

Democratic superstring field theory: gauge fixing

Michael Kroyter

*Center for Theoretical Physics, Massachusetts Institute of Technology,
Cambridge, MA 02139, U.S.A.*

*School of Physics and Astronomy, The Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel Aviv University, Ramat Aviv, 69978, Israel*

E-mail: mikroyt@tau.ac.il

ABSTRACT: We show that a partial gauge fixing of the NS sector of the democratic-picture superstring field theory leads to the non-polynomial theory. Moreover, by partially gauge fixing the Ramond sector we obtain a non-polynomial fully RNS theory at pictures 0 and $\frac{1}{2}$. Within the democratic theory and in the partially gauge fixed theory the equations of motion of both sectors are derived from an action. We also discuss a representation of the non-polynomial theory analogous to a manifestly two-dimensional representation of WZW theory and the action of bosonic pure-gauge solutions.

We further demonstrate that one can consistently gauge fix the NS sector of the democratic theory at picture number -1 . The resulting theory is new. It is a \mathbb{Z}_2 dual of the modified cubic theory. We construct analytical solutions of this theory and show that they possess the desired properties.

KEYWORDS: String Field Theory, Superstrings and Heterotic Strings

ARXIV EPRINT: [1010.1662](https://arxiv.org/abs/1010.1662)

Contents

1	Introduction	1
2	Cubic superstring field theory in the democratic picture	2
2.1	Wrong ways to gauge fix	6
2.1.1	Gauge fixing to a given picture number	7
2.1.2	Reducing the large space to the small (or dual) space	7
2.1.3	Gauge symmetry in Witten’s theory	8
3	Gauge fixing to the non-polynomial theory	9
3.1	Preliminaries	9
3.1.1	Example of using A_Q : the action of bosonic pure-gauge solutions	11
3.2	The gauge fixing	12
3.2.1	Equations of motion	14
3.2.2	Residual gauge symmetry	14
3.2.3	The action	16
3.3	The Ramond sector	17
3.3.1	Direct comparison	17
3.3.2	The non-polynomial RNS action	19
4	Gauge fixing at picture number -1	20
5	Analytical classical solutions of the new theory	22
5.1	Marginal deformation	22
5.2	Tachyon vacuum solutions	25
5.2.1	Trivial cohomology	28
5.2.2	The action of the solution	28
6	Conclusions	30

1 Introduction

Could string field theory serve as a non-perturbative definition of superstring theory? For that to be the case one needs a reliable formulation of superstring field theory. Such a formulation should describe interactions of the world-sheet modes in a consistent way. It should also respect the symmetries of the theory, e.g., be covariant. Moreover, since string theory can be defined around many backgrounds, a good formulation should “look the same”, regardless of the background chosen. This requirement is not as strong as that of “background independence”, since it allows the theory to depend on the details of the background, e.g., on the BRST charge Q . A “universal formalism” would depend only on

“universal objects” such as Q that can be defined for any consistent background, regardless of its dimensionality, specific matter content, etc. Indeed, universality played a major role in the advance of string field theory following Sen’s conjectures [1, 2].¹ Superstring field theory should also be able to describe the Ramond sector, again, in a universal way. An open RNS string field theory, which we call the “democratic theory”, that passes all the above criteria was constructed in [5].

Of course, the above is only a preliminary list of the expected properties of such a theory. First, the theory should support classical solutions describing known physics. Such solutions exist in the democratic theory. Then, for supersymmetric backgrounds, one might expect to obtain a supersymmetric string field theory. This was not fully established yet for the democratic theory. Note, however, that supersymmetry is not a universal property. Moreover, the expectation that the full string field theory is supersymmetric might be too naive.

Finally, it should be possible to gauge fix the theory. Then, one should be able to identify propagators and construct perturbation theory that leads to correct expressions for scattering amplitudes. The gauge structure of the democratic theory was identified and the classical BV master action of it was constructed. However, the issue of gauge fixing the theory was not clear. It is the purpose of this paper to clarify it to some extent. Specifically, we show that different gauge choices for the NS sector of the theory lead to the two known consistent NS string field theories, namely the non-polynomial theory and the modified theory, as well as to a new theory, which is the \mathbb{Z}_2 dual of the modified theory (the non-polynomial theory is self-dual). Furthermore, the equations of motion of the Ramond sector of the democratic theory seem to be equivalent to those proposed for the non-polynomial theory and have the added advantage of being derivable from an action.

The rest of the paper is organized as follows: we begin by recalling the definition of the democratic theory and setting our conventions in section 2. Then, we start section 3 with some algebraic preliminaries related to the representation of the non-polynomial theory and to the action of pure-gauge solutions and also present a simple two-dimensional representation of the WZW theory. We then show that the democratic theory can be partially gauge fixed in a way that leads to the non-polynomial theory. Moreover, we present a partially gauge fixed action generalizing the usual non-polynomial one that includes the Ramond sector. In section 4 we derive a new theory that is obtained by gauge fixing the NS sector of the democratic theory at picture number -1 . Next, we illustrate, in section 5, that this theory supports analytical solutions of familiar form and that the properties of these solutions agree with the expectations. Conclusions and future directions are given in section 6.

2 Cubic superstring field theory in the democratic picture

When a string field theory is constructed, one first identifies the vertex operators of the string theory and generalizes them to an off-shell set of string fields. Then, the equations

¹A review that covers recent developments in the field, including some material that is relevant for this work is [3]. A shorter review of some of these developments is [4].

of motion and gauge symmetry of the theory are constructed with the restriction that their linearized versions agree with the world-sheet conditions of being “closed” and “exact” respectively.

There are many ways for constructing a complete set of vertex operators of the RNS string. The source of the complication is the existence of an infinite number of pictures for each vertex [6]. While the sets of vertex operators at different pictures are equivalent [7, 8], their natural off-shell generalizations are not. Hence, the structure of the resulting string field theory might depend on the choice of picture.

A related subtlety is the existence of different ways for representing vertex operators in the large Hilbert space (henceforth, “the large space”). The space one gets from considering the ghost systems that are adequate for dealing with the world-sheet symmetries of the RNS string is called the small Hilbert space (henceforth, “the small space”). The bosonization of the $\beta\gamma$ superghost system to the $\xi\eta\phi$ variables does not include the ξ zero mode. The large space is the space obtained by including this zero mode. This space relates to the small space in a way that reminds the relation between complex and real numbers:

- The large space is a natural extension of the small space, since the latter is obtained from the former by dropping a single mode of a conformal field.
- It manifests the symmetries of the theory. Specifically, in the large space one can think of the RNS string as a topological $N = 4$ system [9]. In this description, the BRST operator is dual to the η -ghost zero mode η_0 . Both operators are nilpotent and they (anti-)commute. This \mathbb{Z}_2 symmetry can be extended in some cases to act also on the spectrum of the theory [10] (though not in a universal way).
- The operators Q and η_0 have trivial cohomologies in the large space. This means that every closed state has a primitive. Up to OPE singularities, these primitives can be obtained using the contracting homotopy operators of Q and η_0 , which are ξ and $P \equiv -c\xi\xi'e^{-2\phi}$ respectively. While these primitives are “unreal” in the sense that they generically are not part of the small space (they are the analogues of $i \in \mathbb{C}$), they might enable the simplification of various constructions, e.g., that of picture changing operators.

The first attempt towards a universal RNS string field theory, by Witten, was based on picture -1 string fields living in the small space. This construction was shown to be inconsistent [11]. Nonetheless, Berkovits’ construction of a theory that is based on a large space vertex operators at picture number zero (the immediate analogues of the small space picture number -1 operators) led to a consistent, non-polynomial theory [12]. An earlier attempt towards a theory was made by simply modifying the picture number from -1 to zero [13–15]. This theory also seems to be consistent in the NS sector.

Despite the success of the “non-polynomial” and the “modified” theories, one might wonder whether the zero picture is really what one should ultimately use, especially since, as we already mentioned, the off-shell structure is different at different pictures. Other than this “aesthetic” question there exists also the problem of including the Ramond sector: The formulation of the Ramond sector of the modified theory is inconsistent [16], while in

the non-polynomial theory it is not known how to include it in a universal way [17], at least without “tricks” such as imposing constraints after the derivation of the equations of motion [18].

One can address these issues by building a theory upon a different realisation of vertex operators. This realisation is obtained by allowing the vertex operators to live in the large space and to have an arbitrary picture number [19], while modifying the kinetic operator from Q to $Q - \eta_0$. Equivalent vertex operators belong to the same cohomology class regardless of their picture. Since each cohomology class now lives in an infinite dimensional space, inconsistencies might evolve, unless some restrictions are applied. A natural such restriction is that the sum over pictures of the coefficients of each vertex operator is well defined and finite. With this restriction of the space of unbounded picture numbers, it was shown in v2 of [5], that the cohomology of $Q - \eta_0$ is the correct one, i.e., it is identical to that of the same operator over an arbitrary bounded range of picture operators.

A string field theory that generalizes this construction of vertex operators was introduced in [5]. Its string field Ψ has ghost number one (as do the vertex operators), but an arbitrary picture and it lives in the large space. String fields should be restricted in a way that generalizes the restriction on vertex operators mentioned above. However, since there is no canonical norm at our disposal, it is not clear how exactly should this condition be quantified. This is one more manifestation of the problem regarding the definition of a proper space of string fields (see, e.g., [20]). The action of this theory (henceforth “the democratic theory”) is given by,²

$$S = - \oint \mathcal{O} \left(\frac{1}{2} \Psi(Q - \eta_0) \Psi + \frac{1}{3} \Psi^3 \right), \tag{2.1}$$

with \mathcal{O} being a mid-point insertion described below.

This theory addresses well the “aesthetic challenge”, since it lives in the large space and include the off-shell generalizations of all pictures. Moreover, its BV master action takes the same form, only with the ghost number of Ψ being arbitrary. This means that the whole space of NS string fields is used by the theory. The second challenge, that of including the Ramond sector is resolved by allowing Ψ to be a linear combination of all integer picture number NS fields and all half-integer number Ramond fields. We write,

$$\Psi = A + \alpha, \tag{2.2}$$

where A is the NS string field and α is the Ramond string field. In terms of A and α the action takes the form,

$$S = - \oint \mathcal{O} \left(\frac{1}{2} A(Q - \eta_0) A + \frac{1}{3} A^3 + \frac{1}{2} \alpha(Q - \eta_0) \alpha + A \alpha^2 \right), \tag{2.3}$$

which is exactly what one would expect for an RNS theory. Reverting to the aesthetic issue, we see that the theory uses all possible component fields of the RNS theory and

²Here, we introduce the convention of the integration symbol for a large space expectation value, as opposed to the standard integration symbol which refers to either the small space or to the space of the bosonic theory. Also, here and in the rest of the paper, we keep the star product [21] implicit. Another convention that we use is that $[A, B]$ represents a graded (star) commutator.

naturally unifies the NS and Ramond component fields to a single entity Ψ . Moreover, the inclusion of arbitrary D-brane systems can be straightforwardly achieved along the lines of [22] and the theory would then use the maximal set of permissible component fields.

In order to complete the definition of the theory we have to specify the mid-point insertion³ \mathcal{O} . The “physical part” of the vertex operators that we generalize lives in the small space. Thus, \mathcal{O} should include a component that is proportional to ξ . In [5], we showed that gauge invariance of the action together with ξ as an “initial condition” implies that,

$$\mathcal{O} \simeq \xi \sum_{k=-\infty}^{\infty} X_k, \tag{2.4}$$

where X_k is the picture changing operator of order k , e.g.,

$$X_0 = 1, \quad X_1 = X = Q\xi, \quad X_{-1} = Y = \eta_0 P. \tag{2.5}$$

The \simeq symbol in (2.4) reminds us that the expression is not really well defined and should be regularized in order to remove OPE poles. It should further be modified in order to turn \mathcal{O} into a primary conformal field. A more accurate definition of \mathcal{O} would be the statement that it is a weight zero primary conformal field obeying,

$$Q\mathcal{O} = \eta_0\mathcal{O} = \sum_{k=-\infty}^{\infty} X_k. \tag{2.6}$$

One can also decompose this equality according to picture number and write,

$$\mathcal{O} = \sum_{k=-\infty}^{\infty} \mathcal{O}_k, \quad Q\mathcal{O}_k = X_k, \quad \eta_0\mathcal{O}_k = X_{k-1}. \tag{2.7}$$

It was shown in [5] that such an \mathcal{O} exists and that the resulting X_k operators are also zero-weight primaries. Furthermore, it was also shown there (following [23]) that different \mathcal{O} insertions obeying these relations differ by a $Q - \eta_0$ exact terms and they are all (at least) classically equivalent.

