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Moduli spaces of instantons in flag manifold sigma models. Vortices in quiver gauge theories

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Abstract: In this paper, we discuss lumps (sigma model instantons) in flag manifold sigma models. In particular, we focus on the moduli space of BPS lumps in general Kähler flag manifold sigma models. Such a Kähler flag manifold, which takes the form $\frac{\mathrm{U}(n_1+\cdots+n_{L+1})}{\mathrm{U}(n_1)\times\cdots\times\mathrm{U}(n_{L+1})}$, can be realized as a vacuum moduli space of a $\mathrm{U}(N_1)\times\cdots\times\mathrm{U}(N_L)$ quiver gauged linear sigma model. When the gauge coupling constants are finite, the gauged linear sigma model admits BPS vortex configurations, which reduce to BPS lumps in the low energy effective sigma model in the large gauge coupling limit. We derive an ADHM-like quotient construction of the moduli space of BPS vortices and lumps by generalizing the quotient construction in $\mathrm{U}(N)$ gauge theories by Hanany and Tong. As an application, we check the dualities of the 2d models by computing the vortex partition functions using the quotient construction.

Keywords: Solitons Monopoles and Instantons, Sigma Models

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1 Introduction

Since their discovery, nonlinear sigma models (NLσMs) have been studied extensively in diverse subjects, including high energy physics and condensed matter physics. In high energy physics, NLσMs in two dimensions share many non-perturbative properties with gauge theories in four dimensions, such as asymptotic freedom, dynamical mass gap, confinement and instantons [1–3], and thus they are investigated as toy models of gauge theories in four dimensions. NLσMs

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are defined by a map from spacetime to target spaces. Among possible target spaces, the $\mathbb{C}P^{N-1}$ model with the complex projective space $\mathbb{C}P^{N-1} \simeq \mathrm{SU}(N)/[\mathrm{SU}(N-1)\times \mathrm{U}(1)]$ as the target space has been most considered [1, 2, 4–6] together with the $O(N)$ model with $S^{N-1} \simeq O(N)/O(N-1)$ target space. In particular, the $\mathbb{C}P^{N-1}$ model has instanton solutions, which play a central role in the non-perturbative dynamics of the model. The $\mathbb{C}P^{N-1}$ model appears as the effective theory of a single non-Abelian vortex in supersymmetric $\mathrm{U}(N)$ gauge theories [7–14], dense QCD at high density [15–20], and two-Higgs doublets models [21–23]. In the recent development of the resurgence theory, the $\mathbb{C}P^{N-1}$ model on $\mathbb{R}^1 \times S^1$ with a twisted boundary condition along S^1 has been extensively discussed, where a single $\mathbb{C}P^{N-1}$ instanton is decomposed into N fractional instantons with induced domain wall charges that sum to zero [24, 25]. Then, a pair of fractional instanton and anti-instanton called a bion may play an essential role in the resurgence theory [26–34]. Self-consistent non-homogeneous solutions of the $\mathbb{C}P^{N-1}$ model were discussed in the large-N limit in infinite space [35] and a finite interval [36–40]. In condensed matter physics, the $\mathbb{C}P^{N-1}$ model appears in spin chains [41, 42], deconfined criticality [43–45], $\mathrm{SU}(N)$ Heisenberg models [46] and ultracold atomic gases [47, 48].

Recently, yet another class of target spaces, flag manifolds, have attracted great attention from both high energy and condensed matter physics [49]. The flag manifold sigma models are NLσMs whose target space is the generalized flag manifold $\mathcal{F}_{n_1 n_2 \cdots n_{L+1}}$, which is a homogeneous space G/H of the form

$$\mathcal{F}_{n_1 n_2 \cdots n_{L+1}} \equiv G/H \cong \frac{\mathrm{U}(n_1+n_2+\cdots+n_{L+1})}{\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\cdots\times\mathrm{U}(n_{L+1})} \cong \frac{\mathrm{SU}(n_1+n_2+\cdots+n_{L+1})}{S[\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\cdots\times\mathrm{U}(n_{L+1})]}. \tag{1.1}$$

The flag manifold sigma models appear in various fields of physics as low-energy effective theories [50–61]: spin chains [50, 60], flag manifold sigma model on $\mathbb{R}\times S^1$ [54], anomaly and topological θ term [55, 61], world-sheet theories of composite non-Abelian vortices [62, 63], and a non-Abelian vortex lattice [64]. As in other sigma models, the flag manifold sigma models admit topologically non-trivial configurations [57–59]. In particular, there exist sigma model lumps (also called sigma model instantons in two dimensions) characterized by the second homotopy class $\pi_2(G/H)$ of the flag manifolds

$$\pi_2(G/H) = \pi_1(H) = \pi_1\left(S[\mathrm{U}(n_1)\times\mathrm{U}(n_2)\times\cdots\times\mathrm{U}(n_{L+1})]\right) = \mathbb{Z}^L. \tag{1.2}$$

In the case of $L=1$, the target space is a Grassmaniann for which lumps have been studied in refs. [65, 66]. For $L>1$, various properties of lumps have been elucidated in [57, 58]. In the previous works, many authors have focused on the symmetric points in the space of sigma model coupling constants (decay constants), such as the $\mathbb{Z}_3$ symmetric point in the $\mathrm{SU}(3)/\mathrm{U}(1)^2$ sigma model [57, 58].

There is another special subspace in the parameter space related to supersymmetric versions of the flag manifold sigma models [67–72]. When the coupling constants satisfy a certain relation, the target space becomes a Kähler manifold [70, 71] for which the model can be made supersymmetric [73]. In such Kähler sigma models, sigma model lumps are Bogomol'nyi-Prasad-Sommerfield (BPS) objects, whose moduli spaces, in general, have rich structures due to the property that no static force is exerted among BPS objects. In this paper, we study the moduli space of BPS lumps in the flag manifold sigma models.

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A convenient way to describe NLσMs, particularly Kähler sigma models, is to use gauged linear sigma models (GLσMs) whose moduli space of vacua gives the target space [74]. When the gauge coupling constants are finite, the GLσMs admit semi-local vortex solutions [65, 66, 75, 76] characterized by the fundamental group π_1 of the spontaneously broken gauge group. Since they reduce to the sigma model lumps in the large gauge coupling limit (or low-energy limit), the moduli space of BPS vortices is equivalent to that of BPS lumps except for the small lump singularities which are resolved by the finite gauge coupling constants. In the case of $L = 1$ (and the case of local vortices), the moduli space of BPS vortices is conjectured in terms of a D-brane configuration in string theory [7], which is described by half of the Atiyah-Drinfeld-Hitchin-Mannin(ADHM) construction for Yang-Mills instantons. This half-ADHM formalism was shown to coincide [9, 10] with one obtained in a purely field-theoretic way called the moduli matrix approach [9, 10, 12, 77] and has been used to analyze the structure of the vortex moduli spaces [10, 62, 66, 78].

In this paper, we consider GLσMs that realize the Kählercoset manifolds with arbitrary complex structures as its target manifolds: we formulate the flag manifold sigma models by quiver gauge theories [74]. We then construct BPS vortices (lumps, instantons), obtain their moduli space through the moduli matrix approach, and reformulate it from the viewpoint of the ADHM-like construction. As applications of the half ADHM moduli space, we compute vortex partition functions and use them to check the Seiberg-like duality in two dimensions.

This paper is organized as follows. In section 2, we formulate the flag manifold sigma models by quiver GLσMs. In section 3, we construct BPS vortices and NLσM instantons in the flag manifold sigma models. In section 4, the half-ADHM quotient construction of the moduli space of BPS vortices is formulated in the quiver GLσMs and the flag manifold NLσMs, and in section 5, the moduli space of sigma model instantons is discussed. In section 6, we calculate the vortex partition functions and check the Seiberg-like duality. Section 7 is devoted to summary and discussion. In appendix A, we clarify the relation between Kählerand Riemannian flag manifolds. In appendix B, we give comments on the proposition on the existence of the BPS solutions addressed in the main text and on the non-existence of other solutions. Appendix C summarizes coordinate patches of half-ADHM data. In appendix D, we give a condition of non-singular instantons. Appendices C and D focus on the case of L = 1, which forms the foundation of general cases with $L > 1$. We give explicit proofs of the theorems related to the equivalence of the moduli spaces of the moduli matrix and the half-ADHM data, discussed previously in [9, 12, 66], in a more comprehensive manner for the sake of self-containment. In appendix E, we give embeddings of the moduli matrix and the half-ADHM data in the case of $L = 1$ to those in the general cases. Appendix F describes a D-brane configuration in string theory that provides a quotient construction of the moduli space of BPS vortices and flag manifold sigma model instantons. In appendix G, a Lagrange multiplier and its vanishing theorem are described. In appendix H, we summarize the torus action on the Kählerquotient corresponding to the vortex moduli space. In appendix I, we derive the integration formula for the vortex partition function.

2 Quiver gauge theories and flag manifold sigma models

In this section, we present the gauged linear sigma model (GLσM) description of the flag manifold sigma model.

2.1 Flag manifolds

Before describing the GLσM for flag manifolds, we first recapitulate the basics of flag manifolds. Let $\mathcal{V}$ be an N-dimensional complex linear space and $(\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_L, \mathcal{V}_{L+1})$ be a flag, i.e. a sequence of vector spaces such that

$$\{0\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_L \subset \mathcal{V}_{L+1} = \mathcal{V}, \tag{2.1}$$

where $\mathcal{V}_i$ $(i = 0, 1, \cdots, L+1)$ are linear subspaces with $\dim_{\mathbb{C}} \mathcal{V}_i = N_i$ satisfying

$$0 = N_0 < N_1 < \cdots < N_L < N_{L+1} = N. \tag{2.2}$$

A flag manifold is the space of possible configurations of the flag

$$\mathcal{F}_{n_1 n_2 \cdots n_{L+1}} \equiv \Big\{ (\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_{L+1}) \,\Big|\, \mathcal{V}_i : \text{vector space},\ \{0\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_L \subset \mathcal{V}_{L+1} = \mathcal{V} \Big\}. \tag{2.3}$$

In this paper, we label flag manifolds by a sequence of integers $(n_1, n_2, \cdots, n_L)$ defined by

$$n_i \equiv \dim_{\mathbb{C}} \mathcal{W}_i = N_i - N_{i-1}, \qquad \left(N_i = \dim_{\mathbb{C}} \mathcal{V}_i = \sum_{j=1}^{i} n_j \right), \tag{2.4}$$

where $\mathcal{W}_i$ is the orthogonal complements of $\mathcal{V}_{i-1}$ in $\mathcal{V}_i$ $(\mathcal{V}_i = \mathcal{V}_{i-1} \oplus \mathcal{W}_i)$. A point in the flag manifold $\mathcal{F}_{n_1 n_2 \cdots n_{L+1}}$ can be specified by a set of matrices $(\xi_1, \xi_2, \cdots, \xi_L)$, where ξ_i is an N_i-by-N matrix whose rows form a basis of $\mathcal{V}_i$

$$\xi_i = \left(\boldsymbol{v}_i^{(1)}, \boldsymbol{v}_i^{(2)}, \cdots, \boldsymbol{v}_i^{(N_i)} \right)^{\mathrm{T}} \qquad \{\boldsymbol{v}_i^{(a)}\}\text{: basis of } \mathcal{V}_i. \tag{2.5}$$

Since $\mathcal{V}_i$ is a linear subspace of $\mathcal{V}_{i+1}$, the basis vectors of $\mathcal{V}_i$ can be expressed as linear combinations of those of $\mathcal{V}_{i+1}$. Hence, there exist a N_i-by-N_{i+1} matrix q_i such that

$$\xi_i = q_i \xi_{i+1}, \quad (\exists\, q_i : \text{full rank } N_i\text{-by-}N_{i+1} \text{ matrix}), \qquad \xi_{L+1} = \mathbf{1}_{L+1}. \tag{2.6}$$

Note that this condition implies that ξ_i can be written as $\xi_i = q_i q_{i+1} \cdots q_L$. Two different sets of matrices $(\xi_1, \xi_2, \cdots, \xi_L)$ and $(\xi_1', \xi_2', \cdots, \xi_L')$ corresponds to the same flag if they are related by a change of basis of $(\mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_L)$, i.e. by a $\mathrm{GL}(N_1, \mathbb{C}) \times \mathrm{GL}(N_2, \mathbb{C}) \times \cdots \times \mathrm{GL}(N_L, \mathbb{C})$ transformation

$$\xi_i' = V_i \xi_i \quad \Longleftrightarrow \quad (\xi_1, \xi_2, \cdots, \xi_L) \sim (\xi_1', \xi_2', \cdots, \xi_L'), \tag{2.7}$$

where $V_i \in \mathrm{GL}(N_i, \mathbb{C})$ $(i = 1, 2, \cdots, L)$. Therefore, the flag manifold (2.3) can be identified with the space of the equivalence class (2.7) satisfying the condition (2.6)

$$\mathcal{F}_{n_1 n_2 \cdots n_{L+1}} = \left\{ (\xi_1, \xi_2, \cdots, \xi_L) \;\middle|\; \begin{array}{l} \xi_i : \text{full rank } N_i\text{-by-}N \text{ matrix} \\ \xi_i = q_i \xi_{i+1}, \quad \exists\, q_i : \text{full rank } N_i\text{-by-}N_{i+1} \text{ matrix} \end{array} \right\} \Big/ \sim . \tag{2.8}$$

Since $\xi_i = q_i q_{i+1} \cdots q_L$, the flag manifold (2.3) can also be regarded as the space of the equivalence classes of the matrices $(q_1, q_2, \cdots, q_L)$

$$\mathcal{F}_{n_1 n_2 \cdots n_{L+1}} = \Big\{ (q_1, q_2, \cdots, q_L) \;\Big|\; q_i : \text{full rank } N_i\text{-by-}N_{i+1} \text{ matrix} \Big\} \Big/ \sim, \tag{2.9}$$

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where the equivalence relation for $(q_1, q_2, \cdots, q_L)$ is given by

$$q'_i = V_i q_i V_{i+1}^{-1} \quad \Longleftrightarrow \quad (q_1, q_2, \cdots, q_L) \sim (q'_1, q'_2, \cdots, q'_L), \tag{2.10}$$

with $V_i \in \mathrm{GL}(N_i, \mathbb{C})$ $(i = 1, 2, \cdots, L)$ and $V_{L+1} = \mathbf{1}_N$.

We can show that the flag manifold is a homogeneous space given by the coset space

$$\mathcal{F}_{n_1 n_2 \cdots n_{L+1}} \cong \frac{\mathrm{U}(N)}{\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_{L+1})}. \tag{2.11}$$

To show this, let us note that any set of full rank matrices $(q_1, q_2, \cdots, q_L)$ can be rewritten by using the equivalence relation (2.10) as

$$q_i = \begin{cases} q_i^o & \text{for } 1 \le i \le L-1 \\ q_L^o\, U & \text{for } i = L \end{cases} \qquad \text{with} \quad q_i^o \equiv (\mathbf{1}_{N_i}, \mathbf{0}_{N_i \times n_{i+1}}), \tag{2.12}$$

where U is an element of $\mathrm{U}(N)$ and q_i^o are matrices corresponding to the standard flag $(\mathcal{V}_0^o, \mathcal{V}_1^o, \cdots, \mathcal{V}_{L+1}^o)$, i.e. the flag consisting of the vector space $\mathcal{V}_i^o$ spanned by the first i fundamental unit vectors. This indicates that any flag is related to the standard flag by a $\mathrm{U}(N)$ transformation. For a given flag, the corresponding unitary matrix U is not unique since the flag is invariant under $\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{L+1})$ transformations, i.e.

$$(q_1^o, \cdots, q_{L-1}^o, q_L^o U) \sim (q_1^o, \cdots, q_{L-1}^o, q_L^o U' U), \quad \text{with} \quad U' = \begin{pmatrix} U_1 & & \\ & \ddots & \\ & & U_L \end{pmatrix}, \qquad U_i \in \mathrm{U}(n_i). \tag{2.13}$$

The unitary matrices U and $U'U$ give the same flag, and hence the flag manifold is given by the coset space (2.11).

The denominator of the coset space (2.11) implies that if $(n'_1, n'_2, \cdots, n'_{L+1})$ is a permutation of $(n_1, n_2, \cdots, n_{L+1})$, the flag manifolds $\mathcal{F}_{n_1 n_2 \cdots n_{L+1}}$ and $\mathcal{F}_{n'_1 n'_2 \cdots n'_{L+1}}$ are identical as a homogeneous space.[1] However, in general, they have different complex structures and hence they are distinct as complex manifolds. To make the complex structure manifest, let us rewrite an arbitrary set of full rank matrices $(q_1, \cdots, q_L)$ by using the equivalence relation (2.10) as

$$q_i = \begin{cases} q_i^o & \text{for } 1 \le i \le L-1 \\ q_L^o\, \mathcal{G} & \text{for } i = L \end{cases}, \qquad \mathcal{G} \in \mathrm{GL}(N, \mathbb{C}). \tag{2.14}$$

In this case, the isotropy group of $(q_1^o, \cdots, q_L^o)$ is the parabolic subgroup $\hat{H}(n_1, \cdots, n_{L+1}) \subset \mathrm{GL}(N, \mathbb{C})$, i.e. the subgroup whose elements are matrices of the form

$$\hat{h} = \begin{pmatrix} h_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \star & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \star & \cdots & \star & h_{L+1} \end{pmatrix}, \qquad \text{with} \qquad \begin{array}{l} h_i : \text{element of } \mathrm{GL}(n_i, \mathbb{C}) \\ \star : \text{complex block matrix} \end{array}. \tag{2.15}$$

[1] For a permutation $\sigma : (n_1, \cdots, n_{L+1}) \mapsto (n'_1, \cdots, n'_{L+1}) = (n_{\sigma(1)}, \cdots, n_{\sigma(L+1)})$, one can define a diffeomorphism $\mathcal{F}_{n_1 \cdots n_{L+1}} \to \mathcal{F}_{n'_1 \cdots n'_{L+1}}$ as $(\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_{L+1}) \mapsto (\mathcal{V}'_0, \mathcal{V}'_1, \cdots, \mathcal{V}'_{L+1})$ with $\mathcal{V}_i = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_i$ and $\mathcal{V}'_i = \mathcal{W}_{\sigma(1)} \oplus \cdots \oplus \mathcal{W}_{\sigma(i)}$.

Since the matrices $\mathcal{G}$ and $\hat{h}\mathcal{G}$ give the same flag

$$(q_1^o, \cdots, q_{L-1}^o, q_L^o \mathcal{G}) \sim (q_1^o, \cdots, q_{L-1}^o, q_L^o \hat{h}\mathcal{G}), \tag{2.16}$$

the flag manifold can also be written as the coset space

$$\mathcal{F}_{n_1,\cdots,n_{L+1}} \cong \mathrm{GL}(N,\mathbb{C})/\hat{H}(n_1,\cdots,n_{L+1}). \tag{2.17}$$

In general, for different ordering of the integers $(n_1, \cdots, n_L)$ and $(n'_1, \cdots, n'_L)$, the parabolic subgroups are not isomorphic to each other and hence give different complex manifolds. The only exception is the case with $(n'_1, n'_2, \cdots, n'_{L+1}) = (n_{L+1}, \cdots, n_2, n_1)$, for which the map between flags $(\mathcal{V}_0, \mathcal{V}_1, \cdots, \mathcal{V}_{L+1}) \mapsto (\mathcal{V}'_0, \mathcal{V}'_1, \cdots, \mathcal{V}'_{L+1}) = (\mathcal{V}_{L+1}^\perp, \cdots, \mathcal{V}_1^\perp, \mathcal{V}_0^\perp)$ defines a biholomorphic map between $\mathcal{F}_{n_1 n_2 \cdots n_{L+1}}$ and $\mathcal{F}_{n_{L+1} \cdots n_2 n_1}$(see section 2.3). Correspondingly, there exists a duality between the GLσMs for $\mathcal{F}_{n_1 n_2 \cdots n_{L+1}}$ and $\mathcal{F}_{n_{L+1} \cdots n_2 n_1}$.

The holomorphic coordinates of $\mathcal{F}_{n_1,\cdots,n_{L+1}}$ are the coordinates parameterizing the coset defined by $\mathcal{G} \sim \hat{h}\mathcal{G}$ with $\hat{h} \in \hat{H}(n_1, \cdots, n_{L+1})$. For example, in the neighborhood of $\mathcal{G} = \mathbf{1}$, which corresponds to the standard flag $(q_1^o, \cdots, q_L^o)$, we can decompose the matrix $\mathcal{G}$ as

$$\mathcal{G} = \mathcal{L}\mathcal{U}, \qquad \mathcal{U} = \begin{pmatrix} \mathbf{1}_{n_1} & \varphi_{12} & \cdots & \varphi_{1,L+1} \\ \mathbf{0} & \mathbf{1}_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \varphi_{L,L+1} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1}_{n_{L+1}} \end{pmatrix}, \tag{2.18}$$

where $\mathcal{L}$ is an element of the parabolic subgroup $\hat{H}(n_1, \cdots, n_{L+1})$ (lower-triangular block matrix) and $\mathcal{U}$ is an upper-unitriangular block matrix whose blocks φ_{ij} $(1 \leq i < j \leq L+1)$ are n_i-by-n_j complex matrices. The entries of φ_{ij} parameterizes the coset space and hence they can be regarded as the holomorphic coordinates in this coordinate patch. For this matrix $\mathcal{G}$, the set of matrices $(q_1, \cdots, q_L) = (q_1^o, \cdots, q_L^o \mathcal{G})$ can be rewritten by using the equivalence relation (2.10) as

$$(q_1^o, \cdots, q_L^o \mathcal{G}) \sim (\mathcal{U}_1^{-1} q_1^o \mathcal{U}_2, \cdots, \mathcal{U}_L^{-1} q_L^o \mathcal{U}) \quad \text{with} \quad \mathcal{U}_i^{-1} q_i^o \mathcal{U}_{i+1} = \left(\begin{array}{ccc|c} \mathbf{1}_{n_1} & & & \varphi'_{1,i+1} \\ & \ddots & & \vdots \\ & & \mathbf{1}_{n_i} & \varphi'_{i,i+1} \end{array}\right), \tag{2.19}$$

where $\mathcal{U}_i$ are the first N_i-by-N_i submatrices of $\mathcal{U}$ and the n_i-by-n_j block $\varphi'_{ij} = \varphi_{ij} + \mathcal{O}(\varphi^2)$ are certain polynomials of φ's. In general, we can find a representative in each class $[q_1, \cdots, q_L]$ such that the matrices $(q_1, \cdots, q_L)$ are holomorphic in φ's in each coordinate patch.

Although the decomposition (2.18) is not always possible, there exists at least one element of the symmetric group $\sigma : (1, \cdots, N) \mapsto (\sigma(1), \cdots, \sigma(N))$ such that

$$\mathcal{G} = \mathcal{L}_\sigma \mathcal{U}_\sigma P_\sigma, \qquad P_\sigma \in S_N, \tag{2.20}$$

where $\mathcal{L}_\sigma \in H(n_1, \cdots, n_{L+1})$, $\mathcal{U}_\sigma$ is an upper-triangular block matrix and P_σ is the permutation matrix corresponding to the element of the symmetric group σ. For a generic $\mathcal{G}$, the element σ is not unique and hence there are several ways to decompose $\mathcal{G}$

$$\mathcal{G} = \mathcal{L}_\sigma \mathcal{U}_\sigma P_\sigma = \mathcal{L}_{\sigma'} \mathcal{U}_{\sigma'} P_{\sigma'} = \cdots. \tag{2.21}$$

The relation between $\mathcal{U}_\sigma(\varphi_\sigma)$ and $\mathcal{U}_{\sigma'}(\varphi_{\sigma'})$ gives the coordinate transformation $\varphi_\sigma \leftrightarrow \varphi_{\sigma'}$ between the patches specified by σ and σ'. The "origin"of each patch $\mathcal{U}_\sigma = \mathbf{1}_N$ ($\varphi = 0$) corresponds to the flag obtained by the permuting the basis of the standard flag $(\mathcal{V}_0^o, \mathcal{V}_1^o, \cdots, \mathcal{V}_L^o, \mathcal{V})$ by σ. Since the permutations of the basis within $\mathcal{W}_i^o$ (the orthogonal complements of $\mathcal{V}_{i-1}$ in $\mathcal{V}_i$) do not change the standard flag, it is invariant under the subgroup $S_{n_1} \times \cdots \times S_{n_{L+1}} \subset S_N$. Hence the number of the "origins", which is also the number of coordinate patches requiered to cover the whole manifold, is $N!/(n_1!n_2! \cdots n_{L+1}!)$.[2]

Let us see the simplest example of $L = 1$. In this case, the flag manifold is identified with the set of planes in a vector space, i.e. the Grassmaniann

$$\mathcal{F}_{n_1 n_2} = \Big\{\mathcal{V} : \text{vector space in } \mathbb{C}^N \mid \dim_{\mathbb{C}} \mathcal{V} = M\Big\} = G(M, N), \tag{2.22}$$

with $M = n_1,\ N = n_1 + n_2$. In particular, $\mathcal{F}_{n_1=1,n_2=1} = \mathbb{C}P^1$ for $n_1 = n_2 = 1$. To see how q_1 is parametrized by the holomorphic coordinate ϕ, let us consider the decomposition (2.18) for $\mathrm{GL}(2, \mathbb{C})$. Any matrix $\mathcal{G} \in \mathrm{GL}(2, \mathbb{C})$ can be decomposed into at least one of the forms

- $$\mathcal{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathcal{L}\mathcal{U}, \qquad \mathcal{L} = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}, \tag{2.23}$$
- $$\mathcal{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathcal{L}'\mathcal{U}'P, \qquad \mathcal{L}' = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}, \qquad \mathcal{U}' = \begin{pmatrix} 1 & \phi' \\ 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.24}$$

where

$$a = A, \quad c = C, \quad d = \frac{AD - BC}{A}, \qquad a' = B, \quad c' = D, \quad d' = \frac{AD - BC}{B}, \tag{2.25}$$

and ϕ and ϕ' are inhomogeneous coordinates of $\mathbb{C}P^1$

$$\phi = \frac{B}{A}, \qquad \phi' = -\frac{A}{B}. \tag{2.26}$$

The decomposed forms (2.23) and (2.24) exist except for the matrices with $A = 0$ and $B = 0$, respectively. Multiplying these decomposed forms of $\mathcal{G}$ and $q_1^o = (1, 0)$, we find two different forms of q_1, each of which is parametrized by the holomorphic coordinate on the respective coordinate patch

$$q_1 = q_1^o\,\mathcal{G} \sim (1, 0)\,\mathcal{U} = (1, \phi) \qquad \text{or} \qquad q_1 = q_1^o\,\mathcal{G} \sim (1, 0)\,\mathcal{U}'P = (-\phi', 1). \tag{2.27}$$

Similarly, using the decomposition of the matrix $\mathcal{G}$, we can obtain holomorphic parametrizations of q_i also for general L.

2.2 GLσM for flag manifolds

In this subsection, we review the gauged linear sigma models (GLσMs) corresponding to the flag manifold sigma models.

[2]The "origins" correspond to the fixed points of a torus action $\mathrm{U}(1)^N \subset \mathrm{U}(N)$ and their number is given by Euler characteristic of the flag manifold $N!/(n_1!n_2! \cdots n_{L+1}!)$.

As shown in appendix A, the flag manifold $\mathcal{F}_{n_1 n_2 \cdots n_{L+1}}$ becomes a Kähler manifold in an L dimensional subspace of the $L(L+1)/2$ dimensional parameter space of Riemann metric on $\mathcal{F}_{n_1 n_2 \cdots n_{L+1}}$. In such a subspace, the flag manifold sigma model can be described by $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_L)$ GLσM specified by the quiver diagram [74]

$$\overset{N_1}{\bigcirc} \xrightarrow{q_1} \overset{N_2}{\bigcirc} \xrightarrow{q_2} \bigcirc \cdots\cdots \bigcirc \longrightarrow \overset{N_L}{\bigcirc} \xrightarrow{q_L} \overset{N_{L+1}}{\square} \qquad (N_1 < N_2 < \cdots < N_L < N_{L+1} = N), \tag{2.28}$$

where the i-th node corresponds to the $\mathrm{U}(N_i)$ gauge group, the i-th arrow denotes a bifundamental field of $\mathrm{U}(N_i) \times \mathrm{U}(N_{i+1})$ and the last box stands for the $\mathrm{U}(N_{L+1}) = \mathrm{U}(N)$ global (flavor) symmetry.[3] The Lagrangian is written in terms of L bifundamental scalar fields Q_i (N_i-by-N_{i+1} matrix, $i = 1, \cdots, L$), auxiliary $\mathrm{U}(N_i)$ gauge fields A_μ^i ($i = 1, \cdots, L$) and Lagrange multipliers D^i (N_i-by-N_i matrix, $i = 1, \cdots, L$) in the adjoint representation of $\mathrm{U}(N_i)$

$$\mathcal{L}_0 = \sum_{i=1}^{L} \mathrm{Tr}\Big[(\mathcal{D}_\mu Q_i)(\mathcal{D}^\mu Q_i)^\dagger + D^i \left(Q_i Q_i^\dagger - Q_{i-1}^\dagger Q_{i-1} - r_i \mathbf{1}_{N_i}\right)\Big], \tag{2.29}$$

where $Q_0 = 0$ and r_i ($i = 1, \cdots, L$) are positive constants parametrizing the Kähler metric. The gauge group acts on the bifundamental field Q_i as

$$Q_i \to U_i Q_i U_{i+1}^\dagger, \qquad U_i \in \mathrm{U}(N_i), \quad U_{i+1} \in \mathrm{U}(N_{i+1}). \tag{2.30}$$

The covariant derivatives are defined as

$$\mathcal{D}_\mu Q_i = \partial_\mu Q_i + i(A_\mu^i Q_i - Q_i A_\mu^{i+1}), \qquad \mathcal{D}_\mu Q_L = \partial_\mu Q_L + i A_\mu^L Q_L. \tag{2.31}$$

To see that this GLσM describes the flag manifold sigma model, we need to eliminate (A_μ^i, D^i) by solving their equations of motion. The variations of the action with respect to the Lagrange multipliers D^i give the constraints

$$Q_i Q_i^\dagger - Q_{i-1}^\dagger Q_{i-1} = r_i \mathbf{1}_{N_i}. \tag{2.32}$$

To solve these constraints, it is convenient to write Q_i as

$$Q_i = S_i^{-1} q_i S_{i+1}, \qquad (S_{L+1} = \mathbf{1}_N), \tag{2.33}$$

where q_i ($i = 1, \cdots, L$) are N_i-by-N_{i+1} matrices of complex scalar fields and S_i ($i = 1, \cdots, L$) are elements of the complexified gauge group $\mathrm{GL}(N_i, \mathbb{C})$.[4] Then, the constraints (2.32) can be rewritten as

$$q_i \Omega_{i+1} q_i^\dagger \Omega_i^{-1} - \Omega_i q_{i-1}^\dagger \Omega_{i-1}^{-1} q_{i-1} = r_i \mathbf{1}_{N_i}, \qquad \text{with} \qquad \Omega_i = S_i S_i^\dagger. \tag{2.34}$$

[3] The overall U(1) of the global symmetry is unphysical since it can be absorbed into the gauge group $\mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_L)$.

[4] The matrices S_i can be regarded as (the lowest components of) the auxiliary vector superfields $S_i = e^{-V_i}$ in the supersymmetric version of our system,

$$S = \int d^4x \int d^4\theta \sum_{i=1}^{L} \mathrm{Tr}\left[e^{-2V_i} q_i e^{2V_{i+1}} q_i^\dagger + 2 r_i V_i\right].$$

where q_i are chiral superfields, r_i are called Fayet-Iliopoulos parameters in this context.

These equation can be uniquely solved for $\Omega_i \in \mathrm{GL}(N_i\mathbb{C})$ as long as the q_i are full rank matrices. Once we obtain the solution Ω_i for a given set of matrices $(q_1, \cdots, q_L)$, we can determine S_i up to gauge transformations. Note that the expression (2.33) in terms of q_i and S_i is redundant since the scalar fields Q_i do not change under the complexified gauge transformation

$$q_i \to V_i q_i V_{i+1}^{-1}, \qquad S_i \to V_i S_i, \qquad (\Omega_i \to V_i \Omega_i V_i^\dagger), \tag{2.35}$$

where V_i are arbitrary elements of $\mathrm{GL}(N_i, \mathbb{C})$ and $V_{L+1} = 1$. Since S_i are unique (up to gauge transformation) for a given set of matrices $(q_1, \cdots, q_L)$, the moduli space of vacua (the set of solutions of (2.34) modulo gauge transformations) is given by

$$\mathcal{M}_{\text{vac}} = \left\{ (q_1, q_2, \cdots, q_L) \;\middle|\; q_i : \text{full rank } N_i\text{-by-}N_{i+1} \text{ matrix} \right\} / \sim, \tag{2.36}$$

where $\sim$ denotes the equivalence relation $q_i \sim V_i q_i V_{i+1}^{-1}$ $(i = 1, \cdots, L)$. This is nothing but one of the representations of the flag manifold (2.9) and hence the moduli space of vacua is isomorphic to $\mathcal{F}_{n_1 n_2 \cdots n_{L+1}}$

$$\mathcal{M}_{\text{vac}} = \mathcal{F}_{n_1 n_2 \cdots n_{L+1}}. \tag{2.37}$$

The general solution of eq. (2.34) can be obtained as follows. As we have mentioned in eq. (2.12), any set of full rank matrices $(q_1, \cdots, q_L)$ can be rewritten, by using the equivalence relation (2.35), into a unitary transform of the standard flag $(q_1^o, \cdots, q_{L-1}^o, q_L^o U)$ with $U \in \mathrm{U}(N)$. Since eq. (2.34) is invariant under the $\mathrm{U}(N)$ global symmetry, the solution Ω_i to eq. (2.34) for $(q_1, \cdots, q_L) = (q_1^o, \cdots, q_{L-1}^o, q_L^o U)$ is given by the solution Ω_i^o for the standard flag $(q_1^o, \cdots, q_{L-1}^o, q_L^o)$

$$\Omega_i = \Omega_i^o \equiv \mathrm{diag}\left(\frac{1}{a_{1i}} \mathbf{1}_{n_1}, \frac{1}{a_{2i}} \mathbf{1}_{n_2}, \cdots, \frac{1}{a_{ii}} \mathbf{1}_{n_i} \right) \quad \text{with} \quad a_{ij} = \prod_{l=j}^{L} \left(\sum_{m=i}^{l} r_m \right) > 0. \tag{2.38}$$

From these solution $\Omega_i^o = S_i^o (S_i^o)^\dagger$, we obtain solution to eq. (2.32) through eq. (2.33) as

$$Q_i = \begin{cases} \left(\mathcal{Q}_i^o, \mathbf{0}_{N_i \times n_{i+1}}\right) & \text{for } i < L \\ \left(\mathcal{Q}_L^o, \mathbf{0}_{N_L \times n_{L+1}}\right) U & \text{for } i = L \end{cases}, \quad \mathcal{Q}_i^o = \mathrm{diag}(b_{1i} \mathbf{1}_{n_1}, b_{2i} \mathbf{1}_{n_2}, \cdots, b_{ii} \mathbf{1}_{n_i}), \quad b_{ij} = \Big(\sum_{m=i}^{j} r_m \Big)^{\frac{1}{2}}, \tag{2.39}$$

up to $\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_L)$ gauge transformations. Although this is the general solution of the constraint (2.34), the complex structure of $\mathcal{F}_{n_1, \cdots, n_{L+1}}$ is not manifest in this form of the general solution. To describe the Kähler flag manifold sigma model, it is convenient to make the complex structure manifest by parametrizing the matrices $(q_1, \cdots, q_L)$ with holomorphic coordinates. To this end, let us rewrite $(q_1, \cdots, q_L)$ by using the equivalence relation (2.35) into the form $(q_1^o, \cdots, q_{L-1}^o, q_L^o \mathcal{G})$ given in (2.14). For $\mathcal{G} \in \mathrm{GL}(N, \mathbb{C})$, we can find a pair of matrices $(\hat{h}, U)$ such that[5]

$$\mathcal{G} = \hat{h}\, U \quad \text{with} \quad \hat{h} \in \hat{H}(n_1, \cdots, n_{L+1}) \quad \text{and} \quad U \in \mathrm{U}(N), \tag{2.40}$$

[5]For a given $\mathcal{G} \in \mathrm{GL}(N, \mathbb{C})$, the pair $(\hat{h}, U)$ is unique up to $\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{L+1})$ transformations $(\hat{h}, U) \to (\hat{h} U'^\dagger, U' U)$ with $U' \in \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{L+1})$.

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where $\hat{H}(n_1, \cdots, n_{L+1}) \subset \mathrm{GL}(N, \mathbb{C})$ is the parabolic subgroup given in (2.15). Noting that

$$\hat{h}_L^{-1} q_L^o \, \mathcal{G} = q_L^o U, \qquad \hat{h}_i^{-1} q_i^o \, \hat{h}_{i+1} = q_i^o \quad \text{with} \quad \hat{h}_i = \begin{pmatrix} h_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \star & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \star & \cdots & \star & h_i \end{pmatrix} \in \hat{H}(n_1, \cdots, n_i), \tag{2.41}$$

we can rewrite the general solution of the constraint (2.34) by using the equivalence relation (2.35) as

$$(q_1, \cdots, q_L) = (q_1^o, \cdots, q_L^o U), \qquad \Omega_i = \Omega_i^o \tag{2.42}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$(q_1, \cdots, q_L) = (q_1^o, \cdots, q_L^o \mathcal{G}), \qquad \Omega_i = \hat{h}_i \Omega_i^o \hat{h}_i^\dagger, \tag{2.43}$$

where two forms of the solution are related by (2.35) with $V_i = \hat{h}_i$. In this form of the solution, the matrices $\{q_i\}$ are parametrized by the holomorphic coordinates ϕ^α ($\alpha = 1, \cdots, \dim_{\mathbb{C}} \mathcal{F}_{n_1, \cdots, n_{L+1}}$), which are entries of the block matrices φ_{ij} $(1 \le i < j \le L+1)$ given in eq. (2.18).

Next, let us write down the Lagrangian of the NLσM in terms of the complex coordinates $(\phi^\alpha, \bar{\phi}^{\bar{\beta}})$ by regarding them as scalar fields depending on the spacetime coordinates. The auxiliary gauge fields A_μ^i can be eliminated by solving their equations of motion

$$i\Big[Q_i (\mathcal{D}_\mu Q_i)^\dagger - \mathcal{D}_\mu Q_i Q_i^\dagger + Q_{i-1}^\dagger \mathcal{D}_\mu Q_{i-1} - (\mathcal{D}_\mu Q_{i-1})^\dagger Q_{i-1}\Big] = 0. \tag{2.44}$$

These equations can be solved as

$$A_\mu^i = -i S_i^{-1} \left(\partial_\mu - \partial_\mu \phi^\alpha \frac{\partial}{\partial \phi^\alpha} \Omega_i \Omega_i^{-1} \right) S_i = -i \partial_\mu \bar{\phi}^{\bar{\alpha}} S_i^{-1} \frac{\partial S_i}{\partial \bar{\phi}^{\bar{\alpha}}} + i \partial_\mu \phi^\alpha \frac{\partial S_i^\dagger}{\partial \phi^\alpha} S_i^{\dagger -1}. \tag{2.45}$$

Substituting into the original action (2.29), we obtain the NLσM in terms of the complex coordinates $(\phi^\alpha, \bar{\phi}^{\bar{\beta}})$

$$\mathcal{L}_0 = -g_{\alpha\bar{\beta}} \, \partial_\mu \phi^\alpha \overline{\partial^\mu \phi^\beta}, \tag{2.46}$$

where the Kähler metric $g_{\alpha\bar{\beta}}$ is given by the formula

$$g_{\alpha\bar{\beta}} = \sum_{i=1}^L r_i \frac{\partial}{\partial \bar{\phi}^\beta} \mathrm{Tr} \left(\Omega_i^{-1} \frac{\partial q_i}{\partial \phi^\alpha} \Omega_{i+1} q_i^\dagger \right) = \frac{\partial^2}{\partial \phi^\alpha \partial \bar{\phi}^{\bar{\beta}}} \sum_{i=1}^L r_i \log\det \Omega_i = \frac{\partial^2 K}{\partial \phi^\alpha \partial \bar{\phi}^{\bar{\beta}}}. \tag{2.47}$$

This form of the Kähler metric implies that the Kähler potential K takes the form

$$K = \sum_{i=1}^L r_i \log\det \Omega_i. \tag{2.48}$$

Using the solution of the constraint (2.43), we find that

$$\log\det \Omega_i = \log |\det \hat{h}_i|^2 + \log\det \Omega_i^o. \tag{2.49}$$

Although this is the general formula for the Kähler potential for $\mathcal{F}_{n_1,\cdots,n_{L+1}}$, it is more convenient to express $|\det \hat{h}_i|^2$ in terms of holomorphic quantities. Let us consider

$$\xi_i \equiv q_i q_{i+1} \cdots q_{L-1} q_L, \tag{2.50}$$

which takes the form $\xi_i = q_i^o \cdots q_L^o \mathcal{G}$ for the set of matrices $(q_1, \cdots, q_L) = (q_1^o, \cdots, q_L^o \mathcal{G})$. As we have seen in eq. (2.19), the matrices $(q_1, \cdots, q_L)$ are holomorphically parametrized by the coordinates and hence ξ_i are also holomorphic. By using the decomposition (2.40), the relations $q_j^o \hat{h}_{j+1} = \hat{h}_j q_j^o$ and $q_i^o q_i^{o\dagger} = \mathbf{1}_{N_i}$, we can show that

$$\xi_i \xi_i^\dagger = \hat{h}_i \hat{h}_i^\dagger. \tag{2.51}$$

Thus, we find that the Kähler potential is given by

$$K = \sum_{i=1}^{L} r_i \log \det(\xi_i \xi_i^\dagger), \tag{2.52}$$

where we have neglected the unphysical constant term $\log \det \Omega_i^o$. This expression coincides with the Kähler potential constructed in refs. [67–70]. Note that this formula is applicable for any gauge choice other than eq. (2.43) since this Kähler potential is invariant under the complexified gauge transformations (2.35) up to a Kählertransformation.[6]

Example of Kähler potential. Let us see an explicit example of the Kähler potential in the case of $L = 2$, $n_1 = n_2 = n_3 = 1$. Let us introduce inhomogeneous complex coordinates $(\phi_{12}, \phi_{13}, \phi_{23})$ of the target manifold $G^{\mathbb{C}}/\hat{H} = \mathcal{F}_{1,1,1}$. They are contained in the matrix $\mathcal{G}$ in eq. (2.18) as

$$\mathcal{G} = \mathcal{L} \begin{pmatrix} 1 & \phi_{12} & \phi_{13} \\ 0 & 1 & \phi_{23} \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{C}), \tag{2.53}$$

where $\mathcal{L} \in \hat{H}$ is a lower-triangular matrix. Using $\xi_i = q_i^o \cdots q_L^o \mathcal{G}$, we obtain the holomorphic parametrization of ξ_i as

$$\xi_1 = (1, 0, 0)\, \mathcal{G} \sim (1, \phi_{12}, \phi_{13}), \qquad \xi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathcal{G} \sim \begin{pmatrix} 1 & 0 & \phi_{13} - \phi_{12}\phi_{23} \\ 0 & 1 & \phi_{23} \end{pmatrix}, \tag{2.54}$$

where we have used the equivalence relation $\xi_i \sim V_i \xi_i$ with $V_1 \in \mathrm{GL}(1, \mathbb{C})$ and $V_2 \in \mathrm{GL}(2, \mathbb{C})$. Inserting these expressions into (2.52), we obtain

$$K = r_1 \log(1 + |\phi_{12}|^2 + |\phi_{13}|^2) + r_2 \log(1 + |\phi_{23}|^2 + |\phi_{13} - \phi_{12}\phi_{23}|^2). \tag{2.55}$$

In this way, we can obtain the explicit forms of the Kähler potentials for the flag manifolds.

[6]For any gauge choice, we can confirm that $\log \det \Omega_i = \log \det(\xi_i \xi_i^\dagger) + \mathit{const}$ by using the explicit form of the solution of (2.34)

$$\Omega_i = \frac{1}{a_{ii}} \xi_i \xi_i^\dagger + \sum_{j=1}^{i-1} \left(\frac{1}{a_{ji}} - \frac{1}{a_{j+1,i}} \right) \xi_i \xi_j^\dagger (\xi_j \xi_j^\dagger)^{-1} \xi_j \xi_i^\dagger.$$

Coefficients of the beta functions. In two dimensions, the target space metric flows under the renormalization group flow

$$\mu\frac{\partial}{\partial\mu}g_{\alpha\bar\beta}=\frac{1}{2\pi}R_{\alpha\bar\beta}+\mathcal{O}(1/r_i), \tag{2.56}$$

where μ is the renormalization scale and $R_{\alpha\bar\beta}$ is the Ricci curvature. For a Kähler flag manifold, this renormalization group equation for the metric can be rewritten into that for the parameters r_i

$$\mu\frac{\partial}{\partial\mu}r_i=\frac{N_{i+1}-N_{i-1}}{2\pi}+\mathcal{O}(1/r_i). \tag{2.57}$$

At the one-loop order, the solutions can be written as

$$r_i=\frac{\beta_i}{2\pi}\log\frac{\mu}{\Lambda_i}+\cdots, \qquad \beta_i\equiv N_{i+1}-N_{i-1}, \tag{2.58}$$

where Λ_i are dynamically generated scale parameters. Since $\beta_i>0$, the sigma model coupling constants $1/r_i$ become small for $\mu\to\infty$ and hence the system is asymptotically free. As we will see below, the coefficients β_i are also related to the dimension of the moduli space of vortices.

2.3 Duality at classical level

In general, a flag manifold $\mathcal{F}_{n_{\sigma(1)},\cdots,n_{\sigma(L+1)}}$ obtained by permuting the integers $(n_1,\cdots,n_{L+1})\to(n_{\sigma(1)},\cdots,n_{\sigma(L+1)})$ has a different complex structure from that of $\mathcal{F}_{n_1,\cdots,n_{L+1}}$. However, when $(n_{\sigma(1)},\cdots,n_{\sigma(L+1)})=(n_{L+1},\dots,n_2,n_1)$ the two manifold have an identical complex structure, i.e. $\mathcal{F}_{n_1,n_2,\dots,n_{L+1}}$ and $\mathcal{F}_{n_{L+1},\dots,n_2,n_1}$ are identical as a complex manifold

$$\mathcal{F}_{n_1,n_2,\dots,n_{L+1}}=\mathcal{F}_{n_{L+1},\dots,n_2,n_1}. \tag{2.59}$$

This equivalence is explicitly given by the biholomorphic map given in terms of the matrix $\mathcal{G}$

$$\mathcal{G}\in GL(N,\mathbb{C}) \quad\mapsto\quad \mathcal{G}_{\rm dual}=R\,(\mathcal{G}^{\rm T})^{-1}R^\dagger\in GL(N,\mathbb{C}) \tag{2.60}$$

where the matrix R is defined as

$$R=\begin{pmatrix}\mathbf{0}&\cdots&\mathbf{0}&\mathbf{1}_{n_{L+1}}\\ \vdots&\iddots&\iddots&\mathbf{0}\\ \mathbf{0}&\mathbf{1}_{n_2}&\iddots&\vdots\\ \mathbf{1}_{n_1}&\mathbf{0}&\cdots&\mathbf{0}\end{pmatrix}\in U(N). \tag{2.61}$$

This transformation reduces to the map between the equivalence class $\mathcal{G}\sim\hat h\mathcal{G}$ with $\hat h\in\hat H(n_1,n_2,\cdots,n_{L+1})$ and $\mathcal{G}_{\rm dual}\sim\hat h'\mathcal{G}_{\rm dual}$ with $\hat h'=R\,(\hat h^{\rm T})^{-1}R^\dagger$. We can show that matrices of the form $R\,(\hat h^{\rm T})^{-1}R^\dagger$ are elements of $\hat H(n_{L+1},\dots,n_2,n_1)$

$$\hat h' \;\in\; R\,\hat H(n_1,n_2,\dots,n_{L+1})^{\rm T-1}R^\dagger\cong\hat H(n_{L+1},\dots,n_2,n_1). \tag{2.62}$$

Therefore, the transformation (2.60) gives a one to one map between the flag manifolds $GL(N,\mathbb{C})/\hat H(n_1,\cdots,n_{L+1})$ and $GL(N,\mathbb{C})/\hat H(n_{L+1},\cdots,n_1)$. Suppose $\mathcal{G}$ can be decomposed into $\mathcal{L}$ and $\mathcal{U}$ as given in eq. (2.18) ($\mathcal{G} = \mathcal{L}\mathcal{U}$). Then, $\mathcal{G}_{\rm dual}$ can be decomposed into $\mathcal{L}_{\rm dual} \in \hat{H}(n_{L+1}, \dots, n_2, n_1)$ and an upper block-triangular matrix $\mathcal{U}_{\rm dual}$ as

$$\mathcal{G}_{\rm dual} = \mathcal{L}_{\rm dual}\,\mathcal{U}_{\rm dual} \tag{2.63}$$

where $\mathcal{L}_{\rm dual}$ and $\mathcal{U}_{\rm dual}$ are related to $\mathcal{L}$ and $\mathcal{U}$ as

$$\mathcal{L}_{\rm dual} \equiv R\,(\mathcal{L}^{\rm T})^{-1}R^\dagger, \qquad \mathcal{U}_{\rm dual} \equiv R\,(\mathcal{U}^{\rm T})^{-1}R^\dagger. \tag{2.64}$$

Since the complex coordinates are contained in the matrices $\mathcal{U}$ and $\mathcal{U}_{\rm dual}$ (see eq. (2.18)), the relation between $\mathcal{U}$ and $\mathcal{U}_{\rm dual}$ gives an explicit holomorphic coordinate transformation between the complex coordinates of $\mathcal{F}_{n_1,\dots,n_{L+1}}$ and $\mathcal{F}_{n_{L+1},\cdots,n_1}$.

Correspondingly, by replacing the ranks of gauge groups and the FI parameters as

$$N_i \to N_i^{\rm dual} = N - N_{L+1-i}, \qquad r_i \to r_i^{\rm dual} = r_{L+1-i}, \tag{2.65}$$

we obtain a dual GLσM and an effective NLσM whose Kähler potential is identical to the original one up to a Kähler transformation

$$\sum_i^L r_i^{\rm dual} \ln\det \xi_i^{\rm dual}(\xi_i^{\rm dual})^\dagger = \sum_i^L r_i \ln\det \xi_i \xi_i^\dagger + \text{Kähler trf.}, \tag{2.66}$$

with $\xi_i^{\rm dual} = (\mathbf{1}_{N_i^{\rm dual}}, \mathbf{0})\,\mathcal{G}_{\rm dual}$.[7] This shows that two GLσMs are equivalent at the classical level. In section 6, we will check this duality at the quantum level by comparing the vortex partition functions.

As an example, let us consider the $L = 1$ case. In the case of U(M) gauge theory with N fundamentals, the moduli space of vacua $\mathcal{M}_{\rm vac}$ is the Grassmaniann

$$\mathcal{F}_{n_1 n_2} = \Big\{\mathcal{V} : \text{vector space in } \mathbb{C}^N \mid \dim_{\mathbb{C}} \mathcal{V} = M\Big\} = G(M, N), \tag{2.67}$$

with $M = n_1$, $N = n_1 + n_2$. The dual theory is U$(N - M)$ gauge theory with N fundamentals and its $\mathcal{M}_{\rm vac}$ is given by

$$\mathcal{F}_{n_2 n_1} = \Big\{\mathcal{W} : \text{vector space in } \mathbb{C}^N \mid \dim_{\mathbb{C}} \mathcal{W} = N - M\Big\} = G(N - M, N). \tag{2.68}$$

These spaces are identical since any plane $\mathcal{V} \in \mathbb{C}^N$ can also be specified by its orthogonal complement $\mathcal{W} = \mathcal{V}^\perp$. Let us see the explicit coordinate transformation between these spaces. Let φ and $\tilde{\varphi}$ be M-by-$N - M$ and $N - M$-by-M matrices whose entries are inhomogeneous

[7] Here we have used the following identities for the determinants of $\xi_i\xi_i^\dagger$ and $\det \xi_i^{\rm dual}(\xi_i^{\rm dual})^\dagger$

$$\det \xi_i \xi_i^\dagger = \det \hat{h}_i \hat{h}_i^\dagger = \prod_{j=1}^{i} |\det h_j|^2, \quad \det \xi_i^{\rm dual}(\xi_i^{\rm dual})^\dagger = \prod_{j=L-i+2}^{L+1} |\det h_j^{{\rm T}-1}|^2,$$

where h_i are the matrices given in eq. (2.40). Taking determinants of the both sides of eq. (2.40), we find that

$$\sum_{i=1}^{L+1} \ln\det h_i = \ln\det(\mathcal{G})$$

which is holomorphic and thus, can be removed using a Kähler transformation.

coordinates of the Grassmaniann $G(M,N)$ and $G(N-M,M)$, respectively. They are contained in the matrices

$$\mathcal{U} = \begin{pmatrix} \mathbf{1}_{n_1} & \varphi \\ \mathbf{0} & \mathbf{1}_{n_2} \end{pmatrix}, \qquad \mathcal{U}_{\text{dual}} = \begin{pmatrix} \mathbf{1}_{n_2} & \tilde{\varphi} \\ \mathbf{0} & \mathbf{1}_{n_1} \end{pmatrix}. \tag{2.69}$$

From the duality relation $\mathcal{U}_{\text{dual}} = R\,(\mathcal{U}^{\text{T}})^{-1}R^\dagger$, we can read off the coordinate transformation between φ and $\tilde{\varphi}$ as

$$\tilde{\varphi} = -\varphi^{\text{T}}. \tag{2.70}$$

This is the simplest example of the duality of the flag manifold sigma models and the corresponding GLσMs.

