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M5-branes and D4-branes wrapped on a direct product of spindle and Riemann surface

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Abstract: We construct multi-charged $AdS_3 \times \mathbb{\Sigma} \times \Sigma_\mathfrak{g}$ and $AdS_2 \times \mathbb{\Sigma} \times \Sigma_\mathfrak{g}$ solutions from M5-branes and D4-branes wrapped on a direct product of spindle, $\mathbb{\Sigma}$, and Riemann surface, $\Sigma_\mathfrak{g}$. Employing uplift formula, we obtain these solutions by uplifting the multi-charged $AdS_3 \times \mathbb{\Sigma}$ and $AdS_2 \times \mathbb{\Sigma}$ solutions to seven and six dimensions, respectively. We further uplift the solutions to eleven-dimensional and massive type IIA supergravity and calculate the holographic central charge and the Bekenstein-Hawking entropy, respectively. We perform the gravitational block calculations and, for the $AdS_3 \times \mathbb{\Sigma} \times \Sigma_\mathfrak{g}$ solutions, the result precisely matches the holographic central charge from the supergravity solutions.

Keywords: AdS-CFT Correspondence, Supergravity Models

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Contents

1 Introduction

Recently, there was a discovery of novel class of anti-de Sitter solutions obtained from branes wrapped on an orbifold, namely, a spindle, [2]. The spindle, $\mathbb{\Sigma}$, is an orbifold, $\mathbb{WCP}^1_{[n_-,n_+]}$, with conical deficit angles at two poles. The spindle numbers, n_-, n_+, are arbitrary coprime positive integers. Interestingly, these solutions realize the supersymmetry in different ways from very well studied topological twist in field theory, [3], and in gravity, [4]. It was first constructed from D3-branes, [2, 5, 6], and then generalized to other branes: M2-branes, [7–10], M5-branes, [11], and D4-branes, [12, 13]. Furthermore, two possible ways of realizing supersymmetry, topologically topological twist and anti-twist, were studied, [14, 15].

The spindle solutions were then generalized to an orbifold with a single conical deficit angle, namely, a topological disk. These solutions were first constructed from M5-branes, [16, 17], and proposed to be a gravity dual to a class of 4d $\mathcal{N} = 2$ Argyres-Douglas theories, [18]. See also [19] for further generalizations. Brane solutions wrapped on a topological disk were then constructed from D3-branes, [20, 21], M2-branes, [10, 22], D4-branes, [23], and more from M5-branes, [24]. See also [25] for defect solutions from different completion of global solutions.

JHEP02(2024)205

An interesting generalization would be to find AdS solutions from branes wrapped on an orbifold of dimensions more than two. Four-dimensional orbifolds are natural place to look for such constructions and some solutions were found. First, by uplifting $AdS_3 \times \mathbb{Z}$ solutions, where $\mathbb{Z}$ is a spindle, [6], or a disk, [21], $AdS_3 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}} \times S^4$ solutions from M5-branes were obtained where $\Sigma_{\mathfrak{g}}$ is a Riemann surface of genus $\mathfrak{g}$. Also $AdS_2 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}}$ solutions with spindle, $\mathbb{Z}$, from D4-branes were obtained, [12, 13]. More recently, performing and using a consistent truncation on a spindle, $AdS_3 \times \mathbb{Z}_1 \ltimes \mathbb{Z}_2$ solutions from M5-branes wrapped on a spindle fibered over another spindle were found, [26]. Also $AdS_3 \times \mathbb{Z} \ltimes \Sigma_{\mathfrak{g}}$ solutions on a spindle fibered over Riemann surface were found, [26].

In this work, we fill in the gaps in the literature. First, we construct multi-charged $AdS_3 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}}$ solutions from M5-branes. Employing the consistent truncation of [1], we obtain the solutions by uplifting the multi-charged $AdS_3 \times \mathbb{Z}$ solutions, [6], to seven-dimensional gauged supergravity. When the solutions are uplifted to eleven-dimensional supergravity, they precisely match the previously known $AdS_3 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}} \times S^4$ solutions in [6] and [21], which were obtained by uplifting the $AdS_3 \times \mathbb{Z}$ solutions of five-dimensional gauged supergravity. However, it is the first time to construct the $AdS_3 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}}$ solutions in seven-dimensional gauged supergravity.

Second, we construct multi-charged $AdS_2 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}}$ solutions from D4-branes. Inspired by the consistent truncation in [27], we construct them by uplifting the multi-charged $AdS_2 \times \mathbb{Z}$ solutions, [14], to matter coupled $F(4)$ gauged supergravity. Our solutions generalize the minimal $AdS_2 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}}$ solutions in [12] and also the solutions obtained in [13]. We then uplift the solutions to massive type IIA supergravity to obtain $AdS_2 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}} \times \tilde{S}^4$.

Finally, we perform the gravitational block calculations and, for the $AdS_3 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}}$ solutions, the result precisely matches the holographic central charge obtained from the supergravity solutions.