As already mentioned, the action (2.1) has a gauge symmetry, which is a non-linear extension of the exactness property of vertex operators. The infinitesimal form of the gauge transformation is,

$$\delta\Psi = (Q - \eta_0)\Lambda + [\Psi, \Lambda], \tag{2.8}$$

where Λ is the gauge string field,⁴ which is even and includes all integer (NS) and half-integer (R) picture components. The zero-picture small-space component of Λ generates the

³One might worry that the use of a mid-point insertion would lead to inconsistencies as in the cases of Witten’s theory and of the Ramond sector of the modified theory. This is not the case, since the mid-point insertion appears only in the action. It is our understanding that mid-point insertions should never appear on string fields, since this would in any case lead to inconsistencies [20]. It is therefore important that the equations of motion and the gauge symmetry of the theory do not include mid-point insertions. This is indeed the case with the democratic theory, in contrast with the situation of the inconsistent theories.

⁴This terminology might be a bit confusing, since the analogue of the QED gauge field A_μ is Ψ , not Λ . Due to this analogy one can find in the literature the term “gauge parameter” for Λ . We find this terminology even more confusing, since Λ is not a parameter, but a string field.

standard gauge transformations, while all other (NS) components generate picture changing. Thus, “most” of the gauge symmetry is associated with picture changing. The gauge symmetry (2.8) can be exponentiated to give the finite form of a gauge transformation,

$$\Psi \rightarrow e^{-\Lambda}(Q - \eta_0 + \Psi)e^{\Lambda}. \tag{2.9}$$

Note, that the fact that we use an all-picture theory enabled us to write a gauge symmetry with no explicit (mid-point) picture changing operators. This feature is what enables the theory to be consistent, as opposed to previous similar formulations.

The equations of motion and gauge symmetry immediately suggest a simple way of gauge fixing in the NS sector. Requiring that the string field and gauge string field both carry picture number zero and live in the small space leads to the action,

$$S = - \int X_{-2} \left(\frac{1}{2} AQA + \frac{1}{3} A^3 \right), \tag{2.10}$$

where $X_{-2} = Y_{-2}$ is the double inverse picture changing operator and the remaining integration is in the small space. The residual gauge symmetry is,

$$A \rightarrow e^{-\Lambda}(Q + A)e^{\Lambda}. \tag{2.11}$$

We note that we obtained the modified theory. Indeed, this fact lied in the heart of the construction of the democratic theory. We would like to find other ways to partially gauge-fix the theory, where “partially” refers to the picture-changing gauge symmetry, but first, we would like to address some misconceptions regarding this issue.

2.1 Wrong ways to gauge fix

A naive examination of the theory suggests that gauge fixing leads to erroneous results. Specifically, one might claim that the gauge symmetry related to picture number can be fixed to any single picture. Then, it seems that choosing picture -1 for the NS field, leads to Witten’s theory, which is known to be inconsistent. Thus, it seems that the democratic theory should also be inconsistent. This observation is wrong, for at least three reasons:

1. It is not clear at all that fixing the gauge at any particular picture is a legitimate gauge choice.
2. Our theory is defined in the large space. In order to get to a theory such as Witten’s, one has to make the further gauge choice of restricting the string field to the small space. This is in general an inconsistent choice.
3. Even if we could obtain Witten’s theory, we would have not obtained the inconsistent gauge symmetry of this theory.

We discuss these three points in the following three subsections.

2.1.1 Gauge fixing to a given picture number

The democratic theory mixes different picture numbers in the equations of motion and in the gauge transformation. In the latter, generically all picture numbers are intertwined. Furthermore, since the content of the (off-shell) string field depends non-trivially on its picture, it seems unnatural to assume that a single picture number can be used in order to fix the gauge and the possibility to fix the gauge to one picture number does not imply that it is possible to do that at any other picture number. Thus, the different picture numbers should be examined on a case by case basis. Given a picture number n that can be used for a partial gauge fixing, one can form other permissible partially gauge fixed sets by considering an arbitrary gauge string field Λ and defining the set,

$$H_\Lambda \equiv \{A : \exists \hat{A}, \text{pic}(\hat{A} = n), A = e^{-\Lambda}(Q - \eta_0 + \hat{A})e^\Lambda\}. \tag{2.12}$$

These sets generically include all picture numbers. Thus, gauge fixing to a given picture number does not seem, a-priori, to be equivalent to gauge fixings to other picture numbers.

In order to rule out most picture numbers from the list of permissible partial gauge fixings, we note the following. String fields are not allowed to carry mid-point insertions. Hence, when we look at the equations of motion, we should actually demand that the coefficient of each \mathcal{O}_k separately vanishes. Then, if we set the picture numbers to arbitrary values, the linear terms and interaction terms decouple and we are left with,

$$QA = \eta_0 A = A^2 = \alpha^2 = \dots = 0. \tag{2.13}$$

While these equations are not wrong by themselves, they are certainly too restrictive to allow for general solutions of the equations of motion. Hence, they cannot be part of a permissible gauge fixing, since, by definition, a gauge fixing should leave us with a representative of each gauge orbit [24].⁵ The conclusion of this discussion is that, in general, a permissible gauge fixing is not attained by fixing the picture number to some given value.

In order to further understand this issue, we note that fixing the picture number of the string field implies also an appropriate constraining of the gauge string field. Then, in order not to generate all possible pictures, the gauge string field should carry zero picture number. The only contributions to the string field come now at pictures zero and -1 . This fact turns those picture numbers into the only permissible ones for a fixed picture gauge choice in the NS sector. Furthermore, this state of affairs complicates the gauge fixing of the Ramond sector, which carries a half-integer picture.

2.1.2 Reducing the large space to the small (or dual) space

From the point of view of vertex operators, the restriction to the small space using η_0 and the condition of closeness that is imposed by Q , stand on the same footing. The

⁵In the case of a full gauge fixing this representative should also be unique. For partial gauge fixing, as we consider here, there should actually be many representatives of each gauge orbit.

\mathbb{Z}_2 symmetry between Q and η_0 is manifest when we write the closeness condition and exactness relation as,

$$QV = \eta_0 V = 0, \quad V \sim V' \Leftrightarrow V - V' = Q\eta_0 \Upsilon. \quad (2.14)$$

In fact, one can interpret this relations not as saying

$$QV = 0, \quad V \sim V' \Leftrightarrow V - V' = Q\Lambda, \quad (2.15)$$

with V and Λ in the small space, but as saying that,

$$\eta_0 V = 0, \quad V \sim V' \Leftrightarrow V - V' = \eta_0 \Lambda, \quad (2.16)$$

where V and Λ are restricted to live in the space that is obtained by acting on the large space with Q . This space is the \mathbb{Z}_2 dual of the small space and we refer to it as “the dual space”. The fact that one should decompose the equations of motion according to the picture number (\mathcal{O} components), might lead to a situation where the choice of the small space is inconsistent. Indeed, we will see that a simple restriction to picture number -1 leads not to Witten’s theory, but to a new theory that lives in the dual space and is the \mathbb{Z}_2 dual of the modified theory.

2.1.3 Gauge symmetry in Witten’s theory

Finally, we want to understand the nature of the inconsistency of Witten’s theory. In this theory singularities appear in the gauge symmetry, as well as in the evaluation of scattering amplitudes. It might be the case, that the latter ones originate from a bad gauge fixing (e.g., the Siegel gauge might be inadequate in this case). Another problem with Witten’s theory is the absence of a tachyon vacuum, at least as can be seen by using low-level level truncation [25].⁶ The problem with Witten’s theory can be traced to the absence of an analogue of the state $c(0)|0\rangle$ in the -1 picture [27]. This state cannot be picture-changed to the -1 picture, since it is not on-shell and an attempt to “carry it over anyway” using Y leads to zero. The importance of this state is that it is an auxiliary field, which when integrated out gives a quartic potential for the tachyon field.

At any rate, the problem with the gauge symmetry of Witten’s theory is genuine. This gauge symmetry includes mid-point insertions of picture changing operators that collide upon iteration and lead to singularities. Even if we would have gotten this theory as a result of a partial gauge fixing of the democratic one, we would have not gotten this gauge symmetry, since we do not allow mid-point insertions over string fields. Hence, we would have concluded that the theory has no gauge symmetry at all. Such a theory should have

⁶Contrary to what was written in previous versions of this paper, the analysis of [25] is reliable and the choice of internal Chan-Paton factors there is effectively identical to that of [26]. I am grateful to the referee for clarifying to me the conventions of [25].

That Witten’s theory cannot support a tachyon vacuum can be proven without referring at all to level-truncation. The reason for that is that, according to our understanding of the role of mid-point insertions, the equations of motion are $QA = A^2 = 0$. Hence, the action of all solutions must equal zero and tachyon solutions cannot exist (on the other hand, solutions describing marginal deformations might exist).

been also considered as inconsistent, since it would not lead to a correct space of vertex operators upon linearization. However, we do not get this theory by gauge fixing the democratic one. We study the gauge fixing of the democratic theory to picture number -1 in section 4, but first we turn to another gauge fixing, which leads to the non-polynomial theory.

3 Gauge fixing to the non-polynomial theory

We now want to illustrate that the NS sector of the democratic theory reduces to the non-polynomial theory upon a partial gauge fixing. We start by presenting some useful algebraic structures in 3.1, where we also discuss the action of bosonic pure gauge solutions. The main construction follows in 3.2. We then discuss the gauge fixing of the Ramond sector in 3.3.

3.1 Preliminaries

Following [28], we define the operator L as,

$$LA = [\Phi, A]. \tag{3.1}$$

Such an operator can be defined for an arbitrary Φ and thus could be better named L_Φ . We would mostly be using this operator for whatever string field that we call “ Φ ”. Hence, in this case, we avoid the subscript. The operator L is an even derivation. This implies the following integration by parts formula (the same holds for integration in the large space),

$$\int f(L)A_1A_2 = \int A_1f(-L)A_2. \tag{3.2}$$

One can formally represent the derivations Q and η_0 as commutators [29],

$$QA = [A_Q, A], \quad \eta_0A = [A_\eta, A], \tag{3.3}$$

where we define the formal states A_Q and A_η as line integrals over the identity state. In the cylinder coordinates they take the form,

$$A_Q = - \int_{-i\infty}^{i\infty} J_B(z)dz |1\rangle, \tag{3.4a}$$

$$A_\eta = - \int_{-i\infty}^{i\infty} \eta(z)dz |1\rangle. \tag{3.4b}$$

We can also write,

$$Q\Phi = -LA_Q, \quad \eta_0\Phi = -LA_\eta. \tag{3.5}$$

The action of the non-polynomial theory can be cast into the form,⁷

$$S_{\text{NP}} = - \oint \eta_0\Phi \frac{e^{-L} - 1 + L}{L^2} Q\Phi. \tag{3.6}$$

⁷This result was obtained in the course of a work in progress on the gauge structure of the non-polynomial theory, carried out in collaboration with Nathan Berkovits, Yuji Okawa, Martin Schnabl, Shingo Torii and Barton Zwiebach. The starting point for this derivation was the representation of the non-polynomial theory found by Berkovits, Okawa and Zwiebach in [30].

By expanding the exponent and using (3.1), this expression is seen to be equal to,

$$S_{\text{NP}} = - \oint \eta_0 \Phi \left(\frac{1}{2} Q \Phi - \frac{1}{6} [\Phi, Q \Phi] + \frac{1}{24} (\Phi^2 Q \Phi - 2 \Phi Q \Phi \Phi + Q \Phi \Phi^2) + \dots \right), \quad (3.7)$$

which agrees with eq. (2.35) of [22] (up to a global factor of 2 stemming from the change of normalization due to the inclusion of the GSO(-) sector).