2.4 Sigma model instantons

For any Kähler manifold $\mathcal{M}$, the non-linear sigma model with target space $\mathcal{M}$ admits BPS instanton solutions. They are given by holomorphic maps $\partial_{\bar{z}}\phi^i = 0$, which saturates the lower bound of the action

$$\int d^2x\,\mathcal{L} = \int d^2x\, g_{i\bar{j}}\partial_M\phi^i\partial^M\bar{\phi}^j = 4\int d^2x\,\|\partial_{\bar{z}}\phi^i\|^2 + \int_{\mathbb{R}^2} ig_{i\bar{j}}d\phi^i\wedge d\bar{\phi}^j \geq \int_{\mathbb{R}^2} ig_{i\bar{j}}d\phi^i\wedge d\bar{\phi}^j, \tag{2.71}$$

where $z = x_1 + ix_2$ and $\|\partial_{\bar{z}}\phi^i\|^2 = g_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\bar{\phi}^j$ is the norm of $\partial_{\bar{z}}\phi^i$ with respect to the Kähler metric $g_{i\bar{j}}$. The lower bound is given by the topological charge obtained by integrating the pullback of the Kähler form $\phi^*(\omega) = \frac{i}{2}g_{i\bar{j}}d\phi^i\wedge d\bar{\phi}^j$. Once we fix the configuration at the spatial infinity to a point on the target space, ϕ can be viewed as a map $\phi : \mathbb{R}^2 \cup \{\infty\} = S^2 \to \mathcal{M}$ and hence the instanton configurations are classified by $\pi_2(\mathcal{M})$.

In the case of the flag manifold, the topological charge is given by

$$\int_{\mathbb{R}^2} ig_{i\bar{j}}d\phi^i\wedge d\bar{\phi}^j = i\sum_{i=1}^{L} r_i\int_{\mathbb{R}^2}\partial\bar{\partial}\log\det\xi_i(z)\xi_i(z)^\dagger = -\frac{i}{2}\sum_{i=1}^{L} r_i\oint_{S^1_\infty}(dz\partial_z - d\bar{z}\partial_{\bar{z}})\log\det\xi_i(z)\xi_i(z)^\dagger, \tag{2.72}$$

where we have used the explicit form of the Kähler form obtained from the Kähler potential (2.52). Assuming that the asymptotic form of $\det\xi_i(z)\xi_i(z)^\dagger$ for large $|z|$ is given by

$$\det\xi_i(z)\xi_i(z)^\dagger = |z|^{2k_i} + \cdots, \qquad (|z|\to\infty), \tag{2.73}$$

we can determine the topological charge as

$$\int_{\mathbb{R}^2} ig_{i\bar{j}}d\phi^i\wedge d\bar{\phi}^j = 2\pi\sum_{i=1}^{L} r_i k_i, \tag{2.74}$$

where $(k_1,\cdots,k_L) \in \mathbb{Z}^L = \pi_2(\mathcal{F}_{n_1,\ldots,n_{L+1}})$ are topological numbers. The space of instanton solutions satisfying the boundary condition with fixed topological numbers is called the moduli space of sigma model instantons. As is well known, there exist small instanton singularities in the moduli space of sigma model instantons. Such singularities can be resolved by introducing the kinetic terms for the gauge fields in the GLσM. In the next section, we discuss vortex solutions which can be viewed as resolved the sigma model instantons in the GLσM.

3 BPS vortices in GLσM

3.1 BPS equations and moduli matrices

In this section, we discuss BPS vortices in the framework of the GLσM with finite gauge coupling constants g_i

$$\mathcal{L}=\sum_{i=1}^{L}\mathrm{Tr}\left[\frac{1}{2g_i^2}F_{\mu\nu}^{i}F^{i\mu\nu}+\mathcal{D}_\mu Q_i\mathcal{D}^\mu Q_i^\dagger-\frac{1}{g_i^2}D_i^2+D_i\left(Q_iQ_i^\dagger-Q_{i-1}^\dagger Q_{i-1}-r_i\mathbf{1}_{N_i}\right)\right], \tag{3.1}$$

where $F_{\mu\nu}^i=\partial_\mu A_\nu^i-\partial_\nu A_\mu^i+i[A_\mu^i,A_\nu^i]$ are the field strength for the i-th gauge field and g_i are gauge coupling constants. One can go back to the original GLσM (2.29) by taking the $g_i\to\infty$ limit. In the vacua of this system, the gauge symmetry $\mathrm{U}(N_1)\times\cdots\times\mathrm{U}(N_L)$ is spontaneously broken and hence this model admits BPS vortex configurations, satisfying the boundary conditions

$$\lim_{|z|\to\infty}Q_i=U_i(\theta)^\dagger\left(\mathcal{Q}_i^o,\mathbf{0}\right)U_{i+1}(\theta),\qquad \lim_{|z|\to\infty}A_\mu^i dx^\mu=-iU_i(\theta)^\dagger dU_i(\theta), \tag{3.2}$$

where $\mathcal{Q}_i^o$ is the constant square matrices defined in eq. (2.39) and $U_i(\theta)\in\mathrm{U}(N_i)$ are nontrivial elements of the gauge group depending on $\theta=\arg z$. Each $U_i(\theta)$ carries the topological charges of $\pi_1(\mathrm{U}(N_i))=\mathbb{Z}$, which is related to the magnetic flux of i-th overall U(1) factor

$$k_i\equiv-\frac{1}{2\pi}\int d^2x\,\mathrm{Tr}F_{12}^i=\frac{i}{2\pi}\int_0^{2\pi}d\theta\,\mathrm{Tr}\left[U_i(\theta)^\dagger\partial_\theta\mathrm{U}(\theta)\right]\in\mathbb{Z}\quad\text{for } 1\le i\le L. \tag{3.3}$$

In the large gauge coupling limit $g_i\to\infty$, these vortex solutions reduce to the sigma model instantons (or singular configurations). Eliminating the auxiliary field D_i by solving their equations of motion

$$D_i=\frac{g_i^2}{2}\left(Q_iQ_i^\dagger-Q_{i-1}^\dagger Q_{i-1}-r_i\mathbf{1}_{N_i}\right), \tag{3.4}$$

we can rewrite the Lagrangian as

$$\int d^2x\,\mathcal{L}=\sum_{i=1}^{L}\int d^2x\,\mathrm{Tr}\left[\frac{g_i^2}{4}\mathcal{E}_i^2+4\tilde{\mathcal{E}}_i\tilde{\mathcal{E}}_i^\dagger-r_iF_{12}^i\right]\ \ge\ 2\pi\sum_{i=1}^{L}r_ik_i \tag{3.5}$$

where we have defined

$$\tilde{\mathcal{E}}_i\equiv\mathcal{D}_{\bar{z}}Q^i, \tag{3.6}$$

$$\mathcal{E}_i\equiv Q_iQ_i^\dagger-Q_{i-1}^\dagger Q_{i-1}-r_i\mathbf{1}_{N_i}-\frac{2}{g_i^2}F_{12}^i \tag{3.7}$$

with $\mathcal{D}_{\bar{z}}Q^i\equiv\frac{1}{2}(\mathcal{D}_1+i\mathcal{D}_2)Q^i$. For a fixed set of topological charges $(k_1,\cdots,k_L)$, the action is minimized when $\mathcal{E}_i=\tilde{\mathcal{E}}_i=0$, i.e. the following equations are satisfied

$$0=\mathcal{D}_{\bar{z}}Q^i, \tag{3.8}$$

$$0=Q_iQ_i^\dagger-Q_{i-1}^\dagger Q_{i-1}-r_i\mathbf{1}_{N_i}-\frac{2}{g_i^2}F_{12}^i. \tag{3.9}$$

These equations are called the BPS equations for vortices.

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Solutions to these equations describe configurations of vortices, that is, squeezed magnetic fluxes in the Higgs phase. The vortices in this system are classified as *local vortices* or *semi-local vortices* depending on their asymptotic behaviors at the spatial infinity. The local vortex exhibits an exponentially dumping behavior and is obtained by, roughly speaking, embedding well-known Abrikosov-Nielsen-Olesen vortex into the matrix elements. The semi-local vortex has a (decreasing) power-law behavior due to the tails of the massless Nambu-Goldstone fields parameterizing the moduli space of vacua $\mathcal{F}_{n_1,\ldots,n_{L+1}}$. The size of a semi-local vortex is one of moduli parameters of vortices and hence it can become arbitrarily large without changing the energy. However, it has a minimum size of order $\mathcal{O}(1/g\sqrt{r})$. In the small size limit, the semi-local vortex reduces to the local type.

The master equation and the boundary condition. Let us rewrite the BPS equations into a convenient form. The first set of BPS equations (3.8) can be solved as

$$Q_i = S_i^{-1} q_i(z)\, S_{i+1}, \qquad A_{\bar z} \equiv \frac{1}{2}(A_1^i + iA_2^i) = -iS_i^{-1}\partial_{\bar z} S_i, \tag{3.10}$$

where S_i are elements of $\mathrm{GL}(N_i,\mathbb{C})$ ($S_{L+1} = \mathbf{1}$) and $q_i(z)$ are N_i-by-N_{i+1} matrices whose entries are arbitrary polynomials of $z = x_1 + ix_2$.[8] Note that the description in terms of q_i and S_i is redundant since the following transformation does not change the original fields Q_i and A_i:

$$q_i(z) \to V_i(z) q_i(z) V_{i+1}^{-1}(z), \qquad S_i(z,\bar z) \to V_i(z) S_i(z,\bar z), \qquad \text{with} \quad V_i(z) \in \mathrm{GL}(N_i,\mathbb{C}). \tag{3.11}$$

This is a complexified gauge transformation depending on the holomorphic coodinate $z = x_1 + ix_2$. We call this transformation the *V-transformation*. To study the moduli space of vortices, it is convenient to define N_i-by-N matrices $\xi_i(z)$ as

$$\xi_i(z) \equiv q_i(z) q_{i+1}(z) \cdots q_L(z), \qquad \text{for } 1 \le i \le L. \tag{3.12}$$

For a given set of matrices $(q_1(z),\cdots,q_L(z))$, the matrices $S_i(z,\bar z)$ are determined (up to gauge transformations) by solving the second BPS equation (3.9), which can be rewritten in terms of $\Omega_i = S_i S_i^\dagger \in \mathrm{GL}(N_i,\mathbb{C})$ as

$$q_i \Omega_{i+1} q_i^\dagger \Omega_i^{-1} - \Omega_i q_{i-1}^\dagger \Omega_{i-1}^{-1} q_{i-1} = r_i \mathbf{1}_{N_i} - \frac{4}{g_i^2}\partial_{\bar z}\left(\partial_z \Omega_i \Omega_i^{-1}\right), \qquad (q_0 = 0). \tag{3.13}$$

We call this set of equations *the master equation* for vortices. Once we determine Ω_i by solving the master equation, we can obtain the original fields Q_i and A_μ satisfying the BPS equation for vorties. The boundary conditions for Ω_i can be determined as follows. Without loss of generality, we can fix the vacuum at the spatial infinity to the point on $\mathcal{F}_{n_1,\ldots,n_{L+1}}$ corresponding to the standard flag $\mathcal{V}_0^o \subset \mathcal{V}_1^o \subset \cdots \subset \mathcal{V}_N^o \subset \mathcal{V}$ as

$$\xi_i(z) \quad \text{“}\to\text{”} \quad \xi_i^o = q_i^o q_{i+1}^o \cdots q_L^o = (\mathbf{1}_{N_i}, \mathbf{0}_{N_i,N-N_i}), \quad \text{for } |z| \to \infty, \tag{3.14}$$

up to the redundancy of the V-transformation (3.11). To precisely describe what the limit “$\to$” means, let us decompose the N_i-by-N matrix $\xi_i(z)$ into a N_i-by-N_i matrix $\mathfrak{D}(z)$ and

[8] Although these entries are arbitrary entire functions in general, we can assume that they are polynomials without loss of generality as shown in appendix C.1.

a N_i-by-$(N-N_i)$ matrix $\widetilde{\mathfrak{D}}(z)$ as $\xi_i(z) = (\mathfrak{D}_i(z), \widetilde{\mathfrak{D}}_i(z))$. This decomposition corresponds to the orthogonal decomposition of the vector space $\mathcal{V} = \mathcal{V}_i^o \oplus \mathcal{V}_i^{o\perp}$

$$\xi_i(z) = (\mathfrak{D}_i(z), \widetilde{\mathfrak{D}}_i(z)) \quad \leftrightarrow \quad \mathfrak{D}_i(z) \equiv \xi_i(z)\, \xi_i^{o\dagger}, \qquad \widetilde{\mathfrak{D}}_i(z) \equiv \xi_i(z)\, \xi_i^{o\perp\dagger}, \tag{3.15}$$

where $\xi_i^{o\perp} = (\mathbf{0}_{N-N_i,N_i}, \mathbf{1}_{N-N_i})$. Using the N_i-by-N_i matrix $\mathfrak{D}_i(z)$, we set the boundary conditions for $\xi_i(z)$ as

$$\lim_{|z|\to\infty} \{\mathfrak{D}_i(z)^{-1} \xi_i(z)\} = \xi_i^o. \tag{3.16}$$

Correspondingly, the boundary coniditions for $\{q_i, S_i\}$ are given by

$$\lim_{|z|\to\infty} \left\{\mathfrak{D}_i(z)^{-1} q_i(z)\, \mathfrak{D}_{i+1}(z)\right\} = q_i^{\rm o}, \qquad \lim_{|z|\to\infty} \left\{\mathfrak{D}_i(z)^{-1} S_i(z,\bar{z})\right\} = S_i^{\rm o} U_i(\theta), \tag{3.17}$$

where $\mathfrak{D}_{L+1}(z) = \mathbf{1}_N$, $U_i(\theta) \in \mathrm{U}(N_i)$ and S_i^o is a matrix such that $\Omega_i^o = S_i^o (S_i^o)^\dagger$ is the diagonal matrix defined in eq. (2.38). These boundary conditions correspond to those for the original quantities eq. (3.2). Note that the transformations with $\mathfrak{D}_i(z)$ in eq. (3.17) can be regraded as local V-transformations analogous to singular gauge transformations, which are regular in the asymptotic region $|z| \to \infty$. Substituting these settings to the master equation, we find that the asymptotic behavior of Ω_i is given by

$$\mathfrak{D}_i(z)^{-1}\, \Omega_i(z\,\bar{z})\, \mathfrak{D}_i(z)^{\dagger -1} = \Omega_i^o + \mathcal{O}(|z|^{-2}) \quad \text{for} \quad |z| \to \infty. \tag{3.18}$$

These are the boundary conditions for the master equations (3.13).

In terms of these matrices satisfying the boundary conditions given above, the vortex numbers k_i defined in (3.3) are given by

$$k_i = \frac{1}{4\pi i} \oint (dz\partial_z - d\bar{z}\partial_{\bar{z}})\, \log |\det \mathfrak{D}_i(z)|^2, \tag{3.19}$$

where we have used the following formula for the magnetic flux and its asymptotic behavior

$$\mathrm{Tr} F_{12}^i = -2\partial_{\bar{z}}\partial_z \log\det\Omega_i, \qquad \log\det\Omega_i = \log|\det\mathfrak{D}(z)|^2 + \text{const.} + \mathcal{O}(|z|^{-2}) \quad \text{for} \quad |z| \to \infty. \tag{3.20}$$

Eq. (3.19) implies that the matrices $(q_1(z), \cdots, q_L(z))$ must be chosen such that $\det\mathfrak{D}_i(z)$ has k_i zeros in the topological sector with vortex numbers $(k_1, \cdots, k_L)$.[9] The zeros of $\det\mathfrak{D}_i(z)$ can be regarded as the positions of vortices. To see this, let us consider the p-th moment of the i-th magnetic fluxes. If all vortices are well separated from each other, the magnetic flux F_{12}^i is symmetrically localized around k_i distinct points $\{z_{(i,\alpha)} | \alpha = 1, 2, \ldots, k_i\}$. In such a case, the p-th moment is given by

$$\langle z^p \rangle_i \equiv -\frac{1}{2\pi} \int d^2x\, z^p\, \mathrm{Tr}[F_{12}^i] = \sum_{\alpha=1}^{k_i} (z_{(i,\alpha)})^p. \tag{3.21}$$

On the other hand, using (3.20), we can calculate $\langle z^p \rangle_i$ as

$$\langle z^p \rangle_i = \frac{1}{2\pi i} \lim_{R\to\infty} \oint_{|z|=R} dz\, z^p \partial_z \log\det\mathfrak{D}_i(z) = \sum_{\alpha=1}^{k_i} (w_{(i,\alpha)})^p, \tag{3.22}$$

[9] Using the V-transformation, $\det\mathfrak{D}_i(z)$ can be set to be a monic polynomial of degree k_i.

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where $\{w_{(i,\alpha)}|\alpha = 1, 2, \ldots, k_i\}$ are zeros of $\det \mathfrak{D}_i(z)$. From (3.21) and (3.22), the vortex positions $\{z_{i,\alpha}\}$ can be identified with the zeros $\{w_{i,\alpha}\}$ of $\det \mathfrak{D}_i(z)$.[10] Extending this identification, we adopt $\{w_{(i,\alpha)}\}$ as the definition of the vortex positions even if several vortices are in close proximity and their flux profiles are overlapping.

Uniqueness and existence of solution. Under the boundary conditions given above, one can prove the uniqueness of the solution $\{\Omega_i\}$ to the set of the master equations (3.13) with a given set of $\{q_i(z)\}$ (see appendix B for the proof). In the following, we only assume that

> *there exists a solution for the set of the master equations* (3.13) *with a given set of* $q_i = q_i(z)$ *such that* $\Omega_i = \Omega_i(z, \bar{z})$ *is an element of* $\mathrm{GL}(N_i, \mathbb{C})$ *everywhere and all entries of* Ω_i *are smooth functions of* z *and* $\bar{z}$.

At least, this assumptions is true in the large coupling limit $g_i^2 \to \infty$ as will be explained later. The set of the master equations is a generalization of the so-called Taubes equation, where this assumption has been shown to be true [79].

The moduli matrices and the moduli space of vortices. If the above assumption for the existence of the solution is true, there is a one-to-one correspondence between the moduli space of vortices and the set of the equivalence classes defined by the V-transformation

$$q_i(z) \sim V_i(z)\, q_i(z)\, V_{i+1}^{-1}(z). \tag{3.23}$$

Hence the matrices $q_i(z)$ are called *the moduli matrices*. As shown in appendix C, all the entries of the matrices $q_i(z)$ and $V_i(z)$ can be assumeed to be polynomials. From eqs. (3.16) and (3.19), we find that the boundary conditions for vortex configurations carrying the topological charges $\{k_i\}$ are expressed in terms of $\xi_i(z) = (\mathfrak{D}_i(z), \widetilde{\mathfrak{D}}_i(z))$ as

$$\det \mathfrak{D}_i(z) = \mathcal{O}(z^{k_i}), \quad \mathfrak{D}_i(z)^{-1}\widetilde{\mathfrak{D}}_i(z) = \mathcal{O}(z^{-1}). \tag{3.24}$$

Roughly speaking, the first and second conditions in (3.24) specify the vortex numbers $\{k_i\}$ and the vacuum at the spacial infinity, respectively. For fixed vortex numbers and boundary conditions, we define the moduli space of vortices as the space of equivalence classes of the matrices $(q_1(z), \ldots, q_L(z))$ satisfying the condition (3.24)

$$\mathcal{M}_{\mathrm{vtx}}{}^{n_1,n_2,\ldots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = \Big\{(q_1(z), q_2(z), \ldots, q_L(z)) \,\Big|\, q_i(z) : \text{conditions (3.24)}\Big\} / \sim, \tag{3.25}$$

where $\sim$ denotes the equivalence relation (3.23). We can determine the dimension of the moduli space by counting the number of zero modes satisfying the linearized BPS equations around a BPS configuration. As discussed in [7] and [80], the number of zero modes can be determined by the index theorem as

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{vtx}}{}^{n_1,n_2,\ldots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = -\sum_{i=1}^{L} \frac{N_{i+1} - N_{i-1}}{2\pi} \int d^2x \, \mathrm{Tr} F_{12}^i = \sum_{i=1}^{L} (N_{i+1} - N_{i-1}) k_i = \sum_{i=1}^{L} \beta_i k_i, \tag{3.26}$$

where $\beta_i = N_{i+1} - N_{i-1}$ is the first coefficient of the beta function (2.58).

[10] Since $\mathfrak{D}_i(z)$ can be reconstructed as $\det \mathfrak{D}_i(z) = z^{k_i} \exp(-\sum_{p=1}^{\infty} \langle z^p \rangle_i / (p z^p))$, each zero $w_{(i,\alpha)}$ can always be uniquely read from any configuration of the magnetic flux.

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The moduli space of BPS vortices is endowed with a natural complex structure such that the variables ϕ^A $(A = 1, \cdots, \sum_{i=1}^{L} \beta_i k_i)$ holomorphically parametrizing the equivalence class of the V-transformation (3.23) are the complex coordinates of the vortex moduli space. Furthermore, the vortex moduli space is equipped with a Kähler metric that determines classical dynamics of the vortices. As shown in appendix B.4, the Kähler metric on the vortex moduli space is given by the formula

$$g_{A\bar{B}} = \int d^2x \, \frac{\partial}{\partial \bar{\phi}^{\bar{B}}} \sum_{i=1}^{L} \mathrm{Tr}\left[\Omega_i^{-1} \frac{\partial q_i}{\partial \phi^A} \Omega_{i+1} q_i^\dagger\right]_{\Omega=\Omega^{\mathrm{sol}}}. \tag{3.27}$$

Local and semi-local vortices. The first condition in (3.24) implies that $\xi_i(z)$ are full rank matrices at a generic point $z \in \mathbb{C}$. Such a set of full rank matrices $(\xi_1(z), \cdots, \xi_L(z))$ at a point z specifies a flag $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_L \subset \mathcal{V}$ and hence a point in the flag manifold $\mathcal{F}_{n_1,\cdots,n_{L+1}}$. Therefore, if $\xi_i(z)$ are full rank matrices $(\mathrm{rank}(\xi_i(z)) = N_i)$ everywhere on $\mathbb{C}$, that is,

$$\det \xi_i(z)\xi_i(z)^\dagger \neq 0, \quad \forall z \in \mathbb{C}, \quad (i = 1, \cdots, L), \tag{3.28}$$

the set of matrices $(\xi_1(z), \cdots, \xi_L(z))$ gives a holomorphic map $\mathbb{C} \to \mathcal{F}_{n_1,\cdots,n_{L+1}}$. In such a case, we can solve the equation (3.13) in the large gauge coupling limit $g_i \to \infty$ by promoting the vacuum solution satisfying (2.34) into a z-dependent configuration

$$\Omega_i = \frac{1}{a_{ii}} \xi_i(z)\xi_i(z)^\dagger + \sum_{j=1}^{i-1} \left(\frac{1}{a_{ji}} - \frac{1}{a_{j+1,i}}\right) \xi_i(z)\xi_j(z)^\dagger (\xi_j(z)\xi_j(z)^\dagger)^{-1} \xi_j(z)\xi_i(z)^\dagger. \tag{3.29}$$

Comparing physical quantities such as energy density, we can confirm that the vortex configuration reduces to the instanton solution specified by the same set of matrices $(\xi_1, \cdots, \xi_L)$.[11] Therefore, the moduli space of sigma model instantons is given by restricting the vortex moduli space (3.25) with the additional condition (3.28)

$$\mathcal{M}_{\mathrm{inst}}{}^{n_1,n_2,\ldots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = \Big\{ (q_1(z), q_2(z), \ldots, q_L(z)) \,\Big|\, q_i(z) : \text{eq. (3.24), eq. (3.28)} \Big\} / \sim. \tag{3.30}$$

The points removed by the condition (3.28) are configurations with matrices $\xi_i(z)$ whose rank becomes smaller $(\mathrm{rank}(\xi_i(z)) < N_i)$ at some points on $\mathbb{C}$. Such configuration cannot be viewed as a holomprphic map since ξ_i must be full rank matrices on the flag manifold. As we approach the removed points on the instanton moduli space, the sizes of some insntantons become infinitesimally small and hence such points are called the small instanton singularities. Although instanton configurations are singular at such points, the corresponding vortex solutions are regular as long as the gauge coupling constants g_i are finite. Instead of the singular instantons, regular vortices with size $\mathcal{O}(1/g\sqrt{r})$ are located at the points where $\mathrm{rank}(\xi_i) < N_i$ when the gauge coupling constants are finite. Such vortices are the so-called

[11] From these matrices $\xi_i(z)$, we can always construct an instanton solution in terms of the inhomogeneous coordinates by comparing $\xi_i(z)$ and $\mathcal{U} = \mathcal{U}(z)$ in eq. (2.18) at each z as

$$\xi_i(z) = \mathcal{L}_i(z)\, \xi_i^o\, \mathcal{U}(z)$$

where $\mathcal{L}_i(z)$ is an appropriate element of the parabolic subgroup $\hat{H}(n_1, \cdots, n_i)$ except for some singular points corresponding to the zeros of $\det \mathcal{L}_i(z) = \det \mathfrak{D}_i(z) = \mathcal{O}(z^{k_i})$.

local vortices whereas vortices corresponding to regular instantons are called the semi-local vortices. A set of matrices $\{\xi_i\}$ with $\widetilde{\mathfrak{D}}_i = 0$ corresponds to a configuration in which all the vortices are of local type. The corresponding modui subspace is called the local vortex moduli space while we call the subspace with no local vortex the semi-local vortex moduli space. As shown in appendix D.3, there is a one-to-one correspondence between the moduli spaces of semi-local vortices and sigma model instantons

$$\mathcal{M}_{\text{semi}}{}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = \mathcal{M}_{\text{inst}}{}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L}. \tag{3.31}$$

The semi-local vortex moduli space can be obtained from the local one by turning on $\widetilde{\mathfrak{D}}_i$, which corresponds to the fibration on the local vortex moduli space. In the large gauge coupling limit $g_i \to \infty$, the local vortex moduli space shrinks to the small instanton singularity.

For a dual pair of GLσMs, the moduli space of sigma model instantons are identical since the corresponding NLσMs agree in the $g_i \to \infty$ limit. On the other hand, as we will see below, the votex moduli spaces can be viewed as different regularizations of the instanton moduli space

$$\begin{array}{ccc} \mathcal{M}_{\text{inst}}{}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L} & = & \mathcal{M}_{\text{inst}}{}^{n_{L+1},\dots,n_2,n_1}_{k_L,\cdots,k_2,k_1} \\ \cap & & \cap \\ \mathcal{M}_{\text{vtx}}{}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L} & \neq & \mathcal{M}_{\text{vtx}}{}^{n_{L+1},\dots,n_2,n_1}_{k_L,\cdots,k_2,k_1}. \end{array} \tag{3.32}$$

In section 5, we will check the duality of instanton solutions by presenting explicit biholomorphic map on the moduli parameters. The small-instanton singularities are regularized in different ways in the dual pair of GLσMs, i.e. both models have distinct local vortex moduli spaces. Nevertheless, as we will check in section 6, the vortex partition functions computed using the information on the local vortex moduli spaces are in perfect agreement.

3.2 Example 1: review of $L = 1$ case

As the simplest example, let us review the case with $L = 1$ [9] where the target manifold of the NLσM is the complex Grassmaniann $G(N, n)$ ($n_1 = n$, $n_2 = N - n$). Here we omit the index i ($i = 1, \cdots, L$) since there is only one gauge group factor U(n) in this case.

For a k-vortex configuration, the matrices $\mathfrak{D}(z)$ and $\tilde{\mathfrak{D}}(z)$ are n-by-n and n-by-$(N - n)$ matrices satisfying

$$\det \mathfrak{D}(z) = \mathcal{O}(z^k), \qquad \mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1}). \tag{3.33}$$

Two pairs of matrices $(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$ and $(\mathfrak{D}(z)', \widetilde{\mathfrak{D}}(z)')$ are equivalent if there is a matrix $V(z) \in \mathrm{GL}(N, \mathbb{C})$ such that

$$(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z)) = (V(z)\mathfrak{D}(z)', V(z)\widetilde{\mathfrak{D}}(z)') \quad \Longleftrightarrow \quad (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z)) \sim (\mathfrak{D}(z)', \widetilde{\mathfrak{D}}(z)'). \tag{3.34}$$

Therefore, the moduli space of k-vortex configurations is given by

$$\mathcal{M}_{\text{vtx}}{}^{n,N-n}_{k} \cong \left\{ (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z)) \,\middle|\, \det \mathfrak{D}(z) = \mathcal{O}(z^k), \mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1}) \right\} / \sim. \tag{3.35}$$

This moduli space can be parametrized in the following way (see appendix C.1 for more details). Let $\lambda = (l_1, l_2, \cdots, l_n)$ be a set of non-negative integers such that $k = \sum_{b=1}^{n} l_b$.

By using the V-transformation (3.34), a generic matrix $\mathfrak{D}(z)$ with $\det\mathfrak{D}(z) = \mathcal{O}(z^k)$ can be transformed into the following form

$$\mathfrak{D}(z) = \mathfrak{D}_\lambda(z) \equiv \begin{pmatrix} z^{l_1} & & \\ & \ddots & \\ & & z^{l_n} \end{pmatrix} + \begin{pmatrix} P^{11} & \cdots & P^{1n} \\ \vdots & \ddots & \vdots \\ P^{n1} & \cdots & P^{nn} \end{pmatrix}, \qquad P^{ab}(z) = \sum_{m=1}^{l_b} T_m^{ab}\, z^{m-1}. \tag{3.36}$$

For each "gauge choice", i.e. the choice of $(l_1, \cdots, l_n)$, all the degrees of freedom of the V-transformation (3.34) is fixed. Therefore, the coefficients T_m^{ab} can be regarded as part of the complex coordinates of the moduli space of vortices in this coordinate patch, which we call the $(l_1, l_2, \cdots, l_n)$-patch. The other coordinates parameterize the degrees of freedom contained in the matrix $\widetilde{\mathfrak{D}}(z)$ obeying the condition $\mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1})$. To extract such degrees of freedom, let us consider n-component column vectors $j_i(z)$ with polynomial entries satisfying

$$\mathfrak{D}(z)^{-1} j_i(z) = \mathcal{O}(z^{-1}) \quad (z \to \infty). \tag{3.37}$$

We can show that there exist k linearly independent solutions $j_i(z)$ $(i = 1, \cdots, k)$ satisfying this condition. Let $\mathfrak{J}(z)$ be an n-by-k matrix whose columns form a basis of the solutions to (3.37), that is, $\mathfrak{J}(z) = (j_1(z), \cdots, j_k(z))$. For instance, in the $(l_1, l_2, \cdots, l_n)$-patch (3.36), $\mathfrak{J}(z)$ can be chosen as

$$\mathfrak{J}_\lambda = \Big(\boldsymbol{\mathfrak{J}}_1 \boldsymbol{\mathfrak{J}}_2 \cdots \boldsymbol{\mathfrak{J}}_n \Big), \qquad (\boldsymbol{\mathfrak{J}}_\alpha)^a{}_p = \frac{\mathfrak{D}^a{}_\alpha}{z^p}\bigg|_{\mathrm{reg}} = \delta^a{}_\alpha\, z^{l_\alpha - p} + \sum_{m=1}^{l_\alpha - p} T_m^{a\alpha} z^{m-1}, \tag{3.38}$$

where $\boldsymbol{\mathfrak{J}}_\alpha$ are n-by-l_α block matrices and $|_{\mathrm{reg}}$ stands for the regular part. Since $\widetilde{\mathfrak{D}}(z)$ satisfies the condition $\mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1})$, it can be written as linear combinations of $j_i(z)$, that is, the matrix $\widetilde{\mathfrak{D}}(z)$ in the $(l_1, l_2, \cdots, l_n)$-patch can be written as

$$\widetilde{\mathfrak{D}}(z) = \widetilde{\mathfrak{D}}_\lambda(z) \equiv \mathfrak{J}_\lambda \widetilde{\Psi}_\lambda, \tag{3.39}$$

where $\widetilde{\Psi}$ is a k-by-$(N-n)$ matrix. The components of $\widetilde{\Psi}$ parameterize the degrees of freedom of $\widetilde{\mathfrak{D}}$ and hence can be regarded as the remaining coordinates of the moduli space of vortices.

One can check that the number of the coordinates T_m^{ab} $(a, b = 1, \cdots, n,\ m = 1, \cdots, l_b)$ and $\widetilde{\Psi}_{ic}$ $(i = 1, \cdots, k,\ c = 1, \cdots, N-n)$ agrees with the dimension of the moduli space $\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{vtx}}{}^{n,N-n}_{k} = kN$ obtained through the analysis of the index theorem [7] (see eq. (3.26)). There are $(k+n-1)!/(k!(n-1)!)$ patches and the transition functions can be read off from the V-transformation between two different fixed forms $(\mathfrak{D}_{\lambda'}, \widetilde{\mathfrak{D}}_{\lambda'}) = (V\,\mathfrak{D}_\lambda, V\,\widetilde{\mathfrak{D}}_\lambda)$ (see the example below).

Next, let us read off the sigma model instanton solutions in the large coupling limit from $\xi = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$. Let φ is the n-by-$(N-n)$ matrix which appears in the matrix $\mathcal{G}$ in eq. (2.18) as

$$\mathcal{G} = \mathcal{L} \begin{pmatrix} \mathbf{1}_n & \varphi \\ \mathbf{0} & \mathbf{1}_{N-n} \end{pmatrix} \in \mathrm{GL}(N, \mathbb{C}), \tag{3.40}$$

where $\mathcal{L}$ is a lower-triangular block matrix. The components of φ can be regarded as the inhomogeneous coordinates of Grassmannian $G(N,n)$. From the relation $(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z)) \sim$

$(\mathbf{1}_n, \mathbf{0})\,\mathcal{G}$, the matrix $\varphi(z)$ can be read off as

$$\varphi(z) = \mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z). \tag{3.41}$$

This implies that the inhomogeneous coordinate are rational functions of z. Note that only semilocal vortices can be observed as sigma model instantons. To see this, let us assume that if l vortices in $(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$ are of local type. Then, as shown in appendix D.1, the matrices $\mathfrak{D}(z)$ and $\widetilde{\mathfrak{D}}(z)$ have a common factor $\mathfrak{D}_{\mathrm{lc}}(z)$ representing local vortices

$$(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z)) = \mathfrak{D}_{\mathrm{lc}}(z)(\mathfrak{D}_{\mathrm{sm}}(z), \widetilde{\mathfrak{D}}_{\mathrm{sm}}(z)), \quad \det \mathfrak{D}_{\mathrm{lc}}(z) = \mathcal{O}(z^l), \tag{3.42}$$

where $\xi_{\mathrm{sm}}(z) = (\mathfrak{D}_{\mathrm{sm}}(z), \widetilde{\mathfrak{D}}_{\mathrm{sm}}(z))$ satisfies the condition (3.28). From this factorized form, we find that the local vortices do not appear in the sigma model instantons in the large coupling limit:

$$\varphi(z) = \mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathfrak{D}_{\mathrm{sm}}(z)^{-1}\widetilde{\mathfrak{D}}_{\mathrm{sm}}(z). \tag{3.43}$$

Abelian case. For $n = 1$, the gauge group is U(1) and $\mathfrak{D}(z)$ is a monic polynomial of z

$$\mathfrak{D}(z) = \prod_{\alpha}(z - z_\alpha)^{d_\alpha}, \qquad \sum_{\alpha} d_\alpha = k, \tag{3.44}$$

where d_α denotes the multiplicity of the vortex sitting at $z = z_\alpha$. The condition $\mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1})$ for $\widetilde{\mathfrak{D}}(z) = (\widetilde{\mathfrak{D}}_2(z), \widetilde{\mathfrak{D}}_3(z), \cdots, \widetilde{\mathfrak{D}}_N(z))$ can be solved by setting $\widetilde{\mathfrak{D}}_a(z)$ to be polynomials of degree $k-1$. Then the sigma model instanton solution takes the form

$$\frac{\widetilde{\mathfrak{D}}_a(z)}{\mathfrak{D}(z)} = \sum_{\alpha}\sum_{p=1}^{d_\alpha} \frac{c_{a,\alpha,p}}{(z - z_\alpha)^p}, \qquad \text{for } 2 \le a \le N. \tag{3.45}$$

Since the number of the coefficients is k for each flavor a, this solution has kN moduli parameters. For separated vortex configuration ($d_\alpha = 1$ for all α), each vortex has $N-1$ parameters in addition to its position modulus. For example, for $N = 2$ and $d_\alpha = 1$, the absolute values of the coefficient $c_{2,\alpha,1}$ determines the size of the vortex at $z = z_\alpha$, and hence $c_{2,\alpha,1}$ $(\alpha = 1, \cdots, k)$ are called *size moduli*. The phase of $c_{2,\alpha,1}$ corresponds to the Nambu-Goldstone mode of the $H = \mathrm{U}(1)$ global symmetry that is preserved by the vacuum but broken by the vortex. The configurations with $c_{a,\alpha,p=d_\alpha} = 0$ correspond to a small-instanton singularities.

Non-Abelian case. In the case of $n > 1$, vortices possesses another type of moduli parameters that the Abelian vortices do not have. Since $\mathfrak{D}(z)$ has a smaller rank at the center of each vortex ($z = z_\alpha$, $\det \mathfrak{D}(z_\alpha) = 0$), there exists an n-column vector ψ satisfying[12]

$$\mathfrak{D}(z_\alpha)\,\psi_\alpha = 0. \tag{3.46}$$

Each vector ψ_α, defined up to a normalization factor, specifies a point on $\mathbb{C}P^{n-1} = \mathrm{U}(n)/[\mathrm{U}(n-1) \times \mathrm{U}(1)]$. These degrees of freedom correspond to the Nambu-Goldstone

[12] A set of this type of relations for all vortices will be summarized in the equation (4.8) in the next section.

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zero mode of the $\mathrm{U}(n)$ color-flavor locked symmetry broken due to existence of a vortex. Even when a vortex has a vanishing size modulus, this type of *orientational moduli* survives and thus describes internal degrees of freedom of the local vortex.

Next, let us demonstrate how to describe the moduli space of vortices with the simplest example of non-Abelian vortices in the case of $N=3, n=2, k=1$ ($n_1=2, n_2=1$). There are two coordinate patches $(l_1,l_2)=(1,0)$ and $(0,1)$ for which the "gauge-fixed forms" of the matrix $\mathfrak{D}$ in eq. (3.36) are respectively given by

$$\mathfrak{D}_{(1,0)}(z) = \begin{pmatrix} z-a & 0 \\ -b & 1 \end{pmatrix}, \qquad \mathfrak{D}_{(0,1)}(z) = \begin{pmatrix} 1 & -\tilde{b} \\ 0 & z-\tilde{a} \end{pmatrix}. \tag{3.47}$$

The conditions for $\mathfrak{J}_{(1,0)}(z)=(J_1,J_2)^{\mathrm{T}}$ and $\mathfrak{J}_{(0,1)}(z)=(J_1',J_2')^{\mathrm{T}}$ are given by

$$\mathfrak{D}^{-1}_{(1,0)}\,\mathfrak{J}_{(1,0)} = \frac{J_1(z)}{z-a}\begin{pmatrix} 1 \\ b \end{pmatrix} + J_2(z)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathcal{O}(z^{-1}). \tag{3.48}$$

$$\mathfrak{D}^{-1}_{(0,1)}\,\mathfrak{J}_{(0,1)} = \frac{J_2'(z)}{z-a}\begin{pmatrix} \tilde{b} \\ 1 \end{pmatrix} + J_1'(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathcal{O}(z^{-1}). \tag{3.49}$$

By using the solutions to these conditions $\mathfrak{J}_{(1,0)}=(1,0)^T$ and $\mathfrak{J}_{(0,1)}=(0,1)^T$, the matrices $\widetilde{\mathfrak{D}}_{(1,0)}$ and $\widetilde{\mathfrak{D}}_{(0,1)}$ can be written as

$$\widetilde{\mathfrak{D}}_{(1,0)} = c\,\mathfrak{J}_{(1,0)} = \begin{pmatrix} c \\ 0 \end{pmatrix}, \qquad \widetilde{\mathfrak{D}}_{(0,1)} = \tilde{c}\,\mathfrak{J}_{(0,1)} = \begin{pmatrix} 0 \\ \tilde{c} \end{pmatrix}, \tag{3.50}$$

where c and $\tilde{c}$ are constants. The parameters (a,b,c) and $(\tilde{a},\tilde{b},\tilde{c})$ are the coordinates of the moduli space in the (l_1,l_2)-patches with $(l_1,l_2)=(1,0)$ and $(l_1,l_2)=(0,1)$, respectively. They are related by the coordinate transformation

$$(\tilde{a},\tilde{b},\tilde{c}) = (a, b^{-1}, cb), \tag{3.51}$$

which can be determined from the regularity condition for the V-transformation between the two gauge-fixed forms of $\mathfrak{D}$

$$V\,\mathfrak{D}_{(1,0)} = \mathfrak{D}_{(0,1)} \quad\Longrightarrow\quad V(z) = \mathfrak{D}_{(0,1)}(\mathfrak{D}_{(1,0)})^{-1} = \begin{pmatrix} \frac{1-b\tilde{b}}{z-a} & -\tilde{b} \\ b\frac{z-\tilde{a}}{z-a} & z-\tilde{a} \end{pmatrix}, \tag{3.52}$$

and the relation $V\,\widetilde{\mathfrak{D}}_{(1,0)} = \widetilde{\mathfrak{D}}_{(0,1)}$. The parameter a and c are the position and size moduli of the vortex, respectively. The parameter b is the inhomogeneous coordinate of the orientational moduli $\mathbb{C}P^1$. The sigma model solution $\varphi_{12}(z)=\mathfrak{D}^{-1}\widetilde{\mathfrak{D}}$ in the $(1,0)$- and $(0,1)$-patches take the forms

$$\varphi_{12}^{(1,0)}(z) = \frac{c}{z-a}\begin{pmatrix} 1 \\ b \end{pmatrix}, \qquad \varphi_{12}^{(0,1)}(z) = \frac{\tilde{c}}{z-a}\begin{pmatrix} \tilde{b} \\ 1 \end{pmatrix}. \tag{3.53}$$

By using the coordinate transformation (3.51), one can confirm that these are identical $\varphi_{12}^{(1,0)}(z) = \varphi_{12}^{(0,1)}(z)$ on the overlap of the coordinate patches.

Duality and moduli spaces. In the case of $N=3, n=2, k=1$ $(n_1=2, n_2=1)$ discussed above, the total vortex moduli space is given by

$$\mathcal{M}_{\text{vtx}}{}_{k=1}^{n_1=2,n_2=1} = \mathbb{C} \times \mathcal{O}_{\mathbb{C}P^1}(-1), \tag{3.54}$$

where the first factor corresponds to the vortex position $a \in \mathbb{C}$ and the second factor is the total space of the line bundle $\mathcal{O}_{\mathbb{C}P^1}(-1)$ over $\mathbb{C}P^1$

$$\mathcal{O}_{\mathbb{C}P^1}(-1) = \left\{(b_1,b_2,c) \,\middle|\, (b_1,b_2) \in \mathbb{C}^2 \backslash \{(0,0)\}, c \in \mathbb{C}\right\} / \sim \quad \text{with} \quad (b_1,b_2,c) \sim (\lambda b_1, \lambda b_2, \lambda^{-1} c). \tag{3.55}$$

By removing the subspace $c=0$ corresponding to the small-instanton singularity, this space reduces to the moduli space of sigma model instanton

$$\mathcal{M}_{\text{inst}}{}_{k=1}^{n_1=2,n_2=1} = \mathbb{C} \times \left\{(b_1,b_2,c) \,\middle|\, (b_1,b_2) \in \mathbb{C}^2 \backslash \{(0,0)\},\ c \in \mathbb{C} \backslash \{0\}\right\} / \sim = \mathbb{C} \times \left(\mathbb{C}^2 \backslash \{(0,0)\}\right). \tag{3.56}$$

In the dual theory, which is an Abelian theory with $N=3, n=1$ $(n_1=1, n_2=2)$, the vortex and instanton moduli spaces are respectively given by

$$\mathcal{M}_{\text{vtx}}{}_{k=1}^{n_1=1,n_2=2} = \mathbb{C} \times \mathbb{C}^2, \qquad \mathcal{M}_{\text{inst}}{}_{k=1}^{n_1=1,n_2=2} = \mathbb{C} \times \left(\mathbb{C}^2 \backslash \{(0,0)\}\right). \tag{3.57}$$

We can confirm that even though the instanton moduli spaces are identical, the vortex moduli spaces are different in the dual theories

$$\begin{array}{ccc} \mathcal{M}_{\text{inst}}{}_{k=1}^{n_1=2,n_2=1} & = & \mathcal{M}_{\text{inst}}{}_{k=1}^{n_1=1,n_2=2} \\ \cap & & \cap \\ \mathcal{M}_{\text{vtx}}{}_{k=1}^{n_1=2,n_2=1} & \neq & \mathcal{M}_{\text{vtx}}{}_{k=1}^{n_1=1,n_2=2}. \end{array} \tag{3.58}$$

In each theory, the small-instanton singularity is regularized by replacing it with the local vortex moduli space in each theory

$$\mathcal{M}_{\text{vtx}}{}_{k=1}^{n_1=2,n_2=1} \backslash \mathcal{M}_{\text{inst}}{}_{k=1}^{n_1=2,n_2=1} = \mathbb{C} \times \mathbb{C}P^1, \qquad \mathcal{M}_{\text{vtx}}{}_{k=1}^{n_1=1,n_2=2} \backslash \mathcal{M}_{\text{inst}}{}_{k=1}^{n_1=1,n_2=2} = \mathbb{C}. \tag{3.59}$$

In other words, the singularity is blown up in the case of $n_1=1$, $n_2=2$ whereas $\mathbb{C} \times \{(0,0)\}$ is added along the singularity in the case of $n_1=2$, $n_2=1$.

3.3 Example 2: $L=2$ case

Next, we consider the case with $L=2$. As we have seen in the previous example with $L=1$, all the information on the moduli space of vortices is contained in the matrix $\xi = (\mathfrak{D}, \widetilde{\mathfrak{D}})$ obeying the constraints $\det \mathfrak{D}(z) = \mathcal{O}(z^k)$ and $\mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1})$. For $L>1$, the matrices $\xi_i = (\mathfrak{D}_i, \widetilde{\mathfrak{D}}_i)$ must satisfy additional constraints since they are composite quantities obtained from q_i $(L=1,\cdots,N)$. For example, in the case of $L=2$, ξ_1 and ξ_2 are related as $\xi_1(z) = q_1(z)\xi_2(z)$ and hence they are not independent. To read off the information on the vortex moduli space, we need to clarify the constraints for $\xi_i(z)$.

For simplicity, let us focus on the case of vortex numbers $(k_1,k_2)=(1,1)$ in the model with $n_1=n_2=n_3=1$ as the simplest nontrivial example.[13] It follows from the conditions $\det\mathfrak{D}_i=\mathcal{O}(z^{k_i})$ and $\mathfrak{D}_i^{-1}\widetilde{\mathfrak{D}}_i=\mathcal{O}(z^{-1})$ that the matrices $\xi_1(z)$ and $\xi_2(z)$ for $(k_1,k_2)=(1,1)$ take the forms

$$\xi_1(z)=(z-a',c',c''),\qquad \xi_2(z)=(\mathfrak{D}_{(1,0)}(z),\widetilde{\mathfrak{D}}_{(1,0)}(z))=\begin{pmatrix} z-a & 0 & c\\ -b & 1 & 0\end{pmatrix}, \tag{3.60}$$

where we have fixed $\xi_2(z)=(\mathfrak{D}_2,\widetilde{\mathfrak{D}}_2)$ so that it takes the $(l_1,l_2)=(1,0)$ form given in eq. (3.47) and eq. (3.50). These matrices must be related as $\xi_1(z)=q_1(z)\xi_2(z)$ with a certain non-singular 1-by-2 matrix $q_1(z)$. The regularity of $q_1(z)$ requires that the parameters are related as

$$a'=a+bc'\qquad c''=c. \tag{3.61}$$

In the $(0,1)$-patch, $\xi_1(z)$ and $\xi_2(z)=(\mathfrak{D}_{(0,1)}(z),\widetilde{\mathfrak{D}}_{(0,1)}(z))$ take the forms

$$\xi_1(z)=(z-a',c',c''),\qquad \xi_2(z)=(\mathfrak{D}_{(0,1)}(z),\widetilde{\mathfrak{D}}_{(0,1)}(z))=\begin{pmatrix} 1 & -\tilde b & 0\\ 0 & z-\tilde a & \tilde c\end{pmatrix}, \tag{3.62}$$

where the parameters $(\tilde a,\tilde b,\tilde c)$ are related to (a,b,c) in the same way as the case of $L=1$ (3.51). The regularity of $q_1(z)$ requires that

$$c'=\tilde b\,(a'-\tilde a),\qquad c''=\tilde b\,\tilde c. \tag{3.63}$$

This relation is consistent with (3.61) and (3.51). The constraints on (3.60) and (3.62) with (3.51) implies that the moduli space is given by

$$\mathcal{M}_{\mathrm{vtx}}{}^{n_1=1,n_2=1,n_3=1}_{k_1=1,k_2=1}=\mathbb{C}\times(\mathcal{O}_{\mathbb{C}P^1}(-1)\oplus\mathcal{O}_{\mathbb{C}P^1}(-1)), \tag{3.64}$$

where $\mathbb{C}$ is the center of mass position parametrized by $a+a'$ and $\mathcal{O}_{\mathbb{C}P^1}(-1)\oplus\mathcal{O}_{\mathbb{C}P^1}(-1)$ is the space given by

$$\mathcal{O}_{\mathbb{C}P^1}(-1)\oplus\mathcal{O}_{\mathbb{C}P^1}(-1)=\Big\{(b_1,b_2,c_1,c_2)\,\big|\,(b_1,b_2)\in\mathbb{C}^2\backslash\{(0,0)\},\,(c_1,c_2)\in\mathbb{C}^2\Big\}/\sim, \tag{3.65}$$

where the equivalence relation $\sim$ is defined as

$$(b_1,b_2,c_1,c_2)\sim(\lambda b_1,\lambda b_2,\lambda^{-1}c_1,\lambda^{-1}c_2)\quad\text{with}\quad \lambda\in\mathbb{C}^*. \tag{3.66}$$

The coordinates in the $(1,0)$ and $(0,1)$ patches are related to the $\mathbb{C}^*$ invariants as

$$b=\frac{b_1}{b_2},\qquad c=b_2c_2,\qquad c'=b_2c_1, \tag{3.67}$$

$$\tilde b=\frac{b_2}{b_1},\qquad \tilde c=b_1c_2,\quad a'-a=b_1c_1. \tag{3.68}$$

Here, a complex parameter $b(\tilde b)$ parametrizing $\mathbb{C}P^1$ appears like in the non-Abelian case given in eq. (3.47). Its argument, $\arg(b)$, is again the Nambu-Goldstone zero mode due to a broken U(1)-symmetry, but unlike in the non-Abelian case, its absolute value, $|b|$ is no longer a Nambu-Goldstone mode. We can observe that the configurations of the magnetic fluxes depend on $|b|$, as seen in figure 1.

[13] Another simple example is the case where the matrices ξ_i can be obtained by embedding that of the $L=1$ case. A general discussion on the embedding of the $L=1$ case is given in appendix E.

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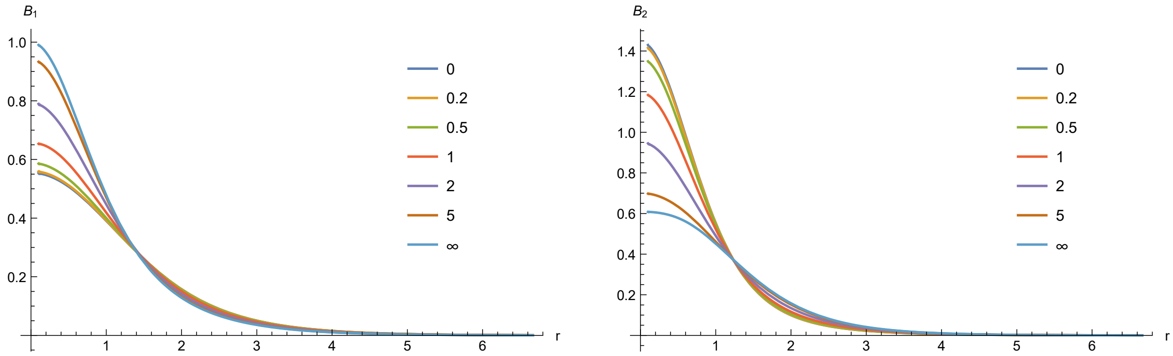

Figure 1. Profiles of the magnetic fields, $B_{i=1,2} \equiv -\mathrm{Tr}[F_{12}^{i=1,2}]$ for local vortices in the $L = 2$ model with setting $r_1 = r_2 = g_1 = g_2 = 1$. Here we use the moduli matrix in eq. (3.60) with setting $a = c = c' = 0$ and varying values of b as $b = 0, 0.2, 0.5, 1, 2, 5, \infty$.

