On the factorization of overlapping singularities at NNLO

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Abstract: Real and virtual corrections in NNLO QCD require multi-dimensional integrals with overlapping singularities. We first review ideas and methods which have been proposed for performing such computations. We then present a new method for the factorization of overlapping singularities based on non-linear integral transformations. We apply this method for the evaluation of all integral topologies which appear in double real radiation corrections in cross-section calculations for the production of a heavy system at hadron colliders. Finally, we demonstrate with typical examples that two-loop virtual corrections are amenable to the same method.

Contents

1 Introduction

Modern accelerator experiments require precise perturbative calculations for the event rate of a variety of physical processes. Jets, electroweak gauge bosons and heavy quarks are being produced copiously at the Tevatron and the LHC. The precision of the measurements of physical masses, coupling parameters and the structure of colliding hadrons depends significantly on theoretical uncertainties which are better controlled at higher orders in perturbation theory. The exclusion of hypotheses for novel particles and interactions is more significant when candidate signal processes are predicted accurately. With the arrival of new discoveries, the nature of physics laws will be deciphered more confidently with the aid of solid quantitative theory predictions.

Our abilities to simulate complicated physical processes beyond the leading order (LO) have been improved dramatically in the last few years. At next-to-leading-order (NLO), previously inaccessible calculations with up to five particles in the final state are now possible [1]. Basic collider processes with fewer particles have also been computed at next-to-next-to-leading (NNLO) order in QCD [2–13]. For hadron colliders, the experimental frontier in particle physics, only cross-sections for $2 \rightarrow 1$ processes have been computed at NNLO. Such computations must be extended to $2 \rightarrow 2$ processes which are relevant to the experimental program. These include top and bottom quark pair production, inclusive jet production, electroweak diboson production, electroweak gauge boson and Higgs production in association with jets, single top production and beyond the Standard Model signals. It is unclear whether existing methods are suited to this task, and refinements of traditional methods or the development of new ones are required in order to face the increased complexity of such calculations.

A fundamental technical difficulty in NNLO calculations is the appearance of multidimensional integrals over the momenta of up to two additional real or virtual particles with respect to the Born process. These integrals are separately infrared divergent and only their sum is finite. Higher order computations are performed almost exclusively within dimensional regularization, where real and virtual corrections are expanded in a dimensional regulator $\epsilon = 2 - \frac{D}{2}$, where D is the number of dimensions. Laurent expansions in ϵ are intricate in the presence of overlapping singularities. In this paper we present a new method for the calculation of the Laurent series in ϵ of multidimensional integrals which typically appear in NNLO computations and generic higher order computations.

Existing methods which tackle or bypass the problem of overlapping singularities are based on differential equations [14–16], Mellin-Barnes representations [17–20] and sector decomposition [21–23]. The first two approaches can be applied to the calculation of virtual or inclusive real radiation corrections. A subtraction method can reduce the problem of fully differential real radiation calculations at NNLO to inclusive phase-space calculations [10–12] enabling the method of differential equations and Mellin-Barnes for fully differential calculations. Sector decomposition can be applied universally, for virtual inclusive phase-space integrations and fully differential integrations of real radiation matrix-elements.

Sector decomposition has been the first successful method for performing fully differential NNLO calculations for hadron collider processes [7, 8]. This is largely attributed to the conceptual simplicity of the method and its algorithmic nature which permits a full automatization. The algorithm eliminates overlapping singularities by slicing the integration volume such that variables which contribute to an overlapping singular limit are ordered. In this way, the singularity is always factorized and it appears as a single singular limit of only the largest variable. While this algorithm leads to numerically stable evaluations, it requires a large number of integrals due to the slicing of the integration volume. This hinders the application of the method to processes with more complicated matrix-elements.

We present here an alternative method for the factorization of overlapping singularities. We have observed that a factorization is possible by means of simple rescaling of the integration variables and non-linear transformations which preserve the geometry of the integration boundaries. Our method leads to a rather small number of numerically stable integrations.

We apply our technique to all singular integral topologies which appear in the evaluation of NNLO double real radiation corrections to production processes of a massive system at hadron colliders. We present suitable phase-space parameterizations, analyze the singularity structure of the matrix-elements, and demonstrate how to obtain their expansion in ϵ with simple changes of integration variables. We then demonstrate how our technique can be applied to very complicated and maximally singular two-loop master integrals. In massless two-loop boxes overlapping singularities are very hard to treat with non-linear transformations, and we have not been able to find suitable ones which factorize them completely. On the other hand, a hybrid approach of non-linear transformations combined with sector decomposition is straightforward and more efficient than applying only sector decomposition.

In section 2, we review existing methods for the Laurent expansion in the dimensional regulator of integrals in higher perturbative orders. In section 3 we present our method and we demonstrate it on typical examples of integrals with overlapping singularities in section 4. In section 5 we discuss phase-space parameterizations and the singularity structure of double real radiation at NNLO. In section 6 we present the numerical evaluation of integrals from all topologies which appear in double real radiation corrections at NNLO. In section 7 we apply our method to maximally singular two-loop integrals, the crossed-triangle and the crossed-box. Finally, we present our conclusions in section 8.

2 Laurent expansion of Feynman integrals in the dimensional regulator

Loop integrals and phase-space integrals for the calculation of production rates of physical processes in quantum field theory are divergent in four space-time dimensions. In dimensional regularization, $d = 4 - 2\epsilon$, all divergent integrals are computed as a Laurent expansion in the dimensional regulator ϵ. This task is tedious due to physical singularities, corresponding to infrared and collinear configurations of real and virtual particles. Singular manifolds in the integration volume are of increasing complexity at higher orders in perturbation theory. We shall consider examples of physical loop and phase-space integrals in later sections of this paper. Here, we shall present illustrative mathematical examples with similar singular behavior as in realistic cases.

The easiest category of singular integrals is when divergences in the integrand occur as poles of a single variable. Consider an integral

$$I = \int_0^1 d^N \vec{x} \frac{f(x_1, \ldots, x_N)}{x_1^{1-\epsilon}}. \tag{2.1}$$

with $f(\vec{x})$ being a finite function in the integration volume. This integral is divergent for $\epsilon = 0$ due to the pole in $x_1 = 0$. To expand in ϵ we use a subtraction technique, isolating the pole contribution,

$$I = \int_0^1 d^N \vec{x} \frac{f(x_1, x_2, \ldots, x_N) - f(0, x_2, \ldots, x_N)}{x_1^{1-\epsilon}} + \int_0^1 d^N \vec{x} \frac{f(0, x_2, \ldots, x_N)}{x_1^{1-\epsilon}}. \tag{2.2}$$

In the first term, we are allowed to perform a Taylor expansion in ϵ, given that the integrand is finite in the limit $x_1 \to 0$. In the second term, we can perform the integration in ϵ easily.

We then have

$$I = \int_0^1 d^N\vec{x} \frac{f(x_1, x_2, \ldots, x_N) - f(0, x_2, \ldots, x_N)}{x_1} \left[\sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \log^n(x_1)\right]$$
$$+ \frac{1}{\epsilon} \int_0^1 d^N\vec{x} f(0, x_2, \ldots, x_N). \tag{2.3}$$

Equivalently, we write

$$\int_0^1 d^N\vec{x} x_1^{-1+\epsilon} f(x_1, \ldots, x_N) =$$
$$\int_0^1 d^N\vec{x} \left\{\frac{\delta(x_1)}{\epsilon} + \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left[\frac{\log^n(x_1)}{x_1}\right]_+\right\} f(x_1, \ldots, x_N). \tag{2.4}$$

All integral coefficients in the Laurent series of the last expression can be evaluated numerically. In case of many factorized singularities,

$$I = \int_0^1 d^N\vec{x} \frac{f(\vec{x})}{x_{i_1}^{1-a_1\epsilon} \ldots x_{i_m}^{1-a_m\epsilon}}, \tag{2.5}$$

we can apply readily the same procedure, and obtain a Laurent expansion in ϵ with the substitution

$$x_i^{-1+a_i\epsilon} = \frac{\delta(x_i)}{a_i\epsilon} + \sum_{n=0}^{\infty} \frac{a_i^n \epsilon^n}{n!} \left[\frac{\log^n(x_i)}{x_i}\right]_+ \tag{2.6}$$

We note that one may also encounter singularities due to poles of second or higher order, as for example in

$$I = \int_0^1 dx \frac{f(x)}{x^{2-\epsilon}}. \tag{2.7}$$

The subtraction method can be also applied here, writing

$$I = \int_0^1 dx \frac{f(x) - f(0) - xf'(0)}{x^{2-\epsilon}} + f(0) \int_0^1 dx x^{-2+\epsilon} + f'(0) \int_0^1 dx x^{-1+\epsilon}. \tag{2.8}$$

The integrals in x of the above expression can be computed numerically (as an expansion in ϵ).

The extraction of divergences is more complicated for integrals with overlapping singularities. Consider as an example the integral

$$I = \int_0^1 dx_1 dx_2 \frac{1}{(x_1 + cx_2)^{2+\epsilon}}. \tag{2.9}$$

For $\epsilon = 0$ the integrand becomes divergent when both $x_1, x_2 \to 0$. Here, it is easy to perform successively both integrations, finding the explicit result

$$I = \frac{1}{\epsilon(1+\epsilon)} \left\{-c^{-1-\epsilon} + \frac{(1+c)^{-\epsilon} - 1}{c}\right\}. \tag{2.10}$$

2.1 The differential equation method

Analogous problems in realistic NNLO calculations are very hard to tackle with direct analytic integrations. A powerful method which has found numerous applications is the method of differential equations. In this approach we find a physical parameter for the integral and formulate a differential equation using integration by parts. In our example, we can write a differential equation with respect to the parameter c, by integrating the total derivative

$$\int_0^1 dx_1 dx_2 \frac{\partial}{\partial x_2} \frac{x_2}{(x_1 + cx_2)^{2+\epsilon}}.$$
(2.11)

This yields the differential equation,

$$c\frac{\partial I}{\partial c} + I = I_{\text{simpler}}$$
(2.12)

The inhomogeneous term on the right side of the above equation is simpler than I. Specifically,

$$I_{\text{simpler}} = \int_0^1 dx_1 \frac{1}{(c + x_1)^{2+\epsilon}}.$$
(2.13)

and we find

$$I_{\text{simpler}} = \frac{-1}{1 + \epsilon} \left[(1 + c)^{-1-\epsilon} - c^{-1-\epsilon} \right].$$
(2.14)

The general solution of eq. (2.12) involves integrals over one variable only,

$$I = \frac{1}{c} \left[\text{const.} + \int^c d\tilde{c}\, I_{\text{simpler}}(\tilde{c}) \right]$$
(2.15)

thus bypassing the problem of overlapping singularities. The constant of integration can be determined from knowing the integral at a special value of c or by exploiting a known limiting behavior or scaling. For example, in our case, we could be using that

$$I(1/c) = c^{2+\epsilon} I(c),$$
(2.16)

which we can easily derive with a change of variables $x_1 \leftrightarrow x_2$ in eq. (2.9).

2.2 The Mellin-Barnes representation method

Mellin-Barnes representations allow a straightforward Laurent expansion of Feynman integrals by using Cauchy's theorem. Such representations are obtained by using the identity,

$$\frac{\Gamma(N)}{(A + B)^N} = \frac{1}{2\pi i} \int_{w_0 - i\infty}^{w_0 + i\infty} dw \Gamma(-w)\Gamma(N + w)A^w B^{-N-w},$$
(2.17)

where the contour of integration is a straight line parallel to the imaginary axis, crossing the real axis at a point w_0 such that the real part of the arguments of the gamma functions are positive. Using eq. (2.17) and integrating x_1 and x_2 for the toy example of eq. (2.9), we obtain the Mellin-Barnes representation

$$I = \frac{1}{2\pi i} \int dw \frac{\Gamma(-w)\Gamma(2 + \epsilon + w)}{\Gamma(2 + \epsilon)} \frac{\Gamma(1 + w)\Gamma(-1 - \epsilon - w)}{\Gamma(2 + w)\Gamma(-\epsilon - w)} c^w,$$
(2.18)

where the representation is valid (all Gamma functions have arguments with positive real part) if we choose, for example, $\epsilon = -0.9$ and $\mathrm{Re}\, w = -0.2$. Notice that we cannot find any value of $\mathrm{Re}\, w$ which renders the integral well defined if we choose $\epsilon = 0$. This means that the integral develops a pole in ϵ. A Laurent expansion can be achieved with an analytic continuation method, moving the value of ϵ from a value that the integral is well defined, $\epsilon = -0.9$ in our example, to $\epsilon = 0$ and isolating with Cauchy's theorem the poles which arise when the arguments of ϵ dependent Gamma functions become zero or negative integers. In our example, we find that $\Gamma(-1 - \epsilon - w) = \Gamma(0)$ develops a pole as $\epsilon = -0.8$ (and $w = w_0$). No other pole is encountered by continuing the value of ϵ further to $\epsilon = 0$. We can then write,

$$I = \mathrm{TaylorExpand}(I)_{\epsilon=0} + \mathrm{Res}_{w=-1-\epsilon}(I). \qquad (2.19)$$

2.3 The subtraction method

The differential equation method and the Mellin-Barnes method bypass the problem of overlapping singularities by integrating out Feynman parameters and phase-space variables and generating equivalent representations where overlapping singularities cannot occur. Both methods rely on the integration volume being well known and free of parameters, other than the space-time dimension. This is the case for loop integrals and inclusive phase-space integrations. An important class of phase-space integrals requires parametric boundaries which are determined according to varied selection criteria for the experimentally measured observables. For such integrals the differential equation and Mellin-Barnes methods are not generally suitable.