Using (3.1) and (3.3), the action (3.6) can be rewritten in the more suggestive form,

$$\begin{aligned} S_{\text{NP}} &= - \oint [A_\eta, \Phi] \frac{e^{-L} - 1 + L}{L^2} [A_Q, \Phi] = - \oint L A_\eta \frac{e^{-L} - 1 + L}{L^2} L A_Q \\ &= \oint A_\eta (e^{-L} - 1 + L) A_Q = \oint A_\eta e^{-L} A_Q. \end{aligned} \quad (3.8)$$

Here, we used (3.2) in the second equality and in the last equality we dropped one term in light of the fact that both integrands of A_Q and A_η live in the small space (3.4) and we dropped another term using,

$$[A_Q, A_\eta] = 0. \quad (3.9)$$

Another way to prove that the action (3.8) equals the standard non-polynomial one is by considering the variation of both actions. Specifically, given Φ , we can consider the one-parameter family connecting it to the trivial configuration, $\Phi(\alpha) \equiv \alpha \Phi$ ($0 \leq \alpha \leq 1$). The action of $\Phi(\alpha)$ is obtained by replacing L in (3.8) by αL . Evaluating the derivative with respect to α leads to,

$$\frac{dS_{\text{NP}}}{d\alpha} = \frac{d}{d\alpha} \oint A_\eta e^{-\alpha L} A_Q = - \oint A_\eta L A_Q = \oint L A_\eta A_Q = - \oint \eta_0 \Phi A_Q = \oint \eta_0 A \Phi. \quad (3.10)$$

The same expression is obtained from the familiar variation of the standard representation of the non-polynomial action

$$\delta S_{\text{NP}} = \oint \eta_0 A e^{-\Phi} \delta e^\Phi, \quad (3.11)$$

upon replacing the variation by a derivative with respect to α . Since both actions obey the same first order differential equation with the same (trivial) initial conditions they must agree.

The representations (3.6) and (3.8) are analogous to manifestly two dimensional representations of WZW theory. The former appears to be less attractive, but can be written directly for the WZW case. All that is needed is to replace the Q and η_0 by the derivatives ∂ and $\bar{\partial}$, to reinterpret the string field as an algebra element in WZW theory (also reinterpret L accordingly) and to replace the integration by a combination of complex plane integration and a trace over group space. The representation (3.8), which looks simpler, includes the string fields A_Q and A_η . Their counterparts in the WZW case are the derivatives ∂ and $\bar{\partial}$, considered as operators, i.e., acting all the way to their right (and further on, when we take the cyclicity property of the trace into account). A manifestly two-dimensional representation of the WZW theory might be counter-intuitive, since it is known that the

coupling constant should be quantized and one might worry that we do not get this condition in the current representation. This is not the case. Given g (the analogue of A in the WZW case) a choice of Φ implies a choice of cohomology class. In order to obtain an action that is independent on the latter choice (up to the addition of $2\pi n$), the usual quantization condition [31] should hold in our new representation. For non-compact groups, as usual, no quantization would arise, since the relevant cohomology class is trivial. This is similar to the string field theory case.

It is not known how to extend the non-polynomial theory to the Ramond sector in a universal way. What is known [18] is a way to write an action with two Ramond string fields and a constraint that should be imposed only after the derivation of the equations of motion. The action with the two Ramond string fields Ξ and Ψ can be cast into the form,

$$S_{\text{NP}} = \oint \hat{A}_\eta e^{-L} \hat{A}_Q, \tag{3.12}$$

where we defined,

$$\hat{A}_\eta \equiv A_\eta + \frac{[A_\eta, \Psi]}{\sqrt{2}}, \quad \hat{A}_Q \equiv A_Q + \frac{[A_Q, \Xi]}{\sqrt{2}}, \tag{3.13}$$

i.e., the operators that generate the derivations should be modified in order to include the influence of the Ramond string field. The factors of $\sqrt{2}$ originate from the fact that we use two Ramond string fields. This is similar to the construction in section 5 of [16]. For completeness we recall that the constraint relating the picture $\frac{1}{2}$ string field Ψ and the picture $-\frac{1}{2}$ string field Ξ is,

$$\eta_0 \Psi = e^{-L} Q \Xi. \tag{3.14}$$

3.1.1 Example of using A_Q : the action of bosonic pure-gauge solutions

It is a “common knowledge” that one cannot prove that the action vanishes for pure-gauge bosonic solutions, in terms of the finite form of the gauge transformation. Specifically, one has to prove that,

$$\int e^{-\Lambda} Q e^\Lambda Q e^{-\Lambda} Q e^\Lambda = 0, \tag{3.15}$$

which cannot be achieved just by integrating by parts and using the trace property. However, if one writes the infinitesimal form of the gauge transformation,

$$\delta A = Q \Lambda + [A, \Lambda] \equiv \mathcal{Q} \Lambda, \tag{3.16}$$

and uses the fact that A is a solution, i.e., that the action equals,

$$S = -\frac{1}{6} \int A Q A, \tag{3.17}$$

one gets

$$\delta S = -\frac{1}{6} \int (\mathcal{Q} \Lambda Q A + A Q \mathcal{Q} \Lambda) = 0. \tag{3.18}$$

Then, the fact that the action of a pure-gauge solution is zero can be established, since S obeys the first order linear differential equation and initial condition,

$$\frac{dS}{d\alpha} = 0, \quad S(0) = 0, \quad (3.19)$$

where we defined,

$$A(\alpha) = e^{-\alpha\Lambda} Q e^{\alpha\Lambda}. \quad (3.20)$$

One usually says that the proof holds provided it is allowed to use the parametrization (3.20). This condition is what distinguishes a genuine gauge solution from a formal one.

Let us now write the pure gauge solution in terms of the formal string field A_Q ,

$$A = e^{-\Lambda} Q e^{\Lambda} = (e^{-L_\Lambda} - 1) A_Q. \quad (3.21)$$

Using the trace property we can write,

$$S = \frac{1}{6} \int A^3 = \frac{1}{6} \int \left((e^{-L_\Lambda} A_Q)^3 + 3(e^{-L_\Lambda} A_Q)^2 A_Q + 3e^{-L_\Lambda} A_Q A_Q^2 + A_Q^3 \right). \quad (3.22)$$

The last two terms vanish in light of the identity,

$$A_Q^2 = 0. \quad (3.23)$$

The fact that L_Λ is a derivation implies that for arbitrary A_n the following holds,⁸

$$e^{L_\Lambda} (A_1 \dots A_n) = e^{L_\Lambda} A_1 \dots e^{L_\Lambda} A_n. \quad (3.24)$$

This identity together with (3.23) imply that the first two terms in (3.22) drop out as well. Thus, we managed to prove that the action of a pure-gauge solution vanishes using the finite form of the gauge transformation. One can now define a formal gauge string field as one for which any of the manipulations/representations used in our construction do not hold.

3.2 The gauge fixing

Let us examine first the case of the linearized theory of the NS sector,

$$S = -\frac{1}{2} \oint \mathcal{O} A (Q - \eta_0) A. \quad (3.25)$$

We want to partially fix the gauge by requiring A to have picture number zero,

$$\text{pic}(A) = 0. \quad (3.26)$$

The equations of motion then reduce to,

$$QA = \eta_0 A = 0. \quad (3.27)$$

⁸The operator e^{L_Λ} generates the ‘‘Taylor expansion’’ for the derivation L_Λ .

The gauge symmetry at this level reads,⁹

$$\delta A = Q\eta_0\Upsilon. \tag{3.28}$$

We can eliminate the gauge symmetry by restricting the string field not to be part of either the small space or the dual space. The two options are equivalent and we find it convenient to choose the second and define,

$$A = Q\Phi. \tag{3.29}$$

Equations (3.26) and (3.29) constitute our gauge conditions. Note, that by introducing Φ as a solution to the gauge condition $QA = 0$, we introduce an extra gauge symmetry, namely

$$\Phi \rightarrow \Phi + Q\hat{\Lambda}. \tag{3.30}$$

With our gauge conditions the action can be written as,

$$S = \frac{1}{2} \oint \mathcal{O}Q\Phi\eta_0Q\Phi = \frac{1}{2} \oint (Q\mathcal{O})Q\Phi\eta_0\Phi = -\frac{1}{2} \oint \eta_0\Phi Q\Phi. \tag{3.31}$$

Here, we integrated by parts in the second equality and used (2.6) and the picture number of the integrand in order to set $Q\mathcal{O}$ to unity in the last equality. We recognize the final result as the linearized action of the non-polynomial theory.

Consider now the interacting NS theory. As our gauge condition we take again (3.26) and for the second condition we choose what might be considered as the natural non-linear generalization of (3.29), namely

$$A = e^{-\Phi}Qe^{\Phi} = (e^{-L} - 1)A_Q. \tag{3.32}$$

Note that this is *not* (even a formal) pure-gauge form, since the kinetic operator of the democratic theory is $Q - \eta_0$ and not Q . In particular, the resulting A does not necessarily obey the equation of motion. Also, we do not restrict Φ to obey the equation of motion of the non-polynomial theory. While such a requirement was necessary in previous constructions of this type, this is not necessary here, since the string field A lives in the large space. Again, writing A in terms of Φ introduces an extra gauge symmetry. This gauge symmetry is generated by,

$$\delta e^{\Phi} = Q\hat{\Lambda}_0 e^{\Phi}, \tag{3.33}$$

with $\hat{\Lambda}_0$ being an arbitrary picture zero odd string field.

We want to prove that this gauge fixing leads to the non-polynomial theory. To that end, we first note, in 3.2.1, that the equation of motion is obeyed simultaneously in both theories. Next, in 3.2.2, we prove that there is a one to one correspondence between the gauge orbits of the non-polynomial theory and the residual gauge orbits of the democratic theory. Finally, in 3.2.3, we prove that the action coincides for both theories *even off-shell*. All that proves that the non-polynomial theory is a partially gauge-fixed version of the (NS sector of the) democratic theory (assuming that our gauge choice is a legitimate one).

⁹This form of the gauge symmetry does not use the linearized equations of motion. Instead, it uses the gauge condition (3.26). See also the discussion in 3.2.2.

Note, that one can choose gauge transformed versions of our gauge condition (3.32). Substituting $\Lambda = -\Phi$ into (2.9), leads to,

$$A = -e^\Phi \eta_0 e^{-\Phi}, \tag{3.34}$$

while substituting $\Lambda = -\frac{\Phi}{2}$ leads to the symmetrized expression,

$$A = e^{-\frac{\Phi}{2}} Q e^{\frac{\Phi}{2}} - e^{\frac{\Phi}{2}} \eta_0 e^{-\frac{\Phi}{2}}. \tag{3.35}$$

These variants have different pictures, but they are gauge equivalent. Hence, in light of the proof mentioned above, they are all equivalent to the non-polynomial theory. Their existence manifests the \mathbb{Z}_2 symmetry of both theories. We now turn to the actual proof.

3.2.1 Equations of motion

The constraint (3.26) reduces the equations of motion to,

$$QA + A^2 = \eta_0 A = 0. \tag{3.36}$$

The first of this equations is automatically obeyed in light of (3.32). Indeed, it seems to us that a consistent gauge fixing of the freedom related to the use of the large space could only be achieved by choosing a gauge that enforces one of these two equations (or a linear combination thereof). Thus, neglecting the possibility of enforcing a linear combination of the equations, a consistent gauge fixing to the zero picture would either lead to the non-polynomial theory, as we do here, or to the modified theory, should the second equation be chosen as the gauge condition.

With our choice of the gauge fixing, the second equation takes the form,

$$\eta_0(e^{-\Phi} Q e^\Phi) = 0, \tag{3.37}$$

which is exactly the equation of motion of the non-polynomial theory. Hence, the equations of motion hold in one theory if and only if they hold in the other.

3.2.2 Residual gauge symmetry

Consider the infinitesimal form of the gauge transformations. This is completely general, since the string fields are not restricted to be infinitesimal. The non-polynomial theory has two types of gauge symmetry,

$$\delta e^\Phi = Q \hat{\Lambda}_0 e^\Phi + e^\Phi \eta_0 \hat{\Lambda}_1, \tag{3.38}$$

with the subscript referring to the picture number and the two gauge string fields are odd. We use the hat in order to distinguish these generators from those of the gauge string fields of the democratic theory. We already noticed that the first of these two transformations is just a new gauge symmetry that is induced by our parametrization (3.32). The second transformation does induce a change of A . Note, that had we chosen (3.34) or (3.35) for our A , the roles of the two types of gauge transformation would have changed, but the final

conclusion would have not. The change of A (3.32) that is induced by a general change of Φ is,

$$\delta A = e^{-\Phi}(Q\delta e^\Phi - \delta e^\Phi A). \quad (3.39)$$

Plugging (3.38) into (3.39) leads to,

$$\delta A = \mathcal{Q}\eta_0\hat{\Lambda}_1, \quad (3.40)$$

where \mathcal{Q} is defined as in (3.16).