Next, let us consider the corresponding sigma model instanton solution. Let us introduce inhomogeneous complex coordinates $(\phi_{12}, \phi_{13}, \phi_{23})$ of the target manifold $G^{\mathbb{C}}/\hat{H} = \mathcal{F}_{1,1,1}$. They are contained in the matrix $\mathcal{G}$ in eq. (2.18) as

$$\mathcal{G} = \mathcal{L} \begin{pmatrix} 1 & \phi_{12} & \phi_{13} \\ 0 & 1 & \phi_{23} \\ 0 & 0 & 1 \end{pmatrix} \in G^{\mathbb{C}}, \tag{3.69}$$

where $\mathcal{L}$ is a lower-triangular matrix. In this parametrization, $\mathfrak{D}_i^{-1}\widetilde{\mathfrak{D}}_i$ are given by

$$\mathfrak{D}_1^{-1}\widetilde{\mathfrak{D}}_1 = (\phi_{12}, \phi_{13}), \qquad \mathfrak{D}_2^{-1}\widetilde{\mathfrak{D}}_2 = \begin{pmatrix} \phi'_{13} \\ \phi_{23} \end{pmatrix}, \tag{3.70}$$

where ϕ'_{13} is related to ϕ_{13} as[14]

$$\phi'_{13} = \phi_{13} - \phi_{12}\phi_{23}. \tag{3.71}$$

If we ignore the constraints $\xi_i = q_i \xi_{i+1}$, each ξ_i can be regarded as a flag for the Grasmannian $G(N_i, N)$. In this case, ξ_1 and ξ_2 specify points on $G(1,3) = \mathbb{C}P^2$ and $G(2,3) = \mathbb{C}P^2$, respectively. In this case, both (ϕ_{12}, ϕ_{13}) and (ϕ_{23}, ϕ'_{13}) are the inhomogenrous coordinates of $\mathbb{C}P^2$, and hence the sigma model instanton solutions with $k_1 = 1$ and $k_2 = 1$ are generally takes the forms (see (3.53))

$$(\phi_{12}, \phi_{13}) = \frac{1}{z - a'}(B, A'), \qquad (\phi_{23}, \phi'_{13}) = \frac{1}{z - a}(C, A). \tag{3.72}$$

The additional condition (3.71) gives rise to the following constaraint on the moduli parameters a', A', B, a, A, C

$$AD = BC, \quad A = A', \quad \text{with} \quad D \equiv a' - a. \tag{3.73}$$

[14]The flag manifold $\mathcal{F}_{n_1,n_2,n_3}$ can be realized as two orthogonal flags, $\xi_1 \in G(n_1, N)$ and $\tilde{\xi} \in G(n_3, N)$. Eq. (3.71) can be regard as the orthogonality condition $\xi_1 \tilde{\xi}^{\mathrm{T}} = 0$ with identifying $\xi_1 = (\mathbf{1}, \phi_{12}, \phi_{13})$ and $\tilde{\xi} = (-\phi'_{13}{}^{\mathrm{T}}, -\phi_{23}^{\mathrm{T}}, \mathbf{1})$.

The space given by the condition $AD = BC$ has singularities but they are covered by the small-instanton singularities located at $(B, A') = (0, 0)$ and $(C, A) = (0, 0)$.,[15] These singularities can be simultaneously removed by requiring $A \neq 0$, and hence the moduli space of instantons is given by

$$\mathcal{M}_{\text{inst}}{}^{n_1=1,n_2=1,n_3=1}_{k_1=1,k_2=1} = \mathbb{C} \times \mathbb{C}^2 \times (\mathbb{C}\backslash\{0\}). \tag{3.74}$$

The vortex configuration given in (3.60) and (3.62) is mapped to $(\phi_{12}, \phi_{23}, \phi_{13}, \phi'_{13})$ through relation (3.70), from which the parameters in (3.72) can be read as,

$$(B, A') = b_2(c_1, c_2), \quad (C, A) = c_2(b_1, b_2) \quad \text{and } a' - a = b_1 c_1, \tag{3.75}$$

with $[b_1 : b_2 : c_1 : c_2] \in \mathcal{O}_{\mathbb{C}P^1}(-1) \oplus \mathcal{O}_{\mathbb{C}P^1}(-1)$. The above mapping can be regarded as a blowup of the space $AD = BC$ along the center $A = C = 0$. Therefore, this resolution of the singularity defines an inclusion map between them as

$$\mathcal{M}_{\text{inst}}{}^{n_1=1,n_2=1,n_3=1}_{k_1=1,k_2=1} = \mathbb{C}\times\mathbb{C}^2\times(\mathbb{C}\backslash\{(0)\}) \hookrightarrow \mathcal{M}_{\text{vtx}}{}^{n_1=1,n_2=1,n_3=1}_{k_1=1,k_2=1} = \mathbb{C}\times(\mathcal{O}_{\mathbb{C}P^1}(-1)\oplus\mathcal{O}_{\mathbb{C}P^1}(-1)). \tag{3.76}$$

Removing the small instanton singularities at $b_2 = 0$ and $c_2 = 0$, we obtain the condition for the moduli space of the instanton

$$\mathcal{M}_{\text{vtx}}{}^{n_1=1,n_2=1,n_3=1}_{k_1=1,k_2=1} \xrightarrow[b_2\neq 0,\ c_2\neq 0]{} \mathcal{M}_{\text{inst}}{}^{n_1=1,n_2=1,n_3=1}_{k_1=1,k_2=1} = \mathbb{C} \times \mathbb{C}^2 \times (\mathbb{C}\backslash\{0\}). \tag{3.77}$$

Next, let us see how the moduli spaces are related under the duality map. In the $n_1 = n_2 = n_3 = 1$ case, the duality theory is identical but the inhomogeneous coordinates $(\phi_{12}, \phi_{23}, \phi_{13})$ and $\phi'_{13} = \phi_{13} - \phi_{12}\phi_{23}$ are swapped as (see eq. (2.64))

$$(\phi_{12}, \phi_{23}, \phi_{13}, \phi'_{13})_{\text{dual}} = -(\phi_{23}, \phi_{12}, \phi'_{13}, \phi_{13}). \tag{3.78}$$

From this relation, we can read off the duality transformation for the moduli parameters as

$$(b_1, b_2, c_1, c_2)_{\text{dual}} = (c_1, c_2, -b_1, -b_2). \tag{3.79}$$

This map is well-defined on $\mathcal{M}_{\text{inst}}{}^{n_1=1,n_2=1,n_3=1}_{k_1=1,k_2=1}$ but ill-defined on $\mathcal{M}_{\text{vtx}}{}^{n_1=1,n_2=1,n_3=1}_{k_1=1,k_2=1}$ since the point $(c_1, c_2)_{\text{dual}} = (0, 0)$ of the vortex moduli space is mapped to the forbidden point $(b_1, b_2) = (0, 0)$ in the original theory. This indicates that there are vortex configurations that do not have corresponding configuration in the dual theory.

This asymmetry of the vortex moduli space becomes manifest if we focus on its subspaces containing local vortices. The vortex described by ξ_1 (ξ_2) becomes a local vortex in the limit of $b_2 \to 0$ ($c_2 \to 0$) Interestingly, there exist two subspaces ($\mathbb{C}^2$ and $\mathbb{C} \times \mathbb{C}P^1$ shown in the bottom row of figure 2) where both of the two vortices becomes local ones.

[15]There are two type of singularity: the former is algebraic singularity where the tangent space is ill-defined, and the latter is "physical" singularity where the NLσM breaks down.

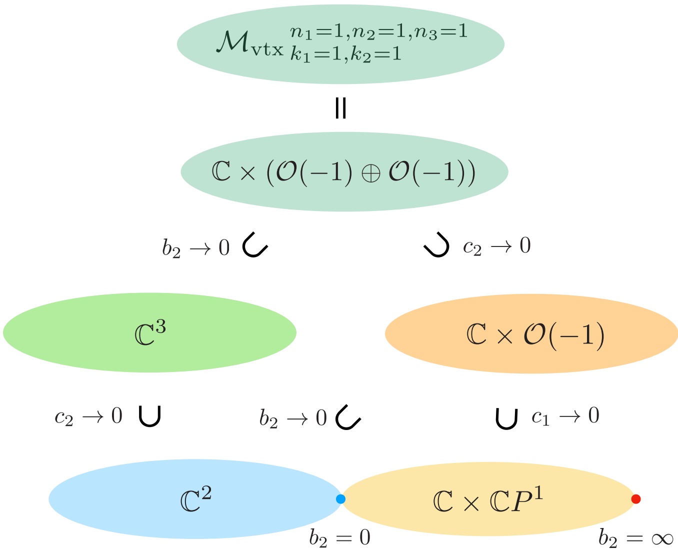

Figure 2. The structure of moduli space of vortices with $n_1 = n_2 = n_3 = 1$, $k_1 = k_2 = 1$.

Fixed points of torus action. The two subspaces $\mathbb{C}^2$ and $\mathbb{C}\times\mathbb{C}P^1$ shown in the bottom row of figure 2 contain two fixed points of a torus action which will be discussed in appendix H. They are also viewed as the origins of the It is convenient to characterize these fixed points by Young tableaux as

$$\left(\,\begin{ytableau} \none[\,] \\ \none[\,] \end{ytableau}\!\!\!\ydiagram{1,1}\,,\mathbf{1}\right):\quad q_1=(1,0),\quad q_2=\begin{pmatrix} z & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\in C_2, \tag{3.80}$$

$$\left(\ydiagram{1}\,,\ydiagram{1}\right):\quad q_1=(z,0),\quad q_2=\begin{pmatrix} 1 & 0 & 0\\ 0 & z & 0\end{pmatrix}\in C_1\cap C_2. \tag{3.81}$$

The height d of each young diagram indicates a composite state of d different types of vortices. For $n_1 = n_2 = n_3 = 1$, the general fixed point and the corresponding set of Young diagrams is given by

$$\left(\begin{ytableau} 1 & \cdots & l \\ \; & \; & \; & 1 & \cdots & m \end{ytableau}\,,\begin{ytableau} 1 & \cdots & n \end{ytableau}\right):\quad q_1=(z^m,0),\quad q_2=\begin{pmatrix} z^l & 0 & 0\\ 0 & z^n & 0\end{pmatrix}, \tag{3.82}$$

with $(k_1, k_2) = (l + m, l + n)$. In section 4.3.1, we will see the way to classify the fixed points in terms Young tableaux.

4 Quotient construction of vortex moduli space

In this section, we discuss a quotient construction of the vortex moduli space. We show that the vortex moduli space (3.25) can be identified with a quotient of a vector space of matrices by a complex Lie group.

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4.1 $L = 1$ case and half-ADHM mapping relation

Let us first review the quotient construction in the $L = 1$ case [9, 10]. We show that the vortex moduli space (3.35) is given by the $\mathrm{GL}(N,\mathbb{C})$ quotient of the vector space of matrices $\{Z, \Psi, \widetilde{\Psi}\}$

$$\mathcal{M}_{\mathrm{vtx}}{}_{k}^{n,N-n} \cong \left\{(Z,\Psi,\widetilde{\Psi}) \,\middle|\, \{Z,\Psi\} \text{ on which } \mathrm{GL}(k,\mathbb{C}) \text{ action is free}\right\} / \mathrm{GL}(k,\mathbb{C}), \qquad (4.1)$$

where Z k-by-k matrix, Ψ is a n-by-k matrix and $\widetilde{\Psi}$ is a k-by-$(N-n)$ matrix on which $\mathrm{GL}(k,\mathbb{C})$ acts

$$Z \to g^{-1} Z g, \qquad \Psi \to \Psi g, \qquad \widetilde{\Psi} \to g^{-1}\widetilde{\Psi}. \qquad (4.2)$$

They are related to the moduli matrix $\xi(z) = q(z) = (\mathfrak{D}, \widetilde{\mathfrak{D}})$ through the relations, which we call *the half-ADHM mapping relation*

$$\mathfrak{D}(z)\Psi = \mathfrak{J}(z)(z\mathbf{1}_k - Z), \qquad \widetilde{\mathfrak{D}}(z) = \mathfrak{J}(z)\widetilde{\Psi}, \qquad (4.3)$$

where $\mathfrak{J}(z) = (j_1(z), \cdots, j_k(z))$ is the n-by-k matrix defined in section 3.2, which is characterized by the property

$$\mathfrak{D}(z)^{-1}\mathfrak{J}(z) = \mathcal{O}(z^{-1}) \quad (z \to \infty). \qquad (4.4)$$

The $\mathrm{GL}(k,\mathbb{C})$ transformation acts on $\mathfrak{J}(z)$ as

$$\mathfrak{J}(z) \to \mathfrak{J}(z)\, g \quad (g \in \mathrm{GL}(k,\mathbb{C})), \qquad (4.5)$$

and can be regarded as the change of basis $\{j_1(z), \cdots, j_k(z)\}$ of the solutions of (4.4).

From the matrices $\mathfrak{D}$ and $\mathfrak{J}$, the matrices (Z,Ψ) can be obtained through the first equation in (4.3). The existence of such constant matrices (Z,Ψ) can be shown by using the following decomposition algorithm. By using $\mathfrak{D}$ and $\mathfrak{J}$, any column vector $\vec{f}(z)$ with arbitrary polynomial entries can be decomposed as

$$\vec{f}(z) = \mathfrak{D}(z)\vec{g}(z) + \mathfrak{J}(z)\boldsymbol{v}, \qquad (4.6)$$

with a column vector $\vec{g}(z)$ with polynomial entries and a constant vector $\boldsymbol{v} \in \mathbb{C}^k$. Note that for a given $\vec{f}(z)$, the column vectors $\vec{g}(z)$ and $\boldsymbol{v}$ are unique since the columns of $\mathfrak{D}(z)$ and $\mathfrak{J}(z)$ are independent in the sense that

$$\vec{0} = \mathfrak{D}(z)\vec{g}(z) + \mathfrak{J}(z)\boldsymbol{v} \quad \Leftrightarrow \quad \vec{g}(z) = \vec{0}, \quad \boldsymbol{v} = \mathbf{0}. \qquad (4.7)$$

Applying the decomposition (4.6) to each column of $z\mathfrak{J}(z)$, we can show that there exist a n-by-k matrix Ψ and a k-by-k matrix Z such that

$$z\mathfrak{J}(z) = \mathfrak{D}(z)\Psi + \mathfrak{J}(z)Z, \qquad \text{or equivalently} \qquad \mathfrak{D}(z)\Psi = \mathfrak{J}(z)(z\mathbf{1}_k - Z). \qquad (4.8)$$

Note that $\Psi = \mathfrak{D}(z)^{-1}\mathfrak{J}(z)(z\mathbf{1}_k - Z)$ must be a constant matrix since it is regular everywhere and $\mathfrak{D}(z)^{-1}(z\mathfrak{J}(z)) = \mathcal{O}(1)$ in the limit $z \to \infty$. Similarly, by applying the decomposition (4.6) to each column of $\widetilde{\mathfrak{D}}(z)$, we obtain the k by $(N-n)$ matrix $\widetilde{\Psi}$ as

$$\widetilde{\mathfrak{D}}(z) = \mathfrak{J}(z)\widetilde{\Psi}. \qquad (4.9)$$

Note that $\widetilde{\mathfrak{D}}(z)$ has no term proportional to $\mathfrak{D}(z)$ since $\widetilde{\mathfrak{D}}(z)$ satisfies the condition $\mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1})$.

As we have seen, the set of matrices $\{Z, \Psi, \widetilde{\Psi}\}$ can be extracted from the moduli matrix $\xi(z) = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$. However, $\{Z, \Psi, \widetilde{\Psi}\}$ is not unique for a given $\xi(z) = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$ due to the degrees of freedom of the change of the basis $\{\vec{j}_a(z)\}$. Thus, for a given $\xi(z) = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$, we obtain a unique equivalence class of matrices defined by

$$\{Z, \Psi, \widetilde{\Psi}\} \sim \{g^{-1}Zg\,,\ \Psi g\,,\ g^{-1}\widetilde{\Psi}\} \quad \text{with} \quad g \in \mathrm{GL}(k, \mathbb{C}). \tag{4.10}$$

We can show that this $\mathrm{GL}(k, \mathbb{C})$ action is free on the part of the data $\{Z, \Psi\}$ obtained from $\mathfrak{D}(z)$; that is, for any infinitesimal $\mathrm{GL}(k, \mathbb{C})$ action $\delta_X Z = [Z, X]$, $\delta_X \Psi = \Psi X$ with $X \in \mathfrak{gl}(k, \mathbb{C})$,

$$\delta_X Z = 0, \quad \delta_X \Psi = 0 \quad \Rightarrow \quad X = 0. \tag{4.11}$$

As shown in appendix C.2, this condition on the data is equivalent to the following statement for a vector $\vec{v}$:

$$\forall z \in \mathbb{C}, \quad \Psi(z\mathbf{1} - Z)^{-1}\vec{v} = 0 \quad \Rightarrow \quad \vec{v} = 0. \tag{4.12}$$

Since the relation (4.8) is rewritten as

$$\Psi(z\mathbf{1} - Z)^{-1} = \mathfrak{D}(z)^{-1}\mathfrak{J}(z), \tag{4.13}$$

the above $\mathrm{GL}(k, \mathbb{C})$-free condition is satisfied when the k columns of $\mathfrak{J}(z)$ are linearly independent. Since this is true by construction, the infinitesimal action of $\mathrm{GL}(k, \mathbb{C})$ on $\{Z, \Psi\}$ is free.

Through the half-ADHM mapping relations (4.3), we can show that there exists a one-to-one map (see appendix C) between the moduli matrix $(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$ and the half-ADHM data $\{Z, \Psi, \widetilde{\Psi}\}$ in each coordinate patch given in eq. (3.24).[16] Thus, we find that the moduli space of BPS vortices turns out to be given by

$$\begin{aligned} &\mathcal{M}_{\mathrm{vtx}}{}_k^{n,N-n} \\ &\cong \left\{ (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z)) \,\middle|\, \det \mathfrak{D}(z) = \mathcal{O}(z^k)\,,\, \mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1})) \right\} / \{V\text{-transf. in eq. (3.34)}\} &(4.14) \\ &\cong \left\{ (Z, \Psi, \widetilde{\Psi}) \,\middle|\, \{Z, \Psi\} \text{ on which } \mathrm{GL}(k, \mathbb{C}) \text{ action is free} \right\} / \mathrm{GL}(k, \mathbb{C}). &(4.15) \end{aligned}$$

Indeed, we can show that the matrices $\{Z, \Psi\}$ have all the $V(z)$ invariant information contained in $\mathfrak{D}(z)$ from the fact that all the invariants under the V-transformation consisting of $\mathfrak{D}(z)$ and $\mathfrak{J}(z)$ can be expressed in terms of $\{Z, \Psi\}$ as[17]

$$\det \mathfrak{D}(z) = \det(z\mathbf{1}_k - Z), \quad \text{and} \quad \mathfrak{D}(z)^{-1}\mathfrak{J}(z) = \Psi(z\mathbf{1}_k - Z)^{-1}. \tag{4.16}$$

Note that the second relation obeys from (4.3) and the first one can be derived as follows. By applying the decomposition (4.6) to each column of the unit matrix, we obtain

$$\mathbf{1}_n = \mathfrak{D}(z)P^{\mathfrak{D}} + \mathfrak{J}(z)P^{\mathfrak{J}}, \tag{4.17}$$

[16] We can confirm that the number of the degrees of freedom of the equivalence class (4.10) coincides with that of the moduli matrices $\#T^a{}_{b,m} = \#Z + \#\Psi - \#g = kn$. See appendix C for more details.

[17] Although any minor determinants of the matrix $(\mathfrak{D}(z), \mathfrak{J}(z))$ are invariants, they are related to $\det \mathfrak{D}(z)$ and $\mathfrak{D}(z)^{-1}\mathfrak{J}(z)$ through the Plücker relations.

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where $P^{\mathfrak{D}}$ and $P^{\mathfrak{J}}$ are n-by-n and k-by-n constant matrices, respectively. Since this equation is not invariant under the V-transformation, $P^{\mathfrak{J}}$ and $P^{\mathfrak{D}}$ depend on the choice of the coordinate patch.[18] Using the half-ADHM mapping relation (4.3) and (4.17), one can show that

$$\begin{pmatrix} \mathfrak{D}(z) & \mathfrak{J}(z) \\ \mathbf{0} & \mathbf{1}_k \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & \mathbf{0} \\ P^{\mathfrak{J}} & z\mathbf{1} - Z \end{pmatrix} \mathcal{N}^{-1}, \quad \text{with} \quad \mathcal{N} \equiv \begin{pmatrix} P^{\mathfrak{D}} & -\Psi \\ P^{\mathfrak{J}} & z\mathbf{1} - Z \end{pmatrix}. \tag{4.18}$$

By taking the determinant of the both sides and counting their degrees, we conclude this polynomial $\det \mathcal{N}$ is $\mathcal{O}(1)$, that is, $\det \mathcal{N} = 1$ when $\det \mathfrak{D}(z)$ is chosen to be a monic polynomial. Thus we find that $\det \mathfrak{D}(z) = \det(z\mathbf{1} - Z)$.

4.2 Quotient construction for general L

For the case of general L, we can define L copies of the matrices (and relations) defined in the previous subsection by attaching an index $i = 1, \cdots, L$. For example, we can define N_i-by-k_i matrix $\mathfrak{J}_i(z)$ with polynomial entries by solving the condition

$$\mathfrak{D}_i(z)^{-1}\mathfrak{J}_i(z) = \mathcal{O}(z^{-1}). \tag{4.19}$$

Then, we can obtain matrices $\{Z_i, \Psi_i, \widetilde{\Psi}_i\}$ from the N_i-by-N matrix $\xi_i(z) = (\mathfrak{D}_i(z), \widetilde{\mathfrak{D}}_i(z))$ thought the realtions

$$\mathfrak{D}_i(z)\Psi_i = \mathfrak{J}_i(z)(z\mathbf{1}_{k_i} - Z_i), \qquad \widetilde{\mathfrak{D}}_i(z) = \mathfrak{J}_i(z)\widetilde{\Psi}_i, \tag{4.20}$$

where $\{Z_i, \Psi_i, \widetilde{\Psi}_i\}$ are k_i-by-k_i, N_i-by-k_i and k_i-by-$(N - N_i)$ matrices, respectively. Conversely, from a given set of constant matrices $\{Z_i, \Psi_i, \widetilde{\Psi}_i | 1 \le i \le L\}$, we can obtain the matrices $\{\xi_i(z) | 1 \le i \le L\}$ up to V-transformations. However, such matrices $\{\xi_i(z)\}$ do not necessarily satisfy the constraint that there must be everywhere non-singular N_i-by-N_{i+1} matrices $q_i(z)$ such that

$$q_i(z)\xi_{i+1}(z) = \xi_i(z). \tag{4.21}$$

To guarantee that these constraints are satisfied, the matrices $\{Z_i, \Psi_i, \widetilde{\Psi}_i\}$ must satisfy some constraints. To write down the constraints, let us decompose the matrices Ψ_i and $\widetilde{\Psi}_i$ as

$$\Psi_{i+1} = \begin{pmatrix} \Upsilon'_{i+1} \\ \Upsilon_{i+1} \end{pmatrix}, \qquad \widetilde{\Psi}_i = (\widetilde{\Upsilon}_i, \, \widetilde{\Upsilon}'_i), \tag{4.22}$$

where Υ'_{i+1} is an N_i-by-k_{i+1} matrix, Υ_{i+1} is an n_{i+1}-by-k_{i+1} matrix, $\widetilde{\Upsilon}_i$ is a k_i-by-$(N_{i+1} - N_i)$ matrix and $\widetilde{\Upsilon}'_i$ is a k_i-by-$(N - N_{i+1})$ matrix. Then, the relation (4.21) can be rewritten as

$$\widetilde{\Upsilon}'_i = W_i \, \widetilde{\Psi}_{i+1}, \tag{4.23}$$

$$\Upsilon'_{i+1} = \Psi_i W_i, \tag{4.24}$$

$$\widetilde{\Upsilon}_i \Upsilon_{i+1} = Z_i W_i - W_i Z_{i+1}, \tag{4.25}$$

[18] For the $(l_1, l_2, \cdots, l_n)$ patch in eq. (3.36) and eq. (3.38), these matrices can be easily obtained as

$$(P^{\mathfrak{D}})^a{}_b = \delta^a_b \, \delta^{l_a}_0, \quad (P^{\mathfrak{J}})^{(a,p)}{}_b = \delta^a_b (1 - \delta^{l_a}_0)\delta^p_{l_a},$$

which are independent of any moduli parameters in $\mathfrak{D}(z)$.

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where W_i is a k_i-by-k_{i+1} matrix such that

$$q_i(z)\mathfrak{J}_{i+1}(z) = \mathfrak{J}_i(z)W_i. \tag{4.26}$$

The constraints (4.23)–(4.25) can be derived as follows.

Constraints on half-ADHM data. Let us first show the half-ADHM data $\{Z_i, \Psi_i, \widetilde{\Psi}_i\}$ obtained through (4.20) satisfy (4.23)–(4.25). Then, we show that the matrices $\{\xi_i(z)\}$ obtained from the half-ADHM data obeying the constraints (4.23)–(4.25) satisfy the relation (4.21) with suitable matrices $q_i(z)$.

(4.21) to (4.23)–(4.25). We can rewrite the relation (4.21) for the matrices $\xi_i(z) = (\mathfrak{D}_i(z), \widetilde{\mathfrak{D}}_i(z)) = (\mathfrak{D}_i(z), \mathfrak{J}_i(z)\widetilde{\Psi}_i)$ as

$$q_i(z)\mathfrak{D}_{i+1}(z) = \left(\mathfrak{D}_i(z),\, \mathfrak{J}_i(z)\widetilde{\Upsilon}_i\right), \tag{4.27}$$

$$q_i(z)\mathfrak{J}_{i+1}(z)\widetilde{\Psi}_{i+1} = \mathfrak{J}_i(z)\widetilde{\Upsilon}_i'. \tag{4.28}$$

It follows from the first equation that $\mathfrak{D}_i(z)^{-1}\left[q_i(z)\mathfrak{J}_{i+1}(z)\right] = \mathcal{O}(z^{-1})$ and hence there exist a k_i-by-k_{i+1} matrix W_i such that[19]

$$q_i(z)\mathfrak{J}_{i+1}(z) = \mathfrak{J}_i(z)W_i. \tag{4.29}$$

Then, by substituting this relation into (4.28) we find that (4.23) is satsified

$$\mathfrak{J}_i(z)W_i\,\widetilde{\Psi}_{i+1} = \mathfrak{J}_i(z)\widetilde{\Upsilon}_i' \quad\Longleftrightarrow\quad \widetilde{\Upsilon}_i' = W_i\,\widetilde{\Psi}_{i+1}, \tag{4.30}$$

where we have used the fact that the columns of $\mathfrak{J}_i(z)$ are linearly independent. We can see that (4.24) and (4.25) are satisfied as follows. By multiplying the both sides of eq. (4.27) by $\Psi_{i+1} = (\Upsilon_{i+1}', \Upsilon_{i+1})^T$ from the right, we obtain

$$q_i(z)\mathfrak{D}_{i+1}(z)\Psi_{i+1} = \mathfrak{D}_i(z)\Upsilon_{i+1}' + \mathfrak{J}_i(z)\widetilde{\Upsilon}_i\Upsilon_{i+1}. \tag{4.31}$$

The left hand side can be rewritten by using the half-ADHM mapping relation and (4.29) as

$$q_i\mathfrak{D}_{i+1}\Psi_{i+1} = q_i\mathfrak{J}_{i+1}(z\mathbf{1}_{k_{i+1}} - Z_{i+1}) = \mathfrak{J}_iW_i(z\mathbf{1}_{k_{i+1}} - Z_{i+1}) = \mathfrak{D}_i\Psi_iW_i + \mathfrak{J}_i(Z_iW_i - W_iZ_{i+1}). \tag{4.32}$$

Comparing this with the right hand side of (4.31) and using the linear independence of $(\mathfrak{D}_i(z), \mathfrak{J}_i(z))$, we find that (4.24) and (4.25) are satisfied.

[19] Since

$$\mathfrak{D}_i^{-1}\left(q_i\mathfrak{J}_{i+1}\right) = \mathfrak{D}_i^{-1}\left(q_i\mathfrak{D}_{i+1}\right)\left(\mathfrak{D}_{i+1}^{-1}\mathfrak{J}_{i+1}\right) = \left(\mathbf{1}_{N_i},\, \mathfrak{D}_i^{-1}\mathfrak{J}_i\widetilde{\Upsilon}_i\right)\left(\mathfrak{D}_{i+1}^{-1}\mathfrak{J}_{i+1}\right) = \mathcal{O}(z^{-1}),$$

the columns of the matrix $q_i(z)\mathfrak{J}_{i+1}(z)$ can be written as linear combinations of the columns of $\mathfrak{J}_i(z)$, and hence there exist a matrix W_i such that $q_i(z)\mathfrak{J}_{i+1}(z) = \mathfrak{J}_i(z)W_i$.

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(4.23)–(4.25) **to** (4.21). Next, let us show that if the half-ADHM data satisfies the constraints (4.23)–(4.25), the corresponding $\xi_i(z)$ related through (4.20) satisfy the relation (4.21) (or equivalently (4.27) and (4.28)), with suitable matrices $q_i(z)$. Although formally the condition (4.27) is always satisfied by adopting $q_i(z) = (\mathfrak{D}_i(z), \mathfrak{J}_i(z)\widetilde{\Upsilon}'_i)\mathfrak{D}_{i+1}(z)^{-1}$, such matrices $q_i(z)$ may not be suitable since they can have some poles. We can show that $q_i(z) = (\mathfrak{D}_i(z), \mathfrak{J}_i(z)\widetilde{\Upsilon}'_i)\mathfrak{D}_{i+1}(z)^{-1}$ do not have any pole if (4.23)–(4.25) are satisfied. To show this, let us rewrite $\mathfrak{D}_{i+1}^{-1}$ as

$$\mathfrak{D}_{i+1}^{-1} = P^{\mathfrak{D}}_{i+1} + \mathfrak{D}_{i+1}^{-1}\mathfrak{J}_{i+1}P^{\mathfrak{J}}_{i+1} = P^{\mathfrak{D}}_{i+1} + \Psi_{i+1}(z\mathbf{1}_{k_{i+1}} - Z_{i+1})^{-1}P^{\mathfrak{J}}_{i+1}, \tag{4.33}$$

where $P^{\mathfrak{D}}_{i+1}$ and $P^{\mathfrak{J}}_{i+1}$ are matrices defined by $\mathbf{1}_{N_{i+1}} = \mathfrak{D}_{i+1}(z)P^{\mathfrak{D}}_{i+1} + \mathfrak{J}_{i+1}(z)P^{\mathfrak{J}}_{i+1}$. Then, $q_i = (\mathfrak{D}_i, \mathfrak{J}_i\widetilde{\Upsilon}'_i)\mathfrak{D}_{i+1}^{-1}$ can be rewritten as

$$q_i(z) = (\mathfrak{D}_i,\, \mathfrak{J}_i\widetilde{\Upsilon}_i)\mathfrak{D}_{i+1}^{-1} = (\mathfrak{D}_i,\, \mathfrak{J}_i\widetilde{\Upsilon}_i)P^{\mathfrak{D}}_{i+1} + (\mathfrak{D}_i,\, \mathfrak{J}_i\widetilde{\Upsilon}_i)\Psi_{i+1}(z\mathbf{1}_{k_{i+1}} - Z_{i+1})^{-1}P^{\mathfrak{J}}_{i+1}. \tag{4.34}$$

Obviously the first term has no pole and the regularity of the second term can be shown by rewriting

$$\begin{aligned}(\mathfrak{D}_i, \mathfrak{J}_i\widetilde{\Upsilon}_i)\Psi_{i+1}(z\mathbf{1}_{k_{i+1}} - Z_{i+1})^{-1} &= (\mathfrak{D}_i\Upsilon'_{i+1} + \mathfrak{J}_i\widetilde{\Upsilon}_i\Upsilon_{i+1})(z\mathbf{1}_{k_{i+1}} - Z_{i+1})^{-1}\\ &= (\mathfrak{D}_i\Psi_iW_i + \mathfrak{J}_i(Z_iW_i - W_iZ_{i+1}))(z\mathbf{1}_{k_{i+1}} - Z_{i+1})^{-1} = \mathfrak{J}_iW_i.\end{aligned} \tag{4.35}$$

Since this has no pole, $q_i(z) = (\mathfrak{D}_i,\, \mathfrak{J}_i\widetilde{\Upsilon}_i)P^{\mathfrak{D}}_{i+1} + \mathfrak{J}_iW_iP^{\mathfrak{J}}_{i+1}$ is regular and hence (4.27) is satisfied. Then we can show that (4.28) is also satisfied as

$$q_i\mathfrak{J}_{i+1}\widetilde{\Psi}_{i+1} = q_i\mathfrak{D}_{i+1}\mathfrak{D}_{i+1}^{-1}\mathfrak{J}_{i+1}\widetilde{\Psi}_{i+1} = (\mathfrak{D}_i, \mathfrak{J}_i\widetilde{\Upsilon}_i)\Psi_{i+1}(z\mathbf{1}_{k_{i+1}} - Z_{i+1})^{-1}\widetilde{\Psi}_{i+1} = \mathfrak{J}_iW_i\widetilde{\Psi}_{i+1} = \mathfrak{J}_i\widetilde{\Upsilon}'_i. \tag{4.36}$$

Thus, we find that *no further constraints* other than (4.23)–(4.25) is needed on a data set $\{Z_i,\, \Upsilon_i,\, \widetilde{\Upsilon}_i,\, W_i\}$ to guarantee that $q_i(z)$ obtained through the half-ADHM mapping relation (4.20) satisfy the relation (4.21).

The constraints (4.23) and (4.24) imply that Υ'_i and $\widetilde{\Upsilon}'_i$ are not independent but can be rewritten in terms of $\{W_i, \Upsilon_i, \widetilde{\Upsilon}_i\}$ as

$$\Psi_i = \Big(\Psi_{i-1}W_{i-1}\,,\ \Upsilon_i\Big)^T = \Big(\Upsilon_1W_1W_2\cdots W_{i-1}\,,\ \Upsilon_2W_2\cdots W_{i-1}\,,\ \cdots,\ \Upsilon_{i-1}W_{i-1}\,,\ \Upsilon_i\Big)^T, \tag{4.37}$$

$$\widetilde{\Psi}_i = \Big(\widetilde{\Upsilon}_i\,,\ W_i\widetilde{\Psi}_{i+1}\Big) = \Big(\widetilde{\Upsilon}_i\,,\ W_i\widetilde{\Upsilon}_{i+1}\,,\ W_iW_{i+1}\widetilde{\Upsilon}_{i+2}\,,\ \cdots,\ W_iW_{i+1}\cdots W_{L-1}\widetilde{\Upsilon}_L\Big), \tag{4.38}$$

with $\Upsilon_1 \equiv \Psi_1$ and $\widetilde{\Upsilon}_L \equiv \widetilde{\Psi}_L$. Therefore, all information describing vortex moduli are contained in the set of matrices $\{Z_i, \Upsilon_i, \widetilde{\Upsilon}_i | 1 \le i \le L\}$ and $\{W_i | 1 \le i \le L-1\}$ obeying the constraints

$$Z_iW_i - W_iZ_{i+1} = \widetilde{\Upsilon}_i\Upsilon_{i+1} \qquad \text{for } 1 \le i \le L-1. \tag{4.39}$$

Since $q_i(z)$ related to $\{Z_i,\, \Upsilon_i,\, \widetilde{\Upsilon}_i,\, W_i\}$ though the half-ADHM mapping relation do not change under the $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_L, \mathbb{C})$ transformation

$$\{Z_i,\, \Upsilon_i,\, \widetilde{\Upsilon}_i,\, W_i\} \simeq \{\mathcal{U}_i^{-1}Z_i\mathcal{U}_i,\, \Upsilon_i\mathcal{U}_i,\, \mathcal{U}_i^{-1}\widetilde{\Upsilon}_i,\, \mathcal{U}_i^{-1}W_i\mathcal{U}_{i+1}\} \quad \text{with } \forall\,\mathcal{U}_i \in \mathrm{GL}(k_i, \mathbb{C}), \tag{4.40}$$

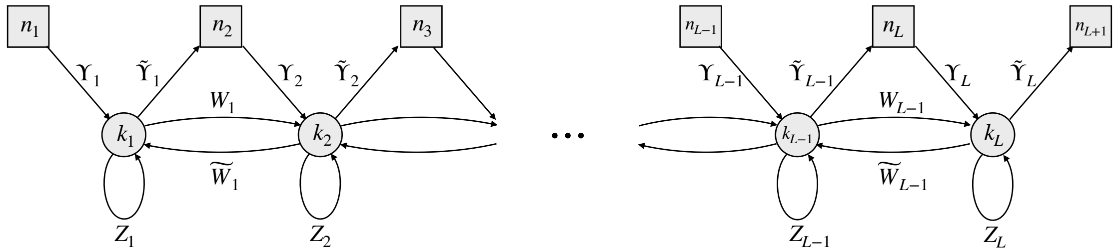

Figure 3. Quiver diagram for vortex moduli. Here $\widetilde{W}_i$ represent Lagrange multipliers for the constraint (4.39), which will be explained in the next section.

the quotient space of the data $\{Z_i,\, \Upsilon_i,\, \tilde{\Upsilon}_i,\, W_i\}$ by $GL(k_1,\mathbb{C}) \times \cdots \times GL(k_L,\mathbb{C})$ can be identified with the vortex moduli space. Note that the action of $GL(k_1,\mathbb{C}) \times \cdots \times GL(k_L,\mathbb{C})$ must be free on Ψ_i, Z_i, that is,

$$\exists\, k_i\text{-column vector } \vec{v}_i : \Upsilon_j W_j W_{j+1} \cdots W_{i-1} (z\mathbf{1}_{k_i} - Z_i)^{-1} \vec{v}_i = 0 \text{ for } 1 \leq \forall j \leq i,\ \forall z \in \mathbb{C} \ \Rightarrow\ \vec{v}_i = 0. \tag{4.41}$$

Note that this set of conditions is a generalization of the condition (4.12) for $L = 1$.

In summary, the moduli space of vortices turns out to be given by the quotient

$$\begin{aligned} &\mathcal{M}_{\text{vtx}}{}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = \\ &\left\{ Z_i,\, \Upsilon_i,\, \tilde{\Upsilon}_i,\, W_j \,\middle|\, \text{constraints}(4.39),\, \text{condition}(4.41) \right\} / GL(k_1,\mathbb{C}) \times \cdots \times GL(k_L,\mathbb{C}). \end{aligned} \tag{4.42}$$

The contents of this quotient are summarized in the quiver diagram figure 3 and the corresponding gauged linear sigma model can also be obtained from the D-brane configuration for BPS vortices (see appendix F).

In appendix G, we prove that all possible singularities due to the constraints (4.39) is removed by the condition (4.41) and the moduli space defined above is smooth everywhere.

4.2.1 An example of $L = 2$: $SU(3)/U(1)^2$ sigma model

Here, we illustrate the moduli space of vortices in the gauged linear sigma model corresponding to the $SU(3)/U(1)^2$ sigma model ($L = 2, N_1 = 1, N_2 = 2, N_3 = 3, n_1 = n_2 = n_3 = 1$). The model is the $U(1) \times U(2)$ gauge theory with an $SU(3)_F$ flavor symmetry. The matter content consists of two scalar fields Q_1 and Q_2 in the $(1, \bar{\mathbf{2}}, \mathbf{0})$ and $(0, \mathbf{2}, \bar{\mathbf{3}})$ of the $U(1) \times U(2) \times SU(3)_F$ symmetry, respectively. The topological sectors are labeled by two integers (k_1, k_2) corresponding to the vortex numbers of $U(1) \times U(2)$ gauge group.

Two coordinate patches in $(k_1, k_2) = (1, 1)$ case. First, let us consider the case of $(k_1, k_2) = (1, 1)$. As we have seen in subsection 3.3, there are two coordinate patches:

- $(1,0)$-patch : $\quad \xi_1(z) = (z - a', c', c), \quad \xi_2(z) = \begin{pmatrix} z-a & 0 & c \\ -b & 1 & 0 \end{pmatrix}, \qquad (4.43)$
- $(0,1)$-patch : $\quad \xi_1(z) = (z - a', c', c), \quad \xi_2(z) = \begin{pmatrix} 1 & -\tilde{b} & 0 \\ 0 & z-a & \tilde{c} \end{pmatrix}, \qquad (4.44)$

where the parameters are realted as

$$b = \tilde{b}^{-1}, \qquad c' = \tilde{b}(a' - a), \qquad c = \tilde{b}\tilde{c}. \tag{4.45}$$

We can move from the $(1, 0)$-patch to the $(0, 1)$-patch by using the V-transformation $\xi_i \to V_i\xi_i$ with

$$V_1 = 1, \qquad V_2 = \begin{pmatrix} 0 & -\tilde{b} \\ b & z - a \end{pmatrix}. \tag{4.46}$$

• **$(1, 0)$-patch.** From the definition $\xi_i = (\mathfrak{D}_i, \widetilde{\mathfrak{D}}_i)$ and (4.43), we find that the matrices $\mathfrak{D}_i$ and $\widetilde{\mathfrak{D}}_i$ are givenby

$$\mathfrak{D}_1 = z - a', \qquad \widetilde{\mathfrak{D}}_1 = (c', c), \qquad \mathfrak{D}_2 = \begin{pmatrix} z - a & 0 \\ -b & 1 \end{pmatrix}, \qquad \widetilde{\mathfrak{D}}_2 = \begin{pmatrix} c \\ 0 \end{pmatrix}. \tag{4.47}$$

From these matrices, the corresponding half-ADHM data can be read off as

$$\mathfrak{D}_i^{-1}\mathfrak{J}_i = \mathcal{O}(z^{-1}) \quad \to \quad \mathfrak{J}_1 = 1, \qquad \mathfrak{J}_2 = (1, 0)^T \tag{4.48}$$

$$\mathfrak{D}_i\Psi_i = \mathfrak{J}_i(z - Z_i) \quad \to \quad Z_1 = a', \qquad Z_2 = a, \qquad \Psi_1 = 1, \quad \Psi_2 = (1, b)^T, \tag{4.49}$$

$$\widetilde{\mathfrak{D}}_i = \mathfrak{J}_i\widetilde{\Psi}_i \quad \to \quad \widetilde{\Psi}_1 = (c', c), \qquad \widetilde{\Psi}_2 = c, \tag{4.50}$$

$$\Psi_i = (\Upsilon_i', \Upsilon_i)^T \quad \to \quad \Upsilon_1 = 1, \qquad \Upsilon_2 = b, \tag{4.51}$$

$$\widetilde{\Psi}_i = (\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_i') \quad \to \quad \widetilde{\Upsilon}_1 = c', \qquad \widetilde{\Upsilon}_2 = c, \tag{4.52}$$

$$\widetilde{\Upsilon}_1\Upsilon_2 = Z_1W_1 - W_1Z_2 \quad \to \quad W_1 = 1. \tag{4.53}$$

• **$(0, 1)$-patch.** From the definition $\xi_i = (\mathfrak{D}_i, \widetilde{\mathfrak{D}}_i)$ and (4.44), we find that the matrices $\mathfrak{D}_i$ and $\widetilde{\mathfrak{D}}_i$ are given by

$$\mathfrak{D}_1 = z - a', \qquad \widetilde{\mathfrak{D}}_1 = (c', c), \qquad \mathfrak{D}_2 = \begin{pmatrix} 1 & -\tilde{b} \\ 0 & z - \tilde{a} \end{pmatrix}, \qquad \widetilde{\mathfrak{D}}_2 = \begin{pmatrix} 0 \\ \tilde{c} \end{pmatrix}. \tag{4.54}$$

From these matrices, the corresponding half-ADHM data can be read off as

$$\mathfrak{D}_i^{-1}\mathfrak{J}_i = \mathcal{O}(z^{-1}) \quad \to \quad \mathfrak{J}_1 = 1, \qquad \mathfrak{J}_2 = (0, 1)^T \tag{4.55}$$

$$\mathfrak{D}_i\Psi_i = \mathfrak{J}_i(z - Z_i) \quad \to \quad Z_1 = a', \qquad Z_2 = a, \qquad \Psi_1 = 1, \quad \Psi_2 = (\tilde{b}, 1)^T, \tag{4.56}$$

$$\widetilde{\mathfrak{D}}_i = \mathfrak{J}_i\widetilde{\Psi}_i \quad \to \quad \widetilde{\Psi}_1 = (c', c), \qquad \widetilde{\Psi}_2 = \tilde{c}, \tag{4.57}$$

$$\Psi_i = (\Upsilon_i', \Upsilon_i)^T \quad \to \quad \Upsilon_1 = 1, \qquad \Upsilon_2 = 1, \tag{4.58}$$

$$\widetilde{\Psi}_i = (\widetilde{\Upsilon}_i, \widetilde{\Upsilon}_i') \quad \to \quad \widetilde{\Upsilon}_1 = c', \qquad \widetilde{\Upsilon}_2 = \tilde{c}, \tag{4.59}$$

$$\widetilde{\Upsilon}_1\Upsilon_2 = Z_1W_1 - W_1Z_2 \quad \to \quad W_1 = \tilde{b}. \tag{4.60}$$

The coordinate transformation from the $(1, 0)$-patch to $(0, 1)$-patch is given by a group element $(g_1, g_2) \in \mathrm{GL}(1, \mathbb{C}) \times \mathrm{GL}(1, \mathbb{C})$

$$\Upsilon_i \to \Upsilon_i g_i, \qquad \widetilde{\Upsilon}_i \to g_i^{-1}\widetilde{\Upsilon}_i, \qquad \text{with} \qquad g_1 = 1, \qquad g_2 = \tilde{b}. \tag{4.61}$$

In appendix E, the half-ADHM data obtained by embedding from the $L = 1$ case is discussed as another example.

4.3 Coordinate patches on moduli space

In this section, we discuss the coordinate patches of the moduli space of the half-ADHM data. To define an analogue of the $(l_1, \cdots, l_k)$-patch for the $L = 1$ case, we first discuss the fixed point of a torus action that plays the role of the "origin" in each coordinate patch.

4.3.1 Torus action and fixed points on vortex moduli space

The torus action we discuss here is a combination of an Abelian subgroup of the flavor symmetry and the spatial rotation (see appendix H for the details of the torus action). Its fixed point configurations can be viewed as the BPS vortex solutions in the presence of the omega background and the mass deformation. Such configurations must satisfy the following conditions in addition to the vortex equations (3.8) and (3.9)

$$i\epsilon(z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}})Q_i + \Sigma_i Q_i - Q_i \Sigma_{i+1} = 0, \qquad \text{with} \qquad \Sigma_{L+1} = -M, \tag{4.62}$$

where Σ_i $(i = 1, \cdots, L)$ are $\mathrm{SU}(N_i)$ adjoint scalar fields,[20] ϵ is the omega deformation parameter and $M = \mathrm{diag}(m^1, \cdots, m^N)$ is the mass matrix. A configuration satisfying (4.62) is invariant under the infinitesimal spatial rotation and flavor rotation up to an infinitesimal gauge transformation Σ_i. For such a fixed point configuration, the magnetic fluxes take the diagonal forms

$$\frac{1}{2\pi}\int F_i = \text{block-diag}(\boldsymbol{l}^{(i,1)}, \cdots, \boldsymbol{l}^{(i,i)}) = \begin{pmatrix} \boldsymbol{l}^{(i,1)} & & \\ & \ddots & \\ & & \boldsymbol{l}^{(i,i)} \end{pmatrix} \qquad \text{with} \qquad \boldsymbol{l}^{(i,j)} = \begin{pmatrix} l^{(i,j,1)} & & \\ & \ddots & \\ & & l^{(i,j,n_j)} \end{pmatrix}, \tag{4.63}$$

where $\boldsymbol{l}^{(i,j)}$ denotes the n_j-by-n_j diagonal block of the $\mathrm{SU}(N_i)$ magnetic flux of the i-th gauge group. The labels (i, j) and (i, j, α) $(i = 1, \cdots, L,\ j = 1, \cdots, i,\ \alpha = 1, \cdots, n_j)$ specify the following subgroup of the gauge group:

- (i, j) : j-th $\mathrm{U}(n_j)$ subgroup of i-th gauge group $\mathrm{U}(N_i) \supset \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_i)$.
- (i, j, α) : α-th Cartan subgroup $\mathrm{U}(1)$ of j-th $\mathrm{U}(n_j)$ subgroup of i-th gauge group $\mathrm{U}(N_i)$.

The magnetic fluxes $l^{(i,j,\alpha)}$ are also related to winding numbers of the scalar fields

$$q_i = \left(\begin{array}{ccc|ccc} z^{\boldsymbol{\nu}^{(i,1)}} & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & z^{\boldsymbol{\nu}^{(i,i)}} & 0 & \cdots & 0 \end{array}\right) \qquad \text{with} \qquad \boldsymbol{\nu}^{(i,j)} \equiv \boldsymbol{l}^{(i,j)} - \boldsymbol{l}^{(i+1,j)}, \tag{4.64}$$

where $\boldsymbol{\nu}^{(i,j)}$ are diagonal matrices of winding numbers. We can confirm that $q_i(z)$ is invariant under the torus action (the Cartan part of the spatial rotation and the flavor rotation) up to V-transformations

$$q_i(z) = V_i\, q_i(e^{i\epsilon} z)\, V_{i+1}^{-1}, \qquad V_i = \exp(i\Sigma_i), \qquad V_{L+1}(z) = \exp(-iM). \tag{4.65}$$

[20] In 2d $\mathcal{N} = (2,2)$ supersymmetric models, Σ_i can be interpreted as the adjoint scalar fileds in the vector multiplets and become auxiliary fields in the nonlinear sigma model limit.

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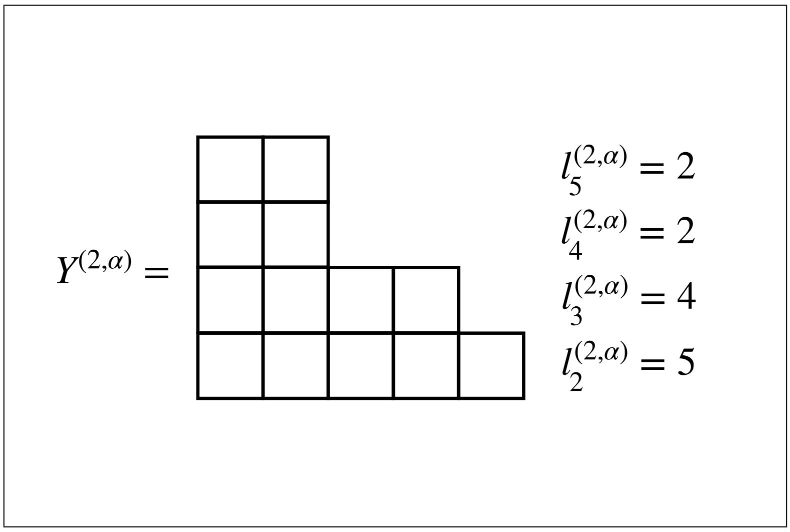

Figure 4. An example of Young tabuleax.

The element of the V-transformations are specified by the fixed point values of the adjoint scalar Σ_i, which take the forms

$$\Sigma_i = \text{block-diag}(\boldsymbol{\sigma}^{(i,1)}, \cdots, \boldsymbol{\sigma}^{(i,i)}), \qquad \boldsymbol{\sigma}^{(i,j)} = \text{diag}(\sigma^{(i,j,1)}, \cdots, \sigma^{(i,j,n_j)}), \tag{4.66}$$

with the eigenvalues

$$\sigma^{(i,j,\alpha)} = -m^{(j,\alpha)} - l^{(i,j,\alpha)}\epsilon, \tag{4.67}$$

where we have labeled the eigenvalues of the mass matrix as

$$\Sigma_{L+1} = M = \text{block-diag}(\boldsymbol{m}^1, \cdots, \boldsymbol{m}^{L+1}), \qquad \boldsymbol{m}^j = \text{diag}(m^{(j,1)}, \cdots, m^{(j,n_j)}). \tag{4.68}$$

Since the winding numbers $\boldsymbol{\nu}^{(i,j)} \equiv \boldsymbol{l}^{(i,j)} - \boldsymbol{l}^{(i+1,j)}$ of the scalar fields q_i must be non-negative integers, the magnetic fluxes must satisfy $l^{(i,j,\alpha)} \geq l^{(i+1,j,\alpha)}$. Therefore, the fixed points are classified by a set of N Young tableaux $Y^{(j,\alpha)}$ where $\alpha = 1, \cdots, n_j$ for each $j = 1, \cdots, L$. The height of $Y^{(j,\alpha)}$ is $L - j + 1$ and we denote the length of i-th row as $l^{(i+j-1,j,\alpha)}$, i.e.

$$Y^{(j,\alpha)} = \left(l^{(j,j,\alpha)}, l^{(j+1,j,\alpha)}, \cdots, l^{(L,j,\alpha)}\right), \qquad l^{(j,j,\alpha)} \geq l^{(j+1,j,\alpha)} \geq \cdots \geq l^{(L,j,\alpha)} \geq 0. \tag{4.69}$$

See figure 4 for an example.