In section 2, we construct $AdS_3 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}}$ solutions from M5-branes. We uplift the solutions to eleven-dimensional supergravity and calculate the holographic central charge. In section 3, we construct $AdS_2 \times \mathbb{Z} \times \Sigma_{\mathfrak{g}}$ solutions from D4-branes. We uplift the solutions to massive type IIA supergravity and calculate the Bekenstein-Hawking entropy. In section 4, we present the gravitational block calculations. In section 5, we conclude. We present the equations of motion in appendix A and briefly review the consistent truncations of [1] in appendix B.

2 M5-branes wrapped on $\mathbb{Z} \times \Sigma_{\mathfrak{g}}$

2.1 $\mathrm{U}(1)^2$-gauged supergravity in seven dimensions

We review $\mathrm{U}(1)^2$-gauged supergravity in seven dimensions, [28], in the conventions of [26]. The bosonic field content is consist of the metric, two U(1) gauge fields, A^{12}, A^{34}, a three-form field, S^5, and two scalar fields, λ_1, λ_2. The Lagrangian is given by

$$\begin{aligned}\mathcal{L} = & \, (R - V)\,\mathrm{vol}_7 - 6 * d\lambda_1 \wedge d\lambda_1 - 6 * d\lambda_2 \wedge d\lambda_2 - 8 * d\lambda_1 \wedge d\lambda_2 \\ & - \frac{1}{2} e^{-4\lambda_1} * F^{12} \wedge F^{12} - \frac{1}{2} e^{-4\lambda_2} * F^{34} \wedge F^{34} - \frac{1}{2} e^{-4\lambda_1 - 4\lambda_2} * S^5 \wedge S^5 \\ & + \frac{1}{2g} S^5 \wedge dS^5 - \frac{1}{g} S^5 \wedge F^{12} \wedge F^{34} + \frac{1}{2g} A^{12} \wedge F^{12} \wedge F^{34} \wedge F^{34} \,, \end{aligned} \tag{2.1}$$

where $F^{12} = dA^{12}$, $F^{34} = dA^{34}$ and the scalar potential is

$$V = g^2 \left[\frac{1}{2} e^{-8(\lambda_1+\lambda_2)} - 4e^{2(\lambda_1+\lambda_2)} - 2e^{-2(2\lambda_1+\lambda_2)} - 2e^{-2(\lambda_1+2\lambda_2)} \right] . \tag{2.2}$$

The equations of motion are presented in appendix A.

2.2 Multi-charged $AdS_3 \times \Sigma$ solutions

We review the $AdS_3 \times \Sigma$ solutions of $\mathrm{U}(1)^3$-gauged $\mathcal{N} = 2$ supergravity in five dimensions, [6]. These solution are obtained from D3-branes wrapped on a spindle, Σ. The metric, gauge fields and scalar fields read

$$\begin{aligned} ds_5^2 &= H^{1/3} \left[ds^2_{AdS_3} + \frac{1}{4P} dy^2 + \frac{P}{H} dz^2 \right] , \\ A^{(I)} &= \frac{y-\alpha}{y+3K_I} dz \,, \qquad X^{(I)} = \frac{H^{1/3}}{y+3K_I} \,, \end{aligned} \tag{2.3}$$

where $I = 1, \ldots, 3$ and the functions are defined to be

$$H = (y+3K_1)(y+3K_2)(y+3K_3) \,, \qquad P = H - (y-\alpha)^2 \,, \tag{2.4}$$

where K_I and α are constant and satisfy the constraint, $K_1 + K_2 + K_3 = 0$.

In the case of three distinct roots, $0 < y_1 < y_2 < y_3$, of cubic polynomial, $P(y)$, the solution is positive and regular in $y \in [y_1, y_2]$. The spindle, Σ, is an orbifold, $\mathbb{WCP}^1_{[n_-,n_+]}$, with conical deficit angles at $y = y_1$, y_2, [6]. The spindle numbers, n_-, n_+, are arbitrary coprime positive integers. The Euler number of the spindle is given by

$$\chi(\Sigma) = \frac{1}{4\pi} \int_\Sigma R_\Sigma \mathrm{vol}_\Sigma = \frac{n_- + n_+}{n_- n_+} \,, \tag{2.5}$$

where R_Σ and vol_Σ are the Ricci scalar and the volume form on the spindle. The magnetic flux through the spindle is given by

$$Q_I = \frac{1}{2\pi} \int_\Sigma F^{(I)} = \frac{(y_2 - y_1)(\alpha + 3K_I)}{(y_1 + 3K_I)(y_2 + 3K_I)} \frac{\Delta z}{2\pi} \equiv \frac{p_I}{n_- n_+} \,, \tag{2.6}$$

and we demand $p_I \in \mathbb{Z}$. One can show that the R-symmetry flux is given by

$$Q_1 + Q_2 + Q_3 = \frac{p_1 + p_2 + p_3}{n_- n_+} = \frac{\eta_1 n_+ - \eta_2 n_-}{n_- n_+} \,, \tag{2.7}$$

where the supersymmetry is realized by, [14],

$$\begin{aligned} &\text{Anti-twist :} \qquad (\eta_1, \eta_2) = (+1, +1) \,, \\ &\text{Twist :} \qquad (\eta_1, \eta_2) = (-1, +1) \,. \end{aligned} \tag{2.8}$$

In minimal gauged supergravity, $K_1 = K_2 = K_3$, only the anti-twist solutions are allowed. Otherwise, both anti-twist and twist are allowed.

