

## FV-type action for $AdS_5$ mixed-symmetry fields

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**ABSTRACT:** We formulate Fradkin-Vasiliev type theory of massless higher spin fields in  $AdS_5$ . The corresponding action functional describes cubic order approximation to gravitational interactions of bosonic mixed-symmetry fields of a particular “hook” symmetry type and totally symmetric bosonic and fermionic fields.

**KEYWORDS:** Supersymmetric gauge theory, Gauge Symmetry, Field Theories in Higher Dimensions, Global Symmetries

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## 1 Introduction

Interacting theories with spectra including graviton along with particles of spin greater than two provide a fascinating playground for exploring the gravity both on classical and quantum levels. For example, string theory describes a dynamics of an infinite collection of massive fields with growing masses and spins and a finite set of massless lower spin fields. An important feature of higher spin models is infinite symmetries which are believed to improve conventional quantum inconsistency of Einstein gravity. Higher spin theories with

massless spectra play a distinguished role because they can be considered as an unbroken phase for massive higher spin theories including string theory itself [1, 2] (see also, e.g. [3–16] for a discussion in the AdS/CFT correspondence context).

The problem of constructing a consistent theory of interactions between higher spin massless fields and the gravity has been first attacked by Aragone and Deser [17]. According to them massless fields of spin  $s > 2$  do not minimally interact with the gravity and therefore no higher spin extension of supergravity theories is possible (see [18, 19] for a review). The solution has been proposed by Fradkin and Vasiliev in [20, 21] who formulated guiding principles to construct a consistent interacting theory of higher spin fields. They identified anti-de Sitter background geometry as a natural background for gravitational higher spin interactions and explicitly constructed higher spin gauge symmetry algebra [22]. It turns out that the presence of additional dimensionful parameter — the cosmological constant  $\lambda$  of anti-de Sitter spacetime — enables one to build various higher derivative interaction terms in the action with overall coefficients proportional to the inverse of  $\lambda$ , and this is quite similar to string theory vertices of massive higher spin fields.<sup>1</sup> Let us mention that a wide class of higher spin cubic (self-)interaction vertices is known in Minkowski space but they do not however contain minimal couplings with the gravity [26–35].

More recently the original FV theory has been extended from  $d = 4$  to  $d = 5$  for both  $\mathcal{N} = 0$  pure bosonic and  $\mathcal{N} = 1$  supersymmetric cubic interactions of totally symmetric (Fronsdal) fields [36, 37]. The  $5d$  theory inherits all basic features of the  $4d$  theory and is governed by the higher spin symmetry superalgebra identified by Fradkin and Linetsky in the context of the  $4d$  higher spin conformal theory [38, 39].<sup>2</sup> The novel feature as compared to  $4d$  FV theory is an infinite degeneracy of the spectrum of excitations: a field of each spin enters in an infinitely many copies. In this respect the spectrum of  $5d$  FV-type theory resembles that of string theory where massive excitations of a given spin appear on different mass levels growing up to infinity.

Going to higher dimensions one encounters a new phenomenon though: there are more than one spin number in  $d > 4$  so fields of mixed-symmetry type described by  $o(d - 1)$  Young diagrams appear. Mixed-symmetry  $AdS_d$  fields may interact to each other and with totally symmetric fields including gravity so it will be interesting to study their interactions. In particular, a FV-type theory for mixed-symmetry fields is still unknown.<sup>3</sup> In this paper we partially fulfill this gap and explicitly construct cubic order interacting theory in  $AdS_5$  that includes mixed-symmetry field vertices.

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<sup>1</sup>In particular, it implies that the straightforward  $\lambda \rightarrow 0$  limit is ill-defined thereby conforming the no-go theorem of [17]. However there exists a tricky limiting procedure that allows one to build some non-minimal couplings of higher spin fields with the gravity [23]. See also recent papers [24, 25] which consider some particular vertices of spin-3 massless field with the gravity. Moreover, using the analogy between massless fields in AdS spacetime and massive fields in Minkowski space these results are extended to interacting massive spin-3 fields in Minkowski spacetime [24, 25].

<sup>2</sup>This algebra was also identified as an algebra of global  $AdS_5$  HS symmetries within  $5d$  unfolded formulation proposed in [40, 41]. More general class of conformal higher spin algebras has been described in [42].

<sup>3</sup>The cubic interaction vertices of mixed-symmetry fields in Minkowski spacetime were analyzed within the light-cone formalism in [43]. Inspired by string field theory some covariant vertices for mixed-symmetry fields in Minkowski space were constructed in [44, 45].

We build  $\mathcal{N} = 2$  FV-type theory thereby extending  $\mathcal{N} = 0$  and  $\mathcal{N} = 1$  results obtained previously [36, 37]. The higher spin algebra that governs consistent interactions in our model is  $\mathcal{N} = 2$  Fradkin-Linetsky superalgebra [38, 39]. It contains  $\mathcal{N} = 2$  extended  $su(2, 2|2)$  superalgebra as a maximal finite-dimensional subalgebra so fields of the theory are organized in  $su(2, 2|2)$  supermultiplets. Obviously,  $AdS_5$  symmetry algebra  $su(2, 2)$  and  $R$ -symmetry algebra  $u(2)$  are bosonic subalgebras of  $su(2, 2|2)$  superalgebra. Contrary to spectra of  $\mathcal{N} = 0, 1$  theories the  $\mathcal{N} = 2$  supermultiplet contains not only totally symmetric fields but also the so-called "hook" fields. The "hooks" are mixed-symmetry fields with particular symmetry type differing from totally symmetric fields by additional row of a single cell in the respective Young diagram. Denoting spins of  $AdS_5$  massless gauge fields by a pair of (half-)integer numbers  $(s_1, s_2)$  we give the content of  $\mathcal{N} = 2$  spin- $s$  supermultiplet

$$\{s\} = (s, 0)_{[1]} \oplus \left(s - \frac{1}{2}\right)_{[2]} \oplus (s - 1)_{[4]} \oplus (s - 1, 1)_{[1]} \oplus \left(s - \frac{3}{2}\right)_{[2]} \oplus (s - 2)_{[1]}, \quad (1.1)$$

where  $s$  is a highest spin, while labels in square brackets denote dimensions of  $u(2)$  algebra representations. Each spin- $s$  supermultiplet possesses equal number of  $16s - 8$  bosonic and fermionic degrees of freedom.

Generally,  $AdS_5$  higher spin models based on Fradkin-Linetsky superalgebra describe an infinite collection of supermultiplets (1.1) with a highest spin  $s$  running up to infinity

$$\sum_{k=0}^L \sum_{s=2}^{\infty} \{s\}^{(k)} \quad (1.2)$$

while  $k$  parameterizes a  $k$ -th copy of a spin- $s$  supermultiplet. The models considered in this paper corresponds to  $L = \infty$  (unreduced model) or  $L = 0$  (reduced model).

According to (1.1) the spectrum of massless excitations in a full  $\mathcal{N} = 2$  supersymmetric theory includes lower spins  $0, \frac{1}{2}, 1$  contained in the spin-2 (graviton) and spin-3 (hypergraviton) supermultiplets. However, we eliminate all these lower spin fields so that the resulting theory is not supersymmetric in a strong sense, i.e. it is not globally supersymmetric. It is legitimate because in the cubic approximation one can set to zero a coupling of any three fields keeping the gauge invariance of the theory intact.<sup>4</sup> This allows one to truncate all vertices with lower spin fields which is equivalent to eliminating them from the spectrum. It greatly simplifies the whole analysis because within the FV-type theory the action functionals for lower spin and higher spin fields are formulated in different terms thereby leading to some technical complications (see, however, [21, 46, 47]).

The paper is organized as follows. In section 2 we extensively discuss the unfolded formulation of higher spin dynamics in the  $AdS_5$  background geometry in spinor language. We

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<sup>4</sup>Indeed, a spin  $s_1-s_2-s_3$  cubic coupling can be represented as  $g \Phi_{a_1 \dots a_{s_1}}^{(s_1)} J^{a_1 \dots a_{s_1}}(\Phi^{(s_2)}, \Phi^{(s_3)})$ , where  $g$  is a coupling constant,  $\Phi^{(s_i)}$  are spin- $s_i$  fields, and  $J^{a_1 \dots a_{s_1}}(\Phi^{(s_2)}, \Phi^{(s_3)})$  are higher spin currents bilinear in the fields and their derivatives. Gauge invariance of the above coupling implies that the currents are conserved  $\mathcal{D}_{a_1} J^{a_1 a_2 \dots a_{s_1}}(\Phi^{(s_2)}, \Phi^{(s_3)}) \approx 0$ , where  $\approx$  means going on-shell while in the cubic order approximation it is sufficient to use free field equations for  $\Phi^{(s_1, 2)}$ . Recall also that Jacobi identities of the gauge algebra are proportional to  $g^2$ . As a result, gauge symmetry do not mix different cubic couplings and one can consistently switch off any of them.

consider totally symmetric fields with integer and half-integer spins and mixed-symmetry fields of the "hook" symmetry type. The respective set of unfolded fields is given by physical, auxiliary and extra fields which play different dynamical roles. We build quadratic Lagrangians and introduce the set of constraints that express all auxiliary and extra fields in terms of the physical ones. These constraints play a crucial role in the analysis of the cubic higher spin interactions. In sections 2.4 and 2.5 we introduce bosonic and fermionic auxiliary variables that enable us to represent higher spin fields as expansion coefficients of polynomials in these variables. Introducing auxiliary variables is not just a technical tool that greatly simplifies the whole consideration but also brings to light such concepts like Howe duality that allows one to formulate group-theoretical properties of higher spin fields in a simple and manifest fashion. Section 2 also serves to set our notations and conventions.

In section 3 we review a construction of Fradkin-Linetsky superalgebra with any number of supersymmetries  $\mathcal{N}$  giving particular emphasis to the  $\mathcal{N} = 2$  case. In section 3.3 we describe a gauging procedure that introduces local symmetry and provides a link to unfolded gauge fields considered in section 2. In section 3.4 we explicitly describe the structure of  $\mathcal{N} = 2$  higher spin supermultiplets.

Higher spin theories of FV-type are reviewed in section 4. We formulate all necessary conditions to be satisfied by the searched-for action in the cubic approximation. In section 4.1 we formulate the final answer and list all coefficients in the action both for unreduced and reduced models. Section 5 contains explicit calculations of the coefficients in the action. Because the total expression for the gauge transformations contains over a hundred terms we split them in groups associated with different gauge supermultiplet parameters and analyze them separately. In section 5.3.1 we explicitly calculate bosonic gauge invariance for the "hook" fields and this sets a pattern for calculating the remaining invariance. In section 5.3.2 we sketch the main steps of how calculation of the remaining invariance develops and give the final result for the coefficient functions collected in section 4.1.

In conclusion 6 we shortly discuss our results and future perspectives. In appendix 7 we collect the explicit expressions for the gauge transformations omitted in the main text.

## 2 Free fields

The isometries of  $AdS_5$  spacetime form  $o(4, 2)$  algebra and the spectrum of local excitations of relativistic fields is arranged in terms of labels of irreducible representations of maximally compact subalgebra  $o(2) \oplus o(4) = o(2) \oplus o(3) \oplus o(3) \subset o(4, 2)$ . The  $o(2)$  quantum number physically means the energy  $E_0$  while  $o(4)$  quantum numbers are spins  $(s_1, s_2)$  associated with two  $o(3)$  factors of  $o(4)$  subalgebra. For massless gauge fields quantum numbers are linearly dependent so one may represent the energy via spin numbers,  $E_0 = E_0(s_1, s_2)$ , thereby expressing the fact that massless fields has less degrees of freedom than massive ones [48–50]. Let  $D(s_1, s_2)$  denote a space of states of an  $AdS_5$  massless gauge field. It is identified with some highest weight unitary irreducible (infinite-dimensional) representation of  $o(4, 2)$  algebra.

A (real) number of local degrees of freedom propagated by massless fields  $\#D(s_1, s_2)$  with  $s_1 > s_2$  has been first calculated in [51, 52] using the light-cone form of higher spin

dynamics and the answer is given by

$$\#D(s_1, s_2) = \begin{cases} 2s_1 + 1, & s_1 = n, s_2 = 0, n \in \mathbb{N}, \\ 4s_1 + 2, & s_1 = n + 1/2, s_2 = 1/2, n \in \mathbb{N}, \\ 4s_1 + 2, & s_1 = n, s_2 = k, n, k \in \mathbb{N}, \\ 4s_1 + 2, & s_1 = n + 1/2, s_2 = k + 1/2, n, k \in \mathbb{N}. \end{cases} \quad (2.1)$$

It is important for our future considerations that non-symmetric bosonic field ( $s_2 \neq 0$ ) have a number of on-shell degrees of freedom twice that of totally symmetric bosonic field ( $s_2 = 0$ ), while fermionic fields have the same degrees of freedom irrespective of a second spin value. Massless fields with equal spins  $s_1 = s_2$  are the so-called doubletons and have no local degrees of freedom [53–55].

In this paper we use the unfolded formulation of higher spin dynamics and describe massless gauge fields as differential 1-forms taking values in some irreducible  $o(4, 2)$  representations (for review see [56]). Moreover, we use the well-known isomorphism

$$o(4, 2) \sim su(2, 2)$$

and develop a spinor form of the unfolded dynamics in  $AdS_5$  spacetime [36, 40, 41, 57, 58]. Fields of the higher spin models under consideration form particular (super)multiplets of massless bosonic spin- $(s, 0)$  fields, fermionic spin- $(s, \frac{1}{2})$  fields, and massless spin- $(s, 1)$  "hook" fields. In what follows we explicitly describe quadratic Lagrangian formulation for these fields giving particular emphasis to description of "hook" fields. We start however from describing  $su(2, 2)$  spinor form of the gravity thus setting a pattern for higher spin generalizations.

## 2.1 Gauge description of $AdS_5$ spacetime

$5d$  gravity with the negative cosmological constant can be formulated in terms of 1-form connection taking values in the  $su(2, 2)$  algebra<sup>5</sup>

$$\Omega(x) = dx^\mu \Omega_\mu^\alpha{}_\beta(x) T_\alpha^\beta, \quad (2.2)$$

where  $T_\alpha^\beta$  are basis elements of  $su(2, 2)$  algebra<sup>6</sup> and the connection is traceless,  $\Omega_\mu^\alpha{}_\alpha = 0$ . As usual, the connection decomposes into the frame field and the Lorentz connection. By virtue of the compensator mechanism for gravity theories this splitting can be done in a manifestly  $su(2, 2)$  covariant fashion [59, 60]. For the case at hand we introduce the compensator as an antisymmetric bispinor field

$$V^{\alpha\beta}(x) = -V^{\beta\alpha}(x), \quad (2.3)$$

normalized so that  $V_{\alpha\gamma}V^{\beta\gamma} = \delta_\alpha^\beta$  and  $V_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\rho}V^{\gamma\rho}$ . The compensator field does not carry local degrees of freedom because it is an auxiliary field with the transformation

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<sup>5</sup>Throughout the paper we work within the mostly minus signature and use notation  $\alpha, \beta = 1, \dots, 4$  for  $su(2, 2)$  spinor indices,  $i, j = 1, \dots, \mathcal{N}$  for  $R$ -symmetry  $u(\mathcal{N})$  indices,  $\mu, \nu = 0, \dots, 4$  for world indices,  $a, b = 0, \dots, 4$  for tangent Lorentz  $o(4, 1)$  vector indices.

<sup>6</sup>An explicit realization of  $su(2, 2)$  algebra is discussed in sections 2.4 and 3.

law of Stueckelberg type (see [36] for more details). The Lorentz subalgebra in  $su(2, 2)$  is identified with stability transformations of the compensator. It follows that the frame field  $E^{\alpha\beta}$  and Lorentz spin connection  $\omega^\alpha_\beta$  are defined as [36]

$$E^{\alpha\beta} = DV^{\alpha\beta} \equiv dV^{\alpha\beta} + \Omega^\alpha_\gamma V^{\gamma\beta} + \Omega^\beta_\gamma V^{\alpha\gamma}, \quad \omega^\alpha_\beta = \Omega^\alpha_\beta + \frac{\lambda}{2} E^{\alpha\gamma} V_{\gamma\beta}, \quad (2.4)$$

where  $\lambda$  is a cosmological parameter,  $\lambda^2 > 0$ , operator  $d = dx^\mu \partial_\mu$  is the de Rham differential, and  $D$  is the  $su(2, 2)$  covariant derivative. Compensator  $V^{\alpha\beta}$  is Lorentz-invariant so it can be treated as a symplectic metric that allows one to raise and lower spinor indices in a Lorentz covariant way as

$$X^\alpha = V^{\alpha\beta} X_\beta, \quad Y_\alpha = Y^\beta V_{\beta\alpha}. \quad (2.5)$$

In particular, it follows that  $E^{\alpha\beta} V_{\alpha\beta} = 0$  and  $\omega^{\alpha\beta} V_{\alpha\beta} = 0$  which implies that the frame and Lorentz connection are irreducible Lorentz tensors.

The 2-form curvature  $R_\alpha^\beta = \frac{1}{2} R_{\mu\nu\alpha}^\beta dx^\mu \wedge dx^\nu$  associated with the connection (2.2) is given by

$$R_\alpha^\beta = d\Omega_\alpha^\beta + \Omega_\alpha^\gamma \wedge \Omega_\gamma^\beta. \quad (2.6)$$

The zero-curvature equation

$$R_\alpha^\beta(\Omega_0) = 0 \quad (2.7)$$

locally describes metric of  $AdS_5$  spacetime of radius  $\lambda^{-1}$ . Indeed, decomposing curvature  $R_\alpha^\beta$  in Lorentz-covariant components one finds the torsion tensor along with Riemann tensor extended by cosmological term proportional to  $\lambda^2$ . Setting these tensors to zero provides a link with Einstein gravity (see [56] for more details). The background gravitational fields will be denoted as  $\Omega_0^\alpha_\beta = (h^{\alpha\beta}, w^{\alpha\beta})$  while the background  $su(2, 2)$  derivative will be denoted as  $D_0$ . From (2.7) it follows that  $D_0$  is nilpotent,  $D_0^2 = 0$ .

## 2.2 Totally symmetric massless fields in $AdS_5$

The metric-like formulation of higher spin dynamics introduces spin- $s$  massless fields as totally symmetric Lorentz tensors  $\phi_{a_1 \dots a_s}(x)$  or spin-tensor  $\psi^{\hat{\alpha}}_{a_1 \dots a_{s-1/2}}(x)$ , where  $\hat{\alpha}$  is a spinor index. These (spin-)tensors are gauge fields and transform as  $\delta\phi_{a_1 \dots a_s} = \mathcal{D}_{(a_1} \xi_{a_2 \dots a_s)}$  and  $\delta\psi^{\hat{\alpha}}_{a_1 \dots a_{s-1/2}} = \mathcal{D}_{(a_1} \xi^{\hat{\alpha}}_{a_2 \dots a_{s-3/2})}$ , where  $\mathcal{D}$  is a background Lorentz derivative and  $\xi_{a_1 \dots a_{s-1}}$  and  $\xi^{\hat{\alpha}}_{a_1 \dots a_{s-5/2}}$  are gauge parameters. Both fields and gauge parameters satisfy certain algebraic conditions, like trace and gamma-transversality constraints [61, 62].

In the framework of the unfolded approach a totally symmetric field of a given spin is represented as a differential 1-form taking values in a definite  $o(4, 2)$  irreducible representation [36, 63–65].<sup>7</sup> The  $su(2, 2)$  spinor realization of the unfolded fields is the following.

- Spin- $s$  bosonic gauge fields [36, 40, 41]:

$$\Omega_\mu^{\alpha_1 \dots \alpha_{s-1}}_{\beta_1 \dots \beta_{s-1}} \quad (2.8)$$

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<sup>7</sup>Let us mention other useful approaches to higher spin dynamics of totally symmetric fields proposed in refs. [66–75].



- Spin- $s$  fermionic gauge fields [40, 41, 57]:

$$\Omega_{\mu}^{\alpha_1 \dots \alpha_{s-1/2}}_{\beta_1 \dots \beta_{s-3/2}} \oplus \Omega_{\mu}^*{}^{\alpha_1 \dots \alpha_{s-3/2}}_{\beta_1 \dots \beta_{s-1/2}} \quad (2.9)$$

Here symbol  $*$  denotes complex conjugation defined by

$$(X_{\alpha})^* = X^{\beta} C_{\beta\alpha}, \quad (Y^{\alpha})^* = C^{\alpha\beta} Y_{\beta}, \quad (2.10)$$

where  $C^{\alpha\beta} = -C^{\beta\alpha}$  and  $C_{\alpha\beta} = -C_{\beta\alpha}$  are some real matrices satisfying

$$C_{\alpha\gamma} C^{\beta\gamma} = \delta_{\alpha}^{\beta}. \quad (2.11)$$

We notice that fermionic fields are described by a pair of mutually conjugated multispinors while bosonic fields are self-conjugated. All multispinors are symmetric in upper and lower groups of indices and traceless with respect to  $su(2, 2)$  invariant tensor  $\delta_{\beta}^{\alpha}$ . The simplest fields in the list above are Maxwell field  $\Omega_{\mu}$ , Rarita-Schwinger field  $\Omega_{\mu}^{\alpha}$  and its conjugated  $\Omega_{\mu}^*{}^{\alpha}$ , and the gravity field  $\Omega_{\mu}^{\alpha\beta}$ , cf. (2.2).