One approach is to use a subtraction method in order to map the problem of fully differential phase-space integrations onto a problem of fully inclusive phase-space. Consider the toy example,

$$I\,[J] = \int_0^1 dx_1 dx_2 \frac{J(x_1, x_2)}{(x_1 + x_2)^{2+\epsilon}}, \qquad (2.20)$$

where the function $J(x_1, x_2)$ plays the role of selecting an arbitrary subregion of the integration volume according to, for example, the wishes of the experimentalists. Using subtraction, we can re-write

$$I\,[J] = \int_0^1 dx_1 dx_2 \frac{J(x_1, x_2) - J(0,0)}{(x_1 + x_2)^{2+\epsilon}} + J(0,0) \int_0^1 dx_1 dx_2 \frac{1}{(x_1 + x_2)^{2+\epsilon}}. \qquad (2.21)$$

The first integral contains only an integrable singularity as $\epsilon \to 0$ and can be computed numerically. The second integral is a "fully inclusive" integral and may be computable with the Mellin-Barnes or differential equation method.

2.4 The sector decomposition method

A different approach is to use sector decomposition. We divide the integration region according to the relative magnitude of the integration variables which are required for the singular limit (in our example $x_1 = x_2 = 0$), by multiplying the integrand with an appropriate unity,

$$1 = \Theta(x_2 > x_1) + \Theta(x_1 > x_2). \qquad (2.22)$$

This gives rise to two integration domains (sectors). In the sector with $x_1 > x_2$ we rescale $x_2 = x_2 x_1$, and in the sector with $x_2 > x_1$ we transform $x_1 = x_1 x_2$. We then obtain

$$I[J] = \int_0^1 dx_1 dx_2 x_1^{-1-\epsilon} \frac{J(x_1, x_1 x_2)}{(1+x_2)^{2+\epsilon}} + \int_0^1 dx_1 dx_2 x_2^{-1-\epsilon} \frac{J(x_1 x_2, x_2)}{(1+x_1)^{2+\epsilon}}. \tag{2.23}$$

The singularities in both integrands are now factorized and a Laurent expansion can be easily achieved with a simple subtraction. The method of sector decomposition is suited for all types of loop and phase-space integrals.

It is instructive to see how the method is used on a physical example. Let us consider the one-loop box scalar integral,

$$I = \int \frac{d^d k}{i\pi^{\frac{d}{2}}} \frac{1}{k^2 (k+p_1)^2 (k+p_1+p_2)^2 (k+p_1+p_2+p_3)^2}. \tag{2.24}$$

The corresponding Feynman parameterization reads,

$$I = \int_0^1 dx_1 \ldots dx_4 \delta(1 - x_1 - \ldots - x_4) f(x_1, \ldots, x_4) \tag{2.25}$$

with

$$f(x_1, \ldots, x_4) \equiv \frac{\Gamma(2+\epsilon)}{[-sx_1 x_3 - tx_2 x_4 - i0]^{2+\epsilon}}. \tag{2.26}$$

To avoid creating poles at the upper limit of the x_i integrations we apply first the method of primary-sectors [21]. We write

$$I = \int dx \int_0^1 \left(\prod_i dx_i\right) \delta\left(1 - \sum x_i\right) f(\{x_i\}) \sum_i \delta(x_i - x) \prod_{j \neq i} \Theta(x_i \geq x_j) \tag{2.27}$$

We now rescale

$$x_k = y_k x, \tag{2.28}$$

and perform the x integration. This yields

$$I = \Gamma(2+\epsilon) \int_0^1 dy_1 \ldots dy_4 \left(\sum_i y_i\right)^{2\epsilon} \frac{\sum_i \delta(1 - y_i)}{[-sy_1 y_3 - ty_2 y_4 - i0]^{2+\epsilon}}. \tag{2.29}$$

All terms in the sum can be computed in exactly the same fashion. For convenience, although not necessary, we use the special symmetry of this problem, $y_1 \leftrightarrow y_3$ and $y_2 \leftrightarrow y_4$, and cast the integral as

$$I = 2\,\Gamma(2+\epsilon) \int_0^1 dy_1 dy_2 dy_3 \, (1 + y_1 + y_2 + y_3)^{2\epsilon}$$
$$\times \left\{ [-sy_1 - ty_2 y_3]^{-2-\epsilon} + [-ty_1 - sy_2 y_3]^{-2-\epsilon} \right\} \tag{2.30}$$

We observe that the integral becomes singular in the following instances

$$y_1 = 0 \text{ and } (y_2 = 0 \text{ or } y_3 = 0). \tag{2.31}$$

We now apply sector decomposition to factorize the entangled singularity structure. We multiply the integrand with

$$1 = \Theta(y_2 < y_1) + \Theta(y_1 < y_2)\left(\Theta(y_1 < y_2 y_3) + \Theta(y_2 y_3 < y_1)\right)$$
$$= \Theta(y_2 < y_1) + \Theta(y_1 < y_2 y_3) + \Theta(y_2 y_3 < y_1 < y_2) \tag{2.32}$$

In each of the three sectors of the above equation we rescale the smallest variables with respect to the large ones, mapping the boundaries of the sectors to the unit cube. Specifically,

$$\Theta(y_1 < y_2 y_3): \quad y_1 \to y_1 y_2 y_3 \tag{2.33}$$
$$\Theta(y_2 y_3 < y_1 < y_2): \quad y_1 \to y_1 y_2 \text{ and } y_3 \to y_3 y_1 \tag{2.34}$$
$$\Theta(y_2 < y_1): \quad y_2 \to y_2 y_1 \tag{2.35}$$

We then obtain a representation of the one-loop box,

$$I = I_1 + I_2 + I_3, \tag{2.36}$$

with a simple, factorized, singularity structure:

$$I_1 = 2\,\Gamma(2+\epsilon) \int_0^1 dy_1 dy_2 dy_3 \left(1 + y_1 y_2 y_3 + y_2 + y_3\right)^{2\epsilon}$$
$$\times \left\{ [-sy_1 - t]^{-2-\epsilon} + [-ty_1 - s]^{-2-\epsilon} \right\} (y_2 y_3)^{-1-\epsilon}, \tag{2.37}$$

$$I_2 = 2\,\Gamma(2+\epsilon) \int_0^1 dy_1 dy_2 dy_3 \left(1 + y_1 y_2 + y_2 + y_3 y_1\right)^{2\epsilon}$$
$$\times \left\{ [-s - ty_3]^{-2-\epsilon} + [-t - sy_3]^{-2-\epsilon} \right\} (y_2 y_1)^{-1-\epsilon}, \tag{2.38}$$

$$I_3 = 2\,\Gamma(2+\epsilon) \int_0^1 dy_1 dy_2 dy_3 \left(1 + y_1 + y_2 y_1 + y_3\right)^{2\epsilon}$$
$$\times \left\{ [-s - ty_2 y_3]^{-2-\epsilon} + [-t - sy_2 y_3]^{-2-\epsilon} \right\} y_1^{-1-\epsilon}. \tag{2.39}$$

The resulting integrals I_1, I_2, I_3 of sector decomposition can all be expanded in ϵ with the subtraction method.

3 Factorization of singularities with non-linear transformations

In this section, we propose a new method for the factorization of overlapping singularities. We consider again the very simple case of an overlapping singularity in the two-dimensional toy example integral

$$I_1 = \int_0^1 dx_1 dx_2 \frac{1}{(c_1 x_1 + c_2 x_2)^{2+\epsilon}} \tag{3.1}$$

We shall perform a rescaling transformation over the entire integration region,

$$x_2 = \lambda x_1, \tag{3.2}$$

This yields,

$$I_1 = \int_0^1 dx_1 x_1^{-1-\epsilon} \int_0^{\frac{1}{x_1}} d\lambda \frac{1}{(c_1 + c_2\lambda)^{2+\epsilon}}. \tag{3.3}$$

We notice that there is a factorized singularity at $x_1 = 0$. In this singular point of the x_1 integration, the variable λ ranges up to $+\infty$. However, the λ integration is convergent at $\epsilon = 0$, since the integrand scales as $\frac{1}{\lambda^2}$ for very large values of λ. Therefore, we could immediately treat the singularity at $x_1 = 0$ with the subtraction method

$$I_1 = \int_0^1 dx_1 x_1^{-1-\epsilon} \int_0^\infty d\lambda \frac{1}{(c_1 + c_2\lambda)^{2+\epsilon}} \left[\Theta(\lambda < \frac{1}{x_1}) - 1 \right] - \frac{1}{\epsilon} \int_0^\infty d\lambda \frac{1}{(c_1 + c_2\lambda)^{2+\epsilon}}. \tag{3.4}$$

We can then evaluate numerically the integrals which are produced after we perform an expansion in ϵ.

Alternatively, we could perform a transformation[1] on λ to bring the integration region back to $[0, 1]$.

$$\lambda = g(u), \tag{3.5}$$

Such a transformation maps the integration region to,

$$\int_0^1 dx_1 dx_2 = \int_0^1 dx\, x \int_{g^{-1}(0)}^{g^{-1}(1/x)} du\, g'(u) \tag{3.6}$$

It is very important to select carefully this transformation. A linear mapping

$$g(u) = \frac{x_1}{u}, \tag{3.7}$$

is clearly ineffective, since it undoes the original rescaling of $x_2 = \lambda x_1$. However, non-linear mappings, such as

$$g(u, x) = \frac{u}{x + \delta(1 - u)} \tag{3.8}$$

$$g(u, x) = \frac{u}{x + \delta_1(1 - u)^{\delta_2}} \tag{3.9}$$

$$g(u, x) = \frac{1 + x}{\sqrt{(1 + 2x)(1 - u) + x^2}} - 1 \tag{3.10}$$

$$\dots$$

are effective. For almost all practical applications in this paper we employ the mapping

$$g(u, x) = \frac{u}{x + \delta(1 - u)}. \tag{3.11}$$

with δ often chosen equal to 1.

Explicitly, the transformation

$$x_2 = \frac{x_1 x_2'}{x_1 + (1 - x_2')}, \qquad 1 - x_2 = \frac{(1 + x_1)(1 - x_2')}{x_1 + (1 - x_2')}, \tag{3.12}$$

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[1]Integrating λ numerically, with Monte Carlo methods, requires a transformation as well, in practice, since one needs to generate λ from some random variable that is produced in $[0, 1]$.

with a Jacobian

$$\frac{\partial x_2}{\partial x_2'} = \frac{x_1(1 + x_1)}{[x_1 + (1 - x_2')]^2} \tag{3.13}$$

disentangles the overlapping singularity, transforming the integral of eq. (3.1) as

$$I_1 = \int_0^1 dx_1 dx_2' x_1^{-1-\epsilon}(1 + x_1)(1 - x_2' + x_1)^\epsilon [c_1(1 - x_2' + x_1) + c_2 x_2']^{-2-\epsilon} \tag{3.14}$$

The singularity in the limit $x_1 = 0$ and $\epsilon = 0$ can be subtracted away, and a Laurent series expansion around $\epsilon = 0$ is achieved using the expansion

$$x^{-1+\epsilon} = \frac{\delta(x)}{\epsilon} + \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left[\frac{\ln^n(x)}{x} \right]_+ . \tag{3.15}$$

In this approach, we have achieved to factorize the overlapping singularity with a simple transformation. In comparison, a factorization with sector decomposition doubles the number of integrals, as we have seen in the previous section. Economizing in the number of integrals is even more significant for physical applications where entanglement of singularities with more variables may take place.

Let us now revisit the one-loop box calculation using the new method instead of sector decomposition. Following the "analytical-transformation" approach, we perform the change of variables on the integral of eq. (2.30)

$$y_1 \rightarrow \frac{y_1 y_2 y_3}{1 - y_1 + y_2 y_3}. \tag{3.16}$$

This yields the integral

$$I = 2\,\Gamma(2+\epsilon) \int_0^1 dy_1 dy_2 dy_3 (y_2 y_3)^{-1-\epsilon}(1-y_1+y_2 y_3)^{-\epsilon}[y_1 y_2 y_3 + (1+y_2+y_3)(1-y_1+y_2 y_3)]^{2\epsilon}$$
$$\times \left\{ [-sy_1 - t(1 - y_1) - t y_2 y_3]^{-2-\epsilon} + [-t y_1 - s(1 - y_1) - s y_2 y_3]^{-2-\epsilon} \right\} \tag{3.17}$$

In this integral, the singularities have nicely factorized in the term $(y_2 y_3)^{-1-\epsilon}$. In comparison to sector decomposition, we now have to perform one integration rather than three.