Half-ADHM data at fixed points. We can show that the invariant vortex data corresponding to the Young tableaux $Y^{(j,\alpha)}$ take the form

$$\mathfrak{D}_i = \text{block-diag}(\boldsymbol{\mathfrak{D}}^{(i,1)}, \cdots, \boldsymbol{\mathfrak{D}}^{(i,i)}) \quad \text{with} \quad \boldsymbol{\mathfrak{D}}^{(i,j)} = \text{diag}(z^{l^{(i,j,1)}}, \cdots, z^{l^{(i,j,n_j)}}), \quad \text{and} \quad \widetilde{\mathfrak{D}}_i = 0. \tag{4.70}$$

This implies that each diagonal component of $\boldsymbol{\mathfrak{D}}^{(i,j)}$ represents axially symmetric Abelian vortices with flux $l_i^{(j,\alpha)}$ and hence all the half-ADHM data can be obtained by embedding those of Abelian vortices. For an axially symmetric Abelian vortex configuration $\mathfrak{D} = z^l$, the vortex data satisfying $\mathfrak{D}\Psi = \mathfrak{J}(z\mathbf{1}_l - Z)$ are given by (see section C.1)

$$\mathfrak{J}(l) = (z^{l-1}\,,\, z^{l-2}\,,\, \cdots\,,\, 1), \qquad \Psi(l) = (1\,,\, 0\,,\, \cdots\,,\, 0), \qquad Z(l) = \left.\left(\begin{array}{c|ccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \hline 0 & 0 & \cdots & 0 \end{array}\right)\right\} l. \tag{4.71}$$

By embedding these matrices, we can construct the matrices satisfying $\mathfrak{D}_i\Psi_i = \mathfrak{J}_i(z\mathbf{1}_{k_i} - Z_i)$ as

$$\mathfrak{J}_i = \text{block-diag}(\boldsymbol{\mathfrak{J}}^{(i,1)},\cdots,\boldsymbol{\mathfrak{J}}^{(i,i)}),\ \ \Psi_i = \text{block-diag}(\boldsymbol{\Psi}^{(i,1)},\cdots,\boldsymbol{\Psi}^{(i,i)}),\ \ Z_i = \text{block-diag}(\boldsymbol{Z}^{(i,1)},\cdots,\boldsymbol{Z}^{(i,i)}), \tag{4.72}$$

with

$$\boldsymbol{\mathfrak{J}}^{(i,j)} = \text{block-diag}\big(\mathfrak{J}(l^{(i,j,1)}),\cdots,\mathfrak{J}(l^{(i,j,n_j)})\big),\ \ \boldsymbol{\Psi}^{(i,j)} = \text{block-diag}\big(\Psi(l^{(i,j,1)}),\cdots,\Psi(l^{(i,j,n_j)})\big), \tag{4.73}$$

$$\boldsymbol{Z}^{(i,j)} = \text{block-diag}\big(Z(l^{(i,j,1)}),\cdots,Z(l^{(i,j,n_j)})\big). \tag{4.74}$$

Note that $\widetilde{\Psi}_i = 0$ since $\widetilde{\mathfrak{D}}_i = 0$ for the fixed point configurations. The matrices Υ_i and $\widetilde{\Upsilon}_i$ defined in (4.22) can be extracted from Ψ_i and $\widetilde{\Psi}_i$ as

$$\Upsilon_i = \left(\mathbf{0}_{n_i,k_{i-1}}\,\boldsymbol{\Psi}^{(i,i)}\Big|\right), \qquad \widetilde{\Upsilon}_i = 0. \tag{4.75}$$

The matrix W_i can be determined by solving the constraint $Z_iW_i - W_iZ_{i+1} = \widetilde{\Upsilon}_i\Upsilon_{i+1}$ as

$$W_i = \left(\begin{array}{ccc|ccc} \boldsymbol{W}^{(i,1)} & & & \mathbf{0} & \cdots & \mathbf{0} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \boldsymbol{W}^{(i,i)} & \mathbf{0} & \cdots & \mathbf{0} \end{array}\right), \qquad \boldsymbol{W}^{(i,j)} = \begin{pmatrix} W\big(l^{(i,j,1)}, l^{(i+1,j,1)}\big) & & \\ & \ddots & \\ & & W\big(l^{(i,j,n_j)}, l^{(i+1,j,n_j)}\big) \end{pmatrix}, \tag{4.76}$$

where $W(l,l')$ is the matrix satisfying $Z(l)W(l,l') - W(l,l')Z(l') = 0$, which takes the form

$$W(l,l') = \begin{pmatrix} \mathbf{1}_{l'} \\ \mathbf{0}_{l-l',l'} \end{pmatrix}. \tag{4.77}$$

Note that these half-ADHM data for the fixed points can also be obtained by solving the fixed point condition for the half-ADHM data. We can check these matrices are invariant under the torus action on the half-ADHM data (see appendix H.2).

4.3.2 Coordinates around fixed points

The coordinate patches around the fixed points discussed above can be obtained by considering fluctuations around the fixed point, eliminating the $\mathrm{GL}(k_1,\mathbb{C})\times\cdots\times\mathrm{GL}(k_L,\mathbb{C})$ gauge degrees of freedom and imposing the constraints (4.39). After fixing the $\mathrm{GL}(k_i,\mathbb{C})$ transformations, we find the following non-zero components of the fluctuations

$$(\delta\Upsilon_i)^{\alpha}{}_{(i,j,\beta,p)} \qquad \text{for } \alpha\in\{\alpha\,|\,l^{(i,i,\alpha)}=0\} \qquad (\delta\widetilde{\Upsilon}_i)^{(i,j,\alpha,p)}{}_{(i+1,\beta)} \tag{4.78}$$

$$(\delta Z_i)^{(i,j,\alpha,p)}{}_{(i,k,\beta,q)} \qquad \text{with } p = l^{(i,j,\alpha)}, \qquad (\delta W_i)^{(i,j,\alpha,p)}{}_{(i+1,k,\beta,q)} \quad \text{for } 2\le p\le l^{(i,j,\alpha)}. \tag{4.79}$$

Not all of these fluctuations independent since they are subject to the constraints (4.39). The total number of the components of (4.39) is given by

$$d_{\rm c} = \sum_{i=1}^{L-1} k_i k_{i+1}. \tag{4.80}$$

In appendix G we show that all components of the constraints (4.39) are linearly independent of each other for all points satisfying the condition (4.41). Fixing $d_{\rm c}$ degrees of freedom by solving the constraints (4.39), we can obtain the coordinates of the moduli space around this fixed point. We can show that in the vicinity of the fixed point the linearized constraint can be solved without any singularity and hence a smooth coordinate patch can be constructed around each fixed point. We can check that the complex dimension of the moduli space of vortices is given by

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{M}_{\rm vtx}{}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L} &= \sum_{i=1}^{L}(\#\Upsilon_i + \#\widetilde{\Upsilon}_i + \#Z_i - \#\mathrm{GL}(k_i,\mathbb{C})) + \sum_{i=1}^{L-1}(\#W_i - \#\widetilde{W}_i) \\
&= \sum_{i=1}^{L}(n_i k_i + k_i n_{i+1} + k_i^2 - k_i^2) + \sum_{i=1}^{L-1}(k_i k_{i+1} - k_{i+1} k_i) \\
&= \sum_{i=1}^{L} k_i (n_i + n_{i+1}),
\end{aligned} \tag{4.81}$$

which agrees with a result given by the index theorem (3.26).

Solutions of the constraints for separated vortices

Here we discuss solutions of the constraints (4.39). For a generic point on the moduli space, we can easily construct a solution in the following way. Let us consider the case of separated vortices given by

$$z_{i,\alpha} \neq z_{i,\beta} \quad \text{for} \quad \alpha \neq \beta \qquad \text{with} \qquad \det(z\mathbf{1}_{k_i} - Z_i) = \prod_{\alpha=1}^{k_i}(z - z_{i,\alpha}) \quad \text{for } 1 \leq i \leq L. \tag{4.82}$$

In this case, the square matrices $\{Z_i \,|\, 1 \leq i \leq L\}$ can be diagonalized as $(Z_i)^\alpha{}_\beta = \delta^\alpha{}_\beta\, z_{i,\alpha}$. In addition, if we assume that

$$z_{i,\alpha} \neq z_{i+1,\beta} \quad \text{for} \quad 1 \leq \alpha \leq k_i, \quad 1 \leq \beta \leq k_{i+1} \quad \text{and} \quad 1 \leq i \leq L-1, \tag{4.83}$$

we find the constraints are solved with respect to $\{W_i\}$ as

$$(W_i)^\alpha{}_\beta = \frac{(\widetilde{\Upsilon}_i \Upsilon_{i+1})^\alpha{}_\beta}{z_{i,\alpha} - z_{i+1,\beta}}. \tag{4.84}$$

This result implies that all the components of the constraints (4.39) are independent and each of them has a solution.

4.4 Metric on the moduli space and Kähler quotient

As we have seen that the vortex modui space is given by the $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_L, \mathbb{C})$ quotient (4.42) of the matrices $(Z_i, \Upsilon_i, \widetilde{\Upsilon}_i, W_i)$ as a complex manifold. One may think that it is also possible to consider the corresponding $\mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_L)$ Kähler quotient by introducing an appropriate Kähler potential on the space of the matrices $(Z_i, \Upsilon_i, \widetilde{\Upsilon}_i, W_i)$. A natural choice of the Kähler potential would be

$$\mathcal{K} = \sum_{i=1}^{L} \mathrm{Tr}\left[Z_i Z_i^\dagger + \Upsilon_i^\dagger \Upsilon_i + \widetilde{\Upsilon}_i \widetilde{\Upsilon}_i^\dagger + W_i W_i^\dagger\right], \tag{4.85}$$

which gives a flat metric on the linear space of the matrices. In addition, we need to impose the constraint (4.39). Following the standard procedure of the Kähler quotient construction, one can obtain a Kähler potential and metric on the moduli space. However, the Käher metric obtained in this way does not agree with the correct metric, shown in appendix B.4, that describes the classical dynamics of the vortices. Nonetheless, the 2d $\mathcal{N} = (2, 2)$ $\mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_L)$ gauge theory constructed based on the above Kähler potential $\mathcal{K}$ and the constraint (4.39) captures some quantum aspects of the original $\mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_L)$ quiver gauge theory. In section 6, we compute the vortex partition function from the viewpoint of the quotient construction as an example of the quantities that do not depend on the detail of the Kähler potential.

5 Sigma model instantons and duality

In this section, we discuss the sigma model solutions in the flag manifold sigma model. We check that the duality of the sigma model (2.65) defined by the relation (2.60) holds even on the moduli space of sigma model instantons, except for the instanton singularities.

5.1 Grassmannian case: $L = 1$

In the Grassmannian case ($L = 1$), the inhomogeneous coordinates ϕ is an n-by-$(N - n)$ matrix related to $\xi = (\mathfrak{D}, \widetilde{\mathfrak{D}})$ as $\varphi = \mathfrak{D}^{-1}\widetilde{\mathfrak{D}}$. Using the half-ADHM mapping relation eq. (4.3), we can show that the sigma model instanon solution $\varphi(z)$ corresponding to the half-ADHM data $\{Z, \Psi, \widetilde{\Psi}\}$ can be written as

$$\varphi(z) = \Psi(z\mathbf{1}_k - Z)^{-1}\widetilde{\Psi}. \tag{5.1}$$

For separated vortices, the marix Z can be diagonalized as $(Z)^\alpha{}_\beta = \delta^\alpha{}_\beta\, z_\alpha$ with $z_\alpha \neq z_\beta$ ($\alpha \neq \beta$) and hence the instanton solution takes the form

$$(\varphi(z))^a{}_b = \sum_{\alpha=1}^{k} \frac{(\Psi)^a{}_\alpha (\widetilde{\Psi})^\alpha{}_b}{z - z_\alpha}. \tag{5.2}$$

The column vectors $(\Psi)^a{}_\alpha$ and row vectors $(\widetilde{\Psi})^\alpha{}_b$ for each α are respectively called the orientational moduli and size moduli of the vortex sitting at $z = z_\alpha$. Here we have partially fixed the $\mathrm{GL}(k, \mathbb{C})$ redundancy by diagonalizing the k-by-k matrix Z. The remnant group which does not change the form of Z is $(\mathrm{U}(1)^{\mathbb{C}})^k \subset \mathrm{GL}(k, \mathbb{C})$ and each $\mathrm{U}(1)^{\mathbb{C}} \cong \mathbb{C}^*$ acts on the orientational moduli ${\Psi^a}_\alpha$. Due to the condition that the GL$(k,\mathbb{C})$ action is free, ${\Psi^a}_\alpha$ cannot be a zero column vector for each α and hence the orientational moduli space of each vortex described by ${\Psi^a}_\alpha$ is a $\mathbb{C}P^{n-1} = (\mathbb{C}^n\backslash\{0\})/\sim$. On the contrary, the "size" moduli ${\widetilde{\Psi}^a}_\alpha$ can be a zero vector, which corresponds to a local vortex when the gauge coupling constant is finite.

As shown in eq. (3.28) in appendix D.2, the condition that all vortices are of semi-local type can be rewritten in terms of the half-ADHM data as

$$\exists\, \vec{v} \in \mathbb{C}^k \text{ (row vector)} \quad \text{s.t.} \quad \vec{v}\,(z\mathbf{1}_k - Z)^{-1}\widetilde{\Psi} = 0 \quad \text{for} \quad \forall z \in \mathbb{C} \quad \Rightarrow \quad \vec{v} = 0. \tag{5.3}$$

This condition requires that the GL$(k,\mathbb{C})$ acts freely not only on $\{Z,\Psi\}$, but also on $\{Z,\widetilde{\Psi}\}$. On the other hand, we can uniquely determine the half-ADHM data satisfying (5.3) corresponding to any given sigma model instanton solution (See appendix D.3). Therefore the moduli space of instanton solutions $\mathcal{M}_{\text{inst}}$ are written in terms of the half-ADHM data as

$$\mathcal{M}_{\text{inst}} \equiv \left\{(Z,\Psi,\widetilde{\Psi}) \,\middle|\, \text{GL}(k,\mathbb{C}) \text{ actions on } \{Z,\Psi\},\{Z,\widetilde{\Psi}\} \text{ are free}\right\}/\text{GL}(k,\mathbb{C}). \tag{5.4}$$

This is a subspace of the total vortex moduli space $\mathcal{M}_{\text{vortex}}$, for which the GL$(k,\mathbb{C})$ free condition is imposed only $\{Z,\Psi\}$

$$\mathcal{M}_{\text{vortex}} \equiv \left\{(Z,\Psi,\widetilde{\Psi}) \,\middle|\, \text{GL}(k,\mathbb{C}) \text{ action on } \{Z,\Psi\} \text{ is free}\right\}/\text{GL}(k,\mathbb{C}). \tag{5.5}$$

Note that, as we have discussed, there is a correspondence between sigma model instanton solutions in the dual pair of theories. The half-ADHM data $\{Z^{\text{dual}},\Psi^{\text{dual}},\widetilde{\Psi}^{\text{dual}}\}$ describing the dual sigma model instanton is given by

$$\{Z^{\text{dual}},\Psi^{\text{dual}},\widetilde{\Psi}^{\text{dual}}\} = \{Z^{\text{T}},\widetilde{\Psi}^{\text{T}},-\Psi^{\text{T}}\}, \tag{5.6}$$

up to GL$(k,\mathbb{C})$ transformations. We can read off this relation from the duality transformation $\mathcal{U}^{\text{dual}} = R\,\mathcal{U}^{\text{T}-1}R^\dagger$ (see eq. (2.64)), which maps a solution $\varphi(z)$ to a dual solution $\varphi^{\text{dual}}(z)$ as

$$\varphi^{\text{dual}}(z) = -\varphi(z)^{\text{T}} = -\widetilde{\Psi}^{\text{T}}(z\mathbf{1}_k - Z^{\text{T}})^{-1}\Psi^{\text{T}}. \tag{5.7}$$

The total moduli space of the dual vorties are given by

$$\mathcal{M}^{\text{dual}}_{\text{vortex}} = \left\{(Z^{\text{dual}},\Psi^{\text{dual}},\widetilde{\Psi}^{\text{dual}}) \,\middle|\, \text{GL}(k,\mathbb{C}) \text{ action on } \{Z^{\text{dual}},\Psi^{\text{dual}}\} \text{ is free}\right\}/\text{GL}(k,\mathbb{C}). \tag{5.8}$$

Here the GL$(k,\mathbb{C})$-free condition on $\{Z^{\text{dual}},\Psi^{\text{dual}}\}$ is nothing but the condition (5.3) through the relation (5.6). Therefore $\mathcal{M}_{\text{inst}}$ is given as an intersection of the original vortex moduli space and the dual one as

$$\mathcal{M}_{\text{inst}} = \mathcal{M}_{\text{vortex}} \cap \mathcal{M}^{\text{dual}}_{\text{vortex}}. \tag{5.9}$$

5.2 General L

For general L, a well-defined sigma model instanton solution is given if and only if

$$\forall z: \quad \det \xi_i(z)\xi_i(z)^\dagger \neq 0, \quad i = 1,2,\cdots,L. \tag{5.10}$$

Repeating the discussion in the case of $L=1$, we can show that the above condition for each i is equivalent to the following two conditions on $\{Z_i,\Psi_i,\widetilde{\Psi}_i\}$

1. $\mathrm{GL}(k_i,\mathbb{C})$ action is free on $\{Z_i,\Psi_i\}$,

2. $\mathrm{GL}(k_i,\mathbb{C})$ action is free on $\{Z_i,\widetilde{\Psi}_i\}$.

From the viewpoint of the original gauge theory, the first condition comes from the definition of the vortex moduli space and the second condition is imposed to avoid small-instanton singularities. On the other hand, from the viewpoint of the dual gauge theory characterized by (2.65), the roles of the above two conditions are interchanged.

For general L, a solution is written in terms of n_i-by-n_j matrices $\varphi_{ij}(z)$ $(1\le i<j\le L+1)$ whose entries are inhomogeneous coordinates (2.18) of $G^{\mathbb{C}}/\hat{H}$

$$\mathcal{G}=\mathcal{G}(\varphi_{ij})\equiv\begin{pmatrix} \mathbf{1}_{n_1} & \varphi_{12} & \varphi_{13} & \cdots & \varphi_{1,L+1}\\ \mathbf{0} & \mathbf{1}_{n_2} & \varphi_{23} & & \\ \mathbf{0} & \mathbf{0} & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \mathbf{1}_{n_L} & \varphi_{L,L+1}\\ \mathbf{0} & \cdots & & \mathbf{0} & \mathbf{1}_{n_{L+1}}\end{pmatrix}. \tag{5.11}$$

For a given half-ADHM data $\{Z_i,\Upsilon_i,\widetilde{\Upsilon}_i,W_j\}$, the corresponding $\varphi_{ij}(z)$ are given by

$$\varphi_{ij}(z)=\Upsilon_i\,(z\mathbf{1}-Z_i)^{-1}\,W_iW_{i+1}\cdots W_{j-2}\widetilde{\Upsilon}_{j-1}\quad\text{for } i<j. \tag{5.12}$$

This can be shown as follows. The matrices $\varphi_{ij}(z)$ $(i=i+1,\cdots,L+1)$ are contained in $\xi_i=\xi_i^p\mathcal{G}$

$$\xi_i=\begin{pmatrix}\mathbf{1}_{N_{i-1}} & \mathcal{A}_i & \widetilde{\mathcal{A}}_i\\ \mathbf{0} & \mathbf{1}_{n_i} & \mathcal{B}_i\end{pmatrix}\sim\begin{pmatrix}\mathbf{1}_{N_i} & \mathbf{0} & \widetilde{\mathcal{A}}_i-\mathcal{A}_i\mathcal{B}_i\\ \mathbf{0} & \mathbf{1}_{n_{i+1}} & \mathcal{B}_i\end{pmatrix}\quad\text{with}\quad \mathcal{B}_i=(\varphi_{i,i+1},\varphi_{i,i+2},\cdots,\varphi_{i,L+1}). \tag{5.13}$$

It follows from $\xi_i\sim(\mathbf{1}_{N_{i+1}},\mathfrak{D}_i^{-1}\mathfrak{D}_i)=(\mathbf{1}_{N_{i+1}},\Psi_i(z\mathbf{1}-Z_i)^{-1}\widetilde{\Psi}_i)$ that

$$\begin{pmatrix}\widetilde{\mathcal{A}}_i-\mathcal{A}_i\mathcal{B}_i\\ \mathcal{B}_i\end{pmatrix}=\begin{pmatrix}\Psi_{i-1}W_{i-1}(z\mathbf{1}-Z_i)^{-1}\widetilde{\Psi}_i\\ \Upsilon_i(z\mathbf{1}-Z_i)^{-1}\widetilde{\Psi}_i\end{pmatrix}. \tag{5.14}$$

From the lower blocks of the both hand sides, we find that

$$\mathcal{B}_i=\Upsilon_i(z\mathbf{1}-Z_i)^{-1}\widetilde{\Psi}_i=\Big(\Upsilon_i(z\mathbf{1}-Z_i)^{-1}\widetilde{\Upsilon}_i\,,\ \cdots,\ \Upsilon_i(z\mathbf{1}-Z_i)^{-1}W_iW_{i+1}\cdots W_{L-1}\widetilde{\Upsilon}_L\Big). \tag{5.15}$$

This indicates the relation (5.12).[21]

[21] One can check the equation for the upper blocks is also satisfied. Since $\xi_{i-1}=p_{i-1}\xi_i\sim(\mathbf{1}_{N_{i-1}},\mathcal{A}_i,\widetilde{\mathcal{A}}_i)=(\mathbf{1}_{N_{i-1}},\mathfrak{D}_{i-1}^{-1}\widetilde{\mathfrak{D}}_{i-1})=(\mathbf{1}_{N_{i-1}},\Psi_{i-1}(z\mathbf{1}-Z_{i-1})^{-1}\widetilde{\Psi}_{i-1})$, one finds that

$$\mathcal{A}_i=\Psi_{i-1}(z\mathbf{1}-Z_{i-1})^{-1}\widetilde{\Upsilon}_{i-1},\qquad \widetilde{\mathcal{A}}_i=\Psi_{i-1}(z\mathbf{1}-Z_{i-1})^{-1}W_{i-1}\widetilde{\Psi}_i.$$

Then, one can show that

$$\begin{aligned}\widetilde{\mathcal{A}}_i-\mathcal{A}_i\mathcal{B}_i&=\widetilde{\mathcal{A}}_i-\Psi_{i-1}(z\mathbf{1}-Z_{i-1})^{-1}\widetilde{\Upsilon}_{i-1}\Upsilon_i(z\mathbf{1}-Z_i)^{-1}\widetilde{\Psi}_i\\ &=\widetilde{\mathcal{A}}_i-\Psi_{i-1}(z\mathbf{1}-Z_{i-1})^{-1}(Z_{i-1}W_{i-1}-W_{i-1}Z_i)(z\mathbf{1}-Z_i)^{-1}\widetilde{\Psi}_i=\Psi_{i-1}W_{i-1}(z\mathbf{1}-Z_i)^{-1}\widetilde{\Psi}_i.\end{aligned}$$

Next, let us discuss how the half-ADHM data transform under the duality transformation. For given half-ADHM data $\{Z_i, \Upsilon_i, \widetilde{\Upsilon}_i, W_i\}$ which give a non-singular lump solution, there exist data $\{Z_i^{\rm dual}, \Upsilon_i^{\rm dual}, \widetilde{\Upsilon}_i^{\rm dual}, W_i^{\rm dual}\}$ in the dual theory which give the same lump solution. They are related as

$$\{Z_i^{\rm dual},\, \Upsilon_i^{\rm dual},\, \widetilde{\Upsilon}_i^{\rm dual},\, W_j^{\rm dual}\} = \{Z_{L+1-i}^{\rm T},\, \widetilde{\Upsilon}_{L+1-i}^{\rm T},\, -\Upsilon_{L+1-i}^{\rm T},\, W_{L-j}^{\rm T}\}. \tag{5.16}$$

This relation can be shown as follows. As shown in (2.60), for the matrix $\mathcal{G} \in G^{\mathbb{C}}$ in (5.11), the corresponding matrix $\mathcal{G}_{\rm dual}$ in the dual theory is given by

$$\mathcal{G}_{\rm dual} = R(\mathcal{G}^{-1})^T R^\dagger \in G^{\mathbb{C}} \quad \text{with} \quad R = \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1}_{n_{L+1}} \\ \vdots & \iddots & \iddots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \iddots & \vdots \\ \mathbf{1}_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \in \mathrm{U}(N). \tag{5.17}$$

One can check that the inverse of $\mathcal{G} = \mathcal{G}(\varphi_{ij}(z))$ takes the form $\mathcal{G}(\varphi_{ij}^{\rm inv}(z))$ with

$$\varphi_{ij}^{\rm inv}(z) = -\Upsilon_i W_i W_{i+1} \cdots W_{j-2} \left(z\mathbf{1} - Z_{j-1}\right)^{-1} \widetilde{\Upsilon}_{j-1} \quad \text{for } i < j. \tag{5.18}$$

Here, one can confirm $\mathcal{G}(\varphi_{ij}(z))\mathcal{G}(\varphi_{ij}^{\rm inv}(z)) = \mathbf{1}$ by using the following identity

$$\begin{aligned}
&\sum_{k=i+1}^{j-1} \varphi_{ik}(z)\varphi_{kj}^{\rm inv}(z) \\
&= -\sum_{k=i+1}^{j-1} \Upsilon_i (z\mathbf{1}-Z_i)^{-1} W_i \cdots W_{k-2} \widetilde{\Upsilon}_{k-1} \Upsilon_k W_k \cdots W_{j-2} (z\mathbf{1}-Z_{j-1})^{-1} \widetilde{\Upsilon}_{j-1} \\
&= -\sum_{k=i+1}^{j-1} \Upsilon_i (z\mathbf{1}-Z_i)^{-1} W_i \cdots W_{k-1} (z\mathbf{1}-Z_k) W_k \cdots W_{j-2} (z\mathbf{1}-Z_{j-1})^{-1} \widetilde{\Upsilon}_{j-1} \\
&\quad + \sum_{k=i+1}^{j-1} \Upsilon_i (z\mathbf{1}-Z_i)^{-1} W_i \cdots W_{k-2} (z\mathbf{1}-Z_{k-1}) W_{k-1} \cdots W_{j-2} (z\mathbf{1}-Z_{j-1})^{-1} \widetilde{\Upsilon}_{j-1} \\
&= -\Upsilon_i (z\mathbf{1}-Z_i)^{-1} W_i \cdots W_{j-2} \widetilde{\Upsilon}_{j-1} + \Upsilon_i W_i \cdots W_{j-2} (z\mathbf{1}-Z_{j-1})^{-1} \widetilde{\Upsilon}_{j-1} \\
&= -\varphi_{ij}(z) - \varphi_{ij}^{\rm inv}(z).
\end{aligned} \tag{5.19}$$

By substituting the inverse $\mathcal{G}(\varphi_{ij}^{\rm inv}(z))$ into eq. (5.17), we find that $\mathcal{G}_{\rm dual}$ takes the form $\mathcal{G}(\varphi_{ij}^{\rm dual}(z))$ with

$$\varphi_{ij}^{\rm dual}(z) = -\widetilde{\Upsilon}_{L+1-i}^{\rm T}(z\mathbf{1} - Z_{L+1-i}^{\rm T})^{-1} W_{L-i}^{\rm T} \cdots W_{L+1-j}^{\rm T} \Upsilon_{L+2-j}^{\rm T}, \tag{5.20}$$

from which we can read the duality transformation

$$\{Z_i,\, \Upsilon_i,\, \widetilde{\Upsilon}_i,\, W_j\} \quad \mapsto \quad \{Z_i^{\rm dual},\, \Upsilon_i^{\rm dual},\, \widetilde{\Upsilon}_i^{\rm dual},\, W_j^{\rm dual}\}. \tag{5.21}$$

Note that the $\mathrm{GL}(k_i, \mathbb{C})$ free condition on $Z_i, \widetilde{\Psi}_i$ for each i

$$\nexists k_i\text{-column vector } \vec{v}: \quad \widetilde{\Upsilon}_j^{\rm T} W_{j-1}^{\rm T} W_{j-2}^{\rm T} \cdots W_i^{\rm T} (z\mathbf{1}_{k_i} - Z_i^{\rm T})^{-1} \vec{v} = 0 \quad \text{for } i \le \forall j \le L, \quad \forall z \in \mathbb{C} \;\Rightarrow\; \vec{v} = 0, \tag{5.22}$$

is equivalent to the GL$(k_i,\mathbb{C})$ free condition on $Z^{\rm dual}_{L+1-i},\Psi^{\rm dual}_{L+1-i}$. By denoting the conditions (4.41) and (5.22) by $\mathcal{C}_i$ and $\widetilde{\mathcal{C}}_i$, respectively, the condition for lump configuration without any small-lump singularity can be written as

$$\bigwedge_i \left(\mathcal{C}_i \wedge \widetilde{\mathcal{C}}_i\right) = \left(\bigwedge_i \mathcal{C}_i\right) \wedge \left(\bigwedge_i \widetilde{\mathcal{C}}_i\right) \tag{5.23}$$

and thus the moduli space for lump configurations is obtained as

$$\mathcal{M}_{\rm lump}{}^{n_1,n_2,\ldots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = \mathcal{M}_{\rm vtx}{}^{n_1,n_2,\ldots,n_{L+1}}_{k_1,k_2,\cdots,k_L} \cap \mathcal{M}_{\rm vtx}{}^{n_{L+1},\ldots,n_2,n_1}_{k_L,\cdots,k_2,k_1}. \tag{5.24}$$

6 Vortex partition function from Kähler quotient

6.1 Vortex effective action

In this section, we consider the vortex partition functions in the quiver GLσMs. In the $L=1$ case, the vortex partition functions have been calculated from the viewpoint of the half-ADHM formalism in [81, 82]. As an application of the half-ADHM formalism for general L, we compute the vortex partition function and check the duality between GLσMs [83] as was done in [84] for the $L=1$ case.

In three dimensions, the dynamics of vortices can be described by the quantum mechanical GLσM specified by the quiver diagram (2.28). Let us introduce chemical potentials for the conserved charges in the vortex effective theory

$$\mathcal{Z}_{\text{hADHM-QM}}(m_a,\epsilon,\mu_f) = \text{Tr}\left[e^{-\beta(H+im_a q^a+i\epsilon J+i\mu_f F)}\right], \tag{6.1}$$

where q^a $(a=1,\cdots,N)$ are the Cartan part of the flavor charge, J is the angular momentum operator, F is the Fermion number operator and (m_a,ϵ,μ_f) are the (imaginary) chemical potentials for the corresponding charges. It is well known that Z at $\mu_f=\pi/\beta$, which we consider in the floowing, is exactly calculable through the supersymmetric localization method. Although it is possible to calculate Z in three dimensions, for simplicity, we focus on the 2d limit $\beta\to 0$ in the following.

The vortex effective theory in the 2d quiver GLσM is described by the 0d half-ADHM GLσM specified by the quiver diagram figure 3. Using its action $S_{\rm eff}$, we can write down the integral expression for the 2d limit of the partition function (6.1)

$$\mathcal{Z}^{n_1,n_2,\cdots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = e^{-\sum_{i=1}^L 2\pi r_i k_i}\int d\mu\, \exp\left(-S_{\rm eff}\right), \tag{6.2}$$

where $d\mu$ stands for the measure for all the degrees of freedom of the 0d half-ADHM GLσM.

Let us focus on the case in which the original model has 2d $\mathcal{N}=(2,2)$ supersymmetry. Since the vortices preserve the half of supersymmetry, the effective theory possesses two (real) supercharges. The supermultiplets in the vortex effective theory are chiral multiplets (φ_I,ψ_I) whose scalar components are $\varphi_I\in(Z_i,\Upsilon_i,\widetilde{\Upsilon}_i,W_i)$, U$(k_i)\times$U$(k_{i+1})$ anti-bifundamental Fermi multiplets $(\tilde\psi_i,\widetilde{W}_i)$ and U(k_i) adjoint gauge multiplets (Φ_i,λ_i,D_i). Their supersymmetry transformations are given by

$$\delta\varphi_I = \varepsilon\psi_I, \qquad \delta\tilde\psi_i = \varepsilon\widetilde{W}_i, \qquad \delta\Phi_i = \delta D_i = 0, \qquad \delta\lambda_i = \varepsilon D_i \tag{6.3}$$

$$\delta\psi_I = \bar\varepsilon\Delta\varphi_I, \qquad \delta\widetilde{W}_i = \bar\varepsilon\Delta\tilde\psi_i, \qquad \delta\bar\Phi_i = i(\bar\varepsilon\lambda_i - \varepsilon\bar\lambda_i), \qquad \delta\bar\lambda_i = \bar\varepsilon D_i, \tag{6.4}$$

where $\Delta\varphi_I$ deenote the infinitesimal transformation of the $\mathrm{U}(k_1)\times\cdots\times\mathrm{U}(k_L)$ gauge, spatial and flavor rotations

$$\Delta Z_i = [\Phi_i, Z_i] + \epsilon Z_i, \qquad \Delta\Upsilon_i = M_i\Upsilon_i - \Upsilon_i\Phi_i, \qquad \Delta\widetilde{\Upsilon}_i = \Phi_i\widetilde{\Upsilon}_i - \widetilde{\Upsilon}_i M_{i+1} + \epsilon\widetilde{\Upsilon}_i, \tag{6.5}$$

$$\Delta W_i = \Phi_i W_i - W_i\Phi_{i+1}, \qquad \Delta\widetilde{W}_i = \Phi_{i+1}\widetilde{W}_i - \widetilde{W}_i\Phi_i - \epsilon\widetilde{W}_i, \tag{6.6}$$

where $M_i = \mathrm{diag}(m^{(i,1)}, \cdots, m^{(i,n_i)})$ and ϵ are the parameters corresponding to the twisted masses and the omega deformation parameter in the original 2d $\mathcal{N} = (2,2)$ model. If we adopt the naive Kähler potential (4.85), the explicit form effective action is given by

$$\begin{aligned} S_{\mathrm{eff}} = &\sum_{i=1}^{L} \mathrm{Tr}\left[\left\|\Delta Z_i\right\|^2 + \left\|\Delta\Upsilon_i\right\|^2 + \left\|\Delta\widetilde{\Upsilon}_i\right\|^2 + \left\|\delta W_i\right\|^2 + \widetilde{W}_i\widetilde{W}_i^\dagger + \left\{\widetilde{W}_i(Z_iW_i - W_iZ_{i+1} - \widetilde{\Upsilon}_i\Upsilon_{i+1}) + (c.c)\right\}\right] \\ &+ \sum_{i=1}^{L} \mathrm{Tr}\left[D_i\left([Z_i, Z_i^\dagger] - \Upsilon_i^\dagger\Upsilon_i + \widetilde{\Upsilon}_i\widetilde{\Upsilon}_i^\dagger + W_iW_i^\dagger - W_{i-1}^\dagger W_{i-1} + \frac{4\pi}{g_i^2}\right)\right] + (\text{fermionic terms}), \end{aligned} \tag{6.7}$$

where $\mathrm{Tr}\left\|\Delta A_I\right\|^2$ stands for the norms of the infinitesimal transformations

$$\mathrm{Tr}\left\|\Delta Z_i\right\|^2 = \mathrm{Tr}\left[\Delta Z_i(\Delta Z_I)^\dagger + \Delta Z_i^\dagger(\Delta Z_i^\dagger)^\dagger)\right], \quad etc. \tag{6.8}$$

The FI parameters $\frac{4\pi}{g^2}$ are chosen so that the areas of the two cycles in the vortex moduli space agree with that calculated from the 2d perspective. Eliminating the auxiliary fields $\widetilde{W}_i$, we obtain a potential whose minimization condition gives the constraints (4.39)

$$\widetilde{W}_i = Z_iW_i - W_iZ_{i+1} - \widetilde{\Upsilon}_i\Upsilon_{i+1} = 0. \tag{6.9}$$

The variations with respect to D_i give the D-term constraints, whose set of solution modulo gauge transformations agrees with the vortex moduli space (4.42). In the presence of the twisted masses and the omega deformation parameters, the conditions $\left\|\delta A_I\right\|^2 = 0$ are satisfied only at the fixed points of the torus action discussed in subsection 4.3.1. When we apply the supersymmetric localization method, the integral (6.2) localizes to those fixed points.

6.2 Contour integral for vortex partition function

Although this explicit effective action (6.7) does not give a correct moduli space metric, it gives the correct vortex partition function since the deformation of the Kähler potential is a Q-exact deformation, which do not change the vortex partition function thanks to the supersymmetric localization.

It is well known that the integral (6.2) can be evaluated by using the localization formula, which relates the partition function to the weights of the torus action at the fixed points

$$\mathcal{Z}_{k_1,k_2,\cdots,k_L}^{n_1,n_2,\cdots,n_{L+1}} = \left(\prod_{i=1}^{L}\Lambda_i^{\beta_i k_i}\right)\sum_{s\in\mathfrak{S}}\frac{1}{\det\mathcal{M}_s}, \tag{6.10}$$

where $\mathfrak{S}$ is the set of the fixed points and $\mathcal{M}_s$ is the generator of the torus action at the fixed point s discussed in subsection 4.3.1. Explicitly, it can be read off from the contour

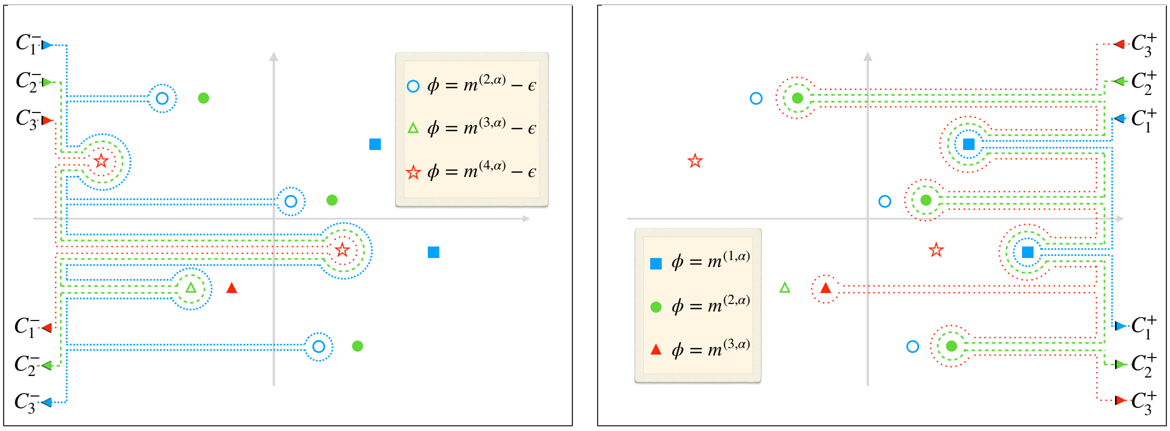

Figure 5. Integration contours. For all $r = 1, \cdots, N_i$, the integration contours for ϕ_i^r are identical path C_i^+ (right panel) on the complex ϕ plane. Each path C_i^+ can be freely deformed as long as it does not cross over the other paths and the poles at $\phi = m^{(i,\alpha)}$ and $\phi = m^{(i+1,\alpha)} - \epsilon$. Following this rule, one can deform the paths C_i^+ to C_i^- (left panel) if there is no pole at the infinity.

integral (see appendix I)

$$\mathcal{Z}^{n_1,n_2,\cdots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = \prod_{i=1}^{L}\left[\frac{1}{k_i!}\Lambda_i^{\beta_{0i}k_i}\prod_{r=1}^{k_i}\oint_{C_i^+}\frac{d\phi_i^r}{2\pi i\epsilon}\right]\left[\prod_{i=1}^{L}\mathcal{Z}_i^{\Upsilon\widetilde{\Upsilon}}\mathcal{Z}_i^{Z\Phi}\prod_{i=1}^{L-1}\mathcal{Z}_i^{W\widetilde{W}}\right], \tag{6.11}$$

where $\mathcal{Z}_i^{\Upsilon\widetilde{\Upsilon}}$, $\mathcal{Z}_i^{Z\Phi}$ and $\mathcal{Z}_i^{W\widetilde{W}}$ are given by

$$\mathcal{Z}_i^{\Upsilon\widetilde{\Upsilon}} \equiv \prod_{r=1}^{k_i}\left[\prod_{\alpha=1}^{n_i}\frac{1}{\phi_i^r - m^{(i,\alpha)}}\prod_{\beta=1}^{n_{i+1}}\frac{1}{m^{(i+1,\beta)} - \phi_i^r - \epsilon}\right], \tag{6.12}$$

$$\mathcal{Z}_i^{Z\Phi} \equiv \prod_{r=1}^{k_i}{\prod_{s=1}^{k_i}}'\frac{\phi_i^r - \phi_i^s}{\phi_i^r - \phi_i^s - \epsilon}, \tag{6.13}$$

$$\mathcal{Z}_i^{W\widetilde{W}} \equiv \prod_{r=1}^{k_i}\prod_{s=1}^{k_{i+1}}\frac{\phi_{i+1}^s - \phi_i^r - \epsilon}{\phi_{i+1}^s - \phi_i^r}, \tag{6.14}$$

where $\prod'$ indicates that the factors with $s = r$ are omitted from the product. The integration contours C_i^+ are paths which are determined from the Jeffrey-Kirwan residue formula to pick up the poles corresponding to the fixed points. Explicitly, C_i^+ are contours on the complex ϕ plane starting at a point $\phi = +\infty$, surrounding $\phi = m^{(j,\alpha)}$ $(j = 1, \cdots, i)$ and C_j^+ $(j = 1, \cdots, i-1)$, and ending at another point $\phi = +\infty$ (see the right panel of figure 5). As shown in appendix I, The contour integral (6.11) is given by the sum of residues at the poles classified by sets of Young tableaux. For set of a Young tableaux

$$\mathbf{Y} = \left\{Y^{(j,\alpha)} = \left(l_j^{(j,\alpha)}, l_{j+1}^{(j,\alpha)}, \cdots, l_L^{(j,\alpha)}\right)\right\}, \tag{6.15}$$

the corresponding pole is located at

$$\phi_i^{(j,\alpha,p)} = m^{(j,\alpha)} + (p-1)\epsilon, \qquad (i = 1, \cdots, L,\ j = 1, \cdots, i,\ \alpha = 1, \cdots, n_j,\ p = 1, \cdots, l_i^{(j,\alpha)}), \tag{6.16}$$

where we have relabeled ϕ_i^r as $\phi_i^{(j,\alpha,p)}$. Performing the contour integral and evaluating the residues, we obtain the partition function of the form

$$\mathcal{Z}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = \left(\prod_{i=1}^{L} \Lambda_i^{\beta_i k_i}\right) \sum_{\mathbf{Y}} Z_{\Upsilon}^{\mathbf{Y}} Z_{\widetilde{\Upsilon}}^{\mathbf{Y}} Z_{Z\Phi}^{\mathbf{Y}} Z_{W\widetilde{W}}^{\mathbf{Y}}, \tag{6.17}$$

where $Z_{\Upsilon}^{\mathbf{Y}}$, $Z_{\widetilde{\Upsilon}}^{\mathbf{Y}}$, $Z_{Z\Phi}^{\mathbf{Y}}$ and $Z_{W\widetilde{W}}^{\mathbf{Y}}$ are contributions from Υ, $\widetilde{\Upsilon}$, (Z,Φ) and $(W,\widetilde{W})$, which are respectively given by

$$Z_{\Upsilon}^{\mathbf{Y}} = \prod_{i=1}^{L}\prod_{\alpha=1}^{n_i}\prod_{k=1}^{i}\prod_{\beta=1}^{n_k}\prod_{q=1}^{l_i^{(k,\beta)}}{}' \frac{1}{m^{(k,\beta)} - m^{(i,\alpha)} + (q-1)\epsilon}, \tag{6.18}$$

$$Z_{\widetilde{\Upsilon}}^{\mathbf{Y}} = \prod_{i=1}^{L}\prod_{j=1}^{i}\prod_{\alpha=1}^{n_j}\prod_{p=1}^{l_i^{(j,\alpha)}}\prod_{\beta=1}^{n_{i+1}}{}' \frac{1}{m^{(i+1,\beta)} - m^{(j,\alpha)} - p\epsilon}, \tag{6.19}$$

$$Z_{Z\Phi}^{\mathbf{Y}} = \prod_{i=1}^{L}\prod_{j=1}^{i}\prod_{\alpha=1}^{n_j}\prod_{p=1}^{l_i^{(j,\alpha)}}\prod_{k=1}^{i}\prod_{\beta=1}^{n_k}\prod_{q=1}^{l_i^{(k,\beta)}}{}' \frac{m^{(k,\beta)} - m^{(j,\alpha)} - (p-q)\epsilon}{m^{(k,\beta)} - m^{(j,\alpha)} - (p-q+1)\epsilon}, \tag{6.20}$$

$$Z_{W\widetilde{W}}^{\mathbf{Y}} = \prod_{i=1}^{L-1}\prod_{j=1}^{i}\prod_{\alpha=1}^{n_j}\prod_{p=1}^{l_i^{(j,\alpha)}}\prod_{k=1}^{i+1}\prod_{\beta=1}^{n_k}\prod_{q=1}^{l_{i+1}^{(k,\beta)}}{}' \frac{m^{(k,\beta)} - m^{(j,\alpha)} - (p-q+1)\epsilon}{m^{(k,\beta)} - m^{(j,\alpha)} - (p-q)\epsilon}, \tag{6.21}$$

where $\prod'$ indicates that the vanishing factors in the denominator and numerator are omitted. We can show that the partition function (6.17) can be rewritten as

$$\mathcal{Z}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L} = \left(\prod_{i=1}^{L} \Lambda_i^{\beta_i k_i}\right) \sum_{\mathbf{Y}} \frac{A^{\mathbf{Y}}}{B^{\mathbf{Y}}}, \tag{6.22}$$

with

$$A^{\mathbf{Y}} = \prod_{i=1}^{L}\prod_{j=1}^{i}\prod_{\alpha=1}^{n_j}\prod_{k=1}^{i}\prod_{\beta=1}^{n_k} (-\epsilon)^{l_i^{(j,\alpha)} - l_i^{(k,\beta)}} \left(\frac{m^{(j,\alpha)} - m^{(k,\beta)}}{\epsilon} + 1\right)_{l_i^{(j,\alpha)} - l_i^{(k,\beta)}}, \tag{6.23}$$

$$B^{\mathbf{Y}} = \prod_{i=1}^{L}\prod_{j=1}^{i}\prod_{\alpha=1}^{n_j}\prod_{k=1}^{i+1}\prod_{\beta=1}^{n_k} (-\epsilon)^{l_i^{(j,\alpha)} - l_i^{(k,\beta)}} \left(\frac{m^{(j,\alpha)} - m^{(k,\beta)}}{\epsilon} + 1\right)_{l_i^{(j,\alpha)} - l_{i+1}^{(k,\beta)}}, \tag{6.24}$$

where $(a)_b$ denotes the Pochhammer symbol

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}. \tag{6.25}$$

We can confirm that the result (6.22) of the contour integral for $\mathcal{Z}^{n_1,n_2,\dots,n_{L+1}}_{k_1,k_2,\cdots,k_L}$ is proportional to the residue of the integrand for the total vortex partition function in the 2d $\mathcal{N}=(2,2)$ theory

$$Z = \int \prod_{i=1}^{L}\prod_{a=1}^{N_i} \left[\frac{d\sigma_i^a}{2\pi i\epsilon} \exp\left(-\frac{2\pi i \sigma_i^a \tau_i}{\epsilon}\right) \prod_{b=1}^{N_i}{}' \Gamma\left(\frac{\sigma_i^a - \sigma_i^b}{\epsilon}\right)^{-1} \prod_{c=1}^{N_{i+1}} \Gamma\left(\frac{\sigma_i^a - \sigma_{i+1}^c}{\epsilon}\right)\right], \tag{6.26}$$

at the pole

$$\sigma_i^{(j,\alpha)} = -m^{(j,\alpha)} - l_i^{(j,\alpha)}\epsilon. \tag{6.27}$$

This is consistent with the fact that the total vortex partition function Z can be written as a sum of the contributions from each topological sectors

$$Z = \mathcal{Z}_{1-\text{loop}}^{n_1,n_2,\ldots,n_{L+1}} \sum_{k_1=0}^{\infty} \cdots \sum_{k_L=0}^{\infty} \left(\prod_{i=1}^{L} \Lambda_i^{\beta_i k_i} \right) \mathcal{Z}^{n_1,n_2,\ldots,n_{L+1}}_{k_1,k_2,\cdots,k_L}. \tag{6.28}$$

Thus, we can confirm that the Kähler quotient construction gives the correct information on the vortex moduli space and the non-perturbative effects in the parent 2d $\mathcal{N} = (2,2)$ theory.

6.3 Duality and partition function

One of the advantages of using the Kähler quotient construction is that it makes the duality manifest in the contour integral expression (6.11). We can show that the partition function agrees with that of the dual theory as follows. In the dual theory, the effective vortex action is described by the same action as (6.7) with the duality map of the degrees of freedom (5.21) and the sign flip of the FI parameters

$$\frac{4\pi}{g_i^2} \to -\frac{4\pi}{g_i^2}. \tag{6.29}$$

As the Jeffrey-Kirwan residue formula implies, the relevant poles become those enclosed by the contour C_i^- in the left panel of figure 5 due to the sign flip of the FI parameters. We can see from (6.12) that if $n_i \neq 0$ ($N_i \neq N_{i+1}$) for all i, the integrand in (6.11) has no pole at the infinity, and hence we can change the contour of integration from C_i^+ to C_i^- without changing the result of the integration (see the left panel of figure 5). The residues for the contours C_i^+ and C_i^- give the vortex partition functions in the original and dual theory, respectively, and hence they are equivalent. More precisely, it is easy to check by changing the variables as $\phi_i^r \to -\tilde{\phi}^r_{L+1-i}$ that

$$\mathcal{Z}^{n_1,n_2,\ldots,n_{L+1}}_{k_1,k_2,\ldots,k_L}(M_1,M_2,\ldots,M_{L+1},\epsilon,\Lambda_i) = \mathcal{Z}^{n_{L+1},n_L\ldots,n_2,n_1}_{k_L,k_{L-1},\ldots,k_1}(\epsilon-M_{L+1},\ldots,\epsilon-M_2,\epsilon-M_1,\epsilon,\Lambda_{L+1-i}), \tag{6.30}$$

and thus the duality of the NLσM holds also for the vortex partition functions.

7 Summary and discussion

In this paper, we analyzed the moduli spaces of the following topological solitons:

1. BPS vortices in the $\mathrm{U}(N_1)\times\cdots\times\mathrm{U}(N_L)$ GLσMs characterized by the linear quiver (2.28).

2. sigma model instantons (lumps) in the Kähler flag manifold sigma models, which can be obtained in the large gauge coupling limit of the GLσMs.

In these theories, vortices and sigma model instantons carrying multiple topological charges $\{k_i\} \in \mathbb{Z}^L$ appear. We analyzed BPS equations for vortices, extracted the data set of the vortex moduli through the moduli matrix method, and exactly determined the moduli spaces of 1/2 BPS vortices in the GLσMs as shown in eq. (3.25). We also discussed how to obtain general exact instanton solutions in the Kähler flag manifold sigma models.

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We showed that the moduli matrix method in the $L = 1$ case can be recast into the ADHM-like quotient construction (the half-ADHM construction) of the moduli space, which has been originally derived from the view point of the D-brane construction in the string theory [7]. Generalizing this technique to the case of general L, we constructed a quotient construction for the vortex moduli space eq. (4.42) specified by the quiver diagram shown in figure 3. In this half-ADHM formalism, various features of the moduli space are manifest. In particular, we have observed that the duality relations are expressed in the simple form (5.16), which is realized by reversing all the arrows in figure 3.

As an application, we have used the quotient construction of the vortex moduli space to calculate the vortex partition functions in 2d $\mathcal{N} = (2,2)$ GLσMs. Applying the localization formula to the half-ADHM system, we have computed the vortex partition from the data of the fixed points of the torus action acting on the vortex moduli space. From the viewpoint of the vortex partition functions, we have confirmed dualities between pairs of quiver gauge theories whose vacuum moduli spaces are identical flag manifolds. We have found that the partition functions agree even though the structures of the vortex moduli spaces, in particular, the fixed point structures, are different between the dual pairs of GLσMs. The half-ADHM formalism has turned out to make the duality manifest even at the quantum level.

One of the future directions is to study the low energy dynamics of vortices and lumps, which are described by geodesic motions on the moduli space equipped with a metric. In the case of $L = 1$ (Grassmannian sigma models) and local non-Abelian vortices, the moduli space metrics have been determined for well-separated vortices [85] and low energy dynamics have been studied [86]. Extending these analyses to the quiver gauge theories and the flag manifold NLσMs discussed in this paper would be interesting.

In this paper, we have extended the half-ADHM formalism for the moduli space of vortices to the gauged linear sigma models characterized by the linear quiver (2.28). One of the future problems is to investigate to what extent the half-ADHM formalism can be generalized to the vortices in arbitrary gauge theories. We have studied flag manifold NLσMs G/H with $G = \mathrm{U}(N)$. Extending our work to other groups G such as $\mathrm{SO}(N)$, $\mathrm{USp}(2N)$ and exceptional groups is an important future work. For $L = 1$, such isometry G can be realized by imposing holomorphic constraints (superpotentials in supersymmetric cases) [87].

Replacing the base space with a space of different geometry and topology would broaden the application of the half-ADHM formalism. However, such a replacement in the base space could drastically change the half-ADHM formalism. In particular, all the proofs in appendices C and D must be reconstructed from scratch. Although research in this direction is challenging, it would deepen our understanding of vortices.

In both theories of the dual pairs discussed in this paper, the quiver diagrams consist of linear chains with all arrows pointing in the same direction. According to [88], the cluster algebra on quiver diagrams produces more dual pairs of theories. That is, the present theory should be dual to various theories characterized by complicated quiver diagrams, such as one involving chain loops or arrows in the opposite direction. It would be interesting to extract the half-ADHM data from the moduli matrices for such models and see how those dualities are expressed at both the classical and quantum levels.

Another important future direction is to study sigma model lumps (instantons) in non-

Kählerflag NLσMs. In this paper, we have studied only BPS lumps (instantons) in Kählerflag NLσMs, where there are no forces among lumps, thus admitting the moduli space. On the other hand, lumps in non-Kählerflag NLσMs are non-BPS; hence there are forces among them. Interaction between non-BPS sigma model instantons would be important when we discuss the non-perturbative aspects of the sigma models from the viewpoint of instantons.