Gauge symmetry for the above fields is defined by bosonic 0-from parameter  $\xi^{\alpha_1 \dots \alpha_{s-1}}_{\beta_1 \dots \beta_{s-1}}$  and mutually conjugated fermionic 0-from parameters  $\xi^{\alpha_1 \dots \alpha_{s-1/2}}_{\beta_1 \dots \beta_{s-3/2}}$  and  $\xi^*{}^{\alpha_1 \dots \alpha_{s-3/2}}_{\beta_1 \dots \beta_{s-1/2}}$ . The respective transformations of 1-from gauge fields are given by

$$\delta\Omega^{\alpha_1 \dots \alpha_{s-1}}_{\beta_1 \dots \beta_{s-1}} = D_0 \xi^{\alpha_1 \dots \alpha_{s-1}}_{\beta_1 \dots \beta_{s-1}}, \quad (2.12)$$

and

$$\delta\Omega^{\alpha_1 \dots \alpha_{s-1/2}}_{\beta_1 \dots \beta_{s-3/2}} = D_0 \xi^{\alpha_1 \dots \alpha_{s-1/2}}_{\beta_1 \dots \beta_{s-3/2}} \quad (2.13)$$

along with the complex conjugated expression.

The metric-like fields discussed in the beginning of the section are encoded into the unfolded field (2.8) and (2.9) as their particular components that can be singled out by imposing particular gauge fixing of the above symmetry. Such a mechanism is similar to that one used in the gravity theory: the frame field contains a component to be identified with the metric after gauge fixing local Lorentz symmetry.

### 2.3 “Hook” massless fields in $AdS_5$

The first non-trivial example of non-symmetric fields is given by ”hooks” which are bosonic spin- $(s, 1)$  massless fields. They can be described as tensor fields  $\phi_{a_1 \dots a_s, b_1}(x)$  with two groups of symmetrized Lorentz indices satisfying Young symmetry condition [76–79].<sup>8</sup> The gauge transformations are  $\delta\phi_{a_1 \dots a_s, b_1} = \mathcal{D}_{a_1} \xi_{a_2 \dots a_s, b_1} + \mathcal{D}_{b_1} \rho_{a_1 \dots a_s} + \dots$ , where the dots denote appropriate Young symmetrizations needed to adjust symmetry properties of both sides. Here the gauge parameters  $\xi_{a_2 \dots a_s, b_1}$  and  $\rho_{a_1 \dots a_s}$  are rank- $(s-1, 1)$  tensor and rank- $(s, 0)$  tensor, respectively. Both fields and gauge parameters satisfy certain trace conditions [77–79].

It is worth noticing that in  $5d$  Minkowski spacetime massless spin- $(s_1, 1)$  fields are dual to massless totally symmetric spin- $(s_1, 0)$  fields while those with  $s_2 > 1$  do not propagate

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<sup>8</sup>Exhaustive discussion of mixed-symmetry bosonic gauge fields both in Minkowski and AdS spacetimes can be found, e.g., in refs. [48–50, 80–94].



local degrees of freedom. This fact is in agreement with that local degrees of freedom of  $5d$  Minkowski fields are described by irreducible tensor representations of little Wigner algebra  $o(3)$ . In  $AdS_5$  spacetime local degrees of freedom of massless fields are classified according to  $o(3) \oplus o(3)$  so mixed-symmetry massless fields with  $s_2 > 1$  are not dynamically trivial.

A remarkable feature of non-symmetric fields is that they have different number of gauge symmetries on Minkowski spacetime and  $AdS_d$  spacetime [48–50]. Namely, given a mixed-symmetry massless field in Minkowski spacetime we observe that only a part of gauge symmetries can be deformed to  $AdS_d$  spacetime. In the case under consideration, the symmetry that survives in  $AdS_d$  corresponds to the gauge parameter  $\xi_{a_2\dots a_s, b_1}$ . Lacking one of gauge symmetries on  $AdS_d$  results in a mismatch between numbers of degrees of freedom propagated by  $\phi_{a_1\dots a_s, b_1}(x)$  in Minkowski and  $AdS_d$  spacetimes.

In  $AdS_5$  the spinor realization of the unfolded spin- $(s, 1)$  bosonic gauge fields is based on the following 1-forms [40, 41, 58, 65]:

$$\Omega_\mu^{\alpha_1\dots\alpha_s}_{\beta_1\dots\beta_{s-2}} \oplus \Omega_\mu^{*\alpha_1\dots\alpha_{s-2}}_{\beta_1\dots\beta_s} \tag{2.14}$$

By analogy with fermionic fields the "hooks" are complex fields described by a pair of mutually conjugated multispinors. All multispinors are symmetric in upper and lower groups of indices and traceless with respect to  $su(2, 2)$  invariant tensor  $\delta_\beta^\alpha$ . Spinor version of gauge symmetry  $\xi_{a_2\dots a_s, b_1}$  for the  $AdS_5$  "hook" fields is defined by mutually conjugated 0-form parameters  $\xi^{\alpha_1\dots\alpha_s}_{\beta_1\dots\beta_{s-2}}$  and  $\xi^{*\alpha_1\dots\alpha_{s-2}}_{\beta_1\dots\beta_s}$  as

$$\delta\Omega^{\alpha_1\dots\alpha_s}_{\beta_1\dots\beta_{s-2}} = D_0\xi^{\alpha_1\dots\alpha_s}_{\beta_1\dots\beta_{s-2}} \tag{2.15}$$

along with the complex conjugated expression.

The simplest example of a non-symmetric field, an antisymmetric tensor, is absent in (2.14). This happens because  $AdS_5$  antisymmetric gauge fields are doubletons which do not carry local degrees of freedom [53–55]. Therefore we set  $s > 1$  and the first non-trivial example is given by spin- $(2, 1)$  field. Its spinor realization is given by symmetric bispinor  $\Omega_\mu^{\alpha\beta}$  along with the complex conjugated  $\Omega_\mu^{*\alpha\beta}$ .

## 2.4 Auxiliary spinor variables

In practice it is convenient to represent higher spin fields considered in the previous sections as expansion coefficients of polynomials with respect to some set of auxiliary spinor variables. It also brings to light a rich algebraic structure known as Howe duality that allows one to control group-theoretical properties of (spin-)tensor fields in a manifest fashion.

Let us introduce two sorts of auxiliary Grassmann even variables  $a_\alpha$  and  $b^\beta$ ,  $\alpha, \beta = 1, \dots, 4$ . It is assumed that  $a_\alpha$ ,  $b^\beta$  and their derivatives  $\frac{\partial}{\partial a_\alpha}$ ,  $\frac{\partial}{\partial b^\beta}$  act in the space  $\mathcal{P}_8$  of polynomials in eight spinor variables

$$F(a, b) = \sum_{m,n=0}^{\infty} F^{\alpha_1\dots\alpha_m}_{\beta_1\dots\beta_n} a_{\alpha_1} \dots a_{\alpha_m} b^{\beta_1} \dots b^{\beta_n}, \tag{2.16}$$

where expansion coefficients are multispinors totally symmetric in the upper and lower groups of indices.

Space  $\mathcal{P}_8$  is a module of  $gl(4)$  algebra realized by the following basis elements

$$G_\alpha^\beta = \frac{1}{2} \left\{ a_\alpha, \frac{\partial}{\partial a_\beta} \right\} + \frac{1}{2} \left\{ b^\beta, \frac{\partial}{\partial b^\alpha} \right\}, \quad (2.17)$$

that produce  $gl(4)$  commutation relations via usual commutator. Algebra  $gl(4)$  acts homogeneously in  $\mathcal{P}_8$  thereby decomposing it into finite-dimensional irreducible submodules. The expansion coefficients in (2.16) are then identified with  $gl(4)$  tensors.

The condition that elements  $F(a, b) \in \mathcal{P}_8$  form an irreducible submodule under  $gl(4)$  transformations is expressed by a set of the following constraints [36],

$$N_a = a_\alpha \frac{\partial}{\partial a_\alpha} : \quad N_a F(a, b) = m F(a, b), \quad (2.18)$$

$$N_b = b^\alpha \frac{\partial}{\partial b^\alpha} : \quad N_b F(a, b) = n F(a, b), \quad (2.19)$$

where  $m$  and  $n$  are some integers, and

$$T^- = \frac{1}{4} \frac{\partial^2}{\partial a_\alpha \partial b^\alpha} : \quad T^- F(a, b) = 0. \quad (2.20)$$

Then one observes that above operators  $N_a, N_b$  and  $T^-$  supplemented by

$$T^+ = a_\alpha b^\alpha \quad (2.21)$$

form  $gl(2)$  algebra. By construction the above  $gl(4)$  and  $gl(2)$  algebras are mutually commuting. It is important that  $gl(4)$  invariant conditions (2.18)–(2.20) are the highest weight (HW) conditions of  $gl(2)$  algebra. Indeed, by an appropriate change of basis one can identify elements  $T^\pm$  with upper- and lower-triangular subalgebras of  $gl(2)$  algebra, while  $N_{a,b}$  are its Cartan elements. Algebra  $gl(2)$  can be decomposed in a standard fashion as  $gl(2) = sl(2) \oplus gl(1)$ , where the  $sl(2)$  part is given by

$$T^\pm, \quad T^0 = \frac{1}{4}(N_a + N_b + 4) \quad (2.22)$$

while the following combination

$$G^0 = N_a - N_b \quad (2.23)$$

is identified with  $gl(1)$  basis element. The commutation relations of  $sl(2)$  subalgebra are given by

$$[T^0, T^\pm] = \pm \frac{1}{2} T^\pm, \quad [T^-, T^+] = T^0. \quad (2.24)$$

By definition, element  $G^0$  is central and therefore commutes with any element of  $sl(2)$ .

The above consideration also remains valid for  $sl(4) \subset gl(4)$  subalgebra. To this end one notes that condition (2.20) still defines HW vector of  $sl(4) \subset gl(4)$  while conditions (2.18) and (2.19) fix some integer weight of  $sl(4)$  via

$$T^0 F(a, b) = \frac{1}{4}(m + n + 4) F(a, b), \quad (2.25)$$

along with the following eigenvalue of  $gl(1)$

$$G^0 F(a, b) = (m - n)F(a, b) . \quad (2.26)$$

We see that  $\mathcal{P}_8$  is in fact a bimodule over  $gl(4)$  and  $gl(2)$  algebras and its structure suggests that the above two algebras form Howe dual pair [95, 96].

In addition to commuting auxiliary variables we introduce auxiliary Grassmann odd variables  $\psi_i$  and  $\bar{\psi}^j$  with  $i, j = 1, \dots, \mathcal{N}$ . It enables us to supersymmetrize the above pure bosonic construction. To this end we introduce a superspace  $\mathcal{P}_{8|2\mathcal{N}}$  of polynomials

$$F(a, b, \psi, \bar{\psi}) = \sum_{m,n=0}^{\infty} \sum_{k,l=0}^{\mathcal{N}} F_{\beta_1 \dots \beta_m | j_1 \dots j_l}^{\alpha_1 \dots \alpha_n} a_{\alpha_1} \dots a_{\alpha_n} b^{\beta_1} \dots b^{\beta_m} \psi_{i_1} \dots \psi_{i_k} \bar{\psi}^{j_1} \dots \bar{\psi}^{j_l} , \quad (2.27)$$

where expansion coefficients are multispinors with two groups of totally symmetric indices and two groups of totally anti-symmetric indices. Superspace  $\mathcal{P}_{8|2\mathcal{N}}$  is a module of  $gl(4|\mathcal{N})$  superalgebra with the following basis elements

$$\begin{aligned} G_{\alpha}^{\beta} &= \frac{1}{2} \left\{ a_{\alpha}, \frac{\partial}{\partial a_{\beta}} \right\} + \frac{1}{2} \left\{ b^{\beta}, \frac{\partial}{\partial b^{\alpha}} \right\} , \\ Q_{\alpha}^i &= a_{\alpha} \bar{\psi}^i + \frac{\partial}{\partial b^{\alpha}} \frac{\partial}{\partial \bar{\psi}^i} , & \bar{Q}_i^{\alpha} &= b^{\alpha} \psi_i + \frac{\partial}{\partial a_{\alpha}} \frac{\partial}{\partial \psi^i} , \\ U_i^j &= \frac{1}{2} \left[ \psi_i, \frac{\partial}{\partial \psi_j} \right] + \frac{1}{2} \left[ \bar{\psi}^j, \frac{\partial}{\partial \bar{\psi}^i} \right] , \end{aligned} \quad (2.28)$$

and  $\mathcal{P}_{8|2\mathcal{N}}$  decomposes into  $gl(4|\mathcal{N})$  invariant submodules.

Introducing Grassmann odd variables enables one to extend the above bosonic realization of  $gl(2)$  algebra. The respective basis elements of  $sl(2)$  are given by [37]

$$P^+ = T^+ - \psi_i \bar{\psi}^i , \quad P^- = T^- + \frac{1}{4} \frac{\partial^2}{\partial \bar{\psi}^i \partial \psi_i} , \quad P^0 = T^0 + \frac{1}{4} (N_{\psi} + N_{\bar{\psi}} - \mathcal{N}) , \quad (2.29)$$

and  $gl(1)$  basis element is

$$Z^0 = G^0 + N_{\psi} - N_{\bar{\psi}} , \quad (2.30)$$

where

$$N_{\psi} = \psi_i \frac{\partial}{\partial \psi_i} , \quad N_{\bar{\psi}} = \bar{\psi}^i \frac{\partial}{\partial \bar{\psi}^i} . \quad (2.31)$$

The respective  $sl(2)$  commutation relations are

$$[P^0, P^{\pm}] = \pm \frac{1}{2} P^{\pm} , \quad [P^-, P^+] = P^0 . \quad (2.32)$$

By construction, the above  $gl(2)$  algebra and  $gl(4|\mathcal{N})$  superalgebra are mutually commuting and form Howe dual pair. It makes possible to study  $gl(4|\mathcal{N})$  irreducible submodules in  $\mathcal{P}_{8|2\mathcal{N}}$  via imposing the following  $sl(2)$  HW condition

$$P^- F(a, b, \psi, \bar{\psi}) = 0 , \quad (2.33)$$

along with some fixed eigenvalues of  $sl(2)$  Cartan element  $P^0$  and  $gl(1)$  element  $Z^0$ . It is worth noting that the present construction describes only particular class of  $gl(4|\mathcal{N})$  irreducible representations.

Up to now we considered complex  $gl(4|\mathcal{N})$  superalgebra. However, we are interested in  $su(2, 2|\mathcal{N})$  superalgebra that is defined as an appropriate real form of  $sl(4|\mathcal{N}) \subset gl(4|\mathcal{N})$ . The respective reality condition are given by

$$(a_\alpha)^* = b^\beta C_{\beta\alpha}, \quad (b^\alpha)^* = C^{\alpha\beta} a_\beta, \quad (\psi_i)^* = \bar{\psi}^i, \quad (\bar{\psi}^i)^* = \psi_i, \quad (2.34)$$

where conjugation matrices are defined by (2.11). Then it follows that  $a_\alpha$  and  $b^\beta$  are in the fundamental and the conjugated fundamental representations of  $su(2, 2)$  while  $\psi_i$  and  $\bar{\psi}^i$  are in the fundamental and the conjugated fundamental representations of  $u(\mathcal{N})$ .

In section 3.1 we discuss a star-product realization of the above construction. Finally, we note that the Howe dual pair  $gl(4|\mathcal{N})$  and  $gl(2)$  coincides with that one discussed in [87] within the BRST framework.

## 2.5 Gauge fields as polynomials in auxiliary variables

The unfolded gauge fields discussed in sections 2.2 and 2.3 can be collectively represented as a pair of mutually conjugated multispinors

$$\Omega_\mu^{\alpha_1 \dots \alpha_{s_1+s_2-1}} \oplus \Omega_\mu^*{}_{\beta_1 \dots \beta_{s_1-s_2-1}}^{\beta_1 \dots \beta_{s_1-s_2-1}}, \quad (2.35)$$

provided that  $s_1 = s$  and  $s_2 = 0, \frac{1}{2}, 1$ . Using spinor auxiliary variables introduced in the previous section we define the above massless gauge fields as follows

$$\Omega(a, b|x) = dx^\mu \Omega_{\mu\beta_1 \dots \beta_{s_1-s_2-1}}^{\alpha_1 \dots \alpha_{s_1+s_2-1}}(x) a_{\alpha_1} \dots a_{\alpha_{s_1+s_2-1}} b^{\beta_1} \dots b^{\beta_{s_1-s_2-1}} \quad (2.36)$$

along with the complex conjugated  $\Omega^*(a, b|x)$ . The associated linearized higher spin curvature is a 2-form  $R_1 = \frac{1}{2} dx^\mu \wedge dx^\nu R_{1\mu\nu}(a, b|x)$  given by

$$R_1 = D_0 \Omega \equiv d \Omega + \Omega_0^\alpha{}_\beta \left( b^\beta \frac{\partial}{\partial b^\alpha} - a_\alpha \frac{\partial}{\partial a_\beta} \right) \wedge \Omega, \quad (2.37)$$

where  $\Omega_0^\alpha{}_\beta$  is the background 1-form connection satisfying the zero-curvature condition (2.7), and the background covariant derivative is given by

$$D_0 = d + \Omega_0^\alpha{}_\beta \left( b^\beta \frac{\partial}{\partial b^\alpha} - a_\alpha \frac{\partial}{\partial a_\beta} \right). \quad (2.38)$$

Subscript 1 indicates that curvature (2.37) is a linearized part of some full non-Abelian curvature introduced in section 3.3. The gauge transformations are

$$\delta \Omega = D_0 \xi, \quad (2.39)$$

where a gauge parameter is a 0-form  $\xi = \xi(a, b|x)$ . As a corollary of  $D_0^2 = 0$  it follows that

$$\delta R_1(a, b|x) = 0, \quad (2.40)$$

while the respective Bianchi identities read as

$$D_0 R_1(a, b|x) = 0 . \tag{2.41}$$

Using  $gl(2)$  basis elements (2.22) one easily formulates algebraic conditions on  $\Omega(a, b|x)$  that single out an irreducible field of a given spin as the respective  $gl(2)$  HW condition

$$T^- \Omega(a, b|x) = 0 , \tag{2.42}$$

along with particular eigenvalues of Cartan elements

$$\begin{aligned} N_a \Omega(a, b|x) &= (s_1 + s_2 - 1) \Omega(a, b|x) , \\ N_b \Omega(a, b|x) &= (s_1 - s_2 - 1) \Omega(a, b|x) . \end{aligned} \tag{2.43}$$

The last two conditions can be equivalently rewritten as

$$\begin{aligned} T^0 \Omega(a, b|x) &= \frac{1}{2} (s_1 + 1) \Omega(a, b|x) , \\ G^0 \Omega(a, b|x) &= 2s_2 \Omega(a, b|x) . \end{aligned} \tag{2.44}$$

It is obvious that the associated curvatures satisfy the same algebraic constraints.

In the subsequent analysis we use the following set of differential operators in auxiliary spinor variables [36]

$$S^- = a_\alpha \frac{\partial}{\partial b^\beta} V^{\alpha\beta} , \quad S^+ = b^\alpha \frac{\partial}{\partial a_\beta} V_{\alpha\beta} , \quad S^0 = N_b - N_a . \tag{2.45}$$

They explicitly involve the compensator field and form  $sl(2)$  algebra

$$[S^0, S^\pm] = \pm \frac{1}{2} S^\pm , \quad [S^-, S^+] = S^0 . \tag{2.46}$$

Note that the above set of  $sl(2)$  elements commute with other  $sl(2)$  elements introduced earlier in section 2.4. It is worth noting that  $sl(2)$  algebra (2.46) can be interpreted as Howe dual algebra for the Lorentz subalgebra of  $su(2, 2)$ . We hope to consider this issue in a more detail elsewhere.

Irreducible  $su(2, 2)$  gauge fields can be further decomposed with respect to Lorentz subalgebra. The resulting Lorentz fields are given by the following collection of differential 1-forms

$$\omega_\mu^t \alpha_1 \dots \alpha_{s_1+s_2+t-1} \beta_1 \dots \beta_{s_1-s_2-t-1} (x) , \quad 0 \leq t \leq s_1 - s_2 - 1 , \tag{2.47}$$

that satisfy the Young symmetry condition and the  $V_{\alpha\beta}$ -transversality condition. Recall that compensator  $V^{\alpha\beta}$  can be used to raise and lower indices in the Lorentz-invariant manner, see (2.5). Fields (2.47) can be described as expansion coefficients of

$$\omega^t(a, b|x) = dx^\mu \omega_\mu^t(a, b|x) . \tag{2.48}$$

Irreducibility conditions imposed on Lorentz-covariant tensors have the form of two  $gl(2)$  HW conditions

$$S^- \omega^t = 0 , \quad T^- \omega^t = 0 . \tag{2.49}$$

The first condition is in fact the Young symmetry condition while the second one tells us that Lorentz tensors are transversal to compensator  $V^{\alpha\beta}$ . The last condition expresses the fact that we describe Lorentz irreps in a manifestly  $su(2, 2)$  covariant manner. Indeed, operators (2.45) enables one to write down a decomposition of an irreducible  $su(2, 2)$  gauge field as

$$\Omega(a, b|x) = \sum_{t=0}^{s_1-s_2-1} (S^+)^t \omega^t(a, b|x). \quad (2.50)$$

Since  $sl(2)$  algebras (2.24) and (2.46) mutually commute one concludes that the second HW condition in (2.49) on  $\omega^t(a, b|x)$  follows from HW condition (2.42) on  $\Omega(a, b|x)$ .