One can express Δz, y_1, y_2, and the parameters, K_I, α, in terms of the spindle numbers, n_-, n_+, p_1, and p_2, [6]. The period of the coordinate, z, is given by

$$\frac{\Delta z}{2\pi} = \frac{(n_- - n_+)(p_1 + p_2) + n_- n_+ - p_1^2 - p_1 p_2 - p_2^2}{n_- n_+ (n_- + n_-)} \,. \tag{2.9}$$

In the special case of

$$K_1 = K_2\,, \qquad X^{(1)} = X^{(2)}\,, \qquad A^{(1)} = A^{(2)}\,, \tag{2.10}$$

expressions of y_1, y_2, and $K_1 = K_2$ are simpler,

$$\begin{aligned} y_1 &= \frac{q\,(n_+ + q)\left[2n_-^2 - 2n_-\,(n_+ + 4q) + q\,(5n_+ + 9q)\right]}{3\left[n_-\,(n_+ + 2q) - q\,(2n_+ + 3q)\right]^2}\,, \\ y_2 &= -\frac{q\,(n_- - q)\left[2n_+^2 - 2n_+\,(n_- - 4q) - q\,(5n_- - 9q)\right]}{3\left[n_-\,(n_+ + 2q) - q\,(2n_+ + 3q)\right]^2}\,, \\ K_1 = K_2 &= \frac{q\,(n_- - n_+ - 3q)\,(n_+ + q)\,(n_- - q)}{9\left[n_-\,(n_+ + 2p) - q\,(2n_+ + 3q)\right]^2}\,, \end{aligned} \tag{2.11}$$

where we define $q \equiv p_1 = p_2$. For the expression of α, we leave the readers to [6]. For this special case, the $AdS_3 \times \mathbb{\Sigma}$ solutions are also solutions of SU(2) × U(1)-gauged $\mathcal{N} = 4$ supergravity in five dimensions, [29]. The solutions can be uplifted to eleven-dimensional supergravity, [30], as it was done for a spindle, [6], and for a disk, [21].

2.3 Multi-charged $AdS_3 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions

A consistent reduction of seven-dimensional maximal gauged supergravity, [31], on a Riemann surface was performed in [1]. Empolying the consistent truncation, we uplift the $AdS_3 \times \mathbb{\Sigma}$ solutions in section 2.2 with

$$K_1 = K_2 \neq K_3\,, \tag{2.12}$$

to $\mathrm{U}(1)^2$-gauged supergravity in seven dimension. We briefly summarize the uplift by consistent truncation in appendix B. As a result, we find the $AdS_3 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions,

$$\begin{aligned} ds_7^2 &= e^{-4\varphi} H^{1/3} \left(ds^2_{AdS_3} + \frac{1}{4P} dy^2 + \frac{P}{H} dz^2 \right) + \frac{1}{g^2} e^{6\varphi} ds_{\Sigma_{\mathfrak{g}}}\,, \\ e^{-\frac{10}{9}\lambda_1} &= 2^{1/3} X\,, \qquad e^{\frac{5}{3}\lambda_2} = 2^{1/3} X\,, \qquad e^{10\varphi} = 2^{1/3} X\,, \\ S^5 &= 2^{2/3}\,(3K + \alpha)\,\mathrm{vol}_{AdS_3}\,, \\ F^{12} &= \frac{1}{g}\frac{d}{dy}\left(\frac{y-\alpha}{y+3K_3}\right) dy \wedge dz + \frac{1}{g}\mathrm{vol}_{\Sigma_{\mathfrak{g}}}\,, \\ F^{34} &= \frac{2}{g}\frac{d}{dy}\left(\frac{y-\alpha}{y+3K_1}\right) dy \wedge dz\,, \end{aligned} \tag{2.13}$$

where $\Sigma_{\mathfrak{g}}$ is a Riemann surface and we define

$$H = (y + 3K_1)^2\,(y + 3K_3)\,, \qquad P = H - (y-\alpha)^2\,, \qquad X = X^{(1)} = X^{(2)} = \frac{H^{1/3}}{y + 3K_1}\,, \tag{2.14}$$

and $g^2 L^2_{AdS_5} = 2^{4/3}$. The gauge coupling and the radius of asymptotic AdS_5 are fixed to be $g = 2^{2/3}$ and $L_{AdS_5} = 1$, respectively.