4 Characteristic forms of entangled singularities

In this section, we present some typical examples of integrals with overlapping singularities and the mappings that we use to disentangle them.

Our second example is the integral

$$I_2 = \int_0^1 dx\, dy\, dz\, \frac{1}{(x + yz)^{2+\epsilon}} \tag{4.1}$$

which is a trivial extension of eq. (3.1). We use the mapping

$$(x + yz) \quad : \quad x \rightarrow \frac{xyz}{1 - x + yz} \tag{4.2}$$

where we have also designated the singularity structure of the integral. This mapping leads to

$$I_2 = \int_0^1 \frac{dx\, dy\, dz}{(yz)^{1+\epsilon}} \frac{(1-x+yz)^\epsilon}{(1+yz)^{1+\epsilon}} \qquad (4.3)$$

where the singularities are factorized in terms of y and z.

Next, let's consider

$$I_3 = \int_0^1 dx\, dy\, dz \frac{1}{(x+y+z)^{3+\epsilon}}. \qquad (4.4)$$

Here we use the simultaneous double mapping

$$(x+y+z) \quad : \qquad y \to \frac{yx}{1-y+x} \quad , \quad z \to \frac{zx}{1-z+x} \qquad (4.5)$$

which leads to

$$I_3 = \int_0^1 \frac{dx\, dy\, dz}{x^{1+\epsilon}} \frac{(1+x)^2(1-y+x)^{1+\epsilon}(1-z+x)^{1+\epsilon}}{((1+x)^2-zy)^{3+\epsilon}}. \qquad (4.6)$$

Next, let's consider the integral

$$I_4 = \int_0^1 dx\, dy\, dz\, dw \frac{1}{(x+y(z+w))^{2+\epsilon}}. \qquad (4.7)$$

Here we use successively

$$(x+y(z+w)) \quad : \qquad z \to \frac{zw}{1-z+w} \quad , \quad x \to \frac{xyw}{1-x+yw}. \qquad (4.8)$$

The integral then becomes

$$I_4 = \int_0^1 \frac{dx\, dy\, dz\, dw}{y^{1+\epsilon}} \frac{(1+w)(1+yw)(1-x+yw)^\epsilon(1-z+w)^\epsilon}{[1-zx+w(1+y+yw)]^{2+\epsilon}}. \qquad (4.9)$$

It is maybe instructive to see how this integral of eq. (4.7) behaves under simple rescaling. Consider $z = \lambda_z w$ and then $x = \lambda_x yw$. We get

$$I_4 = \int_0^1 \frac{dy\, dw}{y^{1+\epsilon}} \int_0^{1/w} d\lambda_z \int_0^{1/yw} d\lambda_x \frac{1}{(\lambda_x+\lambda_z+1)^{2+\epsilon}}. \qquad (4.10)$$

Only the integral over λ_x extends to infinity and in that limit the behavior of the integrand is $d\lambda_x/\lambda_x^2$ which vanishes at infinity.

Next let's consider

$$I_5 = \int_0^1 dx\, dy\, dz\, dw \frac{(xyzw)^\epsilon}{(x+y+zw)^3}. \qquad (4.11)$$

Here we use the successive mappings

$$(x+y+zw) \quad : \qquad x \to \frac{xzw}{1-x+zw} \quad , \quad y \to \frac{yzw}{1-y+zw} \qquad (4.12)$$

which brings the integral to the factorized form

$$I_5 = \int_0^1 dx\, dy\, dz\, dw \frac{(xy)^\epsilon}{z^{1-3\epsilon}w^{1-3\epsilon}} \frac{(1+zw)^2(1-x+zw)^{1-\epsilon}(1-y+zw)^{1-\epsilon}}{(1-xy+2zw+z^2w^2)^3}. \qquad (4.13)$$

When overlapping singularities appear together with factorized singularities in the same variable, a slight complication appears. Consider the integral

$$I_6 = \int_0^1 dxdy \frac{(xy)^\epsilon}{x(x+y)}. \tag{4.14}$$

It has a factorized singularity at $x = 0$ and an overlapping singularity at $x = 0 = y$. Let us call the singularity at $x = 0$ active and the one at $y = 0$ passive. In order to disentangle the singularity we would like to use the same mapping as in the previous examples. It turns out that we can do this, but only as long as we choose to remap the active singularity, i.e.

$$x(x+y) \quad : \quad x \rightarrow \frac{xy}{1-x+y}. \tag{4.15}$$

The integral then becomes

$$I_6 = \int_0^1 dxdy\, x^{-1+\epsilon} y^{-1+2\epsilon} (1-x+y)^{-\epsilon}. \tag{4.16}$$

which can be subtracted easily.

Note that applying the wrong rescaling $y = \lambda x$ one gets

$$I_6 = \int_0^1 dx \int_0^{1/x} d\lambda \frac{(\lambda)^\epsilon}{x^{1-2\epsilon}(1+\lambda)}. \tag{4.17}$$

We can see immediately that the $d\lambda$ integral is logarithmically divergent at the active singularity $x \rightarrow 0$ due to the upper limit of the integration region. On the contrary, applying the correct rescaling $x = \lambda y$ one gets

$$I_6 = \int_0^1 dy \int_0^{1/y} d\lambda \frac{(\lambda)^\epsilon}{y^{1-2\epsilon}\lambda(1+\lambda)} \tag{4.18}$$

which, as $y \rightarrow 0$, behaves as $d\lambda/\lambda^2$ which is finite.

We see that the simple λ-rescaling works as a guideline, showing when a mapping properly factorizes the singularities of an integral. It will be instrumental in more complicated cases presented below.

Let us now consider the integral[2]

$$I_7 = \int_0^1 dxdydz \frac{(xyz)^\epsilon}{xy(xy+z)}. \tag{4.19}$$

Identifying $x = 0 = z$ and $y = 0 = z$ as two independent overlapping singularities, where z is passive and both x and y are active, we know from the previous example that we should not map z from the previous example. So one is left to map $x \rightarrow z$ or $y \rightarrow z$. By the symmetry of the integrand it does not matter which one of these one can choose. Let us choose x and a slightly modified mapping that keeps the expressions simpler:

$$xy(y+z) \quad : \quad x \rightarrow \frac{xz}{1-x+zx}. \tag{4.20}$$

---

[2]We find similar singularity structures in double real radiation.

The integrand then becomes

$$I_7 = \int_0^1 dx\,dy\,dz \frac{(xy^2z)^\epsilon (1-x+zx)^{-\epsilon}}{xyz(x(y+z)+(1-x))}. \tag{4.21}$$

We have now managed to "activate" the singularity at $z = 0$. At the same time the singularities at $y = 0$ and at $x = 0$ have remained active as well. However, we now find an overlapping singularity at $z = 0 = y$ *and* $x = 1$, where the singularity at $x = 1$ is passive. Notice that we started with two independent (partially interfering) overlapping singularities, have treated one of them and are now left with only one, which lies at a different point. We shall remap z and y as follows

$$y \to \frac{y(1-x)}{1-y+(1-x)y} \quad , \quad z \to \frac{z(1-x)}{1-z+(1-x)z}. \tag{4.22}$$

The integrand becomes

$$I_7 = \int_0^1 dx\,dy\,dz \frac{(xy)^{-1+\epsilon}((1-x)z)^{-1+2\epsilon}}{(1-xy)^\epsilon(1-xz)^\epsilon(1-x^2yz)}. \tag{4.23}$$

Note that the remaining singularity of the integrand is integrable.

Let's now explore the potential of a slightly different kind of mapping. We have the integral

$$I = \int_0^1 \prod_i dy_i \frac{dx}{(ax+b)^N} \tag{4.24}$$

with a, b independent of x but potentially dependent on y_i. In the latter case the integral might have overlapping singularities, as $a, b \to 0$ or $x, b \to 0$. We employ

$$x \to \frac{xb/a}{1-x+b/a} \tag{4.25}$$

and get

$$I = \int_0^1 \prod_i dy_i \frac{dx(a(1-x)+b)^{N-2}}{b^{N-1}(a+b)^{N-1}}. \tag{4.26}$$

If $N \geq 2$ this mapping factorizes the singularity at $b \to 0$ and, at the same time, exposes the $a+b$ structure of the overlapping $a, b \to 0$ singularity, making it ready for further mappings.

Let's see, as an example,

$$I_8 = \int_0^1 \frac{dx_1\,dx_2\,dx_3\,dx_4\,dx_5}{(x_1+x_2x_3+x_2x_4+x_4x_5)^{3+\epsilon}}. \tag{4.27}$$

We map:

$$x_1 \to \frac{x_1(x_2x_3+x_2x_4+x_4x_5)}{1-x_1+(x_2x_3+x_2x_4+x_4x_5)} \tag{4.28}$$

to get

$$I_8 = \int_0^1 dx_1\,dx_2\,dx_3\,dx_4\,dx_5 \frac{(1-x_1+x_2x_3+x_2x_4+x_4x_5)^{1+\epsilon}}{(x_2(x_3+x_4)+x_4x_5)^{2+\epsilon}(1+x_2x_3+x_2x_4+x_4x_5)^{2+\epsilon}}. \tag{4.29}$$

We can now use the mapping of eq. (4.25) with $a = x_3 + x_4$ and $b = x_4 x_5$ to get

$$I_8 = \int_0^1 dx_1\, dx_2\, dx_3\, dx_4\, dx_5 \frac{F(x_i)}{x_4^{1+\epsilon} x_5^{1+\epsilon} (x_3 + x_4 + x_4 x_5)^{1+\epsilon}} \tag{4.30}$$

where $F(x_i)$ is a finite function of x_i. Noting that x_4 is an active singularity, we use the mapping of eq. (4.25) again with $a = 1 + x_5$ and $b = x_3$ to get

$$I_8 = \int_0^1 dx_1\, dx_2\, dx_3\, dx_4\, dx_5 \frac{F'(x_i)}{x_4^{1+\epsilon} x_5^{1+\epsilon} x_3^{1+\epsilon}} \,. \tag{4.31}$$

Let us now see some examples where we employ a hybrid method of one-step of sector decomposition and non-linear transformations to factorize overlapping singularities. A similar singularity structure appears in two-loop massless box integrals.

We consider

$$I_9 = \int_0^1 \frac{dx_1\, dx_2\, dx_3\, dx_4\, dx_5}{[x_1 x_3 + x_1 x_2 + x_2 (x_4 + x_5 + x_3 x_4 x_5)]^{3+\epsilon}} \,. \tag{4.32}$$

We split this integral in two sectors

$$x_2 > x_3 \quad : \quad \int_0^1 \frac{1}{x_2^{2+\epsilon}} \frac{dx_1\, dx_2\, dx_3\, dx_4\, dx_5}{[x_1 (1 + x_3) + x_4 + x_5 + x_3 x_4 x_5]^{3+\epsilon}} \tag{4.33}$$

which has the singularity structure of eq. (4.4) and can be factorized by the mapping of eq. (4.5), and

$$x_3 > x_2 \quad : \quad \int_0^1 \frac{1}{x_3^{2+\epsilon}} \frac{dx_1\, dx_2\, dx_3\, dx_4\, dx_5}{[x_1 (1 + x_2) + x_2 (x_4 + x_5 + x_4 x_5)]^{3+\epsilon}} \tag{4.34}$$

which is of the type of eq. (4.7) and can be factorized with the mapping of eq. (4.8).

We now move to the most complicated example of this section, the integral

$$I_{10} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2}{[x_1 B + x_1 x_3 A + x_4 \tau_1 C + x_4 \tau_2 D + x_3 \tau_1 \tau_2 E]^{3+2\epsilon}} \tag{4.35}$$

with A, E finite and B, C, D finite functions of $\tau_{1,2}$.