The background covariant derivative can be cast into explicit Lorentz-covariant form as  $D_0 = \mathcal{D}_0 + \sigma_- + \lambda\sigma_0 + \lambda^2\sigma_+$ , where  $\mathcal{D}_0$  stands for Lorentz derivative constructed with respect to background Lorentz connection  $w^{\alpha\beta}$ , while  $\sigma$ -operators satisfy the relations

$$(\sigma_{\pm})^2 = 0, \quad \{\sigma_0, \sigma_{\pm}\} = 0, \quad \mathcal{D}^2 + \lambda^2 \{\sigma_-, \sigma_+\} + \lambda^2 (\sigma_0)^2 = 0, \quad (2.51)$$

that follow from  $D_0^2 = 0$ . The explicit expressions for  $\sigma$ -operators are given in [58]. It is worth noting that non-trivial  $\sigma_0$  appears not only for fermionic totally symmetric fields but also for bosonic and fermionic mixed-symmetry fields.

Lorentz-covariant fields  $\omega^t$  at different values of parameter  $t$  play different dynamical roles. One distinguishes between physical, auxiliary, and extra fields.

- For integer spin- $(s, 0)$  system: fields with  $t = 0$  are called physical, fields with  $t = 1$  are auxiliary ones, fields with  $t > 1$  are called extra fields.
- For half-integer spin- $(s, \frac{1}{2})$  system: fields with  $t = 0$  are physical ones, fields with  $t > 0$  are extra fields. The absence of auxiliary fields is a manifestation of the first-order form of the fermionic field equations.
- For integer spin- $(s, 1)$  system: fields with  $t = 0$  are physical and auxiliary ones, fields with  $t > 0$  are extra fields. Physical field is identified with  $\text{Re} \omega^t$ , an auxiliary field is identified with  $\text{Im} \omega^t$ . In particular, it allows one to cast the dynamical equations of non-symmetric fields into the first-order form. The analogous decomposition into pure real and imaginary parts duplicates the number of (real) extra fields. For more details see [58].

The unfolded dynamical higher spin equations of motion can be represented as a system of variational equations and certain constraints. Variational equations involve just physical and auxiliary fields, and auxiliary field is expressed via first derivative of the physical field, while the constraints express all extra fields via derivatives of the physical field. The next two sections discuss the action functional and the appropriate constraints.

## 2.6 Higher spin action functionals

One of basic advantages of using the unfolded formulation is that quadratic action functionals for higher spin fields can be represented in a manifestly gauge-invariant fashion.

The actions have the form of a bilinear combination of linearized curvatures so the gauge invariance of the action is a direct consequence of (2.40).

The  $AdS_5$  action functional involves HS fields described as polynomials in two sets of auxiliary variables  $X_1 = (a_{1\alpha}, \beta_1^\beta)$  and  $X_2 = (a_{2\alpha}, \beta_2^\beta)$ . The action functional is built then in the following schematic form

$$S = \int_{\mathcal{M}^5} \widehat{H} \left( E, V, \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2} \right) \wedge R(X_1) \wedge R(X_2) \Big|_{X_1=X_2=0}, \quad (2.52)$$

where  $\widehat{H}$  is a polynomial in the compensator and auxiliary variable derivatives acting on a tensor product of two field strengths  $R(X)$ . Also, since the integrand is 5-form it follows that  $\widehat{H}$  is a 1-form proportional to the frame field  $E_\mu^{\alpha\beta}$ . Expansion coefficients of  $\widehat{H}$  with respect to derivatives in auxiliary variables are some  $su(2, 2)$  covariant tensors built of  $V^{\alpha\beta}$  and  $\delta_\beta^\alpha$  and their combinations parameterize various types of index contractions between curvatures. Any such action is manifestly  $su(2, 2)$  invariant and automatically gauge-invariant with respect to the gauge transformations (2.39).

Generally, actions of the type (2.52) do not describe propagation of a correct number of on-shell degrees of freedom because of redundant dynamical modes associated with the extra fields. In order to eliminate their contribution one should fix the operator  $\widehat{H}$  in an appropriate form by virtue of the extra field decoupling condition. It requires that the variation of the quadratic action with respect to extra fields is identically zero,

$$\frac{\delta S_2}{\delta \omega^{\text{ex}}} \equiv 0. \quad (2.53)$$

Extra fields maintain an explicit gauge invariance of the action functional but the above condition constrains them to fall out of the quadratic action. Having decoupled extra fields, the action can be cast into a minimal form with just two fields, physical and auxiliary ones, but then the residual gauge invariance is implicit. Nonetheless, for both versions of the action, minimal form with two fields and non-minimal with added extra fields, the respective free field equations of motion always have manifestly gauge-invariant form, i.e., they are represented as linear combinations of linearized higher spin curvatures.

The action for spin- $(s_1, s_2)$  massless gauge field is searched in the following form [36, 57, 58]

$$S_2^{(s_1, s_2)} = \int_{\mathcal{M}^5} \widehat{H} \wedge R(a_1, b_1) \wedge R^*(a_2, b_2) \Big|_{a_i=b_i=0}, \quad s_2 = 0, \frac{1}{2}, 1, \quad (2.54)$$

where  $R$  and  $R^*$  are mutually conjugated linearized spin- $(s_1, s_2)$  curvatures (2.37) and  $\widehat{H}$  is the following 1-form differential operator

$$\begin{aligned} \widehat{H} = & \left( \alpha(p, q) E_{\alpha\beta} \frac{\partial^2}{\partial a_{1\alpha} \partial a_{2\beta}} b_{12} + \beta(p, q) E^{\alpha\beta} \frac{\partial^2}{\partial b_1^\alpha \partial b_2^\beta} a_{12} \right. \\ & \left. + \gamma(p, q) E_\alpha^\beta \frac{\partial^2}{\partial a_{2\alpha} \partial b_1^\beta} c_{12} + \zeta(p, q) E_\alpha^\beta \frac{\partial^2}{\partial a_{1\alpha} \partial b_2^\beta} c_{21} \right) (c_{12})^{2s_2}. \end{aligned} \quad (2.55)$$

Here  $E^{\alpha\beta}$  is the frame field (2.4). For quadratic action under consideration the frame field is taken to be background

$$E^{\alpha\beta} = h^{\alpha\beta}, \quad (2.56)$$



so dynamical fields are contained in the linearized curvatures only. The coefficients  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\zeta$  are functions of operators

$$p = a_{12}b_{12}, \quad q = c_{12}c_{21}, \quad (2.57)$$

where

$$\begin{aligned} a_{12} &= V_{\alpha\beta} \frac{\partial^2}{\partial a_{1\alpha} \partial a_{2\beta}}, & b_{12} &= V^{\alpha\beta} \frac{\partial^2}{\partial b_1^\alpha \partial b_2^\beta}, \\ c_{12} &= \frac{\partial^2}{\partial a_{1\alpha} \partial b_2^\alpha}, & c_{21} &= \frac{\partial^2}{\partial a_{2\alpha} \partial b_1^\alpha}. \end{aligned} \quad (2.58)$$

These functions are responsible for various types of index contractions between the background frame field, compensator and curvatures.

Below we list solutions of the extra field decoupling condition for totally symmetric bosonic and fermionic fields, and for bosonic "hook" fields. Note that quadratic actions are defined modulo total derivative contributions.

- Spin- $(s, 0)$  bosons:

$$\begin{aligned} \alpha(p, q) &= 2 \int_0^1 d\tau \left( 1 + q \frac{\partial}{\partial q} \right) \rho(\tau p + q), \\ \zeta(p, q) + \gamma(p, q) &= 0, \quad \beta(p, q) = 0, \quad \gamma(p, q) = \rho(p + q). \end{aligned} \quad (2.59)$$

- Spin- $(s, \frac{1}{2})$  fermions:

$$\begin{aligned} \alpha(p, q) &= - \int_0^1 d\tau \frac{\partial}{\partial p} \rho(p\tau + q), \\ \gamma(p, q) &= 0, \quad \beta(p, q) = 0, \quad q\zeta(p, q) = \rho(p + q). \end{aligned} \quad (2.60)$$

- Spin- $(s, 1)$  bosons:

$$\begin{aligned} \alpha(p, q) &= - \int_0^1 d\tau \frac{\partial}{\partial p} \rho(p\tau + q), \\ \gamma(p, q) &= 0, \quad \beta(p, q) = 0, \quad q\zeta(p, q) = \rho(p + q). \end{aligned} \quad (2.61)$$

We see that the quadratic action for a given spin is fixed unambiguously up to overall factors parameterized by polynomials  $\rho(p + q)$  of fixed order,  $\rho(p + q) = \rho_0(p + q)^{s_1 - 2}$  for  $s_2 = 0$  fields and  $\rho(p + q) = \rho_0(p + q)^{s_1 - s_2 - 1}$  for  $s_2 \neq 0$  fields,  $\rho_0$  are arbitrary constants. Constants  $\rho_0$  cannot be fixed from the free field analysis and represent the leftover ambiguity in the coefficients. On the other hand, requiring gauge invariance in the cubic theory fixes  $\rho(p + q)$  unambiguously, see section 4.

## 2.7 Generalized Weyl tensors and constraints

As discussed in the previous section in order to have a manifest higher spin gauge invariance, the quadratic action is always written down with the extra fields, at least formally. It turns out that on the interaction level variation of the action with respect to extra fields

cannot be consistently required to vanish identically. It follows that proper constraints should be imposed expressing extra fields in terms of physical fields thereby preserving a correct number of gauge symmetries and physical degrees of freedom.

We assume that constraints for extra fields should have the following form [36, 37, 58]

$$\widehat{\Upsilon}_2^+ \wedge r^t = 0, \quad 0 \leq t < s_1 - s_2 - 1, \quad s_2 = 0, \frac{1}{2}, 1, \quad (2.62)$$

where  $\widehat{\Upsilon}_2^+$  is some 2-form operator increasing grading  $t$  by one. It satisfies the condition

$$\sigma_+ \wedge \widehat{\Upsilon}_2^+ = 0 \quad (2.63)$$

that guarantees that the number of independent algebraic relations imposed on the curvature  $r^t$  coincides with the number of components of extra fields  $\omega^{t>0}$  modulo pure gauge components of the form  $\delta\omega^{t+1} = \sigma_- \xi^{t+2}$ . One can show that the operator  $\widehat{\Upsilon}_2^+$  is uniquely fixed in the form

$$\widehat{\Upsilon}_2^+ = \sigma_0 \wedge \sigma_+. \quad (2.64)$$

Constraints (2.62) are described by 4-form which in  $d = 5$  dimensions is dual to 1-form so it follows that the number of equations in (2.62) coincides with the number of components of  $\omega^{t+1}$ . Therefore, field  $\omega^{t+1}$  can be expressed via derivatives of the field  $\omega^t$  for any  $t > 0$ . Finally, one can obtain fields  $\omega^t$  expressed in terms of derivatives of the field  $\omega^0$  with an order of highest derivatives equal to  $t$ . The schematic form of the corresponding expressions is

$$\omega^t \sim \left( \frac{\partial}{\lambda \partial x} \right)^t \omega^0. \quad (2.65)$$

On the non-linear level such expression for extra fields provide a useful parameterization of higher derivatives in the higher spin interaction terms.

Next we cite the proposition known in the literature as the first on-mass-shell theorem, see, e.g., [40, 41, 63, 64, 97].

**Proposition 1.** *Variational equations of motion for spin- $(s_1, 0)$  and spin- $(s_1, 1/2)$  fields supplemented with the constraints for extra fields can be equivalently rewritten as*

$$R_{\beta_1 \dots \beta_{s_1-1}}^{\alpha_1 \dots \alpha_{s_1-1}} = H_2 \delta_\rho C_0^{\alpha_1 \dots \alpha_{s_1-1} \gamma_1 \dots \gamma_{s_1-1} \delta \rho} V_{\gamma_1 \beta_1} \dots V_{\gamma_{s_1-1} \beta_{s_1-1}}, \quad (2.66)$$

and

$$R_{\beta_1 \dots \beta_{s_1-3/2}}^{\alpha_1 \dots \alpha_{s_1-1/2}} = H_2 \delta_\rho C_{1/2}^{\alpha_1 \dots \alpha_{s_1-1/2} \gamma_1 \dots \gamma_{s_1-3/2} \delta \rho} V_{\gamma_1 \beta_1} \dots V_{\gamma_{s_1-3/2} \beta_{s_1-3/2}}, \quad (2.67)$$

plus analogous expression for complex conjugated curvatures. Here  $H_2 \delta_\rho = h_\delta^\gamma \wedge h_{\gamma\rho}$ . Totally symmetric multispinor  $C_0^{\alpha_1 \dots \alpha_{2s_1}}$  is a generalized bosonic Weyl tensor, and totally symmetric multispinors  $C_{1/2}^{\alpha_1 \dots \alpha_{2s_1}}$  and its complex conjugated constitute generalized fermionic Weyl tensor.

In particular, the above proposition tells us that all Lorentz-covariant curvatures except for that with  $t = s_1 - 1$  for bosons and  $t = s_1 - 3/2$  for fermions can be set to zero on-shell provided appropriated constraints are imposed. The proposition generalizes the well-known construction of Weyl tensor in gravity.

Now we formulate and prove the analogous proposition for  $AdS_5$  mixed-symmetry fields of particular integer spin  $(s_1, 1)$ . Actually, higher spin field equations of the form  $R = H_2 C$  are known to describe  $AdS_5$  "hook" field dynamics [40, 41]. Here we prove that these equations do arise as variational equations supplemented by some constraints thus guaranteeing the proposed action functional for "hook" fields (2.54) correctly describes physical degrees of freedom.

**Proposition 2.** *Variational equations of motion for spin- $(s_1, 1)$  fields supplemented with the constraints can be equivalently rewritten as*

$$\begin{aligned} R_{\beta_1 \dots \beta_{s_1-2}}^{\alpha_1 \dots \alpha_{s_1}} &= H_2 \delta_\rho C_1^{\alpha_1 \dots \alpha_{s_1} \gamma_1 \dots \gamma_{s_1-2} \delta \rho} V_{\gamma_1 \beta_1} \dots V_{\gamma_{s_1-2} \beta_{s_1-2}} \\ R_{\beta_1 \dots \beta_{s_1}}^*{}^{\alpha_1 \dots \alpha_{s_1-2}} &= H_2 \delta^\rho C_{1 \beta_1 \dots \beta_{s_1} \gamma_1 \dots \gamma_{s_1-2} \delta \rho} V^{\gamma_1 \alpha_1} \dots V^{\gamma_{s_1-2} \alpha_{s_1-2}} . \end{aligned} \quad (2.68)$$

Here  $H_2 \delta_\rho = h_\delta^\gamma \wedge h_{\gamma\rho}$ . Totally symmetric multispinors  $C_1^{\alpha_1 \dots \alpha_{2s}}$  and  $C_{1 \beta_1 \dots \beta_{2s}}^*$  are mutually conjugated and constitute generalized Weyl tensor for "hook" fields.

*Proof.* The main idea behind the proof is to observe that both the variational equations of motion for the physical and auxiliary fields and constraints for extra fields can be visualized as a system of linear equations imposed on curvature components. The kernel of the linear system should be identified with generalized Weyl tensors so finding it is in fact the content of the above proposition.

More precisely, Lorentz-covariant curvatures can be cast into the following form

$$r_{\mu\nu}^{\alpha_1 \dots \alpha_{s+t}, \beta_1 \dots \beta_{s-t-2}} \Rightarrow r^{(\delta\rho) | \alpha_1 \dots \alpha_{s+t}, \beta_1 \dots \beta_{s-t-2}} , \quad (2.69)$$

where antisymmetric 2-form indices were converted to symmetric spinor indices by virtue of a 2-form composed of the background frame field,  $H_2(\delta\rho) = h_\delta^\gamma \wedge h_{\gamma\rho}$ . The tensor product  $(\delta\rho) \otimes (\alpha_1 \dots \alpha_{s+t}, \beta_1 \dots \beta_{s-t-2})$  contains a set of irreducible Lorentz-covariant multispinor components  $r^{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_l}$  for some definite integers  $k, l$ . Field equations and constraints impose various linear relations on these components.

As a first step we consider the curvature  $t = 0$  and analyze which of its components do not vanish on the equations of motions. The equations of motion have the form  $\widehat{\mathcal{E}} \wedge r^{t=0} = 0$ , where  $\widehat{\mathcal{E}}$  is a 2-form operator satisfying the conditions  $[S^-, \widehat{\mathcal{E}}] = 0$  and  $[T^-, \widehat{\mathcal{E}}] = 0$ , i.e. when acting on the Lorentz-covariant curvature it preserves its Young symmetry and  $V^{\alpha\beta}$ -transversality properties. Operator  $\widehat{\mathcal{E}}$  is proportional to background frame 2-form  $H^{\alpha\beta}$  and is a differential operator in auxiliary spinor variables. The exact expression for  $\widehat{\mathcal{E}}$  can be found in [58].

The explicit analysis of the component form of equations of motion is straightforward but technically involved to be given here in all detail. However, since we work with linear equations it is possible to estimate the lower bound of the respective kernel dimension just by comparing the numbers of variables and equations by  $\#(\text{kernel}) = \#(\text{variables}) - \#(\text{equations})$ . The explicit analysis confirms that a rank of the linear system is maximal and the above formula is exact.

Denoting  $t = 0$  curvature component  $r^{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_l}$  satisfying the Young symmetry and  $V^{\alpha\beta}$ -transversality conditions as a pair  $(k, l)$  we find that the following multispinor components of  $t = 0$  curvature remain non-zero on-shell:  $(s+2, s-2)$ ,  $(s, s-4)$ ,  $(s, s)$ ,  $(s-2, s-2)$ .

Consider the Bianchi identities for  $t = 0$  curvature,  $\mathcal{D}r^{t=0} + \lambda\sigma_0 r^{t=0} + \sigma_- r^{t=1} = 0$ , where  $\sigma$ -operators satisfy (2.51). Projecting these Bianchi identities on components  $(s, s)$  and  $(s - 2, s - 2)$  gives rise to conditions  $\lambda r^{\alpha_1 \dots \alpha_s, \beta_1 \dots \beta_s} = 0$  and  $\lambda r^{\alpha_1 \dots \alpha_{s-2}, \beta_1 \dots \beta_{s-2}} = 0$ . Note that these components originate from the term with  $\sigma_0$  operator.

Then we consider constraints (2.62) at  $t = 0$ . They can be equivalently represented as 1-form taking values in  $(s - 1, s - 3)$  multispinor corresponding to extra field with  $t = 1$ . It implies that by virtue of this constraint the  $t = 1$  extra field can be completely expressed as the first derivative of  $t = 0$  field. Again, considering the constraint as a system of linear equations on  $t = 0$  curvature one finds that its  $(s + 2, s - 2)$ ,  $(s, s - 4)$  components vanish.

To summarize, we proved that equations of motion along with the first of constraints can be equivalently rewritten as  $r^{t=0} = 0$ . The rest of the proof is straightforward and reduces to the observation that curvatures  $r^{t>0}$  satisfy the cohomological equation  $\sigma_- r^{t>0} = 0$  as it follows from the respective Bianchi identities. Modulo exact contributions the general solution is

$$\begin{aligned} r^{\alpha_1 \dots \alpha_{s+t}, \beta_1 \dots \beta_{s-t-2}} &= 0, & 0 < t < s - 2, \\ r^{\alpha_1 \dots \alpha_{2s-2}} &= H_{2\delta\rho} C_1^{\alpha_1 \dots \alpha_{2s-2}\delta\rho}, & t = s - 2, \end{aligned} \tag{2.70}$$

where  $H_{2\delta\rho} = h_\delta^\gamma \wedge h_{\gamma\rho}$  and totally symmetric multispinor  $C_1^{\alpha_1 \dots \alpha_{2s}}$  should be identified with the generalized Weyl tensor. The analogous expression is valid for complex conjugated curvatures. We see that the above expressions can be equivalently cast into the form of the proposition 2. □

Using auxiliary variables expression (2.66), (2.67), (2) can be uniformly cast into the following form

$$R^{s_1, s_2}(a, b|x) = H_{2\alpha}^\beta \frac{\partial^2}{\partial a_\alpha \partial b^\beta} \text{Res}_\mu(\mu^{2s_2} C^{s_1, s_2}(\mu a + \mu^{-1} b|x)), \tag{2.71}$$

$$R^{*s_1, s_2}(a, b|x) = H_{2\alpha}^\beta \frac{\partial^2}{\partial a_\alpha \partial b^\beta} \text{Res}_\mu(\mu^{-2s_2} C^{*s_1, s_2}(\mu a + \mu^{-1} b|x)), \tag{2.72}$$

where  $s_2 = 0, \frac{1}{2}, 1$ , and  $H_{2\alpha\beta} = h_{\alpha\gamma} \wedge h^\gamma_\beta$  and  $\text{Res}_\mu$  singles out the  $\mu$ -independent part of Laurent series in  $\mu$ . A function of one spinor variable

$$C(\mu a + \mu^{-1} b) = \sum_{k, l} \frac{\mu^{k-l}}{k! l!} C^{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_l} a_{\alpha_1} \dots a_{\alpha_k} b_{\beta_1} \dots b_{\beta_l} \tag{2.73}$$

has totally symmetric coefficients  $C^{\alpha_1 \dots \alpha_k \beta_1 \dots \beta_l}$  and  $b_\beta = b^\gamma V_{\gamma\beta}$ .