The flux quantization through the Riemann surface is given by

$$\begin{aligned} \mathfrak{s}_1 &= \frac{g}{2\pi}\int_{\Sigma_{\mathfrak{g}}} F^{12} = 2\,(1-\mathfrak{g}) \in \mathbb{Z}\,, \\ \mathfrak{s}_2 &= \frac{g}{2\pi}\int_{\Sigma_{\mathfrak{g}}} F^{34} = 0\,, \end{aligned} \tag{2.15}$$

where we find $\mathfrak{s}_1+\mathfrak{s}_2=2\,(1-\mathfrak{g})$. Fluxes through the spindle are quantized to be

$$\mathfrak{n}_1 \equiv -\frac{g}{2\pi}\int_\Sigma F^{12} = -\frac{(y_2-y_1)\,(\alpha+3K_3)}{(y_1+3K_3)\,(y_2+3K_3)}\frac{\Delta z}{2\pi} = -\frac{p_3}{n_-n_+}\,,$$
$$2\mathfrak{n}_2 \equiv -\frac{g}{2\pi}\int_\Sigma F^{34} = -2\frac{(y_2-y_1)\,(\alpha+3K_1)}{(y_1+3K_1)\,(y_2+3K_1)}\frac{\Delta z}{2\pi} = -2\frac{p_1}{n_-n_+}\,, \tag{2.16}$$

where p_1 and p_3 are introduced in (2.6) and $p_i\in\mathbb{Z}$. The minus signs in the definition of $\mathfrak{n}_i$ are introduced for later convenience in the gravitational block calculations. By (2.7) the total flux is obtained to be

$$\mathfrak{n}_1+2\mathfrak{n}_2 = -\frac{2p_1+p_3}{n_-n_+} = \frac{\eta_1 n_+ - \eta_2 n_-}{n_-n_+}\,, \tag{2.17}$$

where η_1 and η_2 are given in (2.8) and, thus, both twist and anti-twist solutions are allowed.

2.4 Uplift to eleven-dimensional supergravity

We review the uplift formula, [32], of $\mathrm{U}(1)^2$-gauged supergravity in seven dimensions to eleven-dimensional supergravity, [33], as presented in [26]. The metric is given by

$$L^{-2}ds_{11}^2 = \Delta^{1/3}ds_7^2 + \frac{1}{g^2}\Delta^{-2/3}\left\{e^{4\lambda_1+4\lambda_2}dw_0^2 + e^{-2\lambda_1}\left[dw_1^2+w_1^2\left(d\chi_1-gA^{12}\right)^2\right]\right.$$
$$\left.+\,e^{-2\lambda_2}\left[dw_2^2+w_2^2\left(d\chi_2-gA^{34}\right)^2\right]\right\}\,, \tag{2.18}$$

where

$$\Delta = e^{-4\lambda_1-4\lambda_2}dw_0^2 + e^{2\lambda_1}w_1^2 + e^{2\lambda_2}w_2^2\,, \tag{2.19}$$

and L is a length scale. We employ the parametrizations of coordinates of internal four-sphere by

$$\mu^1+i\mu^2=\cos\xi\cos\theta\,e^{i\chi_1}\,,\qquad \mu^3+i\mu^4=\cos\xi\sin\theta\,e^{i\chi_2}\,,\qquad \mu^5=\sin\xi\,, \tag{2.20}$$

with

$$w_0=\sin\xi\,,\qquad w_1=\cos\xi\cos\theta\,,\qquad w_2=\cos\xi\sin\theta\,, \tag{2.21}$$

where $w_0^2+w_1^2+w_2^2=1$ and $\xi\in[-\pi/2,\pi/2]$, $\theta\in[0,\pi/2]$, χ_1, $\chi_2\in[0,2\pi]$. The four-form flux is given by

$$\begin{aligned}
L^{-3}F_{(4)} =\; & \frac{w_1w_2}{g^3w_0}U\Delta^{-2}dw_1\wedge dw_2\wedge\left(d\chi_1-gA^{12}\right)\wedge\left(d\chi_2-gA^{34}\right)\\
&+\frac{2w_1^2w_2^2}{g^3}\Delta^{-2}e^{2\lambda_1+2\lambda_2}\left(d\lambda_1-d\lambda_2\right)\wedge\left(d\chi_1-gA^{12}\right)\wedge\left(d\chi_2-gA^{34}\right)\wedge dw_0\\
&+\frac{2w_0w_1w_2}{g^3}\Delta^{-2}\left[e^{-2\lambda_1-4\lambda_2}\wedge(3d\lambda_1+2d\lambda_2)-e^{-4\lambda_1-2\lambda_2}w_2dw_1\wedge(2d\lambda_1+3\lambda_2)\right]\\
&\qquad\qquad\wedge\left(d\chi_1-gA^{12}\right)\wedge\left(d\chi_2-gA^{34}\right)\\
&+\frac{1}{g^2}\Delta^{-1}F^{12}\wedge\left[w_0w_2e^{-4\lambda_1-4\lambda_2}dw_2-w_2^2e^{2\lambda_2}dw_0\right]\wedge\left(d\chi_2-gA^{34}\right)\\
&+\frac{1}{g^2}\Delta^{-1}F^{34}\wedge\left[w_0w_1e^{-4\lambda_1-4\lambda_2}dw_1-w_1^2e^{2\lambda_1}dw_0\right]\wedge\left(d\chi_1-gA^{12}\right)\\
&-w_0e^{-4\lambda_1-4\lambda_2}*_7S^5+\frac{1}{g}S^5\wedge dw_0\,,
\end{aligned} \tag{2.22}$$

