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Spontaneous breaking of SU(3) to finite family symmetries — a pedestrian's approach

Christoph Luhn

School of Physics and Astronomy, University of Southampton, Highfield, Southampton, SO17 1BJ, United Kingdom

E-mail: christoph.luhn@soton.ac.uk

Abstract: Non-Abelian discrete family symmetries play a pivotal role in the formulation of models with tri-bimaximal lepton mixing. We discuss how to obtain symmetries such as $\mathcal{A}_4$, $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ and $\Delta(27)$ from an underlying SU(3) gauge symmetry. Higher irreducible representations are required to achieve the spontaneous breaking of the continuous group. We present methods of identifying the required vacuum alignments and discuss in detail the symmetry breaking potentials.

Keywords: Spontaneous Symmetry Breaking, Beyond Standard Model, Discrete and Finite Symmetries

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JHEP03(2011)108

Contents

1 Introduction

The Standard Model of particle physics provides a successful and accurate description of Nature as has been proved in countless experiments over the last few decades. Yet, the observation of neutrino oscillations demands its extension to include massive neutrinos. Due to our ignorance of the absolute neutrino mass scale, the structure of the neutrino mass spectrum is still in the dark with hierarchical and quasi-degenerate scenarios being equally well conceivable. A better clue towards understanding the underlying physics of flavor is given by the observed mixing pattern in the lepton sector. While the quarks mix with three small angles, the lepton mixing features one small and two large angles. Even more intriguing is the fact that the best fit values [1, 2] for the lepton mixing angles are remarkably close to the so-called tri-bimaximal pattern [3, 4],

$$\begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \tag{1.1}$$

corresponding to $\theta_{12} = 35.26^\circ$, $\theta_{23} = 45^\circ$, $\theta_{13} = 0^\circ$. This peculiar mixing pattern suggests a non-Abelian discrete family symmetry $\mathcal{G}$ lurking behind the flavor structure of the chiral fermions. The virtue of imposing such a non-Abelian symmetry is that the irreducible representations (irreps) of $\mathcal{G}$ allow one to collect the families of chiral fermions into multiplets. With three known families it is natural to investigate finite groups with triplet and/or doublet representations. These are found among the finite subgroups of SU(3), SU(2) and SO(3), with popular candidates being $\mathcal{A}_4$ [5], $\mathcal{S}_4$ [6] and $\Delta(27)$ [7–9]. Adopting

JHEP03(2011)108

their preferred finite group, many authors have constructed even more models of flavor, all aiming to explain the remarkable tri-bimaximal mixing pattern. There are two classes of such models: direct and indirect ones [10]. In the former type of models tri-bimaximal lepton mixing is directly linked to the residual symmetries of the neutrino and charged lepton sectors after the original family symmetry gets broken to different subgroups in different sectors. In contrast, in indirect models the vacuum expectation values (VEVs) of the flavons, i.e. the symmetry breaking fields, break the family symmetry completely in both sectors. However, the particular form of the flavon VEVs gives rise tri-bimaximal mixing by effectively restoring the required symmetries of the neutrino and charged lepton mass matrices. For an extensive list of references of family symmetry models we refer the reader to the review by Altarelli and Feruglio [11].

In this paper we wish to address questions relating to a possible gauge origin of the non-Abelian discrete family symmetry. A symmetry $\mathcal{G}$ is called a *discrete gauge symmetry* if it originates from a spontaneously broken (continuous) gauge symmetry G. The assumption of a gauge origin has the advantage that the remnant discrete symmetry $\mathcal{G}$ is protected against violations by quantum gravity effects [12]. Imposing a discrete symmetry without such a gauge origin would thus strictly speaking be meaningless as it would not remain a symmetry of the whole theory once gravity is switched on. Independently of the above it should be interesting and instructive to study the connection between a discrete family symmetry and a continuous one for three reasons. Firstly, the set of free parameters that is used to formulate a model based on a small discrete family symmetry is often rather large. Assuming an underlying bigger or even continuous symmetry allows one to relate some of these parameters so that the theory becomes more predictive. Related to this, a second benefit of having an underlying SU(3) symmetry concerns the set of flavon fields. While these are completely independent in the framework of the discrete symmetry $\mathcal{G}$, one could envisage a scenario where some of them originate form the same SU(3) representation which in turn would relate such subsets of flavons. Thirdly, some of the earlier models of fermion masses and mixings adopt an SU(3) family symmetry, see e.g. [13]. With discrete family symmetries dominating the recent activities is should be interesting to go back and reexamine some of these earlier models and try to connect them with the understanding we have today. At this point it is worth emphasizing that it is actually irrelevant for the purpose of our study whether the SU(3) family symmetry is local or global, though, for the sake of definiteness, we will in the following assume a gauged SU(3) family symmetry.

The idea of gauging a discrete symmetry has been applied to Abelian symmetries [14–17] and is well established and understood. Assuming a gauged U(1) symmetry with integer charge normalization, one obtains a residual Z_N symmetry when a field ϕ with U(1) charge N develops a VEV via a potential of the form

$$V = -m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \ . \tag{1.2}$$

The resulting would-be Goldstone boson of the spontaneously broken U(1) symmetry is then eaten by the U(1) gauge boson's longitudinal polarization.

The situation is much more involved in the non-Abelian case since higher representations of the continuous gauge group G are required to achieve the desired breaking. The

JHEP03(2011)108

breaking patters of $G = \mathrm{SO}(3)$ using low-dimensional representations have been investigated in [18–22]. In the context of flavor models, the most interesting result of these studies is that the tetrahedral group $\mathcal{A}_4$ can originate from an SO(3) symmetric potential involving only the **7** representation. The free parameters of the potential can be chosen without fine-tuning so that the potential is minimized by a VEV which breaks SO(3) but not $\mathcal{A}_4$.

It is the purpose of this paper to similarly examine the case of $G = \mathrm{SU}(3)$. A first attempt in this direction has been undertaken in [23] where the SU(3) representations **3**, **6** and **8** have been considered to achieve the breaking of the continuous symmetry. It is shown there that these small representations are insufficient to generate a remnant discrete symmetry with triplet representations like e.g. $\mathcal{A}_4$. Furthermore, the study stops short of discussing the potential and the relevant order parameters that determine the breaking of SU(3) to the discrete symmetry $\mathcal{G}$. In the present work we show how to overcome these shortcomings and obtain — for the first time — non-Abelian discrete symmetries with triplet representations explicitly from the spontaneous breakdown of SU(3). To this end we need to go beyond [23] by (a) including also higher representations of SU(3) in our discussion and (b) scrutinizing the relevant symmetry breaking potential.

The paper is structured as follows. In section 2 we present a simple way to identify the embedding of a given finite group $\mathcal{G}$ in SU(3). Having worked out the decomposition of SU(3) representations under $\mathcal{G}$, we discuss the procedure of finding the $\mathcal{G}$ singlet directions of the appropriate SU(3) irreps in section 3. Along the way we also comment on the choice of basis of the finite subgroup. The maximal subgroup that is left invariant by a VEV in a particular singlet direction is determined in appendix A. Section 4 is devoted to the study of several symmetry breaking potentials which can give rise to $\mathcal{A}_4$, $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ and $\Delta(27)$, respectively. Finally, we conclude in section 5.

2 Decomposition of SU(3) irreps

In order to break SU(3) spontaneously down to a finite subgroup $\mathcal{G}$ it is necessary to find those SU(3) irreps which contain a singlet of $\mathcal{G}$ in their decomposition. A simple method for obtaining the full decompositions is based on the observation that all SU(3) irreps $\boldsymbol{\rho}$ can be successively generated from the fundamental **3**. The complex conjugate representations $\overline{\boldsymbol{\rho}}$ are directly derived from $\boldsymbol{\rho}$. Table 1 lists the relevant tensor products that can be used to find the irreps up to dimension 27. The last number in each line shows the new irrep that is generated from multiplying already known ones.

Identifying the triplet of SU(3) with a faithful representation of $\mathcal{G}$, one can successively work out the decomposition of all $\boldsymbol{\rho}$ by comparing the SU(3) tensor products with the Kronecker products of $\mathcal{G}$. This method is best illustrated for explicit examples. Let us consider the case of the tetrahedral group $\mathcal{A}_4 = \Delta(12)$ as well as $\Delta(27)$.

(i) $\mathcal{A}_4 = \Delta(12)$ has four irreps $\mathbf{1}, \mathbf{1'}, \overline{\mathbf{1'}}$ and the real **3** which satisfy the following multiplication rules.

JHEP03(2011)108

some SU(3) tensor products
$\mathbf{3}\otimes\mathbf{3} = \overline{\mathbf{3}}+\mathbf{6}$
$\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{1}+\mathbf{8}$
$\mathbf{6}\otimes\mathbf{3} = \mathbf{8}+\mathbf{10}$
$\mathbf{6}\otimes\overline{\mathbf{3}} = \mathbf{3}+\mathbf{15}$
$\mathbf{10}\otimes\mathbf{3} = \mathbf{15}+\mathbf{15}'$
$\overline{\mathbf{10}}\otimes\mathbf{3} = \overline{\mathbf{6}}+\mathbf{24}$
$\overline{\mathbf{10}}\otimes\overline{\mathbf{6}} = \mathbf{15}+\mathbf{24}+\mathbf{21}$
$\mathbf{6}\otimes\overline{\mathbf{6}} = \mathbf{1}+\mathbf{8}+\mathbf{27}$

Table 1. A list of SU(3) tensor products which can be used to successively obtain the SU(3) irreps up to dimension 27.