We split this integral in two sectors, x_1, x_4, and we get

$$x_1 > x_4 \quad : \quad I_{10A} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2\, x_1}{[x_1 (B + A x_3 + C x_4 \tau_1 + D x_4 \tau_2) + E x_3 \tau_1 \tau_2]^{3+2\epsilon}} \tag{4.36}$$

which is of the type of eq. (4.1) and can be immediately factorized with the mapping eq. (4.2). The other sector is

$$x_4 > x_1 \quad : \quad I_{10B} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2\, x_4}{[B x_1 x_4 + A x_1 x_4 x_3 + C x_4 \tau_1 + D x_4 \tau_2 + E x_3 \tau_1 \tau_2]^{3+2\epsilon}}. \tag{4.37}$$

This should be further split in x_3, x_4 to get

$$x_4 > x_3 \quad : \quad I_{10B1} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2}{x_4^{1+2\epsilon}} \frac{1}{[B x_1 + A x_1 x_4 x_3 + C \tau_1 + D \tau_2 + E x_3 \tau_1 \tau_2]^{3+2\epsilon}}. \tag{4.38}$$

which is of the type of eq. (4.4) and we can use the mapping eq. (4.5) to factorize it. The other sector is

$$x_3 > x_4 \ : \ \ I_{10B2} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2}{x_3^{1+2\epsilon}} \frac{x_4}{[Bx_1 x_4 + Ax_1 x_4 x_3 + Cx_4 \tau_1 + Dx_4 \tau_2 + E\tau_1 \tau_2]^{3+2\epsilon}} \tag{4.39}$$

and requires further splitting. We choose to split in the variables x_4, τ_1 to get

$$x_4 > \tau_1 \ : \ \ I_{10B2A} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2}{x_3^{1+2\epsilon} x_4^{1+2\epsilon}} \frac{1}{[B'x_1 + Ax_1 x_3 + Cx_4 \tau_1 + D'\tau_2 + E\tau_1 \tau_2]^{3+2\epsilon}} \tag{4.40}$$

which is of the type of eq. (4.11), and

$$\tau_1 > x_4 \ : \ \ I_{10B2B} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2}{x_3^{1+2\epsilon} \tau_1^{1+2\epsilon}} \frac{x_4}{[Bx_1 x_4 + Ax_1 x_4 x_3 + Cx_4 \tau_1 + Dx_4 \tau_2 + E\tau_2]^{3+2\epsilon}} \tag{4.41}$$

which requires a final split in x_4, τ_2,

$$\tau_2 > x_4 \ : \ \ I_{10B2B1} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2}{x_3^{1+2\epsilon} \tau_1^{1+2\epsilon} \tau_2^{1+2\epsilon}} \frac{x_4}{[Bx_1 x_4 + Ax_1 x_4 x_3 + Cx_4 \tau_1 + Dx_4 \tau_2 + E]^{3+2\epsilon}} \tag{4.42}$$

which is finite and

$$x_4 > \tau_2 \ : \ \ I_{10B2B2} = \int_0^1 \frac{dx_1 dx_3 dx_4 d\tau_1 d\tau_2}{x_3^{1+2\epsilon} \tau_1^{1+2\epsilon} x_4^{1+2\epsilon}} \frac{1}{[Bx_1 + Ax_1 x_3 + C\tau_1 + Dx_4 \tau_2 + E\tau_2]^{3+2\epsilon}} \tag{4.43}$$

which is of the type of eq. (4.7). The original integral can be written in terms of its five sectors as

$$I_{10} = I_{10A} + I_{10B1} + I_{10B2A} + I_{10B2B1} + I_{10B2B2}. \tag{4.44}$$

Finally, let's consider

$$I_{11} = \int_0^1 \frac{dx_1 dx_2 dx_4 d\tau_1 d\tau_2 \ x_2^{1+\epsilon}}{[x_1 A + x_1 x_2 B_1 B_2 + x_2 x_4 \tau_1 B_2 C + x_2 x_4 \tau_2 B_1 D + x_2 \tau_1 \tau_2 E]^{3+2\epsilon}} \tag{4.45}$$

with A, E, C, D finite and $B_{1,2} = 2 - \tau_{1,2}$, also finite for all values of $\tau_{1,2}$. We begin by splitting the integral in τ_1, τ_2. We get

$$\tau_1 > \tau_2 \ : \ \ I_{11A} = \int_0^1 \frac{dx_1 dx_2 dx_4 d\tau_1 d\tau_2 \ x_2^{1+\epsilon}}{[x_1 A + x_1 x_2 B_1 B_{12} + x_2 x_4 \tau_1 B_{12} C + x_2 x_4 \tau_2 \tau_1 B_1 D + x_2 \tau_1^2 \tau_2 E]^{3+2\epsilon}} \tag{4.46}$$

and

$$\tau_2 > \tau_1 \ : \ \ I_{11B} = \int_0^1 \frac{dx_1 dx_2 dx_4 d\tau_1 d\tau_2 \ x_2^{1+\epsilon}}{[x_1 A + x_1 x_2 B_2 B_{12} + x_2 x_4 \tau_1 \tau_2 B_1 C + x_2 x_4 \tau_1 B_{12} D + x_2 \tau_1 \tau_2^2 E]^{3+2\epsilon}} \tag{4.47}$$

where $B_{12} = 2 - \tau_1 \tau_2$. We notice that we can get I_{11B} from I_{11A} if we exchange C and D and rename the dummy integration variables $\tau_1 \leftrightarrow \tau_2$, so that

$$I_{11B}(C, D) = I_{11A}(D, C). \tag{4.48}$$

We will use the same decompositions and mappings to factorize I_{11B} and I_{11A} (with τ_1 and τ_2 interchanged), so we only describe the latter below. We split I_{11A} in two sectors with respect to τ_1 and x_4. We get

$$\tau_1 > x_4 \; : \; I_{11A1} = \int_0^1 \frac{dx_1 dx_2 dx_4 d\tau_1 d\tau_2 \; x_2^{1+\epsilon} \tau_1}{[x_1(A + x_2 B_1 B_{12}) + x_2 \tau_1^2 (x_4 B_{12} C + x_4 \tau_2 B_1 D + \tau_2 E)]^{3+2\epsilon}}. \quad (4.49)$$

We perform the mappings

$$\tau_2 \rightarrow \frac{\tau_2 x_4}{1 - \tau_2 + x_4} \quad (4.50)$$

and then

$$x_1 \rightarrow \frac{x_1 \tau_1^2 \tau_2 x_4}{1 - x_1 + \tau_1^2 \tau_2 x_4} \quad (4.51)$$

which factorize all singularities. We also have

$$x_4 > \tau_1 \; : \; I_{11A2} = \int_0^1 \frac{dx_1 dx_2 dx_4 d\tau_1 d\tau_2 \; x_2^{1+\epsilon} x_4}{[x_1(A + x_2 B_1' B_{12}') + x_2 x_4^2 \tau_1 (B_{12}' C + \tau_2 B_1' D + \tau_1 \tau_2 E)]^{3+2\epsilon}}. \quad (4.52)$$

We will split with respect to τ_1, τ_2 to get

$$\tau_1 > \tau_2 \; : \; I_{11A2a} = \int_0^1 \frac{dx_1 dx_2 dx_4 d\tau_1 d\tau_2 \; x_2^{1+\epsilon} x_4 \tau_1}{[x_1(A + x_2 B_1' B_{12}'') + x_2 x_4^2 \tau_1 (B_{12}'' C + \tau_2 \tau_1 B_1' D + \tau_1^2 \tau_2 E)]^{3+2\epsilon}} \quad (4.53)$$

which can be factorized by

$$x_1 \rightarrow \frac{x_1 x_2 x_4^2 \tau_1}{1 - x_1 + x_2 x_4^2 \tau_1} \quad (4.54)$$

and

$$\tau_2 > \tau_1 \; : \; I_{11A2b} = \int_0^1 \frac{dx_1 dx_2 dx_4 d\tau_1 d\tau_2 \; x_2^{1+\epsilon} x_4 \tau_2}{[x_1(A + x_2 B_1''' B_{12}''') + x_2 x_4^2 \tau_1 \tau_2 (B_{12}''' C + \tau_2 B_1''' D + \tau_1 \tau_2^2 E)]^{3+2\epsilon}} \quad (4.55)$$

which can be factorized by

$$x_1 \rightarrow \frac{x_1 x_2 x_4^2 \tau_1 \tau_2}{1 - x_1 + x_2 x_4^2 \tau_1 \tau_2}. \quad (4.56)$$

The original integral can, therefore, be factorized in six different integrals:

$$I_{11} = I_{11A} + I_{11B} = I_{11A1} + I_{11A2a} + I_{11A2b} + (u \leftrightarrow t) \quad (4.57)$$

5 Double real radiation for final states with massive particles

One of the major challenges at NNLO in QCD has been the computation of the double real emission part of the cross-section. While the computation of the matrix elements with $N+2$ particles in the final state is not a problem per se, difficulties arise when one integrates over the phase space of the two unresolved particles. The corresponding integrals are infrared divergent in the soft and collinear limits and are dimensionally regulated. The divergences have to be subtracted before the integrals can be numerically evaluated. As long as the singularities are factorized, as they usually are at NLO, it is straightforward to use a

Laurent expansion over the singular variables, and evaluate its coefficients numerically. At NNLO, the singularity structure of the integral is more intricate, as line and overlapping singularities appear, and the desired factorization is not straightforward.

The method of sector decomposition has already been applied successfully to achieve this factorization for hadron collider [7, 8] and decay processes [24–26]. A drawback of the method is that it leads to a large number of sectors. The goal of this paper is to replace sector decomposition for double-real radiation integrals with an economical factorization method based on non-linear transformations.

5.1 Infrared singularities in double real radiation

We consider double real emission to a generic NNLO $2 \to n + 2$ process (see Fig 1) with n massive particles and 2 massless partons in the final state. We denote the momenta of the incoming particles by q_1, q_2, those of the outgoing massive particles by $p_1 \ldots p_n$ and those of the two unresolved partons by q_3 and q_4. Infrared singularities in this phase space will occur whenever q_3 and/or q_4 become soft or collinear to q_1, q_2 or to each other. For the case of double real radiation to the production of a single massive particle (e.g. Higgs, W or Z production) potentially singular propagators can be summarized as

$$
\begin{aligned}
s_{34} &= 2q_3.q_4 \\
s_{13} &= -2q_1.q_3 \\
s_{23} &= -2q_2.q_3 \\
s_{14} &= -2q_1.q_4 \\
s_{24} &= -2q_2.q_4
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
s_{134} &= (q_3 + q_4 - q_1)^2 = s_{34} + s_{13} + s_{14} \\
s_{234} &= (q_3 + q_4 - q_2)^2 = s_{34} + s_{23} + s_{24}.
\end{aligned}
\tag{5.2}
$$

Note that s_{123}, s_{124} are bounded from below. Further soft singularities can be found if there are colored massive particles in the final state, which can radiate off soft gluons. One can then get also the following possibly singular denominators:

$$
\begin{aligned}
t_{3i} &= 2q_3.p_i \\
t_{4i} &= 2q_4.p_i
\end{aligned}
\tag{5.3}
$$

and

$$
t_{34i} = (q_3 + q_4 + p_k)^2 - m_k^2 = s_{34} + t_{3i} + t_{4i}
\tag{5.4}
$$

for $i \geq 1$. Since

$$
t_{3i} = 2q_3.p_i = 2E_3(E_i - |\mathbf{p}_i| \cos \theta_{3i}), \qquad E_i > |\mathbf{p}_i|
\tag{5.5}
$$

the soft singularity is factorized in E_3. Whenever some heavy colored state radiates off two gluons we can also get the denominator t_{34i}, it can only become singular in the double
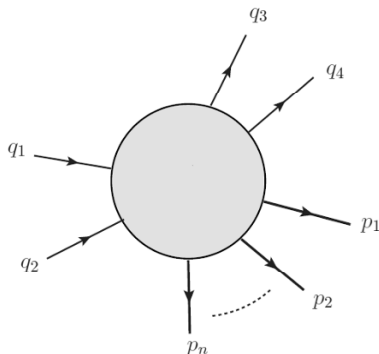
Figure 1. Notational setup: q_i are massless particles, while p_i are massive.

soft limit when $E_3 = 0 = E_4$. However the double soft limit will always be factorized as we will show in the next section.

Let us now discuss the denominator structure of the most singular diagrams which one could expect in double real radiation: those where radiation is emitted by initial state particles. We have illustrated the propagator structure of these topologies using some diagrams containing gluons in figure 2 (diagrams containing just massless quarks correspond to the same topologies).

Diagrams whose propagator structure can be related to the ones in figure 2 by a simple interchange of q_3 with q_4 or of q_1 with q_2 will also fall into the same topology.