where

$$U=\left(e^{-8\lambda_1-8\lambda_2}-2e^{-2\lambda_1-4\lambda_2}-2e^{-4\lambda_1-2\lambda_2}\right)w_0^2 \\ -\left(e^{-2\lambda_1-4\lambda_2}+2e^{2\lambda_1+2\lambda_2}\right)w_1^2-\left(e^{-4\lambda_1-2\lambda_2}+2e^{2\lambda_1+2\lambda_2}\right)w_2^2\,, \tag{2.23}$$

and $*_7$ is a Hodge dual in seven dimensions.

We find a quantization condition of four-form flux through the internal four-sphere,

$$\frac{1}{(2\pi l_p)^3}\int_{S^4}F_{(4)}=\frac{L^3}{(2\pi l_p)^3}\int_{S^4}\frac{w_1w_2}{g^3w_0}U\Delta^{-2}dw_1\wedge dw_2\wedge d\chi_1\wedge d\chi_2 \\ =\frac{L^3}{\pi g^3 l_p^3}\equiv N\in\mathbb{Z}\,, \tag{2.24}$$

where l_p is the Planck length and N is the number of M5-branes wrapping $\mathbb{Z}\times\Sigma_{\mathfrak{g}}$.

For the metric of the form,

$$ds_{11}^2=e^{2A}\left(ds^2_{AdS_3}+ds^2_{M_8}\right)\,, \tag{2.25}$$

the central charge of dual two-dimensional conformal field theory is given by [34, 35], and we follow [6],

$$c=\frac{3}{2G_N^{(3)}}=\frac{3}{2G_N^{(11)}}\int_{M_8}e^{9A}\text{vol}_{M_8}\,, \tag{2.26}$$

where the eleven-dimensional Newton's gravitational constant is $G_N^{(11)}=\frac{(2\pi)^8l_p^9}{16\pi}$. For the solutions, with (2.11), we find the holographic central charge to be

$$c=\frac{L^9\Delta z}{8\pi^5g^6l_p^9}\left(y_2-y_1\right)vol_{\Sigma_{\mathfrak{g}}}=\frac{\Delta z}{2\pi^2}N^3\left(y_1-y_2\right)vol_{\Sigma_{\mathfrak{g}}} \\ =\frac{4q^2\left(n_--n_+-2q\right)}{n_-n_+\left[n_-\left(n_++2q\right)-q\left(2n_++3q\right)\right]}\left(\mathfrak{g}-1\right)N^3\,, \tag{2.27}$$

where $vol_{\Sigma_{\mathfrak{g}}}=4\pi\left(\mathfrak{g}-1\right)$. This precisely matches the result obtained from the solutions by uplifting $AdS_3\times\mathbb{Z}$ to eleven-dimensional supergravity, [6].

3 D4-branes wrapped on $\mathbb{Z}\times\Sigma_{\mathfrak{g}}$

3.1 Matter coupled $F(4)$ gauged supergravity

We review $F(4)$ gauged supergravity, [36], coupled to a vector multiplet in six dimensions, [37, 38], in the conventions of [12]. The bosonic field content is consist of the metric, two U(1) gauge fields, A_i, a two-form field, B, and two scalar fields, φ_i, where $i=1,2$. We introduce a parametrization of the scalar fields,

$$X_i=e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\varphi}}\,,\qquad \vec{a}_1=\left(2^{1/2},2^{-1/2}\right)\,,\qquad \vec{a}_2=\left(-2^{1/2},2^{-1/2}\right)\,, \tag{3.1}$$

with

$$X_0=\left(X_1X_2\right)^{-3/2}\,. \tag{3.2}$$

The field strengths of the gauge fields and two-form field are, respectively,

$$F_i = dA_i \,, \qquad H = dB \,. \tag{3.3}$$

The action is given by

$$\begin{aligned} S = \frac{1}{16\pi G_N^{(6)}} \int d^6x\sqrt{-g}\Bigg[& R - V - \frac{1}{2}|d\vec{\varphi}|^2 - \frac{1}{2}\sum_{i=1}^{2} X_i^{-2}|F_i|^2 - \frac{1}{8}\left(X_1X_2\right)^2|H|^2 \\ & -\frac{m^2}{4}\left(X_1X_2\right)^{-1}|B|^2 - \frac{1}{16}\frac{\varepsilon^{\mu\nu\rho\sigma\tau\lambda}}{\sqrt{-g}}B_{\mu\nu}\left(F_{1\rho\sigma}F_{2\tau\lambda} + \frac{m^2}{12}B_{\rho\sigma}B_{\tau\lambda}\right)\Bigg] \,, \end{aligned} \tag{3.4}$$