$\mathcal{A}_4$ Kronecker products
$\mathbf{1}'\otimes\mathbf{1}' = \overline{\mathbf{1}'}$
$\mathbf{1}'\otimes\overline{\mathbf{1}'} = \mathbf{1}$
$\mathbf{3}\otimes\mathbf{1}' = \mathbf{3}$
$\mathbf{3}\otimes\mathbf{3} = \mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'}+2\cdot\mathbf{3}$

As the $\mathcal{A}_4$ triplet is real, we can identify it with both the $\mathbf{3}$ as well as the $\overline{\mathbf{3}}$ of SU(3). Comparing the products of $\mathbf{3}\otimes\mathbf{3}$ we directly find the decomposition of the sextet, $\mathbf{6}\to\mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'}+\mathbf{3}$. The decomposition of the octet is obtained similarly from $\mathbf{3}\otimes\overline{\mathbf{3}}$, leading to $\mathbf{8}\to\mathbf{1}'+\overline{\mathbf{1}'}+2\cdot\mathbf{3}$. For the $\mathbf{10}$ we consider the SU(3) tensor product $\mathbf{6}\otimes\mathbf{3}=\mathbf{8}+\mathbf{10}$. Plugging in the just determined $\mathcal{A}_4$ decompositions we find

$$\mathbf{10}\to\underbrace{(\mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'}+\mathbf{3})}_{\mathbf{6}}\otimes\mathbf{3}-\underbrace{(\mathbf{1}'+\overline{\mathbf{1}'}+2\cdot\mathbf{3})}_{\mathbf{8}} = \mathbf{1}+3\cdot\mathbf{3}\,,$$

where, in the last step, we have used the $\mathcal{A}_4$ Kronecker products. Continuation of these simple calculations yields the decomposition of any SU(3) irrep. We list the results up to the $\mathbf{27}$, cf. also [24].

SU(3) $\supset$ $\mathcal{A}_4$
$\mathbf{3} = \mathbf{3}$
$\mathbf{6} = \mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'}+\mathbf{3}$
$\mathbf{8} = \mathbf{1}'+\overline{\mathbf{1}'}+2\cdot\mathbf{3}$
$\mathbf{10} = \mathbf{1}+3\cdot\mathbf{3}$
$\mathbf{15} = \mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'}+4\cdot\mathbf{3}$
$\mathbf{15}' = 2\cdot(\mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'})+3\cdot\mathbf{3}$
$\mathbf{21} = \mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'}+6\cdot\mathbf{3}$
$\mathbf{24} = 2\cdot(\mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'})+6\cdot\mathbf{3}$
$\mathbf{27} = 3\cdot(\mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'})+6\cdot\mathbf{3}$

JHEP03(2011)108

This shows that the irreps $\mathbf{6}$, $\mathbf{10}$, $\mathbf{15}$, $\mathbf{15'}$, $\mathbf{21}$, $\mathbf{24}$ and $\mathbf{27}$ contain at least one singlet of $\mathcal{A}_4$ and can thus, in principle, be used to break SU(3) spontaneously down to $\mathcal{A}_4$ or a group that contains $\mathcal{A}_4$ as a subgroup.

(ii) $\Delta(27)$ has nine one-dimensional irreps

$$\begin{array}{lllll} \mathbf{1} = \mathbf{1_{0,0}}\,, & \mathbf{1_1} = \mathbf{1_{0,1}}\,, & \mathbf{1_3} = \mathbf{1_{1,0}}\,, & \mathbf{1_5} = \mathbf{1_{1,1}}\,, & \mathbf{1_7} = \mathbf{1_{1,2}}\,, \\ & \mathbf{1_2} = \overline{\mathbf{1}}_\mathbf{1} = \mathbf{1_{0,2}}\,, & \mathbf{1_4} = \overline{\mathbf{1}}_\mathbf{3} = \mathbf{1_{2,0}}\,, & \mathbf{1_6} = \overline{\mathbf{1}}_\mathbf{5} = \mathbf{1_{2,2}}\,, & \mathbf{1_8} = \overline{\mathbf{1}}_\mathbf{7} = \mathbf{1_{2,1}}\,, \end{array}$$

as well as a triplet $\mathbf{3}$ and its complex conjugate $\overline{\mathbf{3}}$. The Kronecker products read as follows.

$\Delta(27)$ Kronecker products
$\mathbf{1_{r,s}} \otimes \mathbf{1_{r',s'}} = \mathbf{1_{r+r',s+s'}}$
$\mathbf{3} \otimes \mathbf{1_j} = \mathbf{3}$
$\overline{\mathbf{3}} \otimes \mathbf{1_j} = \overline{\mathbf{3}}$
$\mathbf{3} \otimes \mathbf{3} = 3 \cdot \overline{\mathbf{3}}$
$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} + \sum_{j=1}^{8} \mathbf{1_j}$

Here $r, s = 0, 1, 2$ and the sums $r + r'$ and $s + s'$ are taken modulo 3. Without loss of generality we can identify the $\mathbf{3}$ of SU(3) with the $\mathbf{3}$ of $\Delta(27)$. Then also their complex conjugates automatically correspond to each another. Comparing the product $\mathbf{3} \otimes \mathbf{3}$ gives the decomposition of the sextet, $\mathbf{6} \to 2{\cdot}\overline{\mathbf{3}}$. From $\mathbf{3} \otimes \overline{\mathbf{3}}$ we derive the decomposition of the octet, $\mathbf{8} \to \sum_{j=1}^{8} \mathbf{1_j}$. The $\mathbf{10}$ is again obtained from the SU(3) tensor product $\mathbf{6} \otimes \mathbf{3} = \mathbf{8} + \mathbf{10}$.

$$\mathbf{10} \to \underbrace{(2{\cdot}\overline{\mathbf{3}})}_{\mathbf{6}} \otimes \mathbf{3} - \underbrace{\sum_{j=1}^{8} \mathbf{1_j}}_{\mathbf{8}} = 2{\cdot}\mathbf{1} + \sum_{j=1}^{8} \mathbf{1_j}\,.$$

Analogously we get the decomposition for any other SU(3) irrep showing that, for irreps up to dimension 27, only the $\mathbf{10}$ and the $\mathbf{27}$ contain singlets of $\Delta(27)$, cf. [24].

SU(3) $\supset$ $\Delta(27)$
$\mathbf{3} = \mathbf{3}$
$\mathbf{6} = 2 \cdot \overline{\mathbf{3}}$
$\mathbf{8} = \sum_{j=1}^{8} \mathbf{1_j}$
$\mathbf{10} = 2 \cdot \mathbf{1} + \sum_{j=1}^{8} \mathbf{1_j}$
$\mathbf{15} = 5 \cdot \mathbf{3}$
$\mathbf{15'} = 5 \cdot \mathbf{3}$
$\mathbf{21} = 7 \cdot \mathbf{3}$
$\mathbf{24} = 8 \cdot \mathbf{3}$
$\mathbf{27} = 3 \cdot (\mathbf{1} + \sum_{j=1}^{8} \mathbf{1_j})$

JHEP03(2011)108

finite subgroup $\mathcal{G}$	**3**	**6**	**8**	**10**	**15**	**15′**	**21**	**24**	**27**
$\mathcal{A}_4 = \Delta(12)$	–	1	–	1	1	2	1	2	3
$\Delta(27)$	–	–	–	2	–	–	–	–	3
$\mathcal{S}_4 = \Delta(24)$	–	1	–	–	–	2	–	1	2
$\Delta(54)$	–	–	–	–	–	–	–	–	3
$\mathcal{Z}_7 \rtimes \mathcal{Z}_3 = \mathcal{T}_7$	–	–	–	1	1	1	1	1	1
$\mathcal{PSL}_2(7) = \Sigma(168)$	–	–	–	–	–	1	–	–	–

Table 2. The number of singlets of $\mathcal{G}$ within each SU(3) irrep for various finite subgroups.

The same procedure can be repeated for any other finite subgroup $\mathcal{G}$ of SU(3) [25–36]. This way it is possible to identify those irreps which can potentially break SU(3) down to $\mathcal{G}$. Table 2 summarizes these results by listing the number of singlets of $\mathcal{G}$ within each SU(3) irrep for various finite subgroups.

3 Singlet directions

In the previous section we have determined the SU(3) irreps that contain singlets of the finite subgroup $\mathcal{G}$. The next step is to find the directions of these representation which correspond to the singlets. It is worth emphasizing that such singlet VEVs may or may not break SU(3) directly to the desired finite group $\mathcal{G}$. In the latter case, a bigger subgroup of SU(3) is left intact and the breaking to $\mathcal{G}$ can be achieved *sequentially* by adding a second irrep with an appropriate singlet VEV.[1] Focusing on the smallest irreps we confine ourselves to the **6**, **10** and **15** in the following. We construct them using the fundamental triplet.

The three orthonormal states of an SU(3) triplet are denoted by $|\, i\,\rangle$, with $i = 1, 2, 3$. Then we can express a general triplet as a linear combination

$$\sum_{i=1}^{3} \varphi_i |\, i\,\rangle \,, \tag{3.1}$$

with φ_i being the components of the state.

The **6** of SU(3) corresponds to the symmetric product of two triplets. Using the compact notation $|\, ij\,\rangle \equiv |\, i\,\rangle \otimes |\, j\,\rangle$ we can define six orthonormal states $|\, \alpha\,\}$, where $\alpha =$

[1] An example of such a sequential breaking is discussed in appendix A. There we will show that $\mathcal{A}_4$ cannot be obtained directly from the **6** or **10** alone but only their combination.