We observe that generalized Weyl tensor for bosonic non-symmetric spin- $(s_1, s_2)$  field is given by a pair of mutually conjugated generalized Weyl tensors for totally symmetric spin- $s_1$  field. In particular, it implies that the number of physical degrees of freedom is twice that of symmetric spin- $s_1$  field, cf. (2.1). In the flat limit  $\lambda = 0$  the above mixed-symmetry field decomposes into two independent totally symmetric spin- $s_1$  fields. Indeed, there are no mixed-symmetry fields on Minkowski spacetime since the respective little Wigner

algebra  $o(3)$  has just totally symmetric representations. It conforms the Brink-Metsaev-Vasiliev conjecture [50] asserting that an irreducible massless field in  $AdS_d$  decomposes in the flat limit into a collection of irreducible massless fields in Minkowski spacetime.<sup>9</sup> In the case of  $AdS_5$  spacetime a set of Minkowski fields drastically reduces so that a non-symmetric bosonic field decomposes into a pair of equal spin totally symmetric Minkowski fields [51, 52].

To conclude this section one should note that the naive flat limit  $\lambda = 0$  of the unfolded quadratic action (2.54) for  $AdS_5$  massless spin- $(s, 1)$  fields is inconsistent in the sense a half of PDoF is lost [50]. However, such a type of inconsistency may be ignored on the non-linear level since the higher spin interaction terms contain a factor of  $\lambda^{-1}$  so the naive flat limit in the  $AdS_5$  interacting theory is singular. This drawback could be cured within the Stueckelberg-like approach developed for mixed-symmetry fields in [50, 84, 98] thus allowing one to study consistent passages of interacting theory from  $AdS_5$  to Minkowski spacetime.

### 3 Fradkin-Linetsky superalgebra

#### 3.1 Higher spin superalgebra $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$

Let Grassmann even variables  $a_\alpha, b^\beta$  with  $\alpha, \beta = 1, \dots, 4$  and Grassmann odd variables  $\psi_i$  and  $\bar{\psi}^j$  with  $i, j = 1, \dots, \mathcal{N}$  satisfy the following non-vanishing (anti-)commutation relations

$$[a_\alpha, b^\beta]_\star = \delta_\alpha^\beta, \quad \{\psi_i, \bar{\psi}^j\}_\star = \delta_i^j, \quad (3.1)$$

with respect to Weyl star-product

$$(F \star G)(a, b, \psi, \bar{\psi}) = F(a, b, \psi, \bar{\psi}) (\exp \Delta) G(a, b, \psi, \bar{\psi}), \quad (3.2)$$

where

$$\Delta = \frac{1}{2} \left( \overleftarrow{\partial} \frac{\overrightarrow{\partial}}{\partial a_\alpha} - \overleftarrow{\partial} \frac{\overrightarrow{\partial}}{\partial b^\alpha} + \overleftarrow{\partial} \frac{\overrightarrow{\partial}}{\partial \psi_i} + \overleftarrow{\partial} \frac{\overrightarrow{\partial}}{\partial \bar{\psi}^i} \right).$$

Thus we get particular Weyl-Clifford star-product algebra with elements  $F = F(a, b, \psi, \bar{\psi})$  (2.27). The above variables are sufficient to build basis elements of  $\mathcal{N}$ -extended  $gl(4|\mathcal{N})$  superalgebra,

$$T_\alpha^\beta = \frac{1}{2} \{a_\alpha, b^\beta\}_\star, \quad Q_\alpha^i = a_\alpha \bar{\psi}^i, \quad \bar{Q}_i^\beta = b^\beta \psi_i, \quad U_i^j = \frac{1}{2} \{\psi_i, \bar{\psi}^j\}_\star. \quad (3.3)$$

Basis elements  $U_i^j$  form  $R$ -symmetry algebra  $U(\mathcal{N}) \subset gl(4|\mathcal{N})$ . The graded supercommutator has the standard form  $[F, G]_\star = F \star G - (-1)^{\pi(F)\pi(G)} G \star F$ , where the  $Z_2$  grading  $\pi$  is defined by

$$F(-a, -b, \psi, \bar{\psi}) = (-1)^{\pi(F)} F(a, b, \psi, \bar{\psi}), \quad \pi(F) = 0 \text{ or } 1. \quad (3.4)$$

---

<sup>9</sup>The conjecture was originally put forward in the group-theoretical terms while its field-theoretical justification based on unfolded formalism has been proposed in [89, 90] for  $AdS_d$  mixed-symmetry fields of general shape. There, however, the proof could only be provided in full rigor for fields up to four rows, due to technicalities in the manipulation of so-called cell-operators. The proof of the conjecture in the general case was given in [92] where BRST extension of the unfolding formalism was used, that dispensed the authors of [92] with an explicit manipulation of cell-operators.

Factoring out an ideal of  $gl(4|\mathcal{N})$  generated by the central element

$$N = a_\alpha b^\alpha - \psi_i \bar{\psi}^i \tag{3.5}$$

yields subalgebra  $sl(4|\mathcal{N}) \subset gl(4|\mathcal{N})$  and the  $AdS_5$  superalgebra  $su(2, 2|\mathcal{N})$  is defined as a real form of  $sl(4|\mathcal{N})$  singled out by the reality conditions defined below.

Higher spin extension of  $su(2, 2|\mathcal{N})$  introduced in [38] under the name  $shsc^\infty(4|\mathcal{N})$  and called  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  in [42] is associated with the star product algebra of all polynomials  $F(a, b, \psi, \bar{\psi})$  satisfying the condition [39–42]

$$[N, F]_\star = 0 . \tag{3.6}$$

Thus, Fradkin-Linetsky superalgebra is spanned by star-(anti)commutators of the elements of the centralizer of  $N$  in the Weyl-Clifford star product algebra. The above commutator can be equivalently cast into the form

$$[N, F]_\star = (N_a - N_b + N_\psi - N_{\bar{\psi}})F , \tag{3.7}$$

where  $N_{a,b}$  and  $N_{\psi, \bar{\psi}}$  are Euler operators (2.18), (2.19), (2.31). Then condition (3.6) is represented as

$$(N_a + N_\psi)F = (N_b + N_{\bar{\psi}})F , \tag{3.8}$$

so it follows that an element  $F \in cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  depends on equal numbers of even and odd variables with upper and lower indices. Expanding out elements  $F(a, b, \psi, \bar{\psi})$  with respect to both even and odd variables yields expression (2.27). From (3.8) it follows that total numbers of upper and lower indices of expansion coefficients coincide. It is worth to comment that expression (3.7) is in fact an adjoint star product realization of Howe dual  $gl(1)$  basis element  $Z^0$  (2.30).

To single out an appropriate real form of the complex higher spin algebra  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  we impose reality conditions in the following way [42]. Introduce an involution  $\dagger$  defined by the relations

$$(a_\alpha)^\dagger = i b^\beta C_{\beta\alpha} , \quad (b^\alpha)^\dagger = i C^{\alpha\beta} a_\beta , \quad (\psi_i)^\dagger = \bar{\psi}^i , \quad (\bar{\psi}^i)^\dagger = \psi_i , \tag{3.9}$$

where  $C_{\alpha\beta}$  and  $C^{\alpha\beta}$  are some real antisymmetric matrices defining complex conjugation (2.10), (2.11), cf. (2.34). An involution reverses an order of product factors and conjugates complex numbers  $(F \star G)^\dagger = G^\dagger \star F^\dagger$ ,  $(\mu F)^\dagger = \mu^* F^\dagger$ ,  $\mu \in \mathbf{C}$ , where  $*$  denotes complex conjugation. The involution  $\dagger$  leaves invariant the defining star product commutation relations (3.1) and satisfies  $(\dagger)^2 = Id$ . The action (3.9) of  $\dagger$  extends to an arbitrary element  $F$  of the star product algebra.

Using the involution  $\dagger$  enables one to define a real form of the Lie superalgebra built by virtue of a graded commutators of elements by imposing the condition [99]

$$F^\dagger = -i^{\pi(F)} F . \tag{3.10}$$

This condition defines the real higher spin algebra  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  [42]. It contains the  $\mathcal{N}$  extended  $AdS_5$  superalgebra  $su(2, 2|\mathcal{N})$  as its maximal finite-dimensional subalgebra.

### 3.2 Factorized higher spin superalgebra $hu_0(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$

Superalgebra  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  is not simple and contains infinitely many ideals  $I_{P(N)}$ , where  $P(N)$  is any star-polynomial of the central element  $N$ , spanned by the elements of the form  $\{x \in I_{P(N)} : x = P(N) \star F, F \in cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)\}$  [39]. There are different quotient superalgebras

$$cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)/I_{P(N)}. \quad (3.11)$$

In particular, one may consider maximally factorized superalgebra with  $P(N) = N$ . In ref. [42] a real form of this quotient algebra singled out by conditions (3.10) has been denoted as  $hu_0(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$ .

We note that the element  $N$  is in fact the basis element  $P^+$  of  $gl(2)$  algebra realized by (2.29) on the linear space of  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  superalgebra. It follows that factoring out  $N \equiv P^+$  leaves supertraceless elements only, i.e.,

$$P^- F(a, b, \psi, \bar{\psi}) = 0, \quad (3.12)$$

and therefore  $hu_0(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  superalgebra is spanned by elements with supertraceless expansion coefficients in (2.27). Put differently, representatives of the quotient superalgebra are identified with the HW vectors of  $gl(2)$  algebra, cf. (2.33).

The quotient algebra can also be defined using the projecting technique elaborated in [36, 37, 100]. To this end one introduces some element  $\Pi$  that satisfies the following conditions

$$\Pi \star N = N \star \Pi = 0, \quad \Pi \star F = F \star \Pi, \quad \forall F \in cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8). \quad (3.13)$$

In particular, it implies that  $\Pi$  is some function of  $N$

$$\Pi = M(N). \quad (3.14)$$

Obviously, the second condition in (3.13) is satisfied and one can explicitly check that the first condition (3.13) reduces to the following differential equation

$$xM''(x) - (\mathcal{N} - 4)M' - 4xM = 0, \quad (3.15)$$

where  $x$  is an indeterminate variable, and  $M'(x)$ ,  $M''(x)$  are the first and the second derivatives of  $M(x)$ . For  $\mathcal{N} \neq 4$  we obtain that the above equation is solved by

$$M(x) = \sum_{n=0}^{\infty} \frac{2^n}{(2n)!! (2n + 3 - \mathcal{N})!!} x^{2n}, \quad (3.16)$$

while for the exceptional case  $\mathcal{N} = 4$  we find that

$$M(x) = e^{2x}. \quad (3.17)$$

The simple form of  $\Pi$  in the case of  $\mathcal{N} = 4$  may be traced back to that  $su(2, 2|\mathcal{N})$  is not simple and possesses an additional ideal to be factored out to obtain  $psu(2, 2|4)$ . It follows that its higher spin extension  $hu_0(8, 8|8)$  is not simple as well. We hope to consider this issue in more detail elsewhere.



### 3.3 Gauging $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$ superalgebra

The gauging procedure introduces  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  as local symmetry in the corresponding higher spin model. According to a general analysis of [99] we consider basis elements  $e_I$  of Lie superalgebra  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  with definite parities  $\pi(e_I) = 0, 1$ . Then one defines gauge connections of  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  as 1-forms  $\Omega = dx^\mu \Omega_\mu^I e_I$ . Their parities coincide with those of the basis elements,  $\pi(\Omega_\mu^I) = \pi(e_I) = 0, 1$ . Gauge transformation and curvature are defined in a standard fashion

$$R_{\mu\nu} = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + [\Omega_\mu, \Omega_\nu]_\star, \quad (3.18)$$

and

$$\delta \Omega_\mu = D_\mu \xi \equiv \partial_\mu \xi + [\Omega_\mu, \xi]_\star, \quad \delta R_{\mu\nu} = [R_{\mu\nu}, \xi]_\star. \quad (3.19)$$

Here brackets  $[\cdot, \cdot]_\star$  denote commutator and it is assumed that basis elements  $e_I$  commute with gauge connections. On the other hand, gauge connections commute as

$$\Omega_\mu^I \Omega_\nu^J = (-)^{\pi(e_I)\pi(e_J)} \Omega_\nu^J \Omega_\mu^I, \quad (3.20)$$

in accordance with boson-fermion spin statistics. Thus we obtain that gauge fields associated with  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  are 1-forms (2.35) satisfying

$$\Omega_\mu^{\alpha_1 \dots \alpha_m}_{\beta_1 \dots \beta_n} \Omega_\nu^{\alpha_1 \dots \alpha_l}_{\beta_1 \dots \beta_k} = (-)^{(m+n)(k+l)} \Omega_\nu^{\alpha_1 \dots \alpha_l}_{\beta_1 \dots \beta_k} \Omega_\mu^{\alpha_1 \dots \alpha_m}_{\beta_1 \dots \beta_n}. \quad (3.21)$$

$R$ -symmetry algebra indices are implicit here. Let us note that constructing gauge superalgebra  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  involves two mutually commuting Grassmann algebras, one formed by gauge connections and another formed by auxiliary variables themselves. It is worth noting that the above definition replaces a graded commutator by usual commutator. This happens because for  $cu(2^{\mathcal{N}-1}, 2^{\mathcal{N}-1}|8)$  Lie superalgebra we chosen the so-called first-class Grassmann shell [101] (see also [99]).

### 3.4 $\mathcal{N} = 2$ higher spin supermultiplets

From now on we set  $\mathcal{N} = 2$  and confine ourselves to the case of  $cu(2, 2|8)$  superalgebra. Expanding out an arbitrary element of  $cu(2, 2|8)$  with respect to Grassmann odd variables one obtains

$$F = F_{e_1} + F_{o_{11}}^i \psi_i + F_{o_{12}i} \bar{\psi}^i + F_{e_{21}} (\epsilon^{mn} \psi_m \psi_n) + F_{e_{22}} (\epsilon_{mn} \bar{\psi}^m \bar{\psi}^n) + F_{e_{31}} \psi_k \bar{\psi}^k \\ + F_{e_{32}i}^j \psi_j \bar{\psi}^i + F_{o_{21}}^i \psi_i (\psi_k \bar{\psi}^k) + F_{o_{22}i} \bar{\psi}^i (\psi_k \bar{\psi}^k) + F_{e_4} (\psi_k \bar{\psi}^k) (\psi_m \bar{\psi}^m). \quad (3.22)$$

Here expansion coefficients are  $F_{e,o} = F_{e,o}(a, b)$ , subscripts  $e$  (even) and  $o$  (odd) indicate bosons and fermions, while their indices enumerate different fields of the supermultiplet. Expansion coefficients  $F_{e_{32}j}^i$  are traceless  $F_{e_{32}i}^i = 0$ . Fields  $F_{e,o}(a, b)$  do not necessarily have equal numbers of  $a_\alpha$  and  $b^\beta$ , so  $(N_a - N_b)F_{e,o}(a, b) = p F_{e,o}(a, b)$ , where  $p = 0, 1, 2$ .

Expanding out  $F(a, b)$  in  $a_\alpha$  and  $b^\beta$  yields traceful coefficients, i.e.,  $F_{\beta_1 \dots \beta_m \gamma}^{\alpha_1 \dots \alpha_n} = 0$ , and therefore they decompose into a collection of traceless components. Namely, for any fixed  $n$  and  $m$ , a multispinor  $F_{\beta_1 \dots \beta_m}^{\alpha_1 \dots \alpha_n}$  decomposes into the set of irreducible traceless components  $F_{\beta_1 \dots \beta_k}^{\alpha_1 \dots \alpha_k}$ , with all  $k + l \leq m + n$ ,  $k - l = m - n$ ,  $k \geq 0$ ,  $l \geq 0$ .

It follows from (3.22) that the spectrum of  $cu(2,2|8)$  gauge fields is represented by the following sum

$$\Omega =: \sum_{k=0}^{\infty} \sum_{s=2}^{\infty} D_{[1]}^{(k)}(s) \oplus D_{[2]}^{(k)}\left(s - \frac{1}{2}\right) \oplus D_{[4]}^{(k)}(s-1) \oplus D_{[1]}^{(k)}(s-1,1) \oplus D_{[2]}^{(k)}\left(s - \frac{3}{2}\right) \oplus D_{[1]}^{(k)}(s-2), \quad (3.23)$$

where  $D^{(k)}(s_1, s_2)$  denotes a  $k$ -th copy of spin- $(s_1, s_2)$  unitary irreducible representation of  $su(2, 2)$  (2.1). Numbers in square brackets denote dimensions of  $R$ -symmetry algebra  $u(2)$  representations. We note that the difference between highest and lowest spins in a supermultiplet equals 2 and highest spin field in the supermultiplet is always bosonic. Using formula (2.1) one can explicitly verify a balance of bosonic and fermionic degrees of freedom.

By way of an example let us consider  $s = 2$  (graviton) supermultiplet. Modulo infinite degeneracy its field content is given by  $(2_{[1]}, \frac{3}{2}_{[2]}, 1_{[4]}, (1, 1)_{[1]}, \frac{1}{2}_{[2]}, 0)$ . We stress that  $(1, 1)_{[1]}$  representation corresponds to massive not massless antisymmetric field  $B_{\mu\nu}$  [102]. Spin  $s = 3$  (hypergraviton) supermultiplet is given by  $(3_{[1]}, \frac{5}{2}_{[2]}, 2_{[4]}, (2, 1)_{[1]}, \frac{3}{2}_{[2]}, 1_{[1]})$ . It is this supermultiplet where a "hook" field appears for the first time. It is worth to comment that  $\mathcal{N} = 3$  supermultiplet contains the same spin fields as  $\mathcal{N} = 2$  supermultiplet but there appears also a fermionic "hook" field. Spin- $(s, 2)$  field appears in  $\mathcal{N} = 4$  supermultiplet. Generally, it follows from (3.7) that a value of the second spin is given by  $s_2 \leq \mathcal{N}/2$ .

#### 4 A general view of FV-type action

For the analysis of interactions we use perturbation expansion with the dynamical fields  $\Omega_1$  treated as fluctuations above the  $AdS_5$  background

$$\Omega = \Omega_0 + \Omega_1, \quad (4.1)$$

where vacuum gauge fields  $\Omega_0$  satisfy the zero-curvature condition (2.7). Both gauge transformations and non-linear curvatures are given by formulas (3.18) and (3.19). Since  $R(\Omega_0) = 0$ , we have  $R = R_1 + R_2$ , where

$$R_1 = d\Omega_1 + \Omega_0 \star \wedge \Omega_1 + \Omega_1 \star \wedge \Omega_0, \quad R_2 = \Omega_1 \star \wedge \Omega_1. \quad (4.2)$$

It follows that linearized curvatures  $R_1$  are of the first order in fluctuations while  $R_2$  contain their quadratic combinations. Gauge transformations for the first order fields are given by

$$\delta\Omega_1 = D_0\xi + [\Omega_1, \xi]_{\star}, \quad \delta R_1 = [R_1, \xi]_{\star}. \quad (4.3)$$

Let us note that the lowest order part of the above gauge transformation has the form (2.39), (2.40).

Higher spin gravitational interactions in the cubic approximations can be described by FV-type action functional

$$S(\Omega) = \frac{1}{2} \mathcal{A}(R(\Omega), R(\Omega)), \quad (4.4)$$

where  $R(\Omega)$  are 2-form curvatures associated to gauge fields of higher spin superalgebra.  $\mathcal{A}(F, G) = \mathcal{A}(G, F)$  is a bilinear symmetric inner product of the type (2.52) defined for any differential 2-forms  $F$  and  $G$  (for more details see [36, 37, 103, 104]).

It is important that the above action is to be supplemented by off-shell constraints (2.62),

$$\widehat{\Upsilon}(R_1) = 0. \tag{4.5}$$

In other words, to maintain gauge invariance of the action in the cubic approximation one has to add constraints which are some linear combinations of the linearized higher spin curvatures. The constraints express all extra fields via derivatives of physical fields as in (2.65).