where the scalar potential is

$$V = m^2X_0^2 - 4g^2X_1X_2 - 4gmX_0\left(X_1 + X_2\right) \,, \tag{3.5}$$

and $\varepsilon_{012345} = +1$. The norm of form fields are defined by

$$|\omega|^2 = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\omega^{\mu_1\cdots\mu_p} \,. \tag{3.6}$$

The equations of motion are presented in appendix A.

3.2 Multi-charged $AdS_2 \times \Sigma$ solutions

We review the $AdS_2 \times \Sigma$ solutions of $\mathrm{U}(1)^4$-gauged $\mathcal{N} = 2$ supergravity in four dimensions, [10, 14]. These solution are obtained from M2-branes wrapped on a spindle, Σ. The metric, gauge fields and scalar fields read

$$\begin{aligned} ds_4^2 &= H^{1/2}\left[\frac{1}{4}ds^2_{AdS_2} + \frac{1}{P}dy^2 + \frac{P}{4H}dz^2\right] \,, \\ A^{(I)} &= \frac{y}{y+q_I}dz \,, \qquad X^{(I)} = \frac{H^{1/4}}{y+q_I} \,, \end{aligned} \tag{3.7}$$

where $I = 1, \ldots, 4$ and the functions are defined to be

$$H = \left(y+q_1\right)\left(y+q_2\right)\left(y+q_3\right)\left(y+q_4\right) \,, \qquad P = H - 4y^2 \,. \tag{3.8}$$

In the case of four distinct roots, $y_0 < y_1 < y_2 < y_3$, of quartic polynomial, $P(y)$, the solution is positive and regular in $y \in [y_1, y_2]$. The spindle, Σ, is an orbifold, $\mathbb{WCP}^1_{[n_1,n_2]}$, with conical deficit angles at $y = y_1$, y_2, [10, 14]. The spindle numbers, n_1, n_2, are arbitrary coprime positive integers. The Euler number of the spindle is given by

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R_\Sigma \mathrm{vol}_\Sigma = \frac{n_1+n_2}{n_1n_2} \,, \tag{3.9}$$

where R_Σ and vol_Σ are the Ricci scalar and the volume form on the spindle. The magnetic flux through the spindle is given by

$$Q_I = \frac{1}{2\pi}\int_\Sigma F^{(I)} = \left(\frac{y_2}{y_2+q_I} - \frac{y_1}{y_1+q_I}\right)\frac{\Delta z}{2\pi} \equiv \frac{2p_I}{n_1n_2} \,, \tag{3.10}$$

JHEP02(2024)205

and we demand $p_I \in \mathbb{Z}$. One can show that the R-symmetry flux is given by

$$Q^R = \frac{1}{2}\left(Q_1 + Q_2 + Q_3 + Q_4\right) = \frac{p_1 + p_2 + p_2 + p_4}{n_1 n_2} = \frac{\eta_1 n_2 - \eta_2 n_1}{n_1 n_2}\,, \tag{3.11}$$

where the supersymmetry is realized by, [14, 15],

$$\begin{aligned} &\text{Anti-twist}: \qquad (\eta_1, \eta_2) = (+1, +1)\,, \\ &\text{Twist}: \qquad (\eta_1, \eta_2) = (\pm 1, \mp 1)\,. \end{aligned} \tag{3.12}$$

When parameters, q_I, $I = 1, \ldots, 4$, are all identical or identical in pairwise, only the anti-twist solutions are allowed. Otherwise, for all distinct or three identical with one distinct parameters, both the twist and anti-twist solutions are allowed.

Unlike five-dimensional $\mathrm{U}(1)^3$-gauged supergravity which has a unique $\mathrm{U}(1)^2$ subtruncation, there are two distinct $\mathrm{U}(1)^2$ subtruncations from four-dimensional $\mathrm{U}(1)^4$-gauged supergravity,

$$\begin{aligned} &\mathrm{ST}^2 \text{ model}: \qquad A^{(1)} = A^{(2)} \neq A^{(3)} = A^{(4)}\,, \qquad X^{(1)} = X^{(2)} \neq X^{(3)} = X^{(4)}\,, \\ &\mathrm{T}^3 \text{ model}: \qquad A^{(1)} = A^{(2)} = A^{(3)} \neq A^{(4)}\,, \qquad X^{(1)} = X^{(2)} = X^{(3)} \neq X^{(4)}\,, \end{aligned} \tag{3.13}$$

and their permutations.