By considering square and interference terms of the topologies C_1, C_2 and C_3, we obtain the following list of integrals:

1. Topology $C_1 \otimes C_1$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{(s_{13}s_{24})^2}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{13}s_{23}s_{14}s_{24}} \tag{5.6}$$

2. Topology $C_2 \otimes C_2$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{(s_{34}s_{134})^2}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}^2 s_{134}s_{234}} \tag{5.7}$$

3. Topology $C_3 \otimes C_3$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{(s_{13}s_{134})^2}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{13}s_{23}s_{134}s_{234}}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{13}s_{24}s_{134}s_{234}} \tag{5.8}$$

4. Topology $C_1 \otimes C_2$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{234}s_{13}s_{24}} \tag{5.9}$$

5. Topology $C_1 \otimes C_3$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{134}s_{13}s_{23}s_{14}}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{134}s_{13}^2 s_{14}} \tag{5.10}$$

6. Topology $C_2 \otimes C_3$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{134}^2 s_{13}}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{134}s_{234}s_{23}} \tag{5.11}$$
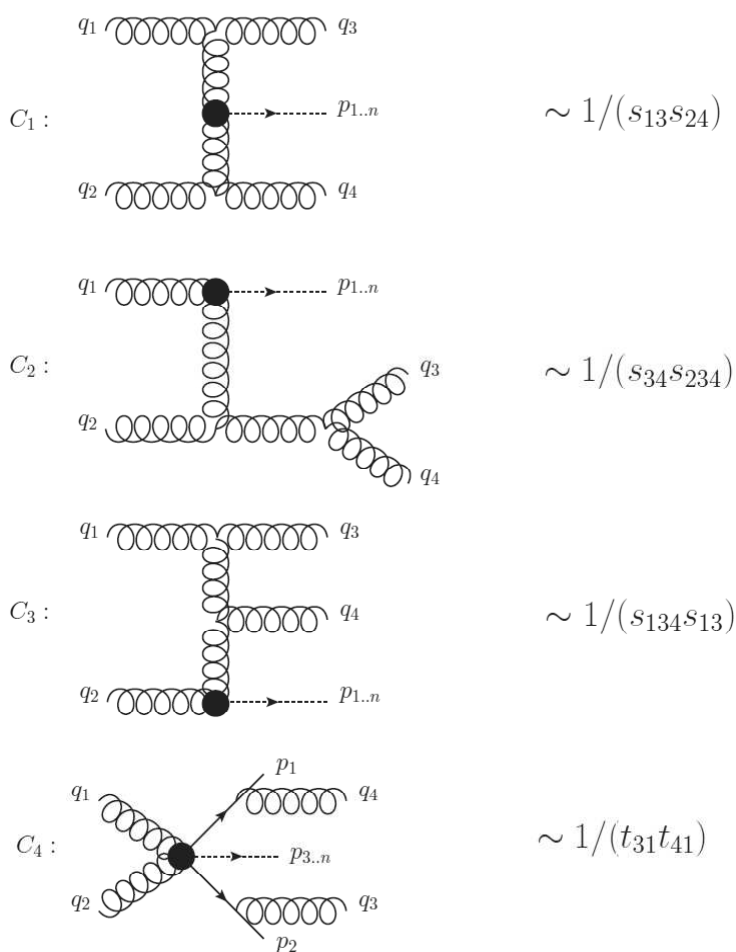
Figure 2. Most singular topologies: C_1, C_2, C_3, C_4.

7. Topology $C_4 \otimes C_4$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{t_{i3}^2 t_{j4}^2}, \int \frac{d\Phi_3 N(\{s_{ij}\})}{t_{i3}t_{j4}t_{j3}t_{i4}} \qquad (5.12)$$

8. Topology $C_4 \otimes C_1$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{t_{i3}t_{j4}s_{13}s_{14}} \qquad (5.13)$$

9. Topology $C_4 \otimes C_2$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{t_{i3}t_{j4}s_{34}s_{134}} \qquad (5.14)$$

10. Topology $C_4 \otimes C_3$:
$$\int \frac{d\Phi_3 N(\{s_{ij}\})}{t_{i3}t_{j4}s_{13}s_{134}} \qquad (5.15)$$

Where $d\Phi_3$ is the differential double emission phase space element for $2 + n$ final state particles, and $N(\{s_{ij}\})$ is in general a finite function of the kinematical invariants.

The topology C_4 contains only soft singularities similar to those in C_1. The topologies $C_4 \otimes C_4$ and $C_4 \otimes C_1$ are, therefore, easier than $C_1 \otimes C_1$. They can be treated exactly like $C_1 \otimes C_1$ and we will not discuss them in what follows.

5.2 Phase-space of double real parton radiation

We would like to point out that different parameterizations of the phase-space can factorize different sets of kinematic invariants. We will derive two such parameterizations which allow for a more convenient numerical evaluation of diverse diagrams, according to their topology.

The phase-space of n massive particles in four dimensions is:

$$d\Phi_n(\sqrt{s}; m_1, \ldots, m_n) = (2\pi)^{4-3n} \left(\prod_{i=1}^n d^4 p_i \delta^+(p_i^2 - m_i^2) \right) \delta^{(4)} \left(q_1 + q_2 - \sum_{i=1}^n p_i \right), \quad (5.16)$$

where $s = (q_1 + q_2)^2$. We assume that a $2 \to n$ process exists at leading order in perturbation theory, and a strictly four-dimensional evaluation is therefore sufficient. At NNLO, the double emission phase space is given by including two further massless particles (whose momenta we denote by q_3 and q_4)

$$d\Phi_{n+2}(\sqrt{s}; m_1, \ldots, m_n, 0, 0) = (2\pi)^{4-3n} \left(\prod_{i=1}^n d^4 p_i \delta^+(p_i^2 - m_i^2) \right) (2\pi)^{2-2d} d^d q_3 \delta^+(q_3^2) d^d q_4 \delta^+(q_4^2)$$

$$\times \delta^{(d)} \left(q_1 + q_2 - \sum_{i=1}^n p_i - q_3 - q_4 \right). \quad (5.17)$$

We factorize the double real phase space into a 3-particle phase space times an n-particle phase space as follows

$$d\Phi_{n+2}(\sqrt{s}; m_1, \ldots, m_n, 0, 0) = \int \frac{ds_{1\ldots n}}{2\pi} d\Phi_3(\sqrt{s}; 0, 0, \sqrt{s_{1\ldots n}}) d\Phi_n(\sqrt{s_{1\ldots n}}; m_1, \ldots, m_n). \quad (5.18)$$

with $s_{1\ldots n} = (\sum_{i=1}^n p_i)^2$ shall denote the center of mass energy (or invariant mass) of the n massive momenta $p_1, \ldots, p_n$. This is depicted graphically in figure 3. The limits of integration of $s_{1\ldots n}$ are

$$s \geq s_{1\ldots n} \geq \left(\sum_{i=1}^n m_i \right)^2 \quad (5.19)$$

and parameterizing $s_{1\ldots n}$ linearly we get

$$s_{1\ldots n} = \left(s - \left(\sum_{i=1}^n m_i \right)^2 \right) x_5 + \left(\sum_{i=1}^n m_i \right)^2. \quad (5.20)$$

The parameter $x_5 \in [0, 1]$ then uniquely defines the double soft limit when $x_5 = 1$. In the following discussion we will use the variable

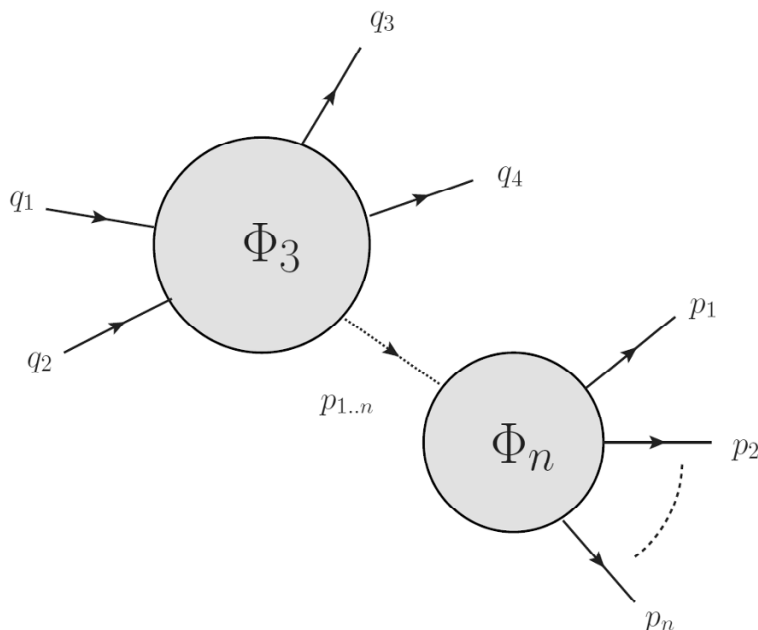$$z = \frac{s_{1\ldots n}}{s} \quad (5.21)$$

Figure 3. Phase space factorization.

which in the special case of $n = 1$ reduces to $z = \frac{m_1^2}{s}$. Then the variable x_5 is trivially removed and the double soft singularity occurs whenever $s = m_1^2$.

In the following we will assume that one can parametrize the n-particle phase space $d\Phi_n$, and we will focus on the phase-space of the potentially unresolved massless partons $d\Phi_3$.

5.3 Energies and angles parameterization

The three particle phase space element $d\Phi_3$ is

$$d\Phi_3(\sqrt{s}; 0, 0, \sqrt{s_{1...n}}) = (2\pi)^{3-2d} d^d q_3 \delta^{(+)}(q_3^2) d^d q_4 \delta^{(+)}(q_4^2) d^d Q$$
$$\times \delta^{(+)}(Q^2 - s_{1...n}) \delta^d(q_1 + q_2 - q_3 - q_4 - Q). \quad (5.22)$$

Integrating out Q and using that $d^d q \delta^{(+)}(q^2) = dE E^{d-3} d\Omega^{(d-1)}/2$ we get

$$d\Phi_3(\sqrt{s}; 0, 0, \sqrt{s_{1...n}}) = (2\pi)^{3-2d} \frac{1}{4} d\Omega_3^{(d-1)} d\Omega_4^{(d-1)} dE_3 dE_4 (E_3 E_4)^{d-3}$$
$$\times \delta^{(+)}(s - s_{1...n} - 2\sqrt{s}(E_3 + E_4) + 2E_3 E_4(1 - \cos\theta_{34})). \quad (5.23)$$

We can solve the delta constraint for the energies in a symmetric way using the following ansatz:

$$E_3 = \frac{1}{2}\sqrt{s}(1 - z)x_1\kappa$$
$$E_4 = \frac{1}{2}\sqrt{s}(1 - z)(1 - x_1)\kappa. \quad (5.24)$$

We find

$$\kappa = \frac{1 - \sqrt{1 - 2(1 - z)x_1(1 - x_1)(1 - \cos\theta_{34})}}{(1 - z)x_1(1 - x_1)(1 - \cos\theta_{34})} \in [1, 2) \quad (5.25)$$

The double soft limit now appears when $z \to 1$, while the single soft singularities occur as $x_1 \to 0, 1$. After this transformation the phase space volume becomes

$$d\Phi_3 = (2\pi)^{3-2d} \frac{d\Omega_3^{(d-1)} d\Omega_4^{(d-1)} dx_1 \kappa(1-z)}{16\sqrt{1-2x_1(1-x_1)(1-z)(1-\cos\theta_{34})}} \left(\frac{s(1-z)^2 \kappa^2 x_1 (1-x_1)}{4} \right)^{d-3}. \quad (5.26)$$

Having solved the energy constraint we move on to parametrize the angles. Choosing the z-axis as the direction of q_1, we directly parameterize the angles which q_3 and q_4 make with the z-axis. Finally we parametrize the angle ϕ between q_3 and q_4 in the x-y plane leading to the following expressions of the solid angles

$$\begin{aligned}
d\Omega_3^{(d-1)} &= d\Omega_3^{(d-2)} d\cos\theta_3 (\sin\theta_3)^{d-4} \\
d\Omega_4^{(d-1)} &= d\Omega_4^{(d-3)} d\cos\theta_4 (\sin\theta_4)^{d-4} d\cos\phi (\sin\phi)^{d-5}.
\end{aligned} \quad (5.27)$$

Suppressing any extra dimensional components our 4-vectors are then fully parametrized as $q_3 = E_3(1, \sin\theta_3, 0, \cos\theta_3)$ and $q_4 = E_4(1, \sin\theta_4 \sin\phi, \sin\theta_4 \cos\phi, \cos\theta_4)$. Mapping the remaining angles linearly, i.e. $\cos\theta_3 = 2x_3 - 1$, $\cos\theta_4 = 2x_4 - 1$ and $\phi = x_2\pi$, one obtains

$$\begin{aligned}
\int d\Phi_3 &= \frac{(2\pi)^{-3+2\epsilon}}{16\Gamma(1-2\epsilon)} \int_0^1 dx_1 dx_2 dx_3 dx_4 \left(\frac{s(1-z)^3 \kappa^4 x_1 (1-x_1)}{2-\kappa} \right) \\
&\quad \times \left(s^2(1-z)^4 \kappa^4 x_1^2 (1-x_1)^2 x_3 (1-x_3) x_4 (1-x_4) \sin^2(\pi x_2) \right)^{-\epsilon} \quad (5.28)
\end{aligned}$$

The following lists the propagators of massless partons in this parameterization:

$$\begin{aligned}
s_{13} &= -s(1-z)\kappa x_1 x_3 \\
s_{23} &= -s(1-z)\kappa x_1 (1-x_3) \\
s_{14} &= -s(1-z)\kappa (1-x_1) x_4 \\
s_{24} &= -s(1-z)\kappa (1-x_1)(1-x_4)
\end{aligned} \quad (5.29)$$

and

$$\begin{aligned}
s_{34} &= s(1-z)^2 \kappa^2 x_1 (1-x_1) \tilde{x}_{34} \\
s_{134} &= s(1-z)\kappa \left[(1-z)\kappa x_1 (1-x_1) \tilde{x}_{34} - x_1 x_3 - (1-x_1) x_4 \right] \\
s_{234} &= s(1-z)\kappa \left[(1-z)\kappa x_1 (1-x_1) \tilde{x}_{34} - x_1(1-x_3) - (1-x_1)(1-x_4) \right] \quad (5.30)
\end{aligned}$$

where

$$\tilde{x}_{34} = x_3 + x_4 - 2x_3 x_4 - 2\cos(x_2\pi)\sqrt{x_3(1-x_3)x_4(1-x_4)} \quad (5.31)$$

and

$$\kappa = \frac{1 - \sqrt{1 - 4(1-z)x_1(1-x_1)\tilde{x}_{34}}}{2(1-z)x_1(1-x_1)\tilde{x}_{34}}. \quad (5.32)$$

The angle between q_3 and q_4 is related to

$$\tilde{x}_{34} = \frac{1 - \cos\theta_{34}}{2} = \frac{1 - \cos\theta_3 \cos\theta_4 - \cos\phi \sin\theta_3 \sin\theta_4}{2}. \quad (5.33)$$

This expression exposes the weak point of this parameterization: it gives rise to an overlapping line singularity when $\phi = 0$ and $\theta_3 = \theta_4$ i.e. when q_3 is parallel to q_4. Nevertheless

the above construction can be used to fully subtract all phase space integrals which do not contain singularities in $\tilde{x}_{34}$, i.e. which do not contain s_{34}, s_{134}, s_{234}.