Before explicitly constructing cubic order theory for  $AdS_5$  higher spin fields it will be useful to consider the general scheme of how to prove establish gauge invariance of the FV-type coupling. For a more detailed discussion see [20, 21, 36, 37]. The gauge invariance of the action can be achieved by attributing to fields  $\Omega_1$  a suitable transformation law. Indeed, the action can be made invariant provided  $\Omega_1$  transform as

$$\delta\Omega_1 = D\xi + \Delta(R, \xi), \tag{4.6}$$

where  $\Delta(R, \xi)$  denotes some  $R$ -dependent deformation of the original transformation law (3.19) such that  $\Delta(0, \xi) = 0$ . These deformations are the so-called improved diffeomorphisms which are intrinsic to all theories containing propagating gravity [105]. In what follows we denote the undeformed transformation (3.19) as  $\delta^{\text{alg}}\Omega_1$  thus emphasizing its origin in  $cu(2, 2|8)$  superalgebra.

Within the perturbation scheme both the action and the gauge transformations are expanded as

$$\begin{aligned} S(\Omega_1) &= S_2(\Omega_1) + S_3(\Omega_1) + \dots, \\ \delta\Omega_1 &= \delta_0\Omega_1 + \delta_1\Omega_1 + \dots. \end{aligned} \tag{4.7}$$

Here zeroth order transformation  $\delta_0\Omega_1$  is given by expression (2.39). Since quadratic action is invariant under linearized transformations,  $\delta_0 S_2 = 0$ , it follows that the action in the cubic approximation stays invariant against deformed transformations (4.6) if

$$\delta^{\text{alg}}S + \Delta S_2 + \dots = 0, \tag{4.8}$$

where the dots stand for higher order corrections  $\mathcal{O}(\Omega_1^3\xi)$ . Recalling that the quadratic action does not depend on extra fields and auxiliary fields are expressed via derivatives of physical fields, one obtains  $\Delta S_2 = \frac{\delta S_2}{\delta\omega^0}\Delta\omega^0$ , where  $\omega^0$  denote physical fields.<sup>10</sup> Let us note that both  $\frac{\delta S_2}{\delta\omega^0}$  and deformation  $\Delta$  are proportional to linearized curvature  $R_1$ . According to (4.8) a deformation of the original gauge transformation (4.6) guaranteeing the cubic order gauge invariance of the action does exist provided that  $\delta^{\text{alg}}S$  is a definite bilinear combination of curvatures and the gauge parameter  $\xi$ , i.e.,

$$\delta^{\text{alg}}S \sim R_1 R_1 \xi + \dots. \tag{4.9}$$

We observe that up to higher order corrections  $\delta^{\text{alg}}S$  vanishes provided that free field equations are fulfilled,  $\frac{\delta S_2}{\delta\omega^0} = 0$ . Using constraints (4.5) and propositions 1 and 2 one reformulates the gauge invariance condition in the cubic approximation as follows

$$\delta^{\text{alg}}S \Big|_{R_1=C} = 0, \tag{4.10}$$

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<sup>10</sup>Recall that the physical field  $\omega^0$  is the Lorentz field  $\omega^t$  (2.48) at  $t = 0$  and for hook fields it is identified with  $\text{Re}\omega^0$ , see the discussion in the end of section 2.5.

where  $C$  are generalized Weyl tensors. In particular, fulfilling the invariance condition (4.10) guarantees the existence of an appropriate deformation  $\Delta$  of the algebraic gauge transformation law for the physical field.

Note that algebraic gauge variations of auxiliary and extra fields are also deformed but these corrections are irrelevant for the action variation in the cubic approximation. Indeed, auxiliary and extra fields contribute both to the cubic action and to constraints (4.5) but due to the extra field decoupling condition they enter the action only in trilinear combinations  $\Omega_1\Omega_1\Omega_1$ . The cubic approximation variation of the action is given by bilinear combinations  $\Omega_1\Omega_1$ . It immediately follows that first order corrections of the gauge transformation law for auxiliary and extra fields are irrelevant in the gauge variation of the action and it is sufficient to know just their zeroth order part. On the other hand, because linearized curvatures  $R_1$  transform homogeneously (4.3) the gauge variation of constraints (4.5) is of the first order in  $\Omega_1$ . Therefore, to maintain gauge invariance of the constraints one deforms extra field gauge transformations by terms linear in  $\Omega_1$ . However, they do not contribute to the variation of the action.

The above consideration provides a general scheme of how to achieve a gauge invariance in FV-type theories. However, higher spin models in question possess several peculiar features as local supersymmetry and an infinite degeneracy of the spectrum. It follows that the action should fulfill additional conditions.

- *R-symmetry invariance.* The  $\mathcal{N} = 2$  superalgebra  $cu(2, 2|8)$  is invariant under global  $u(2)$  rotations of supercharges (3.3). Therefore, a corresponding field theory should also exhibit such a global symmetry, referred to as  $R$ -symmetry.
- *Factorization condition.* Superalgebra  $cu(2, 2|8)$  gives rise to an infinite set of copies for a given spin field. The factorization condition diagonalizes a quadratic part of the action (4.4) so that different copies of the same spin field do not mix up in the quadratic action.
- *C-invariance condition.* The action possesses a cyclic property with respect to the central element  $N$  of  $cu(2, 2|8)$  superalgebra,

$$\mathcal{A}(N \star F, G) = \mathcal{A}(F, G \star N) , \tag{4.11}$$

where  $F, G$  are  $cu(2, 2|8)$  elements and hence they commute with  $N$  (3.6).

In the subsequent sections we consider each of the above conditions. Note that the factorization condition and the  $C$ -invariance condition were originally formulated in [36] for pure bosonic theory while their  $\mathcal{N} = 1$  extension was considered in [37].

The full action (4.4) is naturally split into a sum of bosonic and fermionic parts

$$\mathcal{A}(F, G) = \mathcal{B}(F_e, G_e) + \mathcal{F}(F_o, G_o) , \tag{4.12}$$

where subscripts  $e$  (even) and  $o$  (odd) indicate bosonic and fermionic components of  $F$  and  $G$ , while  $\mathcal{B}$  and  $\mathcal{F}$  are bosonic and fermionic actions, respectively.

The fermionic part is a sum of actions for two  $su(2)$ -valued totally symmetric fermions

$$\mathcal{F}(F_o, G_o) = \mathcal{F}_1(F_{o_1}, G_{o_1}) + \mathcal{F}_2(F_{o_2}, G_{o_2}), \quad (4.13)$$

where

$$\mathcal{F}_1(F_{o_1}, G_{o_1}) = \frac{1}{2} \int \widehat{H}_{o_1} \wedge G_{o_{12}i} \wedge F_{o_{11}}^i + \frac{1}{2} \int \widehat{H}_{o_1} \wedge F_{o_{12}i} \wedge G_{o_{11}}^i \quad (4.14)$$

$$\mathcal{F}_2(F_{o_2}, G_{o_2}) = \frac{1}{2} \int \widehat{H}_{o_2} \wedge G_{o_{22}i} \wedge F_{o_{21}}^i + \frac{1}{2} \int \widehat{H}_{o_2} \wedge F_{o_{22}i} \wedge G_{o_{21}}^i. \quad (4.15)$$

The bosonic part is a sum of actions for five  $u(2)$ -valued bosonic fields

$$\begin{aligned} \mathcal{B}(F_e, G_e) = & \mathcal{B}_1(F_{e_1}, G_{e_1}) + \mathcal{B}_{31}(F_{e_{31}}, G_{e_{31}}) \\ & + \mathcal{B}_{32}(F_{e_{32}}, G_{e_{32}}) + \mathcal{B}_4(F_{e_4}, G_{e_4}) + \mathcal{B}_2(F_{e_2}, G_{e_2}), \end{aligned} \quad (4.16)$$

where each term is defined as follows. Actions for totally symmetric fields are

$$\begin{aligned} \mathcal{B}_1(F_{e_1}, G_{e_1}) &= \int \widehat{H}_{e_1} \wedge F_{e_1} \wedge G_{e_1}, & \mathcal{B}_{31}(F_{e_{31}}, G_{e_{31}}) &= \int \widehat{H}_{e_{31}} \wedge F_{e_{31}} \wedge G_{e_{31}}, \\ \mathcal{B}_{32}(F_{e_{32}}, G_{e_{32}}) &= \int \widehat{H}_{e_{31}} \wedge F_{e_{31}i}^j \wedge G_{e_{31}j}^i, & \mathcal{B}_4(F_{e_4}, G_{e_4}) &= \int \widehat{H}_{e_4} \wedge F_{e_4} \wedge G_{e_4}, \end{aligned} \quad (4.17)$$

while the action for non-symmetric fields is

$$\mathcal{B}_2(F_{e_2}, G_{e_2}) = \frac{1}{2} \int \widehat{H}_{e_2} \wedge F_{e_{22}} \wedge G_{e_{21}} + \frac{1}{2} \int \widehat{H}_{e_2} \wedge G_{e_{22}} \wedge F_{e_{21}}. \quad (4.18)$$

From now on the symbol of exterior product  $\wedge$  will be systematically omitted. By construction, all the above actions are invariant under  $R$ -symmetry transformations  $u(2)$ . They are of the type (2.54) defined by operators  $\widehat{H}_{e,o} = \widehat{H}_{e,o}(E)$  (2.55) which depend on dynamical gravitation field described by the frame  $E^{\alpha\beta}$ . To construct the cubic order action we use the following anzats for operators  $\widehat{H}_{e,o} = \widehat{H}_{e,o}(E)$ . Namely, we set a part of coefficients or their linear combinations to zero

$$\beta_e(p, q) = 0, \quad \zeta_e(p, q) = -\gamma_e(p, q), \quad (4.19)$$

for spin- $s$  bosonic fields, and

$$\beta_{e,o}(p, q) = 0, \quad \gamma_{e,o}(p, q) = 0, \quad (4.20)$$

for spin- $s$  fermionic fields and spin- $(s, 1)$  bosonic fields. Note that the above choice is consistent with the quadratic action coefficients (2.59)–(2.61).

It is important to comment that describing gauge fields as differential forms and using the compensator mechanism that makes  $su(2, 2)$  symmetry manifest guarantees that the full action (4.12) is explicitly  $su(2, 2)$  covariant and diffeomorphism invariant. Note that we treat gravitational fields appearing in the full action in two different setups, as the frame field  $E^{\alpha\beta}$  that explicitly enters operators  $\widehat{H}_{e,o} = \widehat{H}_{e,o}(E)$  and gauge connection  $\Omega^{\alpha\beta}$  of  $su(2, 2) \subset cu(2, 2|8)$ . As a result, the gauge variation  $\delta^{\text{alg}} S$  of the full action (4.12)

involves two types of terms resulting from varying operators  $\widehat{H}_{e,o}(E)$  and curvatures  $R(\Omega_1)$ . The invariance of the first type results from the explicit  $su(2,2)$  covariance and diffeomorphism invariance of the whole setup. The invariance of the second type gives rise to the condition (4.10) which now takes the form

$$\mathcal{A}(R_1, [R_1, \xi]_\star) \approx 0. \tag{4.21}$$

here  $\mathcal{A}$  is given by (4.12) and  $\approx$  means that all linearized curvatures  $R_1$  are replaced by generalized Weyl tensors according to propositions 1 and 2. Gauge parameter  $\xi \in cu(2,2|8)$  is arbitrary.

The above discussion of the gauge invariance in the cubic approximation is valid for a higher spin model with  $cu(2,2|8)$  local symmetry but the same methods are also applied for a reduced system governed by factorized superalgebra  $hu_0(2,2|8)$ . To build a reduced model we use the approach elaborated for  $\mathcal{N} = 0$  pure bosonic system in [36] and for  $\mathcal{N} = 1$  system in [37] which consists of inserting the projecting operator  $\Pi$  (3.13) into the action of  $cu(2,2|8)$  system as

$$\mathcal{A}(F, G) \rightarrow \mathcal{A}_0(F, G) = \mathcal{A}(F, \Pi \star G), \tag{4.22}$$

where  $\mathcal{A}(F, G)$  is given by (4.12). Then  $\mathcal{A}_0(F, G)$  defines an action of the reduced model. Because the projecting operator  $\Pi(N)$  is some fixed function of  $N$  (3.16) it follows that the  $C$ -invariance condition guarantees

$$\mathcal{A}(F, \Pi \star G) = \mathcal{A}(F \star \Pi, G), \tag{4.23}$$

so the bilinear form in the action with  $\Pi$  inserted remains symmetric. The idea is that all terms in  $F$  and  $G$  proportional to  $N$  do not contribute to the action (4.22) which therefore is defined on the quotient subalgebra  $hu_0(2,2|8)$ . Note that  $\mathcal{A}_0(F, G)$  is well-defined as a functional of polynomial functions  $F$  and  $G$  because for polynomial  $F$  and  $G$  only a finite number of terms in the expansion of  $\Pi$  in auxiliary variables contributes. The explicit expression for  $\mathcal{N} = 2$  projecting operator  $\Pi$  is given in section 4.1.

### 4.1 Summary of results

In this section we list all the coefficients in the action for  $cu(2,2|8)$  model.

- Spin- $(s_1, 0)$  sector is given by

$$\alpha_{e_1}(p, q) = 2\gamma_{e_1}(p + q) - \frac{1}{2}\Phi_0 \int_0^1 d\tau \text{Res}_\nu \nu e^{\frac{1}{2}(-\nu^{-1} + \nu(\tau p + q))} \tag{4.24}$$

$$\gamma_{e_1}(p) = -\zeta_{e_1}(p) = -\frac{\Phi_0}{4} \int_0^1 d\tau \tau \text{Res}_\nu \nu e^{\frac{1}{2}(-\nu^{-1} + \nu\tau p)}$$

$$\begin{aligned} \alpha_{e_{31}}(p, q) &= \frac{1}{2}\alpha_{e_1}(p, q) & \gamma_{e_{31}}(p) &= \frac{1}{2}\gamma_{e_1}(p) \\ \alpha_{e_4}(p, q) &= \frac{1}{4}\alpha_{e_1}(p, q) & \gamma_{e_4}(p) &= \frac{1}{4}\gamma_{e_1}(p) \end{aligned} \tag{4.25}$$

$$\begin{aligned}
 \alpha_{e_{32}}(p, q) &= 2\gamma_{e_{32}}(p+q) - \frac{1}{8}\Phi_0 \int_0^1 d\tau \text{Res}_\nu \nu e^{\frac{1}{2}(-\nu^{-1} + \nu(\tau p + q))} \\
 \gamma_{e_{32}}(p) &= -\zeta_{e_{32}}(p) = -\frac{\Phi_0}{16} \int_0^1 d\tau \tau \text{Res}_\nu \nu e^{\frac{1}{2}(-\nu^{-1} + \nu\tau p)}
 \end{aligned} \tag{4.26}$$

According to (4.19) all coefficients  $\beta_{e_1}(p, q) = \beta_{e_{31}}(p, q) = \beta_{e_{32}}(p, q) = \beta_{e_4}(p, q) = 0$ .

- Spin- $(s_1, 1)$  sector is given by

$$\begin{aligned}
 \alpha_{e_2}(p, q) &= \zeta_{e_2}(p, q) + \frac{\Phi_0}{q} \int_0^1 d\tau \text{Res}_\nu \nu^{-1} e^{\frac{1}{2}(-\nu^{-1} + \nu(\tau p + q))} \\
 \zeta_{e_2}(p, q) &= -\frac{\Phi_0}{q(p+q)} \int_0^1 d\tau \text{Res}_\nu \nu^{-1} e^{\frac{1}{2}(-\nu^{-1} + \nu\tau(p+q))}
 \end{aligned} \tag{4.27}$$

According to (4.20) coefficients  $\beta_{e_2}(p, q) = 0$  and  $\gamma_{e_2}(p, q) = 0$ .

- Spin- $(s_1, \frac{1}{2})$  sector is given by

$$\begin{aligned}
 \alpha_{o_1}(p, q) &= \zeta_{o_1}(p, q) + \frac{\Phi_0}{2q} \int_0^1 d\tau \text{Res}_\nu e^{\frac{1}{2}(-\nu^{-1} + \nu(p\tau + q))} \\
 \zeta_{o_1}(p, q) &= -\frac{\Phi_0}{2q(p+q)} \int_0^1 d\tau \text{Res}_\nu e^{\frac{1}{2}(-\nu^{-1} + \nu(p+q)\tau)}
 \end{aligned} \tag{4.28}$$

$$\alpha_{o_2}(p, q) = \frac{1}{4}\alpha_{o_1}(p, q) \quad \zeta_{o_2}(p, q) = \frac{1}{4}\zeta_{o_1}(p, q) \tag{4.29}$$

According to (4.20) coefficients  $\beta_{o_1}(p, q) = \beta_{o_2}(p, q) = 0$  and  $\gamma_{o_1}(p, q) = \gamma_{o_2}(p, q) = 0$ .

Here  $\Phi_0$  is an arbitrary factor properly normalized in terms of the cosmological constant  $\lambda$  and the gravitational constant  $\kappa$ .

The action of the reduced  $hu(2, 2|8)$  model is defined according to (4.22), where the form of the projecting operator is read off from the general expression (3.16) at  $\mathcal{N} = 2$

$$\Pi(N) = \sum_{n=0}^{\infty} \frac{2^n}{(2n)!!(2n+1)!!} N^{2n}. \tag{4.30}$$

## 5 Calculation of gauge invariance

The novel feature of  $\mathcal{N} = 2$  analysis compared to  $\mathcal{N} = 0, 1$  case is the appearance of "hook" fields. In this section we study the invariance condition (4.21) giving particular emphasis to calculations involving fields of "hook" symmetry type. Our analysis of the cubic order interaction vertices is heavily based on the technique elaborated in the previous papers on  $\mathcal{N} = 0, 1$  FV-type theory [36, 37]. In particular, we do not repeat here calculations related to totally symmetric fields and use results obtained in [36, 37].



### 5.1 Factorization condition for “hook” fields

We begin by noting that due to (super)traces of  $cu(2, 2|8)$  gauge fields  $su(2, 2|2)$  supermultiplets are not irreducible and decompose into (super)traceless components (see section 3.4). Having in mind (2.33) we call a gauge field  $\Omega(a, b, \psi, \bar{\psi}|x)$  supertraceless if it fulfills the following algebraic constraint

$$P^- \Omega(a, b, \psi, \bar{\psi}|x) = 0, \tag{5.1}$$

where operator  $P^-$  is given by (2.29). It follows that using operators  $P^-$  and  $P^+$  allows one to decompose any element  $\Omega(a, b, \psi, \bar{\psi}|x)$  of  $cu(2, 2|8)$  superalgebra into irreducible  $su(2, 2|2)$  supermultiplets as

$$\Omega(a, b, \psi, \bar{\psi}|x) = \sum_{k=0}^{\infty} \sum_{s_1=2}^{\infty} \chi(k, s_1) (P^+)^k \Omega^{k, s_1}(a, b, \psi, \bar{\psi}|x), \tag{5.2}$$

where  $\chi(k, s_1)$  are arbitrary coefficients,  $s_1$  denotes the highest integer spin in a supermultiplet and  $\Omega^{k, s_1}$  are supertraceless (5.1). The supertraceless decomposition can be equivalently rewritten (modulo finite field redefinitions) in the manifest  $su(2, 2)$  fashion with all multispinors being traceless rather than supertraceless

$$\Omega(a, b|x) = \sum_{k=0}^{\infty} \sum_{s_1=2}^{\infty} v(n, s_1) (T^+)^n \Omega^{n, s_1}(a, b|x), \tag{5.3}$$

where  $v(n, s_1)$  are arbitrary coefficients and  $\Omega^{n, s_1}(a, b|x)$  describe an  $n$ -th copy of irreducible field of a given spin  $(s_1, s_2)$  (2.43). Note that  $s_2 = 0, \frac{1}{2}, 1$  is implicit in the above decompositions. The decomposition analogous to (5.3) is valid for the curvatures

$$R(a, b|x) = \sum_{n, s_1=0}^{\infty} v(n, s_1) (T^+)^n R^{n, s_1}(a, b|x), \tag{5.4}$$

where  $R^{n, s_1}(a, b|x)$  are associated with irreducible fields  $\Omega^{n, s_1}(a, b|x)$ .

The factorization condition requires

$$S_2(\Omega) = \sum_{n=0}^{\infty} \sum_{s_1=2}^{\infty} \sum_{s_2} S_2^{n, s_1, s_2}(\Omega^{n, s_1+2, s_2}), \tag{5.5}$$

where  $S_2$  is a quadratic part of (4.4) and  $S_2^{n, s_1, s_2}$  is a quadratic action for a  $n$ -th copy of a given spin field (recall that it may take values in  $u(2)$  irreps). The condition diagonalizes  $S_2$ , i.e. the terms containing products of the fields  $\Omega^{n, s_1, s_2}$  and  $\Omega^{m, s_1, s_2}$  with  $n \neq m$  in the trace decomposition (5.3) should all vanish. Note that normalization coefficients  $v_n(T^0)$  in expansion (5.3) can be chosen in such a way that all copies of the same spin in the quadratic actions enter (5.5) with the same overall factor. The factorization condition for totally symmetric fields has been explicitly calculated in refs. [36, 37]. In this section we perform the analogous analysis for “hook” fields.