JHEP03(2011)108

$1, \ldots, 6$, as follows

$$\begin{aligned} |\,1\,\} &= |\,11\,\rangle\,, \qquad |\,2\,\} = |\,22\,\rangle\,, \qquad |\,3\,\} = |\,33\,\rangle\,, \\ |\,4\,\} &= \frac{1}{\sqrt{2}}(|\,12\,\rangle + |\,21\,\rangle)\,, \\ |\,5\,\} &= \frac{1}{\sqrt{2}}(|\,23\,\rangle + |\,32\,\rangle)\,, \\ |\,6\,\} &= \frac{1}{\sqrt{2}}(|\,31\,\rangle + |\,13\,\rangle)\,. \end{aligned} \tag{3.2}$$

A general sextet state is then given by

$$\sum_{\alpha=1}^{6} \chi_\alpha |\,\alpha\,\} = \sum_{i,j=1}^{3} T_{ij}\,|\,ij\,\rangle\,, \tag{3.3}$$

where χ_α denotes the six independent components of the sextet state[2] and T_{ij} is the corresponding symmetric tensor. T_{ij} and χ_α are related via eqs. (3.2), (3.3). For example, $T_{11} = \chi_1$ and $T_{12} = T_{21} = \frac{1}{\sqrt{2}}\chi_2$.

The **10** of SU(3) corresponds to the symmetric product of three triplets. We can define its orthonormal basis $|\,a \succ$, with $a = 1, \ldots, 10$, by

$$\begin{aligned} |\,1 \succ &= |\,111\,\rangle\,, \qquad |\,2 \succ = |\,222\,\rangle\,, \qquad |\,3 \succ = |\,333\,\rangle\,, \\ |\,4 \succ &= \frac{1}{\sqrt{3}}(|\,112\,\rangle + |\,121\,\rangle + |\,211\,\rangle)\,, \qquad |\,5 \succ = \frac{1}{\sqrt{3}}(|\,113\,\rangle + |\,131\,\rangle + |\,311\,\rangle)\,, \\ |\,6 \succ &= \frac{1}{\sqrt{3}}(|\,221\,\rangle + |\,212\,\rangle + |\,122\,\rangle)\,, \qquad |\,7 \succ = \frac{1}{\sqrt{3}}(|\,223\,\rangle + |\,232\,\rangle + |\,322\,\rangle)\,, \\ |\,8 \succ &= \frac{1}{\sqrt{3}}(|\,331\,\rangle + |\,313\,\rangle + |\,133\,\rangle)\,, \qquad |\,9 \succ = \frac{1}{\sqrt{3}}(|\,332\,\rangle + |\,323\,\rangle + |\,233\,\rangle)\,, \\ |\,10 \succ &= \frac{1}{\sqrt{6}}(|\,123\,\rangle + |\,231\,\rangle + |\,312\,\rangle + |\,321\,\rangle + |\,213\,\rangle + |\,132\,\rangle)\,. \end{aligned} \tag{3.4}$$

Again, the most general state reads

$$\sum_{a=1}^{10} \psi_a |\,a \succ \; = \sum_{i,j,k=1}^{3} T_{ijk}\,|\,ijk\,\rangle\,, \tag{3.5}$$

with eqs. (3.4), (3.5) relating ψ_a and T_{ijk}, e.g. $T_{112} = T_{121} = T_{211} = \frac{1}{\sqrt{3}}\psi_4$.

[2]Note that the definition of χ_α as well as the components of other higher SU(3) representations depends on the chosen basis. While the particular choice must necessarily be irrelevant, fixing the basis is crucial for doing explicit calculations like minimizing the flavon potentials in section 4.

JHEP03(2011)108

Turning to the **15** of SU(3) we define its orthonormal basis $|\, A\,)$, with $A = 1, \ldots, 15$, as

$$
\begin{aligned}
|\,1\,) &= \frac{1}{\sqrt{3}}\,(|\,11\bar{1}\,\rangle - |\,12\bar{2}\,\rangle - |\,21\bar{2}\,\rangle)\,,\\
|\,2\,) &= \frac{1}{2\sqrt{6}}\,(2\cdot|\,11\bar{1}\,\rangle + |\,12\bar{2}\,\rangle + |\,21\bar{2}\,\rangle - 3\cdot|\,13\bar{3}\,\rangle - 3\cdot|\,31\bar{3}\,\rangle)\,,\\
|\,3\,) &= \frac{1}{\sqrt{3}}\,(|\,22\bar{2}\,\rangle - |\,23\bar{3}\,\rangle - |\,32\bar{3}\,\rangle)\,,\\
|\,4\,) &= \frac{1}{2\sqrt{6}}\,(2\cdot|\,22\bar{2}\,\rangle + |\,23\bar{3}\,\rangle + |\,32\bar{3}\,\rangle - 3\cdot|\,21\bar{1}\,\rangle - 3\cdot|\,12\bar{1}\,\rangle)\,,\\
|\,5\,) &= \frac{1}{\sqrt{3}}\,(|\,33\bar{3}\,\rangle - |\,31\bar{1}\,\rangle - |\,13\bar{1}\,\rangle)\,,\\
|\,6\,) &= \frac{1}{2\sqrt{6}}\,(2\cdot|\,33\bar{3}\,\rangle + |\,31\bar{1}\,\rangle + |\,13\bar{1}\,\rangle - 3\cdot|\,32\bar{2}\,\rangle - 3\cdot|\,23\bar{2}\,\rangle)\,,\\
|\,7\,) &= |\,11\bar{2}\,\rangle\,, \qquad |\,8\,) = |\,11\bar{3}\,\rangle\,, \qquad |\,9\,) = |\,22\bar{3}\,\rangle\,,\\
|\,10\,) &= |\,22\bar{1}\,\rangle\,, \qquad |\,11\,) = |\,33\bar{1}\,\rangle\,, \qquad |\,12\,) = |\,33\bar{2}\,\rangle\,,\\
|\,13\,) &= \frac{1}{\sqrt{2}}\,(|\,12\bar{3}\,\rangle + |\,21\bar{3}\,\rangle)\,,\\
|\,14\,) &= \frac{1}{\sqrt{2}}\,(|\,23\bar{1}\,\rangle + |\,32\bar{1}\,\rangle)\,,\\
|\,15\,) &= \frac{1}{\sqrt{2}}\,(|\,31\bar{2}\,\rangle + |\,13\bar{2}\,\rangle)\,.
\end{aligned}
\tag{3.6}
$$

The most general state is now given by

$$\sum_{A=1}^{15} \Sigma_A |\, A\,) \;=\; \sum_{i,j,k=1}^{3} T^k_{ij}\, |\, ij\bar{k}\,\rangle\;. \tag{3.7}$$

The fifteen independent components Σ_A of the **15** are related to the tensor T^k_{ij} via eqs. (3.6), (3.7), e.g. $T^2_{12} = T^2_{21} = -\frac{1}{\sqrt{3}}\Sigma_1 + \frac{1}{2\sqrt{6}}\Sigma_2$. Note that T^k_{ij} is symmetric in i, j as well as traceless, i.e. $\sum_{k=1}^{3} T^k_{ik} = 0$.

Having defined the SU(3) irreps $\boldsymbol{\rho}$ in terms of triplets and anti-triplets, we now have to fix the basis of the triplet generators of the finite subgroup $\mathcal{G}$ in order to see which direction of $\boldsymbol{\rho}$ is left invariant under $\mathcal{G}$. A particularly simple basis for the triplets of $\Delta(3n^2)$, $\Delta(6n^2)$ as well as $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ is based on the matrices [10]

$$D \;=\; \begin{pmatrix} e^{i\vartheta_1} & 0 & 0 \\ 0 & e^{i\vartheta_2} & 0 \\ 0 & 0 & e^{-i(\vartheta_1+\vartheta_2)} \end{pmatrix}, \quad A \;=\; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B \;=\; -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

The generators of $\Delta(3n^2)$ are given by A and D with $\vartheta_1 = 0$ and $\vartheta_2 = 2\pi l/n$, where $l \in \mathbb{N}$. Adding the generator B yields the group $\Delta(6n^2)$. The triplet representation of $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ can be defined via A and D with $\vartheta_1 = \vartheta_2/2 = 2\pi/7$.

In the following we consider the SU(3) irreps **6**, **10** and **15** and determine the singlet directions for the respective groups as shown in table 2.

