathcal{L}_{\mathrm{FI}}$ cannot.

More extended supersymmetry. By combining a vectormultiplet with an adjoint chiral multiplet we expect to get a gauge theory with more extended supersymmetry. In the following we will write the Lagrangians for this extended multiplet using a new set of fields,

$$
\hat{\phi} = \sqrt{2}\phi, \quad \hat{\psi} = -i\sqrt{2}\psi, \quad \hat{\bar{\psi}} = -i\sqrt{2}\bar{\psi}, \quad \hat{F} = \sqrt{2}F, \quad \hat{D} = D + i[\phi,\bar{\phi}]. \tag{3.25}
$$

To get the Lagrangian with $\mathcal{N} = 4$ extended supersymmetry, it turns out one has to take a linear combination of YM, matter and CS terms. The total Lagrangian becomes (hereafter the hats for the new set of fields are omitted),

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YM}} &+ \mathcal{L}_{\mathrm{mat}} - \frac{1}{2\ell}\mathcal{L}_{\mathrm{CS}} \\
= \mathrm{Tr}\bigg(&\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\ell\sqrt{g}}\varepsilon^{\mu\nu\lambda}\bigg(A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda\bigg) + \frac{1}{2}D^2 + \frac{1}{2}\bar{F}F \\
&+ \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}D_\mu\bar{\phi}D^\mu\phi + \frac{1}{2\ell^2}(\sigma^2 + \bar{\phi}\phi) - \frac{1}{2}[\sigma,\phi][\sigma,\bar{\phi}] + \frac{1}{8}[\phi,\bar{\phi}]^2 + \frac{i}{2\ell}\sigma[\phi,\bar{\phi}] \\
&+ \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar{\psi}\gamma^\mu D_\mu\psi + \frac{1}{4\ell}(\bar{\lambda}\lambda + \bar{\psi}\psi) \\
&+ \frac{i}{2}\bar{\lambda}[\sigma,\lambda] - \frac{i}{2}\bar{\psi}[\sigma,\psi] + \frac{1}{2}\bar{\psi}[\phi,\lambda] - \frac{1}{2}\bar{\lambda}[\bar{\phi},\psi]\bigg).
\end{aligned}
\tag{3.26}
$$

This Lagrangian has an $SO(4) \simeq SU(2) \times SU(2)$ enlarged R-symmetry and therefore $\mathcal{N} = 4$ supersymmetry. The two $SU(2)$'s act on the triplet of scalars $(\sigma, \phi, \bar{\phi})$ and the auxiliary fields $(D, F, \bar{F})$ respectively.

By combining the complex mass term (F-term) for the adjoint chiral field

$$
\mathcal{L}_F + \mathcal{L}_{\bar{F}} \;=\; \mathrm{Tr}\bigg(F\phi + \frac{1}{2}\psi\psi + \bar{F}\bar{\phi} + \frac{1}{2}\bar{\psi}\bar{\psi}\bigg)
\tag{3.27}
$$

with the CS action for the vectormultiplet one obtains an action

$$
\mathcal{L}_{\mathrm{CS}} + \mathcal{L}_F + \mathcal{L}_{\bar{F}} = \mathrm{Tr}\bigg(\frac{1}{\sqrt{g}}\varepsilon^{\mu\nu\lambda}\bigg(A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda\bigg) - \bar{\lambda}\lambda + \frac{1}{2}\psi\psi + \frac{1}{2}\bar{\psi}\bar{\psi} \\
+ 2D\sigma + F\phi + \bar{F}\bar{\phi} - i[\phi,\bar{\phi}]\sigma\bigg),
\tag{3.28}
$$

which has an $SO(3)$ extended R-symmetry which rotates the scalars, auxiliary fields and three of the four Majorana fermions simultaneously, and therefore $\mathcal{N} = 3$ supersymmetry.

The above observations are reminiscent of the well-known fact that, on 3D flat space-time, gauge theories with YM and CS terms can have at most $\mathcal{N} = 3$ supersymmetry.

4 Localization

Here we discuss the computation of partition function of the supersymmetric gauge theories on S^3 based on the localization principle. As has been explained in [5, 8], the path integral localizes onto the saddle points characterized by

$$
A_\mu = \phi = 0, \quad \sigma = -\ell D = \text{constant}.
\tag{4.1}
$$

So the calculation of partition function amounts to evaluating the one-loop determinant at each saddle point and then integrating over the space of saddle points parametrized by σ.