From the above discussion it follows that the factorization condition in the spin- $(s_1, 1)$  sector is valid provided that

$$\mathcal{B}_2(F_{e_2}, T^+ G_{e_2}) = \widetilde{\mathcal{B}}_2(T^- F_{e_2}, G_{e_2}), \quad (5.6)$$

where action  $\widetilde{\mathcal{B}}_2$  is defined for some set of new parameters  $(\widetilde{\alpha}_{e_2}, \widetilde{\zeta}_{e_2})$  expressed in terms of old parameters  $(\alpha_{e_2}, \zeta_{e_2})$ , see (2.55) and (2.61). Then one finds that two actions differ from each other by the following term

$$\int Q_{e_2}(p, q) E_\alpha^\beta \frac{\partial^2}{\partial a_{2\alpha} \partial b_2^\beta} (c_{12})^2 F_{e_{21}}(a_1, b_1) G_{e_{22}}(a_2, b_2), \quad (5.7)$$

which is required to vanish,

$$Q_{e_2}(p, q) \equiv \left(1 + p \frac{\partial}{\partial p}\right) \alpha_{e_2}(p, q) + \left(1 + q \frac{\partial}{\partial q}\right) \zeta_{e_2}(p, q) = 0. \quad (5.8)$$

The new coefficients are expressed through the old ones as follows

$$\begin{aligned} \widetilde{\alpha}_{e_2}(p, q) &= 4 \left( \left(2 + p \frac{\partial}{\partial p}\right) \frac{\partial}{\partial p} + \left(3 + q \frac{\partial}{\partial q}\right) \frac{\partial}{\partial q} \right) \alpha_{e_2}(p, q), \\ \widetilde{\zeta}_{e_2}(p, q) &= 4 \left( \frac{2}{q} + \left(1 + p \frac{\partial}{\partial p}\right) \frac{\partial}{\partial p} + \left(4 + q \frac{\partial}{\partial q}\right) \frac{\partial}{\partial q} \right) \zeta_{e_2}(p, q). \end{aligned} \quad (5.9)$$

They will be further constrained by the  $C$ -invariance condition discussed below. One can show that the factorization condition (5.8) and the extra field decoupling condition (2.53) are compatible and the solution is given by (2.61). Quite analogously one considers totally symmetric fields and proves that the coefficients are fixed by the factorization and extra field decoupling conditions as in (2.59) and (2.60), see [36, 37].

## 5.2 The $C$ -invariance condition

Let us discuss the  $C$ -invariance condition (4.11). The exact formula for  $N \star F$  reads

$$\begin{aligned} N \star F &= (T^+ - T^-)F - F_{e_1}(\psi_k \bar{\psi}^k) - F_{o_{11}^i} \psi_i(\psi_k \bar{\psi}^k) - F_{o_{12}^i} \bar{\psi}^i(\psi_k \bar{\psi}^k) \\ &\quad - F_{e_{31}}(\psi_k \bar{\psi}^k)(\psi_m \bar{\psi}^m) - \frac{1}{2} F_{e_{31}} - \frac{1}{4} F_{o_{21}^i} \psi_i - \frac{1}{4} F_{o_{22}^i} \bar{\psi}^i - \frac{1}{2} F_{e_4}(\psi_m \bar{\psi}^m). \end{aligned} \quad (5.10)$$

where  $F$  is given by (3.22). Imposing the  $C$ -invariance condition results in the mutual conjugation of the trace creation operator  $T^+$  and trace annihilation operator  $T^-$  with respect to the inner product  $\mathcal{A}$ :

$$\mathcal{A}(T^\pm F, G) = -\mathcal{A}(F, T^\mp G), \quad (5.11)$$

while the relative coefficients between different type actions are fixed as

$$\mathcal{B}_1 = 2\mathcal{B}_{31}, \quad \mathcal{B}_1 = 4\mathcal{B}_4, \quad \mathcal{F}_1 = 4\mathcal{F}_2. \quad (5.12)$$

In particular, condition (5.11) implemented in the "hook" field sector along with the factorization condition yields additional relations for coefficients (5.9),

$$\alpha(p, q) + \widetilde{\alpha}(p, q) = 0, \quad \zeta(p, q) + \widetilde{\zeta}(p, q) = 0. \quad (5.13)$$

It is worth noting that the factorization condition is implemented on the free field level only while the C-invariance conditions is valid for the non-linear action as well. In particular, we see that condition (5.11) for free fields is a stronger version of the factorization condition. Also, conditions (5.12) for free fields are too restrictive because they relate normalization constants in front of different spin quadratic actions.

The C-invariance condition also implies that it is sufficient to consider the invariance condition (4.10) only for the fields satisfying the tracelessness condition (2.42). Because curvatures decompose into traceless components as (5.4) we single out the zeroth order terms in  $T^+$  and denote them as

$$\mathcal{R}(a, b|x) \equiv \sum_{s_1=2}^{\infty} \mathcal{R}^{s_1}(a, b|x) \tag{5.14}$$

By definition, each term in this expansion is traceless,  $T^-\mathcal{R}^{s_1} = 0$ . Recall that both the second spin value  $s_2 = 0, \frac{1}{2}, 1$  and  $u(2)$  indices are implicit here. One may explicitly prove that the invariance condition (4.10) is now takes the form

$$\mathcal{A}(\mathcal{R}, [\mathcal{R}, \xi]_{\star}) \approx 0, \tag{5.15}$$

where  $\approx$  means that all linearized curvatures are replaced by generalized Weyl tensors according to propositions 1 and 2. The idea of the proof is to consider the variation  $\mathcal{A}(R, [R, \xi]_{\star})$  with curvatures decomposed according to the trace decomposition (5.4). Then using formula [36]

$$T^+F(a, b) = T^+ \star F(a, b) + \left(T^- - \frac{1}{2}G^0\right)F(a, b), \tag{5.16}$$

where  $T^{\pm}$  and  $G^0$  are given by (2.22), (2.23), along with the C-invariance condition in the form (5.11) enables one reduce step by step a degree in  $T^+$  thereby ending up with pure traceless curvatures  $\mathcal{R}$  and new gauge parameter  $\xi \rightarrow T^+ \star \xi$ . More detailed exposition can be found in [36, 37].

### 5.3 Cubic order gauge invariance

Gauge transformations of  $cu(2, 2|8)$  superalgebra are defined by 0-form parameter  $\xi = \xi(a, b, \psi, \bar{\psi}|x)$  expanded out analogously to (3.22),

$$\begin{aligned} \xi = & \xi_{e_1} + \xi_{o_{11}}^i \psi_i + \xi_{o_{12}i} \bar{\psi}^i + \xi_{e_{21}} (\epsilon^{mn} \psi_m \psi_n) + \xi_{e_{22}} (\epsilon_{mn} \bar{\psi}^m \bar{\psi}^n) + \xi_{e_{31}} \psi_k \bar{\psi}^k \\ & + \xi_{e_{32}j}^i \psi_i \bar{\psi}^j + \xi_{o_{21}}^i \psi_i (\psi_k \bar{\psi}^k) + \xi_{o_{22}i} \bar{\psi}^i (\psi_k \bar{\psi}^k) + \xi_{e_4} (\psi_k \bar{\psi}^k) (\psi_m \bar{\psi}^m). \end{aligned} \tag{5.17}$$

Because the curvatures  $R(a, b, \psi, \bar{\psi}|x)$  are transformed homogeneously (3.19) it follows that the component form of  $\delta R(a, b, \psi, \bar{\psi}|x)$  comprises over a hundred terms. In what follows we consider invariance with respect to each type of gauge transformations associated with supermultiplet parameters (5.17), but explicit calculations are too lengthy to present them here. Instead, we explicitly analyze the invariance with respect to bosonic symmetry defined by  $\xi_{e_1}$ , while the rest of gauge invariance analysis is given schematically just emphasizing key points. Explicit expressions for gauge transformations are relegated to appendix 7.

### 5.3.1 Cubic order invariance for “hook” fields

In this section we study the gauge invariance with respect to bosonic parameter  $\xi_{e_1} = \xi_{e_1}(a, b)$  in the “hook” field sector. Let us note that the respective symmetry does not mix different type fields, see (7.4). The gauge invariance for totally symmetric fields was analyzed in [36, 37].

A general variation of the action for “hook” fields (4.18) is given by

$$\delta\mathcal{B}_2 = \int \hat{H}_{e_2} \delta R_{e_{22}} R_{e_{21}} + \int \hat{H}_{e_2} R_{e_{22}} \delta R_{e_{21}}. \quad (5.18)$$

Substituting  $\delta R_{e_{21}} = [R_{e_{21}}, \xi_{e_1}]_\star$  and  $\delta R_{e_{22}} = [R_{e_{22}}, \xi_{e_1}]_\star$  from (7.4) we obtain

$$\begin{aligned} \delta\mathcal{B}_2 = & \int \hat{H}_{e_2} (R_{e_{22}} \star \xi_{e_1}) R_{e_{21}} - \int \hat{H}_{e_2} (\xi_{e_1} \star R_{e_{22}}) R_{e_{21}} \\ & + \int \hat{H}_{e_2} R_{e_{22}} (R_{e_{21}} \star \xi_{e_1}) - \int \hat{H}_{e_2} R_{e_{22}} (\xi_{e_1} \star R_{e_{21}}). \end{aligned} \quad (5.19)$$

In order to calculate the above variation in the form (5.15) we set all traces in  $R_{e_{21}}$  and  $R_{e_{22}}$  to zero and for respective traceless components use the following representation in terms of Weyl tensors, cf. (2.71) and (2.72),

$$\begin{aligned} \mathcal{R}_{e_{21}}(a, b) &= \text{Res}_\nu \nu^{-2} e^{\nu^{-1} a_\alpha \frac{\partial}{\partial c_\alpha} + \nu b^\beta \frac{\partial}{\partial c^\beta}} H_2^{\gamma\rho} \frac{\partial^2}{\partial c^\gamma \partial c^\rho} C_{e_{21}}(c) \Big|_{c=0}, \\ \mathcal{R}_{e_{22}}(a, b) &= \text{Res}_\nu \nu^{-2} e^{\nu a_\alpha \frac{\partial}{\partial c_\alpha} + \nu^{-1} b^\beta \frac{\partial}{\partial c^\beta}} H_2^{\gamma\rho} \frac{\partial^2}{\partial c^\gamma \partial c^\rho} C_{e_{22}}(c) \Big|_{c=0}. \end{aligned} \quad (5.20)$$

We find that up to non-zero multiplicative constant variation  $\delta\mathcal{B}_2$  is given by

$$\begin{aligned} & \int H_5 \bar{k}^2 \text{Res}_\nu e^{\frac{1}{2}(\nu \bar{v}_1 - \nu^{-1} \bar{u}_1)} \nu^{-2} (\nu \bar{k} + \bar{u}_2)^2 \Phi(Z) C_{e_{22}}(c_1) C_{e_{21}}(c_2) \xi(a_3, b_3) \\ & - \int H_5 \bar{k}^2 \text{Res}_\nu e^{\frac{1}{2}(-\nu \bar{v}_1 + \nu^{-1} \bar{u}_1)} \nu^{-2} (\nu \bar{k} + \bar{u}_2)^2 \Phi(Z) C_{e_{22}}(c_1) C_{e_{21}}(c_2) \xi(a_3, b_3) + \\ & + \int H_5 \bar{k}^2 \text{Res}_\nu e^{\frac{1}{2}(\nu^{-1} \bar{v}_2 - \nu \bar{u}_2)} \nu^{-2} (\nu \bar{k} + \bar{v}_1)^2 \Phi(Y) C_{e_{22}}(c_1) C_{e_{21}}(c_2) \xi(a_3, b_3) \\ & - \int H_5 \bar{k}^2 \text{Res}_\nu e^{\frac{1}{2}(-\nu^{-1} \bar{v}_2 + \nu \bar{u}_2)} \nu^{-2} (\nu \bar{k} + \bar{v}_1)^2 \Phi(Y) C_{e_{22}}(c_1) C_{e_{21}}(c_2) \xi(a_3, b_3), \end{aligned} \quad (5.21)$$

where we used the following notation

$$\bar{k} = \frac{\partial^2}{\partial c_{1\alpha} \partial c_2^\alpha}, \quad \bar{u}_i = \frac{\partial^2}{\partial c_i^\alpha \partial a_{3\alpha}}, \quad \bar{v}_i = \frac{\partial^2}{\partial c_{i\alpha} \partial b_3^\alpha}, \quad (5.22)$$

and

$$Z \equiv AB = (\nu \bar{k} + \bar{u}_2)(\nu^{-1} \bar{k} - \bar{v}_2), \quad Y \equiv FD = (\nu \bar{k} + \bar{v}_1)(\nu^{-1} \bar{k} - \bar{u}_1), \quad (5.23)$$

while the function  $\Phi(Z)$  is given by

$$\Phi(Z) = Z(\alpha_{e_2}(Z, -Z) - \zeta_{e_2}(Z, -Z)). \quad (5.24)$$

Quantity  $H_5$  is a 5-form defined as  $H_5 = h_\alpha^\beta h_\beta^\gamma h_\gamma^\rho h_\rho^\delta h_\delta^\alpha$  [36]. The invariance condition (5.15) requires the above variation to vanish. Because it is legitimate to omit generalized Weyl tensors and  $H_5 \bar{k}^2$  in the left-hand-side of (5.21) we obtain the following equation

$$\begin{aligned} & \text{Res}_\nu e^{\frac{1}{2}(\nu \bar{v}_1 - \nu^{-1} \bar{u}_1)} \nu^{-2} A^2 \Phi(AB) - \text{Res}_\nu e^{\frac{1}{2}(-\nu \bar{v}_1 + \nu^{-1} \bar{u}_1)} \nu^{-2} A^2 \Phi(AB) + \\ & + \text{Res}_\nu e^{\frac{1}{2}(\nu^{-1} \bar{v}_2 - \nu \bar{u}_2)} \nu^{-2} F^2 \Phi(FD) - \text{Res}_\nu e^{\frac{1}{2}(-\nu^{-1} \bar{v}_2 + \nu \bar{u}_2)} \nu^{-2} F^2 \Phi(FD) = 0. \end{aligned} \quad (5.25)$$

Let us define a function  $\tilde{\Phi}(A, B) = A^2 \Phi(AB)$  and rewrite the above equation as follows

$$\begin{aligned} & \text{Res}_\nu \nu^{-2} \left( e^{\frac{1}{2}(\nu \bar{v}_1 - \nu^{-1} \bar{u}_1)} \tilde{\Phi}(A, B) - e^{\frac{1}{2}(-\nu \bar{v}_1 + \nu^{-1} \bar{u}_1)} \tilde{\Phi}(A, B) + \right. \\ & \left. e^{\frac{1}{2}(\nu^{-1} \bar{v}_2 - \nu \bar{u}_2)} \tilde{\Phi}(F, D) - e^{\frac{1}{2}(-\nu^{-1} \bar{v}_2 + \nu \bar{u}_2)} \tilde{\Phi}(F, D) \right) = 0. \end{aligned} \quad (5.26)$$

An educated guess is that the function  $\tilde{\Phi}(A, B) = \Phi_0^{e_2} \text{Res}_\mu (\mu^{-2} e^{\frac{1}{2}(\mu A + \mu^{-1} B)})$ , where  $\Phi_0^{e_2}$  is an arbitrary constant, is a solution to the above equation. Indeed, substituting this function back into (5.26) gives

$$\begin{aligned} & \text{Res}_\nu \nu^{-2} \mu^{-2} \left( e^{\frac{1}{2}(\nu \bar{v}_1 - \nu^{-1} \bar{u}_1) + \frac{1}{2}(\mu A + \mu^{-1} B)} - e^{\frac{1}{2}(-\nu \bar{v}_1 + \nu^{-1} \bar{u}_1) + \frac{1}{2}(\mu A + \mu^{-1} B)} + \right. \\ & \left. e^{\frac{1}{2}(\nu^{-1} \bar{v}_2 - \nu \bar{u}_2) + \frac{1}{2}(\mu F + \mu^{-1} D)} - e^{\frac{1}{2}(-\nu^{-1} \bar{v}_2 + \nu \bar{u}_2) + \frac{1}{2}(\mu F + \mu^{-1} D)} \right) = 0, \end{aligned} \quad (5.27)$$

or

$$\begin{aligned} & \text{Res}_\nu \nu^{-2} \mu^{-2} \left( e^{\frac{1}{2}(\nu \bar{v}_1 - \nu^{-1} \bar{u}_1) + \frac{1}{2}(\mu(\nu \bar{k} + \bar{u}_2) + \mu^{-1}(\nu^{-1} \bar{k} - \bar{v}_2))} + \right. \\ & - e^{\frac{1}{2}(-\nu \bar{v}_1 + \nu^{-1} \bar{u}_1) + \frac{1}{2}(\mu(\nu \bar{k} + \bar{u}_2) + \mu^{-1}(\nu^{-1} \bar{k} - \bar{v}_2))} + \\ & + e^{\frac{1}{2}(\nu^{-1} \bar{v}_2 - \nu \bar{u}_2) + \frac{1}{2}(\mu(\nu \bar{k} + \bar{v}_1) + \mu^{-1}(\nu^{-1} \bar{k} - \bar{u}_1))} + \\ & \left. - e^{\frac{1}{2}(-\nu^{-1} \bar{v}_2 + \nu \bar{u}_2) + \frac{1}{2}(\mu(\nu \bar{k} + \bar{v}_1) + \mu^{-1}(\nu^{-1} \bar{k} - \bar{u}_1))} \right) = 0. \end{aligned} \quad (5.28)$$

The first and the fourth terms are equal to each other under  $\nu \leftrightarrow \mu$ , while the second and the third terms are equal to each other under  $\nu \leftrightarrow -\mu$ . Therefore, we conclude that the function

$$\Phi(A) = \Phi_0^{e_2} A^{-2} \text{Res}_\mu \left( \mu^{-2} \exp \frac{1}{2}(\mu A + \mu^{-1}) \right), \quad (5.29)$$

where  $A$  is some indeterminate variable, solves the invariance condition in the sector of "hook" fields. As a result, we arrive at the following equation on the coefficient functions

$$A(\alpha(A, -A) - \zeta(A, -A)) = \Phi_0 A^{-2} \text{Res}_\mu \left( \mu^{-2} \exp \frac{1}{2}(\mu A + \mu^{-1}) \right). \quad (5.30)$$

The left-hand-side of the above equation does not vanish at  $A = 0$  because the coefficient  $\zeta(A, -A)$  is not necessarily polynomial and contains poles in  $A$ . Contrary, the right-hand-side is polynomial but the zeroth order in  $A$  is not generally zero so the equation is consistent at  $A = 0$ . Let us note that though the above equation involves the coefficients which are functions of two variables  $p$  and  $q$  it defines dependence on just one variable. Actually this is due to the fact that equation (5.30) involves a function of a single variable  $\rho(p + q)$  which defines normalization constants in front of quadratic actions (2.59)–(2.61).

Equation (5.30) can be cast into the following convenient integral form

$$\alpha(A, -A) - \zeta(A, -A) = \frac{\Phi_0^{e_2}}{2} A^{-2} \int_0^1 d\tau \text{Res}_\nu \nu^{-1} e^{\frac{1}{2}(\nu^{-1} + \nu\tau A)}. \quad (5.31)$$

We write down the answer in terms of function

$$\rho(p) = -\frac{\Phi_0}{2p} \int_0^1 d\tau \text{Res}_\nu \nu^{-1} e^{\frac{1}{2}(-\nu^{-1} + \nu\tau p)}. \quad (5.32)$$

It follows that the coefficient functions take the the form

$$\zeta(p, q) = \frac{\rho(p+q)}{q}, \quad (5.33)$$

$$\alpha(p, q) = \frac{\rho(p+q)}{q} + \frac{\Phi_0}{2q} \int_0^1 d\tau \text{Res}_\nu \nu^{-1} e^{\frac{1}{2}(-\nu^{-1} + \nu(\tau p + q))}. \quad (5.34)$$

One can explicitly check that the above formal series satisfy the following identities

$$\left( p \frac{\partial^2}{\partial p^2} + 3 \frac{\partial}{\partial p} + \frac{1}{4} \right) \rho(p) = 0, \quad (5.35)$$

$$\left( \left( 2 + p \frac{\partial}{\partial p} \right) \frac{\partial}{\partial p} + \left( 3 + q \frac{\partial}{\partial q} \right) \frac{\partial}{\partial q} + \frac{1}{4} \right) \alpha(p, q) = 0, \quad (5.36)$$

which are in fact conditions (5.9), (5.13). Thus it is shown that the coefficient functions for "hook" fields satisfy the factorization condition, the  $C$ -invariance condition, extra field decoupling condition and the invariance condition (5.15). One concludes that the action for "hook" fields is consistently defined both on the free field and interaction levels.

### 5.3.2 The remaining invariance

Gauge invariance of actions for totally symmetric bosonic and fermionic fields with respect to  $\xi_{e_1}(a, b)$  has been considered in [36, 37]. The common feature of the variation in different field sectors of the full action is that coefficient functions  $\alpha(p, q)$ ,  $\beta(p, q)$ ,  $\gamma(p, q)$ , and  $\zeta(p, q)$  in (2.55) appear only through particular combinations identified with functions  $\Phi(X)$  of the type (5.24); exact expressions are collected in (7.1)–(7.3). It follows that considering the gauge variation is more convenient in terms of functions  $\Phi(X)$ . Taking into account the results obtained in the previous section we list functions  $\Phi(X)$  for spin- $s_1$  fields and for spin- $(s_1, 1)$  in the following manner

$$\Phi(X) = \Phi_0 \Psi(X), \quad \Psi(X) = X^{-2s_2} \text{Res}_\nu \left( \nu^{-2s_2} \exp \frac{1}{2}(\nu^{-1} + \nu X) \right), \quad (5.37)$$

where  $X$  is an indeterminate variable, normalization constants  $\Phi_0$  are arbitrary, and  $s_2 = 0, \frac{1}{2}, 1$ . This result tells us that gauge invariance with parameter  $\xi_{e_1}$  fixes all coefficients inside actions for each type of supermultiplet fields and leaves arbitrary overall constants. The remaining gauge invariance imposes on them some linear relations so that all these constants are expressed via a single normalization constant.