In the case of the democratic theory, restricting only to transformations that do not alter (3.26), leads to,

$$\delta A = \mathcal{Q}\Lambda_0 - \eta_0\Lambda_1, \quad (3.41)$$

where the gauge string fields Λ_k are restricted to obey,

$$\mathcal{Q}\Lambda_k = \eta_0\Lambda_{k+1}, \quad -\infty < k < \infty, \quad k \neq 0. \quad (3.42)$$

While the operator \mathcal{Q} is often named “the cohomology operator around the solution A ”, we remind that in the case at hand A is generally not a solution. On the other hand, had we been thinking of A as a solution of a theory whose kinetic operator is Q , it would have been a pure-gauge solution. The variation δA should be further restricted in order to maintain (3.32). A necessary condition for that is

$$\delta(QA + A^2) = Q\delta A = 0. \quad (3.43)$$

From the definition (3.16) and the gauge choice (3.32) it follows that,

$$\mathcal{Q}^2 = 0. \quad (3.44)$$

Hence, the general gauge transformation induced by the non-polynomial theory (3.40) obeys (3.43). In fact, the simplest resolution of (3.41) and (3.43) is to set

$$\Lambda_k = 0 \quad \forall k \neq 0. \quad (3.45)$$

The remaining condition of (3.42) is then,

$$\eta_0\Lambda_0 = 0, \quad (3.46)$$

which can be immediately solved to give (3.40), provided we identify,

$$\Lambda_0 = \eta_0\hat{\Lambda}_1. \quad (3.47)$$

The condition (3.45) can be thought of as a gauge condition for the gauge string fields. This is sensible, since the gauge symmetry of the theory is reducible and includes “a gauge for gauge symmetry”, “a gauge for gauge for gauge symmetry”, and so on, ad infinitum. However, the gauge for gauge symmetry of the theory uses the equations of motion. Hence, it might be the case that the condition can be justified only on-shell. This should be enough

for our purposes, since “Frobenius’ theorem” implies that gauge orbits are generally well defined only on-shell [24, 32, 33].¹⁰ Imposing the equations of motion leads to,

$$[\eta_0, \mathcal{Q}] = 0. \tag{3.48}$$

The operator \mathcal{Q} is trivial in the large space since,

$$\mathcal{Q}B = 0 \Rightarrow \mathcal{Q}e^L B = 0 \Rightarrow e^L B = \mathcal{Q}\Upsilon \Rightarrow B = \mathcal{Q}e^{-L}\Upsilon. \tag{3.49}$$

These two facts are enough to enable pushing the Λ_k ’s all the way to the gauge choice (3.45), in a similar manner to the proof of the equivalence of \mathcal{Q} in the small space at a given picture and $\mathcal{Q} - \eta_0$ in the large space with arbitrary picture (assuming a bounded or decaying behaviour as a function of picture number) [19]. We conclude that the gauge orbits associated with the residual gauge symmetry agree with those of the non-polynomial theory.

3.2.3 The action

Using (3.32), the partially gauge fixed action of the democratic theory can be written as,

$$S = - \oint \mathcal{O} \left(\frac{1}{6} AQA - \frac{1}{2} A\eta_0 A \right). \tag{3.50}$$

Note, that for this form of the action the allowed variation for A is not the most general one, due to the constraint (3.43). Thus, the generalization of the fundamental lemma of the calculus of variations does not hold and the action cannot be used for deriving the equation of motion. The way out is either to add to the action Lagrangian multipliers enforcing the constraint or to solve the constraint and rewrite the action in terms of Φ . The former option cannot be achieved covariantly without generating new unphysical degrees of freedom for the multiplier. The latter option, on the other hand, leads exactly to the non-polynomial action as we now prove.

One could have hoped to use the formal string fields A_Q and A_η in order to prove the equality of the actions. However, the presence of the \mathcal{O} mid-point insertion complicates the story, as it induces poles that should somehow be regularized. Ignoring this issue would lead to erroneous results. From the example in 3.1.1, we can infer that if we are to avoid the use of A_Q and A_η , we should not hope to be able to derive the results with the finite form of A . However, the same example also suggests a way out. We should write,

$$A(\alpha) = e^{-\alpha\Phi} \mathcal{Q}e^{\alpha\Phi}, \tag{3.51}$$

and prove that $\frac{dS}{d\alpha}$ obtains the same value in both theories. The equality of the initial condition is trivial, since in both cases the string field at $\alpha = 0$ vanishes and so does the action.

We first note that,

$$\frac{d}{d\alpha} A = \mathcal{Q}\Phi. \tag{3.52}$$

¹⁰In our case gauge orbits can be defined off-shell, but the gauge for gauge orbits cannot be defined, since these gauge symmetries do use the equations of motion.

Explicitly deriving the first term in the action with respect to α leads to,

$$\frac{d}{d\alpha} \left(-\frac{1}{6} \oint \mathcal{O} A Q A \right) = \frac{d}{d\alpha} \left(\frac{1}{6} \oint \mathcal{O} A^3 \right) = \frac{1}{2} \oint \mathcal{O} A^2 Q \Phi = -\frac{1}{2} \oint \mathcal{O} Q A Q \Phi. \quad (3.53)$$

Substituting we get,

$$\begin{aligned} \frac{dS}{d\alpha} &= \frac{1}{2} \oint \mathcal{O} \left(Q \Phi \eta_0 A + A \eta_0 Q \Phi - Q A Q \Phi \right) \\ &= \frac{1}{2} \oint \mathcal{O} \left(Q \Phi \eta_0 A + A \eta_0 Q \Phi - Q A Q \Phi + 2 Q A \eta_0 \Phi \right), \end{aligned} \quad (3.54)$$

where in the second line we have rewritten \mathcal{Q} in terms of Q and used

$$\oint \mathcal{O} A [B, C] = \oint \mathcal{O} [A, B] C. \quad (3.55)$$

Integrating by parts the first and third terms in (3.54) gives,

$$\frac{dS}{d\alpha} = \oint \mathcal{O} (Q A \eta_0 \Phi - A Q \eta_0 \Phi) = \oint (Q \mathcal{O}) A \eta_0 \Phi. \quad (3.56)$$

We know from (2.6) that $Q \mathcal{O}$ is just a sum of picture changing operators. Inspecting the rest of the integrand reveals that the only one that contributes is $X_0 = 1$. Hence we obtain,

$$\frac{dS}{d\alpha} = \oint A \eta_0 \Phi, \quad (3.57)$$

in agreement with the expression obtained for the non-polynomial theory (3.10). We conclude that the action is the same for both theories *even off-shell*. We thus finished establishing that the non-polynomial theory is a partially gauge fixed version of the democratic one.

3.3 The Ramond sector

The equations of motion for the Ramond sector of the non-polynomial theory were established by Berkovits in [17], where it was claimed that they cannot be derived from an action. However, it was explicitly assumed there that the action does not include any mid-point insertions. We would like to show that we can get these equations of motion by a partial gauge fixing of the democratic theory. We repeat the analysis twice. First, in 3.3.1, we compare the equations of motion and on-shell gauge symmetries of both sides. Then, in 3.3.2, we explicitly write down the partially gauge (picture) fixed action, which is a non-polynomial, fully RNS action, with string fields at pictures 0 and $\frac{1}{2}$.

3.3.1 Direct comparison

Before attempting to compare the two theories, we have to settle some conventions. As presented here, the democratic theory has $Q - \eta_0$ as its kinetic operator. The conventions of [17], on the other hand, are adequate for the kinetic operator $Q + \eta_0$. It is easy to see that a theory based on this operator is dual to the one we are using here. The transformations between the two are established by appending a minus sign to all the NS string fields, whose picture number is odd as well as to all even picture number mid-point insertions.

For the Ramond string fields it is immaterial which ones get the minus sign, as long as the signs alternate as a function of the picture number. To summarize, one can choose,

$$\Psi_k \rightarrow (-1)^{|k|} \Psi_k, \quad \mathcal{O}_k \rightarrow -(-1)^k \mathcal{O}_k, \quad \eta_0 \rightarrow -\eta_0. \quad (3.58)$$

With our conventions for the kinetic operator and with (3.58) in mind, the equations of motion of [17] take the form,

$$\eta_0(e^{-\Phi} Q e^{\Phi}) - (\eta_0 \Xi)^2 = 0, \quad \mathcal{Q}(\eta_0 \Xi) = 0, \quad (3.59)$$

where Ξ is a picture $\frac{1}{2}$ even Ramond string field. The fact that these equations include only $\eta_0 \Xi$ implies the existence of a gauge symmetry associated with this string field, which can be solved by defining,

$$\alpha = \eta_0 \Xi. \quad (3.60)$$

Now, α is the (odd) Ramond string field. It carries picture number $-\frac{1}{2}$,

$$\text{pic}(\alpha) = -\frac{1}{2}, \quad (3.61a)$$

and is constrained to obey,

$$\eta_0 \alpha = 0. \quad (3.61b)$$

The equations of motion are,

$$\eta_0(e^{-\Phi} Q e^{\Phi}) - \alpha^2 = 0, \quad \mathcal{Q}\alpha = 0. \quad (3.62)$$

Consider now the democratic theory and restrict the NS fields, as before, to obey (3.26) and (3.32). For the Ramond string field we choose the gauge conditions (3.61). Decomposing the equations of motion according to picture number leads exactly to (3.62). This is hardly a surprise, since it was shown already in [17] that the sum of the Ramond and NS string fields obey an equation of motion with $Q + \eta_0$ as the kinetic operator. What is new, is the realisation that these equations of motion *can* be derived from an action.

The gauge symmetry of (3.62), identified in [17], can be rewritten as,

$$\delta A = \mathcal{Q}\eta_0 \hat{\Lambda}_1 + [\alpha, \mathcal{Q}\hat{\Lambda}_{\frac{1}{2}}], \quad (3.63a)$$

$$\delta \alpha = [\alpha, \eta_0 \hat{\Lambda}_1] - \eta_0 \mathcal{Q}\hat{\Lambda}_{\frac{1}{2}}. \quad (3.63b)$$

We already saw that the residual NS gauge symmetry of A takes just this form in the democratic theory. In our case, this gauge symmetry produces also a picture $-\frac{1}{2}$ piece, which takes exactly the form of the first term in the r.h.s. of (3.63b). Hence, we should only consider the Ramond gauge symmetry, which is,

$$\delta A = [\alpha, \chi], \quad \delta \alpha = \mathcal{Q}\chi - \eta_0 \chi. \quad (3.64a)$$

The requirement that the gauge conditions are invariant under these transformations are,

$$[\alpha, \mathcal{Q}\chi] = \eta_0 \mathcal{Q}\chi = 0. \quad (3.64b)$$

The last of these conditions can be solved resulting in,

$$\delta\alpha = \eta_0 \hat{\chi}. \quad (3.65)$$

It seems reasonable that one can get this expression by gauge fixing the gauge for gauge symmetry as,

$$\mathcal{Q}\chi = 0, \quad (3.66)$$

which also completely resolves (3.64b). Solving this condition leads exactly to (3.63) with the identification,

$$\chi = \mathcal{Q}\hat{\Lambda}_{\frac{1}{2}}. \quad (3.67)$$

3.3.2 The non-polynomial RNS action

For the purpose of deriving the equations of motion from the action, we have to recast the action in terms of unconstrained string fields. Using the gauge conditions (3.26), (3.32) and (3.61), the action (2.3) can be written as,

$$S_{\text{NP}} = \oint A_\eta e^{-L} A_Q - \oint P \left(\frac{1}{2} \alpha Q \alpha + e^{-\Phi} Q e^\Phi \alpha^2 \right). \quad (3.68)$$

Here, we refrained from using A_Q in the part that includes the P mid-point insertion in order to avoid potential ambiguities. The form of the action (3.68) is still not satisfactory, since it depends on the constrained string field α . The problematic constraint (3.61b) can be resolved by reverting to the even Ξ field (3.60). The resulting action,

$$S_{\text{NP}} = \oint A_\eta e^{-L} A_Q - \oint P \left(\frac{1}{2} \eta_0 \Xi Q \eta_0 \Xi + e^{-\Phi} Q e^\Phi (\eta_0 \Xi)^2 \right), \quad (3.69)$$

is unconstrained and can be used for deriving the equations of motion. We can use the \mathbb{Z}_2 symmetry, to be discussed in the following sections, in order to define yet another fully RNS action,

$$\tilde{S}_{\text{NP}} = \oint A_Q e^L A_\eta - \oint \xi \left(\frac{1}{2} Q \Xi \eta_0 Q \Xi + e^\Phi \eta_0 e^{-\Phi} (Q \Xi)^2 \right), \quad (3.70)$$

where now Ξ carries picture number $-\frac{1}{2}$. This action should share the properties of (3.69). For concreteness and in order to continue the discussion so far, we stick to the action (3.69).