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A Riemannian manifolds and Kähler manifolds

A.1 General Riemannian metric for flag manifolds

The homogeneous Riemannian metric of the generalized flag manifolds $G/H = \mathrm{U}(N)/[\mathrm{U}(n_1)\times \cdots \times \mathrm{U}(n_{L+1})]$ has $L(L+1)/2$ parameters (decay constants) and it becomes a Kählermetric on a L dimensional subspace parameterized by the FI parameters. Here we explain the relations between various expressions for the flag manifold sigma models. Using n_i-by-N matrix valued fields v_i $(i = 1, \cdots, L, L+1)$, the flag manifold sigma model can be given in the following form

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{L+1} f_{ij}\mathrm{Tr}\Big[(\mathcal{D}_\mu v_i v_j^\dagger)(\mathcal{D}_\mu v_i v_j^\dagger)^\dagger\Big] + \sum_{i,j=1}^{L+1}\mathrm{Tr}\Big[\lambda_{ij}(v_i v_i^\dagger - \delta_{ij}\mathbf{1}_{n_i})\Big], \tag{A.1}$$

where $f_{ij} = f_{ji} > 0$ are coupling constants and the covariant derivative on v_i is defined as $\mathcal{D}_\mu v_i = \partial_\mu v_i + i a_\mu^i v_i$ with $\mathrm{U}(n_i)$ gauge fields a_μ^i. The n_i-by-n_j matrices λ_{ij} are Lagrange multipliers which gives the constraints

$$v_i v_i^\dagger = \delta_{ij}\mathbf{1}_{n_i} \quad \Leftrightarrow \quad U^\dagger = (v_1^\dagger, v_2^\dagger, \cdots, v_{L+1}^\dagger) \in \mathrm{U}(N) \quad \text{with} \quad N = \sum_{i=1}^{L+1} n_i, \tag{A.2}$$

and thus the target manifold becomes the generalized flag manifold when the auxiliary gauge fields are eliminated

$$a_\mu^i = i\partial_\mu v_i v_i^\dagger. \tag{A.3}$$

Substituting these into the Lagrangian, we obtain

$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{L}\sum_{j=i+1}^{L+1} f_{ij}\mathrm{Tr}[\partial_\mu P_i \partial^\mu P_j], \tag{A.4}$$

where P_i are projection operators

$$P_i \equiv v_i^\dagger v_i, \qquad (P_i P_j = \delta_{ij} P_i). \tag{A.5}$$

Note that the terms proportional to diagonal elements of f_{ij} are introduced only for stability of the auxiliary fields and disappear in the above Lagrangian since $\mathcal{D}_\mu v_i v_i^\dagger = 0$.

For some purposes, it might be convenient to express the sigma model with quadratic kinetic terms, which reduces to certain special cases of the above model

$$\sum_i f_i \mathrm{Tr}\Big[(\mathcal{D}_\mu v_i)(\mathcal{D}^\mu v_i)^\dagger\Big] = \sum_{i,j} f_i \mathrm{Tr}\Big[(\mathcal{D}_\mu v_i v_j^\dagger)(v_j \mathcal{D}^\mu v_i^\dagger)\Big] = \frac{1}{2}\sum_{i,j}(f_i+f_j)\mathrm{Tr}\Big[(\mathcal{D}_\mu v_i v_j^\dagger)(\mathcal{D}^\mu v_i v_j^\dagger)^\dagger\Big], \tag{A.6}$$

where the completeness condition $\sum_j v_j v_j^\dagger = \mathbf{1}_N$ is used. For $L > 2$, this model does not cover whole space of the homogeneous Riemannian metric, whereas for $L = 1, 2$, it reproduces the Riemannian metric with arbitrary decay constants since $\# f_i = L + 1 \geq L(L+1)/2 = \# f_{ij}$.

Each field configuration can be viewed as a map form $\mathbb{R}^2 \cup \{\infty\} = S^2$ to the target space $\mathcal{M} = G/H$ and defines the topological charges

$$\pi_2(G/H) = \pi_1(H) = \pi_1\left(S[\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \cdots \times \mathrm{U}(n_{L+1})]\right) = \mathbb{Z}^L. \tag{A.7}$$

Explicitly, the topological numbers m_i $(i = 1, \cdots, L+1)$ are given by

$$m_i \equiv -\frac{1}{2\pi}\int_{\mathbb{R}^2} dx^2\, \mathrm{Tr}[f_{12}^i] = \frac{i}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}[dv_i \wedge dv_i^\dagger] = \frac{1}{2\pi i}\oint_{S^1} \mathrm{Tr}[dv_i v_i^\dagger] \in \mathbb{Z}, \tag{A.8}$$

where $f_{\mu\nu}^i = \partial_\mu a_\nu^i - \partial_\nu a_\mu^i + i[a_\mu^i, a_\nu^j]$. Note that there are only L independent charges since the total charge vanishes as

$$\sum_{i=1}^{L+1} m_i = \frac{i}{2\pi}\int_{\mathbb{R}^2} \sum_{i=1}^{L+1} \mathrm{Tr}[dv_i \wedge dv_i^\dagger] = -\frac{i}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}[dUU^\dagger \wedge dUU^\dagger] = 0. \tag{A.9}$$

A.2 Kähler condition on decay constants

Let us discuss the relation between the model with a Rieamannian metric introduced above and the nonlinear sigma model discussed in the main text. As we have seen in (2.52), the Kählerpotential for the Kählermetric is given by

$$K = \sum_{i=1}^{L} r_i \ln\det(\xi_i \xi_i^\dagger). \tag{A.10}$$

Note that the set of matrices $\{\xi_i\}$ and $\{v_i\}$ are related as

$$\left(v_1^\dagger, v_2^\dagger, \cdots, v_i^\dagger\right)^\dagger = \xi_i^o U = \xi_i^o \hat{h}^{-1}\mathcal{G} = \hat{h}_i^{-1}\xi_i^o \mathcal{G} = \hat{h}_i^{-1}\xi_i, \tag{A.11}$$

where $\mathcal{G} = \hat{h}U$ $(\hat{h} \in \hat{H})$, $\hat{h}_i = \xi_i^o \hat{h}(\xi_i^o)^\dagger \in \mathrm{GL}(N_i, \mathbb{C})$. Using the Kählermetric $g_{A\bar{B}}(X) = \partial_A \bar{\partial}_B K$, we find that the Lagrangian for the Kählerflag manifold sigma model is given by

$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{L} r_i \sum_{j,l}^{j\leq i<l} \mathrm{Tr}[\partial_\mu P_j \partial^\mu P_l]. \tag{A.12}$$

Comparing with (A.4), we find that the Riemaniann model reduces to the Kählermodel when the decay constants f_{ij} are given by

$$f_{ij} = \sum_{k=i}^{j-1} r_k \quad \text{or equivalently} \quad f_{i,i+1} = r_i, \quad f_{ij} = f_{ik} + f_{kj} \quad \text{for } i < k < j. \tag{A.13}$$

The vortex numbers $\{k_i\}$ defined in eq. (3.19) are related to topological charges defined in eq. (A.8) as

$$k_i \equiv \frac{1}{2\pi i}\int_{\mathbb{R}^2} \bar{\partial}\wedge\partial \ln\det(\xi_i\xi_i^\dagger) = \sum_{j=1}^{i} m_j. \tag{A.14}$$

B Comments on the master equations

In this appendix, we discuss the existence and the uniqueness of the solution of the set of the master equations (3.13)

$$\hat{\mathcal{E}}_i \equiv q_i\Omega_{i+1}q_i^\dagger\Omega_i^{-1} - \Omega_i q_{i-1}^\dagger\Omega_{i-1}^{-1}q_{i-1} + \frac{4}{g_i^2}\partial_{\bar z}\left(\partial_z\Omega_i\Omega_i^{-1}\right) - r_i\mathbf{1}_{N_i} = 0. \tag{B.1}$$

These are equations for the positive definite Hermitian matrices $\Omega_i \in \mathrm{GL}(N_i,\mathbb{C})$ determined by a given set of the moduli matrices $(q_1,\cdots,q_L)$. The master equations $\hat{\mathcal{E}}_i = 0$ are related to the original BPS equations $\mathcal{E}_i = 0$ for the magnetic flux as

$$\mathcal{E}_i = S_i^{-1}\hat{\mathcal{E}}_i S_i \quad \text{with} \quad \mathcal{E}_i \equiv Q_iQ_i^\dagger - Q_{i-1}^\dagger Q_{i-1} - \frac{2}{g_i^2}F_{12}^i - r_i\mathbf{1}_{N_i}, \tag{B.2}$$

where $S_i \in \mathrm{GL}(N_i,\mathbb{C})$ are the matrices that can be obtained from Ω_i by the Cholesky decomposition $\Omega_i = S_iS_i^\dagger$. The matrices q_i and S_i are related to Q_i and $\mathcal{D}_\mu = \partial_\mu + iA_\mu^i$ as

$$Q_i = S_i^{-1}q_i(z)\,S_{i+1}, \qquad A_{\bar z} = \frac{1}{2}(A_1^i + iA_2^i) = -iS_i^{-1}\partial_{\bar z}S_i. \tag{B.3}$$

The boundary condition for Ω_i is

$$\lim_{|z|\to\infty}\left\{\mathfrak{D}_i(z)^{-1}\Omega_i(z,\bar z)\,\mathfrak{D}_i(z)^{\dagger-1}\right\} = \Omega_i^{\mathrm{o}} \in \mathrm{GL}(N_i,\mathbb{C}), \tag{B.4}$$

where $\mathfrak{D}_i(z)$ is the N_i-by-N_i matrix defined through the relation (see eq. (3.1))

$$\xi_i(z) \equiv q_i(z)\,q_{i+1}(z)\cdots q_L(z) = (\mathfrak{D}_i(z), \widetilde{\mathfrak{D}}_i(z)) \tag{B.5}$$

and Ω_i^{o} is the constant positive-definite matrix corresponding to the vacuum configuration (see eq. (2.38)). By examining the master equation for large z, we find that the deviation of Ω_i from the large coupling limit Ω_i^∞ (see eq. (3.29)) is given by,

$$\mathfrak{D}_i(z)^{-1}(\Omega_i - \Omega_i^\infty)\,\mathfrak{D}_i(z)^{\dagger-1} = \frac{\partial_z\partial_{\bar z}}{g_i^2}\mathcal{O}(|z|^{-2}) = \mathcal{O}\left(|z|^{-4}\right). \tag{B.6}$$

Here, let $\mathcal{F}_\Omega(\mathbb{R}^2)$ denote the space of configurations $\mathbf{\Omega} = \{\Omega_1,\Omega_2,\ldots,\Omega_L\}$ where Ω_i are smooth maps from $\mathbb{R}^2$ to the space of positive definite N_i-by-N_i Hermitian matrices satisfying the boundary condition (B.4) and the asymptotic behavior (B.6) with a given set of moduli matrices $\{q_i(z)\,|\,i=1,2,\ldots,L\}$. Let us take a reference point $\mathbf{\Omega}^{\mathrm{ref}}$ in $\mathcal{F}_\Omega(\mathbb{R}^2)$ and denote its components as $\Omega_i^{\mathrm{ref}} = S_i^{\mathrm{ref}}S_i^{\mathrm{ref}\dagger}$. Since Ω_i is a positive definite Hermitian matrix, we can define an N_i-by-N_i Hermitian matrix ω_i depending on the reference point as

$$\omega_i = \log\left[(S_i^{\mathrm{ref}})^{-1}\Omega_i(S_i^{\mathrm{ref}\dagger})^{-1}\right] \quad \Leftrightarrow \quad \Omega_i = S_i^{\mathrm{ref}}e^{\omega_i}S_i^{\mathrm{ref}\dagger}. \tag{B.7}$$

This relation defines a one-to-one map between $\mathcal{F}_\Omega$ and the functional space $\mathcal{F}_\omega$: the space of $\boldsymbol{\omega} = (\omega_1, \omega_2, \cdots, \omega_L)$ whose components ω_i are smooth bounded functions from $\mathbb{R}^2$ to the space of Hermitian matrices of order N_i satisfying the boundary condition and the asymptotic behavior

$$\lim_{|z|\to\infty} \omega_i = 0, \qquad \omega_i = \mathcal{O}(|z|^{-4}). \tag{B.8}$$

It is worth noting that the asymptotic behaviors in eqs. (B.6) and (B.8) implies that the following square-integrable conditions are satisfied

$$\langle \mathcal{E}^2 \rangle < \infty \quad \text{for } \forall \boldsymbol{\Omega} \in \mathcal{F}_\Omega, \qquad \langle \boldsymbol{\omega}^2 \rangle < \infty \quad \text{for } \forall \boldsymbol{\omega} \in \mathcal{F}_\omega, \tag{B.9}$$

where we have used the following bracket notation

$$\langle \mathcal{O} \rangle \equiv \int d^2x \sum_{i=1}^{L} \mathrm{Tr}[\mathcal{O}_i]. \tag{B.10}$$

B.1 $\mathcal{H}$: linearization of master equations

First, let us define a linear operator $\mathcal{H} = \mathcal{H}(q, S)$ by

$$(\mathcal{H}\mathbf{v})_i \equiv -\frac{4}{g_i^2}\mathcal{D}_{\bar z}\mathcal{D}_z v_i + Q_i \left(Q_i^\dagger v_i - v_{i+1} Q_i^\dagger\right) + \left(v_i Q_{i-1}^\dagger - Q_{i-1}^\dagger v_{i-1}\right) Q_{i-1}, \tag{B.11}$$

where $\mathbf{v} = (v_1, v_2, \cdots, v_L)$ is an element of $\mathcal{F}_\omega^{\mathbb{C}} \equiv \{\mathbf{u} + i\mathbf{w} | \mathbf{u}, \mathbf{w} \in \mathcal{F}_\omega\}$. This operator $\mathcal{H}$, which depends on q_i, S_i, appears in the linearized master equations and plays a central role in the subsequent subsections. Explicitly, one can show that the variation $\delta\hat{\mathcal{E}}_i$ of $\hat{\mathcal{E}}_i$ under the infinitesimal shift $\delta\Omega_i \equiv S_i \omega_i S_i^\dagger$ of $\Omega_i = S_i S_i^\dagger \in \mathcal{F}_\Omega$ with $\omega_i = \omega_i(z, \bar z) \in \mathcal{F}_\omega$ is given in terms of $\mathcal{H}$ as

$$S_i^{-1} \delta\hat{\mathcal{E}}_i S_i = -(\mathcal{H}\,\boldsymbol{\omega})_i. \tag{B.12}$$

This operator $\mathcal{H}$ is Hermitian and positive semi-definite

$$\langle \mathbf{v}\mathcal{H}\mathbf{v}^\dagger \rangle = \int d^2x \sum_{i=1}^{L} \mathrm{Tr}\left[\frac{4}{g_i^2}\mathcal{D}_{\bar z} v_i \mathcal{D}_z v_i^\dagger + (v_i Q_i - Q_i v_{i+1})(v_i Q_i - Q_i v_{i+1})^\dagger\right] \geq 0 \qquad \text{for arbitrary } \mathbf{v} \in \mathcal{F}_\omega^{\mathbb{C}}. \tag{B.13}$$

Furthermore, we can show that there exists a gap in the spectrum of the linear operator $\mathcal{H}$ as follows. Suppose that $\mathcal{H}\mathbf{v}^\dagger = 0$. Then, the above inner product vanishes and hence

$$0 = \mathcal{D}_z v_i^\dagger = \mathcal{D}_{\bar z} v_i, \quad v_i Q_i = Q_i v_{i+1} \quad (v_{L+1} = 0) \quad \text{for } i = 1, 2, \ldots, L. \tag{B.14}$$

It follows from the second equation that

$$v_i Q_i Q_{i+1} \cdots Q_L = Q_i v_{i+1} Q_{i+1} \cdots Q_L = \cdots = Q_i Q_{i+1} \cdots Q_L v_{L+1} = 0. \tag{B.15}$$

This equation implies that v_i must vanish since $Q_i Q_{i+1} \cdots Q_L$ has the maximal rank except for a finite number of points.[22] Therefore, we find that $\mathcal{H}$ has no zero mode

$$\ker \mathcal{H}(q, S) = \{0\} \quad \text{for } \forall (q_i, S_i). \tag{B.16}$$

From this property, we can immediately conclude that $\boldsymbol{\Omega}$ has no additional moduli parameter and all the moduli parameters of vortex solutions are contained in $\{q_i(z)\}$.

[22] The matrix $\xi_i = q_i q_{i+1} \cdots q_L$ has the maximal rank except for a finite number of points. The matrix $Q_i Q_{i+1} \cdots Q_L = S_i^{-1} \xi_i$ has the same property since $S_i(z, \bar z) \in \mathrm{GL}(N_i, \mathbb{C})$.

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B.2 Vortex action and proof of the uniqueness

For a given set of moduli matrices $\{q_i(z)\}$ (with an appropriate gauge fixing of the V-transformations), we can show that there exists a functional $\mathfrak{S}^{\rm vtx}$ of $\boldsymbol{\Omega}$ such that the variation of $\mathfrak{S}^{\rm vtx}$ with respect to Ω_i is given by

$$\delta\mathfrak{S}^{\rm vtx} = -\int d^2x \sum_{i=1}^{L} {\rm Tr}\left[\delta\Omega_i\Omega_i^{-1}\hat{\mathcal{E}}_i\right] = -\int d^2x \sum_{i=1}^{L} {\rm Tr}\left[\omega_i\mathcal{E}_i\right] = -\langle\boldsymbol{\omega}\boldsymbol{\mathcal{E}}\rangle. \tag{B.17}$$

That is, $\mathfrak{S}^{\rm vtx}$ is the action which gives the full set of the master equations $\{\hat{\mathcal{E}}_i = 0\}$.

Existence of $\mathfrak{S}^{\rm vtx}$. Although such an $\mathfrak{S}^{\rm vtx}$ may not be unique due to constant and total derivative terms, we can show that the following functional gives the full set of the master equations

$$\mathfrak{S}^{\rm vtx}[\boldsymbol{\Omega},\boldsymbol{\Omega}^{\rm ref}] \equiv \int d^2x \left\{\mathcal{L}(\boldsymbol{\Omega}) - \mathcal{L}(\boldsymbol{\Omega}^{\rm ref})\right\}, \qquad \mathcal{L}(\boldsymbol{\Omega}) = \sum_{i=1}^{L}\left\{\mathcal{L}_i^D + \frac{1}{g_i^2}\mathcal{L}_i^K\right\}. \tag{B.18}$$

Here $\mathcal{L}_i^D$ and $\mathcal{L}_i^K$,[23] are given by

$$\mathcal{L}_i^D = {\rm Tr}\left[\Omega_i^{-1}q_i\Omega_{i+1}q_i^\dagger + r_i\log\Omega_i\right], \tag{B.19}$$

$$\mathcal{L}_i^K = {\rm Tr}\left[4|\partial_z\psi_i|^2 + 2e^{-2\psi_i}(L_i^{-1}\partial_{\bar z}L_i)\,e^{2\psi_i}\,(L_i^{-1}\partial_{\bar z}L_i)^\dagger\right], \tag{B.20}$$

where ψ_i is a N_i-by-N_i diagonal matrix and L_i is a lower unitriangular matrix obtained by the Cholesky decomposition

$$\Omega_i = L_i\, e^{2\psi_i} L_i^\dagger. \tag{B.21}$$

Note that although $\mathfrak{S}^{\rm vtx}$ is not invariant under the V-transformations, the shift is independent of Ω. Hence, the variation of $\mathfrak{S}^{\rm vtx}$ with respect to Ω_i reproduce the master equations, which are covariant under the V-transformations. The term $\mathcal{L}(\boldsymbol{\Omega}^{\rm ref})$ in the integrand is added to make the integral finite.

To confirm that surface terms vanish in the r.h.s. of (B.17), we need to discuss the boundary conditions for $\{\psi_i, L_i\}$. Note that, as discussed in appendix C.1, an arbitrary given matrix $\mathfrak{D}_i(z)$ can always be transformed into a lower triangular form by a V-transformation as

$$\mathfrak{D}_i(z) = \begin{pmatrix} p_{i,1}(z) & 0 & \cdots & 0 \\ \star & p_{i,2}(z) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \star & \cdots & \star & p_{i,N_i}(z) \end{pmatrix} = \left(\mathbf{1}_{N_i} + \mathcal{O}(z^{-1})\right)\mathbf{p}_i(z) \tag{B.22}$$

where "$\star$" stand for polynomials and $\mathbf{p}_i(z)$ is a diagonal matrix

$$\mathbf{p}_i(z) = {\rm diag}\left(p_{i,1}(z), p_{i,2}(z), \cdots, p_{i,N_i}(z)\right) \tag{B.23}$$

[23] These terms are equivarent to the normal kinetic terms plus the Wess-Zumino-Witten terms up to total derivative terms.

such that the a-th diagonal entry $p_{i,a}(z)$ is a monic polynomial of degree $l_{i,a}$ and $\deg(\det \mathbf{p}_i(z)) = \sum_a l_{i,a} = k_i$. Under this gauge choice, the boundary condition (3.18) for Ω_i can be rewritten in terms of $\{\psi_i, L_i\}$ as

$$\psi_i \to \frac{1}{2}\log\Omega_i^o + \frac{1}{2}\log|\mathbf{p}_i(z)|^2 + \mathcal{O}(|z|^{-2}), \qquad L_i \to \mathfrak{D}_i(z)(\mathbf{1}_{N_i} + \mathcal{O}(|z|^{-2}))\mathbf{p}_i(z)^{-1} \quad \text{for } |z| \to \infty. \tag{B.24}$$

From these behaviors, we conclude that the contributions from the surface terms of $\delta\mathfrak{S}^{\mathrm{vtx}}$ vanishes as

$$\begin{aligned}
&\lim_{R\to\infty} \mathrm{Im} \oint_{|z|=R} dz \sum_i \mathrm{Tr}\left[2\delta\psi_i\partial_z\psi_i + e^{-2\psi_i}(L_i^{-1}\delta L_i)\, e^{2\psi_i}\partial_z L_i^\dagger L_i^{\dagger -1}\right] \\
&= \lim_{R\to\infty} \mathrm{Im} \oint_{|z|=R} dz \sum_i \mathrm{Tr}\left[\delta\psi_i \frac{\partial_z \mathbf{p}_i(z)}{\mathbf{p}_i(z)}\right] \\
&= 0
\end{aligned} \tag{B.25}$$

where we have used $\lim_{|z|\to\infty}\delta\psi_i = 0$. From this property, we conclude that eq. (B.17) holds. Furthermore, this property implies that terms in $\mathcal{L}(\mathbf{\Omega})$ that could diverge when integrated are independent of Ω_i and thus the counter term $\mathcal{L}(\mathbf{\Omega}^{\mathrm{ref}})$ in the action cancels such terms. This cancellation makes $\mathfrak{S}^{\mathrm{vtx}}$ finite and hence its convexity, discussed in the next paragraph, is well-defined. Although $\mathfrak{S}^{\mathrm{vtx}}$ depends on the choice of the coordinate patch of the moduli space (the choice of the label $\lambda_i = (l_{i,1}, l_{i,2}, \ldots, l_{i,N_i})$), there is no problem with the arguments for existence and uniqueness of the solution with fixed moduli parameters.

Convexity of $\mathfrak{S}^{\mathrm{vtx}}$. The action $\mathfrak{S}^{\mathrm{vtx}}$ constructed above is always a convex functional. To see this, let us take a pair of configurations $\Omega_i^{(a)} = S_i^{(a)}(S_i^{(a)})^\dagger (a = 1, 2)$ which satisfy the same boundary conditions with a given $\{q_i = q_i(z)\}$. For such a pair of configurations, let us define an Hermitian matrix $\omega_i \in \mathfrak{u}(N_i)$ by

$$e^{\omega_i} \equiv (S_i^{(1)})^{-1}\Omega_i^{(2)}(S_i^{(1)})^{\dagger -1}, \tag{B.26}$$

which satisfies the boundary condition (B.4). Using these quantities $\{\omega_i\}$, we obtain a set of functions of a parameter τ

$$\Omega_i(\tau\boldsymbol{\omega}) = S_i(\tau\boldsymbol{\omega})S_i(\tau\boldsymbol{\omega})^\dagger \equiv S_i^{(1)} e^{\tau\,\omega_i}(S_i^{(1)})^\dagger \tag{B.27}$$

which continuously interpolates two given configurations:

$$\Omega_i(0) = \Omega_i^{(1)} \quad \text{and} \quad \Omega_i(\boldsymbol{\omega}) = \Omega_i^{(2)}. \tag{B.28}$$

Then, substituting $\Omega_i(\tau\boldsymbol{\omega})$ to the action and using

$$\frac{d\Omega_i(\tau\boldsymbol{\omega})}{d\tau} = S_i(\tau\boldsymbol{\omega})\omega_i S_i(\tau\boldsymbol{\omega})^\dagger, \qquad \frac{d}{d\tau}\left\{\frac{d\Omega_i(\tau\boldsymbol{\omega})}{d\tau}\Omega_i(\tau\boldsymbol{\omega})^{-1}\right\} = 0, \tag{B.29}$$

we find that $\mathfrak{S}^{\mathrm{vtx}}$ is always a convex functions of τ:

$$\forall\tau \in \mathbb{R}: \quad \frac{d^2\mathfrak{S}^{\mathrm{vtx}}[\mathbf{\Omega}(\tau\boldsymbol{\omega}), \mathbf{\Omega}^{\mathrm{ref}}]}{d\tau^2} = \langle\boldsymbol{\omega}\mathcal{H}\boldsymbol{\omega}\rangle\Big|_{S_i \to S_i(\tau\boldsymbol{\omega})} > 0. \tag{B.30}$$

Therefore, if $\{\Omega_i^{(1)}\}$ is the solution of the master equations, then the action takes the minimum at $\tau = 0$ and can never have other extrema since the derivative $\partial_\tau \mathfrak{S}^{\rm vtx}$ is monotonically increasing function for any choice of $\{\Omega_i^{(2)}\}$. Therefore, *the solution of the master equations must be unique if it exists.*

B.3 Comments on existence of solutions

In the main text, it is assumed that a solution to the master equations exists for an arbitrarily given set of moduli matrices $\{q_i(z)\}$. While the existence of the Abelian vortex solutions has been proven in [79], for non-Abelian vortices, however, it is generally difficult to prove the existence of solutions except for some limited cases [89–91]. To the best of our knowledge, the proof of existence of vortex solutions in the general systems is not known. Let us try to give a circumstantial evidence that the solution exists, using a rough argument that is not necessarily mathematically rigorous.

First, let $\|\cdot\|$ be a norm defined in the functional space $\mathcal{F}_\omega$. Let us write an arbitrary element $\boldsymbol{\omega} \in \mathcal{F}_\omega$ as $\boldsymbol{\omega} = \tau\hat{\boldsymbol{\omega}}$ by using its norm $\tau \equiv \|\boldsymbol{\omega}\|$ and the normalized element $\hat{\boldsymbol{\omega}} \in \mathcal{F}_\omega$ satisfying $\|\hat{\boldsymbol{\omega}}\| = 1$. Then an arbitrary $\boldsymbol{\Omega} \in \mathcal{F}_{\boldsymbol{\Omega}}$ can be expressed using eq. (B.7) as

$$\Omega_i = \Omega_i(\tau\hat{\boldsymbol{\omega}}) = S_i(\tau\hat{\boldsymbol{\omega}})(S_i(\tau\hat{\boldsymbol{\omega}}))^\dagger \equiv S_i^{\rm ref} e^{\tau\hat{\omega}_i}(S_i^{\rm ref})^\dagger, \qquad \Omega_i(0) = \Omega_i^{\rm ref}, \tag{B.31}$$

and the vortex action $\mathfrak{S}^{\rm vtx}$ defined in eq. (B.18) can be regarded as a function of τ,

$$\mathfrak{S}_{\hat{\omega}}(\tau) \equiv \mathfrak{S}^{\rm vtx}[\boldsymbol{\Omega}(\tau\hat{\boldsymbol{\omega}}), \boldsymbol{\Omega}^{\rm ref}]. \tag{B.32}$$

Note that this function is convex everywhere as discussed in the previous subsection. In the following, we show that for each choice of $\hat{\boldsymbol{\omega}}$, the function $\mathfrak{S}_{\hat{\omega}}(\tau)$ has a minimum at some point with $\tau = \tau_{\hat{\omega}} < \infty$. Then, tracking the decreasing sequence of the function $\mathfrak{S}_{\hat{\omega}}(\tau_{\hat{\omega}})$ in the space of normalized $\hat{\boldsymbol{\omega}}$, we can find the solution of the master equation. Thus, roughly speaking, showing the existence of a solution to the master equations is equivalent to showing that $\mathfrak{S}_{\hat{\omega}}(\tau)$ has a minimum for an arbitrary normalized configuration $\hat{\boldsymbol{\omega}} \in \mathcal{F}_\omega$ and the space of normalized $\hat{\boldsymbol{\omega}}$ is a complete metric space.

Coerciveness of $\mathfrak{S}^{\rm vtx}$. Here, we show that for an arbitrary nonzero element $\hat{\boldsymbol{\omega}} \in \mathcal{F}_\omega$

$$\lim_{\tau\to\infty} \frac{d\mathfrak{S}_{\hat{\omega}}(\tau)}{d\tau} = \infty \qquad \text{and} \qquad \frac{d^2\mathfrak{S}_{\hat{\omega}}(\tau)}{d\tau^2} > 0 \quad \text{for arbitrary } \tau, \tag{B.33}$$

i.e. $\mathfrak{S}_{\hat{\omega}}(\tau)$ is a coercive and convex function of τ for an arbitrary $\hat{\boldsymbol{\omega}} \in \mathcal{F}_\omega \backslash \{0\}$.

The derivative of $\mathfrak{S}_{\hat{\omega}}(\tau)$ with respect to τ is given by[24]

$$\frac{d\mathfrak{S}_{\hat{\omega}}(\tau)}{d\tau} = -\langle\hat{\boldsymbol{\omega}}\mathcal{E}\rangle\Big|_{S_i = S_i(\tau\hat{\omega})} = -\langle\hat{\boldsymbol{\omega}}\mathcal{E}\rangle\Big|_{S_i = S_i^{\rm ref}} + \int_0^\tau ds \langle\hat{\boldsymbol{\omega}}\mathcal{H}\hat{\boldsymbol{\omega}}\rangle\Big|_{S_i = S_i(s\hat{\omega})}, \qquad \mathfrak{S}_{\hat{\omega}}(0) = 0. \tag{B.34}$$

[24] The above property can be viewed as the definition of the function $\mathfrak{S}_{\hat{\omega}}(\tau)$, which is independent of the details of the original defining equation (B.18). We can confirm that $\mathfrak{S}_{\hat{\omega}}(\tau)$ is finite for an arbitrary $\tau \in \mathbb{R}$ since $\langle\hat{\boldsymbol{\omega}}\mathcal{E}\rangle^2 \leq \langle\hat{\boldsymbol{\omega}}^2\rangle\langle\mathcal{E}^2\rangle < \infty$.

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Note that $\langle \hat{\omega} \mathcal{H} \hat{\omega} \rangle$ implicitly depends on τ, since $\mathcal{H}$ depends on $S_i = S_i(\tau\hat{\omega})$. Let us show that $\langle \hat{\omega} \mathcal{H} \hat{\omega} \rangle$ cannot vanish even in the limit $\tau \to \infty$, whereas it must be positive definite for a finite τ as discussed in section B.1. First, we show that

$$\lim_{\tau\to\infty} \langle \hat{\omega} \mathcal{H} \hat{\omega} \rangle \Big|_{S_i=S_i(\tau\hat{\omega})} = 0 \quad \Longrightarrow \quad \hat{\omega}_i(z,\bar{z}) = 0. \tag{B.35}$$

From the assumption given in left-hand side of the above statement, we obtain

$$\lim_{\tau\to\infty} \mathcal{D}_z \hat{\omega}_i \Big|_{S_i=S_i(\tau\hat{\omega})} = \lim_{\tau\to\infty} \mathcal{D}_{\bar{z}} \hat{\omega} \Big|_{S_i=S_i(\tau\hat{\omega})} = 0. \tag{B.36}$$

Since entries of $\hat{\omega} \in \mathcal{F}_\omega$ are continuous functions, it follows that

$$\partial_z \mathrm{Tr}\hat{\omega}_i^p = p \lim_{\tau\to\infty} \mathrm{Tr}[\hat{\omega}_i^{p-1}\mathcal{D}_z\hat{\omega}_i] = 0 \qquad \text{and hence} \qquad \text{eigenvalues of } \hat{\omega}_i = \text{constant}. \tag{B.37}$$

Combining the above result with the boundary condition $\lim_{|z|\to\infty} \hat{\omega}_i = 0$, we find all $\hat{\omega}_i$ must vanish everywhere

$$\forall i, \forall z, \qquad \hat{\omega}_i = \hat{\omega}_i(z,\bar{z}) = 0. \tag{B.38}$$

This implies eq. (B.35) and its contraposition

$$\forall \hat{\omega} \in \mathcal{F}_\omega \backslash \{0\}: \quad \lim_{\tau\to\infty} \langle \hat{\omega} \mathcal{H} \hat{\omega} \rangle \Big|_{S_i=S_i(\tau\hat{\omega})} > 0, \tag{B.39}$$

where we have used the fact that $\mathcal{H}$ is a positive semi-definite operator. Applying this statement and the convexity of $\mathfrak{S}_{\hat{\omega}}$ to eq. (B.34), we conclude that

$$\forall \hat{\omega} \in \mathcal{F}_\omega \backslash \{0\}: \qquad \lim_{\tau\to\infty} \frac{d\mathfrak{S}_{\hat{\omega}}(\tau)}{d\tau} = \infty, \qquad \forall \tau: \frac{d^2\mathfrak{S}_{\hat{\omega}}(\tau)}{d\tau^2} > 0 \tag{B.40}$$

and thus $\mathfrak{S}_{\hat{\omega}}(\tau)$ is a coercive and convex function of τ for arbitrary $\hat{\omega}_i \in \mathcal{F}_\omega \backslash \{0\}$. This result implies that $\mathfrak{S}_{\hat{\omega}}(\tau)$ has a minimum, $\mathfrak{S}_{\hat{\omega}}(\tau_{\hat{\omega}}) (\leq 0)$, with a certain $\tau = \tau_{\hat{\omega}}$ for each $\hat{\omega}$. Thus, by collecting these minimum, we can define a map from a hypersurface $\hat{\mathcal{F}}_\omega \equiv \{\hat{\omega} \in \mathcal{F}_\omega |\ \|\hat{\omega}\| = 1\}$ to $\mathcal{F}_\omega$ as, $\hat{\omega} \in \hat{\mathcal{F}}_\omega \mapsto \tau_{\hat{\omega}}\hat{\omega} \in \mathcal{F}_\omega$. Since

$$\inf_{\mathbf{\Omega}\in\mathcal{F}_\Omega} \mathfrak{S}^{\mathrm{vtx}}[\mathbf{\Omega}, \mathbf{\Omega}^{\mathrm{ref}}] = \inf_{\hat{\omega}\in\hat{\mathcal{F}}_\omega} \mathfrak{S}_{\hat{\omega}}(\tau_{\hat{\omega}}), \tag{B.41}$$

assuming $\mathcal{F}_\omega$ is a complete metric space, the coereciveness and convexity of $\mathfrak{S}_{\hat{\omega}}(\tau)$ implies that $\mathfrak{S}^{\mathrm{vtx}}$ has a minimum with a certain $\boldsymbol{\omega} = \boldsymbol{\omega}_{\min}$, which gives a solution $\mathbf{\Omega}^{\mathrm{sol}} \equiv \mathbf{\Omega}(\boldsymbol{\omega}_{\min})$ to the master equations. This "proof" for the existence of the solution is, however, not mathematically rigorous, since in this argument the "solution" obtained using a decreasing Cauchy sequence $\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}, \cdots \in \mathcal{F}_\omega$, the limit $\boldsymbol{\omega}_{\min} = \lim_{a\to\infty} \boldsymbol{\omega}^{(a)}$ is not guaranteed to consists of bounded, smooth functions, $\boldsymbol{\omega}_{\min} \in \mathcal{F}_\omega$. For a complete proof, therefore, we need to give more mathematically precise arguments. Nevertheless, the above arguments, especially Eq (B.39), are expected to be useful for an intuitive understanding of the existence of a solution, and can actually distinguish our system from those where the master equations do not have solutions (see the example below).

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Compact cases and Bradlow bound. Most of the above discussion and results can be applied to the models defined on a compact base space Σ with a finite area A as long as $\ker \mathcal{H} = \emptyset$. It is, however, well known that there is a lower bound $A_{\rm Bb}$, the so-called Bradlow bound, on the area for the existence of solutions. In our case, the lower bound is given by the following set of inequalities

$$0 \leq \int_\Sigma d\mathrm{vol}\, \mathrm{Tr} Q_i Q_i^\dagger = \sum_{j=1}^{i} \left(r_j N_j A - \frac{4\pi k_j}{g_j^2} \right), \qquad A = \int_\Sigma d\mathrm{vol}, \qquad \text{for } i = 1, 2, \cdots, L, \tag{B.42}$$

where $d$vol is the volume form on Σ. In the Bradlow limit saturating the above bound, at least one of Q_i must vanish everywhere and thus the operator $\mathcal{H}$ has a non-trivial kernel ($\ker \mathcal{H} \neq \emptyset$), which implies the argument above is no longer applicable. The most significant difference between the compact and non-compact cases is that in the compact case, a (covariantly) non-zero constant $\boldsymbol{\omega}$, for which $\langle \boldsymbol{\omega} \mathcal{H} \boldsymbol{\omega} \rangle$ may vanish in the limit of $\tau \to \infty$, is allowed since the area is finite and the condition (B.8) is absent. The following $\hat{\boldsymbol{\omega}}^{\rm c}_{(i)} \in \mathcal{F}_\omega$ is the simplest example of such a $\hat{\boldsymbol{\omega}}$

$$\hat{\boldsymbol{\omega}}^{\rm c}_{(i)} = \boldsymbol{\omega}^{\rm c}_{(i)} / \|\boldsymbol{\omega}^{\rm c}_{(i)}\|, \qquad \boldsymbol{\omega}^{\rm c}_{(i)} \equiv (\mathbf{1}_{N_1}, \mathbf{1}_{N_2}, \cdots, \mathbf{1}_{N_i}, \mathbf{0}, \mathbf{0}, \cdots), \qquad i \in \{1, 2, \cdots, L\}. \tag{B.43}$$

For this configuration, the i-th gauge group is restored in the limit $\tau \to \infty$

$$\lim_{\tau\to\infty} Q_i(\tau \hat{\boldsymbol{\omega}}^{\rm c}_{(i)}) = \lim_{\tau\to\infty} e^{-\tau/\|\boldsymbol{\omega}^{\rm c}_{(i)}\|} Q_i^{\rm ref} = 0, \qquad \lim_{\tau\to\infty} \langle \hat{\boldsymbol{\omega}} \mathcal{H} \hat{\boldsymbol{\omega}} \rangle \big|_{\hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}^{\rm c}_{(i)}} = 0, \tag{B.44}$$

whereas the other quantities, $Q_j (j \neq i), A_{\bar z}^j$ remain invariant. Thus, we obtain

$$\lim_{\tau\to\infty} \frac{d\mathfrak{S}_{\hat{\omega}}(\tau)}{d\tau} = \frac{1}{\|\boldsymbol{\omega}^{\rm c}_{(i)}\|} \sum_{j=1}^{i} \left(r_j N_j A - \frac{4\pi k_j}{g_j^2} \right), \quad \text{for } \hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}^{\rm c}_{(i)}, \quad i = 1, 2, \cdots, L, \tag{B.45}$$

which shows that $\mathfrak{S}_{\hat{\omega}}(\tau)$ is no longer coercive for a sufficiently small area A. Note that for $\mathfrak{S}^{\rm vtx}[\boldsymbol{\Omega}, \boldsymbol{\Omega}^{\rm ref}]$ to be coercive, it is necessary that $\mathfrak{S}_{\hat{\omega}}(\tau)$ for $\hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}^{\rm c}_{(i)}$ must be coercive for all $i = 1, \cdots, L$. Therefore, $\mathfrak{S}^{\rm vtx}[\boldsymbol{\Omega}, \boldsymbol{\Omega}^{\rm ref}]$ is coercive only when the area A is larger than the Bradlow bound $A > A_{\rm Bb}$. In this way, we can show that the discussion on the existence of solutions based on the coerciveness of $\mathfrak{S}^{\rm vtx}[\boldsymbol{\Omega}, \boldsymbol{\Omega}^{\rm ref}]$ is consistent with the Bradlow bound.

The Bradlow bound, $A \geq A_{\rm Bb}$ is only a necessary condition on the area A for the existence of solutions and a necessary and sufficient condition, $A \geq A_{\rm tb}$ might be stronger than this condition, $A_{\rm tb} \geq A_{\rm Bb}$ and $A_{\rm tb}$ might depend on a point of the moduli space. For a generic moduli point, the condition $A \geq A_{\rm tb}$ would be found indirectly from the non-vanishing requirement for the vortex moduli space volume, which has been computed by the localization method [92–95]. By refining the above argument on coerciveness of the functional $\mathfrak{S}^{\rm vtx}$, we expect that it is possible to prove the condition $A \geq A_{\rm tb}$ directly.

Relaxation method. It is important and useful to provide an explicit procedure for obtaining a numerical solution that minimizing $\mathfrak{S}^{\rm vtx}$ by discretizing the system. To find the minimum of $\mathfrak{S}^{\rm vtx}$, let us consider the following recurcive relation

$$S_i^{(n+1)} = S_i^{(n)} e^{\delta\omega_i^{(n)}}, \quad S_i^{(0)} = S_i^{\rm ref}. \tag{B.46}$$

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If we choose $\delta\omega_i^{(n)} = \mathcal{H}^{-1}\mathcal{E}_i|_{S\to S^{(n)}}$, this procedure can be regarded as Newton's method, which is, however, impractical since the calculation of $\mathcal{H}^{-1}$ is known to be very costly. One simple and effective method is the relaxation method, where $\delta\omega_i$ in each step is given by $\delta\omega_i^{(n)} = \alpha\mathcal{E}_i|_{S_i\to S_i^{(n)}}$ with an appropriate step size $\alpha \in \mathbb{R}_{>0}$. The parameter α must be a sufficiently small to satisfy the Courant-Friedrichs-Lewy condition, $\alpha/a_{\text{lat}}^2 < \mathcal{O}(1)$, where a_{lat} is the spatial lattice spacing. With such small α, the convexity and coerciveness of $\mathfrak{S}^{\text{vtx}}$ guarantee that this sequence converges without being trapped by meta-stable points. To see that $\mathfrak{S}^{\text{vtx}}$ decreases at each step with a sufficiently small step size α, let us take the continuous limit of $\alpha \to 0$ and rewrite the recursive relation into a differential equation by introducing a fictitious time t. Suppose that S_i(or Ω_i) is a function of t and define its time evolution as follows

$$\frac{\partial S_i(t)}{\partial t} = S_i(t)\mathcal{E}_i(t) \quad \left(= \hat{\mathcal{E}}_i(t)S_i(t)\right) \quad \text{with } q_i(z) \text{ fixed} \tag{B.47}$$

where $\mathcal{E}_i(t)$ is the quantity obtained by substituting $S_i = S_i(t)$ into $\mathcal{E}_i$. Under this time evolution, $\mathfrak{S}^{\text{vtx}}$ monotonically decrease as

$$\frac{d}{dt}\mathfrak{S}^{\text{vtx}}[\mathbf{\Omega}(t), \mathbf{\Omega}^{\text{ref}}] = -2\langle\mathcal{E}(t)^2\rangle, \tag{B.48}$$

and this gradient flow stops only when $\mathfrak{S}^{\text{vtx}}$ takes the minimum value with $\mathcal{E}_i = 0$. The relaxation time needed to obtain a numerical solution with a given accuracy can be estimated as follows. From eq. (B.47), we can derive the time evolution of $\mathcal{E}_i(t)$ as

$$\frac{\partial\mathcal{E}_i(t)}{\partial t} = -2\mathcal{H}\mathcal{E}_i(t) \qquad \text{with} \quad \mathcal{D}_{\bar{z}}Q_i = 0. \tag{B.49}$$

The operator $\mathcal{H}$ hss positive definite eigenvalues, and hence if $\mathcal{E}_i$ is expanded in terms of the eigenmodes of $\mathcal{H}$, each eigenmode decays exponentially. After a sufficient time of relaxation, the deviation from the true solution is dominated by the lowest eigenmode $\mathcal{E}_i^\star$ and decreases exponentially as

$$\mathcal{E}_i(t) \approx \mathcal{E}_i^\star \exp(-2\Delta_\star t) \qquad \text{with} \quad \mathcal{H}\mathcal{E}_i^\star = \Delta_\star\mathcal{E}_i^\star, \tag{B.50}$$

where $\Delta_\star \in \mathbb{R}_{>0}$ is the lowest eigenvalue. Thus, we can estimate the accuracy of the numerical solution using the relaxation time t, as long as calculation errors can be ignored. In the limit $t \to \infty$, $S_i(t)$ converges to the solution S_i^{sol}

$$\lim_{t\to\infty}\mathcal{E}_i(t) = 0 \qquad \text{with} \qquad \lim_{t\to\infty}S_i(t) = S_i^{\text{sol}}. \tag{B.51}$$

B.4 Kähler metric and potential for the vortex moduli space

By using the vortex action $\mathfrak{S}^{\text{vtx}}$, the Kähler potential $\mathcal{K}^{\text{vtx}}$ giving the metric for the vortex moduli space can be naturally introduced. Note that since the counter term $\mathcal{L}(\mathbf{\Omega}^{\text{ref}})$ introduced in eq. (B.18) has moduli-dependence, for the definition of $\mathcal{K}^{\text{vtx}}$, it is more natural to regularize the integral by introducing a spatial cut-off $R \in \mathbb{R}_{>0}$. The Kähler potential on the moduli space $\mathcal{K}^{\text{vtx}}$ can be obtained by substituting the solution $\mathbf{\Omega} = \mathbf{\Omega}^{\text{sol}}$ of the master equations to the vortex action with such a regularization

$$\mathcal{K}^{\text{vtx}} = \mathcal{K}^{\text{vtx}}(\phi^A, \bar{\phi}^{\bar{A}}) \equiv \int_{D_R} d^2x\, \mathcal{L}(\mathbf{\Omega}^{\text{sol}}), \quad D_R \equiv \left\{z = x^1 + ix^2 \in \mathbb{C} \,\middle|\, |z| \leq R\right\}. \tag{B.52}$$

Here, $\phi^A \in \mathbb{C}$ are moduli parameters, which linearly appear in the moduli matrices $q_i(z) = q_i(z, \phi^A)$ when the V-transformations are properly fixed. There is a convenient formula for the Kähler metric which is calculable without going back to the definition of $\mathcal{K}^{\rm vtx}$. If the solution $\Omega_i = \Omega_i^{\rm sol}$ is given, the Kähler metric can be calculated by using the formula

$$g_{A\bar{B}} \equiv \frac{\partial^2 \mathcal{K}^{\rm vtx}}{\partial \bar{\phi}^{\bar{B}} \partial \phi^A} = \int d^2x \frac{\partial}{\partial \bar{\phi}^{\bar{B}}} \sum_{i=1}^{L} \mathrm{Tr}\left[\Omega_i^{-1} \frac{\partial q_i}{\partial \phi^A} \Omega_{i+1} q_i^\dagger\right]_{\Omega = \Omega^{\rm sol}}. \tag{B.53}$$

In this formula, we can chenck the invariance under the V-transformations, which were fixed to define $\mathcal{K}^{\rm vtx}$. Since the V-transformations naturally induce coordinate transitions on the vortex moduli space as explaind in appendix C, the invariance under the V-transformations allows us to choose an arbitrary coordinate patch to describe the moduli space metric. This metric $g_{A\bar{B}}$ turns out to be positive definite and thus invertible, as will be shown in the next paragraph. Furthermore, we can show the regularity of the Riemann curvature tensor, ${R^A}_{BC\bar{D}}$, using the fact that ${R^A}_{BC\bar{D}}$ can be expressed in terms of the higher derivatives of Ω_i such as $\partial_{\phi_A}\partial_{\phi_B}\partial_{\bar{\phi}_C}\Omega_i$, which can be determined through the differentiated forms of the master equations. Thanks to the existence of $\mathcal{H}^{-1}$, those equations are algebraically solvable. Therefore, the above formula implies that *the Kähler manifold defined by this metric is regular everywhere.*

Moduli space approximation. The Kähler metric $g_{A\bar{B}}$ defined above is equivalent to that describing the dynamics of vortices. In the moduli space approximation [96], moduli parameters are promoted to slowly varying functions of time t

$$\phi^A \quad \to \quad \phi^A(t) \tag{B.54}$$

The physical fields Q_i and $A_{\bar{z}}^i$ depend on t only through the moduli parameters

$$Q_i = Q_i^{\rm sol}(z, \bar{z}, \phi^A(t), \bar{\phi}^A(t)), \quad A_{\bar{z}}^i = A_{\bar{z}}^{i,\rm sol}(z, \bar{z}, \phi^A(t), \bar{\phi}^A(t)). \tag{B.55}$$

The gauge potentials A_t^i are given by eq. (2.45), for which the linearized equations of motion are satisfied. By substituting these approximations to the original action, we obtain the fowlloing terms form the kinetic term:

$$\int d^2x \sum_{i=1}^{L} \mathrm{Tr}\left[\mathcal{D}_t Q_i \mathcal{D}_t Q_i^\dagger + \frac{4}{g_i^2} F_{tz}^i F_{t\bar{z}}^i\right] = g_{A\bar{B}} \frac{d\phi^A}{dt} \frac{d\bar{\phi}^{\bar{B}}}{dt}, \tag{B.56}$$

where $g_{A\bar{B}}$ is the Kähler metric defined in eq. (B.53). Note that this equation show that the metric defined in eq. (B.53) is positive definite. The coincidence of the two different definitions of the metric $g_{A\bar{B}}$ is not accidental, but is due to the supersymmetry behind the system as shown in [97].

Large coupling limit. In the large coupling limit $g_i \to \infty$ for all i, the Kähler potential becomes

$$\lim_{g_i \to \infty} \mathcal{K}^{\rm vtx} = \int_{D_R} d^2x \sum_{i=1}^{L} r_i \log\det(\xi_i(z)\xi_i(z)^\dagger) + \mathrm{const.}, \tag{B.57}$$

since $\det\Omega_i \sim \log\det(\xi_i(z)\xi_i(z)^\dagger)$ in this limit. This result is consistent with the moduli space approximation for instantons in the sigma models. Note that the above quantity diverges and thus a divergent constants must be subtracted to obtain a finite quantity by introducing IR cut-off R and using Kähler transformation. It is convenient to decompose the integrand into the two parts as

$$\log\det(\xi_i(z)\xi_i(z)^\dagger) = \log|\det\mathfrak{D}_i(z)|^2 + \log\det\left(\mathbf{1}_{N_i} + \varphi_i(z)\varphi_i(z)^\dagger\right), \tag{B.58}$$

where we have used $\mathfrak{D}_i(z)^{-1}\xi_i(z) = (\mathbf{1}_{N_i}, \varphi_i(z))$, and $\varphi_i(z)$ is an instanton solution. These two terms are calculated separately in the subsequent paragraphs.

Position moduli for vortices. At first glance, the contribution from the first term may seem to disappear since it can be cancelled by a Kähler transformation. However, after the regularization, careful calculations lead to the following important term

$$\int_{D_R} d^2x\, r_i \log|\det\mathfrak{D}_i(z)|^2 = k_i\pi r_i R^2 \log\frac{R^2}{e} + \pi r_i \sum_{\alpha=1}^{k_i} |z_{(i,\alpha)}|^2 \tag{B.59}$$

where $\{z_{(i,\alpha)}\}$ are zeros of $\det\mathfrak{D}_i(z)$. Here, the translational invariance is broken due to the regularization. Note that since $2\pi r_i$ is the tension (mass) of a vortex, the second term in the r.h.s. of the above equation gives the (dominant parts of) kinetic terms of the position moduli $\{z_{(i,\alpha)}\}$, whereas the first term is divergent in the limit of $R\to\infty$ and eliminated by the Kähler transformation. It should be noted that this contribution cannot be ignored even in the sigma model.

Non-normalizable moduli. The contribution from the second term gives the following divergent term in the limit of $R = \infty$

$$\int_{D_R} d^2x\, r_i \log\det\left(\mathbf{1}_{N_i} + \varphi_i(z)\varphi_i(z)^\dagger\right) = \pi r_i \mathrm{Tr}[\Phi_i^{\mathrm{non}}(\Phi_i^{\mathrm{non}})^\dagger]\log R + \mathcal{O}\left((R)^0\right), \tag{B.60}$$

where an N_i-by-$(N-N_i)$ matrix $\Phi_i^{\mathrm{non}} \equiv \Psi_i\widetilde{\Psi}_i$ appears in the dominant term of $\varphi_i(z)$ for large $|z|$ as

$$\varphi_i(z) = \mathfrak{D}_i(z)^{-1}\widetilde{\mathfrak{D}}_i(z) = \Psi_i(z\mathbf{1}_{k_i} - Z_i)^{-1}\widetilde{\Psi}_i = \frac{1}{z}\Psi_i\widetilde{\Psi}_i + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{B.61}$$

where we used the half ADHM data $\{Z_i, \Psi_i, \widetilde{\Psi}_i\}$ discussed in section 4. The above divergent term can not be eliminated by a Kähler transformation and gives a divergent kinetic terms for Φ_i^{non}. Thus, the entries of Φ_i^{non} are non-normalizable moduli. Intuitively, this divergence is due to the fact that there is no mass gap in the bulk and any fluctuations of Φ_i^{non} excite zero modes in the bulk where the moduli approximation is invalid. To describe the dynamics of these moduli, we need to go back to the full field equation.