Prior discussing the remaining gauge invariance let us make the following observation. By virtue of the  $C$ -invariance condition the invariance with respect to  $\xi_{e_1}(a, b)$  yields

the invariance with respect to bosonic parameters  $\xi_{e_{31}}(a, b)$  and  $\xi_{e_4}(a, b)$ . Indeed, suppose we proved invariance of the action with respect to  $\xi_{e_1}$ , i.e. the condition (4.10) is satisfied,  $\mathcal{A}(R, [R, \xi_{e_1}]_\star) \approx 0$ . It follows that the same is true also for another element  $R' = N \star R = R \star N$  of gauge  $cu(2, 2|8)$  superalgebra, i.e.  $\mathcal{A}(N \star R, [N \star R, \xi_{e_1}]_\star) \approx 0$ . Since  $N$  is central element of  $cu(2, 2|8)$  and by virtue of the  $C$ -invariance condition one obtains  $\mathcal{A}(R, [R, N \star N \star \xi_{e_1}]) \approx 0$  for some new gauge parameter  $\zeta = N \star N \star \xi_{e_1}$ . In fact, parameter  $\zeta$  is a combination of  $\xi_{e_1}$ ,  $\xi_{e_{31}}$ , and  $\xi_{e_4}$ , expressed via  $T^+$  and  $T^-$  acting on original  $\xi_{e_1}$ . The invariance with respect to  $\xi_{e_1}$ ,  $\xi_{e_{31}}$ , and  $\xi_{e_4}$  can also be checked by direct calculation: varying with respect to  $\xi_{e_{31}}$  and  $\xi_{e_4}$  gives the same relation between the respective normalization constants  $\Phi_0$  as guaranteed by the  $C$ -invariance condition (5.12) and gives equations on coefficient functions equivalent to those that follow from the variation with respect to  $\xi_{e_1}$ .

Analogous reasoning is also applied to the gauge transformations with fermionic parameters  $\xi_{o_1}$  and  $\xi_{o_2}$  and it follows that gauge invariance  $\xi_{o_2}$  is guaranteed by gauge invariance  $\xi_{o_1}$  and the  $C$ -invariance condition. As a result, we obtain that it is sufficient to check gauge invariance for three bosonic parameters  $\xi_{e_1}$ ,  $\xi_{e_{21}}$ ,  $\xi_{e_{32}i^j}$  and for one fermionic parameter  $\xi_{o_{11}i}$ . Expression for these gauge transformations are given in appendix 7. The invariance associated with other gauge parameters is guaranteed through the  $C$ -invariance condition. In fact, imposing the gauge invariance with respect the above parameters leaves just four independent constants  $\Phi_0$  (5.37) in front of actions  $\mathcal{B}_1$ ,  $\mathcal{B}_2$ ,  $\mathcal{B}_{32}$ , and  $\mathcal{F}_1$ . They will be respectively denoted as  $\Phi_0^{e_1}$ ,  $\Phi_0^{e_2}$ ,  $\Phi_0^{e_{32}}$ , and  $\Phi_0^{o_1}$ .

Now we discuss the gauge invariance and linear relations on four normalization constants imposed by each type of gauge symmetry. In order to find these relations one needs to use the following identities between functions  $\Psi_0(X)$ ,  $\Psi_{\frac{1}{2}}(X)$ ,  $\Psi_1(X)$  and their derivatives with different values of a second spin

$$\begin{aligned} X \frac{\partial \Psi_{\frac{1}{2}}(X)}{\partial X} + \Psi_{\frac{1}{2}}(X) &= \frac{1}{2} \Psi_0(X), & \frac{\partial \Psi_0(X)}{\partial X} &= \frac{1}{2} \Psi_{\frac{1}{2}}(X), \\ X \frac{\partial \Psi_1(X)}{\partial X} + 2\Psi_1(X) &= \frac{1}{2} \Psi_{\frac{1}{2}}(X), & \frac{\partial \Psi_{\frac{1}{2}}(X)}{\partial X} &= \frac{1}{2} \Psi_1(X). \end{aligned} \tag{5.38}$$

Let us shortly discuss each of four types of gauge symmetry. Firstly, consider gauge symmetry with parameter  $\xi_{e_{21}} = \xi_{e_{21}}(a, b)$  and its conjugated cousin. Because this symmetry is bosonic it follows that fermionic and bosonic sectors of the full action (4.12) transform independently. In the fermionic sector the gauge symmetry mixes up fields  $\Omega_{o_1}$  and  $\Omega_{o_2}$  (7.5), (7.6) and by direct calculation one obtains that fermionic sector is invariant provided normalization constants are related as  $\mathcal{F}_1 = 4\mathcal{F}_2$ , cf. (5.12). In the bosonic sector the gauge symmetry mixes up four fields  $\Omega_{e_1}$ ,  $\Omega_{e_2}$ ,  $\Omega_{e_{31}}$ , and  $\Omega_{e_4}$  (7.7). Calculating the respective action's variation, using identities (5.38) and the  $C$ -invariance condition one obtains

$$\Phi_0^{e_2} = 2\Phi_0^{e_1}, \tag{5.39}$$

while  $\mathcal{B}_{31} = \frac{1}{2}\mathcal{B}_1$  and  $\mathcal{B}_4 = \frac{1}{4}\mathcal{B}_1$ . It follows that normalization constants in this sector of fields are totally fixed in terms of  $\Phi_0^{e_1}$ .

Quite analogously we consider gauge symmetry with  $su(2)$  matrix-valued parameter  $\xi_{e_{32}j^i} = \xi_{e_{32}j^i}(a, b)$ . Since this symmetry is bosonic it follows that fermionic and bosonic



sectors of the full action (4.12) transform independently. In the fermionic sector the gauge symmetry mixes up fields  $\Omega_{o_1}$  and  $\Omega_{o_2}$  (7.8) and by direct calculation one obtains that fermionic sector is invariant provided normalization constants are related as  $\mathcal{F}_1 = 4\mathcal{F}_2$ , cf. (5.12). In the bosonic sector the gauge symmetry mixes up four fields  $\Omega_{e_1}$ ,  $\Omega_{e_{32}i^j}$ , and  $\Omega_{e_4}$  (7.9). Calculating the respective action's variation, using identities (5.38) and the  $C$ -invariance condition one obtains

$$\Phi_0^{e_{32}} = \frac{1}{4}\Phi_0^{e_1}, \tag{5.40}$$

while  $\mathcal{B}_4 = \frac{1}{4}\mathcal{B}_1$ . It follows that normalization constants in this sector of fields are completely fixed in terms of  $\Phi_0^{e_1}$ . It also implies that all bosonic coefficients are fixed uniquely and the overall normalization constant is  $\Phi_0^{e_1}$ .

Finally, we analyze fermionic  $su(2)$  vector-valued parameter  $\xi_{o_{11}}^i = \xi_{o_{11}}^i(a, b)$  and its conjugated one. The respective gauge transformation is supersymmetric and mixes up all bosonic fields and all fermionic fields, see (7.10). Calculating the respective action's variation, using identities (5.38) and the  $C$ -invariance condition one obtains

$$\Phi_0^{o_1} = -\Phi_0^{e_1}, \tag{5.41}$$

$\mathcal{B}_{31} = \frac{1}{2}\mathcal{B}_1$ ,  $\mathcal{B}_4 = \frac{1}{4}\mathcal{B}_1$ , and  $\mathcal{F}_1 = 4\mathcal{F}_2$ , cf. (5.12). It follows that all normalization constants are fixed uniquely and expressed in terms of  $\Phi_0^{e_1}$  to be denoted as

$$\Phi_0 \equiv \Phi_0^{e_1}. \tag{5.42}$$

The final expressions for coefficient functions are collected in section 4.1.

## 6 Conclusion

In this paper we built and analyzed FV-type formulation of  $AdS_5$  totally symmetric and mixed-symmetry massless fields interacting between themselves and with the gravity. Our consideration is performed in the cubic order approximation. We considered two models with gauge symmetry corresponding to reduced and unreduced  $\mathcal{N} = 2$  Fradkin-Linetsky higher spin superalgebras,  $cu(2, 2|8)$  and  $hu_0(2, 2|8)$ . We have built the projecting operator that explicitly factorizes unreduced superalgebra  $cu(2, 2|8)$  to obtain reduced superalgebra  $hu_0(2, 2|8)$ . Moreover, we have found projecting operators for any  $\mathcal{N}$ .

It is worth noting that constructing the interaction vertices brings to light very powerful algebraic tools like Howe dual pairs of classical Lie (super)algebras realized on a superspace of auxiliary variables. One of the most important implications of Howe duality is the  $gl(1)$  invariance condition referred to as the  $C$ -invariance condition for the action functional (4.11). This condition is the direct analog of the  $sp(2)$  invariance for Vasiliev equations for totally symmetric fields [106]. Indeed,  $N$  is the basis element of  $gl(1)$  considered as Howe dual algebra to  $su(2, 2|2)$  superalgebra in the star product realization. Then the condition  $[N, F]_\star = 0$  (3.6) tells us that fields are  $gl(1)$  invariants and this invariance should be retained on the action level via the  $C$ -invariance condition.

Let us now discuss some future research directions. First of all, it would be worth pursuing our analysis to  $\mathcal{N} > 2$  thereby including mixed-symmetry fields of any value of the second spin  $s_2$  and not only "hook" fields with  $s_2 = 1$ . Further progress depends on establishing for spins  $s_2 > 1$  the proposition analogous to those of section 2.7. Namely, it is necessary to formulate a proper set of constraints for unfolded fields such that one obtains correct on-shell dynamics. We hope to return to this problem elsewhere.

Much more important and difficult task however is to construct nonlinear equations of motion for mixed-symmetry fields in all orders thereby extending Vasiliev equations for totally symmetric fields [106]. Contrary to the on-shell theory one may consider also the so-called off-shell formulation of higher spin dynamics that introduces higher spin fields and their non-linear gauge symmetries without imposing any field equations. It will be interesting to develop the off-shell nonlinear formulation for mixed-symmetry fields both in Minkowski and AdS spacetimes as it has been done in the case of totally symmetric fields [107–109].

It would be useful to extend results of the present paper to higher dimensions  $d > 5$  and consider a FV-type theory based on the higher spin algebra  $hu(1|(1, 2) : [M, 2])$  from [100]. Gauging this algebra yields generalized "hook" massless fields in  $AdS_d$  spacetime, which are fields with one row of any length and one column of any height (in fact, the height is bounded from below by a dimension  $d$ ).

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## 7 Coefficients and gauge transformations

**Coefficient functions.** Spin- $(s, 0)$  case, see [36]:

$$\Phi(X) = -X(\alpha(X, -X) - 2\gamma(X, -X)) . \tag{7.1}$$

Spin- $(s, \frac{1}{2})$  case, see [37]:

$$\Phi(X) = X(\alpha(X, -X) + \zeta(X, -X)) . \tag{7.2}$$

Spin- $(s, 1)$  case, see (5.24):

$$\Phi(X) = X(\alpha(X, -X) - \zeta(X, -X)) . \tag{7.3}$$

In what follows we list explicit expressions for gauge transformations. We use commutators  $[F, G]_\star = F \star G - G \star F$  and anticommutators  $\{F, G\}_\star = F \star G + G \star F$ .

The gauge symmetry with parameter  $\xi_{e_1}(a, b)$ .

$$\begin{aligned}
 \delta R_{e_1} &= [R_{e_1}, \xi_{e_1}]_\star, & \delta R_{e_4} &= [R_{e_4}, \xi_{e_1}]_\star, \\
 \delta R_{e_{21}} &= [R_{o_{21}}, \xi_{e_1}]_\star, & \delta R_{e_{22}} &= [R_{o_{22}}, \xi_{e_1}]_\star, \\
 \delta R_{e_{31}} &= [R_{e_{31}}, \xi_{e_1}]_\star, & \delta R_{e_{32}^i j} &= [R_{e_{32}^i j}, \xi_{e_1}]_\star, \\
 \delta R_{o_{11}}^i &= [R_{o_{11}}^i, \xi_{e_1}]_\star, & \delta R_{o_{12} i} &= [R_{o_{12} i}, \xi_{e_1}]_\star, \\
 \delta R_{o_{21}}^i &= [R_{o_{21}}^i, \xi_{e_1}]_\star, & \delta R_{o_{22} i} &= [R_{o_{22} i}, \xi_{e_1}]_\star,
 \end{aligned} \tag{7.4}$$

The gauge symmetry with parameter  $\xi_{e_{21}}(a, b)$  in the fermionic sector.

$$\delta R_{o_{11}}^i = -\epsilon^{ij} \{R_{o_{12} j}, \xi_{e_{21}}\}_\star - \frac{1}{2} \epsilon^{ij} [R_{o_{22} j}, \xi_{e_{21}}]_\star, \quad \delta R_{o_{12} i} = 0, \tag{7.5}$$

and

$$\delta R_{o_{21}}^i = 2\epsilon^{ij} [R_{o_{12} j}, \xi_{e_{21}}]_\star + \epsilon^{ij} \{R_{o_{22} j}, \xi_{e_{21}}\}_\star, \quad \delta R_{o_{22} i} = 0. \tag{7.6}$$

The analogous transformations hold for the conjugated gauge parameter  $\xi_{e_{22}}$ .

The gauge symmetry with parameter  $\xi_{e_{21}}(a, b)$  in the bosonic sector.

$$\begin{aligned}
 \delta R_{e_1} &= -[R_{e_{22}}, \xi_{e_{21}}]_\star, \\
 \delta R_{e_{21}} &= [R_{e_1}, \xi_{e_{21}}]_\star + \{R_{e_{31}}, \xi_{e_{21}}\}_\star + \frac{1}{2} [R_{e_4}, \xi_{e_{21}}]_\star, \\
 \delta R_{e_{22}} &= 0, & \delta R_{e_{32}^i j} &= 0, \\
 \delta R_{e_{31}} &= 2\{R_{e_{22}}, \xi_{e_{21}}\}_\star, & \delta R_{e_4} &= -2[R_{e_{22}}, \xi_{e_{21}}]_\star.
 \end{aligned} \tag{7.7}$$

The gauge symmetry with parameter  $\xi_{e_{32}}(a, b)$  in the fermionic sector. The symmetry associated with parameter  $\xi_{e_{32}}(a, b, \psi) = \xi_{e_{32}^i j}(a, b)(\psi_i \bar{\psi}^j)$ , where we assume that all  $su(2)$  traces are zero, has the following form

$$\begin{aligned}
 \delta R_{o_{11}}^i &= -\frac{1}{2} \{R_{o_{11}}^j, \xi_j^i\}_\star - \frac{1}{4} [R_{o_{21}}^j, \xi_j^i]_\star, \\
 \delta R_{o_{12} i} &= \frac{1}{2} \{R_{o_{12} j}, \xi_j^i\}_\star - \frac{1}{4} [R_{o_{22} j}, \xi_j^i]_\star, \\
 \delta R_{o_{21}}^i &= -[R_{o_{11}}^j, \xi_j^i]_\star - \frac{1}{2} \{R_{o_{21}}^j, \xi_j^i\}_\star, \\
 \delta R_{o_{22} i} &= -[R_{o_{12} j}, \xi_j^i]_\star + \frac{1}{2} \{R_{o_{22} j}, \xi_j^i\}_\star.
 \end{aligned} \tag{7.8}$$

The gauge symmetry for  $\xi_{e_{32}}(a, b)$  in the bosonic sector. The symmetry associated with parameter  $\xi_{e_{32}}(a, b, \psi) = \xi_{e_{32}^i j}(a, b)(\psi_i \bar{\psi}^j)$ , where we assume that all  $su(2)$  traces are zero, has the following form

$$\begin{aligned}
 \delta R_{e_1} &= \frac{1}{4} [R_{e_{32}^m n}, \xi_{e_{32}^n m}]_\star, & \delta R_{e_{21}} &= 0, & \delta R_{e_{31}} &= 0, \\
 \delta R_{e_{32}^i j} &= [R_{e_1}, \xi_{e_{32}^i j}]_\star - \frac{1}{2} [R_{e_4}, \xi_{e_{32}^i j}]_\star + \\
 &+ \frac{1}{2} \left( \{R_{e_{32}^i n}, \xi_{e_{32}^n j}\}_\star - \frac{1}{2} \delta_j^i \{R_{e_{32}^m n}, \xi_{e_{32}^n m}\}_\star \right) - \\
 &- \frac{1}{2} \left( \{R_{e_{32}^m j}, \xi_{e_{32}^i m}\}_\star - \frac{1}{2} \delta_j^i \{R_{e_{32}^m n}, \xi_{e_{32}^n m}\}_\star \right), \\
 \delta R_{e_4} &= -\frac{1}{2} [R_{e_{32}^m n}, \xi_{e_{32}^n m}]_\star.
 \end{aligned} \tag{7.9}$$

**Supersymmetry transformations.** Let us choose supersymmetric parameter in the form  $\xi_{o_{12}} = \xi_i(a, b)\bar{\psi}^i$ .

$$\begin{aligned}
 \delta R_{e_1} &= \frac{1}{2}[R_{o_{11}}^i, \xi_i]_\star, & \delta R_{e_4} &= \frac{1}{2}\{R_{o_{21}}^i, \xi_i\}_\star, \\
 \delta R_{o_{11}}^i &= \epsilon^{ij}\{R_{e_{21}}, \xi_j\}_\star, & \delta R_{o_{12}i} &= [R_{e_1}, \xi_i]_\star - \frac{1}{2}\{R_{e_{32}}{}^m{}_i, \xi_m\}_\star - \frac{1}{2}\{R_{e_{31}}, \xi_i\}_\star, \\
 \delta R_{e_{21}} &= 0, & \delta R_{e_{22}} &= \frac{1}{2}\epsilon^{ij}\{R_{o_{12}i}, \xi_j\}_\star - \frac{1}{4}\epsilon^{ij}[R_{o_{22}i}, \xi_j]_\star, \\
 \delta R_{e_{31}} &= \frac{1}{4}[R_{o_{21}}^i, \xi_i]_\star + \frac{1}{2}\{R_{o_{11}}^m, \xi_m\}_\star, \\
 \delta R_{e_{32}}{}^i{}_j &= \{R_{o_{11}}^i, \xi_j\}_\star - \frac{1}{2}\left([R_{o_{21}}^i, \xi_j]_\star - \frac{1}{2}\delta_j^i [R_{o_{21}}^m, \xi_m]_\star\right), & (7.10) \\
 \delta R_{o_{21}}^i &= 2\epsilon^{ij}[R_{e_{21}}, \xi_j]_\star, & \delta R_{o_{22}i} &= [R_{e_{31}}, \chi_i]_\star - [R_{e_{32}}{}^m{}_i, \xi_m]_\star - [R_{e_4}, \xi_i]_\star.
 \end{aligned}$$

## References

- [1] M.A. Vasiliev, *Higher spin gauge theories: star-product and AdS space*, [hep-th/9910096](#) [[SPIRES](#)].
- [2] D.J. Gross, *High-energy symmetries of string theory*, *Phys. Rev. Lett.* **60** (1988) 1229 [[SPIRES](#)].
- [3] B. Sundborg, *Stringy gravity, interacting tensionless strings and massless higher spins*, *Nucl. Phys. Proc. Suppl.* **102** (2001) 113 [[hep-th/0103247](#)] [[SPIRES](#)].
- [4] I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical  $O(N)$  vector model*, *Phys. Lett.* **B 550** (2002) 213 [[hep-th/0210114](#)] [[SPIRES](#)].
- [5] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl. Phys.* **B 644** (2002) 303 [*Erratum ibid.* **B 660** (2003) 403] [[hep-th/0205131](#)] [[SPIRES](#)].
- [6] G. Bonelli, *On the tensionless limit of bosonic strings, infinite symmetries and higher spins*, *Nucl. Phys.* **B 669** (2003) 159 [[hep-th/0305155](#)] [[SPIRES](#)].
- [7] G. Bonelli, *On the covariant quantization of tensionless bosonic strings in AdS spacetime*, *JHEP* **11** (2003) 028 [[hep-th/0309222](#)] [[SPIRES](#)].
- [8] G. Bonelli, *On the boundary gauge dual of closed tensionless free strings in AdS*, *JHEP* **11** (2004) 059 [[hep-th/0407144](#)] [[SPIRES](#)].
- [9] M. Bianchi, *Higher spins and stringy  $AdS_5 \times S^5$* , *Fortsch. Phys.* **53** (2005) 665 [[hep-th/0409304](#)] [[SPIRES](#)].
- [10] S. Giombi and X. Yin, *Higher spin gauge theory and holography: the three-point functions*, *JHEP* **09** (2010) 115 [[arXiv:0912.3462](#)] [[SPIRES](#)].
- [11] M. Henneaux and S.-J. Rey, *Nonlinear  $W(\infty)$  algebra as asymptotic symmetry of three-dimensional higher spin anti-de Sitter gravity*, *JHEP* **12** (2010) 007 [[arXiv:1008.4579](#)] [[SPIRES](#)].
- [12] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [[arXiv:1008.4744](#)] [[SPIRES](#)].