For deriving the equations of motion from (3.69), we have to evaluate the variations with respect to Φ and Ξ . The former variation leads to,

$$\delta_1 S_{\text{NP}} = \oint \left[(\eta_0 A - \alpha^2) + P Q (\alpha^2) \right] e^{-\Phi} \delta e^\Phi, \quad (3.71a)$$

while the latter leads to,

$$\delta_2 S_{\text{NP}} = \oint \left[Y Q \alpha - P [\eta_0 A, \alpha] \right] \delta \Xi. \quad (3.71b)$$

Here, we wrote the final expressions in terms of A and α , which should be interpreted as functions of Φ and Ξ . The fact that we used the unconstrained fields in the derivation of

this expression implies that the “fundamental lemma of the calculus of variations” holds and as a result, the expressions inside the square brackets vanish. These expressions include various distinct mid-point insertions and would, thus, vanish, if and only if the coefficients of each of those mid-point insertions separately vanish. The resulting equations of motion are therefore,

$$\eta_0 A - \alpha^2 = 0, \tag{3.72a}$$

$$\mathcal{Q}(\alpha^2) = 0, \tag{3.72b}$$

$$\mathcal{Q}\alpha = 0, \tag{3.72c}$$

$$[\eta_0 A, \alpha] = 0. \tag{3.72d}$$

These equations are not independent, since (3.72b) follows from (3.72c), while (3.72a) implies (3.72d). Hence, the independent equations of motion are exactly those of (3.62), as predicted by Berkovits.

The transformations (3.33) and

$$\delta \Xi = \eta_0 \hat{\Lambda}_{3/2}, \tag{3.73}$$

are easily recognized as gauge symmetries of the action (3.69). Of the other two expected transformations (3.63), the one associated with $\hat{\Lambda}_1$ can also be seen to hold, e.g., using (3.71). The transformation induced by $\hat{\Lambda}_{1/2}$, on the other hand, leaves the action invariant only on-shell. It might be possible that some generalization of it holds. However, it might also be the case that the partial gauge fixing that we performed treats differently the on-shell and off-shell cases. This seems plausible, since the various gauge symmetries are intertwined in the original, democratic, theory. At any rate, one can try to further gauge fix the action (3.69) and use it as a starting point for deriving the RNS perturbation theory. It would be important in this case to treat the mid-point insertion P as part of the definition of the R-sector measure, instead of trying to “invert” it. This might lead to some subtleties, similar to the ones described in section 6 of [5]. We leave the issues of further gauge fixing and the construction of perturbation theory based on this action to future work.

4 Gauge fixing at picture number -1

Standard gauge fixing leads to the modified theory. This is dual to the non-polynomial theory in the sense that the role of the gauge fixing and the equation of motion is interchanged,

$$QA + A^2 = 0 \quad \iff \quad \eta_0 A = 0. \tag{4.1}$$

There is also another duality, the \mathbb{Z}_2 duality of [9, 10], which we encounter below.

Recall first the consequences of fixing the theory to picture numbers zero and $-\frac{1}{2}$. Collecting the components of the equations of motion at different pictures leads to,

$$QA + A^2 = 0, \tag{4.2a}$$

$$-\eta_0 A + \alpha^2 = 0, \tag{4.2b}$$

$$Q\alpha + [A, \alpha] = 0, \tag{4.2c}$$

$$-\eta_0 \alpha = 0. \tag{4.2d}$$

These equations are not quite those of the modified cubic theory. For one thing, we separated terms at different picture numbers, while in the former interpretation explicit mid-point insertions of picture changing operators were used. However, there is also another difference: the “gauge choice” that fixes A to the small space is inconsistent, since it would set $\alpha^2 = 0$. We shall not try here to resolve the Ramond sector gauge fixing associated with the small space $\text{pic}(A) = 0$ gauge choice, since we already gave an alternative in the previous section. Instead we want to examine the NS sector here and in the analogous $\text{pic}(A) = -1$ case.

Restricting to the NS sector, the equations of motion (4.2) reduce to the familiar,

$$QA + A^2 = 0, \tag{4.3a}$$

$$\eta_0 A = 0. \tag{4.3b}$$

Now, not only it is possible to fix the string field to the small space, but it is forced upon us. The linearized gauge symmetry is,

$$\delta A = Q\Lambda, \tag{4.4a}$$

$$\eta_0 \Lambda = 0. \tag{4.4b}$$

Again, the familiar expression of the modified cubic theory.

Consider now the NS sector at -1 picture. The equations of motion are,

$$QA = 0, \tag{4.5a}$$

$$-\eta_0 A + A^2 = 0. \tag{4.5b}$$

These equations are identical to those of the previous case under a \mathbb{Z}_2 symmetry that exchange Q with $-\eta_0$. Alternatively, the \mathbb{Z}_2 can be interpreted as exchanging Q with η_0 , while sending Ψ to $-\Psi$. This is exactly the way this symmetry is realised in the non-polynomial theory [12]. However, that case is different, since there the ghost and picture numbers of the string field equal zero.

The equations of motion (4.5) are very different from those of Witten’s theory. This is a new theory, in which the string field is restricted to the dual space. The cohomology of η_0 in the dual space is the same as that of Q in the usual small space. If we take (4.5a) and $\text{pic}(A) = -1$ as the gauge fixing conditions for the action (2.1) it reduces to,

$$S = \oint \mathcal{O}_2 \left(\frac{1}{2} A \eta_0 A + \frac{1}{3} A^3 \right). \tag{4.6}$$

Note, that we redefined $A \rightarrow -A$. An alternative representation for the theory is,

$$S = \int X_2 \left(\frac{1}{2} A \eta_0 A + \frac{1}{3} A^3 \right), \tag{4.7}$$

where the string field A is implicitly assumed to live in the dual space and where the new integration symbol means that the CFT expectation value is to be evaluated in this space. The double picture changing operator X_2 was obtained using

$$Q\mathcal{O}_2 = X_2. \tag{4.8}$$

In practice, it is more convenient to work with the representation (4.6), since together with (4.5a) it forms the simplest definition we have for (4.7).

The linearized gauge symmetry takes the form,

$$\delta A = \eta_0 \Lambda + [A, \Lambda], \tag{4.9a}$$

$$Q\Lambda = 0. \tag{4.9b}$$

Again, Λ is restricted to the dual space and the gauge transformations take the expected form in this space. The \mathbb{Z}_2 symmetry between the new theory and the modified cubic theory is naturally extended to cover the gauge symmetry.

As we mentioned in section 2.1, an important property of the modified cubic theory is the existence of the operator $c(0)|0\rangle$. Do we have an analogue of this operator in the new theory? Without it, it would be hard to believe that this theory could be consistent even at the classical level with only the NS string field. The existence of this operator is implied by the \mathbb{Z}_2 symmetry. Indeed, a unique operator with picture number -1 , ghost number 1 and conformal dimension -1 exists,

$$\hat{c} \equiv -\xi c c' e^{-2\phi}. \tag{4.10}$$

The chosen normalization and sign are justified below by (5.30). This operator belongs to the dual space in light of,

$$\hat{c} = Q(\xi c e^{-2\phi}). \tag{4.11}$$

This is a reassuring evidence for the consistency of the new theory. In section 5.2 below we show that this operator plays an important role in the construction of a tachyon vacuum solution.

5 Analytical classical solutions of the new theory

We illustrate the classical consistency of the new theory by finding analytical solutions thereof. We start with the case of marginal deformations in 5.1 and follow with tachyon vacuum solutions in 5.2.

5.1 Marginal deformation

The first analytical solution describing marginal deformations were found in [34, 35]. Their RNS counterparts for the non-polynomial theory were given in [36–38]. To get the analogue solutions in the modified cubic theory one can either map these solutions using (the mapping is described, e.g., in [39, 40]),

$$A = e^{-\Phi} Q e^{\Phi}, \tag{5.1}$$

or use some formal pure-gauge expression of the bosonic solution and reinterpret them in the RNS theory. A common limitation of the above constructions are that the marginal deformations should have regular OPE's.

A construction of solutions for singular marginal deformations was given in [41, 42]. Another construction was given in [43, 44]. The former method relies on the existence of a

formal pure-gauge form for the solutions, while the latter, which treats the singularities in a more systematic way, represents the solutions in terms of the integrated vertex operators.¹¹ While it was not proven, we believe that the two methods are equivalent [3]. Here, we use the former one and restrict ourselves to the photon marginal deformation for simplicity.

In all the constructions, the solution was given as a sum,

$$A = \sum_{n=1}^{\infty} \lambda^n A_n, \tag{5.2}$$

where λ is a parameter describing the strength of the marginal deformation. The leading order term is known from CFT considerations,

$$A_1 = V(0) |0\rangle, \tag{5.3}$$

where V is the unintegrated vertex operator associated with the marginal deformation.

The building blocks of the construction [41, 42], were V and its formal Q -primitive W , i.e., W should obey,

$$QW_{\text{old}} = V_{\text{old}}, \tag{5.4}$$

but should not be considered as part of the Hilbert space. In our case, the string field carries picture -1 . Hence, it follows from (5.3) that V should also carry this picture. The unintegrated photon vertex operator in this picture is,

$$V = c\psi e^{-\phi}, \quad \psi \equiv a_\mu \psi^\mu, \tag{5.5}$$

and we assume that the vector a was chosen such that,

$$\psi(z)\psi(0) \sim \frac{1}{z^2}. \tag{5.6}$$

The case of a light-like deformation is simpler. One should merely follow the discussion below, excluding the last step.

In order to derive W we recall that the (integrand of the) integrated vertex operator U obeys,

$$QU_{\text{old}} = \partial V_{\text{old}}, \tag{5.7}$$

which together with (5.4) implies,

$$\partial W_{\text{old}} = U_{\text{old}}. \tag{5.8}$$

The fact that we work in the dual space suggests that the equations that we should solve are instead,

$$\eta_0 U = \partial V, \tag{5.9}$$

$$\partial W = U. \tag{5.10}$$

¹¹A novel representation, based on boundary changing operators was recently presented in [45].

In addition, U should live in the dual space, i.e., it should obey,

$$QU = 0. \quad (5.11)$$

The constraints (5.9) and (5.11) together with the requirement of a fixed picture number for U fix it completely,

$$U = \partial(\xi c e^{-\phi} \psi) + i\partial X, \quad X \equiv a_\mu X^\mu. \quad (5.12)$$

From this expression we can read,

$$W = \xi c e^{-\phi} \psi + iX, \quad (5.13)$$

and the formal nature of W comes from its dependence on x_0 , the zero mode of $X(z)$, which is not defined if the space is compactified, i.e., if the deformation is indeed a non-trivial marginal deformation.

We write U and W in terms of $X(z)$, the holomorphic part of $X(z, \bar{z})$, since we would have to work with holomorphic expressions.¹² Moreover, in this way we can avoid subtleties related to boundary normal ordering. In the general case, holomorphicity implies that the marginal deformation is exactly marginal [46], as should be the case for the photon.

The first term in the definition of W (5.13) has regular OPE with all its powers. Hence, all the subtleties related to singularities of the OPE come from the X insertion. It follows that the singularity structure here is identical to that of the photon solution of the modified theory. Thus, we can immediately write down the solution,

$$A = \eta_0 \Lambda \frac{1}{1 - \Lambda}, \quad (5.14a)$$

$$\Lambda = \sum_{n=1}^{\infty} \lambda^n \Lambda_n, \quad (5.14b)$$

$$\Lambda_n = \frac{(-1)^{n-1}}{n!} (W^n | 0) \Omega^{n-1}. \quad (5.14c)$$

The W^n are implicitly normal ordered.