The one-loop determinant of vectormultiplets was worked out thoroughly in [5], so we focus on the chiral matter multiplets with arbitrary R-charge assignments. We will

focus on the determinant of a single chiral multiplet which has a unit charge under a U(1) gauge symmetry, as the generalization to arbitrary gauge groups and representations is straightforward.

The matter kinetic Lagrangian (3.21) of the previous section was shown to be a total superderivative, so that we can use it as the regulator Lagrangian. This choice of regulator is slightly different from the one in [5], and it simplifies the computation of one-loop determinant a lot since the $SU(2) \times SU(2)$ isometry of S^3 remains unbroken.

The matter kinetic term on the saddle points is $\mathcal{L}_\phi + \mathcal{L}_\psi$, with

$$\mathcal{L}_\phi = g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\phi + \bar{\phi}\sigma^2\phi + \frac{2i(q-1)}{\ell}\bar{\phi}\sigma\phi + \frac{q(2-q)}{\ell^2}\bar{\phi}\phi,$$
$$\mathcal{L}_\psi = -i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\bar{\psi}\sigma\psi - \frac{q-2}{\ell}\bar{\psi}\psi. \tag{4.2}$$

For round S^3 of radius ℓ we substitute $g^{\mu\nu} = \ell^{-2}\mathcal{R}^{a\mu}\mathcal{R}^{a\nu}$, $e^{a\mu} = \ell^{-1}\mathcal{R}^{a\mu}$ and get

$$\mathcal{L}_\phi = \ell^{-2}\left\{\mathcal{R}^a\bar{\phi}\cdot\mathcal{R}^a\phi - \bar{\phi}(q-i\ell\sigma)(q-2-i\ell\sigma)\phi\right\},$$
$$\mathcal{L}_\psi = \ell^{-1}\bar{\psi}\left\{-i\gamma^a\mathcal{R}^a + i\ell\sigma + 2 - q\right\}\psi. \tag{4.3}$$

One can rewrite them in terms of orbital and spin angular momentum operators $J^a \equiv \frac{1}{2i}\mathcal{R}^a$ and $S^a \equiv \frac{1}{2}\gamma^a$ satisfying standard $SU(2)$ commutation relations. Then, all we need to do is to work out the spectrum of the Laplace and Dirac operators Δ_ϕ and Δ_ψ,

$$\Delta_\phi = \frac{1}{\ell^2}\left\{4J^aJ^a - (q-i\ell\sigma)(q-2-i\ell\sigma)\right\},$$
$$\Delta_\psi = \frac{1}{\ell}\left\{4J^aS^a + i\ell\sigma + 2 - q\right\}. \tag{4.4}$$

These operators are diagonalized by spherical harmonics on S^3. First, scalar spherical harmonics sit in the spin (j,j) representations of $SU(2)_{\mathcal{L}} \times SU(2)_{\mathcal{R}}$, with $2j \in \mathbb{Z}_{\geq 0}$. One therefore gets the eigenvalues

$$\Delta_\phi = \ell^{-2}\Big(4j(j+1) - (q-i\ell\sigma)(q-2-i\ell\sigma)\Big)$$
$$= \ell^{-2}(2j+2+i\ell\sigma-q)(2j-i\ell\sigma+q) \tag{4.5}$$

with multiplicity $(2j+1)^2$. Second, spinor spherical harmonics sit in the spin $(j,j,\frac{1}{2})$ representation of $SU(2)_{\mathcal{L}} \times SU(2)_{\mathcal{R}} \times SU(2)_S$, which can be reorganized into the direct sum $(j, j+\frac{1}{2}) \oplus (j, j-\frac{1}{2})$ of the subgroup $SU(2)_{\mathcal{L}} \times SU(2)_{\mathcal{R}+S}$. The Dirac operator can be easily shown to take values

$$\Delta_\psi = \ell^{-1}\left(2\left(j\pm\frac{1}{2}\right)\left(j\pm\frac{1}{2}+1\right) - 2j(j+1) - \frac{3}{2} + i\ell\sigma + 2 - q\right)$$
$$= \ell^{-1}(2j+2+i\ell\sigma-q), \quad \ell^{-1}(-2j+i\ell\sigma-q) \tag{4.6}$$

on these representations, and the multiplicities are $(2j+2)(2j+1)$ and $2j(2j+1)$ respectively. Denoting $n = 2j+1$, one can express the one-loop determinant as an infinite product,

$$\frac{\det\Delta_\psi}{\det\Delta_\phi} = \prod_{n>0}\left(\frac{n+1-q+i\ell\sigma}{n-1+q-i\ell\sigma}\right)^n = s_{b=1}(i-iq-\ell\sigma). \tag{4.7}$$