3.3 Multi-charged $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions

A consistent reduction of matter coupled $F(4)$ gauged supergravity on a Riemann surface was performed in [27]. Inspired by the consistent truncation in [27], the $AdS_3 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions in (2.13), and the minimal $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions in [12], we construct the $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions. However, only the T^3 model is obtained from the truncation of $F(4)$ gauged supergravity and not the ST^2 model. Thus, we only find solutions by uplifting multi-charged $AdS_2 \times \mathbb{\Sigma}$ solutions in section 3.2 with

$$q_1 = q_2 = q_3 \neq q_4\,, \tag{3.14}$$

to six dimensions. After some trial and error we find the solutions to be

$$\begin{aligned} ds_6^2 &= e^{-2C} L_{AdS_4}^2 H^{1/2}\left[\frac{1}{4} ds_{AdS_2}^2 + \frac{1}{P} dy^2 + \frac{P}{4H} dz^2\right] + e^{2C} ds_{\Sigma_{\mathfrak{g}}}^2\,, \\ X_1 &= k_8^{1/8} k_2^{1/2} \frac{H^{1/4}}{y+q_1}\,, \qquad X_2 = k_8^{1/8} k_2^{-1/2} \frac{H^{1/4}}{y+q_1}\,, \qquad e^{-2C} = m^2 k_8^{1/4} k_4 \frac{H^{1/4}}{y+q_1} \\ B &= q_1 \frac{9 k_8^{1/2}}{4g^2} \mathrm{vol}_{AdS_2}\,, \\ F_1 &= \frac{3 k_8^{1/2} k_2^{1/2}}{2g} \frac{q_1}{(y+q_1)^2} dy \wedge dz + \frac{\kappa + \mathbf{z}}{2g} \mathrm{vol}_{\Sigma_{\mathfrak{g}}}\,, \\ F_2 &= \frac{3 k_8^{1/2} k_2^{-1/2}}{2g} \frac{q_4}{(y+q_4)^2} dy \wedge dz + \frac{\kappa - \mathbf{z}}{2g} \mathrm{vol}_{\Sigma_{\mathfrak{g}}}\,, \end{aligned} \tag{3.15}$$

where we define

$$H = (y+q_1)^3 (y+q_4)\,, \qquad P = H - 4y^2\,, \tag{3.16}$$

and

$$g=\frac{3m}{2}\,,\qquad L_{AdS_4}=\frac{k_8^{1/4}k_4^{-1/2}}{m^2}\,. \tag{3.17}$$

There are parameters, $\kappa=0,\pm1$, for the curvature of Riemann surface, and, $\mathtt{z}$, which define

$$k_2=\frac{3\mathtt{z}+\sqrt{\kappa^2+8\mathtt{z}^2}}{\mathtt{z}-\kappa}\,,\qquad k_8=\frac{16k_2}{9\left(1+k_2\right)^2}\,,\qquad k_4=\frac{18}{-3\kappa+\sqrt{\kappa^2+8\mathtt{z}^2}}\,. \tag{3.18}$$

If we set $q_1=q_2=q_3=q_4$, it reduces to the minimal $AdS_2\times\mathbb{Z}\times\Sigma_{\mathfrak{g}}$ solutions in [12]. For our solutions, in order to satisfy the equations of motion, we find that we should choose

$$\kappa=-1\,,\qquad \mathtt{z}=1\,, \tag{3.19}$$

and we find $k_2=k_4=k_8^{-1}=3$. Then the solutions are given by

$$\begin{aligned}
ds_6^2&=e^{-2C}L_{AdS_4}^2H^{1/2}\left[\frac{1}{4}ds_{AdS_2}^2+\frac{1}{P}dy^2+\frac{P}{4H}dz^2\right]+e^{2C}ds_{\Sigma_{\mathfrak{g}}}^2\,,\\
X_1&=3^{3/8}\frac{H^{1/4}}{y+q_1}\,,\qquad X_2=3^{-5/8}\frac{H^{1/4}}{y+q_1}\,,\qquad e^{-2C}=\frac{4g^2}{3^{5/4}}\frac{H^{1/4}}{y+q_1}\\
B&=q_1\frac{3\sqrt{3}}{4g^2}\mathrm{vol}_{AdS_2}\,,\\
F_1&=\frac{3}{2g}\frac{q_1}{\left(y+q_1\right)^2}dy\wedge dz\,,\\
F_2&=\frac{1}{2g}\frac{q_4}{\left(y+q_4\right)^2}dy\wedge dz-\frac{1}{g}\mathrm{vol}_{\Sigma_{\mathfrak{g}}}\,,
\end{aligned} \tag{3.20}$$

where we have

$$g=\frac{3m}{2}\,,\qquad L_{AdS_4}=\frac{3^{5/4}}{4g^2}\,. \tag{3.21}$$

Notice that the components of F_1 on the Riemann surface is turned off by the choice of (3.19).