The proof that the solution (5.14a) is a genuine one, i.e., that A is x_0 -independent, follows exactly as in the bosonic case. We assume as the induction hypothesis that $A_{<n}$ are x_0 -independent. We then use (5.2) and (5.14) to write,

$$A_n = \eta_0 \Lambda_n + \sum_{k=1}^{n-1} A_{n-k} \Lambda_k. \quad (5.15)$$

We see that,

$$\partial_{x_0} A_n = \eta_0 \partial_{x_0} \Lambda_n + \sum_{k=1}^{n-1} A_{n-k} \partial_{x_0} \Lambda_k, \quad (5.16)$$

¹²Strictly speaking, $X(z)$ is not holomorphic, since it contains a logarithmic component. However, this component drops out from all the computations relevant to the current discussion and can be ignored. Everything works exactly the same as in the usual case, described in [41] and in 6.2 of [3].

where we dropped out the terms that vanish according to the induction hypothesis. One can see that this expression equals zero, provided that the following holds,

$$\partial_{x_0}\Lambda_n = -i\Lambda_{n-1}\Omega. \tag{5.17}$$

This identity follows immediately from the definition (5.14c).

In order to explicitly evaluate various coefficients, it might be desirable to write the solution as a sum of fully normal ordered expressions. To that end we write,

$$W^n = (iX)^n + n(iX)^{n-1}(\xi c e^{-\phi}\psi) + \frac{n(n-1)}{2}(iX)^{n-2}(\xi\xi'cc'e^{-2\phi}). \tag{5.18}$$

The terms $\xi c e^{-\phi}\psi$ and $\xi\xi'cc'e^{-2\phi}$ are regular and primary. The X^n are not primary but their mutual OPE's can be deduced from those of e^{kX} . The absence of x_0 in the expression before normal ordering implies its absence also in the fully normal ordered result. From this fact it follows that the solution can be written in terms of integrals of (powers of) ∂X . We see that the solution now depends on U , just as in [43].

In some cases one might wish to impose the reality condition. This can be achieved in a way that is very similar to the bosonic case, i.e., we refer to the solution described so far as the “left solution” and define also the “right solution”. These two solutions are gauge equivalent and the explicit gauge transformation between the two can be easily derived [41]. The real solution is then defined by going half-way along the gauge orbit connecting the two solutions [43].

5.2 Tachyon vacuum solutions

Schnabl’s solution [47] for the tachyon vacuum of the bosonic theory,¹³ was generalized by Erler to the case of the modified theory [52]. The construction uses the formal pure gauge form of the solution and the split string notations of [53, 54], which we also employ. A \mathbb{Z}_2 dual of this solution should exist in the dual theory.

Erler’s solution is given in a formal gauge form using “the same” gauge string field Λ as in the bosonic case,

$$\Lambda_E = Bc|0\rangle, \tag{5.19}$$

with B defined as,

$$B = -\int_{-i\infty}^{i\infty} b(z)dz. \tag{5.20}$$

The solution itself is then given by,

$$A_E = Q\Lambda_E \frac{1}{1 - \Lambda_E}. \tag{5.21}$$

The most natural generalization for our case would be,

$$\Lambda = \hat{B}\hat{c}|0\rangle, \tag{5.22}$$

$$A = \eta_0\Lambda \frac{1}{1 - \Lambda}. \tag{5.23}$$

¹³See also [48–51].

Again we want to have,

$$\hat{B} = - \int_{-i\infty}^{i\infty} \hat{b}(z) dz. \quad (5.24)$$

What is needed, is the identification of the operators \hat{b} and \hat{c} , which for consistency should both live in the dual space. We already identified \hat{c} in (4.10). The identification of \hat{b} becomes simple when one examines the $N = 4$ generators of [9] (eq. (5.1) there),

$$G^+ = J_B, \quad \tilde{G}^+ = \eta, \quad G^- = b, \quad (5.25)$$

where J_B is (up to a total derivative) the BRST current. From the above one can immediately deduce that our \hat{b} should be the \tilde{G}^- generator of [9]. The simplest representation for this operator is using the identity,

$$\hat{b} = \tilde{G}^- = [G^+, J^{--}] = Q(b\xi) = \xi T_{\text{tot}} - bX + \xi''. \quad (5.26)$$

In this representation it is clear that b lives in the dual space, since it is Q exact in the large space. It is also easy to see that another $N = 4$ identity holds,

$$\eta_0 \hat{b} = T_{\text{tot}}. \quad (5.27)$$

This is the analogy for our case of $Qb = T_{\text{tot}}$, which plays an important role in Erler's construction. From (5.27) we immediately see that,

$$\eta_0 \hat{B} = K, \quad (5.28)$$

where K equals that of the modified theory and thus requires no “hat”,

$$K = - \int_{-i\infty}^{i\infty} T(z) dz. \quad (5.29)$$

Another pair of identities that are easily verified are,

$$\hat{b}(z) \hat{c}(0) \sim \frac{1}{z}, \quad (5.30)$$

$$[\hat{B}, \hat{c}] = 1. \quad (5.31)$$

Substituting all the ingredients and using the relations they obey, (5.23) reduces to,

$$A = F \hat{c} \frac{K \hat{B}}{1 - \Omega} \hat{c} F + F (\hat{c} K \hat{c} - \eta_0 \hat{c}) \hat{B} F. \quad (5.32)$$

Direct evaluation of the second term using the OPE leads to,

$$\hat{c} K \hat{c} - \eta_0 \hat{c} = cc' e^{-2\phi} + \frac{1}{12} \xi \xi' cc' c'' c''' e^{-4\phi}. \quad (5.33)$$

One can see that this expression has regular OPE's with \hat{b} and \hat{c} . Comparison with Erler's solution suggests that we should think of this expression as $\hat{\gamma}^2$. We can make this statement more precise. In the small space, $\gamma = \eta e^\phi$ is the unique universal, i.e., matter-independent, operator with conformal weight $-\frac{1}{2}$, ghost number one and picture number

zero. A permissible $\hat{\gamma}$ should also be universal, it should live in the dual space and should have conformal weight $-\frac{1}{2}$, ghost number one and picture number -1 . These requirements fix $\hat{\gamma}$ up to normalization,

$$\hat{\gamma} = ce^{-\phi} + \frac{1}{4}\xi\xi'cc'e^{-3\phi}. \quad (5.34)$$

Had we used just the inverse picture changing operator to find $\hat{\gamma}$, we would have gotten only the first term. Indeed, for an on-shell tachyon this would be enough and it is only on-shell vertices that can be manipulated using picture changing operators. For general values of the momentum (in flat space) our previous considerations imply that the \mathbb{Z}_2 symmetry fixes $\frac{1+2k^2}{4}$ as the coefficient of the second term. This expression reduces to $\frac{1}{4}$ and 0 for $k^2 = 0$ and $k^2 = -\frac{1}{2}$ respectively.

Using the OPE again, one finds,

$$\hat{\gamma}^2 = cc'e^{-2\phi} + \frac{1}{12}\xi\xi'cc'e^{-4\phi}, \quad (5.35)$$

in agreement with (5.33). Then, evaluation of one further OPE gives the full algebraic structure,

$$\begin{aligned} \eta_0\hat{B} &= K, & \eta_0\hat{c} &= \hat{c}K\hat{c} - \hat{\gamma}^2, & \eta_0\hat{\gamma}^2 &= \hat{c}K\hat{\gamma}^2 - \hat{\gamma}^2K\hat{c}, \\ \hat{B}^2 &= \hat{c}^2 = [\hat{\gamma}^2, \hat{c}] = [\hat{\gamma}^2, \hat{B}] = 0, & [\hat{B}, \hat{c}] &= 1. \end{aligned} \quad (5.36)$$

We recognize that the algebraic structure is identical to the standard one. Moreover, using it we see that the solution can now be also written in a form, which is manifestly the \mathbb{Z}_2 image of Erler's solution,

$$A = F\hat{c}\frac{K\hat{B}}{1-\Omega}\hat{c}F + F\hat{\gamma}^2\hat{B}F. \quad (5.37)$$

Instead of proving Sen's conjectures for this solution, we would like to generalize it and prove the conjectures for a simpler solution. One option for a generalization would be to study the one-parameter family of solutions of [40, 55]. However, if we want to get a simpler form of the solution, it would be better to find the counterpart of [56] (see also [57]), which generalizes to the RNS case the simple solutions of Erler and Schnabl [58]. The benefit of studying such solutions is that no phantom terms are needed for the evaluation of the action. The new solution is,

$$A = (-\hat{c} + \eta_0(\hat{B}\hat{c}))\frac{1}{1-K}. \quad (5.38)$$

The equations of motion easily follow. The kinetic term is,

$$\eta_0A = -\eta_0\hat{c}\frac{1}{1-K}. \quad (5.39)$$

Writing the interaction term as,

$$A^2 = \left[(-\hat{c} + \eta_0(\hat{B}\hat{c}))\frac{1}{1-K}(-\hat{c} + \eta_0(\hat{B}\hat{c})) \right] \frac{1}{1-K}, \quad (5.40)$$

we see that the expression inside the square brackets equals,

$$[\dots] = (\hat{c} + \eta_0(\hat{c}\hat{B}))\left(\hat{c} + \frac{\hat{B}}{1-K}\eta_0\hat{c}\right) = \hat{c}\hat{B}\eta_0\hat{c} + \eta_0(\hat{c}\hat{B})\hat{c} = \eta_0(\hat{c}\hat{B}\hat{c}) = \eta_0\hat{c}, \quad (5.41)$$

as it should.

We now turn to proving Sen’s conjectures, namely, we want to prove the triviality of the cohomology around the solution and show that the action of the solution equals minus the volume times the D-brane tension.

5.2.1 Trivial cohomology

The, by now standard [50, 52, 59], method for proving the triviality of the cohomology is to establish the existence of a contracting homotopy operator, i.e., a string field \mathcal{A} obeying $\mathcal{Q}\mathcal{A} = 1$, where 1 is the identity string field. Of course, in our case we have to show,

$$\eta_0\mathcal{A} + [A_0, \mathcal{A}] = 1. \tag{5.42}$$

Again, \mathcal{A} can immediately be guessed using the \mathbb{Z}_2 symmetry,

$$\mathcal{A} = -\hat{B} \frac{1}{1-K}. \tag{5.43}$$

Indeed,

$$\eta_0\mathcal{A} + [A_0, \mathcal{A}] = \left(-K + \hat{c}\hat{B} \frac{1}{1-K} + \frac{1}{1-K} \hat{B}\hat{c} - \hat{c} \frac{\hat{B}K}{1-K} - \frac{\hat{B}K}{1-K} \hat{c} \right) \frac{1}{1-K} = 1. \tag{5.44}$$

5.2.2 The action of the solution

In order to evaluate the action, we could presumably use once again the \mathbb{Z}_2 symmetry, defining explicitly the expectation values in the dual space for the $\hat{b}\hat{c}$ sector, for the PQ ($\hat{\xi}\hat{\eta}$) sector, etc. To that end, one would have to define a few more dual operators and show that the expectation values factor properly. Then, one would get to expressions that precisely match those that were already evaluated in the literature. Instead of going in this direction, we would like to evaluate the expectation values directly. The expressions that one sees in this way differ from those found before. The end result is, however, the same: Sen’s conjecture holds for our solution.

The energy per unit volume, i.e., ignoring the $\delta(0)$ factor that comes from the zero modes of the X sector, should equal $-\frac{1}{2\pi^2}$. Since our solution does not depend on the matter sector, we can simply define the integration as the expectation value in the ghost sectors. Then, we have to prove that,

$$E = -S = -\frac{1}{2\pi^2}. \tag{5.45}$$

We can use the equations of motion to write,

$$S = \frac{1}{6} \oint \mathcal{O}_2 A \eta_0 A. \tag{5.46}$$

The picture number two mid-point insertion \mathcal{O}_2 could be any regularized and primary version of ξX . We choose,

$$\mathcal{O}_2 = \xi(i)X(-i), \tag{5.47}$$

where the $\pm i$ are upper half plane coordinates. It was shown in [5] that the specific choice of \mathcal{O} does not change the value of the action for solutions. Moreover, for the specific choice (5.47), one can exchange the location of the two insertions even without using the fact that A is a solution,

$$\oint \xi(i)X(-i)A\eta_0A = \oint \xi(i)Q\xi(-i)A\eta_0A = \oint X(i)\xi(-i)A\eta_0A. \quad (5.48)$$

Here, we used the definition of X , integrated by parts and used the fact that A lives in the dual space.