C Coordinate patches of moduli space for local votices

In this appendix, we present more details of the vortex moduli space. We show the equivalence of the two expressions of the vortex moduli space: one is written in terms of the moduli

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matrix (3.25) and the other is written in terms of the half ADHM data (4.42). In particular, we focus on the $L = 1$, $N = n$ case where the two expressions of the vortex moduli space take the forms (see below for more precise definitions)

$$\mathcal{M} \equiv \ \{\mathfrak{D}(z)|\det\mathfrak{D}(z) = \mathcal{O}(z^k)\}/V\text{-trf.} \tag{C.1}$$

$$\widetilde{\mathcal{M}} \equiv \ \{(Z,\Psi)|\{Z,\Psi\} \text{ on which } \mathrm{GL}(k,\mathbb{C}) \text{ action is free }\}/\mathrm{GL}(k,\mathbb{C}). \tag{C.2}$$

We will show that both of these two spaces correspond to the vortex moduli space ${\mathcal{M}_{\mathrm{vtx}}}_k^{n,0}$ in the $\mathrm{U}(n)$ gauge theory with n flavors. The space ${\mathcal{M}_{\mathrm{vtx}}}_k^{n,0}$ can also be viewed as the local vortex moduli subspace in the total moduli space ${\mathcal{M}_{\mathrm{vtx}}}_k^{n,N-n}$ for a general flavor number $N \geq n$. Once the equivalence of the local vortex moduli subspace is shown, the equivalence of the total moduli space immediately follows.[25] Hence, we focus on the local vortex moduli subspace. The equivalence the two spaces above play a fundamental and important role also in the general case with $L \geq 1$, where the moduli space can be viewed as a set of L copies of the $L = 1$ moduli space subject to the additional conditions.

C.1 Moduli space $\mathcal{M}$ of moduli matrix $\mathfrak{D}(z)$

Let $\mathbb{C}_{n,k}[z]$ denote the set of n-by-n matrices with polynomial entries whose determinants are degree k polynomials. The definition of the moduli space (3.35) can be rephrased as

$$\mathcal{M} \ \equiv \ \mathbb{C}_{n,k}[z]/\mathbb{C}_{n,0}[z] \ = \ \mathbb{C}_{n,k}[z]/\sim, \tag{C.3}$$

where the equivalence relation "$\sim$" for $\mathfrak{D}(z)$, $\mathfrak{D}'(z) \in \mathbb{C}_{n,k}[z]$ is defined as

$$\mathfrak{D}(z) \sim \mathfrak{D}'(z) \quad \Longleftrightarrow \quad {}^{\exists}V(z) \in \mathbb{C}_{n,0}[z] \quad \text{such that} \quad \mathfrak{D}'(z) = V(z)\mathfrak{D}(z). \tag{C.4}$$

In this subsection, we provide some details on the coordinates of the moduli space.

C.1.1 Atlas $\{(\phi_\lambda, \mathcal{M}_\lambda)\}$ of $\mathcal{M}$

For a given set of non-negative integers $\lambda = (l_1, l_2, \cdots, l_n)$ such that $l_1 + \cdots + l_n = k$, let $\mathcal{M}_\lambda$ be the space of matrices of the form

$$\mathfrak{D}_\lambda(z) \equiv \begin{pmatrix} z^{l_1} & & \\ & \ddots & \\ & & z^{l_n} \end{pmatrix} - \begin{pmatrix} P^{11} & \dots & P^{1n} \\ \vdots & \ddots & \vdots \\ P^{n1} & \dots & P^{nn} \end{pmatrix}, \qquad P^{ab} = \sum_{m=1}^{l_b} T_m^{ab} z^{m-1}. \tag{C.5}$$

Note that if two matrices of the form (C.5) are V-equivalent $\mathfrak{D}_\lambda(z) \sim \mathfrak{D}'_\lambda(z)$, it follows that they are actually identical matrices $\mathfrak{D}_\lambda(z) = \mathfrak{D}'_\lambda(z)$. This is because if a V-equivalent pair $\mathfrak{D}(z)$ and $\mathfrak{D}'(z)$ have the same leading order behavior as (C.5), the V-transformation relating the pair $V(z) = \mathfrak{D}'(z)\,\mathfrak{D}(z)^{-1}$ behaves as $V(z) = \mathbf{1}_n + \mathcal{O}(z^{-1})$ for large $|z|$ and hence its regularity implies that $V(z) = \mathfrak{D}'(z)\,\mathfrak{D}(z)^{-1} = \mathbf{1}_n$. Therefore, each element of $\mathcal{M}_\lambda$ specifies a

[25] The total moduli space ${\mathcal{M}_{\mathrm{vtx}}}_k^{n,N-n}$ has additional directions described by $\widetilde{\mathfrak{D}}(z)$ and $\widetilde{\Psi}$, which are fibered over the local vortex moduli subspace. The equivalence such fiber directions in (3.25) and (4.42) follows from the one-to-one relation $\widetilde{\mathfrak{D}}(z) = \mathfrak{J}(z)\widetilde{\Psi}$.

distinct V-equivalence class and hence $\mathcal{M}_\lambda$ can be viewed as a subspace of $\mathcal{M}$. The local coordinate system on the coordinate patch (chart) $\mathcal{M}_\lambda$ with $\lambda = (l_1, \cdots, l_n)$ is given by

$$\phi_\lambda(\mathcal{M}_\lambda) \simeq \left\{ T_m^{ab} \,\middle|\, 1 \le a, b \le n, 1 \le m \le l_b \right\} \simeq \mathbb{C}^{kn}, \tag{C.6}$$

where T_m^{ab} are coefficients of the polynomials $P^{ab}(z)$. Two coordinate patches $\mathcal{M}_\lambda$ and $\mathcal{M}_{\lambda'}$ with $\lambda \neq \lambda'$ are glued by the coordinate transformation $\tau_{\lambda',\lambda} = \phi_{\lambda'} \circ \phi_\lambda^{-1}$ which can be read off from the V-transformation relating $\mathfrak{D}_\lambda(z) \in \mathcal{M}_\lambda$ and $\mathfrak{D}_{\lambda'}(z) \in \mathcal{M}_{\lambda'}$:

$$\mathfrak{D}_{\lambda'}(z) = V_{\lambda',\lambda}(z)\, \mathfrak{D}_\lambda(z), \qquad V_{\lambda',\lambda}(z) \in \mathbb{C}_{n,0}[z], \tag{C.7}$$

the explicit form of the coordinate transformation can be determined by requiring that all the entries of $V_{\lambda',\lambda}(z) = \mathfrak{D}_{\lambda'}(z)\mathfrak{D}_\lambda(z)^{-1}$ are regular. Gluing all the coordinate patches $\mathcal{M}_\lambda$, we obtain a complex manifold

$$\mathcal{M}' = \bigcup_{\lambda\in\Lambda} \mathcal{M}_\lambda \quad \text{with} \quad \Lambda = \left\{ (l_1, l_2, \cdots, l_n) \,\middle|\, l_a \in \mathbb{Z}_{\ge 0}, \ \sum_{a=1}^{n} l_a = k \right\}. \tag{C.8}$$

This is a submanifold of $\mathcal{M}$ ($\mathcal{M}' \subseteq \mathcal{M}$), since $\mathcal{M}_\lambda \subset \mathcal{M}$ for all $\lambda \in \Lambda$. In subsection C.1.2, we show that $\mathcal{M}$ can be decomposed into a disjoint union of subspaces $\mathcal{M}_\lambda^{\rm tri}$ such that $\mathcal{M}_\lambda^{\rm tri} \subset \mathcal{M}_\lambda$. This fact implies that $\mathcal{M}$ is a subspace of $\mathcal{M}'$:

$$\mathcal{M} = \bigsqcup_{\lambda\in\Lambda} \mathcal{M}_\lambda^{\rm tri} \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{M}_\lambda = \mathcal{M}'. \tag{C.9}$$

Since $\mathcal{M}' \subseteq \mathcal{M}$ and $\mathcal{M}' \supseteq \mathcal{M}$, we conclude that $\mathcal{M} = \mathcal{M}'$ and $\{(\phi_\lambda, \mathcal{M}_\lambda)\}$ gives an atlas of $\mathcal{M}$.

C.1.2 Decomposition into disjoint union

Here we show the decomposition of the moduli space $\mathcal{M}$ into a disjoint union of subspaces $\mathcal{M}_\lambda^{\rm tri}$ such that $\mathcal{M}_\lambda^{\rm tri} \subset \mathcal{M}_\lambda$, which we have used to show that $\mathcal{M}' \supseteq \mathcal{M}$. The moduli space $\mathcal{M}$ is the space of the V-equivalence classes of the matrix $\mathfrak{D}(z)$. We can pick up a representative in each V-equivalence class by fixing the "gauge redundancy" of the V-transformation (C.4) in the following way. Here we focus on the case of $n = 2$ for simplicity. Let $\mathfrak{D}(z)$ be a generic element in $\mathbb{C}_{2,k}[z]$

$$\mathfrak{D}(z) = \begin{pmatrix} f(z) & h(z) \\ g(z) & i(z) \end{pmatrix} \in \mathbb{C}_{2,k}[z], \tag{C.10}$$

where $f(z), g(z), h(z)$ and $i(z)$ are polynomials. Using the Euclidean algorithm (Bézout's identity), we can show that there exist polynomials $\tilde{f}(z)$ and $\tilde{g}(z)$ such that

$$f(z)\tilde{f}(z) + g(z)\tilde{g}(z) = p(z), \tag{C.11}$$

where $p(z)$ is the polynomial greatest common divisor of $f(z)$ and $g(z)$. Using $\tilde{f}(z), \tilde{g}(z)$ and $p(z)$, we can construct $\mathcal{V}(z) \in \mathbb{C}_{2,0}[z]$ with which $\mathfrak{D}(z)$ is transformed into an upper triangular form

$$\mathfrak{D}(z) \to \mathcal{V}(z)\, \mathfrak{D}(z) = \begin{pmatrix} \tilde{f}(z) & \tilde{g}(z) \\ -q_g(z) & q_f(z) \end{pmatrix} \begin{pmatrix} f(z) & h(z) \\ g(z) & i(z) \end{pmatrix} = \begin{pmatrix} p(z) & h'(z) \\ 0 & i'(z) \end{pmatrix}, \tag{C.12}$$

where $q_f(z)$ and $q_g(z)$ are the polynomials defined by $f(z) = q_f(z)p(z)$ and $g(z) = q_g(z)p(z)$. Note that $\tilde{f}(z)$ and $\tilde{g}(z)$ are not unique, that is, we can further multiply another V-transformation without changing the upper-triangular form

$$\mathcal{V}(z)\,\mathfrak{D}(z) \to \mathcal{V}'(z)\mathcal{V}(z)\,\mathfrak{D}(z) = \begin{pmatrix} a_1 & j(z) \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} p(z) & h'(z) \\ 0 & i'(z) \end{pmatrix}, \tag{C.13}$$

where a_1 and a_2 are constants and $j(z)$ is a polynomial. This redundancy can be fixed by requiring that the diagonal entries are monic polynomials and minimizing the degree of the upper-right element

$$\mathcal{V}'(z)\mathcal{V}(z)\,\mathfrak{D}(z) = \begin{pmatrix} z^{l_1} & 0 \\ 0 & z^{l_2} \end{pmatrix} + \begin{pmatrix} P_{11}(z) & P_{12}(z) \\ 0 & P_{22}(z) \end{pmatrix}, \tag{C.14}$$

where l_i $(i = 1, 2)$ are the degrees of the diagonal engries ($l_1 = \deg p(z)$, $l_2 = \deg i'(z)$) and P_{ab} $(1 \le a \le b \le 2)$ are polynomials of degree less than l_b. In each $\mathcal{V}$-equivalence class, the form (C.14) is unique and hence the gauge redundancy is completely fixed. This procedure can be generalized to the case of general n; for any $\mathfrak{D}(z) \in \mathbb{C}_{n,k}[z]$, we can find $\mathcal{V}(z) \in \mathbb{C}_{n,0}[z]$ such that

$$\mathfrak{D}(z) \to \mathcal{V}(z)\mathfrak{D}(z) = \mathfrak{D}^{\text{tri}}_{\lambda}(z) = \begin{pmatrix} z^{l_1} & & & \\ & z^{l_2} & & \\ & & \ddots & \\ & & & z^{l_n} \end{pmatrix} - \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ 0 & P_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & P_{n-1,n} \\ 0 & \cdots & 0 & P_{nn} \end{pmatrix}, \tag{C.15}$$

where $\lambda = (l_1, l_2, \cdots, l_n)$ is a set of non-negative integers such that $l_1 + \cdots + l_n = k$ and $P_{ab}(z)$ $(1 \le a \le b \le n)$ are polynomials of degree less than l_b. Since any $\mathcal{V}$-equivalence class has a unique representative of the form (C.15), the moduli space (C.19) can be decomposed into the disjoint union of the subspaces

$$\mathcal{M} = \bigsqcup_{\lambda\in\Lambda} \mathcal{M}^{\text{tri}}_{\lambda}, \tag{C.16}$$

where $\mathcal{M}^{\text{tri}}_{\lambda}$ are the sets of matrices $\mathfrak{D}^{\text{tri}}_{\lambda}(z)$ of the form (C.15) specified by the set of non-negative integers $\lambda = (l_1, \cdots, l_n)$. Note that there is no overlap between them, i.e. $\mathcal{M}^{\text{tri}}_{\lambda} \cap \mathcal{M}^{\text{tri}}_{\lambda'} = \emptyset$ for $\lambda \ne \lambda'$. Since the form of the matrix (C.15) is a special case of (C.5), it follows that $\mathcal{M}^{\text{tri}}_{\lambda}$ is a subspace of $\mathcal{M}_{\lambda}$. Therefore, we conclude that $\mathcal{M}$ is a subspace of $\mathcal{M}'$

$$\mathcal{M} = \bigsqcup_{\lambda\in\Lambda} \mathcal{M}^{\text{tri}}_{\lambda} \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{M}_{\lambda} = \mathcal{M}'. \tag{C.17}$$

C.1.3 Matrices with non-polynomial entries

So far, entries of the matrices $\mathfrak{D}(z)$ and $\mathcal{V}(z)$ are assumed to be polynomials for simplicity. Strictly speaking, the most general solution can have arbitrary entire functions of $z \in \mathbb{C}$ as their entries. That is, $\mathfrak{D}(z)$ and $\mathcal{V}(z)$ are not necessarily elements of $\mathbb{C}_{n,k}[z]$ and $\mathbb{C}_{n,0}[z]$ but they can be elements of larger spaces $\mathcal{G}_{n,k}[z]$ and $\mathcal{G}_{n,0}[z]$:

$$\mathfrak{D}(z) \in \mathcal{G}_{n,k}[z], \qquad \mathcal{V}(z) \in \mathcal{G}_{n,0}[z], \tag{C.18}$$

where $\mathcal{G}_{n,k}[z]$ are the space of maps from the complex z-plane $\mathbb{C}$ to the space of n-by-n square matrices that have the following properties

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- If $X(z) \in \mathcal{G}_{n,k}[z]$, all the entries of $X(z)$ are entire functions of $z \in \mathbb{C}$ ($\partial_{\bar{z}} X(z) = 0$),
- If $X(z) \in \mathcal{G}_{n,k}[z]$, $\det X(z)$ has k zeros on $\mathbb{C}$ and $\operatorname{rank} X(z) = n$ except at the zeros of $\det X(z)$.

It worth noting that $\mathcal{G}_{n,0}[z]$ forms a group under matrix multiplication.[26]

Definition of $\mathcal{M}_{\mathcal{G}}$. We define the space $\mathcal{M}_{\mathcal{G}}$ as

$$\mathcal{M}_{\mathcal{G}} \equiv \mathcal{G}_{n,k}[z]/\mathcal{G}_{n,0}[z] = \mathcal{G}_{n,k}[z]/\sim, \tag{C.19}$$

where the equivalence relation "$\sim$" for elements $\mathfrak{D}(z)$ and $\mathfrak{D}'(z)$ in $\mathcal{G}_{n,k}[z]$ is defined by

$$\mathfrak{D}(z) \sim \mathfrak{D}'(z) \quad \Longleftrightarrow \quad {}^{\exists}V(z) \in \mathcal{G}_{n,0}[z] \quad \text{such that} \quad \mathfrak{D}'(z) = V(z)\mathfrak{D}(z). \tag{C.20}$$

Replacing $\mathcal{G}_{n,k}[z]$ with the subspace $\mathbb{C}_{n,k}[z] \subset \mathcal{G}_{n,k}[z]$ consisting of matrices with polynomial entries, we can go back to the definition of the moduli space $\mathcal{M}$ given in (C.3). Although (C.19) is the most general definition of the moduli space, it actually gives the same space as (C.3), that is, $\mathcal{M}_{\mathcal{G}} = \mathcal{M}$. Therefore, we can use the simpler definition of the moduli space $\mathcal{M}$ based on the space of matrices with polynomial entries $\mathbb{C}_{n,k}[z]$.

To see that $\mathcal{M}_{\mathcal{G}} = \mathcal{M}$, let us show that any matrix $\mathfrak{D}(z) \in \mathcal{G}_{n,k}[z]$ can be fixed into the upper-triangular form (C.15) with polynomial entries by an element of the V-transformation $\mathcal{V}(z) \in \mathcal{G}_{n,0}[z]$. We will use the following two theorems, which will be proven in subsections C.1.4 and C.1.5:

Theorem C.1. *Let $(f(z), g(z))$ an arbitrary pair of entire functions. If $g(z)$ has m zeros, then, there is a pair of a polynomial $p(z)$ of degree less than m and an entire function $h(z)$ such that,*

$$f(z) = p(z) + h(z)g(z). \tag{C.21}$$

Theorem C.2. *For a pair of entire functions $(f(z), g(z))$, there is an element $V(z)$ of $\mathcal{G}_{2,0}[z]$ such that*

$$\exists V(z) \in \mathcal{G}_{2,0}[z], \exists p(z), \quad V(z)\begin{pmatrix} f(z) \\ g(z) \end{pmatrix} = \begin{pmatrix} p(z) \\ 0 \end{pmatrix}, \tag{C.22}$$

where $p(z)$ is a certain entire function whose a set of zeros is the intersection of those of $f(z)$ and $g(z)$ including their multiplicities. In particular, the first row of the above equation indicates that their is a pair of entire functions, $(\tilde{f}(z), \tilde{g}(z))$ such that

$$f(z)\tilde{f}(z) + g(z)\tilde{g}(z) = p(z). \tag{C.23}$$

[26] For general integers $p \geq 0$ and $q \geq 0$, multiplication of elements of $\mathcal{G}_{n,p}[z]$ and $\mathcal{G}_{n,q}[z]$ defines a map $\mathcal{G}_{n,p}[z] \times \mathcal{G}_{n,q}[z] \to \mathcal{G}_{n,p+q}[z]$

$$(X(z), Y(z)) \in \mathcal{G}_{n,p}[z] \times \mathcal{G}_{n,q}[z] \quad \mapsto \quad X(z)\,Y(z) \in \mathcal{G}_{n,p+q}[z].$$

Decomposition into disjoint union. Replacing the polynomials in subsubsection C.1.2 with entire functions and using Theorems C.1 and C.2, we can show that $\mathcal{M}_{\mathcal{G}}$ can be decomposed into the disjoint union of $\mathcal{M}^{\rm tri}_{\lambda}$. We again focus on the case of $n = 2$ for simplicity. Let $\mathfrak{D}(z)$ be a generic element in $\mathcal{G}_{2,k}[z]$

$$\mathfrak{D}(z) = \begin{pmatrix} f(z) & h(z) \\ g(z) & i(z) \end{pmatrix} \in \mathcal{G}_{2,k}[z], \tag{C.24}$$

where $f(z), g(z), h(z)$ and $i(z)$ are entire functions. Using Theorem C.2, we can find $\tilde{f}(z), \tilde{g}(z)$ and $p(z)$ satisfying eq. (C.23). Using such $\tilde{f}(z)$, $\tilde{g}(z)$ and $p(z)$ we can construct $\mathcal{V}(z) \in \mathcal{G}_{2,0}[z]$ with which $\mathfrak{D}(z)$ is transformed into an upper triangular form

$$\mathfrak{D}(z) \to \mathcal{V}(z)\,\mathfrak{D}(z) = \begin{pmatrix} \tilde{f}(z) & \tilde{g}(z) \\ -q_g(z) & q_f(z) \end{pmatrix} \begin{pmatrix} f(z) & h(z) \\ g(z) & i(z) \end{pmatrix} = \begin{pmatrix} p(z) & h'(z) \\ 0 & i'(z) \end{pmatrix}, \tag{C.25}$$

where $q_f(z)$ and $q_g(z)$ are the entire functions defined by $f(z) = q_f(z)p(z)$ and $g(z) = q_g(z)p(z)$. Note that $p(z)$ and $i'(z)$ have finite number of zeros since $\mathfrak{D}(z) \in \mathcal{G}_{n,k}[z]$ with finite k. The functions $\tilde{f}(z)$ and $\tilde{g}(z)$ are not unique, that is, we can further multiply another V-transformation without changing the upper-triangular form

$$\mathcal{V}(z)\,\mathfrak{D}(z) \to \mathcal{V}'(z)\mathcal{V}(z)\,\mathfrak{D}(z) = \begin{pmatrix} a_1(z) & j(z) \\ 0 & a_2(z) \end{pmatrix} \begin{pmatrix} p(z) & h'(z) \\ 0 & i'(z) \end{pmatrix}, \tag{C.26}$$

where $j(z)$ is an arbitrary entire functions and $a_1(z)$, $a_2(z)$ are entire functions without zero. This redundancy can be fixed by requiring that the diagonal entries are monic polynomials and fixing the upper-right element to be the minimum degree polynomial using Theorem C.1

$$\mathcal{V}'(z)\mathcal{V}(z)\,\mathfrak{D}(z) = \begin{pmatrix} z^{l_1} & 0 \\ 0 & z^{l_2} \end{pmatrix} + \begin{pmatrix} P_{11}(z) & P_{12}(z) \\ 0 & P_{22}(z) \end{pmatrix}, \tag{C.27}$$

where l_i $(i = 1, 2)$ are the numbers of zeros of the diagonal engries and P_{ab} $(1 \leq a \leq b \leq 2)$ are polynomials of degree less than l_b. In each $\mathcal{V}$-equivalence class, the form (C.27) is unique and hence the gauge redundancy is completely fixed. This procedure can be generalized to the case of general n

$$\mathfrak{D}(z) \to \mathcal{V}(z)\mathfrak{D}(z) = \mathfrak{D}^{\rm tri}_{\lambda}(z) = \begin{pmatrix} z^{l_1} & & & \\ & z^{l_2} & & \\ & & \ddots & \\ & & & z^{l_n} \end{pmatrix} - \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ 0 & P_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & P_{n-1,n} \\ 0 & \cdots & 0 & P_{nn} \end{pmatrix}, \tag{C.28}$$

where $\lambda = (l_1, l_2, \cdots, l_n)$ is a set of non-negative integers such that $l_1 + \cdots + l_n = k$ and $P_{ab}(z)$ $(1 \leq a \leq b \leq n)$ are polynomials of degree less than l_b. Since any $\mathcal{V}$-equivalence class has a unique representative of the form (C.28), the moduli space (C.19) can be decomposed into the disjoint union in a similar way as $\mathcal{M}$

$$\mathcal{M}_{\mathcal{G}} = \bigsqcup_{\lambda \in \Lambda} \mathcal{M}^{\rm tri}_{\lambda}, \tag{C.29}$$

where $\mathcal{M}^{\rm tri}_{\lambda}$ are the sets of matrices $\mathfrak{D}^{\rm tri}_{\lambda}(z)$ of the form (C.15) specified by the set of non-negative integers $\lambda = (l_1, \cdots, l_n)$.

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The decomposition implies that $\mathcal{M}_G$ is a subspace of $\mathcal{M}'$ defined in eq. (C.8)

$$\mathcal{M}_{\mathcal{G}} = \bigsqcup_{\lambda\in\Lambda} \mathcal{M}_\lambda^{\rm tri} \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{M}_\lambda = \mathcal{M}'. \tag{C.30}$$

On the other hand, $\mathcal{M}'$ is obviously a submanifold of $\mathcal{M}_{\mathcal{G}}$. Therefore, $\mathcal{M}_{\mathcal{G}} = \mathcal{M}'$ and hence definition of the moduli space $\mathcal{M} = \mathcal{M}'$ defined in terms of polynomials and $\mathcal{M}_{\mathcal{G}}$ defined in terms of entire functions are equivalent.

C.1.4 Proof of Theorem C.1

In the previous subsection we have used Theorem C.1 to show that $\mathfrak{D}(z)$ can always be fixed as (C.15). Here we give the proof of the theorem.

Theorem C.1. *Let $(f(z), g(z))$ an arbitrary pair of entire functions on $\mathbb{C}$. If $g(z)$ has m zeros, then, there is a pair of a polynomial $p(z)$ of degree less than m and an entire function $h(z)$* such that,

$$f(z) = p(z) + h(z)g(z). \tag{C.31}$$

Proof. We first assume that $g(z)$ is a polynomial of degree m. Suppose $z = a$ is a zero of $g(z)$ with multiplicity l_a. Let $p_a(z)$ be the polynomial related to the principal part of the Laurent series of $f(z)/g(z)$ at $z = a$ as

$$\frac{f(z)}{g(z)} = \frac{1}{(z-a)^{l_a}} p_a(z) + \text{regular term}. \tag{C.32}$$

Using $p_a(z)$ defined for the all zeros of $g(z)$, we can define an entire function $h(z)$ as

$$h(z) \equiv \frac{f(z)}{g(z)} - \sum_{a\in Z(g)} \frac{p_a(z)}{(z-a)^{l_a}}, \tag{C.33}$$

where $Z(g)$ denotes the set of zeros of $g(z)$. Multiplying $g(z)$ to the both sides of the above, we find that

$$f(z) = h(z)g(z) + p(z), \tag{C.34}$$

where $p(z)$ is given by

$$p(z) = \sum_{a\in Z(g)} p_a(z) \frac{g(z)}{(z-a)^{l_a}}. \tag{C.35}$$

Note that $p(z)$ is the polynomial of degree less than m

$$p(z) = \sum_a \sum_{n=1}^{l_a} \frac{f^{(n-1)}(a)}{(n-1)!} e_{a,n}(z) \tag{C.36}$$

where $\{e_{a,n}\}$ is the basis of polynomials of degree less than m defined by

$$e_{a,n}(z) = \sum_{q=0}^{l_a-n} \frac{1}{q!} \partial_z^q \left[\frac{(z-a)^{l_a}}{g(z)} \right]_{z=a} \times (z-a)^{q+n-1-l_a} g(z), \quad n = 1, \cdots, l_a. \tag{C.37}$$

If $g(z)$ is not a polynomial but a generic entire function with m zeros, there exists a polynomial $\tilde{g}(z)$ of degree m and a entire function $X(z)$ such that $g(z) = e^{X(z)}\tilde{g}(z)$. As shown above, there exist a polynomial $p(z)$ of degree less than m and a entire function $\tilde{h}(z)$ such that $f(z) = \tilde{h}(z)\tilde{g}(z) + p(z)$. Writing $h(z) = \tilde{h}(z)e^{-X(z)}$, we find that

$$f(z) = h(z)g(z) + p(z). \tag{C.38}$$

□

C.1.5 Proof of Theorem C.2

To prove Theorem C.2, let us first show the following lemma

Lemma C.1. *If $(f(z), g(z))$ is a pair of entire functions of z which have no common zero, then there is a pair of entire functions, $(\tilde{f}(z), \tilde{g}(z))$ such that*

$$f(z)\tilde{f}(z) + g(z)\tilde{g}(z) = 1. \tag{C.39}$$

Proof. Let $A_f = \{a_n | n \in \mathbb{N}\}$ and $A_g = \{b_n | n \in \mathbb{N}\}$ be the ordered sets of zeros of $f(z)$ and $g(z)$, respectively. According to Mittag-Leffler's theorem, we can construct a function $h_f(z)$ such that the set of poles of $h_f(z)$ is in one-to-one correspondence with the set of zeros of $f(z)$, and the principal part at $z = a_n \in A_f$ is $P_{a_n}(z)$

$$h_f(z) = P_{a_n}(z) + \{\,\text{terms regular at } z = a_n\,\}, \qquad {}^{\forall}a_n \in A_f, \tag{C.40}$$

where $P_a(z)$ is the principal part of the function $h(z) = (f(z)g(z))^{-1}$ at $z = a \in A_f \cup A_g$

$$h(z) = \frac{1}{f(z)g(z)} = P_a(z) + \{\,\text{terms regular at } z = a\,\}. \tag{C.41}$$

Similarly, we can construct a function $h_g(z)$. The, the function $r(z)$ defined by

$$r(z) \equiv h(z) - h_f(z) - h_g(z) \tag{C.42}$$

is an entire function since the set of poles of $h(z)$ is $A_f \cup A_g$ and all the poles in the r.h.s. are exactly cancelled. In addition, the following two functions are also entire functions:

$$\tilde{f}(z) \equiv g(z)(h_g(z) + r(z)), \quad \tilde{g}(z) \equiv f(z)h_f(z) \tag{C.43}$$

where all poles of $h_f(z)$ and $h_g(z)$ are cancelled with the corresponding zeros of $f(z)$ and $g(z)$, respectively. By multiplying $h(z)^{-1} = f(z)g(z)$ to the both side of eq. (C.42), we find that

$$1 = f(z)\tilde{f}(z) + g(z)\tilde{g}(z). \tag{C.44}$$

Here the pair $(\tilde{f}(z), \tilde{g}(z))$ constructed above is not the general solution but a special solution. Different point sequences A'_f and A'_g obtained by switching the order infinitely many times can give a different solution. □

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As an example, let us consider $(f(z), g(z)) = (\sin^3(z), \cos(z))$, $h_f(z)$. The functions $h_f(z)$, $h_g(z)$ and $r(z)$ are given by

$$h_f(z)=\frac{1}{z^3}+\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{(z-n\pi)^3}+\frac{1}{z-n\pi}+\frac{1}{(z+n\pi)^3}+\frac{1}{z+n\pi}\right)=\frac{1}{\tan z}\left(1+\frac{1}{\sin^2 z}\right),$$
$$h_g(z)=-\sum_{n=1}^{\infty}\left(\frac{1}{z-\left(n-\frac{1}{2}\right)\pi}+\frac{1}{z+\left(n-\frac{1}{2}\right)\pi}\right)=\tan z, \qquad r(z)=\frac{1}{\sin^3(z)\cos(z)}-h_f(z)-h_g(z)=0, \tag{C.45}$$

and thus a special solution of $(\tilde{f}(z), \tilde{g}(z))$ is given as

$$\tilde{f}(z) = \cos(z)h_g(z) = \sin(z), \qquad \tilde{g}(z) = \sin^3(z)h_f(z) = \cos(z)(1+\sin^2(z)). \tag{C.46}$$

Theorem C.2. *For a pair of entire functions $(f(z), g(z))$, there is an element $V(z)$ of $\mathcal{G}_{2,0}[z]$ such that*

$$^\exists V(z) \in \mathcal{G}_{2,0}[z], \ ^\exists p(z), \qquad V(z)\begin{pmatrix} f(z) \\ g(z) \end{pmatrix} = \begin{pmatrix} p(z) \\ 0 \end{pmatrix}, \tag{C.47}$$

where $p(z)$ is a certain entire function whose set of zeros is the intersection of those of $f(z)$ and $g(z)$ including their multiplicities. In particular, the first row of the above equation indicates that their is a pair of entire functions, $(\tilde{f}(z), \tilde{g}(z))$ such that

$$f(z)\tilde{f}(z) + g(z)\tilde{g}(z) = p(z). \tag{C.48}$$

Proof. Let $A(f,g)$ be an intersection of sets of zeros of $f(z)$ and $g(z)$ including their multiplicities. According to Weierstrass factorization theorem, there exists an entire function $p(z)$ whose set of zeros is $A(f,g)$. Then, functions $f_0(z)$ and $g_0(z)$ defined by

$$f_0(z) \equiv \frac{f(z)}{p(z)}, \quad g_0(z) \equiv \frac{g(z)}{p(z)}, \tag{C.49}$$

are entire functions without common zero, and hence we can apply the above lemma to find a pair of entire functions $(\tilde{f}(z), \tilde{g}(z))$ satisfying

$$f_0(z)\tilde{f}(z) + g_0(z)\tilde{g}(z) = 1 \quad \Rightarrow \quad f(z)\tilde{f}(z) + g(z)\tilde{g}(z) = p(z). \tag{C.50}$$

Then, we can construct the matrix $V(z) \in \mathcal{G}_{2,0}[z]$ satisfying (C.47) as

$$V(z) = \begin{pmatrix} \tilde{f}(z) & \tilde{g}(z) \\ -g_0(z) & f_0(z) \end{pmatrix}. \tag{C.51}$$

$\square$

C.2 Moduli space $\widetilde{\mathcal{M}}$ of the half-ADHM data

In this subsection, we discuss the moduli space of the half-ADHM data, which is given by the $\mathrm{GL}(N,\mathbb{C})$ quotient of the vector space of matrices $\{Z, \Psi\}$

$$\widetilde{\mathcal{M}} \cong \left\{(Z,\Psi) \,\middle|\, \{Z,\Psi\} \text{ on which } \mathrm{GL}(k,\mathbb{C}) \text{ action is free}\right\} / \mathrm{GL}(k,\mathbb{C}), \tag{C.52}$$

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where Z k-by-k matrix and Ψ is a n-by-k matrix on which $\mathrm{GL}(k,\mathbb{C})$ acts

$$Z \to g^{-1} Z g, \qquad \Psi \to \Psi g. \tag{C.53}$$

The condition that the $\mathrm{GL}(k,\mathbb{C})$ action is free means that there is no non-trivial $g \in \mathrm{GL}(k,\mathbb{C})$ that fixes (Z,Ψ). To examine the moduli space $\widetilde{\mathcal{M}}$, it is convenient to rewrite the $\mathrm{GL}(k,\mathbb{C})$ free condition as we show below.

C.2.1 Two expressions of $\mathrm{GL}(k,\mathbb{C})$ free condition

The moduli space of the half-ADHM data is the $\mathrm{GL}(k,\mathbb{C})$ quotient of the space of matrices $\{\Psi, Z\}$ on which the $\mathrm{GL}(k,\mathbb{C})$ action is free. There are two equivalent conditions for (Ψ, Z) to be a pair of matrices on which $\mathrm{GL}(k,\mathbb{C})$ action is free:[27]

$$\mathcal{C}_1(\Psi,Z): \text{ for } X \in \mathfrak{gl}(k,\mathbb{C}), \qquad \Psi X = 0,\ [Z,X]=0 \ \Rightarrow\ X = 0, \tag{C.54}$$

$$\mathcal{C}_2(\Psi,Z): \text{ for } \vec{v} \in \mathbb{C}^k, \qquad \Psi Z^a \vec{v} = 0 \ \text{ for } \ a = 0,1,\dots k-1 \ \Rightarrow\ \vec{v} = 0. \tag{C.55}$$

To show the equivalence of these conditions, let us consider the inclusion relation between the following two sets

$$F_1 \equiv \{(\Psi,Z)\,|\,\mathcal{C}_1(\Psi,Z)\}, \quad F_2 \equiv \{(\Psi,Z)\,|\,\mathcal{C}_2(\Psi,Z)\}. \tag{C.56}$$

Proof of $F_2 \subseteq F_1$. If $\Psi X = 0$ and $[Z,X]=0$ for $X \in \mathfrak{gl}(k,\mathbb{C})$, it follows that $\Psi Z^a X = 0$ for $a = 1,2,\cdots$. For an element $(\Psi,Z) \in F_2$, $\Psi Z^a X = 0$ implies that $X=0$ and hence

$$\Psi X = 0,\ [Z,X]=0 \quad \Rightarrow \quad \Psi Z^a X = 0 \ \text{ for } \ a = 1,2,\dots \quad \Rightarrow \quad X = 0. \tag{C.57}$$

This shows that $(\Psi,Z) \in F_2 \Rightarrow (\Psi,Z) \in F_1$, or equivalently $F_2 \subseteq F_1$. □

Proof of $F_2 \supseteq F_1$. Here we prove that $F_2 \supseteq F_1$ by showing that $\overline{F_2} \subseteq \overline{F_1}$, where $\overline{F_1}$ and $\overline{F_2}$ are the complements of F_1 and F_2, respectively. To show $\overline{F_2} \subseteq \overline{F_1}$, we show that there exists a nontrivial $X \in \mathfrak{gl}(k,\mathbb{C})$ satisfying $\Psi X = 0$ and $[Z,X]=0$ for any element $(\Psi,Z) \in \overline{F_2} = \{(\Psi,Z)\} \backslash F_2$. If $(\Psi,Z) \in \overline{F_2}$, there exist a set of l linearly independent column vectors $\{\vec{v}_p | p = 1,\dots,l \le k\}$ satisfying $\Psi Z^a \vec{v}_p = 0$ for all $a \in \mathbb{Z}_{\ge 0}$. After an appropriate $\mathrm{GL}(k,\mathbb{C})$ transformation, therefore, ΨZ^a take the following form

$$\Psi Z^a = \left(\star \,\middle|\, \mathbf{0}_{k\text{-by-}l} \right) \quad \text{for } a = 0,1,\dots,k-1. \tag{C.58}$$

Under this gauge choice, Z takes the following form

$$Z = \begin{pmatrix} Z_+ & \mathbf{0} \\ W & Z_- \end{pmatrix} \tag{C.59}$$

[27] It is convenient to rewrite the second condition, $\mathcal{C}_2(\Psi,Z)$, as

$$\exists \vec{v}: \text{ a column vector }, \forall z \in \mathbb{C}, \quad \Psi(z\mathbf{1}_n - Z)^{-1}\vec{v} = 0 \quad \Rightarrow \quad \vec{v} = 0.$$

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where $Z_+ \in M_{k-l,k-l}$, $Z_- \in M_{l,l}$ and $W \in M_{l,k-l}$ ($M_{n,m}$: space of n-by-m matrices). Let h be the endomorphism on $M_{l,k-l}$ given by

$$h : B \quad \mapsto \quad h(B) = Z_- B - B Z_+. \tag{C.60}$$

If $\dim(\mathrm{Ker}(h)) \neq 0$, then a non-trivial $X \in \mathfrak{gl}(k, \mathbb{C})$ can be constructed as

$$X = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B_0 & \mathbf{0} \end{pmatrix} \quad \text{with} \quad B_0 \in \mathrm{Ker}(h) \backslash \{\mathbf{0}\}. \tag{C.61}$$

This non-trivial element $X \in \mathfrak{gl}(k, \mathbb{C})$ satisfies

$$\Psi X = 0, \quad [Z, X] = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ h(B_0) & \mathbf{0} \end{pmatrix} = \mathbf{0}. \tag{C.62}$$

If $\dim(\mathrm{Ker}(h)) = 0$, a squared matrix X can be constructed by using the inverse map h^{-1}. For example,

$$X = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ h^{-1}(W) & \mathbf{1} \end{pmatrix} \neq \mathbf{0} \tag{C.63}$$

satisfies

$$\Psi X = 0, \quad [Z, X] = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ Z_- h^{-1}(W) - h^{-1}(W) Z_+ - W & \mathbf{0} \end{pmatrix} = \mathbf{0}. \tag{C.64}$$

Therefore, we can always construct a nontrivial $X \in \mathfrak{gl}(k, \mathbb{C})$ satisfying $\Psi X = 0$ and $[Z, X] = 0$ for any element $(\Psi, Z) \in \overline{F_2}$. Thus we conclude that $\overline{F_2} \subseteq \overline{F_1}$, that is, $F_2 \supseteq F_1$. □

Combining these two facts, $F_1 \supseteq F_2$ and $F_2 \supseteq F_1$, we conclude that $F_1 = F_2$.

C.2.2 Atlas $\{(\widetilde{\phi}_\lambda, \widetilde{\mathcal{M}}_\lambda)\}$ of $\widetilde{\mathcal{M}}$ from $\{Z, \Psi\}$

Using the spaces F_1 or F_2 given in eq. (C.56), we can rewrite the definition of the manifold $\widetilde{\mathcal{M}}$ as

$$\widetilde{\mathcal{M}} \equiv F_1 / \mathrm{GL}(k, \mathbb{C}) = F_2 / \mathrm{GL}(k, \mathbb{C}). \tag{C.65}$$

The atlas of this manifold is given as follows. Let Λ be the same index set Λ as that given in eq. (C.8)

$$\Lambda = \left\{ (l_1, l_2, \cdots, l_n) \,\middle|\, l_a \in \mathbb{Z}_{\geq 0}, \ \sum_{a=1}^{n} l_a = k \right\}. \tag{C.66}$$

For $\lambda \in \Lambda$, let $\widetilde{\mathcal{M}}_\lambda$ be the subspace of $\widetilde{\mathcal{M}}$ given by the equivalence classes of the data (Ψ, Z) of the form

$$\Psi_\lambda = \begin{pmatrix} \psi_1 & & \\ & \ddots & \\ & & \psi_n \end{pmatrix} + \begin{pmatrix} \Psi_{11} & \cdots & \Psi_{1n} \\ \vdots & \ddots & \vdots \\ \Psi_{n1} & \cdots & \Psi_{nn} \end{pmatrix}, \qquad Z_\lambda = \begin{pmatrix} Z_1 & & \\ & \ddots & \\ & & Z_n \end{pmatrix} + \begin{pmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{n1} & \cdots & Z_{nn} \end{pmatrix}, \tag{C.67}$$

where ψ_a are l_a-component row vectors, Ψ_{ab} are l_b-component row vectors, Z_a are l_a-by-l_a matrices and Z_{ab} are l_a-by-l_b matrices such that

$$\begin{pmatrix} \psi_a \\ Z_a \end{pmatrix} = \begin{array}{c} l_a \\ \left(\begin{array}{ccc} 1 & & \\ & \ddots & \\ & & 1 \\ \hline 0 & \cdots & 0 \end{array}\right) \end{array} l_a{+}1, \qquad \begin{pmatrix} \Psi_{ab} \\ Z_{ab} \end{pmatrix} = \begin{array}{c} l_b \\ \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \hline T_{ab,1} & \cdots & T_{ab,l_b} \end{array}\right) \end{array} l_a{+}1. \tag{C.68}$$

To check that $\mathrm{GL}(k,\mathbb{C})$ action is free on (Ψ, Z) given above, it is convenient to map (Ψ, Z) into the infinite dimensional Grassmannian $G(k,\infty)$ given by the set of an infinite number of k-component row vectors constructed from the rows of ΨZ^{p-1} with $p \in \mathbb{N}$. As shown in section C.2.1, the $\mathrm{GL}(k, C)$ action is free on (Ψ, Z) if and only if the image of the mapping to $G(k,\infty)$ contains a bases of the k-dimensional vector space. Let us define k-component row vectors $\{\mathbf{e}_{a,p}\}$ as

$$\mathbf{e}_{a,p} \equiv \text{the } a\text{-th row of } \Psi Z^{p-1}, \quad a = 1, \cdots, n, \quad p = 1, \cdots, k. \tag{C.69}$$

From (Ψ, Z) given in eq. (C.68), we can construct the following (l_a+1)-by-k matrix for each a,

$$\left(\begin{array}{c} \mathbf{e}_{a,1} \\ \vdots \\ \mathbf{e}_{a,l_a} \\ \hline \mathbf{e}_{a,l_a+1} \end{array}\right) = \left(\begin{array}{c|c|c|c|c|c|c} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1}_{l_a} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \vec{T}_{a,1} & \cdots & \vec{T}_{a,a-1} & \vec{T}_{a,a} & \vec{T}_{a,a+1} & \cdots & \vec{T}_{a,n} \end{array}\right), \tag{C.70}$$

with a l_b-component row vector $\vec{T}_{ab} = (T_{ab,1}, \cdots, T_{ab,l_b})$. By correcting the first l_a rows for all a, we find the identity matrix

$$E_\lambda = \mathbf{1}_k \quad \text{with} \quad E_\lambda \equiv \begin{pmatrix} \hat{\mathbf{e}}_1(l_1) \\ \hat{\mathbf{e}}_2(l_2) \\ \vdots \\ \hat{\mathbf{e}}_n(l_n) \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{e}}_a(m) \equiv \begin{pmatrix} \mathbf{e}_{a,1} \\ \mathbf{e}_{a,2} \\ \vdots \\ \mathbf{e}_{a,m} \end{pmatrix} \tag{C.71}$$

which immediately indicate that the $\mathrm{GL}(k,\mathbb{C})$ action is free. All the moduli parameters in (Ψ, Z) are contained in $\{\mathbf{e}_{a,l_a+1} \,|\, a = 1, \ldots, n\} \simeq \mathbb{C}^{kn}$ as $\mathbf{e}_{a,l_a+1} = (\vec{T}_{a,1} \cdots \vec{T}_{a,n})$ and all the entries are independent. Therefore, the local coordinate system on the coordinate patch (chart) $\widetilde{\mathcal{M}}_\lambda$ with $\lambda = (l_1, \cdots, l_n)$ is given by

$$\tilde{\phi}_\lambda(\widetilde{\mathcal{M}}_\lambda) \simeq \left\{ T^{ab}_m \,\middle|\, 1 \le a, b \le n, 1 \le m \le l_b \right\} \simeq \mathbb{C}^{kn}. \tag{C.72}$$

The coordinate transformation to another patch $\widetilde{\mathcal{M}}_{\lambda'}$ can be constructed by using the matrix $g \in \mathrm{GL}(k,\mathbb{C})$ defined by

$$g = E_{\lambda'} \quad \text{with} \quad \lambda' = (l'_1, l'_2, \cdots, l'_n). \tag{C.73}$$

Using the matrix g, we can read off the coordinate transformation $\tilde{f}_{\lambda'\lambda} = \tilde{\phi}_{\lambda'} \circ \tilde{\phi}_\lambda^{-1}$ from the relation

$$(\Psi', Z') = (\Psi g^{-1}, g Z g^{-1}) \in \widetilde{\mathcal{M}}_\lambda \cap \widetilde{\mathcal{M}}_{\lambda'}. \tag{C.74}$$

Gluing all the coordinate patches $\widetilde{\mathcal{M}}_\lambda$, we obtain a complex manifold

$$\widetilde{\mathcal{M}}' = \bigcup_{\lambda\in\Lambda} \widetilde{\mathcal{M}}_\lambda, \quad \text{with} \quad \Lambda = \left\{ (l_1, l_2, \cdots, l_n) \,\middle|\, l_a \in \mathbb{Z}_{\geq 0},\ \sum_{a=1}^{n} l_a = k \right\}. \tag{C.75}$$

This is a submanifold of $\widetilde{\mathcal{M}}$ since $\widetilde{\mathcal{M}}_\lambda \subset \widetilde{\mathcal{M}}$ for all $\lambda \in \Lambda$. In subsection C.2.3, we show that $\widetilde{\mathcal{M}}$ can be decomposed into a disjoint union of subspaces $\widetilde{\mathcal{M}}^{\rm tri}_\lambda$ such that $\widetilde{\mathcal{M}}^{\rm tri}_\lambda \subset \widetilde{\mathcal{M}}_\lambda$. This fact implies that $\widetilde{\mathcal{M}}$ is a subspace of $\widetilde{\mathcal{M}}'$:

$$\widetilde{\mathcal{M}} = \bigsqcup_{\lambda\in\Lambda} \widetilde{\mathcal{M}}^{\rm tri}_\lambda \subseteq \bigcup_{\lambda\in\Lambda} \widetilde{\mathcal{M}}_\lambda = \widetilde{\mathcal{M}}'. \tag{C.76}$$

Since $\widetilde{\mathcal{M}}' \subseteq \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}' \supseteq \widetilde{\mathcal{M}}$, we conclude that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}'$ and $\{(\tilde{\phi}_\lambda, \widetilde{\mathcal{M}}_\lambda)\}$ gives an atlas of $\widetilde{\mathcal{M}}$.

C.2.3 Decomposition into disjoint union

Here, we show that $\widetilde{\mathcal{M}}$ can be decomposed into a disjoint union of subspaces $\widetilde{\mathcal{M}}^{\rm tri}_\lambda$ such that $\widetilde{\mathcal{M}}^{\rm tri}_\lambda \subset \widetilde{\mathcal{M}}_\lambda$.

The manifold $\widetilde{\mathcal{M}}$ is the space of equivalent classes of the matrices $[(\Psi, Z)]$ satisfying the $GL(k,\mathbb{C})$-free condition (C.55). For each equivalence class, we can associate a set of integers $\lambda = (l_1, l_2, \cdots, l_n)$ as follows. Let (Ψ, Z) is an representative of a equivalence class and $\mathbf{e}_{a,p}$ be the k component row vectors defined by

$$\mathbf{e}_{a,p} \equiv \text{the } a\text{-th row of } \Psi Z^{p-1}. \tag{C.77}$$

Let $\widetilde{\mathcal{V}}_a$ $(a = 1, \cdots, N)$ be the vector spaces spanned by $\mathbf{e}_{b,p}$ with $b = 1, \cdots, a$

$$\widetilde{\mathcal{V}}_a = \text{span}\left\{ \mathbf{e}_{b,p} \,|\, p \in \mathbb{N},\ b = 1, \cdots, a \right\}. \tag{C.78}$$

These vector spaces form a flag

$$\{0\} = \widetilde{\mathcal{V}}_0 \subseteq \widetilde{\mathcal{V}}_1 \subseteq \widetilde{\mathcal{V}}_2 \subseteq \cdots \subseteq \widetilde{\mathcal{V}}_n. \tag{C.79}$$

Then, we define l_a as

$$l_a \ = \ \dim_{\mathbb{C}} \widetilde{\mathcal{V}}_a - \dim_{\mathbb{C}} \widetilde{\mathcal{V}}_{a-1}. \tag{C.80}$$

Since (Ψ, Z) satisfies the $GL(k,\mathbb{C})$-free condition (C.55), it follows that

$$\dim_{\mathbb{C}} \widetilde{\mathcal{V}}_n = l_1 + \cdots l_n = k. \tag{C.81}$$

Since the set of integers $\lambda = (l_1, \cdots, l_n)$ is invariant under the $GL(k,\mathbb{C})$ transformation, each equivalent class $[(\Psi, Z)]$ has unique λ. Therefore, $\widetilde{\mathcal{M}}$ can be decomposed into the disjoint union of the spaces of equivalence classes $\widetilde{\mathcal{M}}^{\rm tri}_\lambda$ classified by $\lambda = (l_1, \cdots, l_n)$

$$\widetilde{\mathcal{M}} = \bigsqcup_{\lambda\in\Lambda} \widetilde{\mathcal{M}}^{\rm tri}_\lambda. \tag{C.82}$$

We can determine $\dim_{\mathbb{C}} \widetilde{\mathcal{V}}_a$ by constructing the basis of $\widetilde{\mathcal{V}}_a$. It can be obtained inductively from the basis of $\widetilde{\mathcal{V}}_{a-1}$ by adding the vectors $\mathbf{e}_{a,1}, \cdots, \mathbf{e}_{a,l_a}$. Here, l_a is the maximum number

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such that $\mathbf{e}_{a,l_a}$ is linearly independent of $\{\mathbf{e}_{a,1},\cdots,\mathbf{e}_{a,l_a-1}\}$ and any element of $\widetilde{\mathcal{V}}_{a-1}$, or equivalently, l_a is the minimum number such that

$$\mathbf{e}_{a,l_a+q} \ \in \ \widetilde{\mathcal{V}}_{a-1} \cup \mathrm{span}(\{\mathbf{e}_{a,p}\,|\,p=1,\cdots,l_a\}) \quad \text{for } q=1,2,\cdots. \tag{C.83}$$

Next, let us show that $\widetilde{\mathcal{M}}^{\mathrm{tri}}_{\lambda}$ is a subspace of $\widetilde{\mathcal{M}}_{\lambda}$. Let E_λ be the k-by-k matrix whose row vectors are $\mathbf{e}_{a,p}$

$$E_\lambda \equiv \begin{pmatrix} \hat{\mathbf{e}}_1(l_1) \\ \hat{\mathbf{e}}_2(l_2) \\ \vdots \\ \hat{\mathbf{e}}_n(l_n) \end{pmatrix} \qquad \text{with} \qquad \hat{\mathbf{e}}_a(m) \equiv \begin{pmatrix} \mathbf{e}_{a,1} \\ \mathbf{e}_{a,2} \\ \vdots \\ \mathbf{e}_{a,m} \end{pmatrix} \tag{C.84}$$

Since E_λ is the basis of $\widetilde{\mathcal{V}}_n \cong \mathbb{C}^k$, there is an element of $\mathrm{GL}(k,\mathbb{C})$ such that

$$E_\lambda \to E_\lambda\, g = \mathbf{1}_k, \qquad g \in \mathrm{GL}(k,\mathbb{C}). \tag{C.85}$$

After fixing the $\mathrm{GL}(k,\mathbb{C})$ redundancy as $E_\lambda = \mathbf{1}_k$, $\mathbf{e}_{a,p}$ take the form

$$\left(\begin{array}{c} \mathbf{e}_{a,1} \\ \vdots \\ \mathbf{e}_{a,l_a} \\ \hline \mathbf{e}_{a,l_a+1} \end{array}\right) = \left(\begin{array}{c|c|c|c|c|c|c} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1}_{l_a} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline \vec{T}_{a,1} & \cdots & \vec{T}_{a,a-1} & \vec{T}_{a,a} & \mathbf{0} & \cdots & \mathbf{0} \end{array}\right), \tag{C.86}$$

where "$\mathbf{0}$"s in $\mathbf{e}_{a,l_a+1}$ are due to the property in eq. (C.83). From the above set of row vectors $\{\mathbf{e}_{a,p}\,|\,a=1,\ldots,n;\, p=1,\ldots,l_a+1\}$, the two matrices Ψ, Z can be reconstructed as

$$\Psi = \begin{pmatrix} \mathbf{e}_{1,1} \\ \mathbf{e}_{2,1} \\ \vdots \\ \mathbf{e}_{n,1} \end{pmatrix}, \qquad Z = E_\lambda Z = \begin{pmatrix} \hat{\mathbf{e}}_1(l_1)\, Z \\ \hat{\mathbf{e}}_2(l_2)\, Z \\ \vdots \\ \hat{\mathbf{e}}_n(l_n)\, Z \end{pmatrix} \quad \text{with} \quad \hat{\mathbf{e}}_a(l_a)\, Z = \begin{pmatrix} \mathbf{e}_{a,2} \\ \vdots \\ \mathbf{e}_{a,l_a} \\ \mathbf{e}_{a,l_a+1} \end{pmatrix}. \tag{C.87}$$

From these expression, we find that Ψ and Z take the block lower triangular form $\Psi = \Psi^{\mathrm{tri}}_\lambda$ and $Z = Z^{\mathrm{tri}}_\lambda$ with

$$\Psi^{\mathrm{tri}}_\lambda = \begin{pmatrix} \psi_1 & & \\ & \ddots & \\ & & \psi_n \end{pmatrix} + \begin{pmatrix} \Psi_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \Psi_{21} & \Psi_{22} & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \Psi_{n1} & \cdots & \cdots & \Psi_{nn} \end{pmatrix}, \quad Z^{\mathrm{tri}}_\lambda = \begin{pmatrix} Z_1 & & \\ & \ddots & \\ & & Z_n \end{pmatrix} + \begin{pmatrix} Z_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ Z_{21} & Z_{22} & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ Z_{n1} & \cdots & \cdots & Z_{nn} \end{pmatrix}, \tag{C.88}$$

with $\psi_a, \Psi_{ab}, Z_a, Z_{ab}$ given in eq. (C.68). Since $\widetilde{\mathcal{M}}^{\mathrm{tri}}_\lambda$ is the space of (Ψ, Z) of these forms, it is a subspace of $\widetilde{\mathcal{M}}_\lambda$, the space of matrices of the form (C.67)

$$\widetilde{\mathcal{M}}^{\mathrm{tri}}_\lambda \subseteq \widetilde{\mathcal{M}}_\lambda. \tag{C.89}$$

This fact implies that $\widetilde{\mathcal{M}}$ is a subspace of $\widetilde{\mathcal{M}}'$:

$$\widetilde{\mathcal{M}} = \bigsqcup_{\lambda\in\Lambda} \widetilde{\mathcal{M}}^{\mathrm{tri}}_\lambda \subseteq \bigcup_{\lambda\in\Lambda} \widetilde{\mathcal{M}}_\lambda = \widetilde{\mathcal{M}}'. \tag{C.90}$$

Since $\widetilde{\mathcal{M}}' \subseteq \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}' \supseteq \widetilde{\mathcal{M}}$, we conclude that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}'$ and $\{(\widetilde{\phi}_\lambda, \widetilde{\mathcal{M}}_\lambda)\}$ gives an atlas of $\widetilde{\mathcal{M}}$.