– 9 –

This is our main result. Here $s_b(x)$ is the double sine function which has poles at $x = i(m + \frac{1}{2})b + i(n + \frac{1}{2})b^{-1}$ ($m, n \in \mathbb{Z}_{\geq 0}$) and satisfies $s_b(x) = s_{1/b}(x) = s_b(-x)^{-1}$. It also satisfies the equality

$$s_b\left(\frac{ib}{2} - \sigma\right) s_b\left(\frac{ib}{2} + \sigma\right) = \frac{1}{2\cosh \pi b \sigma}. \tag{4.8}$$

For more detailed explanation on this function, we refer to [19, 20].

5 Integral formula for partition function

Combining the result of the previous section with those for vectormultiplet given in [5], one can write down an integral formula for partition functions of general 3D $\mathcal{N} = 2$ gauge theories.

Since the path integral generally localizes onto saddle points characterized by (4.1), the formula involves an integral over the Lie algebra of gauge group G corresponding to the constant mode of σ. Using gauge symmetry, one can reduce the integration domain further to its Cartan part at the cost of having an extra Vandermonde determinant factor in the integrand. We introduce a dimensionless quantity $\hat{\sigma} \equiv \ell\sigma$, and write it as a linear combination of Cartan generators H_i,

$$\hat{\sigma} = \sum_{i=1}^{r} \hat{\sigma}_i H_i, \tag{5.1}$$

where r is the rank of the gauge group.

For non-abelian gauge groups, there is a nontrivial integrand arising from the Vandermonde and one-loop determinants. The contribution of vectormultiplets is given by [5]

$$Z_{\text{vec}} = \frac{1}{|\mathcal{W}|} \int d^r \hat{\sigma} \prod_{\alpha \in \Delta_+} \left(2\sinh(\pi\alpha_i\hat{\sigma}_i)\right)^2. \tag{5.2}$$

Here α labels the positive roots, the corresponding generator E_α satisfies $[H_i, E_\alpha] = \alpha_i E_\alpha$ and $|\mathcal{W}|$ denotes the order of the Weyl group.

Matter chiral multiplets contribute a one-loop determinant which is a generalization of (4.7). Assume they have R-charge q and belong to the representation R of the gauge group. Then for each weight vector ρ of R, there is a matter chiral multiplet labelled by ρ carrying the H_i-charge ρ_i, and its conjugate anti-chiral multiplet with the H_i-charge $-\rho_i$. Collecting their one-loop determinants we obtain

$$\prod_{\rho \in R} s_{b=1}(i - iq - \rho_i\hat{\sigma}_i) \tag{5.3}$$

from a chiral multiplet belonging to R. As a special case, when $q = \frac{1}{2}$ and $R = \mathrm{r} \oplus \bar{\mathrm{r}}$ the matter one-loop determinant becomes [5]

$$\prod_{\rho \in \mathrm{r}} s_{b=1}\left(\frac{i}{2} - \rho_i\hat{\sigma}_i\right) \cdot s_{b=1}\left(\frac{i}{2} + \rho_i\hat{\sigma}_i\right) = \prod_{\rho \in \mathrm{r}} \frac{1}{2\cosh \pi\rho_i\hat{\sigma}_i}. \tag{5.4}$$

The Chern-Simons and FI terms have nonzero classical values at the saddle points. In the standard convention, the Chern-Simons Lagrangian (3.15) appears in the Euclidean action multiplied by $\frac{ik}{4\pi}$, where k is the Chern-Simons coupling. Also, the trace in (3.15) is that of adjoint representation divided by twice the dual Coxeter number. Its nonzero classical value shifts the integrand by a factor

$$\exp\left(\frac{ik}{4\pi}\int d^3\xi\sqrt{g}\mathcal{L}_{\rm CS}\right) \;=\; \exp(-ik\pi\hat{\sigma}_i\hat{\sigma}_i). \tag{5.5}$$

The FI term (3.23) in our convention appears in the action multiplied by a factor $\frac{i\zeta}{\pi\ell}$, where ζ is the FI coupling. For U(1) gauge theory, the shift of the integrand due to the FI coupling is therefore

$$\exp\left(-\frac{i\zeta}{\pi\ell}\int d^3\xi\sqrt{g}\mathcal{L}_{\rm FI}\right) \;=\; \exp(4\pi i\zeta\hat{\sigma}). \tag{5.6}$$