Substituting (5.38) in (5.46) we get,

$$S = \frac{1}{6} \oint \mathcal{O}_2(\hat{c} - \eta_0(\hat{B}\hat{c})) \frac{1}{1-K} \eta_0 \hat{c} \frac{1}{1-K} = \frac{1}{6} \oint \mathcal{O}_2 \hat{c} \frac{1}{1-K} \eta_0 \hat{c} \frac{1}{1-K}, \quad (5.49)$$

where we dropped the second term, which is a total η_0 derivative in the dual space. One could also understand why does this term vanish in the large space by integrating the η_0 by parts, thus killing the ξ insertion, and then writing $X = Q\xi$ and integrating Q by parts. Next, we substitute,

$$\frac{1}{1-K} = \int_0^\infty e^{-t(1-K)} = \int_0^\infty e^{-t} W_t, \quad (5.50)$$

where W_t are wedge states.

We see that in contrast to many similar expressions, previously evaluated in the literature, in the case at hand we are left with no line integrals. Hence, we transform all expressions to the upper half-plane (in accord with our use of $\pm i$ for the mid-point insertions), instead of evaluating the expression in the cylinder coordinates with their funny looking correlators. Both, \hat{c} and $\eta_0\hat{c}$ are primaries of weight -1 . In the cylinder coordinates these insertions are located at 0 and t . The conformal transformation to the upper half plane takes the form,

$$\zeta(z) = \tan\left(\frac{\pi z}{t+s}\right), \quad (5.51)$$

where ζ is the upper half plane coordinate and z is the cylinder coordinate, where the lines $\Re(z) = 0$ and $\Re(z) = t+s$ are identified. Applying the conformal transformation leads to,

$$S = \int_0^\infty dt ds \frac{e^{-t-s}(t+s)^2}{6\pi^2(1+\zeta(t)^2)} \cdot \left\langle \xi(i)(2e^{2\phi}b\eta' + e^{2\phi}b'\eta + (e^{2\phi})'b\eta)(-i)(\xi c' e^{-2\phi})(0)(c' e^{-2\phi})(\zeta(t)) \right\rangle \quad (5.52)$$

We can write the expectation value as,

$$\left\langle \dots \right\rangle = \partial_{\zeta \circ} \left(\left\langle \xi(i)\eta(\zeta)\xi(0) \right\rangle_{\xi\eta} \left\langle b(\zeta)(c')(0)(c')(\zeta(t)) \right\rangle_{bc} \left\langle e^{2\phi}(\zeta)e^{-2\phi}(0)e^{-2\phi}(\zeta(t)) \right\rangle_{\phi} \right) \Big|_{\zeta=-i}. \quad (5.53)$$

Here, we define the derivative to act in the correct way for reproducing the X insertion, i.e., $\partial_{\zeta \circ}$ represents the sum of the derivatives acting on the three correlators, with a factor of

two multiplying the one acting on the first correlator. Substitution of standard expressions reveals,¹⁴

$$\langle \dots \rangle_{\xi\eta} = \frac{i}{\zeta(i-\zeta)}, \quad \langle \dots \rangle_{bc} = -\frac{\zeta(t)^4}{\zeta^2(\zeta(t)-\zeta)^2}, \quad \langle \dots \rangle_{\phi} = \frac{\zeta^4(\zeta(t)-\zeta)^4}{\zeta(t)^4}. \quad (5.54)$$

Applying the derivative and putting it all together leads to a very simple expression,

$$S = \int_0^\infty \frac{e^{-t-s}(t+s)^2}{12\pi^2} dt ds = \frac{1}{2\pi^2}, \quad (5.55)$$

in agreement with (5.45).

6 Conclusions

In this paper we demonstrated the consistency of the democratic theory by showing that it can be reduced to the reliable non-polynomial theory by a specific partial gauge fixing. Moreover, we managed to extend this partial gauge fixing and obtained a string field theory action, at fixed picture numbers for the open RNS string. We further explained, that in contrast with some naive expectations, the democratic theory cannot be gauge fixed to produce Witten’s theory. A gauge fixing to the -1 picture of the NS string field is possible. However, it leads to a new theory, which is the \mathbb{Z}_2 counterpart of the modified theory. The \mathbb{Z}_2 symmetry was also used for generating another variant of the non-polynomial RNS string field theory action.

One could criticize our construction on several grounds. An obvious criticism is that we did not prove that our gauge choice (3.32) is globally permissible. On the other hand, we do not know also whether Siegel’s gauge or Schnabl’s gauge are globally permissible. We do know that at the linearized level (with respect to the string field) our choice is adequate. Our expectation is that if there are issues related to its global validity, they would probably imply that some legitimate configurations become singular when represented in the non-polynomial theory and not that there is a problem with the democratic theory itself.

The problem with showing that any of our gauge choices is globally permissible is also related to the problem of defining the space of string fields. This is another potential source of criticism on our construction. While there is a lack of understanding regarding the definition of these spaces in all existing variants of string field theory, there are two specific points that are particular to our construction. The first issue is related to the use of mid-point insertions in the action, which we resolved by demanding that the space of string fields does not include string fields that can be interpreted as having mid-point insertions. This resolution is most probably correct. The reason is that if it does not hold, then singularities would emerge from star multiplying string fields in general, regardless of the existence of mid-point insertions or their regularity. Thus, if this constraint cannot be enforced, then all string field theories that rely on Witten’s star product ought to be inconsistent, i.e., not only our theory, but also those with “regular” mid-point insertions in the action [20, 60, 61]

¹⁴We follow the conventions $\langle cc'c'' \rangle = 2$, in accord with our choice of a minus sign for the action (2.1).

and those with actions that lack mid-point insertions altogether [12, 21]. An encouraging observation is that the algebra of finite star products of states with no mid-point insertions closes on states with no mid-point insertions.¹⁵

The other delicate point regarding the space of string fields is our requirement that the string field components decay fast enough as a function of the picture number. Again, we do not know how to quantify this condition, except for vertex operators. For those we know that we are dealing with an infinite number of copies of the same object. We therefore have to require that the sum of coefficients (as calculated from any base representation using multi-picture changing operators and while ignoring exact terms) of any vertex operator is absolutely convergent. This condition invalidates the use of the contracting homotopy operators of [19] and leads to a correct cohomology problem [5]. We believe that some sort of a generalization of this condition to off-shell states should exist. What could have resolved this issue is a positive definite norm that would have allowed us to compare the “size” of different vertex operators and to include also non-closed states. Unfortunately, a canonical norm of this sort does not exist. The lack of the norm and the inability to compare different vertex operators is exactly the usual problem with defining the space of string fields. Thus, as usual, we do not have a definition for the desired space, but we know some properties thereof and it seems that a proper definition should exist. This is in contrast to, e.g., the situation with the pure-spinor string field theory, where it seems that there is no hope of obtaining a sensible space of string fields using the current formulation of the theory [65, 66].¹⁶

Another ground for criticizing the democratic theory could be the presence of operators of arbitrarily negative conformal weight. We believe that this should not be considered a problem of principle, since these operators correspond to auxiliary fields. Nonetheless, this state of affairs can pose a difficulty to some numerical, e.g., lattice [68] studies of the theory. One might try to overcome this by modifying the democratic theory into a “semi-democratic” one along the lines of “the big picture” [69]. It might be interesting to study this possibility.

There is still much more to be done regarding the formulation of the democratic theory. Other than the issues mentioned already the most salient point is the understanding of supersymmetry. Despite some ideas presented in [5], we don’t know if and how does supersymmetry act on off-shell states. Note, that this is not the “usual” problem with supersymmetry that does not close off-shell. Here, we don’t even know how to define it off-shell. Hence, we cannot even claim that it is a symmetry. The problem with the construction of supersymmetry is that singular OPEs can occur between the mid-point insertion \mathcal{O} and the supersymmetry current, regardless of its picture. It might happen that a specific combination of pictures for the current resolves this problem. Another possibility is to use the freedom of adding exact terms to \mathcal{O} . While these terms are of no

¹⁵An alternative to the constructions based on the star product exists, namely the one that includes stubs, similarly to the case of closed string field theory [62]. Constructions of stub-based open and open-closed string field theories appeared in [63, 64]. These theories tend to be non-polynomial.

¹⁶It should be noted, however, that this theory makes at least some sense at the classical level, since it supports analytical solutions for marginal deformations [67].

importance on-shell, they might lead to different results off-shell. It would be interesting to find out whether the construction of supersymmetry constraints these terms. While such a restriction on the form of the exact terms would come from supersymmetry, the derived form of \mathcal{O} would be universal.

There are several other challenges and possible directions for future research. Devising other gauge fixings, in particular fixings that do not include a projection to given picture numbers (other than the trivial cases mentioned in (2.12)) would be very interesting. Deriving the surface states associated with a classical solutions along the lines of [70] would also be useful. Generalization of the construction to some of the other theories studied in [9] by replacing Q and η_0 by the more general G^\pm might lead to string field theories around new backgrounds. In the cases in which these operators possess non-trivial cohomologies there would be no contracting homotopy operators and hence no picture changing operators. There would be no gauge symmetry associated with picture changing, still the action might be correct, provided that the \mathcal{O} insertion can be found. A related issue is the understanding of the correspondence, if any, between the democratic theory and pure-spinor string field theory.

Another possible research direction relates to the recent observation that the modified theory and the non-polynomial theory support different classes of classical solutions [71], at least if one does not impose the reality condition. This observation appears to contradict our claim that both these theories can be obtained from the NS sector of the democratic theory by a partial gauge fixing. It seems that this contradiction can be resolved in one of a few ways. It might be the case that one of the gauge fixings that we employed breaks down at some finite value of the string field. Another possibility is that not all of the assumptions of [71] hold, e.g., it might be the case that the \mathcal{L}^- expansion is not a legitimate one or that the c, B, K, G, γ subalgebra considered there is essentially different from the complete string field algebra. At any rate, we believe that the democratic theory before gauge fixing is the more reliable one. The resolution of this puzzle might shed new light on issues such as gauge fixing in string field theory and the construction of string field spaces.

As a last idea regarding future directions we would like to suggest the construction of heterotic and closed RNS string field theories along the lines of the democratic theory. Explicit insertions of picture changing operators might be useful for resolving the difficulties with the Ramond sectors of these theories. This idea gets complicated by the fact that closed strings have no “mid-points”. Nonetheless, one can look for other special points for the insertion of operators. In fact, such a construction was already carried out successfully by Saroja and Sen [72]. However, it was limited to the NS sector. It might well be the case that adding some of the element of the democratic theory to their construction would allow for the inclusion of the Ramond sectors. We are currently studying this issue.

Acknowledgments

I would like to thank Nathan Berkovits, Stefan Fredenhagen, Udi Fuchs, Michael Kiermaier, Martin Schnabl, Cobi Sonnenschein, Leonardo Rastelli, Adam Schwimmer, Warren Siegel and Barton Zwiebach for discussions.

This work was supported by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-FG0205ER41360. My research is supported by an Outgoing International Marie Curie Fellowship of the European Community. The views presented in this work are those of the author and do not necessarily reflect those of the European Community.