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C.3 Equivalence of $\mathcal{M}$ and $\widetilde{\mathcal{M}}$

C.3.1 Mapping from $\mathcal{M}_\lambda$ to $\widetilde{\mathcal{M}}_\lambda$ through the half-ADHM mapping relation

Let us recapitulate how to extract the data (Ψ, Z) from $\mathfrak{D}(z), \mathfrak{J}(z)$ through the half-ADHM mapping relation. The half-ADHM mapping relation is given by

$$\mathfrak{D}(z)\Psi = \mathfrak{J}(z)(z\mathbf{1}_k - Z), \tag{C.91}$$

which is covariant under both of the V transformation and $\mathrm{GL}(k, \mathbb{C})$ transformation. Therefore, this relation defines a mapping between the equivalence classes $[\mathfrak{D}(z)] \in \mathcal{M} \mapsto [(\Psi, Z)] \in \widetilde{\mathcal{M}}$. For a point in the $\lambda = (l_1, l_2, \cdots, l_n)$-patch of $\mathcal{M}$, where $\mathfrak{D}(z)$ takes the form (C.5), the corresponding matrix $\mathfrak{J}(z)$ is given by

$$\mathfrak{J}_\lambda(z) = \begin{pmatrix} j_1 & & \\ & \ddots & \\ & & j_n \end{pmatrix} + \begin{pmatrix} \mathfrak{J}_{11} & \cdots & \mathfrak{J}_{1n} \\ \vdots & \ddots & \vdots \\ \mathfrak{J}_{n1} & \cdots & \mathfrak{J}_{nn} \end{pmatrix}, \tag{C.92}$$

where j_a and $\mathfrak{J}_{ab}$ are the following block matrices (row vectors with l_a and l_b components, respectively)

$$j_a = \left(z^{l_a-1}, z^{l_a-2}, \cdots, 1\right), \qquad \mathfrak{J}_{ab} = -\left(\sum_{m=1}^{l_b-1} T_{ab,m+1} z^{m-1}, \sum_{m=1}^{l_b-2} T_{ab,m+2} z^{m-1}, \cdots, T_{ab,l_b}, 0\right). \tag{C.93}$$

Since $\mathfrak{J}(z)$ also depends on the gauge choice of $\mathrm{GL}(k, \mathbb{C})$, taking $\mathfrak{J}(z)$ in this form implicitly means that we have chosen a certain coordinate patch for $\widetilde{\mathcal{M}}$. Noting that $\mathfrak{J}_{ab}$ satisfies the relation

$$z\mathfrak{J}_{ab} = -P_{ab}(1, 0, \cdots, 0) + (T_{ab,1}, T_{ab,2}, \cdots, T_{ab,l_b}) + \mathfrak{J}_{ab} \left(\begin{array}{c|ccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \hline 0 & \cdots & \cdots & 0 \end{array}\right), \tag{C.94}$$

we can read off (Ψ, Z) from the half-ADHM mapping relation (C.91) and the resulting (Ψ, Z) turns out to be exactly equal to those of $\widetilde{\phi}_\lambda(\widetilde{\mathcal{M}}_\lambda)$ given in eq. (C.67). Therefore the half-ADHM mapping relation defines a one-to-one map between the coordinate patches

$$i_\lambda : \phi_\lambda(\mathcal{M}_\lambda) \quad \mapsto \quad \widetilde{\phi}_\lambda(\widetilde{\mathcal{M}}_\lambda), \tag{C.95}$$

$$\widetilde{\phi}_\lambda^{-1} \circ i_\lambda \circ \phi_\lambda : \quad \mathcal{M}_\lambda \quad \to \quad \widetilde{\mathcal{M}}_\lambda \tag{C.96}$$

for all $\lambda \in \Lambda$. Furthermore, since the half-ADHM mapping relation is covariant under the V-transformation $\mathfrak{D}'(z) = V(z)\mathfrak{D}(z)$ and the $\mathrm{GL}(k, \mathbb{C})$ transformation, $(\Psi', Z') = (\Psi g, g^{-1} Z g)$ as

$$\mathfrak{D}(z)\Psi = \mathfrak{J}(z)(z\mathbf{1} - Z) \quad \Rightarrow \quad \mathfrak{D}'(z)\Psi' = \mathfrak{J}'(z)(z\mathbf{1} - Z'), \quad \text{with} \quad \mathfrak{J}'(z) = V(z)\mathfrak{J}(z)g, \tag{C.97}$$

we find that the following diagram commutes: for $\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'$,

$$\begin{array}{ccc} \mathfrak{D}(z) \in \phi_\lambda(\mathcal{M}_\lambda \cap \mathcal{M}_{\lambda'}) & \overset{i_\lambda}{\to} & (\Psi, Z) \in \widetilde{\phi}_\lambda(\widetilde{\mathcal{M}}_\lambda \cap \widetilde{\mathcal{M}}_{\lambda'}) \\ f_{\lambda'\lambda} = \phi_{\lambda'} \circ \phi_\lambda^{-1} \downarrow & & \downarrow \widetilde{f}_{\lambda'\lambda} = \widetilde{\phi}_{\lambda'} \circ \widetilde{\phi}_\lambda^{-1} \\ \mathfrak{D}'(z) \in \phi_\lambda(\mathcal{M}_\lambda \cap \mathcal{M}_{\lambda'}) & \overset{i_{\lambda'}}{\to} & (\Psi', Z') \in \widetilde{\phi}_{\lambda'}(\widetilde{\mathcal{M}}_\lambda \cap \widetilde{\mathcal{M}}_{\lambda'}) \end{array} \tag{C.98}$$

This fact indicates that two different maps $i_\lambda, i_{\lambda'}$ with the domain $\mathcal{M}_\lambda \cap \mathcal{M}_{\lambda'}$ are consistent

$$\widetilde{\phi}_\lambda^{-1} \circ i_\lambda \circ \phi_\lambda = \widetilde{\phi}_{\lambda'}^{-1} \circ i_{\lambda'} \circ \phi_{\lambda'} \quad : \quad \mathcal{M}_\lambda \cap \mathcal{M}_{\lambda'} \quad \to \quad \widetilde{\mathcal{M}}_\lambda \cap \widetilde{\mathcal{M}}_{\lambda'} \tag{C.99}$$

and the transition function $\widetilde{f}_{\lambda'\lambda}$ on the manifold $\widetilde{\mathcal{M}}$ induced by the $\mathrm{GL}(k, \mathbb{C})$ transformation is consistent with $f_{\lambda'\lambda}$ on the manifold $\mathcal{M}$ induced by the V-transformation,

$$\widetilde{f}_{\lambda'\lambda} = i_{\lambda'} \circ f_{\lambda'\lambda} \circ i_\lambda^{-1} : \quad \widetilde{\phi}_\lambda(\widetilde{\mathcal{M}}_\lambda \cap \widetilde{\mathcal{M}}_{\lambda'}) \quad \to \quad \widetilde{\phi}_{\lambda'}(\widetilde{\mathcal{M}}_\lambda \cap \widetilde{\mathcal{M}}_{\lambda'}). \tag{C.100}$$

Therefore, we conclude that the two complex manifolds $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are biholomorphically equivalent

$$\mathcal{M} \simeq \widetilde{\mathcal{M}}, \tag{C.101}$$

and the ADHM relation defines the unique one-to-one map between them.

C.3.2 Examples

Let us see an example in the case of $k = 2$ and $n = 2$. In the $(2,0)$ patch ($\lambda = (2,0)$), the matrix $\mathfrak{D}(z)$ and the corresponding matrix $\mathfrak{J}(z)$ are given by (see eqs. (C.5) and (C.92))

$$\mathfrak{D}_{(2,0)}(z) = \begin{pmatrix} z^2 - a_2 z - a_1 & 0 \\ -b_2 z - b_1 & 1 \end{pmatrix}, \qquad \mathfrak{J}_{(2,0)}(z) = \begin{pmatrix} z - a_2 & 1 \\ -b_2 & 0 \end{pmatrix}. \tag{C.102}$$

From the half-ADHM mapping relation, the data (Z, Ψ) can be read off as

$$\Psi_{(2,0)} = \begin{pmatrix} 1 & 0 \\ b_1 & b_2 \end{pmatrix}, \qquad Z_{(2,0)} = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix}. \tag{C.103}$$

These matrices correspond to those in eq. (C.67) with

$$\psi_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \Psi_{11} = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad \Psi_{21} = \begin{pmatrix} b_1 & b_2 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z_{11} = \begin{pmatrix} 0 & 0 \\ a_1 & a_2 \end{pmatrix}. \tag{C.104}$$

One can move to the $(1,1)$ patch ($\lambda = (1,1)$) by performing the transformation

$$\mathfrak{D}_{(1,1)}(z) = V(z)\mathfrak{D}_{(2,0)}(z), \qquad \mathfrak{J}_{(1,1)}(z) = V(z)\mathfrak{J}_{(2,0)}(z)g, \qquad V(z) = \begin{pmatrix} 0 & -v \\ b_2 & z - \tilde{u} \end{pmatrix}, \tag{C.105}$$

as

$$\mathfrak{D}_{(1,1)}(z) = \begin{pmatrix} z - u & -v \\ -\tilde{v} & z - \tilde{u} \end{pmatrix}, \qquad \mathfrak{J}_{(1,1)}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{C.106}$$

where

$$u = -\frac{b_1}{b_2}, \qquad \tilde{u} = a_2 + \frac{b_1}{b_2}, \qquad v = \frac{1}{b_2}, \qquad \tilde{v} = a_1 b_2 - b_1 \left(a_2 + \frac{b_1}{b_2} \right). \tag{C.107}$$

The corresponding half-ADHM data are given by

$$\Psi_{(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z_{(1,1)} = \begin{pmatrix} u & v \\ \tilde{v} & \tilde{u} \end{pmatrix}. \tag{C.108}$$

These matrices correspond to those in eq. (C.67) with

$$\psi_1 = \psi_2 = 1, \quad \Psi_{ij} = 0, \quad Z_i = 0 \ \ (i = 1, 2), \quad Z_{11} = u, \quad Z_{12} = v, \quad Z_{21} = \tilde{v}, \quad Z_{22} = \tilde{u}. \tag{C.109}$$

These data in the $(1, 1)$ patch is related to that $(2, 0)$ patch by a $\mathrm{GL}(2, \mathbb{C})$ transformation

$$\Psi_{(1,1)} = \Psi_{(2,0)} g, \qquad Z_{(1,1)} = g^{-1} Z_{(2,0)} g, \qquad g = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}). \tag{C.110}$$

Similarly, we can show that the data $(0, 2)$ patch is also related to $(Z_{(2,0)}, \Psi_{(2,0)})$ and $(Z_{(1,1)}, \Psi_{(1,1)})$ by $\mathrm{GL}(2, \mathbb{C})$ transformation.

D Condition of non-singular instanton solution

In this appendix, we discuss the condition for $\xi(z)$ and $(Z, \Psi, \widetilde{\Psi})$ to be the data for non-singular sigma model instanton solutions. Throughout this section, we focus on the $L = 1$ case.

D.1 Local-semilocal decomposition

Let us define the moduli space of semi-local vortices $\mathcal{M}_{\mathrm{semi}}{}_{k}^{n,m}$ as the following subspace of the moduli space of vortices $\mathcal{M}_{\mathrm{vtx}}{}_{k}^{n,m}$

$$\mathcal{M}_{\mathrm{semi}}{}_{k}^{n,m} \equiv \{\, [\xi(z)] \in \mathcal{M}_{\mathrm{vtx}}{}_{k}^{n,m} \,|\, \mathrm{rank}(\xi(z)) = n \text{ for } \forall z \in \mathbb{C} \,\} \subset \mathcal{M}_{\mathrm{vtx}}{}_{k}^{n,m}. \tag{D.1}$$

In this subsection, we prove four lemmas D.1–D.4 which will be used in the later subsections.

Lemma D.1. *Any matrix $\xi(z)$ such that $[\xi(z)] \in \mathcal{M}_{\mathrm{vtx}}{}_{k}^{n,m}$ can be decomposed as*

$$\xi(z) = \mathfrak{D}^{\mathrm{lc}}(z)\, \xi^{\mathrm{sm}}(z) \quad \textit{with} \quad \mathfrak{D}^{\mathrm{lc}}(z) \in \mathcal{G}_{n,k-l}[z] \quad \textit{and} \quad [\xi^{\mathrm{sm}}(z)] \in \mathcal{M}_{\mathrm{semi}}{}_{l}^{n,m} \tag{D.2}$$

where l is an integer such that $0 \leq l \leq k$.

Proof of Lemma D.1. Applying the method used in section C.1 to the rows of $\xi(z) = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$ instead of columns, we can find matrices $V(z) \in \mathcal{G}_{n+m,0}[z]$ and $\mathfrak{D}^{\mathrm{lc}}(z) \in \mathcal{G}_{n,l}[z]$ $(0 \leq l \leq k)$ such that

$$\xi(z) = (\mathfrak{D}^{\mathrm{lc}}(z), \mathbf{0})\, V(z). \tag{D.3}$$

Then, this matrix $\xi(z)$ can be rewritten as

$$\xi(z) = \mathfrak{D}^{\mathrm{lc}}(z)\, \xi^{\mathrm{sm}}(z) \quad \text{with} \quad \xi^{\mathrm{sm}}(z) = (\mathbf{1}_n, \mathbf{0})\, V(z). \tag{D.4}$$

Since $\det V(z) \neq 0$ for all $z \in \mathbb{C}$, it follows that $\mathrm{rank}(\xi^{\mathrm{sm}}(z)) = n$ for all $z \in \mathbb{C}$ and hence $[\xi^{\mathrm{sm}}(z)] \in \mathcal{M}_{\mathrm{semi}}{}_{k-l}^{n,m}$. □

Note that the decomposition is not unique since the following transformation does not change the matrix $\xi(z)$

$$\mathfrak{D}^{\mathrm{lc}}(z) \to \mathfrak{D}^{\mathrm{lc}}(z) V'(z)^{-1}, \qquad \xi_{\mathrm{sm}}(z) \to V'(z) \xi_{\mathrm{sm}}(z) \quad \text{with} \quad V'(z) \in \mathcal{G}_{n,0}[z]. \tag{D.5}$$

We can show the equivalence class $[\xi_{\mathrm{sm}}(z)]$ is unique.

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Lemma D.2. *The decomposition in Lemma D.1 is unique up to V-equivalence relations* (D.5).

Proof of Lemma D.2. Let us assume that $[\xi(z)] \in \mathcal{M}_{\mathrm{semi}}{}_k^{n,m}$ is rewritten in the two different ways as

$$\xi(z) = \mathfrak{D}_1^{\mathrm{lc}}(z)\xi_1^{\mathrm{sm}}(z) = \mathfrak{D}_2^{\mathrm{lc}}(z)\xi_2^{\mathrm{sm}}(z) \quad \text{with} \quad \xi_i^{\mathrm{sm}}(z) = (\mathbf{1}_n, \mathbf{0})V_i(z) \quad (i=1,2). \tag{D.6}$$

The above equation can be rewritten as

$$(\mathfrak{D}_1^{\mathrm{lc}}(z), \mathbf{0}) = (\mathfrak{D}_2^{\mathrm{lc}}(z), \mathbf{0})\, V_2(z)V_1(z)^{-1}, \quad \Big(V_2(z)V_1(z)^{-1} \in \mathcal{G}_{n+m,0}[z]\Big). \tag{D.7}$$

This implies that $V_2(z)V_1(z)^{-1}$ takes the form

$$V_2(z)V_1(z)^{-1} = \begin{pmatrix} V_{12}(z) & \mathbf{0} \\ * & * \end{pmatrix}, \tag{D.8}$$

where $V_{12}(z) \in \mathcal{G}_{n,0}[z]$ and $*$'s are undetermined entries. From this equation, we find that $\mathfrak{D}_1^{\mathrm{lc}}(z) = \mathfrak{D}_2(z)^{\mathrm{lc}}\, V_{12}(z)$ and hence $\xi_1^{\mathrm{sm}}(z)$ and $\xi_2^{\mathrm{sm}}(z)$ are related as

$$\xi_1^{\mathrm{sm}}(z) = V_{12}(z)^{-1}\xi_2^{\mathrm{sm}}(z). \tag{D.9}$$

Therefore, the decomposition $\xi(z) = \mathfrak{D}^{\mathrm{lc}}(z)\xi^{\mathrm{sm}}(z)$ defines an unique equivalent class $[\xi^{\mathrm{sm}}(z)] \in \mathcal{M}_{\mathrm{semi}}{}_k^{n,m}$. □

Lemma D.3. *Any element $[\xi^{\mathrm{sm}}(z)] \in \mathcal{M}_{\mathrm{semi}}{}_k^{n,m}$ can be written as*

$$\xi^{\mathrm{sm}}(z) = (\mathbf{1}_n, \mathbf{0})\, V(z) \quad \text{with } V(z) \in \mathcal{G}_{n+m,0}[z]. \tag{D.10}$$

Proof of Lemma D.3. If $\det \mathfrak{D}^{\mathrm{lc}}(z)$ has a zero, then rank of $\xi(z)$ decreases at that point. Equivalently, if $\xi(z)$ has the maximal rank everywhere ($[\xi(z)] \in \mathcal{M}_{\mathrm{semi}}{}_k^{n,m}$), then $\det \mathfrak{D}^{\mathrm{lc}}(z)$ must have no zero ($\mathfrak{D}^{\mathrm{lc}}(z) \in \mathcal{G}_{n,0}[z]$) and hence $\mathfrak{D}^{\mathrm{lc}}(z)$ can be absorbed by the V-transformation $V(z) \in \mathcal{G}_{n,0}[z]$. Therefore, for any $[\xi(z)] \in \mathcal{M}_{\mathrm{semi}}{}_k^{n,m}$, there is a representative $\xi(z)$ of the form

$$\xi(z) = (\mathbf{1}_n, \mathbf{0})\, V(z). \tag{D.11}$$

□

Lemma D.4. *If two equivalent classes $[\xi_i(z)] = [(\mathfrak{D}_i(z), \widetilde{\mathfrak{D}}_i(z)] \in \mathcal{M}_{\mathrm{semi}}{}_{k_i}^{n,m}$ $(i=1,2)$ satisfy*

$$\mathfrak{D}_1^{-1}(z)\,\widetilde{\mathfrak{D}}_1(z) = \mathfrak{D}_2^{-1}(z)\,\widetilde{\mathfrak{D}}_2(z) \quad \textit{or equivalently} \quad \mathfrak{D}_1^{-1}(z)\,\xi_1(z) = \mathfrak{D}_2^{-1}(z)\,\xi_2(z), \tag{D.12}$$

then, they are equivalent

$$k_1 = k_2, \qquad [\xi_1(z)] = [\xi_2(z)]. \tag{D.13}$$

Proof of Lemma D.4. By applying Lemma D.3 to $[\xi_i(z)] \in \mathcal{M}_{\mathrm{semi}}{}_{k_i}^{n,m}$, the assumption (D.12) can be further rewritten as

$$\mathfrak{D}_1(z)^{-1}(\mathbf{1}_n, \mathbf{0})V_1(z) = \mathfrak{D}_2(z)^{-1}(\mathbf{1}_n, \mathbf{0})V_2(z). \tag{D.14}$$

This equation can be rewritten as

$$\mathfrak{D}_2(z)\mathfrak{D}_1(z)^{-1}(\mathbf{1}_n, \mathbf{0}) = (\mathbf{1}_n, \mathbf{0})V_2(z)V_1(z)^{-1}, \quad \Big(V_2(z)V_1(z)^{-1} \in \mathcal{G}_{n+m,0}[z]\Big). \tag{D.15}$$

Applying the same argument as Lemma D.2, we conclude that $\mathfrak{D}_2(z) = V_{12}(z)\mathfrak{D}_1(z)$ with $V_{12}(z) \in \mathcal{G}_{n,0}[z]$, and hence $\xi_2(z) = V(z)\xi_1(z)$, which means that they belong to the same equivalent class. □

D.2 Condition of semilocal vortices

For $[\xi(z)] \in \mathcal{M}_{\text{vtx}}{}_k^{n,m}$, the condition that $[\xi(z)]$ belongs to the semilocal vortex moduli space $\mathcal{M}_{\text{semi}}{}_k^{n,m}$ is given by

$$\mathcal{C}_{\text{semi}}(\xi(z)) \ : \ \det \xi(z)\xi(z)^\dagger \neq 0 \quad \text{for} \quad \forall z \in \mathbb{C}. \tag{D.16}$$

This condition on $\xi(z)$ can be translated into the following condition on the corresponding half-ADHM data $(Z, \Psi, \widetilde{\Psi})$

$$\mathcal{C}_{\text{free}}(Z, \widetilde{\Psi}) \ : \ \text{If } \exists \vec{v} \in \mathbb{C}^k \text{ (row vector) s.t. } \vec{v} Z^{p-1} \widetilde{\Psi} = 0 \text{ for } \forall p \in \mathbb{N} \ \Rightarrow \ \vec{v} = 0. \tag{D.17}$$

In this subsection, we prove the equivalence of the two conditions $\mathcal{C}_{\text{semi}}(\xi(z))$ and $\mathcal{C}_{\text{free}}(Z, \widetilde{\Psi})$.[28]

Theorem D.1. *The conditions $\mathcal{C}_{\text{semi}}(\xi(z))$ and $\mathcal{C}_{\text{free}}(Z, \widetilde{\Psi})$ are equivalent.*

Proof of $\mathcal{C}_{\text{free}}(Z, \widetilde{\Psi}) \rightarrow \mathcal{C}_{\text{semi}}(\xi(z))$. Let us prove the contrapositive $\neg\mathcal{C}_{\text{semi}}(\xi(z)) \rightarrow \neg\mathcal{C}_{\text{free}}(Z, \widetilde{\Psi})$. If $[\xi(z)] \in \mathcal{M}_{\text{vtx}}{}_k^{n,m}$ does not satisfy the condition $\mathcal{C}_{\text{semi}}(\xi(z))$, then Lemma D.1 in appendix D.1 implies that the matrix $\xi(z)$ can be decomposed as

$$\xi(z) = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z)) = \mathfrak{D}_{\text{lc}}(z)(\mathfrak{D}_{\text{sm}}(z), \widetilde{\mathfrak{D}}_{\text{sm}}(z)) = \mathfrak{D}_{\text{lc}}(z)\xi_{\text{sm}}(z), \tag{D.18}$$

where the n-by-n matrices $\mathfrak{D}(z), \mathfrak{D}_{\text{lc}}(z), \mathfrak{D}_{\text{sm}}(z)$ and the n-by-m matrices $\widetilde{\mathfrak{D}}(z), \widetilde{\mathfrak{D}}_{\text{sm}}(z)$ satisfy

$$\begin{aligned} \det \mathfrak{D}(z) &= \mathcal{O}(z^k), \qquad \det \mathfrak{D}_{\text{lc}}(z) = \mathcal{O}(z^l), \qquad \det \mathfrak{D}_{\text{sm}}(z) = \mathcal{O}(z^{k'}), \\ \det \mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) &= \mathcal{O}(z^{-1}), \qquad \det \mathfrak{D}_{\text{sm}}(z)^{-1}\widetilde{\mathfrak{D}}_{\text{sm}}(z) = \mathcal{O}(z^{-1}), \end{aligned} \tag{D.19}$$

with a certain nonzero positive integer $l > 0$ and $k' = k - l \geq 0$. For these matrices, we can obtain $(Z, \Psi, \widetilde{\Psi})$, $(Z_{\text{lc}}, \Psi_{\text{lc}})$ and $(Z_{\text{sm}}, \Psi_{\text{sm}}, \widetilde{\Psi}_{\text{sm}})$ through the half-ADHM mapping relations

$$z\mathfrak{J}(z) = \mathfrak{D}(z)\Psi + \mathfrak{J}(z)Z, \qquad \widetilde{\mathfrak{D}}(z) = \mathfrak{J}(z)\widetilde{\Psi}, \tag{D.20}$$

$$z\mathfrak{J}_{\text{lc}}(z) = \mathfrak{D}_{\text{lc}}(z)\Psi_{\text{lc}} + \mathfrak{J}_{\text{lc}}(z)Z_{\text{lc}}, \tag{D.21}$$

$$z\mathfrak{J}_{\text{sm}}(z) = \mathfrak{D}_{\text{sm}}(z)\Psi_{\text{sm}} + \mathfrak{J}_{\text{sm}}(z)Z_{\text{sm}}, \qquad \widetilde{\mathfrak{D}}_{\text{sm}}(z) = \mathfrak{J}_{\text{sm}}(z)\widetilde{\Psi}_{\text{sm}}, \tag{D.22}$$

where $\mathfrak{J}(z)$, $\mathfrak{J}_{\text{lc}}(z)$ and $\mathfrak{J}_{\text{sm}}(z)$ are matrices satisfying

$$\mathfrak{D}(z)^{-1}\mathfrak{J}(z) = \mathcal{O}(z^{-1}), \quad \mathfrak{D}_{\text{lc}}(z)^{-1}\mathfrak{J}_{\text{lc}}(z) = \mathcal{O}(z^{-1}), \quad \mathfrak{D}_{\text{sm}}(z)^{-1}\mathfrak{J}_{\text{sm}}(z) = \mathcal{O}(z^{-1}). \tag{D.23}$$

Note that the matrices are related as

$$\mathfrak{D}(z) = \mathfrak{D}_{\text{lc}}(z)\mathfrak{D}_{\text{sm}}(z), \quad \widetilde{\mathfrak{D}}(z) = \mathfrak{D}_{\text{lc}}(z)\widetilde{\mathfrak{D}}_{\text{sm}}(z). \tag{D.24}$$

The matrix $\mathfrak{J}(z)$ can always be chosen as

$$\mathfrak{J}(z) = (\mathfrak{J}_{\text{lc}}(z), \mathfrak{D}_{\text{lc}}(z)\mathfrak{J}_{\text{sm}}(z)). \tag{D.25}$$

[28] The proof here is a more concise version of the one given in appendix C of [66].

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We can check the condition $\mathfrak{D}(z)^{-1}\mathfrak{J}(z) = \mathcal{O}(z^{-1})$ as

$$\mathfrak{D}(z)^{-1}\mathfrak{J}_{\rm lc}(z) = \mathfrak{D}_{\rm sm}(z)^{-1}\mathcal{O}(z^{-1}) = \mathcal{O}(z^{-1}), \qquad \mathfrak{D}(z)^{-1}\mathfrak{D}_{\rm lc}(z)\mathfrak{J}_{\rm sm}(z) = \mathfrak{D}_{\rm sm}(z)^{-1}\mathfrak{J}_{\rm sm}(z) = \mathcal{O}(z^{-1}), \tag{D.26}$$

where we have used the transformation (D.5) to fix $\mathfrak{D}_{\rm sm}(z)$ to the form (C.5) so that it satisfies $\mathfrak{D}_{\rm sm}(z)^{-1} = \mathcal{O}(1)$. From the half-ADHM mapping relations eqs. (D.20)–(D.22) we find that

$$z\mathfrak{J}(z) = (z\mathfrak{J}_{\rm lc}(z), \mathfrak{D}_{\rm lc}(z)(z\mathfrak{J}_{\rm sm}(z))) = (\mathfrak{D}_{\rm lc}(z)\Psi_{\rm lc} + \mathfrak{J}_{\rm lc}(z)Z_{\rm lc}, \mathfrak{D}_{\rm lc}(z)\mathfrak{D}_{\rm sm}(z)\Psi_{\rm sm} + \mathfrak{D}_{\rm lc}(z)\mathfrak{J}_{\rm sm}(z)Z_{\rm sm}). \tag{D.27}$$

Using the constant matrices $P^{\mathfrak{D}}_{\rm sm}, P^{\mathfrak{J}}_{\rm sm}$ defined by

$$\mathbf{1}_n = \mathfrak{D}_{\rm sm}(z)P^{\mathfrak{D}}_{\rm sm} + \mathfrak{J}_{\rm sm}(z)P^{\mathfrak{J}}_{\rm sm}. \tag{D.28}$$

Eq. (D.27) can be further rewritten as

$$z\mathfrak{J}(z) = \mathfrak{D}_{\rm lc}(z)\mathfrak{D}_{\rm sm}(z)(P^{\mathfrak{D}}_{\rm sm}\Psi_{\rm lc},\, \Psi_{\rm sm}) + (\mathfrak{J}_{\rm lc}(z), \mathfrak{D}_{\rm lc}(z)\mathfrak{J}_{\rm sm}(z))\begin{pmatrix} Z_{\rm lc} & \mathbf{0} \\ P^{\mathfrak{J}}_{\rm sm}\Psi_{\rm lc} & Z_{\rm sm}\end{pmatrix}. \tag{D.29}$$

In addition, we find the following relation for $\widetilde{\mathfrak{D}}(z)$

$$\mathfrak{J}(z)\widetilde{\Psi} = \widetilde{\mathfrak{D}}(z) = \mathfrak{D}_{\rm lc}(z)\widetilde{\mathfrak{D}}_{\rm sm}(z) = \mathfrak{D}_{\rm lc}(z)\mathfrak{J}_{\rm sm}(z)\widetilde{\Psi}_{\rm sm} = (\mathfrak{J}_{\rm lc}(z), \mathfrak{D}_{\rm lc}(z)\mathfrak{J}_{\rm sm}(z))\begin{pmatrix}\mathbf{0} \\ \widetilde{\Psi}_{\rm sm}\end{pmatrix}. \tag{D.30}$$

From eqs. (D.29) and (D.30), the half-ADHM data $(Z, \Psi, \widetilde{\Psi})$ can be read off as

$$\Psi = (P^{\mathfrak{D}}_{\rm sm}\Psi_{\rm lc},\, \Psi_{\rm sm}), \quad Z = \begin{pmatrix} Z_{\rm lc} & \mathbf{0} \\ P^{\mathfrak{J}}_{\rm sm}\Psi_{\rm lc} & Z_{\rm sm}\end{pmatrix}, \quad \widetilde{\Psi} = \begin{pmatrix}\mathbf{0} \\ \widetilde{\Psi}_{\rm sm}\end{pmatrix}. \tag{D.31}$$

Using these data, we find that

$$Z^{p-1}\widetilde{\Psi} = \begin{pmatrix}\mathbf{0} \\ Z^{p-1}_{\rm sm}\widetilde{\Psi}_{\rm sm}\end{pmatrix}, \quad \text{with } p = 1, 2, \cdots. \tag{D.32}$$

From this expression, we find that there are nonzero row vectors such that $\vec{v}Z^{p-1}\widetilde{\Psi} = 0$. Therefore, we find that the contrapositive "$\neg\mathcal{C}_{\rm semi}(\xi(z)) \to \neg\mathcal{C}_{\rm free}(Z, \widetilde{\Psi})$" is true and hence the lemma "$\mathcal{C}_{\rm free}(Z, \widetilde{\Psi}) \to \mathcal{C}_{\rm semi}(\xi(z))$" is also true. □

Proof of $\mathcal{C}_{\rm semi}(\xi(z)) \to \mathcal{C}_{\rm free}(Z, \widetilde{\Psi})$. Let us prove the contrapositive $\neg\mathcal{C}_{\rm free}(Z, \widetilde{\Psi}) \to \neg\mathcal{C}_{\rm semi}(\xi(z))$. If $GL(k, \mathbb{C})$ does not freely act on $(Z, \widetilde{\Psi})$, that is, there exists a certain non-zero row vector $\vec{v}$ satisfying

$$\vec{v}Z^{p-1}\widetilde{\Psi} = 0 \quad \text{for} \quad \forall p \in \mathbb{N} \quad (\neg\mathcal{C}_{\rm free}(Z, \widetilde{\Psi}) \text{ is true}), \tag{D.33}$$

$(Z, \Psi, \widetilde{\Psi})$ can be transformed by a $GL(k, \mathbb{C})$ transf into the form

$$\Psi = (\Psi_{\rm lc}, \Psi_{\rm sm}), \quad Z = \begin{pmatrix} Z_{\rm lc} & \mathbf{0} \\ W_{\rm ls} & Z_{\rm sm}\end{pmatrix}, \quad \widetilde{\Psi} = \begin{pmatrix}\mathbf{0} \\ \widetilde{\Psi}_{\rm sm}\end{pmatrix}, \tag{D.34}$$

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where $Z_{\rm sm}, \Psi_{\rm sm}$ and $\widetilde{\Psi}_{\rm sm}$ are k'-by-k' $(k' < k)$, n-by-k' and k'-by-m matrices, respectively. The set of matrices $(Z_{\rm sm}, \Psi_{\rm sm}, \widetilde{\Psi}_{\rm sm})$ can be regarded as the half-ADHM data satisfying $\mathcal{C}_{\rm free}(Z_{\rm sm}, \widetilde{\Psi}_{\rm sm})$ with a smaller vortex number $k'(< k)$ since there is no k'-component column vector $\vec{v}$ such that $\Psi_{\rm sm} Z_{\rm sm}^{p-1} \vec{v} = 0$

$$\Psi_{\rm sm} Z_{\rm sm}^{p-1} \vec{v} = 0 \quad \rightarrow \quad \Psi Z^{p-1} \begin{pmatrix} \mathbf{0} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \Psi_{\rm sm} Z_{\rm sm}^{p-1} \vec{v} \end{pmatrix} = \mathbf{0} \quad \rightarrow \quad \vec{v} = 0. \tag{D.35}$$

According to the lemma "$\mathcal{C}_{\rm free}(Z, \widetilde{\Psi}) \rightarrow \mathcal{C}_{\rm semi}(\xi(z))$" shown above, the condition $\mathcal{C}_{\rm free}(Z_{\rm sm}, \widetilde{\Psi}_{\rm sm})$ immediately indicates that the equivalence class of the corresponding matrix $[\xi_{\rm sm}(z)]$ is an element of $\mathcal{M}_{\rm semi}{}_{k'}^{n,m}$. Furthermore, we can show that for $\xi_{\rm sm}(z) = (\mathfrak{D}_{\rm sm}(z), \widetilde{\mathfrak{D}}_{\rm sm}(z))$ and $\xi = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$ corresponding to $(Z, \Psi, \widetilde{\Psi})$,

$$\mathfrak{D}_{\rm sm}(z)^{-1} \widetilde{\mathfrak{D}}_{\rm sm}(z) = \Psi_{\rm sm}(z \mathbf{1}_{k'} - Z_{\rm sm})^{-1} \widetilde{\Psi}_{\rm sm} = \Psi(z \mathbf{1}_k - Z)^{-1} \widetilde{\Psi} = \mathfrak{D}(z)^{-1} \widetilde{\mathfrak{D}}(z), \tag{D.36}$$

where we have used eqs. (D.20), (D.21) and the relation $\Psi Z^{p-1} \widetilde{\Psi} = \Psi_{\rm sm} Z_{\rm sm}^{p-1} \widetilde{\Psi}_{\rm sm}$ which follows from eq. (D.34). The relation $\mathfrak{D}_{\rm sm}(z)^{-1} \widetilde{\mathfrak{D}}_{\rm sm}(z) = \mathfrak{D}(z)^{-1} \widetilde{\mathfrak{D}}(z)$ implies that $[\xi(z)] \notin \mathcal{M}_{\rm semi}{}_k^{n,m}$. This is because if the opposite is true ($[\xi(z)] \in \mathcal{M}_{\rm semi}{}_k^{n,m}$), Lemma D.4 implies that the relation (D.36) leads to $[\xi(z)] = [\xi_{\rm sm}(z)]$ and $k = k'$, which is inconsistent with $k' < k$. Thus, we find that "$\neg \mathcal{C}_{\rm free}(Z, \widetilde{\Psi}) \rightarrow \neg \mathcal{C}_{\rm semi}(\xi(z))$", and hence the lemma "$\mathcal{C}_{\rm semi}(\xi(z)) \rightarrow \mathcal{C}_{\rm free}(Z, \widetilde{\Psi})$" is shown. □

D.3 Instanton solutions in the Grassmannian sigma model

Any semilocal vortex solution becomes a instanton solution in the sigma model limit $g \rightarrow \infty$. In this subsection, we show that there is actually a one-to-one correspondence between the semilocal vortex and instanton solutions.

Theorem D.2. *Let $\varphi = \varphi(z, \bar{z})$ be an n-by-m matrix valued field (inhomogeneous coordinates of $G(n, n+m)$) on the base space $\mathbb{C}$. If φ satisfies the BPS instanton equation*

$$\partial_{\bar{z}} \varphi(z, \bar{z}) = 0, \qquad \lim_{|z| \rightarrow \infty} \varphi(z, \bar{z}) = 0, \tag{D.37}$$

and each matrix entry of $\varphi(z)$ has a finite number of poles, one can uniquely determine the corresponding equivalent class $[\xi(z)] = [(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))] \in \mathcal{M}_{\rm semi}{}_k^{n,m}$ and the half-ADHM data $[(Z, \Psi, \widetilde{\Psi})]$. Explicitly, the instanton solution $\varphi(z)$ can be always written as

$$\varphi(z) = \mathfrak{D}(z)^{-1} \widetilde{\mathfrak{D}}(z) = \Psi(z \mathbf{1}_k - Z)^{-1} \widetilde{\Psi}. \tag{D.38}$$

Proof. The solution of eq. (D.37) can always be written in the following form

$$\varphi(z) = \sum_\alpha \sum_{p=1}^{k_\alpha} \frac{C_{\alpha,p}}{(z - z_\alpha)^p} \tag{D.39}$$

with n-by-m constant matrices $C_{\alpha,p}$. For this solution, let us consider an n-by-$(n+m)$ matrix $\xi'(z)$ given by

$$\xi'(z) = (p(z) \mathbf{1}_n, p(z) \varphi(z)) \quad \text{with} \quad p(z) = \prod_\alpha (z - z_\alpha)^{k_\alpha}, \quad k' = \sum_\alpha k_\alpha. \tag{D.40}$$

Note that all the entries of a matrix $p(z)\varphi(z)$ are polynomials since all the poles in $\varphi(z)$ are cancelled with zeros of $p(z)$. The equivalence class $[\xi'(z)]$ is an element of $\mathcal{M}_{\text{vtx}}{}^{n,m}_{nk'}$ and hence, according to Lemma D.1, there exists a unique equivalence class $[\xi(z)] = [(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))] \in \mathcal{M}_{\text{semi}}{}^{n,m}_{k}$ with an integer $k \in \mathbb{Z}_{\geq 0}$ such that

$$\xi'(z) = \mathfrak{D}_{\text{lc}}(z)\xi(z) = \mathfrak{D}_{\text{lc}}(z)(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z)), \quad 0 \leq k \leq nk', \tag{D.41}$$

with $\mathfrak{D}_{\text{lc}}(z) \in \mathcal{G}_{n,nk'-k}[z]$ and $\mathfrak{D}(z) \in \mathcal{G}_{n,k}[z]$. These matrices are related as

$$p(z)\mathbf{1}_n = \mathfrak{D}_{\text{lc}}(z)\mathfrak{D}(z), \quad p(z)\varphi(z) = \mathfrak{D}_{\text{lc}}(z)\widetilde{\mathfrak{D}}(z). \tag{D.42}$$

In terms of the matrices $(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$, the instanton solution $\varphi(z)$ can always be rewritten as

$$\varphi(z) = \frac{1}{p(z)}(p(z)\varphi(z)) = (\mathfrak{D}_{\text{lc}}(z)\mathfrak{D}(z))^{-1}\mathfrak{D}_{\text{lc}}(z)\widetilde{\mathfrak{D}}(z) = \mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z). \tag{D.43}$$

Furthermore, using the half-ADHM mapping relation, we can rewrite $\varphi(z)$ in terms of the corresponding half-ADHM data $(Z, \Psi, \widetilde{\Psi})$ as,

$$\varphi(z) = \mathfrak{D}(z)^{-1}\widetilde{\mathfrak{D}}(z) = \mathfrak{D}(z)^{-1}\mathfrak{J}(z)\widetilde{\Psi} = \Psi(z\mathbf{1}_k - Z)^{-1}\widetilde{\Psi}. \tag{D.44}$$

According to Lemma D.4, for a given $\varphi(z)$, the equivalent class $[(\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))] \in \mathcal{M}_{\text{semi}}{}^{n,m}_{k}$ satisfying the above is unique, and the equivalent class of the half-ADHM data $[(Z, \Psi, \widetilde{\Psi})]$ satisfying $\mathcal{C}_{\text{free}}(Z, \widetilde{\Psi})$ is also unique. □

E Embedding of Grassmannian case

In this appendix, we discuss vortices obtained by embedding from the $L = 1$ case.

E.1 Embedding of vortices from $L = 1$ to $L = 2$

Let us consider first consider the embedding of the matrix ξ from $L = 1$ to $L = 2$. For example, in the case with $(k_1, k_2) = (k, 0)$, ξ_i are given by,

$$\xi_1(z) = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z), \mathbf{0}), \qquad \xi_2(z) = \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \mathbf{0} \end{pmatrix}, \tag{E.1}$$

where $\xi = (\mathfrak{D}(z), \widetilde{\mathfrak{D}}(z))$ is the matrix for the $L = 1$ case with $n = N_1 = n_1$ and $N = N_2 = n_1 + n_2$. For the case with $(k_1, k_2) = (0, k)$, one can find that the general solution turns out to be

$$\xi_1(z) = (\mathbf{1}_{n_1}, \mathbf{0}, \mathbf{0}), \qquad \xi_2(z) = \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathfrak{D}(z) & \widetilde{\mathfrak{D}}(z) \end{pmatrix}, \tag{E.2}$$

where $\mathfrak{D}(z)$ and $\widetilde{\mathfrak{D}}(z)$ are those for the $L = 1$ case with $n = n_2$ and $N = n_2 + n_3$. Thus, we find that the moduli spaces of vortices for $(k_1, k_2) = (k_1, 0)$ and $(k_1, k_2) = (0, k_2)$ are identical with those of the $L = 1$ case

$$\mathcal{M}_{\text{vtx}}{}^{n_1,n_2,n_3}_{k_1,k_2=0} \simeq \mathcal{M}_{\text{vtx}}{}^{n_1,n_2}_{k_1}, \qquad \mathcal{M}_{\text{vtx}}{}^{n_1,n_2,n_3}_{k_1=0,k_2} \simeq \mathcal{M}_{\text{vtx}}{}^{n_2,n_3}_{k_2}. \tag{E.3}$$

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For $(k_1,k_2)=(k,k)$, there exist a subspace in the moduli space where ξ of the $L=1$ case can be embedded as

$$\xi_1(z)=(\mathfrak{D}(z),\mathbf{0},\widetilde{\mathfrak{D}}(z)),\quad \xi_2(z)=\begin{pmatrix}\mathfrak{D}(z) & \mathbf{0} & \widetilde{\mathfrak{D}}(z)\\ 0 & \mathbf{1}_{n_2} & 0\end{pmatrix}, \tag{E.4}$$

where $\mathfrak{D}(z)$ and $\widetilde{\mathfrak{D}}(z)$ are those for the $L=1$ case with taking $n=n_1$ and $N=n_1+n_3$. This means that the moduli space of vortices with $(k_1,k_2)=(k,k)$ contains the $L=1$, k-vortex moduli space

$$\mathcal{M}_{\text{vtx}}{}^{n_1,n_2,n_3}_{k_1=k,k_2=k}\supset\mathcal{M}_{\text{vtx}}{}^{n_1,n_3}_{k}. \tag{E.5}$$

Embedding of half-ADHM data. The half-ADHM data can also be obatained by embedding that of the $L=1$ case.

- $\boldsymbol{(k_1,k_2)=(k,0)}$**.** If $(k_1,k_2)=(k,0)$, we can determine $\mathfrak{D}_1$ and $\mathfrak{D}_2$ from the condition $\deg(\det\mathfrak{D}_i)=k_i$

$$\mathfrak{D}_1=z^k+\sum_{n=0}^{k-1}a_nz^n,\qquad \mathfrak{D}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad \text{(up to V-transformations)}. \tag{E.6}$$

From the relations

$$\xi_1=(\mathfrak{D}_1,\widetilde{\mathfrak{D}}_1)=q_1q_2,\qquad \xi_2=(\mathfrak{D}_2,\widetilde{\mathfrak{D}}_2)=q_2,\qquad \mathfrak{D}_i^{-1}\widetilde{\mathfrak{D}}_i=\mathcal{O}(z^{-1}), \tag{E.7}$$

the matrices q_1, q_2, $\widetilde{\mathfrak{D}}_1$ and $\widetilde{\mathfrak{D}}_2$ can be determined as

$$q_1=(P(z),Q(z)),\qquad q_2=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\qquad \widetilde{\mathfrak{D}}_1=(Q(z),0),\qquad \widetilde{\mathfrak{D}}_2=\begin{pmatrix}0\\0\end{pmatrix}, \tag{E.8}$$

where $P(z)$ and $Q(z)$ are polynomials of the forms

$$P(z)=z^k+\sum_{n=0}^{k}a_nz^n,\qquad Q(z)=\sum_{n=0}^{k-1}b_nz^n. \tag{E.9}$$

The k-component row vector $\mathfrak{J}_1(z)$ can be determined from the condition $\mathfrak{D}_1^{-1}\mathfrak{J}_1=\mathcal{O}(z^{-1})$ as

$$\mathfrak{J}_1=(\tilde{P}_{k-1},\tilde{P}_{k-2},\cdots,\tilde{P}_0),\quad \text{with}\quad \tilde{P}_l=\sum_{n=0}^{l}a_{n+k-l}z^n. \tag{E.10}$$

Note that the N_2-by-k_2 matrix $\mathfrak{J}_2$ does not exist since $k_2=0$ in this case. From $\mathfrak{D}_1$, $\widetilde{\mathfrak{D}}_1$ and $\mathfrak{J}_1$, the matrices Z_1, Υ_1 and Υ_2 can be determined as

$$Z_1=\left(\begin{array}{c|ccc}0&1&&\\ \vdots&&\ddots&\\ 0&&&1\\ \hline a_0&a_1&\cdots&a_{k-1}\end{array}\right),\qquad \Upsilon_1=\begin{pmatrix}1&0&\cdots&0\end{pmatrix},\qquad \tilde{\Upsilon}_1=\begin{pmatrix}\tilde{b}_1\\ \tilde{b}_2\\ \vdots\\ \tilde{b}_k\end{pmatrix}, \tag{E.11}$$

where $\tilde{b}_l$ $(l = 1, \cdots, k)$ are constants such that

$$Q(z) = \sum_{l=1}^{k} \tilde{b}_l \tilde{P}_l. \tag{E.12}$$

Note that $(Z_2, \Upsilon_2, \widetilde{\Upsilon}_2)$ and $(W_1 \widetilde{W}_1)$ do not exist since $k_2 = 0$ in this case. All these moduli data are identical to those of U(1) semi-local vortex ($L = 1,\ N_1 = 1,\ N_{\rm F} = 2$) with $q_1 = (P(z), Q(z))$.

- $\boldsymbol{(k_1, k_2) = (0, k)}$. For $(k_1, k_2) = (0, k)$, the matrices q_1 and q_2 are given by

$$q_1 = (1, 0) \qquad q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P(z) & Q(z) \end{pmatrix}, \tag{E.13}$$

where $P(z)$ and $Q(z)$ are polynomials of the same form as the previous case (E.9). The matrices $\xi_i = (\mathfrak{D}_i, \widetilde{\mathfrak{D}}_i)$ take the forms

$$\xi_1 = (1,0,0), \qquad \xi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & P(z) & Q(z) \end{pmatrix}, \quad \left(\mathfrak{D}_1 = 1,\ \widetilde{\mathfrak{D}}_1 = (0,0),\ \mathfrak{D}_2 = \begin{pmatrix} 1 & 0 \\ 0 & P(z) \end{pmatrix},\ \widetilde{\mathfrak{D}}_2 = \begin{pmatrix} 0 \\ Q(z) \end{pmatrix}\right). \tag{E.14}$$

The matrix $\mathfrak{J}_1(z)$ does not exist and $\mathfrak{J}_2$ can be determined as

$$J_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \tilde{P}_{k-1} & \tilde{P}_{k-2} & \cdots & \tilde{P}_0 \end{pmatrix}, \tag{E.15}$$

where $\tilde{P}_l$ $(l = 0, \cdots, k-1)$ are the same polynomials as (E.10). Since $k_1 = 0$, the matrices $(Z_1, \Upsilon_1, \widetilde{\Upsilon}_1)$ and $(W_1, \widetilde{W}_1)$ do not exist and $(Z_2, \Upsilon_2, \widetilde{\Upsilon}_2)$ are given by

$$Z_2 = \left(\begin{array}{c|ccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \hline a_0 & a_1 & \cdots & a_{k-1} \end{array}\right), \qquad \Upsilon_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad \widetilde{\Upsilon}_2 = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_k \end{pmatrix}, \tag{E.16}$$

where $\tilde{b}_l$ $(l = 1, \cdots, k)$ are constants defined in (E.12). Again, the moduli data are identical to those of U(1) semi-local vortex ($L = 1,\ N_1 = 1,\ N_{\rm F} = 2$) with $q_1 = (P(z), Q(z))$.

E.2 Embedding of vortices from $L = 1$ to general L

Let I, J be integers such that $1 \leq I < J \leq L + 1$ and prepare a n_I-by-n_I matrix $\mathfrak{D}(z)$ and a n_I-by-n_J matrix $\widetilde{\mathfrak{D}}(z)$ satisfying $\det \mathfrak{D}(z) = \mathcal{O}(z^k)$ and $\mathfrak{D}(z)^{-1} \widetilde{\mathfrak{D}}(z) = \mathcal{O}(z^{-1})$. Next, let us embed these matrices into q_{J-1} as

$$q_{J-1}(z) = \left(\begin{array}{ccc|c} \mathbf{1}_{N_{I-1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathfrak{D}(z) & \mathbf{0} & \widetilde{\mathfrak{D}}(z) \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{N_{J-1} - N_I} & \mathbf{0} \end{array}\right), \tag{E.17}$$

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and set the other q_i to be trivial $q_i = (\mathbf{1}_{N_i}, \mathbf{0})$ for $i \neq J-1$. This setting gives

$$\xi_i(z) = \begin{array}{c} \begin{array}{cccc} \phantom{\mathbf{1}_{N_{I-1}}\ \mathfrak{D}(z)\ \mathbf{1}_{N_i-N_I}} & \overbrace{}^{N_{J-1}-N_i} & \overbrace{}^{N-N_J} \end{array} \\ \left(\begin{array}{ccc|ccc} \mathbf{1}_{N_{I-1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathfrak{D}(z) & \mathbf{0} & \mathbf{0} & \widetilde{\mathfrak{D}}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{N_i-N_I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right) \end{array} \quad \text{for} \quad i \in [I, J-1], \tag{E.18}$$

and $\xi_i(z) = (\mathbf{1}_{N_i}, \mathbf{0})$ for $i \notin [I, J-1]$. The vortex numbers are given by

$$(k_1, k_2, \cdots, k_L) = (\underbrace{0, \cdots, 0}_{I-1}, \underbrace{k, k, \cdots, k}_{J-I}, 0, \cdots, 0). \tag{E.19}$$

In the case of $d = 1$, this construction gives general configurations with k elementary vortices. In cases with $d \geq 2$, however, the above construction gives special configurations where each of k objects can be regarded as a composite state of $J - I$ types of elementary vortices. This construction also gives the following sigma model instanton solution

$$\varphi_{ij} = \delta_i^I \delta_j^J \mathfrak{D}(z)^{-1} \widetilde{\mathfrak{D}}(z), \tag{E.20}$$

where φ_{ij} are inhomogeneous coordinates of the flag manifold defined in eq. (2.18).