JHEP03(2011)108

- Starting with the **6** as given in eq. (3.2) we see that a state with $\chi_1 = \chi_2 = \chi_3$ and $\chi_4 = \chi_5 = \chi_6 = 0$ remains invariant under A, B and $D_{(\vartheta_1=0,\vartheta_2=\pi)}$. Therefore the singlet of $\mathcal{A}_4$ as well as $\mathcal{S}_4$ within the **6** of SU(3) points into the direction

$$\mathcal{A}_4\,,\,\mathcal{S}_4 \text{ singlet within the } \mathbf{6}: \quad \propto (1,1,1,0,0,0)^T\ . \tag{3.9}$$

- For the **10**, see eq. (3.4), we can easily identify a singlet direction which is common to all groups generated by A and D with arbitrary angles ϑ_i. It is given by $\psi_a = 0$ for $a = 1, \ldots, 9$,

$$\mathcal{A}_4\,,\,\Delta(27)\,,\,\mathcal{Z}_7 \rtimes \mathcal{Z}_3 \text{ singlet within the } \mathbf{10}: \quad \propto (0,0,0,0,0,0,0,0,0,1)^T\ . \tag{3.10}$$

  Additionally, there exists a second $\Delta(27)$ singlet defined by $\psi_1 = \psi_2 = \psi_3$ and $\psi_a = 0$ for $a = 4, \ldots, 10$,

$$\Delta(27) \text{ singlet within the } \mathbf{10}: \quad \propto (1,1,1,0,0,0,0,0,0,0)^T\ . \tag{3.11}$$

- Finally, the **15**, see eq. (3.6), contains a singlet of $\mathcal{A}_4$, given by $\Sigma_{13} = \Sigma_{14} = \Sigma_{15}$ and $\Sigma_A = 0$ for $A = 1, \ldots, 12$,

$$\mathcal{A}_4 \text{ singlet within the } \mathbf{15}: \quad \propto (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1)^T\ . \tag{3.12}$$

  The $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ singlet is obtained by setting all components of the fifteen to zero except for $\Sigma_7 = \Sigma_9 = \Sigma_{11}$

$$\mathcal{Z}_7 \rtimes \mathcal{Z}_3 \text{ singlet within the } \mathbf{15}: \quad \propto (0,0,0,0,0,0,1,0,1,0,1,0,0,0,0)^T\ . \tag{3.13}$$

It is clear from this list that not all singlet directions of a given group $\mathcal{G}$ break SU(3) uniquely down to the very group. For instance, the sextet VEV of eq. (3.9) leaves invariant $\mathcal{A}_4$ as well as $\mathcal{S}_4$. In fact, it is straightforward to see that this particular VEV respects even a continuous SO(3) symmetry: as the **6** of SU(3) decomposes into $\mathbf{1}+\mathbf{5}$ of SO(3) which in turn decomposes into $\mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'}+\mathbf{3}$ under $\mathcal{A}_4$, the singlet directions of SO(3) and $\mathcal{A}_4$ coincide. Likewise it can be shown that the VEV of the **10** of eq. (3.10) does not break SU(3) down to a discrete symmetry. These examples show that it is necessary to carefully determine the *maximal* unbroken subgroup in each case. This investigation shows that, see appendix A,

- $\mathcal{A}_4$ can be obtained from either a single **15** of SU(3) or else from a combination of a **6** and a **10**,

- $\mathcal{Z}_7 \rtimes Z_3$ can be generated using a single **15**,

- $\Delta(27)$ can be obtained from a single **10**.

JHEP03(2011)108

4 SU(3) invariant potentials

We have seen in the previous section that certain VEV configurations of SU(3) irreps can break the continuous symmetry to a finite subgroup $\mathcal{G}$. In the following we discuss that these VEVs correspond to minima of particular SU(3) invariant scalar potentials; this exemplifies how discrete non-Abelian symmetries can arise from the spontaneous breakdown of SU(3). As higher irreps seem to be more powerful to break SU(3) uniquely to a specific finite subgroup $\mathcal{G}$, we begin our discussion with the $\mathbf{15}$ which gives rise to either $\mathcal{A}_4$ or $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$. Then we consider the irrep $\mathbf{10}$ which by itself leaves the symmetry $\Delta(27)$ unbroken. Finally we also present the case of a potential that couples the $\mathbf{6}$ and the $\mathbf{10}$ to generate an $\mathcal{A}_4$ symmetry.

4.1 The case of a single 15

Let us consider a potential with a quadratic term $\mathbf{15} \otimes \overline{\mathbf{15}}$ as well as quartic interactions of type $\mathbf{15} \otimes \mathbf{15} \otimes \overline{\mathbf{15}} \otimes \overline{\mathbf{15}}$. As the symmetric product

$$(\mathbf{15} \otimes \mathbf{15})_s = \mathbf{6} + \overline{\mathbf{15}} + \overline{\mathbf{15}'} + \overline{\mathbf{24}} + \overline{\mathbf{60}} \,, \tag{4.1}$$

contains five distinct irreps, we expect five independent quartic invariants. Therefore, the relevant potential for the $\mathbf{15}$ reads

$$V_{\mathbf{15}} \;=\; -\, m_{\mathbf{15}}^2 \, \mathcal{I}_{\mathbf{15}}^{(0)} + \lambda_{\mathbf{15}} \, \mathcal{I}_{\mathbf{15}}^{(1)} + \kappa_{\mathbf{15}} \, \mathcal{I}_{\mathbf{15}}^{(2)} + \rho_{\mathbf{15}} \, \mathcal{I}_{\mathbf{15}}^{(3)} + \tau_{\mathbf{15}} \, \mathcal{I}_{\mathbf{15}}^{(4)} + \eta_{\mathbf{15}} \, \mathcal{I}_{\mathbf{15}}^{(5)} \,, \tag{4.2}$$

where the invariants are obtained from different index contractions of the tensors T^k_{ij} for the $\mathbf{15}$ and $\overline{T}^{ij}_k$ for the $\overline{\mathbf{15}}$. Summing over repeated indices we define

$$\mathcal{I}_{\mathbf{15}}^{(0)} = T^k_{ij} \, \overline{T}^{ij}_k \,, \tag{4.3}$$

$$\mathcal{I}_{\mathbf{15}}^{(1)} = T^k_{ij} \, \overline{T}^{ij}_k \, T^l_{mn} \, \overline{T}^{mn}_l \,, \tag{4.4}$$

$$\mathcal{I}_{\mathbf{15}}^{(2)} = T^i_{jm} \, \overline{T}^{jn}_i \, T^k_{ln} \, \overline{T}^{lm}_k \,, \tag{4.5}$$

$$\mathcal{I}_{\mathbf{15}}^{(3)} = T^i_{jm} \, \overline{T}^{jn}_i \, T^m_{kl} \, \overline{T}^{kl}_n \,, \tag{4.6}$$

$$\mathcal{I}_{\mathbf{15}}^{(4)} = T^m_{ij} \, \overline{T}^{ij}_n \, T^n_{kl} \, \overline{T}^{kl}_m \,, \tag{4.7}$$

$$\mathcal{I}_{\mathbf{15}}^{(5)} = T^i_{jm} \, T^j_{in} \, \overline{T}^{km}_l \, \overline{T}^{ln}_k \,. \tag{4.8}$$

Expressing the quartic invariants in terms of the fifteen components Σ_A, cf. eqs. (3.6), (3.7), we obtain polynomials of the form $c^{AB}_{CD} \, \Sigma_A \Sigma_B \overline{\Sigma}^C \overline{\Sigma}^D$. It is then straightforward to check that all five quartic invariants are linearly independent by comparing (a subset of) the coefficients c^{AB}_{CD} of these polynomials. Eq. (4.2) is thus the most general potential of a single $\mathbf{15}$ with quadratic and quartic terms.

In order to see if such a potential can be minimized by the VEVs of eqs. (3.12), (3.13), we calculate the first and second derivatives of $V_{\mathbf{15}}$ and insert the desired VEV alignments. In general, setting the first derivatives to zero determines the overall scale of the VEV in terms of the parameters of the potential, $m_{\mathbf{15}}, \lambda_{\mathbf{15}}, \kappa_{\mathbf{15}}, \rho_{\mathbf{15}}, \tau_{\mathbf{15}}, \eta_{\mathbf{15}}$. Subsequently, we calculate the Hessian, i.e. the matrix of second derivatives. A positive definite Hessian

JHEP03(2011)108

corresponds to a minimum of the potential. Requiring positive eigenvalues then constrains the parameters of the potential. The so obtained potential is now minimized by a VEV which breaks SU(3) down to the finite subgroup $\mathcal{G}$.

Before presenting the details for the two VEV configurations of eqs. (3.12), (3.13), a comment on the existence of zero eigenvalues of the Hessian is in order. The potential of eq. (4.2) is symmetric under SU(3) as well as a U(1).[3] Both of these symmetries are completely broken. Therefore the Hessian will automatically have 8+1 zero eigenvalues. This means that the minimum of the potential is assumed not only for the VEV alignments of eqs. (3.12), (3.13) but also their SU(3) transformed configurations. These alternative VEV alignments are still invariant under the transformations of the finite subgroup $\mathcal{G}$, however, not in the basis of eq. (3.8) but rather

$$D' = VDV^\dagger\,, \qquad A' = VAV^\dagger\,, \qquad B' = VBV^\dagger\,, \tag{4.9}$$

where V denotes the SU(3) transformation to the alternative VEV alignments.

Let us now turn to the explicit examples.

- Inserting the VEV alignment of eq. (3.12) into the first derivatives fixes the scale of the VEV to

$$|\langle\Sigma\rangle| \;=\; \sqrt{\frac{m_{\mathbf{15}}^2}{2\,F_{\mathbf{15}}}}\cdot(0,0,0,0,0,0,0,0,0,0,0,0,1,1,1)^T\,, \tag{4.10}$$

with

$$F_{\mathbf{15}} \;=\; 3\,\lambda_{\mathbf{15}} + \kappa_{\mathbf{15}} + \rho_{\mathbf{15}} + \tau_{\mathbf{15}} + \eta_{\mathbf{15}}\,. \tag{4.11}$$

As for any Higgs potential which yields a non-trivial vacuum configuration, the coefficient $-m_{\mathbf{15}}^2$ of the quadratic term must be negative, while the "effective" coefficient $F_{\mathbf{15}}$ of the quartic term has to be positive. Hence we get our first conditions

$$0 \;<\; m_{\mathbf{15}}^2\,, \quad 0 \;<\; F_{\mathbf{15}}\,. \tag{4.12}$$

Additional constraints on the parameters of the potential in eq. (4.2) arise from the Hessian H. This 30×30 matrix of second derivatives falls into a block diagonal structure,

$$H \;=\; h_{3\times3} \;\oplus\; 3\times h_{4\times4} \;\oplus\; 3\times h'_{4\times4} \;\oplus\; 0_{3\times3}\,, \tag{4.13}$$

where $h_{3\times3}$ has three non-zero eigenvalues,

$$4\,m_{\mathbf{15}}^2\,, \quad \text{and} \quad 2\;\times\;\frac{m_{\mathbf{15}}^2}{F_{\mathbf{15}}}\,(\kappa_{\mathbf{15}} - 2\,\eta_{\mathbf{15}} - 2\,\rho_{\mathbf{15}} + 4\,\tau_{\mathbf{15}})\,. \tag{4.14}$$

The 4×4 matrices $h_{4\times4}$ and $h'_{4\times4}$ both have one zero eigenvalue as well as

$$-\,3\,\frac{m_{\mathbf{15}}^2}{F_{\mathbf{15}}}\,\eta_{\mathbf{15}}\;; \tag{4.15}$$

[3] One may impose this U(1) symmetry to forbid a potential cubic term in $V_{\mathbf{15}}$. The cosmological implications of such an $\mathrm{SU}(3)\times\mathrm{U}(1)$ setup have been discussed, e.g., in [37].