Let us now analyze the singularities in this parameterization. While s_{13}, s_{23}, s_{14} and s_{24} are fully factorized, there is a overlapping line singularity in s_{34} when $\tilde{x}_{34} = 0$. Furthermore there are overlapping singularities in s_{134} and s_{234}. For s_{134} there are 3 different possibilities

$$
\begin{aligned}
a) \, & x_3 = 0 \text{ and } x_4 = 0 \\
b) \, & x_3 = 0 \text{ and } x_1 = 1 \\
c) \, & x_4 = 0 \text{ and } x_1 = 0
\end{aligned}
\tag{5.34}
$$

while for s_{234} the singularities are located at

$$
\begin{aligned}
a) \, & x_3 = 1 \text{ and } x_4 = 1 \\
b) \, & x_3 = 1 \text{ and } x_1 = 1 \\
c) \, & x_4 = 1 \text{ and } x_1 = 0.
\end{aligned}
\tag{5.35}
$$

We can now apply this parameterization to all integrals of type $C_1 \otimes C_1, C_3 \otimes C_3$ and $C_1 \otimes C_3$.

5.3.1 Line singularities in the energy and angles parameterization

One can use a non-linear transformation to get rid of the overlapping structure in $\tilde{x}_{34}$ [27]. A convenient way to derive such a mapping is remapping $\tilde{x}_{34}$ from $\tilde{x}_{34}^- = \tilde{x}_{34}(\phi = 0)$ to $\tilde{x}_{34}^+ = \tilde{x}_{34}(\phi = 1)$ using

$$
\tilde{x}_{34} = \frac{\tilde{x}_{34}^- \tilde{x}_{34}^+}{\tilde{x}_{34}^+ - x_2(\tilde{x}_{34}^+ - \tilde{x}_{34}^-)}
\tag{5.36}
$$

It is then apparent that $\tilde{x}_{34}$ will vanish whenever $\tilde{x}_{34}^-$ or $\tilde{x}_{34}^+$ will, for any value of x_2.

And that the overlapping line singularity is then re-casted into just a line singularity. To aid numerical stability we perform the mapping $x_2 \rightarrow (1 - \cos(x_2\pi))/2$, such that $\tilde{x}_{34}$ becomes

$$
\tilde{x}_{34} = \frac{(x_3 - x_4)^2}{x_3 + x_4 - 2x_3x_4 + 2\cos(x_2\pi)\sqrt{x_3(1-x_3)x_4(1-x_4)}}.
\tag{5.37}
$$

This is in fact identical to the mapping in [28]. The phase space volume then becomes

$$
\Phi_3 = \frac{(2\pi)^{-3+2\epsilon}}{16\Gamma(1-2\epsilon)} \int_0^1 dx_1 dx_2 dx_3 dx_4 \left(\frac{s(1-z)^3 \kappa^4 x_1(1-x_1)}{2-\kappa} \right)
$$
$$
\times \left(s^2(1-z)^4 \kappa^4 x_1^2(1-x_1)^2 x_3(1-x_3)x_4(1-x_4)\sin^2(\pi x_2) \right)^{-\epsilon} \left(\frac{\tilde{x}_{34}}{|x_3 - x_4|} \right)^{1-2\epsilon}.
\tag{5.38}
$$

To factorize the line singularity in s_{34} (at $x_3 = x_4$) we are forced to split the integration region in two, separating $x_3 < x_4$ from $x_4 < x_3$.

5.4 Hierarchical parameterization

Since in the energy and angles parameterization the invariants s_{34}, s_{134}, s_{234} had line and overlapping singularities, it is worth having a second parameterization which factorizes these, but may not factorize the others. Our second parameterization closely resembles the features of the rapidity parameterization published in [7], however it is somewhat simpler. In this parameterization the three particle phase space element $d\Phi_3$ is

$$d\Phi_3(\sqrt{s}; 0, 0, \sqrt{s_{1...n}}) = (2\pi)^{3-2d} d^d q_3 \delta^{(+)}(q_3^2) d^d q_4 \delta^{(+)}(q_4^2) d^d Q$$
$$\times \delta^{(+)}(Q^2 - s_{1...n}) \delta^d(q_1 + q_2 - q_3 - q_4 - Q) \quad (5.39)$$

is first factorized into a product of two 2-particle phase spaces

$$d\Phi_3(\sqrt{s}, 0, 0, \sqrt{s_{1...n}}) = \int \frac{ds_{34}}{2\pi} d\Phi_2(\sqrt{s}, \sqrt{s_{34}}, \sqrt{s_{1...n}}) d\Phi_2(\sqrt{s_{34}}, 0, 0). \quad (5.40)$$

with

$$d\Phi_2(\sqrt{s}, \sqrt{s_{34}}, \sqrt{s_{1...n}}) = (2\pi)^{2-d} d^d Q \delta^{(+)}(Q^2 - s_{1...n}) d^d \tilde{Q} \delta^{(+)}(\tilde{Q}^2 - s_{34}) \delta^d(q_1 + q_2 - \tilde{Q} - Q) \quad (5.41)$$

and

$$d\Phi_2(\sqrt{s_{34}}, 0, 0) = (2\pi)^{2-d} d^d q_3 \delta^{(+)}(q_3^2) d^d q_4 \delta^{(+)}(q_4^2) \delta^d(\tilde{Q} - q_3 - q_4). \quad (5.42)$$

We can parameterize $d\Phi_2(\sqrt{s}, \sqrt{s_{34}}, \sqrt{s_{1...n}})$ in terms of s_{134}, yielding

$$d\Phi_2(\sqrt{s}, \sqrt{s_{34}}, \sqrt{s_{1...n}}) = (2\pi)^{2-d} \frac{1}{4s} d\Omega^{d-2}(\tilde{Q}_\perp)^{d-4} ds_{134}. \quad (5.43)$$

To satisfy $\tilde{Q}_\perp \geq 0$, we take

$$0 \leq s_{34} \leq \frac{s_{134}(s + s_{134} - s_{1...n})}{s_{134} - s_{1...n}}$$
$$0 \geq s_{134} \geq (s_{1...n} - s). \quad (5.44)$$

$d\Phi_2(\sqrt{s_{34}}; 0, 0)$ can be parameterized in terms of the invariants s_{13} and s_{23} yielding

$$d\Phi_2(\sqrt{s_{34}}; 0, 0) = (2\pi)^{2-d} \frac{1}{8\tilde{Q}_\perp s} ds_{13} ds_{23} d\Omega^{d-3} [(p_3)_\perp \sin\phi]^{d-5} \quad (5.45)$$

where ϕ is the angle between $(p_3)_\perp$ and $\tilde{Q}_\perp$. We fulfil the constraint $(p_3)_\perp \sin\phi \geq 0$ to find the limits of integration for s_{13} and then for s_{23}.

Parameterizing s_{134}, s_{34}, s_{13} and s_{23} linearly we arrive at

$$\int d\Phi_3 = \frac{(2\pi)^{-3+2\epsilon}}{16\Gamma(1-2\epsilon)} \int_0^1 dx_1 dx_2 dx_3 dx_4 \left(\frac{s(1-z)^3 x_1(1-x_1)}{z + x_1(1-z)} \right)$$
$$\times \left(\frac{s^2(1-z)^4 x_1^2(1-x_1)^2 x_2(1-x_2) x_3(1-x_3) \sin^2(\pi x_4)}{z + x_1(1-z)} \right)^{-\epsilon}. \quad (5.46)$$

The invariants in this parameterization are

$$s_{34} = \frac{s(1-z)^2 x_1(1-x_1)x_2}{z + x_1(1-z)}$$

$$s_{134} = -s(1-z)x_1$$

$$s_{234} = -s(1-z)(1-x_1)\left[\frac{z + x_1(1-x_2)(1-z)}{z + x_1(1-z)}\right]$$

$$s_{23} = -s(1-z)(1-x_1)x_3$$

$$s_{24} = -s(1-z)(1-x_1)(1-x_3) \tag{5.47}$$

and

$$s_{13} = -s(1-z)x_1\left[x_3(1-x_2) + \frac{x_2(1-x_3)}{z + x_1(1-z)} - 2\cos(\pi x_4)\sqrt{\frac{x_2(1-x_2)x_3(1-x_3)}{z + x_1(1-z)}}\right]$$

$$s_{14} = -s(1-z)x_1\left[(1-x_3)(1-x_2) + \frac{x_2 x_3}{z + x_1(1-z)} + 2\cos(\pi x_4)\sqrt{\frac{x_2(1-x_2)x_3(1-x_3)}{z + x_1(1-z)}}\right].$$

We see that the only invariants which are not factorized are s_{13} and s_{14}. The variable s_{13} contains overlapping singularities at $x_3 = 0 = x_2$ and $x_3 = 1 = x_2$ as well as an overlapping line singularity at $x_4 = 0, x_1 = 1, x_3 = x_2$, while s_{14} contains overlapping singularities at $x_3 = 0, x_2 = 1$ and $x_3 = 1, x_2 = 1$ as well as an overlapping line singularity at $x_4 = 1, x_1 = 1, x_3 = 1 - x_2$.

5.4.1 Line singularities in the hierarchical parameterization

Consider the expressions

$$\frac{J(p_1, p_2, p_3, p_4)}{s_{13}s_{24}}, \quad \frac{J(p_1, p_2, p_3, p_4)}{s_{13}s_{23}} \tag{5.48}$$

with $J(p_1, p_2, p_3, p_4)$ a finite numerator function. They both contain a line singularity due to s_{13} in the denominator. We now use the partial fractioning identities,

$$\frac{1}{s_{13}s_{24}} = \frac{1}{s_{13}s_{234} + s_{134}s_{24}}\left(\frac{s_{134}}{s_{13}} + \frac{s_{234}}{s_{24}}\right), \tag{5.49}$$

$$\frac{1}{s_{13}s_{23}} = \frac{1}{s_{13}s_{234} + s_{23}s_{134}}\left(\frac{s_{134}}{s_{13}} + \frac{s_{234}}{s_{23}}\right). \tag{5.50}$$

The term $s_{13}s_{234} + s_{134}s_{24}$ has an overlapping singularity at $x_3 = 1 = x_2$, while the term $s_{13}s_{234} + s_{134}s_{23}$ has an overlapping singularity at $x_3 = 0 = x_2$. Then we exchange $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ in the term containing s_{13} to rotate the line singularity out, i.e.

$$\frac{J(p_1, p_2, p_3, p_4)}{s_{13}s_{24}} = \frac{J(p_1, p_2, p_3, p_4) + J(p_2, p_1, p_4, p_3)}{s_{13}s_{234} + s_{134}s_{24}}\frac{s_{234}}{s_{24}} \tag{5.51}$$

$$\frac{J(p_1, p_2, p_3, p_4)}{s_{13}s_{23}} = \frac{J(p_1, p_2, p_3, p_4) + J(p_2, p_1, p_3, p_4)}{s_{13}s_{234} + s_{23}s_{134}}\frac{s_{234}}{s_{23}}. \tag{5.52}$$

and we are left with just overlapping singularities, which can be treated as explained in the following section. This trick was first discovered by Frank Petriello [29] and it has been used in the implementation of the program FEHiP described in [7], it has been also been used in the evaluation of doublereal counterterms in [30].

6 Numerical evaluation of double-real radiation phase-space integrals

In this section, we present a numerical evaluation of all types of scalar phase-space integrals which appear in NNLO calculations. To evaluate our integrals numerically we choose the point $(s = 1, z = 0.1)$. We will use the notation

$$\bar{x}_i = 1 - x_i, \tag{6.1}$$

where the $x_i \in [0, 1]$ are parameters of integration.