- [13] M.R. Gaberdiel and R. Gopakumar, *An AdS<sub>3</sub> dual for minimal model CFTs*, [arXiv:1011.2986](#) [SPIRES].
- [14] S.R. Das and A. Jevicki, *Large-N collective fields and holography*, *Phys. Rev. D* **68** (2003) 044011 [[hep-th/0304093](#)] [SPIRES].
- [15] R.d.M. Koch, A. Jevicki, K. Jin and J.P. Rodrigues, *AdS<sub>4</sub>/CFT<sub>3</sub> construction from collective fields*, *Phys. Rev. D* **83** (2011) 025006 [[arXiv:1008.0633](#)] [SPIRES].
- [16] M.R. Douglas, L. Mazzucato and S.S. Razamat, *Holographic dual of free field theory*, [arXiv:1011.4926](#) [SPIRES].
- [17] C. Aragone and S. Deser, *Consistency problems of hypergravity*, *Phys. Lett. B* **86** (1979) 161 [SPIRES].
- [18] D. Sorokin, *Introduction to the classical theory of higher spins*, *AIP Conf. Proc.* **767** (2005) 172 [[hep-th/0405069](#)] [SPIRES].
- [19] X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, [arXiv:1007.0435](#) [SPIRES].
- [20] E.S. Fradkin and M.A. Vasiliev, *On the gravitational interaction of massless higher spin fields*, *Phys. Lett. B* **189** (1987) 89 [SPIRES].
- [21] E.S. Fradkin and M.A. Vasiliev, *Cubic interaction in extended theories of massless higher spin fields*, *Nucl. Phys. B* **291** (1987) 141 [SPIRES].
- [22] E.S. Fradkin and M.A. Vasiliev, *Candidate to the role of higher spin symmetry*, *Ann. Phys.* **177** (1987) 63 [SPIRES].
- [23] N. Boulanger, S. Leclercq and P. Sundell, *On the uniqueness of minimal coupling in higher-spin gauge theory*, *JHEP* **08** (2008) 056 [[arXiv:0805.2764](#)] [SPIRES].
- [24] Y.M. Zinoviev, *On spin 3 interacting with gravity*, *Class. Quant. Grav.* **26** (2009) 035022 [[arXiv:0805.2226](#)] [SPIRES].
- [25] Y.M. Zinoviev, *Spin 3 cubic vertices in a frame-like formalism*, *JHEP* **08** (2010) 084 [[arXiv:1007.0158](#)] [SPIRES].
- [26] F.A. Berends, G.J.H. Burgers and H. Van Dam, *On spin three selfinteractions*, *Z. Phys. C* **24** (1984) 247 [SPIRES].
- [27] A.K.H. Bengtsson, I. Bengtsson and N. Linden, *Interacting higher spin gauge fields on the light front*, *Class. Quant. Grav.* **4** (1987) 1333 [SPIRES].
- [28] S. Deser and Z. Yang, *Inconsistency of spin 4-spin 2 gauge field couplings*, *Class. Quant. Grav.* **7** (1990) 1491 [SPIRES].
- [29] E.S. Fradkin and R.R. Metsaev, *A cubic interaction of totally symmetric massless representations of the Lorentz group in arbitrary dimensions*, *Class. Quant. Grav.* **8** (1991) L89 [SPIRES].
- [30] X. Bekaert, N. Boulanger and S. Cnockaert, *Spin three gauge theory revisited*, *JHEP* **01** (2006) 052 [[hep-th/0508048](#)] [SPIRES].
- [31] N. Boulanger, S. Leclercq and S. Cnockaert, *Parity violating vertices for spin-3 gauge fields*, *Phys. Rev. D* **73** (2006) 065019 [[hep-th/0509118](#)] [SPIRES].
- [32] N. Boulanger and S. Leclercq, *Consistent couplings between spin-2 and spin-3 massless fields*, *JHEP* **11** (2006) 034 [[hep-th/0609221](#)] [SPIRES].

- [33] R.R. Metsaev, *Cubic interaction vertices for massive and massless higher spin fields*, *Nucl. Phys. B* **759** (2006) 147 [[hep-th/0512342](#)] [[SPIRES](#)].
- [34] R.R. Metsaev, *Cubic interaction vertices for fermionic and bosonic arbitrary spin fields*, [arXiv:0712.3526](#) [[SPIRES](#)].
- [35] R. Manvelyan, K. Mkrtchyan and W. Rühl, *General trilinear interaction for arbitrary even higher spin gauge fields*, *Nucl. Phys. B* **836** (2010) 204 [[arXiv:1003.2877](#)] [[SPIRES](#)].
- [36] M.A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in  $AdS_5$* , *Nucl. Phys. B* **616** (2001) 106 [*Erratum ibid.* **652** (2003) 407] [[hep-th/0106200](#)] [[SPIRES](#)].
- [37] K.B. Alkalaev and M.A. Vasiliev,  *$\mathcal{N} = 1$  supersymmetric theory of higher spin gauge fields in  $AdS_5$  at the cubic level*, *Nucl. Phys. B* **655** (2003) 57 [[hep-th/0206068](#)] [[SPIRES](#)].
- [38] E.S. Fradkin and V.Y. Linetsky, *Conformal superalgebras of higher spins*, *Mod. Phys. Lett. A* **4** (1989) 2363 [[SPIRES](#)].
- [39] E.S. Fradkin and V.Y. Linetsky, *Cubic interaction in conformal theory of integer higher spin fields in four-dimensional space-time*, *Phys. Lett. B* **231** (1989) 97 [[SPIRES](#)].
- [40] E. Sezgin and P. Sundell, *Doubletons and 5D higher spin gauge theory*, *JHEP* **09** (2001) 036 [[hep-th/0105001](#)] [[SPIRES](#)].
- [41] E. Sezgin and P. Sundell, *Towards massless higher spin extension of  $D = 5$ ,  $\mathcal{N} = 8$  gauged supergravity*, *JHEP* **09** (2001) 025 [[hep-th/0107186](#)] [[SPIRES](#)].
- [42] M.A. Vasiliev, *Conformal higher spin symmetries of 4D massless supermultiplets and  $osp(L, 2M)$  invariant equations in generalized (super)space*, *Phys. Rev. D* **66** (2002) 066006 [[hep-th/0106149](#)] [[SPIRES](#)].
- [43] E.S. Fradkin and R.R. Metsaev, *Cubic scattering amplitudes for all massless representations of the Poincaré group in any space-time dimension*, *Phys. Rev. D* **52** (1995) 4660 [[SPIRES](#)].
- [44] A. Fotopoulos and M. Tsulaia, *Interacting higher spins and the high energy limit of the bosonic string*, *Phys. Rev. D* **76** (2007) 025014 [[arXiv:0705.2939](#)] [[SPIRES](#)].
- [45] A. Fotopoulos and M. Tsulaia, *On the tensionless limit of string theory, off-shell higher spin interaction vertices and BCFW recursion relations*, *JHEP* **11** (2010) 086 [[arXiv:1009.0727](#)] [[SPIRES](#)].
- [46] X. Bekaert, E. Joung and J. Mourad, *On higher spin interactions with matter*, *JHEP* **05** (2009) 126 [[arXiv:0903.3338](#)] [[SPIRES](#)].
- [47] X. Bekaert and E. Meunier, *Higher spin interactions with scalar matter on constant curvature spacetimes: conserved current and cubic coupling generating functions*, *JHEP* **11** (2010) 116 [[arXiv:1007.4384](#)] [[SPIRES](#)].
- [48] R.R. Metsaev, *Massless mixed symmetry bosonic free fields in  $d$ -dimensional anti-de Sitter space-time*, *Phys. Lett. B* **354** (1995) 78 [[SPIRES](#)].
- [49] R.R. Metsaev, *Fermionic fields in the  $d$ -dimensional anti-de Sitter spacetime*, *Phys. Lett. B* **419** (1998) 49 [[hep-th/9802097](#)] [[SPIRES](#)].
- [50] L. Brink, R.R. Metsaev and M.A. Vasiliev, *How massless are massless fields in  $AdS_d$* , *Nucl. Phys. B* **586** (2000) 183 [[hep-th/0005136](#)] [[SPIRES](#)].
- [51] R.R. Metsaev, *Massless arbitrary spin fields in  $AdS_5$* , *Phys. Lett. B* **531** (2002) 152 [[hep-th/0201226](#)] [[SPIRES](#)].



- [52] R.R. Metsaev, *Mixed symmetry massive fields in AdS<sub>5</sub>*, *Class. Quant. Grav.* **22** (2005) 2777 [[hep-th/0412311](#)] [[SPIRES](#)].
- [53] M. Günaydin, P. van Nieuwenhuizen and N.P. Warner, *General construction of the unitary representations of anti-de Sitter superalgebras and the spectrum of the S<sup>4</sup> compactification of eleven-dimensional supergravity*, *Nucl. Phys. B* **255** (1985) 63 [[SPIRES](#)].
- [54] M. Günaydin, D. Minic and M. Zagermann, *4D doubleton conformal theories, CPT and II B string on AdS<sub>5</sub> × S<sup>5</sup>*, *Nucl. Phys. B* **534** (1998) 96 [Erratum *ibid.* **538** (1999) 531] [[hep-th/9806042](#)] [[SPIRES](#)].
- [55] S. Ferrara and C. Fronsdal, *Gauge fields and singletons of AdS(2p + 1)*, *Lett. Math. Phys.* **46** (1998) 157 [[hep-th/9806072](#)] [[SPIRES](#)].
- [56] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, [hep-th/0503128](#) [[SPIRES](#)].
- [57] K.B. Alkalaev, *Free fermionic higher spin fields in AdS<sub>5</sub>*, *Phys. Lett. B* **519** (2001) 121 [[hep-th/0107040](#)] [[SPIRES](#)].
- [58] K.B. Alkalaev, *Mixed-symmetry massless gauge fields in AdS<sub>5</sub>*, *Theor. Math. Phys.* **149** (2006) 1338 [*Teor. Mat. Fiz.* **149** (2006) 47] [[hep-th/0501105](#)] [[SPIRES](#)].
- [59] K.S. Stelle and P.C. West, *Spontaneously broken de Sitter symmetry and the gravitational holonomy group*, *Phys. Rev. D* **21** (1980) 1466 [[SPIRES](#)].
- [60] C.R. Preitschopf and M.A. Vasiliev, *The superalgebraic approach to supergravity*, [hep-th/9805127](#) [[SPIRES](#)].
- [61] C. Fronsdal, *Massless fields with integer spin*, *Phys. Rev. D* **18** (1978) 3624 [[SPIRES](#)].
- [62] J. Fang and C. Fronsdal, *Massless fields with half integral spin*, *Phys. Rev. D* **18** (1978) 3630 [[SPIRES](#)].
- [63] V.E. Lopatin and M.A. Vasiliev, *Free massless bosonic fields of arbitrary spin in d-dimensional de Sitter space*, *Mod. Phys. Lett. A* **3** (1988) 257 [[SPIRES](#)].
- [64] M.A. Vasiliev, *Free massless fermionic fields of arbitrary spin in d-dimensional de Sitter space*, *Nucl. Phys. B* **301** (1988) 26 [[SPIRES](#)].
- [65] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *On the frame-like formulation of mixed-symmetry massless fields in (A)dS<sub>d</sub>*, *Nucl. Phys. B* **692** (2004) 363 [[hep-th/0311164](#)] [[SPIRES](#)].
- [66] R.R. Metsaev, *Light cone form of field dynamics in anti-de Sitter spacetime and AdS/CFT correspondence*, *Nucl. Phys. B* **563** (1999) 295 [[hep-th/9906217](#)] [[SPIRES](#)].
- [67] A.Y. Segal, *Point particle in general background fields vs. free gauge theories of traceless symmetric tensors*, *Int. J. Mod. Phys. A* **18** (2003) 4999 [[hep-th/0110056](#)] [[SPIRES](#)].
- [68] M.A. Vasiliev, *Relativity, causality, locality, quantization and duality in the Sp(2M) invariant generalized space-time*, [hep-th/0111119](#) [[SPIRES](#)].
- [69] Y.M. Zinoviev, *On massive high spin particles in (A)dS*, [hep-th/0108192](#) [[SPIRES](#)].
- [70] D. Francia and A. Sagnotti, *Free geometric equations for higher spins*, *Phys. Lett. B* **543** (2002) 303 [[hep-th/0207002](#)] [[SPIRES](#)].

- [71] I.L. Buchbinder, A. Pashnev and M. Tsulaia, *Massless higher spin fields in the AdS background and BRST constructions for nonlinear algebras*, [hep-th/0206026](#) [SPIRES].
- [72] G. Barnich, M. Grigoriev, A. Semikhatov and I. Tipunin, *Parent field theory and unfolding in BRST first-quantized terms*, *Commun. Math. Phys.* **260** (2005) 147 [[hep-th/0406192](#)] [SPIRES].
- [73] G. Barnich, G. Bonelli and M. Grigoriev, *From BRST to light-cone description of higher spin gauge fields*, [hep-th/0502232](#) [SPIRES].
- [74] G. Barnich and M. Grigoriev, *Parent form for higher spin fields on anti-de Sitter space*, *JHEP* **08** (2006) 013 [[hep-th/0602166](#)] [SPIRES].
- [75] I.L. Buchbinder, A.V. Galajinsky and V.A. Krykhtin, *Quartet unconstrained formulation for massless higher spin fields*, *Nucl. Phys. B* **779** (2007) 155 [[hep-th/0702161](#)] [SPIRES].
- [76] S. Ouvry and J. Stern, *Gauge fields of any spin and symmetry*, *Phys. Lett. B* **177** (1986) 335 [SPIRES].
- [77] J.M.F. Labastida, *Massless bosonic free fields*, *Phys. Rev. Lett.* **58** (1987) 531 [SPIRES].
- [78] J.M.F. Labastida and T.R. Morris, *Massless mixed symmetry bosonic free fields*, *Phys. Lett. B* **180** (1986) 101 [SPIRES].
- [79] J.M.F. Labastida, *Massless particles in arbitrary representations of the Lorentz group*, *Nucl. Phys. B* **322** (1989) 185 [SPIRES].
- [80] W. Siegel and B. Zwiebach, *Gauge string fields from the light cone*, *Nucl. Phys. B* **282** (1987) 125 [SPIRES].
- [81] A. Pashnev and M. Tsulaia, *Description of the higher massless irreducible integer spins in the BRST approach*, *Mod. Phys. Lett. A* **13** (1998) 1853 [[hep-th/9803207](#)] [SPIRES].
- [82] K.B. Alkalaev, *Two-column higher spin massless fields in AdS<sub>d</sub>*, *Theor. Math. Phys.* **140** (2004) 1253 [*Teor. Mat. Fiz.* **140** (2004) 424] [[hep-th/0311212](#)] [SPIRES].
- [83] X. Bekaert and N. Boulanger, *Tensor gauge fields in arbitrary representations of GL(D, R). II: quadratic actions*, *Commun. Math. Phys.* **271** (2007) 723 [[hep-th/0606198](#)] [SPIRES].
- [84] Y.M. Zinoviev, *Toward frame-like gauge invariant formulation for massive mixed symmetry bosonic fields*, *Nucl. Phys. B* **812** (2009) 46 [[arXiv:0809.3287](#)] [SPIRES].
- [85] E.D. Skvortsov, *Mixed-symmetry massless fields in Minkowski space unfolded*, *JHEP* **07** (2008) 004 [[arXiv:0801.2268](#)] [SPIRES].
- [86] E.D. Skvortsov, *Frame-like actions for massless mixed-symmetry fields in Minkowski space*, *Nucl. Phys. B* **808** (2009) 569 [[arXiv:0807.0903](#)] [SPIRES].
- [87] K.B. Alkalaev, M. Grigoriev and I.Y. Tipunin, *Massless Poincaré modules and gauge invariant equations*, *Nucl. Phys. B* **823** (2009) 509 [[arXiv:0811.3999](#)] [SPIRES].
- [88] A. Campoleoni, D. Francia, J. Mourad and A. Sagnotti, *Unconstrained higher spins of mixed symmetry. I. Bose fields*, *Nucl. Phys. B* **815** (2009) 289 [[arXiv:0810.4350](#)] [SPIRES].
- [89] N. Boulanger, C. Iazeolla and P. Sundell, *Unfolding mixed-symmetry fields in AdS and the BMV conjecture: I. General formalism*, *JHEP* **07** (2009) 013 [[arXiv:0812.3615](#)] [SPIRES].
- [90] N. Boulanger, C. Iazeolla and P. Sundell, *Unfolding mixed-symmetry fields in AdS and the BMV conjecture: II. Oscillator realization*, *JHEP* **07** (2009) 014 [[arXiv:0812.4438](#)] [SPIRES].



- [91] I.L. Buchbinder, V.A. Krykhtin and L.L. Ryskina, *BRST approach to Lagrangian formulation of bosonic totally antisymmetric tensor fields in curved space*, *Mod. Phys. Lett. A* **24** (2009) 401 [[arXiv:0810.3467](#)] [[SPIRES](#)].
- [92] K.B. Alkalaev and M. Grigoriev, *Unified BRST description of AdS gauge fields*, *Nucl. Phys. B* **835** (2010) 197 [[arXiv:0910.2690](#)] [[SPIRES](#)].
- [93] F. Bastianelli, O. Corradini and A. Waldron, *Detours and paths: BRST complexes and worldline formalism*, *JHEP* **05** (2009) 017 [[arXiv:0902.0530](#)] [[SPIRES](#)].
- [94] E.D. Skvortsov, *Gauge fields in (anti)-de Sitter space and connections of its symmetry algebra*, *J. Phys. A* **42** (2009) 385401 [[arXiv:0904.2919](#)] [[SPIRES](#)].
- [95] R. Howe, *Transcending classical invariant theory*, *J. Amer. Math. Soc.* **2** (1989) 535.
- [96] R. Howe, *Remarks on classical invariant theory*, *Trans. Amer. Math. Soc.* **313** (1989) 539 [*Erratum ibid.* **318** (1990) 823].
- [97] M.A. Vasiliev, *Free massless fields of arbitrary spin in the de Sitter space and initial data for a higher spin superalgebra*, *Fortsch. Phys.* **35** (1987) 741 [*Yad. Fiz.* **45** (1987) 1784] [[SPIRES](#)].
- [98] Y.M. Zinoviev, *Towards frame-like gauge invariant formulation for massive mixed symmetry bosonic fields. II. General Young tableau with two rows*, *Nucl. Phys. B* **826** (2010) 490 [[arXiv:0907.2140](#)] [[SPIRES](#)].
- [99] M.A. Vasiliev, *Extended higher spin superalgebras and their realizations in terms of quantum operators*, *Fortsch. Phys.* **36** (1988) 33 [[SPIRES](#)].
- [100] M.A. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, *JHEP* **12** (2004) 046 [[hep-th/0404124](#)] [[SPIRES](#)].
- [101] F.A. Berezin, A.A. Kirillov and D. Leites eds., *Introduction to superanalysis*, *Mathematical Physics and Applied Mathematics* **9**, Reidel, Dordrecht The Netherlands (1987) [[SPIRES](#)].
- [102] L.J. Romans, *Gauged  $\mathcal{N} = 4$  supergravities in five-dimensions and their magnetovac backgrounds*, *Nucl. Phys. B* **267** (1986) 433 [[SPIRES](#)].
- [103] K.B. Alkalaev, *On manifestly  $Sp(2)$  invariant formulation of quadratic higher spin Lagrangians*, *JHEP* **06** (2008) 081 [[arXiv:0711.3639](#)] [[SPIRES](#)].
- [104] D.P. Sorokin and M.A. Vasiliev, *Reducible higher-spin multiplets in flat and AdS spaces and their geometric frame-like formulation*, *Nucl. Phys. B* **809** (2009) 110 [[arXiv:0807.0206](#)] [[SPIRES](#)].
- [105] R. Jackiw, *Gauge covariant conformal transformations*, *Phys. Rev. Lett.* **41** (1978) 1635 [[SPIRES](#)].
- [106] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in  $(A)dS_d$* , *Phys. Lett. B* **567** (2003) 139 [[hep-th/0304049](#)] [[SPIRES](#)].
- [107] A. Sagnotti, E. Sezgin and P. Sundell, *On higher spins with a strong  $Sp(2, R)$  condition*, [hep-th/0501156](#) [[SPIRES](#)].
- [108] M.A. Vasiliev, *Actions, charges and off-shell fields in the unfolded dynamics approach*, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 37 [[hep-th/0504090](#)] [[SPIRES](#)].
- [109] M. Grigoriev, *Off-shell gauge fields from BRST quantization*, [hep-th/0605089](#) [[SPIRES](#)].