JHEP03(2011)108

the remaining two eigenvalues are

$$\frac{m_{\mathbf{15}}^2}{4\,F_{\mathbf{15}}}\Big\{5\kappa_{\mathbf{15}}+2\rho_{\mathbf{15}}+4\tau_{\mathbf{15}} \mp\sqrt{(4\tau_{\mathbf{15}}+2\rho_{\mathbf{15}}-3\kappa_{\mathbf{15}})^2+16(\rho_{\mathbf{15}}+\kappa_{\mathbf{15}}+2\eta_{\mathbf{15}})^2}\Big\}, \tag{4.16}$$

for $h_{4\times4}$ and

$$\frac{m_{\mathbf{15}}^2}{2\,F_{\mathbf{15}}}\Big\{3\kappa_{\mathbf{15}}-5\eta_{\mathbf{15}}-2\rho_{\mathbf{15}}+4\tau_{\mathbf{15}} \mp\frac{1}{3}\sqrt{(9\eta_{\mathbf{15}}-7\kappa_{\mathbf{15}}+10\rho_{\mathbf{15}}-4\tau_{\mathbf{15}})^2+8(\rho_{\mathbf{15}}+2\kappa_{\mathbf{15}}-4\tau_{\mathbf{15}})^2}\Big\}, \tag{4.17}$$

for $h'_{4\times4}$.

This shows that there are — as expected — nine zero eigenvalues.[4] Requiring all other eigenvalues of the Hessian to be positive defines the set of parameters which ensures a spontaneous breaking of SU(3) to $\mathcal{A}_4$. From eq. (4.15) we immediately see that $\eta_{\mathbf{15}} < 0$. The other conditions for having positive eigenvalues are less trivial. We therefore consider the special situation in which $\lambda_{\mathbf{15}} = \rho_{\mathbf{15}} = \tau_{\mathbf{15}} = 0$.[5] In this case it is straightforward to obtain the condition for the remaining order parameter $\kappa_{\mathbf{15}}$; we find

$$0 < -\eta_{\mathbf{15}} < \kappa_{\mathbf{15}}\ . \tag{4.18}$$

- In order to break SU(3) down to $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ it is necessary to construct a potential of the type of eq. (4.2) which is minimized by the VEV alignment of eq. (3.13). Requiring vanishing first derivatives sets the scale of the VEV to

$$|\langle\Sigma'\rangle| \;=\; \sqrt{\frac{m_{\mathbf{15}}^2}{2\,F'_{\mathbf{15}}}}\cdot(0,0,0,0,0,0,1,0,1,0,1,0,0,0,0)^T\,, \tag{4.19}$$

with

$$F'_{\mathbf{15}} \;=\; 3\,\lambda_{\mathbf{15}}+\kappa_{\mathbf{15}}+\rho_{\mathbf{15}}+\tau_{\mathbf{15}}\ . \tag{4.20}$$

Both, $m_{\mathbf{15}}^2$ and $F'_{\mathbf{15}}$ must be positive. As before, the Hessian breaks into a block diagonal structure as given in eq. (4.13), with nine zero eigenvalues corresponding to the SU(3) and U(1) transformations. The three eigenvalues of $h_{3\times3}$ read

$$4\,m_{\mathbf{15}}^2\,,\quad \text{and}\quad 2\;\times\;\frac{m_{\mathbf{15}}^2}{F'_{\mathbf{15}}}\,(4\kappa_{\mathbf{15}}-2\,\rho_{\mathbf{15}}+4\,\tau_{\mathbf{15}})\ . \tag{4.21}$$

The submatrices $h_{4\times4}$ and $h'_{4\times4}$ turn out to be identical up to a trivial sign change,

$$h_{4\times4} \;=\; \mathrm{Diag}(1,1,-1,-1)\cdot h'_{4\times4}\cdot\mathrm{Diag}(1,1,-1,-1)\,, \tag{4.22}$$

[4]We have checked explicitly that the corresponding eigenvectors point into the directions of the SU(3) and U(1) transformations.

[5]We emphasize that the existence of a positive semi-definite Hessian does not rely on this particular choice of parameters.

JHEP03(2011)108

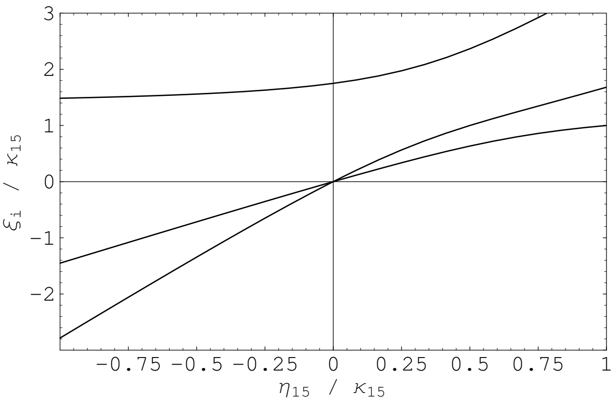

Figure 1. $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ from the **15** of SU(3): the three non-vanishing scaled eigenvalues $\frac{\xi_i}{\kappa_{\mathbf{15}}}$ of the sub-Hessian $h_{4\times 4}$ are shown as functions of $\frac{\eta_{\mathbf{15}}}{\kappa_{\mathbf{15}}}$ in the case where $\lambda_{\mathbf{15}} = \rho_{\mathbf{15}} = \tau_{\mathbf{15}} = 0$.

so that their eigenvalues are identical. One of the four eigenvalues is always zero while, in general, the other three eigenvalues x_i are non-vanishing. They can be determined as the solutions to the following cubic polynomial

$$\begin{aligned} &4\xi_i^3 - \xi_i^2 \left(18\eta_{\mathbf{15}} + 7\kappa_{\mathbf{15}} - 2\rho_{\mathbf{15}} + 12\tau_{\mathbf{15}}\right) \\ &\quad + \xi_i \left(18\eta_{\mathbf{15}}^2 + 27\eta_{\mathbf{15}}\kappa_{\mathbf{15}} - 18\eta_{\mathbf{15}}\rho_{\mathbf{15}} - 5\rho_{\mathbf{15}}^2 + 36\eta_{\mathbf{15}}\tau_{\mathbf{15}} + 20\kappa_{\mathbf{15}}\tau_{\mathbf{15}}\right) \\ &\quad - 3\left(8\eta_{\mathbf{15}}^2\kappa_{\mathbf{15}} - 8\eta_{\mathbf{15}}^2\rho_{\mathbf{15}} - 3\eta_{\mathbf{15}}\rho_{\mathbf{15}}^2 + 8\eta_{\mathbf{15}}^2\tau_{\mathbf{15}} + 12\eta_{\mathbf{15}}\kappa_{\mathbf{15}}\tau_{\mathbf{15}}\right) \;=\; 0\,, \end{aligned} \tag{4.23}$$

where $\xi_i = \frac{F'_{\mathbf{15}}}{m_{\mathbf{15}}^2} x_i$. Note that ξ_i and x_i have identical signs. To present a scenario in which all non-vanishing eigenvalues of the Hessian are positive let us again consider the special case with $\lambda_{\mathbf{15}} = \rho_{\mathbf{15}} = \tau_{\mathbf{15}} = 0$. The condition $0 < F'_{\mathbf{15}}$ as well as eq. (4.21) demand positive $\kappa_{\mathbf{15}}$ in that case. With these assumptions the cubic polynomial simplifies and we can calculate the three roots. However, as the analytic expressions are rather lengthy, we show the results graphically in figure 1. In order to have a minimum all three eigenvalues must be positive. This immediately implies positive $\eta_{\mathbf{15}}$. So in the case where $\lambda_{\mathbf{15}} = \rho_{\mathbf{15}} = \tau_{\mathbf{15}} = 0$, the conditions to get a VEV that breaks SU(3) down to $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ are

$$0 < \kappa_{\mathbf{15}}\,, \quad 0 < \eta_{\mathbf{15}}\,. \tag{4.24}$$

4.2 The case of a single 10

Similar to the previous case, we consider a potential of a single **10** which has a mass term $\mathbf{10} \times \overline{\mathbf{10}}$ as well as quartic interactions of type $\mathbf{10} \times \mathbf{10} \times \overline{\mathbf{10}} \times \overline{\mathbf{10}}$. The symmetric product

$$(\mathbf{10} \times \mathbf{10})_s \;=\; \mathbf{27} + \mathbf{28}\,, \tag{4.25}$$

shows that we can only write down two independent quartic SU(3) invariants. Hence, the potential for the **10** takes the form

$$V_{\mathbf{10}} \;=\; -\, m_{\mathbf{10}}^2\, \mathcal{I}_{\mathbf{10}}^{(0)} + \lambda_{\mathbf{10}}\, \mathcal{I}_{\mathbf{10}}^{(1)} + \kappa_{\mathbf{10}}\, \mathcal{I}_{\mathbf{10}}^{(2)}\,, \tag{4.26}$$