It is now straightforward to write down the partition function for any $\mathcal{N}=2$ gauge theories using the building blocks given above. For example, the partition function for U(N) $\mathcal{N}=2$ Chern-Simons theory at level k coupled to N_f fundamental and $\bar{N}_f$ anti-fundamental chiral matters of R-charge q is,

$$Z = \frac{1}{N!}\int d^N\sigma \prod_{j=1}^{N} e^{-i\pi k\sigma_j^2}\prod_{i<j}^{N}(2\sinh\pi(\sigma_i-\sigma_j))^2$$
$$\cdot\left(\prod_{j=1}^{N}s_{b=1}(i-iq-\sigma_j)\right)^{N_f}\left(\prod_{j=1}^{N}s_{b=1}(i-iq+\sigma_j)\right)^{\bar{N}_f}. \tag{5.7}$$

6 An application

As a simple application of our result, we consider here an $\mathcal{N}=4$ SQED with two electron hypermultiplets, called $T[\mathrm{SU}(2)]$, which is long known to be self-mirror [21, 22]. In terms of $\mathcal{N}=2$ supermultiplets, the theory consists of one abelian vectormultiplet V, one neutral chiral multiplet ϕ and four chiral multiplets $(q_1,q_2,\tilde{q}^1,\tilde{q}^2)$ with charges $(+1,+1,-1,-1)$. The fields $q_i,\tilde{q}^i$ have the R-charge $1/2$ whereas ϕ has R-charge 1. In addition to the kinetic Lagrangians, a superpotential $W=\sqrt{2}\tilde{q}^i\phi q_i$ needs to be introduced to get $\mathcal{N}=4$ supersymmetry.

The theory has an SU(2) flavor symmetry which rotates q_i and $\tilde{q}^i$ as doublets, and the matter fields $(q_1,q_2,\tilde{q}^1,\tilde{q}^2)$ have charges $(+1,-1,-1,+1)$ under its U(1) subgroup. One can turn on the mass for the charged chiral matters via gauging this U(1) flavor symmetry, so that the mass parameter μ appears as the expectation value of a background vectormultiplet scalar. Under the mirror symmetry, the mass parameter μ is mapped to the FI parameter ζ and vice versa. In a recent work [14], further mass deformation of this model has been considered by gauging the U(1) symmetry under which $q_i,\tilde{q}^i$ all have charge -1 and ϕ has charge 2. Since this symmetry is identified with the difference of two U(1)'s in the SU(2) × SU(2) R-symmetry, the corresponding mass parameter m is sign-flipped under the mirror symmetry [23].

The partition function of mass-deformed theory on S^3 is thus given by an integral over the scalar σ in the vectormultiplet V (here we set $\ell = 1$ for simplicity),

$$Z(m, \zeta, \mu) = \int d\sigma e^{4\pi i \zeta \sigma} s_{b=1}(-m) \cdot s_{b=1}\left(\frac{i}{2} - \sigma - \mu + \frac{m}{2}\right) s_{b=1}\left(\frac{i}{2} - \sigma + \mu + \frac{m}{2}\right)$$
$$\cdot s_{b=1}\left(\frac{i}{2} + \sigma + \mu + \frac{m}{2}\right) s_{b=1}\left(\frac{i}{2} + \sigma - \mu + \frac{m}{2}\right). \quad (6.1)$$

By using the following formula given in the appendix of [20],

$$\int dx e^{-2\pi i p x} s_b\left(x + \frac{m}{2} + \frac{iQ}{4}\right) s_b\left(-x + \frac{m}{2} + \frac{iQ}{4}\right)$$
$$= s_b(m) s_b\left(p - \frac{m}{2} + \frac{iQ}{4}\right) s_b\left(-p - \frac{m}{2} + \frac{iQ}{4}\right), \quad (6.2)$$

one can easily show that the partition function satisfies $Z(m, \zeta, \mu) = Z(-m, \mu, \zeta)$, namely it transforms as expected under mirror symmetry.

The mass-deformed $T[SU(2)]$ theory is known to describe the S-duality domain wall of 4D $\mathcal{N} = 2^*$ SYM theory [13]. As has been observed in [14], the partition function $Z(m, \zeta, \mu)$ coincides with the S-duality transformation coefficient of Virasoro torus one-point conformal blocks.

Note added.   When our paper was ready for submission to the arXiv, there appeared a paper by D.L. Jafferis [24] which has significant overlap with ours.

Acknowledgments

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