References

- [1] A. Sen, *Descent relations among bosonic D-branes*, *Int. J. Mod. Phys. A* **14** (1999) 4061 [[hep-th/9902105](#)] [[SPIRES](#)].
- [2] A. Sen, *Universality of the tachyon potential*, *JHEP* **12** (1999) 027 [[hep-th/9911116](#)] [[SPIRES](#)].
- [3] E. Fuchs and M. Kroyter, *Analytical solutions of open string field theory*, [arXiv:0807.4722](#) [[SPIRES](#)].
- [4] M. Schnabl, *Algebraic solutions in open string field theory — a lightning review*, [arXiv:1004.4858](#) [[SPIRES](#)].
- [5] M. Kroyter, *Superstring field theory in the democratic picture*, [arXiv:0911.2962](#) [[SPIRES](#)].
- [6] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys. B* **271** (1986) 93 [[SPIRES](#)].
- [7] G.T. Horowitz, R.C. Myers and S.P. Martin, *BRST cohomology of the superstring at arbitrary ghost number*, *Phys. Lett. B* **218** (1989) 309 [[SPIRES](#)].
- [8] B.H. Lian and G.J. Zuckerman, *BRST cohomology of the supervirasoro algebras*, *Commun. Math. Phys.* **125** (1989) 301 [[SPIRES](#)].
- [9] N. Berkovits and C. Vafa, *$N = 4$ topological strings*, *Nucl. Phys. B* **433** (1995) 123 [[hep-th/9407190](#)] [[SPIRES](#)].
- [10] N. Berkovits, *A new description of the superstring*, [hep-th/9604123](#) [[SPIRES](#)].
- [11] C. Wendt, *Scattering amplitudes and contact interactions in Witten's superstring field theory*, *Nucl. Phys. B* **314** (1989) 209 [[SPIRES](#)].
- [12] N. Berkovits, *Super-Poincaré invariant superstring field theory*, *Nucl. Phys. B* **450** (1995) 90 [[hep-th/9503099](#)] [[SPIRES](#)].
- [13] I.Y. Arefeva, P.B. Medvedev and A.P. Zubarev, *Background formalism for superstring field theory*, *Phys. Lett. B* **240** (1990) 356 [[SPIRES](#)].
- [14] I.Y. Arefeva, P.B. Medvedev and A.P. Zubarev, *New representation for string field solves the consistency problem for open superstring field theory*, *Nucl. Phys. B* **341** (1990) 464 [[SPIRES](#)].
- [15] C.R. Preitschopf, C.B. Thorn and S.A. Yost, *Superstring field theory*, *Nucl. Phys. B* **337** (1990) 363 [[SPIRES](#)].
- [16] M. Kroyter, *Superstring field theory equivalence: Ramond sector*, *JHEP* **10** (2009) 044 [[arXiv:0905.1168](#)] [[SPIRES](#)].
- [17] N. Berkovits, *The Ramond sector of open superstring field theory*, *JHEP* **11** (2001) 047 [[hep-th/0109100](#)] [[SPIRES](#)].

- [18] Y. Michishita, *A covariant action with a constraint and Feynman rules for fermions in open superstring field theory*, *JHEP* **01** (2005) 012 [[hep-th/0412215](#)] [[SPIRES](#)].
- [19] N. Berkovits, *Relating the RNS and pure spinor formalisms for the superstring*, *JHEP* **08** (2001) 026 [[hep-th/0104247](#)] [[SPIRES](#)].
- [20] M. Kroyter, *On string fields and superstring field theories*, *JHEP* **08** (2009) 044 [[arXiv:0905.1170](#)] [[SPIRES](#)].
- [21] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys. B* **268** (1986) 253 [[SPIRES](#)].
- [22] N. Berkovits, A. Sen and B. Zwiebach, *Tachyon condensation in superstring field theory*, *Nucl. Phys. B* **587** (2000) 147 [[hep-th/0002211](#)] [[SPIRES](#)].
- [23] M. Kroyter, *Comments on superstring field theory and its vacuum solution*, *JHEP* **08** (2009) 048 [[arXiv:0905.3501](#)] [[SPIRES](#)].
- [24] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, Princeton U.S.A. (1992) [[SPIRES](#)].
- [25] P.-J. De Smet and J. Raeymaekers, *The tachyon potential in Witten's superstring field theory*, *JHEP* **08** (2000) 020 [[hep-th/0004112](#)] [[SPIRES](#)].
- [26] I.Y. Arefeva, D.M. Belov and A.A. Giriyavets, *Construction of the vacuum string field theory on a non-BPS brane*, *JHEP* **09** (2002) 050 [[hep-th/0201197](#)] [[SPIRES](#)].
- [27] I.Y. Aref'eva, A.S. Koshelev, D.M. Belov and P.B. Medvedev, *Tachyon condensation in cubic superstring field theory*, *Nucl. Phys. B* **638** (2002) 3 [[hep-th/0011117](#)] [[SPIRES](#)].
- [28] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [[hep-th/0108200](#)] [[SPIRES](#)].
- [29] G.T. Horowitz, J.D. Lykken, R. Rohm and A. Strominger, *A purely cubic action for string field theory*, *Phys. Rev. Lett.* **57** (1986) 283 [[SPIRES](#)].
- [30] N. Berkovits, Y. Okawa and B. Zwiebach, *WZW-like action for heterotic string field theory*, *JHEP* **11** (2004) 038 [[hep-th/0409018](#)] [[SPIRES](#)].
- [31] E. Witten, *Nonabelian bosonization in two-dimensions*, *Commun. Math. Phys.* **92** (1984) 455 [[SPIRES](#)].
- [32] M. Henneaux, *Lectures on the antifield-BRST formalism for gauge theories*, *Nucl. Phys. Proc. Suppl.* **18A** (1990) 47 [[SPIRES](#)].
- [33] J. Gomis, J. Paris and S. Samuel, *Antibracket, antifields and gauge theory quantization*, *Phys. Rept.* **259** (1995) 1 [[hep-th/9412228](#)] [[SPIRES](#)].
- [34] M. Schnabl, *Comments on marginal deformations in open string field theory*, *Phys. Lett. B* **654** (2007) 194 [[hep-th/0701248](#)] [[SPIRES](#)].
- [35] M. Kiermaier, Y. Okawa, L. Rastelli and B. Zwiebach, *Analytic solutions for marginal deformations in open string field theory*, *JHEP* **01** (2008) 028 [[hep-th/0701249](#)] [[SPIRES](#)].
- [36] T. Erler, *Marginal solutions for the superstring*, *JHEP* **07** (2007) 050 [[arXiv:0704.0930](#)] [[SPIRES](#)].
- [37] Y. Okawa, *Analytic solutions for marginal deformations in open superstring field theory*, *JHEP* **09** (2007) 084 [[arXiv:0704.0936](#)] [[SPIRES](#)].

- [38] Y. Okawa, *Real analytic solutions for marginal deformations in open superstring field theory*, *JHEP* **09** (2007) 082 [[arXiv:0704.3612](#)] [[SPIRES](#)].
- [39] A. Kling, O. Lechtenfeld, A.D. Popov and S. Uhlmann, *Solving string field equations: new uses for old tools*, *Fortsch. Phys.* **51** (2003) 775 [[hep-th/0212335](#)] [[SPIRES](#)].
- [40] E. Fuchs and M. Kroyter, *On the classical equivalence of superstring field theories*, *JHEP* **10** (2008) 054 [[arXiv:0805.4386](#)] [[SPIRES](#)].
- [41] E. Fuchs, M. Kroyter and R. Potting, *Marginal deformations in string field theory*, *JHEP* **09** (2007) 101 [[arXiv:0704.2222](#)] [[SPIRES](#)].
- [42] E. Fuchs and M. Kroyter, *Marginal deformation for the photon in superstring field theory*, *JHEP* **11** (2007) 005 [[arXiv:0706.0717](#)] [[SPIRES](#)].
- [43] M. Kiermaier and Y. Okawa, *Exact marginality in open string field theory: a general framework*, *JHEP* **11** (2009) 041 [[arXiv:0707.4472](#)] [[SPIRES](#)].
- [44] M. Kiermaier and Y. Okawa, *General marginal deformations in open superstring field theory*, *JHEP* **11** (2009) 042 [[arXiv:0708.3394](#)] [[SPIRES](#)].
- [45] M. Kiermaier, Y. Okawa and P. Soler, *Solutions from boundary condition changing operators in open string field theory*, [arXiv:1009.6185](#) [[SPIRES](#)].
- [46] A. Recknagel and V. Schomerus, *Boundary deformation theory and moduli spaces of D-branes*, *Nucl. Phys. B* **545** (1999) 233 [[hep-th/9811237](#)] [[SPIRES](#)].
- [47] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory*, *Adv. Theor. Math. Phys.* **10** (2006) 433 [[hep-th/0511286](#)] [[SPIRES](#)].
- [48] Y. Okawa, *Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory*, *JHEP* **04** (2006) 055 [[hep-th/0603159](#)] [[SPIRES](#)].
- [49] E. Fuchs and M. Kroyter, *On the validity of the solution of string field theory*, *JHEP* **05** (2006) 006 [[hep-th/0603195](#)] [[SPIRES](#)].
- [50] I. Ellwood and M. Schnabl, *Proof of vanishing cohomology at the tachyon vacuum*, *JHEP* **02** (2007) 096 [[hep-th/0606142](#)] [[SPIRES](#)].
- [51] L. Bonora, C. Maccaferri and D.D. Tolla, *Relevant deformations in open string field theory: a simple solution for lumps*, [arXiv:1009.4158](#) [[SPIRES](#)].
- [52] T. Erler, *Tachyon vacuum in cubic superstring field theory*, *JHEP* **01** (2008) 013 [[arXiv:0707.4591](#)] [[SPIRES](#)].
- [53] T. Erler, *Split string formalism and the closed string vacuum*, *JHEP* **05** (2007) 083 [[hep-th/0611200](#)] [[SPIRES](#)].
- [54] T. Erler, *Split string formalism and the closed string vacuum. II*, *JHEP* **05** (2007) 084 [[hep-th/0612050](#)] [[SPIRES](#)].
- [55] I.Y. Aref'eva, R.V. Gorbachev and P.B. Medvedev, *Tachyon solution in cubic Neveu-Schwarz string field theory*, *Theor. Math. Phys.* **158** (2009) 320 [[arXiv:0804.2017](#)] [[SPIRES](#)].
- [56] R.V. Gorbachev, *New solution of the superstring equation of motion*, *Theor. Math. Phys.* **162** (2010) 90 [[SPIRES](#)].
- [57] E.A. Arroyo, *Generating Erler-Schnabl-type solution for tachyon vacuum in cubic superstring field theory*, *J. Phys. A* **43** (2010) 445403 [[arXiv:1004.3030](#)] [[SPIRES](#)].

- [58] T. Erler and M. Schnabl, *A simple analytic solution for tachyon condensation*, *JHEP* **10** (2009) 066 [[arXiv:0906.0979](#)] [[SPIRES](#)].
- [59] I. Ellwood, B. Feng, Y.-H. He and N. Moeller, *The identity string field and the tachyon vacuum*, *JHEP* **07** (2001) 016 [[hep-th/0105024](#)] [[SPIRES](#)].
- [60] N. Berkovits, *Pure spinor formalism as an $N = 2$ topological string*, *JHEP* **10** (2005) 089 [[hep-th/0509120](#)] [[SPIRES](#)].
- [61] N. Berkovits and W. Siegel, *Regularizing cubic open Neveu-Schwarz string field theory*, *JHEP* **11** (2009) 021 [[arXiv:0901.3386](#)] [[SPIRES](#)].
- [62] B. Zwiebach, *Closed string field theory: quantum action and the B-V master equation*, *Nucl. Phys. B* **390** (1993) 33 [[hep-th/9206084](#)] [[SPIRES](#)].
- [63] B. Zwiebach, *Interpolating string field theories*, *Mod. Phys. Lett. A* **7** (1992) 1079 [[hep-th/9202015](#)] [[SPIRES](#)].
- [64] B. Zwiebach, *Oriented open-closed string theory revisited*, *Annals Phys.* **267** (1998) 193 [[hep-th/9705241](#)] [[SPIRES](#)].
- [65] Y. Aisaka, E.A. Arroyo, N. Berkovits and N. Nekrasov, *Pure spinor partition function and the massive superstring spectrum*, *JHEP* **08** (2008) 050 [[arXiv:0806.0584](#)] [[SPIRES](#)].
- [66] O.A. Bedoya and N. Berkovits, *GGI lectures on the pure spinor formalism of the superstring*, [arXiv:0910.2254](#) [[SPIRES](#)].
- [67] M. Kroyter, *Analytical solutions of pure-spinor superstring field theory*, to be published.
- [68] F. Bursa and M. Kroyter, *Lattice string field theory*, *PoS(Lattice 2010)047* [[arXiv:1009.4414](#)] [[SPIRES](#)].
- [69] N. Berkovits, M.T. Hatsuda and W. Siegel, *The big picture*, *Nucl. Phys. B* **371** (1992) 434 [[hep-th/9108021](#)] [[SPIRES](#)].
- [70] M. Kiermaier, Y. Okawa and B. Zwiebach, *The boundary state from open string fields*, [arXiv:0810.1737](#) [[SPIRES](#)].
- [71] T. Erler, *Exotic universal solutions in cubic superstring field theory*, [arXiv:1009.1865](#) [[SPIRES](#)].
- [72] R. Saroja and A. Sen, *Picture changing operators in closed fermionic string field theory*, *Phys. Lett. B* **286** (1992) 256 [[hep-th/9202087](#)] [[SPIRES](#)].