1. Topology $C_1 \otimes C_1$:

   (a) The integral

   $$I_{11a} = \int \frac{d\Phi_3}{s_{13}s_{23}s_{14}s_{24}} \tag{6.2}$$

   fully factorizes in the energies and angles parameterization (section 5.3), we obtain

   $$I_{11a} = 0.09400(2) + \frac{0.010951(4)}{\epsilon} - \frac{0.0035586(5)}{\epsilon^2} - \frac{0.001119844946(1)}{\epsilon^3}. \tag{6.3}$$

   (b) The integral

   $$I_{11b} = \int d\Phi_3 \frac{(s_{34}s - s_{14}s_{23})^2}{s_{13}^2 s_{24}^2} \tag{6.4}$$

   with the numerator structure as in [7], factorizes in the energies and angles parameterization (section 5.3), we get

   $$I_{11b} = 0.023885(3) + \frac{0.0041606(3)}{\epsilon} + \frac{0.00036930(4)}{\epsilon^2}. \tag{6.5}$$

2. Topology $C_2 \otimes C_2$:

   (a) The integral

   $$I_{22a} = \int \frac{d\Phi_3 N(\{s_{ij}\})}{(s_{34}s_{134})^2} \tag{6.6}$$

   factorizes in the hierarchical parameterization (section 5.4). The numerator function has the scaling behavior $N(\{s_{ij}\}) \sim s_{34}s_{134}$ [7]. We obtain

   $$I_{22a} = \int \frac{d\Phi_3}{s_{34}s_{134}} = 0.0011728(1) - \frac{0.00050726(1)}{\epsilon} - \frac{0.000125982556(0)}{\epsilon^2}. \tag{6.7}$$

   (b) The integral

   $$I_{22b} = \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}^2 s_{134}s_{234}} \tag{6.8}$$

   factorizes in the hierarchical parameterization (section 5.4). The numerator scales as $N(\{s_{ij}\}) \sim s_{34}$. We obtain

   $$I_{22b} = \int \frac{d\Phi_3}{s_{34}s_{134}s_{234}} = -0.0015003(2) + \frac{0.00112726(5)}{\epsilon} + \frac{0.000279961236(1)}{\epsilon^2}. \tag{6.9}$$

3. Topology $C_3 \otimes C_3$:

(a) The integral

$$I_{33a} = \int d\Phi_3 \frac{(s_{34}s - s_{14}s_{23})^2}{s_{234}^2 s_{24}^2} \qquad (6.10)$$

factorizes in the hierarchical parameterization (section 5.4). The numerator structure can be found in [7]. We obtain

$$I_{33a} = -0.003841(2) + \frac{0.0007814(4)}{\epsilon} + \frac{0.00018465(1)}{\epsilon^2}. \qquad (6.11)$$

(b) The integral

$$I_{33b} = \int \frac{d\Phi_3}{s_{134}s_{234}s_{13}s_{23}} \qquad (6.12)$$

neither factorizes in energies and angles nor in the hierarchical parameterization. We use the hierarchical parameterization (section 5.4), since fewer overlapping singularities are present there. Using partial fractions, as described in section 5.4.1 we can rewrite the integral as

$$I_{33b} = \int \frac{2d\Phi_3}{s_{23}s_{134}(s_{134}s_{23} + s_{234}s_{13})}. \qquad (6.13)$$

This contains the following substructure

$$\frac{1}{x_3} \frac{1}{x_3 A + x_2 \bar{x}_3 B + C\sqrt{x_2 \bar{x}_2 x_3 \bar{x}_2}} \qquad (6.14)$$

with A, B, C finite. This becomes singular when $x_3 = 0 = x_2$ where x_3 is active. We factorize this singularity by applying

$$x_3 \rightarrow \frac{x_3 x_2}{(1 - x_3) + x_2} \qquad (6.15)$$

and obtain

$$I_{33b} = 0.023155(3) + \frac{0.0076371(1)}{\epsilon} + \frac{0.00007730(6)}{\epsilon^2} - \frac{0.000279961236(1)}{\epsilon^3}. \quad (6.16)$$

(c) The integral

$$I_{33c} = \int \frac{d\Phi_3}{s_{134}s_{234}s_{13}s_{24}} \qquad (6.17)$$

is similar to I_{33b} in the hierarchical parameterization (section 5.4). Partial fractioning as before we get

$$I_{33c} = \int \frac{2d\Phi_3}{s_{24}s_{134}(s_{134}s_{24} + s_{234}s_{13})}. \qquad (6.18)$$

This contains the substructure

$$\frac{1}{\bar{x}_3} \frac{1}{\bar{x}_2 A + \bar{x}_3 x_2 B + C\sqrt{x_2 \bar{x}_2 x_3 \bar{x}_3}} \qquad (6.19)$$

with A, B, C finite. This becomes singular when $\bar{x}_3 = 0 = \bar{x}_2$ with $\bar{x}_3$ being active. We disentangle this singularity by applying

$$\bar{x}_3 \rightarrow \frac{\bar{x}_3 \bar{x}_2}{(1 - \bar{x}_3) + \bar{x}_2}. \tag{6.20}$$

We then obtain

$$I_{33c} = 0.12567(9) - \frac{0.03645(1)}{\epsilon} - \frac{0.018566(1)}{\epsilon^2} + \frac{0.002799612364(0)}{\epsilon^3}. \tag{6.21}$$

4. Topology $C_1 \otimes C_3$:

   (a) The integral

$$I_{13a} = \int d\Phi_3 \frac{(s_{34}s - s_{14}s_{23})^2}{s_{234}s_{24}^2 s_{13}} \tag{6.22}$$

   with the numerator structure as in [7]. The singularities factorize in energies and angles. We immediately obtain

$$I_{13a} = -0.0040885(4) - \frac{0.00036930(1)}{\epsilon}. \tag{6.23}$$

   (b) The integral

$$I_{13b} = \int d\Phi_3 \frac{N(\{s_{ij}\})}{s_{134}s_{13}s_{14}s_{23}} \tag{6.24}$$

   has a quadratic divergence due to the term $s_{134}s_{13}s_{14}$. This means that $N(\{s_{ij}\}) \sim \{s_{134}, s_{13}, s_{14}\}$. Such that

$$I_{13b} = \left\{ \int \frac{d\Phi_3}{s_{13}s_{14}s_{23}}, \int \frac{d\Phi_3}{s_{134}s_{14}s_{23}} \right\} \tag{6.25}$$

   the first of which is a sub-topology of C_1^2 while the second is a sub-topology of C_3^2.

5. Topology $C_1 \otimes C_2$:

   (a) The integral

$$I_{12} = \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{234}s_{13}s_{24}} \tag{6.26}$$

   has a quadratic divergence due to the term $s_{34}s_{234}s_{24}$. The numerator can have the following scalings: $N(\{s_{ij}\}) \sim \{s_{34}, s_{234}, s_{24}\}$. We therefore consider the following possibilities

$$I_{12} = \left\{ \int \frac{d\Phi_3}{s_{34}s_{13}s_{24}}, \int \frac{d\Phi_3}{s_{34}s_{234}s_{13}}, \int \frac{d\Phi_3}{s_{234}s_{13}s_{24}}. \right\} \tag{6.27}$$

   The last of these is a sub-topology of C_3^2 and does not merit further attention. We will evaluate the other two in the hierarchical parameterization. For

$$I_{12a} = \int \frac{d\Phi_3}{s_{34}s_{13}s_{24}} \tag{6.28}$$

we use the same strategy as we used for I_{33c}. We obtain

$$I_{12a} = 0.07115(1) + \frac{0.006996(1)}{\epsilon} - \frac{0.0029912(1)}{\epsilon^2} - \frac{0.000839883709(0)}{\epsilon^3}. \quad (6.29)$$

The integral

$$I_{12b} = \int \frac{d\Phi_3}{s_{34}s_{234}s_{13}} \quad (6.30)$$

factorizes in the hierarchical parameterization. We obtain

$$I_{12b} = 0.0198554(9) + \frac{0.0023667(2)}{\epsilon} - \frac{0.00088965(4)}{\epsilon^2} - \frac{0.000279961236(1)}{\epsilon^3}. \quad (6.31)$$

6. Topology $C_2 \otimes C_3$:

   (a) The integral

   $$I_{23a} = \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{234}^2 s_{23}} \quad (6.32)$$

   factorizes in the hierarchical parameterization, but carries a cubic divergence in $(1 - x_1) \sim s_{234}$. Taking $N(\{s_{ij}\}) \sim s_{234}^2$, we get

   $$I_{23a} = \int \frac{d\Phi_3}{s_{34}s_{23}} \quad (6.33)$$

   which just is a sub-topology of I_{12a}. While other numerators are possible these do not give different singularity structures.

   (b) The integral

   $$I_{23a} = \int \frac{d\Phi_3 N(\{s_{ij}\})}{s_{34}s_{134}s_{234}s_{23}} \quad (6.34)$$

   factorizes in the hierarchical parameterization, but carries a quadratic divergence in $(1 - x_1) \sim s_{234}$. A minimal choice for the numerator is $N(\{s_{ij}\}) \sim s_{234}$ in which case we recover I_{22b}. Hence no new singularity structures can be obtained from this topology.

7. We will now consider interferences of C_4 with C_2 and C_3. One can evaluate these interferences in the energies and angles parameterization. In the following we will use $t_{13} \sim E_3 \sim (s_{13} + s_{23})$ and $t_{24} \sim E_4 \sim (s_{14} + s_{24})$.

   (a) Topology $C_2 \otimes C_4$:
   The integral

   $$I_{24} = \int \frac{d\Phi_3}{s_{34}s_{134}(s_{13} + s_{23})(s_{14} + s_{24})} \quad (6.35)$$

   has the following singularity structure

   $$\frac{1}{x_1 \bar{x}_1 (x_3 - x_4)} \frac{1}{A x_1 \bar{x}_1 (x_3 - x_4)^2 + B x_1 x_3 + C \bar{x}_1 x_4} \quad (6.36)$$

   in the energy and angle parameterization after the mapping (see 5.3.1) is applied. We first split the integration region into two sectors which we define as $x_3 < x_4$

(sector 1) and $x_4 < x_3$ (sector 2). After this sector decomposition we are still left with overlapping singularities at $x_3 = 0 = \bar{x}_1$ in sector 1 and at $x_4 = 0 = x_1$ in sector 2. These can be disentangled using

$$\bar{x}_1 \rightarrow \frac{\bar{x}_1 x_3}{(1 - \bar{x}_1) + x_3} \tag{6.37}$$

in sector 1 and

$$x_1 \rightarrow \frac{x_1 x_4}{(1 - x_1) + x_4} \tag{6.38}$$

in sector 2. We then obtain

$$I_{24} = -0.006956(3) - \frac{0.0010708(3)}{\epsilon} + \frac{0.00065900(1)}{\epsilon^2} + \frac{0.000207378694(0)}{\epsilon^3}. \tag{6.39}$$

(b) Topology $C_3 \otimes C_4$:
   The integral

$$I_{34} = \int \frac{d\Phi_3}{s_{13} s_{134} (s_{13} + s_{23})(s_{14} + s_{24})} \tag{6.40}$$

has the following singularity structure

$$\frac{1}{x_1 x_3} \frac{1}{A x_1 \bar{x}_1 (x_3 - x_4)^2 + B x_1 x_3 + C \bar{x}_1 x_4}. \tag{6.41}$$

It contains no line singularity but several overlapping ones located at $x_3 = 0 = x_4$, $x_4 = 0 = x_1$ and at $x_3 = 0, \bar{x}_1 = 0$. To separate the two singularities we first partial fraction the soft singularities by multiplying by $1 = x_1 + \bar{x}_1$. We then treat the two terms with different nonlinear transformations. For the first term we apply the mapping

$$x_3 \rightarrow \frac{x_3 x_4 \bar{x}_1}{(1 - x_3) + x_4 \bar{x}_1}. \tag{6.42}$$

since x_3 is the only active singularity, it is clear that we had to remap it. The second term is more difficult, since both x_3 and x_1 are now active. We apply the following sequence of mappings:
First let

$$x_3 \rightarrow \frac{x_3 x_4}{(1 - x_3) + x_4} \tag{6.43}$$

and then

$$x_1 \rightarrow \frac{x_1 \bar{x}_3}{(1 - x_1) + \bar{x}_3}$$

$$x_4 \rightarrow \frac{x_4 \bar{x}_3}{(1 - x_4) + \bar{x}_3}. \tag{6.44}$$

$$\tag{6.45}$$

We obtain

$$I_{34} = -0.32519(4) - \frac{0.048942(2)}{\epsilon} - \frac{0.0062917(3)}{\epsilon^2} - \frac{0.000559922473(3)}{\epsilon^3}. \tag{6.46}$$
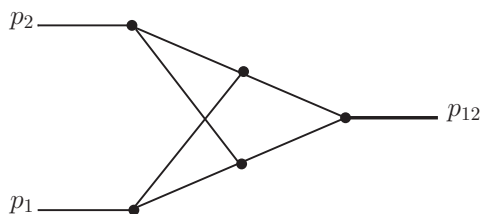
Figure 4. The massless non-planar two-loop triangle with one legs off-shell.

7 Two loop examples

In what follows we show how one can use non-linear mappings to disentangle singularities in two-loop integrals appearing in NNLO virtual amplitudes. We treat only two indicative cases, the massless non-planar triangle with one leg off-shell and the massless non-planar box with all legs on-shell, due to their particularly intricate singularity structure. Integrals involving masses are in general simpler as far as factorization of singularities is concerned.