JHEP03(2011)108

with

$$\mathcal{I}^{(0)}_{\mathbf{10}} = T_{ijk}\,\overline{T}^{ijk}\,, \tag{4.27}$$

$$\mathcal{I}^{(1)}_{\mathbf{10}} = T_{ijk}\,\overline{T}^{ijk}\,T_{lmn}\,\overline{T}^{lmn}\,, \tag{4.28}$$

$$\mathcal{I}^{(2)}_{\mathbf{10}} = T_{ijm}\,\overline{T}^{ijn}\,T_{kln}\,\overline{T}^{klm}\,. \tag{4.29}$$

Using the VEV configuration of eq. (3.11) which breaks SU(3) uniquely down to $\Delta(27)$, we can determine the scale of the VEV alignment by setting the first derivatives to zero. We obtain

$$|\langle\psi\rangle| = \sqrt{\frac{m^2_{\mathbf{10}}}{2\,F_{\mathbf{10}}}}\cdot(1,1,1,0,0,0,0,0,0,0)^T\,, \tag{4.30}$$

with

$$F_{\mathbf{10}} = 3\,\lambda_{\mathbf{10}} + \kappa_{\mathbf{10}}\,. \tag{4.31}$$

Having a minimum requires positive values for $m^2_{\mathbf{10}}$ and $F_{\mathbf{10}}$. The other constraints on the parameters of the potential arise from the Hessian. The 20×20 matrix can be calculated analytically, yielding eleven zero eigenvalues as well as

$$4m^2_{\mathbf{10}}\,,\qquad 6\;\times\;\frac{4m^2_{\mathbf{10}}\kappa_{\mathbf{10}}}{3F_{\mathbf{10}}}\;,\quad\text{and}\quad 2\;\times\;\frac{4m^2_{\mathbf{10}}\kappa_{\mathbf{10}}}{F_{\mathbf{10}}}\;. \tag{4.32}$$

Consequently, we need positive $\kappa_{\mathbf{10}}$ in order to have a potential which is minimized by the VEV alignment of eq. (3.11). The number of zero eigenvalues of the Hessian can be understood by noticing that the potential $V_{\mathbf{10}}$, due to its simple form, possesses a larger symmetry than SU(3). While $\mathcal{I}^{(0)}_{\mathbf{10}}$ and $\mathcal{I}^{(1)}_{\mathbf{10}}$ are invariant under an SU(10) transformation, the invariant $\mathcal{I}^{(2)}_{\mathbf{10}}$ respects an SU(6) symmetry. The VEV in eq. (4.30) breaks SU(6) down to SU(5) leading to $35-24=11$ Goldstone directions.[6] Nine vanishing eigenvalues of the Hessian correspond to the broken generators of $\mathrm{SU}(3)\times\mathrm{U}(1)$; the additional two zeros correspond to the directions of the real and the imaginary part of ψ_{10}. This is exactly the direction of the second $\Delta(27)$ singlet within the **10**. Any linear combination of the VEV alignments in eq. (3.10) and eq. (3.11) leaves the group $\Delta(27)$ intact. Sliding along the ψ_{10} direction, the residual symmetry will remain $\Delta(27)$ as long as $\langle\psi_{1,2,3}\rangle\neq 0$. Only in the special vacuum where the first three components of the **10** vanish identically, we end up with the bigger group given in eq. (A.8). This can be avoided by small deformations of the potential. A simple scenario could consist in adding a second **10** which is aligned as in eq. (3.10), cf. section 4.3. We can then introduce a quartic term which couples the two different **10**s as follows,

$$\sum_{a,b=1}^{10}(\psi_a\,\overline{\psi'_a})\,(\overline{\psi_b}\,\psi'_b)\,. \tag{4.33}$$

Note that such a term is always positive or zero. Assuming this term to enter the potential with a positive coupling constant, the minimum arises if $\sum_{a=1}^{10}\langle\psi_a\rangle\,\langle\overline{\psi'_a}\rangle = 0$. With $\langle\psi'_a\rangle = 0$ for $a=1,2,\ldots,9$, this entails vanishing $\langle\psi_{10}\rangle$. Therefore, the VEV of ψ is driven to the alignment of eq. (3.11) which breaks SU(3) uniquely down to $\Delta(27)$.

[6] I thank Tom Kephart for pointing out this connection.

JHEP03(2011)108

4.3 The case of a 10 and a 6

We have seen that the combination of a **6** and a **10** with alignments along the directions of eqs. (3.9), (3.10) gives rise to a residual $\mathcal{A}_4$ symmetry. In the following we show that there exists a potential which assumes its minimum for exactly these VEV alignments. The most general renormalizable potential of one **6** and one **10** consists of thirteen invariants. It reads

$$\begin{aligned} V_{\mathbf{6}+\mathbf{10}} = & -m_{\mathbf{6}}^2\,\mathcal{I}_{\mathbf{6}}^{(0)} + \lambda_{\mathbf{6}}\,\mathcal{I}_{\mathbf{6}}^{(1)} + \kappa_{\mathbf{6}}\,\mathcal{I}_{\mathbf{6}}^{(2)} + \rho_{\mathbf{6}}\,\mathcal{I}_{\mathbf{6}}^{(3)} \\ & -m_{\mathbf{10}}^2\,\mathcal{I}_{\mathbf{10}}^{(0)} + \lambda_{\mathbf{10}}\,\mathcal{I}_{\mathbf{10}}^{(1)} + \kappa_{\mathbf{10}}\,\mathcal{I}_{\mathbf{10}}^{(2)} + \rho_{\mathbf{10}}\,\mathcal{I}_{\mathbf{10}}^{(3)} + \tau_{\mathbf{10}}\,\mathcal{I}_{\mathbf{10}}^{(4)} \\ & + \eta_1\,\mathcal{I}_{\mathbf{6}+\mathbf{10}}^{(1)} + \eta_2\,\mathcal{I}_{\mathbf{6}+\mathbf{10}}^{(2)} + \eta_3\,\mathcal{I}_{\mathbf{6}+\mathbf{10}}^{(3)} + \eta_4\,\mathcal{I}_{\mathbf{6}+\mathbf{10}}^{(4)}\,, \end{aligned} \tag{4.34}$$

with

$$\mathcal{I}_{\mathbf{6}}^{(0)} = T_{ij}\,\overline{T}^{ij}\,, \tag{4.35}$$

$$\mathcal{I}_{\mathbf{6}}^{(1)} = T_{ij}\,\overline{T}^{ij}\,T_{kl}\,\overline{T}^{kl}\,, \tag{4.36}$$

$$\mathcal{I}_{\mathbf{6}}^{(2)} = T_{ik}\,\overline{T}^{il}\,T_{jl}\,\overline{T}^{jk}\,, \tag{4.37}$$

$$\mathcal{I}_{\mathbf{6}}^{(3)} = \epsilon^{ijk}\,T_{1i}\,T_{2j}\,T_{3k}\ +\ \text{h.c.}\,, \tag{4.38}$$

$$\mathcal{I}_{\mathbf{10}}^{(3)} = \epsilon^{xx'k}\epsilon^{yy'l}\,T_{ixy}\,T_{jx'y'}\,T_{mkl}\,\overline{T}^{ijm}\ +\ \text{h.c.}\,, \tag{4.39}$$

$$\mathcal{I}_{\mathbf{10}}^{(4)} = \epsilon^{xx'k}\epsilon^{yy'l}\,T_{ixy}\,T_{jx'y'}\,\epsilon^{vv'i}\epsilon^{ww'j}\,T_{kvw}\,T_{lv'w'}\ +\ \text{h.c.}\,, \tag{4.40}$$

$$\mathcal{I}_{\mathbf{6}+\mathbf{10}}^{(1)} = T_{ij}\,\overline{T}^{ij}\,T_{klm}\,\overline{T}^{klm}\,, \tag{4.41}$$

$$\mathcal{I}_{\mathbf{6}+\mathbf{10}}^{(2)} = T_{ijm}\,\overline{T}^{ij}\,T_{kl}\,\overline{T}^{klm}\,, \tag{4.42}$$

$$\mathcal{I}_{\mathbf{6}+\mathbf{10}}^{(3)} = T_{ijm}\,\overline{T}^{ijn}\,T_{kn}\,\overline{T}^{km}\,, \tag{4.43}$$

$$\mathcal{I}_{\mathbf{6}+\mathbf{10}}^{(4)} = \epsilon^{xx'k}\epsilon^{yy'l}\,T_{ixy}\,T_{jx'y'}\,T_{kl}\,\overline{T}^{ij}\ +\ \text{h.c.}\,, \tag{4.44}$$

and $\mathcal{I}_{\mathbf{10}}^{(0)}$, $\mathcal{I}_{\mathbf{10}}^{(1)}$, $\mathcal{I}_{\mathbf{10}}^{(2)}$ as given in eqs. (4.27)–(4.29). The tensors $T_{...}$ with three indices correspond to the **10** while those with two indices stand for the **6**; a bar indicates complex conjugate representations. ϵ^{ijk} denotes the totally antisymmetric tensor with $\epsilon^{123} = 1$. Note that all invariants which contain this ϵ tensor are *not* symmetric under a general U(1) while all other invariants feature such a U(1) symmetry.