The corresponding half-ADHM data takes the form

$$\Upsilon_i = \mathbf{0} \quad \text{for } i \neq I, \quad \widetilde{\Upsilon}_i = \mathbf{0} \quad \text{for } i \neq J-1, \tag{E.21}$$

$$Z_I = Z_{I+1} = \cdots = Z_{J-1}, \tag{E.22}$$

$$W_I = W_{I+1} = \cdots = W_{J-2} = \mathbf{1}_k \tag{E.23}$$

Here $d = J - I$ implies a level of compression of vortices and turns out to corresponds to height of the Young tableaux.

F Brane construction of vortices

In this appendix, we discuss the D-brane construction of BPS vortices. By embedding our system into a 4d $\mathcal{N} = 2$ supersymmetric gauge theory, we can identify the D-brane configuration corresponding to the BPS vortex configurations. For $L = 1$, the D-brane construction of the vortex moduli space has been mentioned in [7] and for $L > 1$ with $N_1 = \cdots = N_L = N$, the D-brane configuration for the local vortices has been discussed in [98]. The left figure in figure 6 shows the brane configuration for the Coulomb branch of the model. The 4d $\mathcal{N} = 2$ quiver gauge theory corresponding to our system can be realized as the worldvolume effective theory on D4-branes attached to NS5-branes. There are N_i D4-branes between neighboring NS5 branes and they correspond to the $\mathrm{U}(N_i)$ subgroup of the gauge group. The gauge coupling constants $1/g_i^2$ $(i = 1, \cdots, L)$ are proportional to the separations of the NS5 branes $\Delta x^6_{\mathrm{NS5},i} = x^6_{\mathrm{NS5},i+1} - x^6_{\mathrm{NS5},i}$. The bi-fundamental fields Q_i $(i = 1, \cdots, L)$ (hypermultiplets) corresponds to the fundamental strings stretched between D4-branes in the i-th and $i+1$ intervals. In the presence of the FI parameters, which correspond to $\Delta x^{7,8,9}_{\mathrm{NS5},i} = x^{7,8,9}_{\mathrm{NS5},i+1} - x^{7,8,9}_{\mathrm{NS5},i}$ $(i = 1, \cdots, L)$, the vacuum is in the Higgs phase

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	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D4	×	×	×	×			×			
D6	×	×	×	×				×	×	×
NS5	×	×	×	×	×	×				
D2	×	×								×

Table 1. Brane configuration (×'s indicate the directions in which the branes extend).

as shown in the right figure of figure 6. There are $n_i = N_i - N_{i-1}$ D4-branes attached to the i-th NS5-brane. Figure 7 shows an example of the D-brane configurations for BPS vortices. The vortices with i-th magnetic flux correspond to D2-branes stretched between the i-th and $(i+1)$-th D4-branes. The vortex worldsheet theory is a 2d $\mathcal{N} = (2,2)$ quiver gauge theory on the (x^0, x^1) plane in table 1. The matrices $(Z_i, \Upsilon_i, \widetilde{\Upsilon}_i, W, \widetilde{W})$ are identified with the component fields of the chiral multiplets which are identified with the degrees of freedom in the brane configuration as follows

- Z_i : positions of i-th D2-branes on (x^2, x^3) plane.
- Υ_i : F1 strings between i-th D4-branes and i-th D2-branes.
- $\widetilde{\Upsilon}_i$: F1 strings between $(i+1)$-th D4-branes and i-th D2-branes.
- $W, \widetilde{W}$: F1 strings between i-th and $(i+1)$-th D2-branes.

The moduli space of BPS vortices are identified with that of vacua of this quiver gauge theory determined by solving the D-term condition and the F-term constraint coming from the cubic superpotential

$$\mathcal{W} = \sum_{i=1}^{L-1} \mathrm{Tr}\left[\widetilde{W}_i \left(\widetilde{\Upsilon}_i \Upsilon_{i+1} - Z_i W_i + W_i Z_{i+1}\right)\right]. \tag{F.1}$$

If we turn on hypermultiplets masses, which correspond to the positions of D6-branes on the (x^4, x^5) plane, only the fixed points of the SU(N) flavor symmetry are left as stable BPS configurations. Figure 8 shows an example of the D-brane configurations for such fixed point configurations. In the presence of the Ω-deformation on the (x^0, x^1) plane, all D2-branes are localized at the origin and they form clusters which are characterized by Young tableaux. Such configurations are the fixed points of the torus action, which are relevant to the supersymemtric localization.

G Smoothness of the moduli space

The moduli space of vortices for $L > 1$ is constructed from the space of matrices satisfying the constaraitns (4.39). In this appendix, we show that those constraints do not cause any singularities on the moduli space.

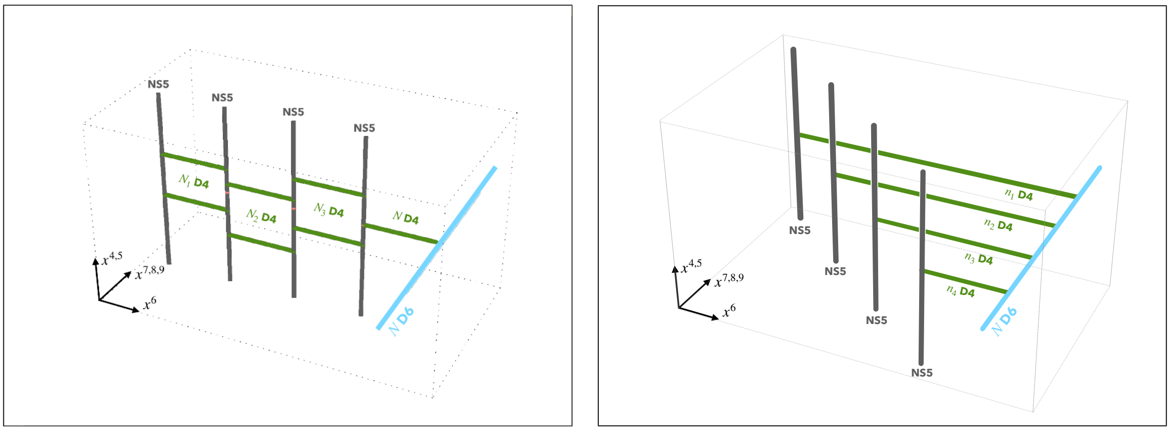

Figure 6. D-brane configurations for the Coulomb branch (left) and the Higgs phase in the presence of FI parameters (right).

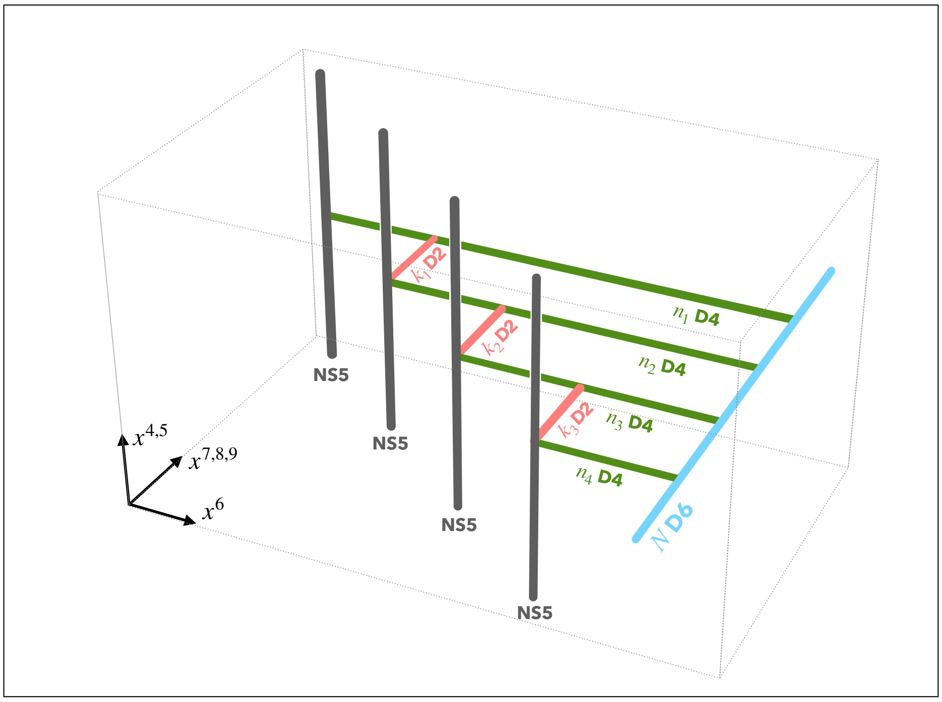

Figure 7. D-brane configuration for BPS vortices. This example shows (k_1, k_2, k_3) vortices in 4d $\mathcal{N} = 2$ $\mathrm{U}(n_1) \times \mathrm{U}(n_1+n_2) \times \mathrm{U}(n_1+n_2+n_3)$ gauge theory with $N_\mathrm{F} = n_1+n_2+n_3+n_4$ hypermultiplets.

G.1 Singular points on algebraic varieties

Let us first recall that a singularity on an algebraic variety is a point where a tangent space is ill-defined. For example, for a subspace $\mathcal{M}_0$ in $\mathbb{C}^K = \{\phi_1, \cdots, \phi_K\}$ defined as the intersection of the zero loci of polynomials $F_I(\phi_1, \cdots, \phi_K)$ $(I = 1, 2, \cdots, n')$, a singularity on $\mathcal{M}_0$ is defined as a point where the rank of the matrix $(J_F)_I{}^i \equiv \partial F_I / \partial \phi_i$ decreases. Let H_{ij} be

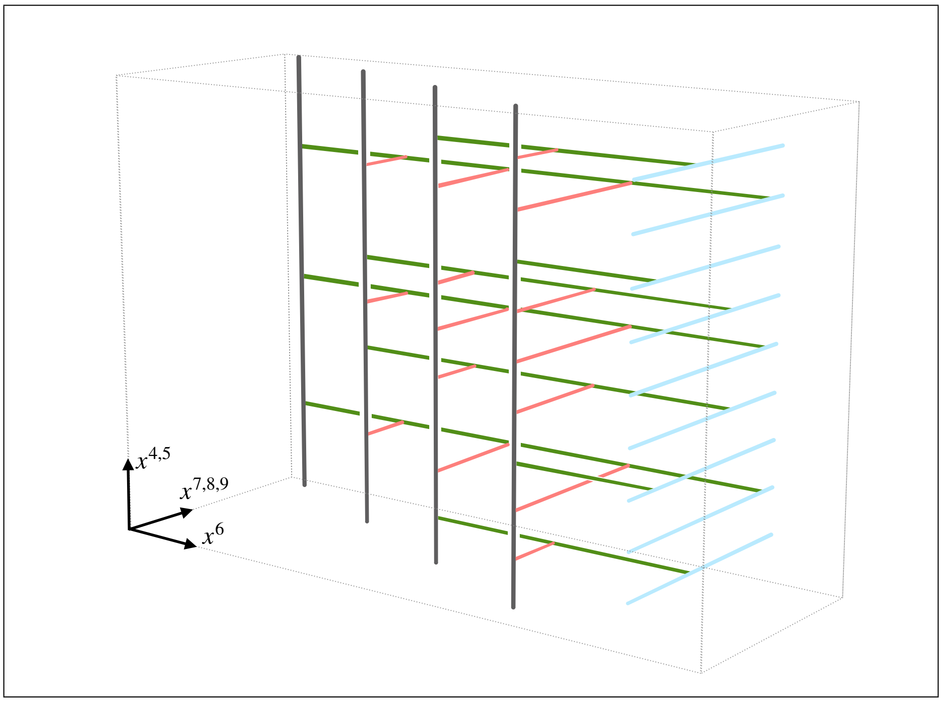

Figure 8. D-brane configuration for BPS vortices in the massive theory. The positions of D6-branes in the x_4 and x_5 directions correspond to hypermultiplet masses. In the presence of the Ω-background, each cluster of D2-branes (pink line) corresponds to a composite of vortices characterized by a Young tableau.

the hessian of the function V defined as

$$V = \sum_I |F_I|^2, \qquad H_{ij} = \frac{\partial^2 V}{\partial \phi_i \partial \bar{\phi}_j} = \sum_I (J_F)_I^i (J_F^*)_I^j. \tag{G.1}$$

Then, at the singular point, an extra flat directions (zero eigenvectors) appears since H_{ij} has a lower rank. If there is a symmetry group G that preserves V, the equation $F = 0$ reduces to a constraint equation $\tilde{F} = 0$ that defines a subspace $\mathcal{M}$ in the quotient space $(\mathbb{C}^K - \{0\})/G$. Since $\operatorname{rank} J_{\tilde{F}} = \operatorname{rank} J_F$ for any smooth quotient space, singularities of $\mathcal{M}$ can be determined by looking at the rank of J_F on $\mathcal{M}_0$. In particular, $\mathcal{M}$ is smooth if the rank of J_F is constant everywhere. If J_F has the maximal rank n' everywhere, that is,

$$0 = \Lambda^I (J_F)_I{}^i = \frac{\partial \Lambda^I F_I}{\partial \phi^i} \quad \text{implies} \quad \Lambda^I = 0 \quad \text{for all points on } \mathcal{M}_0, \tag{G.2}$$

all the constraints $F_I = 0$ are linearly independent and hence $\mathcal{M}$ is a $(K - \dim G - n')$-dimensional smooth manifold.

G.2 Constraints and smoothness of vortex moduli space

The constraints in eq. (4.39) implies that the vortex moduli space for $L > 1$ is the intersection of the zero loci of $F_I = \widetilde{\Upsilon}_i \Upsilon_{i+1} - Z_i W_i + W_i Z_{i+1}$. These constraints can be introduced by turning on the potential

$$\mathcal{V} = \sum_{i=1}^{L-1} \mathrm{Tr}\left[|\widetilde{W}_i|^2 + i\widetilde{W}_i\left(\widetilde{\Upsilon}_i\Upsilon_{i+1} - Z_iW_i + W_iZ_{i+1}\right) + (c.c.)\right], \tag{G.3}$$

where the k_{i+1}-by-k_i matrix $\widetilde{W}_i$ are an auxiliary fields which give the on-shell potential

$$V = \sum_{i=1}^{L-1} \mathrm{Tr}\left|\widetilde{W}_i\right|^2 = \sum_{i=1}^{L-1} \mathrm{Tr}\left|\widetilde{\Upsilon}_i\Upsilon_{i+1} - Z_iW_i + W_iZ_{i+1}\right|^2. \tag{G.4}$$

The variations with respect to the other degrees of freedom give

$$0 = \frac{\partial\mathcal{V}}{\partial\Upsilon_{i+1}} = \widetilde{W}_i\widetilde{\Upsilon}_i, \tag{G.5}$$

$$0 = \frac{\partial\mathcal{V}}{\partial\widetilde{\Upsilon}_i} = \Upsilon_{i+1}\widetilde{W}_i, \tag{G.6}$$

$$0 = \frac{\partial\mathcal{V}}{\partial W_i} = Z_{i+1}\widetilde{W}_i - \widetilde{W}_iZ_i, \tag{G.7}$$

for $1 \le i \le L-1$ and

$$0 = \frac{\partial\mathcal{V}}{\partial Z_i} = \widetilde{W}_{i-1}W_{i-1} - W_i\widetilde{W}_i \quad \text{with} \quad W_{0,L} = \widetilde{W}_{0,L} = 0, \tag{G.8}$$

for $1 \le i \le L$. As we have seen in eq. (G.2), the moduli space has no singularity if and only if $\widetilde{W}_i$ always vanishes when eqs. (G.5)–(G.8) are satisfied. For any solution, we can show

$$\begin{aligned}\left(\Upsilon_jW_jW_{j+1}\cdots W_iZ_{i+1}^{p-1}\right)\widetilde{W}_i &\overset{\text{eq.(G.7)}}{=} \Upsilon_jW_jW_{j+1}\cdots W_i\widetilde{W}_iZ_i^{p-1}\\ &\overset{\text{eq.(G.8)}}{=} \Upsilon_j\widetilde{W}_{j-1}W_{j-1}W_j\cdots W_{i-1}Z_i^{p-1}\\ &\overset{\text{eq.(G.6)}}{=} 0,\end{aligned} \tag{G.9}$$

for $1 \le j \le i$ and $1 \le p \le k$. Under the $\prod_{i=1}^{L}\mathrm{GL}(k_i,\mathbb{C})$ free condition (4.41), this equation implies that $\widetilde{W}_i = 0$ for all i. Therefore, the vortex moduli space is smooth and all the elements of the constraint (4.39) are independent, that is, the number of degrees of freedom suppressed by the constraints is the same as that of the components of $\{\widetilde{W}_i\}$.

H The torus action on the moduli spaces and on the Kähler quotient

In this appendix, we summarize the BPS vortex solutions in the presence of the omega background and the mass deformation. In such a case, BPS configurations have to minimize the deformation terms induced by the omega background ϵ and mass parameters $M = \mathrm{diag}(m^1,\cdots,m^N)$

$$\delta\mathcal{L} \;=\; \sum_{i=1}^{L}\left|i\epsilon(z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}})q_i + \Sigma_iq_i - q_i\Sigma_{i+1}\right|^2 \quad \text{with} \quad \Sigma_{L+1} = -M, \tag{H.1}$$

where Σ_i $(i = 1, \cdots, L)$ are $\mathrm{SU}(N_i)$ adjoint scalar fields.[29] Since δL is positive semi-definite, it is minimized when $\delta\mathcal{L} = 0$, that is

$$i\epsilon(z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}})q_i + \Sigma_i q_i - q_i\Sigma_{i+1} = 0, \qquad (i = 1, \cdots, L,\ \Sigma_{L+1} = -M). \tag{H.2}$$

This condition implies that the vortex configuration must be invariant under the (infinitesimal) spatial rotation and the flavor rotation up to gauge transformations Σ_i. Such fixed points are classified by a set of N Young tableaux $Y^{(j,\alpha)}$ where $\alpha = 1, \cdots, n_i$ for each $j = 1, \cdots, L$. The height of $Y^{(j,\alpha)}$ is $L - j + 1$ and we denote the length of i-th row as $l^{(j,\alpha)}_{i+j-1}$, i.e.

$$Y^{(j,\alpha)} = \left(l^{(j,\alpha)}_j, l^{(j,\alpha)}_{j+1}, \cdots, l^{(j,\alpha)}_L\right), \qquad l^{(j,\alpha)}_j > l^{(j,\alpha)}_{j+1} > \cdots > l^{(j,\alpha)}_L > 0. \tag{H.3}$$

The integers $l^{(j,\alpha)}_i$ are related to the magnetic flux at the fixed point

$$\frac{1}{2\pi}\int F_i \;=\; \text{block-diag}(\boldsymbol{l}^1_i, \cdots, \boldsymbol{l}^i_i) \qquad \text{with} \qquad \boldsymbol{l}^j_i = \text{diag}\left(l^{(j,1)}_i, \cdots, l^{(j,n_j)}_i\right), \tag{H.4}$$

where $\boldsymbol{l}^j_i$ is the n_j-by-n_j diagonal block of the $\mathrm{SU}(N_i)$ magnetic flux of the i-th gauge group. They are also related to the winding numbers of the scalar fields

$$q_i = \left(\begin{array}{ccc|ccc} \boldsymbol{q}^1_i & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \boldsymbol{q}^i_i & 0 & \cdots & 0 \end{array}\right), \qquad \text{with} \qquad \boldsymbol{q}^j_i = \boldsymbol{f}^j_i(r)\exp\left(i\boldsymbol{\nu}^j_i\theta\right), \qquad \boldsymbol{\nu}^j_i \equiv \boldsymbol{l}^j_i - \boldsymbol{l}^j_{i+1}, \tag{H.5}$$

where $\boldsymbol{f}^j_i(r)$ and $\boldsymbol{\nu}^j_i$ are diagonal matrices of profile functions and winding numbers, respectively. We can confirm that $q_i(z)$ is invariant under the torus action (the combination of the spatial rotation and the Cartan part of the flavor rotation) up to V-transformations

$$q_i(z) = V_i\, q_i(e^{i\epsilon}z)\, V^{-1}_{i+1}, \qquad V_i = \exp(i\Sigma_i), \qquad V_{L+1}(z) = \exp(-iM). \tag{H.6}$$

Note that the left hand side of the fixed point condition (H.2) is the infinitesimal version of this transformation. The element of the V-transformations are specified by the fixed point values of the adjoint scalar Σ_i, which take the forms

$$\Sigma_i = \text{block-diag}\,(\boldsymbol{\sigma}^1_i, \cdots, \boldsymbol{\sigma}^i_i) \qquad \boldsymbol{\sigma}^j_i = \text{diag}\,(\sigma^{(j,1)}_i, \cdots, \sigma^{(j,n_j)}_i), \tag{H.7}$$

with the eigenvalues

$$\sigma^{(j,\alpha)}_i = -m^{(j,\alpha)} - l^{(j,\alpha)}_i\epsilon, \tag{H.8}$$

where we have labeled the eigenvalues of the mass matrix as

$$M = \text{block-diag}\,(\boldsymbol{m}^1, \cdots, \boldsymbol{m}^{L+1}), \qquad \boldsymbol{m}^j = \text{diag}\,(m^{(j,1)}, \cdots m^{(j,n_i)}). \tag{H.9}$$

[29] In 2d $\mathcal{N} = (2,2)$ models, Σ_i can be interpreted as the adjoint scalar fileds in the vector multiplets and become auxiliary fields in the nonlinear sigma model limit.

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H.1 Half-ADHM data at fixed points

We can show that the vortex data corresponding to the fixed point specified by the Young tableaux $Y^{(j,\alpha)}$ take the form

$$\mathfrak{D}_i = \text{block-diag}\,(\boldsymbol{\mathfrak{D}}_i^1, \cdots, \boldsymbol{\mathfrak{D}}_i^i) \qquad \text{with} \qquad \boldsymbol{\mathfrak{D}}_i^j = \text{diag}\,(z^{l_i^{(j,1)}}, \cdots, z^{l_i^{(j,n_j)}}) \qquad \text{and} \qquad \widetilde{\mathfrak{D}}_i = 0. \tag{H.10}$$

This implies that each diagonal component represents axially symmetric Abelian vortices with flux $l_i^{(j,\alpha)}$ and hence all the matrix data can be obtained by embedding those of Abelian vortices. For an axially symmetric Abelian vortex configuration $\mathfrak{D} = z^l$, the vortex data satisfying $\mathfrak{D}\Psi = \mathfrak{J}(z\mathbf{1}_l - Z)$ are given by (see section C.1)

$$\mathfrak{J}(l) = (z^{l-1}\,,\, z^{l-2}\,,\, \cdots\,,\, 1), \qquad \Psi(l) = (1\,,\, 0\,,\, \cdots\,,\, 0), \qquad Z(l) = \left.\left(\begin{array}{c|ccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \hline 0 & 0 & \cdots & 0 \end{array}\right)\right\} l. \tag{H.11}$$

By embedding these matrices, we can construct the matrices satisfying $\mathfrak{D}_i\Psi_i = \mathfrak{J}_i(z\mathbf{1}_{k_i} - Z_i)$ as

$$\mathfrak{J}_i = \text{block-diag}\,(\boldsymbol{\mathfrak{J}}_i^1, \cdots, \boldsymbol{\mathfrak{J}}_i^i), \;\; \Psi_i = \text{block-diag}\,(\boldsymbol{\Psi}_i^1, \cdots, \boldsymbol{\Psi}_i^i), \;\; Z_i = \text{block-diag}\,(\boldsymbol{Z}_i^1, \cdots, \boldsymbol{Z}_i^i), \tag{H.12}$$

with

$$\boldsymbol{\mathfrak{J}}_i^j = \text{diag}(\mathfrak{J}(l_i^{(j,1)}), \cdots, \mathfrak{J}(l_i^{(j,n_j)})), \;\; \boldsymbol{\Psi}_i^j = \text{diag}(\Psi(l_i^{(j,1)}), \cdots, \Psi(l_i^{(j,n_j)})), \;\; \boldsymbol{Z}_i^j = \text{diag}(Z(l_i^{(j,1)}), \cdots, Z(l_i^{(j,n_j)})). \tag{H.13}$$

Note that $\widetilde{\Psi}_i = 0$ since $\widetilde{\mathfrak{D}}_i = 0$ for the fixed point configurations. The matrices Υ_i and $\widetilde{\Upsilon}_i$ defined in (4.22) can be extracted from Ψ_i and $\widetilde{\Psi}_i$ as

$$\Upsilon_i = \left(\mathbf{0}_{n_i,k_{i-1}}\, \boldsymbol{\Psi}_i^i \middle| \right), \qquad \widetilde{\Upsilon}_i = 0. \tag{H.14}$$

The matrix W_i can be determined by solving the constraint $Z_iW_i - W_iZ_{i+1} = \widetilde{\Upsilon}_i\Upsilon_{i+1}$ as

$$W_i = \left(\begin{array}{ccc|ccc} \boldsymbol{W}_i^1 & & & \mathbf{0} & \cdots & \mathbf{0} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & \boldsymbol{W}_i^i & \mathbf{0} & \cdots & \mathbf{0} \end{array}\right), \qquad \boldsymbol{W}_i^j = \begin{pmatrix} W(l_i^{(j,1)}, l_{i+1}^{(j,1)}) & & \\ & \ddots & \\ & & W(l_i^{(j,n_j)}, l_{i+1}^{(j,n_j)}) \end{pmatrix} \tag{H.15}$$

where $W(l, l')$ is the matrix satisfying $Z(l)W(l, l') - W(l, l')Z(l') = 0$, which takes the form

$$W(l, l') = \begin{pmatrix} \mathbf{1}_{l'} \\ \mathbf{0}_{l-l',l'} \end{pmatrix}. \tag{H.16}$$

Note that $\widetilde{W}_i = 0$ as shown above.

H.2 Torus action on half-ADHM data

The above set of matrices $\{Z_i, \Upsilon_i, \tilde{\Upsilon}_i, W_i, \widetilde{W}_i\}$ corresponds to the BPS configuration in the presence of the deformations. This satisfies the fixed point condition of the torus action

$$\{Z_i, \Upsilon_i, \widetilde{\Upsilon}_i, W_i, \widetilde{W}_i\} \to \{e^{-i\Phi_i - i\epsilon}Z_ie^{i\Phi_i}, e^{-iY_i}\Upsilon_ie^{i\Phi_i}, e^{-i\Phi_i - i\epsilon}\widetilde{\Upsilon}_ie^{iY_{i+1}}, e^{-i\Phi_i}W_ie^{i\Phi_{i+1}}, e^{-i\Phi_{i+1}}\widetilde{W}_ie^{i\Phi_i + i\epsilon}\}, \tag{H.17}$$

where Φ_i are the elements of $\mathfrak{gl}(k_i)$ given by

$$\Phi_i = \text{block-diag}(\boldsymbol{\Phi}_i^1, \cdots, \boldsymbol{\Phi}_i^i), \;\; \boldsymbol{\Phi}_i^j = \text{block-diag}(\Phi_i^{(j,1)}, \cdots, \Phi_i^{(j,n_j)}), \;\; \Phi_i^{(j,\alpha)} = \text{diag}(\phi_i^{(j,\alpha,1)}, \cdots, \phi_i^{(j,\alpha,p)}), \tag{H.18}$$

with the eigenvalues[30]

$$\phi_i^{(j,\alpha,p)} = m^{(j,\alpha)} + (p-1)\epsilon. \tag{H.19}$$

These eigenvalues at the fixed point correspond to the poles of the integrand for the vortex partition function (6.11), whose residue give the contribution of the fixed point configuration. The fixed point condition can also be rewritten by using the infinitesimal form of the torus action as

$$[\Phi_i, Z_i] - \epsilon Z_i = 0, \qquad M_i \Upsilon_i - \Upsilon_i \Phi_i = 0, \qquad \Phi_i \widetilde{\Upsilon}_i - \widetilde{\Upsilon}_i M_{i+1} + \epsilon \widetilde{\Upsilon}_i = 0, \tag{H.20}$$

$$\Phi_i W_i - W_i \Phi_{i+1} = 0, \qquad \Phi_{i+1} \widetilde{W}_i - \widetilde{W}_i \Phi_i + \epsilon \widetilde{W}_i = 0. \tag{H.21}$$

One can explicitly check that the torus action on the half-ADHM data is consistent with that on $(\mathfrak{D}_i(z), \widetilde{\mathfrak{D}}_i(z))$ as follows. With $\hat{M}_j$ and $\widetilde{M}_j$ defined by

$$\hat{M}_j = \text{diag}(\boldsymbol{m}_1, \boldsymbol{m}_2, \cdots, \boldsymbol{m}_j), \qquad \widetilde{M}_j = \text{diag}(\boldsymbol{m}_{j+1}, \cdots, \boldsymbol{m}_L, \boldsymbol{m}_{L+1}), \tag{H.22}$$

the torus action on $(\mathfrak{D}_j(z), \widetilde{\mathfrak{D}}_j(z))$ can be read off from that on q_i as

$$(\mathfrak{D}_j(z), \widetilde{\mathfrak{D}}_j(z)) \quad \to \quad (\mathfrak{D}'_j(z), \widetilde{\mathfrak{D}}'_j(z)) = V_j(z) \left(\mathfrak{D}_j(e^{i\epsilon} z) e^{i\hat{M}_j}, \widetilde{\mathfrak{D}}_j(e^{i\epsilon} z) e^{i\widetilde{M}_j} \right). \tag{H.23}$$

Since $\mathfrak{J}'_j(z)$ must satsify

$$\mathcal{O}(z^{-1}) = \mathfrak{D}'_j(z)^{-1} \mathfrak{J}'_j(z) = e^{-i\hat{M}_j} \mathfrak{D}_j(e^{i\epsilon} z)^{-1} (V_j(z)^{-1} \mathfrak{J}'_j(z)), \tag{H.24}$$

we find that $\mathfrak{J}'_j(z)$ is given by

$$\mathfrak{J}'_j(z) = V_j(z) \mathfrak{J}_j(e^{i\epsilon} z) e^{i\Phi'_j}, \qquad \left(\because \mathfrak{D}_j(z)^{-1} \mathfrak{J}_j(z) = \mathcal{O}(z^{-1}) \right), \tag{H.25}$$

where $\Phi'_j \in \mathfrak{gl}(k_j, \mathbb{C})$ is a certain constant square matrix. Since the torus action on $(Z_j, \Psi_j, \widetilde{\Psi}_j) \to (Z'_j, \Psi'_j, \widetilde{\Psi}'_j)$ must be consistent with the half-ADHM mapping relation

$$\mathfrak{D}'_j(z) \Psi'_j = \mathfrak{J}'_j(z)(z\mathbf{1} - Z'_j), \quad \widetilde{\mathfrak{D}}'_j(z) = \mathfrak{J}'_j(z) \widetilde{\Psi}'_j, \tag{H.26}$$

it follows that

$$\mathfrak{D}_j(e^{i\epsilon} z) e^{i\hat{M}_j} \Psi'_j e^{-i\Phi'_j} = \mathfrak{J}_j(e^{i\epsilon} z)(z\mathbf{1} - e^{i\Phi'_j} Z'_j e^{-i\Phi'_j}), \qquad \widetilde{\mathfrak{D}}_j(e^{i\epsilon} z) = \mathfrak{J}_j(e^{i\epsilon} z) e^{i\Phi'_j} \widetilde{\Psi}'_j e^{-i\widetilde{M}_j}. \tag{H.27}$$

[30]The matrices $\Phi_i^{(j,\alpha)}$ can be determined by solving the equations

$$[\Phi_i^{(j,\alpha)}, Z_i^{(j,\alpha)}] + \epsilon Z_i^{(j,\alpha)} = 0, \qquad m^{(i,\alpha)} \Psi_i^{(i,\alpha)} - \Psi_i^{(i,\alpha)} \Phi_i^{(i,\alpha)} = 0, \qquad \Phi_i^{(j,\alpha)} - \Phi_{i+1}^{(j,\alpha)} = 0 \quad (\text{for } j = 1, \cdots, i).$$

Comparing with the original half-ADHM mapping relation $\mathfrak{D}_j(z)\Psi_j = \mathfrak{J}_j(z)(z\mathbf{1} - Z_j)$ and $\widetilde{\mathfrak{D}}_j(z) = \mathfrak{J}_j(z)\widetilde{\Psi}_j$, we obtain the torus action on $\{Z_j,\, \Psi_j, \widetilde{\Psi}_j\}$ as

$$(Z_j,\, \Psi_j, \widetilde{\Psi}_j) \quad \to \quad (Z'_j,\, \Psi'_j, \widetilde{\Psi}'_j) = \{e^{-i\Phi_j - i\epsilon} Z e^{i\Phi_j},\, e^{-i\hat{M}_j}\Psi_j e^{i\Phi_j},\, e^{-i\Phi_j - i\epsilon}\widetilde{\Psi}_j e^{i\widetilde{M}_j}\}, \quad \text{(H.28)}$$

where we have defined $\Phi_j = \Phi'_j - \epsilon\mathbf{1}$. The torus action on $(\Upsilon_i,\, \widetilde{\Upsilon}_i)$ can be read off from that on $(\Psi_j, \widetilde{\Psi}_j)$

$$(\Upsilon_i,\, \widetilde{\Upsilon}_i) \quad \to \quad (Z'_i,\, \Upsilon'_i,\, \widetilde{\Upsilon}'_i) = (e^{-i\Phi_i - i\epsilon} Z_i e^{i\Phi_i},\, e^{-iM_i}\Upsilon_i e^{i\Phi_i},\, e^{-i\Phi_i - i\epsilon}\widetilde{\Upsilon}_i e^{iM_{i+1}}). \quad \text{(H.29)}$$

The torus action on $(W_i, \widetilde{W}_i)$ can be obtained from $q'_i(z)\mathfrak{J}'_{i+1}(z) = \mathfrak{J}'_i(z)W'_i$, which can be rewritten as

$$q_i(e^{i\epsilon}z)\mathfrak{J}_{i+1}(e^{i\epsilon}z)e^{i\Phi_{i+1}+i\epsilon} = \mathfrak{J}_i(e^{i\epsilon}z)e^{i\Phi_i + i\epsilon}W'_i. \quad \text{(H.30)}$$

Comparing with $q_i(z)\mathfrak{J}_{i+1}(z) = \mathfrak{J}_i(z)W_i$, we find that

$$(W_i,\, \widetilde{W}_i) \quad \to \quad (W'_i,\, \widetilde{W}'_i) = (e^{-i\Phi_i}W_i e^{i\Phi_{i+1}},\, e^{-i\Phi_{i+1}}\widetilde{W}_i e^{i\Phi_i + i\epsilon}), \quad \text{(H.31)}$$

where we have determined the torus action on $\widetilde{W}_i$ so that $\mathcal{W}$ in eq. (G.3) is invariant.

H.3 Fluctuation around the fixed points

Next, let us consider the fluctuation around the fixed point configuration discussed in the previous subsection. Let us label the fluctuations of q_i around the fixed point as

$$\delta q_i = \left(\begin{array}{ccc|c} \delta\boldsymbol{q}_i^{11} & \cdots & \delta\boldsymbol{q}_i^{1j} & \delta\boldsymbol{q}_i^{1,i+1} \\ \vdots & \ddots & \vdots & \vdots \\ \delta\boldsymbol{q}_i^{i1} & \cdots & \delta\boldsymbol{q}_i^{ii} & \delta\boldsymbol{q}_i^{i,i+1} \end{array}\right) \qquad \text{with} \qquad \delta\boldsymbol{q}_i^{jk} = \begin{pmatrix} \delta q_i^{(j,1),(k,1)} & \cdots & \delta q_i^{(j,1),(k,n_k)} \\ \vdots & \ddots & \vdots \\ \delta q_i^{(j,n_j),(k,1)} & \cdots & \delta q_i^{(j,n_j),(k,n_k)} \end{pmatrix}, \quad \text{(H.32)}$$

where $\delta q_i^{(j,\alpha),(k,\beta)}$ are polynomials of z. Similarly, we label the fluctuations of ξ_i as

$$\delta\xi_i = \left(\begin{array}{ccc|ccc} \delta\boldsymbol{\xi}_i^{11} & \cdots & \delta\boldsymbol{\xi}_i^{1j} & \delta\boldsymbol{\xi}_i^{1,i+1} & \cdots & \delta\boldsymbol{\xi}_i^{1,L+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \delta\boldsymbol{\xi}_i^{i1} & \cdots & \delta\boldsymbol{\xi}_i^{ii} & \delta\boldsymbol{\xi}_i^{i,i+1} & \cdots & \delta\boldsymbol{\xi}_i^{i,L+1} \end{array}\right) \quad \text{with} \quad \delta\boldsymbol{\xi}_i^{jk} = \begin{pmatrix} \delta\xi_i^{(j,1),(k,1)} & \cdots & \delta\xi_i^{(j,1),(k,n_k)} \\ \vdots & \ddots & \vdots \\ \delta\xi_i^{(j,n_j),(k,1)} & \cdots & \delta\xi_i^{(j,n_j),(k,n_k)} \end{pmatrix}, \quad \text{(H.33)}$$

where $\delta\xi_i^{(j,\alpha),(k,\beta)}$ are polynomials of z, which we denote

$$\delta\xi_i^{(j,\alpha),(k,\beta)} = \begin{cases} \delta\mathfrak{D}_i^{(j,\alpha),(k,\beta)} & \text{for } k \le i \\ \delta\widetilde{\mathfrak{D}}_i^{(j,\alpha),(k,\beta)} & \text{for } k \ge i+1 \end{cases}. \quad \text{(H.34)}$$

For a fixed point spacified by the Young tableaux $Y^{(j,\alpha)} = (l_j^{(j,\alpha)}, l_{j+1}^{(j,\alpha)}, \cdots, l_L^{(j,\alpha)})$, $\delta\mathfrak{D}_i^{(j,\alpha),(k,\beta)}$ and $\widetilde{\delta\mathfrak{D}}_i^{(j,\alpha),(k,\beta)}$ are polynomial of degree $l_i^{(k,\beta)} - 1$ and $l_i^{(j,\alpha)} - 1$, respectively.

Since $\xi_i = q_i\xi_{i+1}$, the fluctuations of ξ_i must satisfy the recursive relations

$$\delta\xi_i^{(j,\alpha),(k,\beta)} = z^{l_i^{(j,\alpha)} - l_{i+1}^{(j,\alpha)}}\delta\xi_{i+1}^{(j,\alpha),(k,\beta)} + \delta q_i^{(j,\alpha),(k,\beta)} z^{l_{i+1}^{(k,\beta)}}. \quad \text{(H.35)}$$

This condition gives a constraint to the fluctuations δq_i. To find such constraints, let us write

$$\delta\xi_i^{(j,\alpha),(k,\beta)} = \sum_n c_i^{(j,\alpha),(k,\beta),n} z^n, \tag{H.36}$$

$$\delta q_i^{(j,\alpha),(k,\beta)} = \sum_n a_i^{(j,\alpha),(k,\beta),n} z^n. \tag{H.37}$$

Then the recursive relation (H.35) can be written as

$$c_i^{(j,\alpha),(k,\beta),n} = c_{i+1}^{(j,\alpha),(k,\beta),n-l_i^{(j,\alpha)}+l_{i+1}^{(j,\alpha)}} + a_i^{(j,\alpha),(k,\beta),n-l_{i+1}^{(k,\beta)}}. \tag{H.38}$$

Solving these equation, we find that

$$c_i^{(j,\alpha),(k,\beta),n} = 0, \qquad (\text{for } k \le i \text{ and } n \ge l_i^{(k,\beta)}) \tag{H.39}$$

$$a_i^{(j,\alpha),(k,\beta),n} = 0, \qquad (\text{for } k \ge i+2, \text{ and } n \ge l_i^{(j,\alpha)} - l_{i+1}^{(k,\beta)}) \tag{H.40}$$

$$a_l^{(j,\alpha),(k,\beta),n} = -c_{l+1}^{(j,\alpha),(k,\beta),n+l_{l+1}^{(k,\beta)}-l_l^{(j,\alpha)}+l_{l+1}^{(j,\alpha)}}, \qquad (\text{for } l_l^{(j,\alpha)} - l_{l+1}^{(k,\beta)} \le n < l_l^{(k,\beta)} - l_{l-1}^{(j,\alpha)} + l_l^{(j,\alpha)} - l_{l+1}^{(k,\beta)}). \tag{H.41}$$

The coefficients $a_i^{(j,\alpha),(k,\beta),p}$ satisfying these conditions can be regarded as the coordinates around the fixed point. We can check that the number of the degrees of freedom agrees with the dimension of the moduli space. They transform under the torus action as

$$\begin{aligned} a_i^{(j,\alpha),(k,\beta),p} &\to a_i^{(j,\alpha),(k,\beta),p} \exp i\left[\sigma_i^{(j,\alpha)} - \sigma_{i+1}^{(k,\beta)} + p\epsilon\right] \\ &= a_i^{(j,\alpha),(k,\beta),p} \exp i\left[m^{(j,\alpha)} - m^{(k,\beta)} - (l_i^{(j,\alpha)} - l_i^{(k,\beta)} + p)\epsilon\right]. \end{aligned} \tag{H.42}$$

It is worth noting that the vortex partition function (6.26) can be obtained from these transformation properties. Having solved the constraints for the fluctuations of $(\mathfrak{D}, \widetilde{\mathfrak{D}})$, we can determine those of the half ADHM data satisfying the constraints through the linearized version of the half-ADHM mapping relation.

I Vortex partition function

In this appendix, we derive the integration formula for the vortex partition function (6.11). The vortex partition function is given by the determinant of the torus action on the moduli space (6.10), which can also be obtained from the torus action on the fluctuation (H.42) or that on the half-ADHM matrices.

Let us first consider the character of the torus action on the fluctuations around the fixed point specified by each Young tableaux (H.3). The contributions of $(\Upsilon_i, \widetilde{\Upsilon}_i, Z_i, W_i)$ to the character can be read off from the torus action (H.17) as

$$\chi(\delta\Upsilon_i) = \sum_{\alpha\in\lambda_i} \sum_{j=1}^{i} \sum_{\beta=1}^{n_j} \sum_{p=1}^{l_i^{(j,\beta)}} \exp\left[im^{(j,\beta)} + i(p-1)\epsilon - im^{(i,\alpha)}\right],$$

$$\chi(\delta\widetilde{\Upsilon}_i) = \sum_{j=1}^{i} \sum_{\alpha=1}^{n_i} \sum_{p=1}^{l_i^{(j,\alpha)}} \sum_{\beta=1}^{n_{i+1}} \exp\left[im^{(i+1,\beta)} - im^{(j,\alpha)} - ip\epsilon\right],$$

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$$\chi(\delta Z_i) = \sum_{j=1}^{i} \sum_{\alpha\in\bar{\lambda}_i} \sum_{k=1}^{i} \sum_{b=1}^{n_k} \sum_{q=1}^{l_i(k,\beta)} \exp\left[im^{(k,\beta)} - im^{(j,\alpha)} + i(q - l_i^{(j,\alpha)} - 1)\epsilon\right],$$

$$\chi(\delta W_i) = \sum_{j=1}^{i} \sum_{\alpha=1}^{n_j} \sum_{p=2}^{l_i^{(j,\alpha)}} \sum_{k=1}^{i+1} \sum_{\beta=1}^{n_k} \sum_{q=1}^{l_{i+1}^{(j,\beta)}} \exp\left[im^{(k,\beta)} - im^{(j,\alpha)} + i(q-p)\epsilon\right], \tag{I.1}$$

where $\lambda_i = \{\alpha \,|\, 1 \le \alpha \le n_i, l_i^{(i,\alpha)} = 0\}$ and $\bar{\lambda}_i = \{\alpha \,|\, 1 \le \alpha \le n_i, l_i^{(i,\alpha)} \neq 0\}$. Note that contributions eliminated by the constraints (4.39) must be removed from these characters. We can see from the "superpotential term" in eq. (G.3) that the contributions eliminated by the constraints can be identified with that of $\widetilde{W}_i^\dagger$

$$\chi(\widetilde{W}_i^\dagger) = \sum_{j=1}^{i} \sum_{\alpha=1}^{n_j} \sum_{p=1}^{l_i^{(j,\alpha)}} \sum_{k=1}^{i+1} \sum_{\beta=1}^{n_k} \sum_{q=1}^{l_{i+1}^{(k,\beta)}} \exp\left[im^{(k,\beta)} - im^{(j,\alpha)} + i(q-p-1)\epsilon\right]. \tag{I.2}$$

In total, the character is given by

$$\begin{aligned}
\chi_{\text{fp}} &= \left[\sum_{i=1}^{L}\Big(\chi(\delta\Upsilon_i) + \chi(\delta\widetilde{\Upsilon}_i) + \chi(\delta Z_i)\Big) + \sum_{i=1}^{L-1}(\chi(\delta W_i))\right]_{\text{constrained}} \\
&= \sum_{i=1}^{L}\Big(\chi(\delta\Upsilon_i) + \chi(\delta\widetilde{\Upsilon}_i) + \chi(\delta Z_i)\Big) + \sum_{i=1}^{L-1}\Big(\chi(\delta W_i) - \chi(\widetilde{W}_i^\dagger)\Big) \\
&= \sum_{i=1}^{L}\Big(\chi(\Upsilon_i) + \chi(\widetilde{\Upsilon}_i) + \chi(Z_i) - \chi(\mathcal{U}_i)\Big) + \sum_{i=1}^{L-1}\Big(\chi(W_i) - \chi(\widetilde{W}_i^\dagger)\Big) \\
&= \sum_{i=1}^{L}\Big(\mathrm{Tr}[e^{i\Phi_i}]\mathrm{Tr}[e^{-iM_i}] + e^{-i\epsilon}\mathrm{Tr}[e^{-i\Phi_i}]\mathrm{Tr}[e^{iM_{i+1}}] + (e^{-i\epsilon}-1)\mathrm{Tr}[e^{i\Phi_i}]\mathrm{Tr}[e^{-i\Phi_i}]\Big) \\
&\quad + \sum_{i=1}^{L-1}(1-e^{-i\epsilon})\mathrm{Tr}[e^{i\Phi_{i+1}}]\mathrm{Tr}[e^{-i\Phi_i}],
\end{aligned} \tag{I.3}$$

where we have rewritten the characters of the fluctuations $(\chi(\delta\Upsilon_i), \cdots)$ into those of the matrices $(\chi(\Upsilon_i), \cdots)$ by subtracting the contributions eliminated by the gauge $\mathrm{GL}(k_i,\mathbb{C})$ action $\chi(\mathcal{U}_i) = \mathrm{Tr}[e^{i\Phi_i}]\mathrm{Tr}[e^{-i\Phi_i}]$. From the character, we can read off the determinant as

$$\chi_\sigma = \sum_{a=1}^{d} e^{i\omega_a} \implies \frac{1}{\det \mathcal{M}_\sigma} = \prod_{a=1}^{d}\frac{1}{\omega_a}, \tag{I.4}$$

where d is the dimension of the moduli space. Furthermore, by using the relation

$$f(\phi_\sigma) = \oint \frac{d\phi}{2\pi i}\frac{1}{\phi - \phi_\sigma} f(\phi), \tag{I.5}$$

the determinant can be rewritten into a contour integral as

$$\frac{1}{\det \mathcal{M}_\sigma} = \oint_{C_\sigma} \prod_{i=1}^{L}\prod_{r=1}^{k_i} \frac{d\phi_i^r}{2\pi i\epsilon}\left[\prod_{i=1}^{L} \mathcal{Z}_i^{\Upsilon\widetilde{\Upsilon}} \mathcal{Z}_i^{Z\Phi} \prod_{i=1}^{L-1} \mathcal{Z}_i^{W\widetilde{W}}\right], \tag{I.6}$$

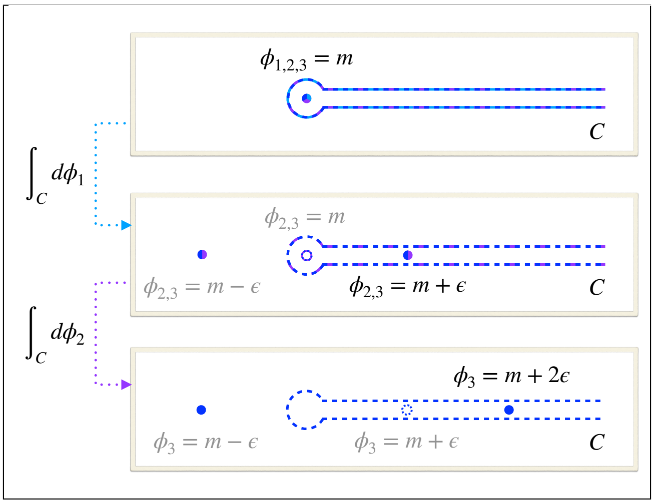

Figure 9. Integration contours for the vortex partition function in the Abelian gauge theory with a single charged scalar field ($L = 1, N_1 = 1, N_2 = 0, k = 3$).

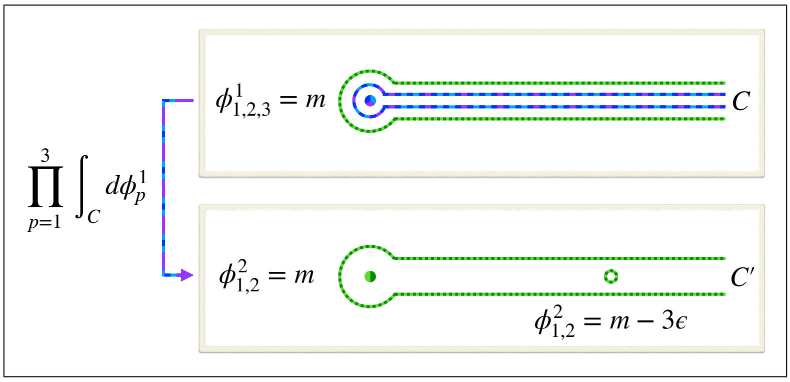

Figure 10. Integration contours in the case of $L = 2, N_1 = 1, N_2 = 1, k_1 = 3, k_2 = 2$).

where $\mathcal{Z}_i^{\Upsilon\widetilde{\Upsilon}}$, $\mathcal{Z}_i^{Z\Phi}$ and $\mathcal{Z}_i^{W\widetilde{W}}$ are given by

$$\mathcal{Z}_i^{\Upsilon\widetilde{\Upsilon}} \equiv \prod_{r=1}^{k_i} \left[\prod_{\alpha=1}^{n_i} \frac{1}{\phi_i^r - m^{(i,\alpha)}} \prod_{\beta=1}^{n_{i+1}} \frac{1}{m^{(i+1,\beta)} - \phi_i^r - \epsilon} \right], \tag{I.7}$$

$$\mathcal{Z}_i^{Z\Phi} \equiv \prod_{r=1}^{k_i} \prod_{s=1}^{k_i}{}' \frac{\phi_i^r - \phi_i^s}{\phi_i^r - \phi_i^s - \epsilon}, \tag{I.8}$$

$$\mathcal{Z}_i^{W\widetilde{W}} \equiv \prod_{r=1}^{k_i} \prod_{s=1}^{k_{i+1}} \frac{\phi_{i+1}^s - \phi_i^r - \epsilon}{\phi_{i+1}^s - \phi_i^r}, \tag{I.9}$$

where $\prod'$ indicates that the factors with $\alpha = \beta$ are omitted. The integration contour C_σ is the path surrounding the poles corresponding to the fixed point values of ϕ. Since the integrand is common for all the fixed points, the vortex partition function can be obtained by integrating the same integrand along the contour surrounding all the poles corresponding to the fixed points. We can check that such contour is given by $C_i^{\pm}$ (figure 5) as follows.

First, let us consider the case of $L = 1, N_1 = 1, N_2 = 0, k = 3$ (figure 9). In this case, the contour C_1^+ is the path surrounding the pole of $\mathcal{Z}_i^{\Upsilon\widetilde{\Upsilon}}$ located at $\phi = m$ $(= m^{(1,1)})$. If we first integrate ϕ_1, the residue at the pole $\phi = m$ gives the poles at $\phi = m \pm \epsilon$ and the pole at $\phi = m$ is eliminated due to the factor $\mathcal{Z}_1^{Z\Phi}$. Then, the integration of ϕ_2 is given by the residue at the pole $\phi = m + \epsilon$. which has a pole at $\phi = m + 2\epsilon$ whose residue gives the final result of the integration. In this way, we can show that C_1^+ is the contour surrounding all the poles corresponding to the fixed points.

We can generalize the discussion to the case of $L > 1$. the contour C_1^+ can be decomposed into the paths surrounding the poles at $\phi = m^{(1,\alpha)}$. Then, we can repeat the same discussion as in the case of $L = 1$ to show that the integration of $\phi_1^{(1,\alpha,p)}$ $(\alpha = 1, \cdots, n_1,\ p = 1, \cdots, l_1^{(1,\alpha)})$ is given by the residues at the poles $\phi_1^{(1,\alpha,p)} = m^{(1,\alpha)} + (p-1)\epsilon$. The only new ingredient for $L > 1$ is the factors $\mathcal{Z}_i^{W\widetilde{W}}$, which have zeros at $\phi = m^{(1,\alpha)} + l_1^{(1,\alpha)}\epsilon$ (see figure 10). Due to these zeros the integrations of $\phi_i^{(1,\alpha,p)}$ $(i > 1)$, which are again given by the residues at $\phi_i^{(1,\alpha,p)} = m^{(1,\alpha)} + (p-1)\epsilon$, vanish if $l_i^{(1,\alpha)} > l_1^{(1,\alpha)}$. Repeating this argument, we can show that the contributions are nonzero only when $l_i^{(j,\alpha)} < l_{i'}^{(j,\alpha)}$ for $i > i'$. In other words, the nonzero contributions can be classified by the same Young tableaux corresponding to the fixed points. In this way, we can show that C_i^+ are the contours surrounding all the poles corresponding to the fixed points.

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