7.1 The massless non-planar triangle with one leg off-shell

The two-loop, non-planar triangle with one off-shell leg (see figure 4) and momenta p_1, p_2

$$Xtri = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{d^d k_2}{i\pi^{d/2}} \frac{1}{k_1^2(k_1+p_1)^2 k_2^2(k_2+p_2)^2(k_1+k_2)^2(k_1+k_2+p_1+p_2)^2} \quad (7.1)$$

A Feynman parameterization reads:

$$Xtri = 4^{2+2\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{zy^{1+\epsilon}(1-y)^{-1-\epsilon}(1-z)^{-1-\epsilon}}{[x(1-x)+yz(x-x_1)(x-x_2)]^{2+2\epsilon}} \quad (7.2)$$

The first overlapping singularity is at $x = 0$ or $x = 1$ and $y = 0$. We also notice that there is a singularity at $y = 1$. To avoid infinite looping we must first guarantee, as in sector decomposition, that no singularities occur at the upper limit of integration.

We split x in the two intervals $R_a = [0, 1/2]$ and $R_b = [1/2, 1]$ and map the integration region back to the unit hypercube. In R_a,

$$x \rightarrow x/2,$$

and in R_b,

$$x \rightarrow 1 - x/2, \quad x_1 \rightarrow 1 - x_1, \quad x_2 \rightarrow 1 - x_2.$$

This gives two identical integrals (the integral in eq. (7.2) is invariant under the combined $x \rightarrow 1-x$, $x_1 \rightarrow 1-x_1$, $x_2 \rightarrow 1-x_2$), and we can write

$$Xtri = 4^{2+2\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{zy^{1+\epsilon}(1-y)^{-1-\epsilon}(1-z)^{-1-\epsilon}}{[x(2-x)+yz(2x_1-x)(2x_2-x)]^{2+2\epsilon}} \quad (7.3)$$

Note here that the denominator of eq. (7.3) has the same singularity structure as

$$x + yzx_1x_2 - yzx(x_1+x_2) \quad (7.4)$$

In particular, there are still singularities at upper corners of the hypercube, when $y \to 1$ and $z \to 1$. This leads us to split the y integration region, $y \to y/2$ and $y \to 1 - y/2$ and the $z \to z/2$ and $z \to 1 - z/2$.

We obtain

$$Xtri = Xtri_{11} + Xtri_{12} + Xtri_{21} + Xtri_{22}, \tag{7.5}$$

where

$$Xtri_{11} = 2^{6+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{zy^{1+\epsilon}(2-y)^{-1-\epsilon}(2-z)^{-1-\epsilon}}{[4x(2-x) + yz(2x_1-x)(2x_2-x)]^{2+2\epsilon}} \tag{7.6}$$

$$Xtri_{12} = 2^{6+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{(2-z)y^{1+\epsilon}(2-y)^{-1-\epsilon}z^{-1-\epsilon}}{[4x(2-x) + y(2-z)(2x_1-x)(2x_2-x)]^{2+2\epsilon}} \tag{7.7}$$

$$Xtri_{21} = 2^{6+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{z(2-y)^{1+\epsilon}y^{-1-\epsilon}(2-z)^{-1-\epsilon}}{[4x(2-x) + (2-y)z(2x_1-x)(2x_2-x)]^{2+2\epsilon}} \tag{7.8}$$

$$Xtri_{22} = 2^{6+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{(2-z)(2-y)^{1+\epsilon}y^{-1-\epsilon}z^{-1-\epsilon}}{[4x(2-x) + (2-y)(2-z)(2x_1-x)(2x_2-x)]^{2+2\epsilon}} \tag{7.9}$$

The first three sectors, $Xtri_{11}, Xtri_{12}, Xtri_{21}$ are free of singularities at 1. They can be directly treated by a non-linear mapping.

$Xtri_{11}$ has a singularity structure equivalent to that of $x + yzx_1x_2$, similar to eq. (4.1) and we use the mapping (directly analogous to eq. (4.2))

$$x \to \frac{xyzx_1x_2}{1 - x + yzx_1x_2}. \tag{7.10}$$

$Xtri_{12}$ has a singularity structure equivalent to that of $x + yx_1x_2$, also similar to eq. (4.1) and we use the mapping

$$x \to \frac{xyx_1x_2}{1 - x + yx_1x_2}. \tag{7.11}$$

$Xtri_{21}$ has a singularity structure equivalent to that of $x + zx_1x_2$, also similar to eq. (4.1) and we use the mapping

$$x \to \frac{xzx_1x_2}{1 - x + zx_1x_2}. \tag{7.12}$$

$Xtri_{22}$ is a bit more complicated. Its singularity structure is the one of

$$x + x_1x_2 - x(x_1 + x_2) \tag{7.13}$$

It retains singularities at $x_{1,2} \to 1$. We therefore split this integral further in $x_1 \to x_1/2$ and $x_1 \to 1 - x_1/2$ as well as $x2 \to x2/2$ and $x_2 \to 1 - x_2/2$.

We obtain

$$Xtri_{2211} = 2^{5+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{(2-z)(2-y)^{1+\epsilon}y^{-1-\epsilon}z^{-1-\epsilon}}{[4x(2-x) + (2-y)(2-z)(x_1-x)(x_2-x)]^{2+2\epsilon}} \tag{7.14}$$

with a singularity structure equivalent to $x + x_1 x_2$ (i.e. eq. (4.1)) for which we will use the mapping

$$x \to \frac{x x_1 x_2}{1 - x + x_1 x_2}. \tag{7.15}$$

$$Xtri_{2212} = 2^{5+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{(2-z)(2-y)^{1+\epsilon} y^{-1-\epsilon} z^{-1-\epsilon}}{[4x(2-x) + (2-y)(2-z)(x_1-x)(2-x_2-x)]^{2+2\epsilon}} \tag{7.16}$$

with a singularity structure equivalent to $x_1 + x(x_1 + y + z + x_2)$, similar to eq. (4.7) for which we will use the following sequence of mappings

$$x_1 \to \frac{x_1 x}{1 - x_1 + x} \ , \quad y \to \frac{y x_2}{1 - y + x_2} \ , \quad z \to \frac{z x_1}{1 - z + x_1} \ , \quad x_1 \to \frac{x_1 x_2}{1 - x_1 + x_2} \tag{7.17}$$

$$Xtri_{2221} = 2^{5+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{(2-z)(2-y)^{1+\epsilon} y^{-1-\epsilon} z^{-1-\epsilon}}{[4x(2-x) + (2-y)(2-z)(2-x_1-x)(x_2-x)]^{2+2\epsilon}} \tag{7.18}$$

with a singularity structure equivalent to $x_2 + x(x_2 + y + z + x_1)$, similar to eq. (4.7), for which we will use the following sequence of mappings

$$x_2 \to \frac{x_2 x}{1 - x_2 + x} \ , \quad y \to \frac{y x_1}{1 - y + x_1} y \ , \quad z \to \frac{z x_2}{1 - z + x_2} \ , \quad x_2 \to \frac{x_2 x_1}{1 - x_2 + x_1} \tag{7.19}$$

$$Xtri_{2222} = 2^{5+9\epsilon} \int_0^1 dx_1 dx_2 dz dy dx \frac{(2-z)(2-y)^{1+\epsilon} y^{-1-\epsilon} z^{-1-\epsilon}}{[4x(2-x) + (2-y)(2-z)(2-x_1-x)(2-x_2-x)]^{2+2\epsilon}} \tag{7.20}$$

which is finite!

We therefore end up with 7 different integrals to be numerically evaluated. This should be contrasted with the 64 number of sectors one arrives using sector decomposition. The numerical convergence of these integrals poses no additional problems and we have checked that the numerical result agrees with the analytic result known in the literature.

7.2 The non-planar double box

Using the representation of ref. [31] for the two loop non-planar box (see figure 5), we get the expression

$$\mathrm{Xbox} = C_\epsilon \int \frac{dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4) d\tau_1 d\tau_2 \ x_2^{1+\epsilon}}{(x_1 x_3 s + x_2 x_4 t_c + x_1 x_2 Q^2 + x_2 x_3 Q_t^2)^{3+2\epsilon}} \tag{7.21}$$

where

$$Q_t^2 = (1-\tau_1)(1-\tau_2)s, \quad Q^2 = \tau_1 \tau_2 s, \quad t_c = \tau_2(1-\tau_1)u + (1-\tau_2)\tau_1 t \tag{7.22}$$

and

$$C_\epsilon = \frac{2\Gamma(3+2\epsilon)\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1+\epsilon)^2 \Gamma(1-2\epsilon)} \tag{7.23}$$
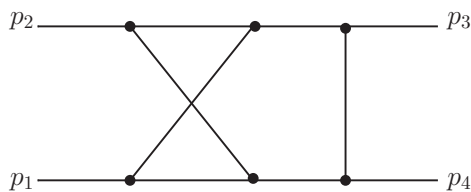
– 33 –

Figure 5. The massless non-planar double box with all legs on-shell.

In order to avoid the singularities at the upper corners of the integration region we split the integral in four, mapping $\tau_1 \to \tau_1/2$, $\tau_1 \to 1 - \tau_1/2$ and then $\tau_2 \to \tau_2/2$ and $\tau_2 \to 1-\tau_2/2$. Two of the resulting integrals can be mapped to the other two by exchanging $x1$ and x_3, so we end up with

$$\text{Xbox}_a = C_\epsilon \frac{4^{3+2\epsilon}}{2} \int \frac{dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4) d\tau_1 d\tau_2 \; x_2^{1+\epsilon}}{(4x_1 x_3 s + x_1 x_2 B_1 B_2 s + x_2 x_3 \tau_1 \tau_2 s + x_2 x_4 (\tau_2 B_1 t + B_2 \tau_1 u))^{3+2\epsilon}} \tag{7.24}$$

$$\text{Xbox}_b = C_\epsilon \frac{4^{3+2\epsilon}}{2} \int \frac{dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4) d\tau_1 d\tau_2 \; x_2^{1+\epsilon}}{(4x_1 x_3 s + x_1 x_2 \tau_1 B_2 s + x_2 x_3 B_1 \tau_2 s + x_2 x_4 (B_1 B_2 u + \tau_1 \tau_2 t))^{3+2\epsilon}} \tag{7.25}$$

where

$$B_{1,2} \equiv 2 - \tau_{1,2} \tag{7.26}$$

Subsequently, we use the method of primary sectors on the variables x_1, x_2, x_3, x_4 on each of the above integrals, to get eight primary sectors:

1. Xbox_{a1}: Xbox_a where $x_1 > x_{2,3,4}$ has the singularity structure of $x_2 + x_3$, and we use the mapping of eq. (3.12).

2. Xbox_{a2}: Xbox_a where $x_2 > x_{1,3,4}$ has the rather intricate singularity structure of the example eq. (4.45). We follow the discussion given there and decompose it to six integrals.

3. Xbox_{a3}: Xbox_a where $x_3 > x_{1,2,4}$ has the rather intricate singularity structure of the example eq. (4.35). We follow the discussion given there and decompose it to five integrals.

4. Xbox_{a4}: Xbox_a where $x_4 > x_{1,2,3}$ has the singularity structure of eq. (4.32). We follow the discussion given there and decompose it in two sub-sectors, in x_2, x_3, each of which can be factorized.

5. Xbox_{b1}: Xbox_b where $x_1 > x_{2,3,4}$ has the singularity structure of eq. (4.7) and we use the mapping of eq. (4.8).

6. Xbox_{b2}: Xbox_b where $x_2 > x_{1,3,4}$ has the singularity structure of eq. (4.27) and we follow the mappings described there to factorize it .

7. Xbox_{b3}: Xbox_b where $x_3 > x_{1,2,4}$ has the singularity structure of eq. (4.7) and we use the mapping of eq. (4.8).

8. Xbox_{b4}: Xbox_b where $x_4 > x_{1,2,3}$ has the singularity structure of eq. (4.1) and we use the mapping of eq. (4.2).

We end up with 18 integrals to be evaluated numerically. This should be contrasted with the 119 sectors that are necessary if one factorizes the non-planar double box with sector decomposition.

8   Conclusions

Higher order perturbative calculations are very important for precision phenomenology at modern accelerator experiments. We believe that NNLO computations will be particularly relevant for signals of yet undiscovered physics, such as a Higgs boson or candidates of dark matter, in $2 \to 1$ and $2 \to 2$ processes. This motivates the development of powerful integration methods of matrix-elements of up to two virtual or real, potentially unresolved, partons. Such integrations entail the disentanglement of overlapping singularities.

In this paper, we have introduced a method for the factorization of singularities based on non-linear transformations. As proof of principle, we presented the most singular integral topologies which appear in NNLO double-real radiation processes with massive particles in the final state. We find that all overlapping singularities can be factorized with our method, which yields a small number of numerically stable integrals. We have also applied our method to complicated crossed two-loop master integrals for massless QCD scattering processes. We find that we can factorize most of the overlapping singularities with non-linear transformations. However, some remaining singularities are cumbersome to be treated purely with our method. In such situations, we employ a hybrid of our method and sector decomposition. This is more efficient than employing a pure sector decomposition approach.

The reduction of the number of integrals which emerge in higher order corrections should facilitate NNLO computations. We are looking forward to applying our method for precision phenomenological studies of basic collider processes.

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