Evaluation of the first derivatives using the alignment directions of eqs. (3.9), (3.10) fixes the scale of the VEVs,

$$\langle\chi\rangle = R_{\mathbf{6}}\,(1,1,1,0,0,0)^T\,, \qquad \langle\psi\rangle = R_{\mathbf{10}}\,(0,0,0,0,0,0,0,0,0,1)^T\,. \tag{4.45}$$

Despite the lack of a general U(1) symmetry we can assume real VEVs $R_{\mathbf{6}}$ and $R_{\mathbf{10}}$ for our purposes, because any potential V' which is minimized by complex VEVs corresponds to a modified potential V in which the coupling constants absorb the phases of the complex

JHEP03(2011)108

VEVs, thus rendering the latter real. With this assumption we obtain the following two conditions on $R_{\mathbf{6}}$ and $R_{\mathbf{10}}$,

$$
\begin{aligned}
0 &= -3m_{\mathbf{6}}^2 + R_{\mathbf{10}}^2(3\eta_1+\eta_3-2\eta_4) + 3R_{\mathbf{6}}(6R_{\mathbf{6}}\lambda_{\mathbf{6}} + 2R_{\mathbf{6}}\kappa_{\mathbf{6}} + \rho_{\mathbf{6}})\,,\\
0 &= -3m_{\mathbf{10}}^2 + 3R_{\mathbf{6}}^2(3\eta_1+\eta_3-2\eta_4) + 2R_{\mathbf{10}}^2(3\lambda_{\mathbf{10}} + \kappa_{\mathbf{10}} + 2\rho_{\mathbf{10}} + 4\tau_{\mathbf{10}})\ .
\end{aligned}
$$

For the sake of simplicity we assume $\rho_{\mathbf{6}} = 0$.[7] Then the above conditions are satisfied for

$$
R_{\mathbf{6}}^2 = \frac{2m_{\mathbf{6}}^2(3\lambda_{\mathbf{10}}+\kappa_{\mathbf{10}}+2\rho_{\mathbf{10}}+4\tau_{\mathbf{10}}) - m_{\mathbf{10}}^2(3\eta_1+\eta_3-2\eta_4)}{4(3\lambda_{\mathbf{6}}+\kappa_{\mathbf{6}})(3\lambda_{\mathbf{10}}+\kappa_{\mathbf{10}}+2\rho_{\mathbf{10}}+4\tau_{\mathbf{10}}) - (3\eta_1+\eta_3-2\eta_4)^2}\,, \tag{4.46}
$$

$$
R_{\mathbf{10}}^2 = \frac{6m_{\mathbf{10}}^2(3\lambda_{\mathbf{6}}+\kappa_{\mathbf{6}}) - 3m_{\mathbf{6}}^2(3\eta_1+\eta_3-2\eta_4)}{4(3\lambda_{\mathbf{6}}+\kappa_{\mathbf{6}})(3\lambda_{\mathbf{10}}+\kappa_{\mathbf{10}}+2\rho_{\mathbf{10}}+4\tau_{\mathbf{10}}) - (3\eta_1+\eta_3-2\eta_4)^2}\ . \tag{4.47}
$$

Evaluating the second derivatives for these VEVs yields a block diagonal structure for the 32×32 Hessian

$$
H \;=\; h_{1\times1} \;\oplus\; h_{4\times4} \;\oplus\; 3\times h'_{4\times4} \;\oplus\; 3\times h''_{4\times4} \;\oplus\; 0_{3\times3}\ . \tag{4.48}
$$

In general, $h_{1\times1}$ and $h_{4\times4}$ have no vanishing eigenvalue, while $h'_{4\times4}$ and $h''_{4\times4}$ each have one zero eigenvalue. Therefore the full Hessian exhibits nine zero eigenvalues corresponding to the directions of the eight SU(3) transformations plus an extra U(1) transformation. Notice that there exists only one U(1) symmetry and not two because the charge of the **10** is fixed to be neutral. In order to have a minimum we need the remaining 23 eigenvalues to be positive. This constrains the set of parameters of the potential $V_{\mathbf{6}+\mathbf{10}}$ in eq. (4.34). As an example we discuss the special case where

$$
m_{\mathbf{6}} = m_{\mathbf{10}} = m\,, \qquad \kappa_{\mathbf{6}} = \kappa_{\mathbf{10}} = \kappa\,, \qquad \eta_4 = \eta\,, \tag{4.49}
$$

$$
\lambda_{\mathbf{6}} = \lambda_{\mathbf{10}} = \rho_{\mathbf{6}} = \rho_{\mathbf{10}} = \tau_{\mathbf{10}} = \eta_1 = \eta_2 = \eta_3 = 0\ . \tag{4.50}
$$

Then the VEVs simplify to

$$
R_{\mathbf{6}}^2 = \frac{m^2}{2(\kappa-\eta)}\,, \qquad R_{\mathbf{10}}^2 = \frac{3m^2}{2(\kappa-\eta)}\,, \tag{4.51}
$$

requiring positive m^2 as well as $\eta < \kappa$. The eigenvalues of the sub-Hessians are calculated to be

$$
h_{1\times1}:\ \frac{4m^2\eta}{\kappa-\eta}\,, \tag{4.52}
$$

$$
h_{4\times4}:\ 4m^2\,,\qquad \frac{4m^2(\kappa+\eta)}{\kappa-\eta}\qquad 2\;\times\;\frac{4m^2\kappa}{\kappa-\eta}\,, \tag{4.53}
$$

$$
h'_{4\times4}:\ x_1\,,\quad x_2\,,\quad x_3\,,\quad 0\,, \tag{4.54}
$$

$$
h''_{4\times4}:\ \frac{4m^2(2\kappa+3\eta)}{3(\kappa-\eta)}\,,\qquad \frac{m^2\eta(13\pm\sqrt{109})}{3(\kappa-\eta)}\,,\qquad 0\,, \tag{4.55}
$$

[7] This could be enforced by a U(1) symmetry under which the **6** carries non-vanishing charge while the **10** is neutral.

JHEP03(2011)108

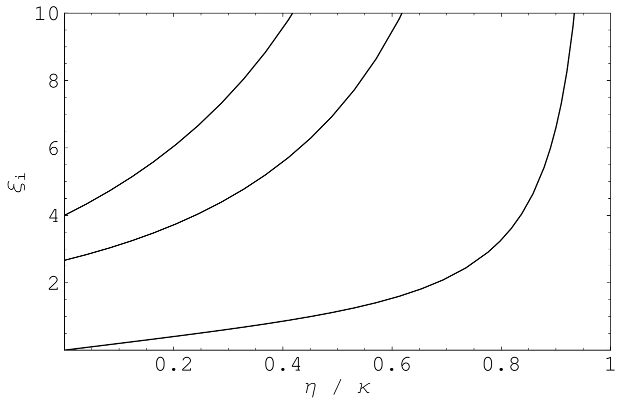

Figure 2. $\mathcal{A}_4$ from a **6** and a **10** of SU(3): the three non-vanishing scaled eigenvalues ξ_i of the sub-Hessian $h'_{4\times4}$ are shown as functions of $\frac{\eta}{\kappa}$ in the special case of eqs. (4.49), (4.50).

where x_i are the solutions to the cubic polynomial

$$3\xi_i^3(\eta-\kappa)^3 + 2\xi_i^2(\eta-\kappa)^2(11\eta+10\kappa) \\ + 4\xi_i(\eta-\kappa)(7\eta^2+22\eta\kappa+8\kappa^2) - 16\eta(\eta^2-2\eta\kappa-4\kappa^2) \;=\; 0\,, \quad (4.56)$$

with $\xi_i = \frac{x_i}{m^2}$. Figure 2 presents the results graphically for the relevant region

$$0 \;<\; \eta \;<\; \kappa\,, \quad (4.57)$$

which is obtained from requiring positive values for the other eigenvalues of the Hessian. From this example it is clear that parameter ranges exist in which the potential $V_{\mathbf{6}+\mathbf{10}}$ of eq. (4.34) is minimized by the alignments of eqs. (3.9), (3.10). Hence $\mathcal{A}_4$ can result as the discrete remnant of a spontaneously broken SU(3) symmetry.

5 Conclusion

In this paper we have investigated the possibility of obtaining a non-Abelian discrete family symmetry $\mathcal{G}$ from an underlying SU(3) gauge symmetry. Such a scenario is appealing in the sense that the residual discrete symmetry is protected against violations by quantum gravity effects. Other motivations for imposing a continuous SU(3) family symmetry which gets broken to a discrete non-Abelian family symmetry $\mathcal{G}$ in flavor model building include the possibility of correlations between free parameters of the $\mathcal{G}$ symmetric theory as well as the possibility to unify two or more $\mathcal{G}$ flavons in a single SU(3) representation. Thus a model of flavor based on an underlying SU(3) family symmetry should be more predictive.

In this work we have first identified the higher SU(3) representations which contain singlets under various discrete subgroups. These are potential candidates of fields that are capable of breaking SU(3) down to $\mathcal{G}$. Fixing the basis of the subgroup, we have determined the $\mathcal{G}$ singlet directions and checked whether these vacuum alignments leave invariant the desired subgroups or something bigger. Scrutinizing various SU(3) invariant potentials which involve higher representations comprises the central part of the paper. Constraining

JHEP03(2011)108

ourselves to the irreps **6**, **10** and **15** we found that $\mathcal{A}_4$, undoubtedly the most popular family symmetry, can be generated from either a single **15** or alternatively a combination of a **6** and a **10**. Similarly, the group $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ is obtained from a single **15**, however using different numerical values for the coupling constants of the potential. Finally, a single **10** allows to break SU(3) down to the group $\Delta(27)$. These results show that an SU(3) gauge symmetry can give rise to non-Abelian discrete family symmetries like e.g. $\mathcal{A}_4$, $\Delta(27)$ and $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$, sometimes adopting only one SU(3) breaking multiplet. We emphasize that the so obtained family symmetries feature — for the first time — irreducible *triplet* representations. We hope that our work will encourage the exploration of possible constraints, arising from the assumption of an underlying SU(3) symmetry, on flavor models adopting non-Abelian discrete symmetries which can guide us through the plethora of different setups.

Having discussed the above examples in great detail, it should be clear how to proceed in the case of other discrete symmetries $\mathcal{G}$. For instance, the family symmetry $\mathcal{PSL}_2(7)$ is expected to arise from an appropriate vacuum alignment of the $\mathbf{15'}$ of SU(3). This case will be treated elsewhere. In the context of a concrete model [38] we hope to find a solution to an unexplained tuning which is required to generate the correct vacuum structure of the flavon sextets.

We conclude by pointing out that our work does not address the question of how the breaking of the continuous symmetry is communicated to the Yukawa sector. In general this is a very model dependent problem as there are different choices for assigning the Standard Model fermions as well as the $\mathcal{G}$ breaking flavons to irreps of the underlying SU(3) symmetry. Depending on this choice the product rules constrain the allowed interactions of the SU(3) breaking field(s) to the chiral fermions and flavons. Such an investigation should be carried out within the context of a specific flavor model and is therefore beyond the scope of our paper.

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A Unbroken subgroups

In this appendix we determine the maximal subgroups that are left unbroken by the VEVs of section 3. We have already argued that the sextet VEV of eq. (3.9) leaves intact a continuous SO(3) symmetry. In general it is however more difficult to extract the *maximal* unbroken subgroups, even though the reverse, i.e. checking if a given symmetry is respected, is much simpler. To systematically analyze this question, let us parameterize a general SU(3) transformation U in the standard way

$$U = P_1 \cdot \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \cdot P_2\,, \tag{A.1}$$

JHEP03(2011)108

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. In addition to the three angles θ_{ij} there are five phases: δ as well as α_i and β_i as given in the phase matrices

$$P_1 = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{-i(\alpha_1+\alpha_2)} \end{pmatrix}, \qquad P_2 = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{-i(\beta_1+\beta_2)} \end{pmatrix}. \tag{A.2}$$

A general SU(3) transformation of a triplet state $|\,i\,\rangle$ now takes the form

$$|\,i\,\rangle \;\to\; \sum_{j=1}^{3} U_{ij}\,|\,j\,\rangle\,. \tag{A.3}$$

- Let us apply this method to the sextet VEV of eq. (3.9) first. In order to determine the subgroup that is left invariant we have to find the most general U which satisfies

$$\sum_{i=1}^{3} |\,ii\,\rangle \;\to\; \sum_{i,j,k=1}^{3} U_{ij}\,U_{ik}\,|\,jk\,\rangle \;\stackrel{!}{=}\; \sum_{i=1}^{3} |\,ii\,\rangle\,. \tag{A.4}$$

  This condition can be reformulated as

$$\sum_{i=1}^{3} U_{ij}\,U_{ik} \;=\; \sum_{i=1}^{3} U^T_{ji}\,U_{ik} \;=\; \delta_{jk}\,,$$

  showing that a continuous SO(3) symmetry is left unbroken by the sextet VEV of eq. (3.9), which confirms our earlier findings. We thus conclude that the sextet by itself is not suitable to break SU(3) down to any of the finite groups of table 2.

- In the case of the **10** we have two interesting directions. The VEV of eq. (3.10) is left invariant under transformations U which satisfy

$$|\,123\,\rangle + \text{perm.} \;\to\; \sum_{i,j,k=1}^{3} U_{1i}\,U_{2j}\,U_{3k}\,|\,ijk\,\rangle + \text{perm.} \;\stackrel{!}{=}\; |\,123\,\rangle + \text{perm.}\,. \tag{A.5}$$

  The ten resulting conditions constrain the parameters of the SU(3) transformation in eq. (A.1). One of these conditions is obtained from the fact that there must not be a $|\,333\,\rangle$ contribution to the transformed state. This translates to

$$U_{13}U_{23}U_{33} \;=\; s_{13}c_{13}^2 s_{23}c_{23}e^{-i(3\beta_1+3\beta_2+\delta)} \;=\; 0\,, \tag{A.6}$$

  requiring $\theta_{13} = 0, \frac{\pi}{2}$ or $\theta_{23} = 0, \frac{\pi}{2}$. Choosing $\theta_{13} = 0$, we continue with the condition arising from the $|\,123\,\rangle$ part of the transformed state. A straightforward calculation yields

$$\cos(2\theta_{12})\cos(2\theta_{23}) \;=\; 1\,. \tag{A.7}$$

  This can only be satisfied if both angles are either zero or $\frac{\pi}{2}$. In that case, all remaining eight conditions are automatically satisfied. Thus the unbroken symmetry includes a continuous phase transformation of type D, see eq. (3.8), as well as $A \cdot D$.

JHEP03(2011)108

Other elements of the unbroken group arise from setting either $\theta_{13} = \frac{\pi}{2}$ or $\theta_{23} = 0, \frac{\pi}{2}$. The resulting unbroken group is generated by A and D and hence given by all elements of the form

$$\left\{D\,,\ A\cdot D\,,\ A^2\cdot D\right\}\,, \tag{A.8}$$

for all possible diagonal phase matrices D with arbitrary ϑ_i. In particular the groups $\Delta(3n^2)$ and $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ are left unbroken. Therefore the VEV of eq. (3.10) alone is not suitable to break SU(3) down to any of the finite groups of table 2. However, combining a **6** and a **10** which respectively develop VEVs in the directions of eqs. (3.9), (3.10), we end up with $\mathcal{A}_4$ as the maximal unbroken symmetry.

The second VEV direction of interest is eq. (3.11). The corresponding unbroken subgroup can be determined from

$$\sum_{i=1}^{3} |\,iii\,\rangle \ \rightarrow \sum_{i,j,k,l=1}^{3} U_{ij}\,U_{ik}\,U_{il}\,|\,jkl\,\rangle \ \stackrel{!}{=}\ \sum_{i=1}^{3} |\,iii\,\rangle\ . \tag{A.9}$$

We have already seen that $\Delta(27)$ is unbroken. The question arises if there exists a symmetry transformation U which is not an element of $\Delta(27)$. In order to find an answer we study the $|\,331\,\rangle$ and $|\,332\,\rangle$ contributions of the transformed state. Since both of them must vanish, also any linear combination has to be zero. Therefore, as a starting point, we can solve the following equation

$$\sum_{i=1}^{3} U_{i3}\,U_{i3}\,U_{i1}\,s_{12}\,e^{-i\beta_1} \ -\ \sum_{i=1}^{3} U_{i3}\,U_{i3}\,U_{i2}\,c_{12}\,e^{-i\beta_2} \ =\ 0\ . \tag{A.10}$$

Evaluating the left-hand side leads to the condition

$$c_{13}^2\,c_{23}\,s_{23}\,(c_{23} - e^{3i(\alpha_1+2\alpha_2)}\,s_{23}) \ =\ 0\,, \tag{A.11}$$

which has solutions for $\theta_{13} = \frac{\pi}{2}$, $\theta_{23} = 0, \frac{\pi}{2}$, as well as $\theta_{23} = \frac{\pi}{4}$ with $(\alpha_1 + 2\alpha_2) = \frac{2\pi}{3}\cdot\mathbb{Z}$. Each of these four cases has to be investigated using the remaining nine conditions. Doing so it is possible to show that $\Delta(27)$ is indeed the maximal subgroup which remains intact in this case. Hence a VEV of the form of eq. (3.11) breaks SU(3) uniquely down to $\Delta(27)$.

- The two interesting directions of the **15** are shown in eqs. (3.12), (3.13). They are left invariant under transformations which satisfy

$$|\,12\bar{3}\,\rangle + \text{perm.} \ \rightarrow \sum_{i,j,k=1}^{3} U_{1i}\,U_{2j}\,U_{3k}^*\,|\,ij\bar{k}\,\rangle + \text{perm.} \ \stackrel{!}{=}\ |\,12\bar{3}\,\rangle + \text{perm.}\,, \tag{A.12}$$

and

$$\begin{aligned} &|\,11\bar{2}\,\rangle\ +\ |\,22\bar{3}\,\rangle\ +\ |\,33\bar{1}\,\rangle \ \ \rightarrow \\ &\sum_{i,j,k=1}^{3} \left(U_{1i}\,U_{1j}\,U_{2k}^* + U_{2i}\,U_{2j}\,U_{3k}^* + U_{3i}\,U_{3j}\,U_{1k}^*\right)|\,ij\bar{k}\,\rangle \stackrel{!}{=} |\,11\bar{2}\,\rangle + |\,22\bar{3}\,\rangle + |\,33\bar{1}\,\rangle\,, \end{aligned} \tag{A.13}$$

JHEP03(2011)108

respectively. Note that the anti-triplet transforms with the complex conjugated matrix U^*. Following the same strategy as before, it is possible to show that the maximal unbroken symmetries are $\mathcal{A}_4$ in the case of eq. (3.12) as well as $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$ for a VEV that is aligned in the direction of eq. (3.13).[8] Hence depending on the VEV alignment, the **15** can break SU(3) uniquely to either $\mathcal{A}_4$ or $\mathcal{Z}_7 \rtimes \mathcal{Z}_3$.

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JHEP03(2011)108