The flux quantization through the Riemann surface is given by

$$\begin{aligned}
\mathfrak{s}_1&=\frac{g}{2\pi}\int_{\Sigma_{\mathfrak{g}}}F_1=0\,,\\
\mathfrak{s}_2&=\frac{g}{2\pi}\int_{\Sigma_{\mathfrak{g}}}F_2=2\left(1-\mathfrak{g}\right)\,\in\,\mathbb{Z}\,,
\end{aligned} \tag{3.22}$$

where we find $\mathfrak{s}_1+\mathfrak{s}_2=2\left(1-\mathfrak{g}\right)$. Fluxes through the spindle are quantized to be

$$\begin{aligned}
3\mathfrak{n}_1&\equiv\frac{g}{2\pi}\int_{\mathbb{Z}}F_1=\frac{3}{2}\left(\frac{y_2}{y_2+q_1}-\frac{y_1}{y_1+q_1}\right)\frac{\Delta z}{2\pi}=\frac{3p_1}{n_1n_2}\,,\\
\mathfrak{n}_2&\equiv\frac{g}{2\pi}\int_{\mathbb{Z}}F_2=\frac{1}{2}\left(\frac{y_2}{y_2+q_4}-\frac{y_1}{y_1+q_4}\right)\frac{\Delta z}{2\pi}=\frac{p_4}{n_1n_2}\,,
\end{aligned} \tag{3.23}$$

where p_1 and p_4 are introduced in (3.10) and $p_i\in\mathbb{Z}$. By (3.11) the total flux is obtained to be

$$3\mathfrak{n}_1+\mathfrak{n}_2=\frac{3p_1+p_4}{n_1n_2}=\frac{\eta_1n_2-\eta_2n_1}{n_1n_2}\,, \tag{3.24}$$

and both the twist and anti-twist solutions are allowed.[1]

[1] We would like to thank Chris Couzens for discussion on this.

3.4 Uplift to massive type IIA supergravity

We review the uplift formula, [39], of matter coupled $F(4)$ gauged supergravity to massive type IIA supergravity, [40], presented in [12]. Although the uplift formula is only given for vanishing of two-form field, B, in $F(4)$ gauged supergravity, it correctly reproduces the metric, the dilaton and the internal four-sphere part of four-form flux. The metric in the string frame and the dilaton field are

$$\begin{aligned} ds^2_{\text{s.f.}} =& \lambda^2 \mu_0^{-1/3} \left(X_1 X_2\right)^{-1/4} \Big\{ \Delta^{1/2} ds_6^2 \\ &+ g^{-2} \Delta^{-1/2} \Big[X_0^{-1} d\mu_0^2 + X_1^{-1} \left(d\mu_1^2 + \mu_1^2 \sigma_1^2 \right) + X_2^{-1} \left(d\mu_2^2 + \mu_2^2 \sigma_2^2 \right) \Big] \Big\} \,, \end{aligned} \tag{3.25}$$

$$e^\Phi = \lambda^2 \mu_0^{-5/6} \Delta^{1/4} \left(X_1 X_2\right)^{-5/8} \,, \tag{3.26}$$

where the function, Δ, is defined by

$$\Delta = \sum_{a=0}^{2} X_a \mu_a^2 \,, \tag{3.27}$$

and the one-forms are $\sigma_i = d\phi_i - gA_i$. The angular coordinates, ϕ_1, ϕ_2, have canonical periodicities of 2π. We employ the parametrization of coordinates,

$$\mu_0 = \sin\xi \,, \qquad \mu_1 = \cos\xi \sin\eta \,, \qquad \mu_2 = \cos\xi \cos\eta \,, \tag{3.28}$$

where $\sum_{a=0}^2 \mu_a^2 = 1$ and $\eta \in [0, \pi/2]$, $\xi \in (0, \pi/2]$. The internal space is a squashed four-hemisphere which has a singularity on the boundary, $\xi \to 0$. The four-form flux is given by

$$\lambda^{-1} * F_{(4)} = gU \text{vol}_6 - \frac{1}{g^2} \sum_{i=1}^{2} X_i^{-2} \mu_i \left(*_6 F_i\right) \wedge d\mu_i \wedge \sigma_i + \frac{1}{g} \sum_{a=0}^{2} X_a^{-1} \mu_a \left(*_6 dX_a\right) \wedge d\mu_a \,, \tag{3.29}$$

where the function, U, is defined by

$$U = 2 \sum_{a=0}^{2} X_a^2 \mu_a^2 - \left[\frac{4}{3} X_0 + 2\left(X_1 + X_2\right) \right] \Delta \,, \tag{3.30}$$

and $*_6$ is a Hodge dual in six dimensions. The Romans mass is given by

$$F_{(0)} = \frac{2g}{3\lambda^3} \,. \tag{3.31}$$

The positive constant, λ, is introduced from the scaling symmetry of the theory. It plays an important role to have regular solutions with proper flux quantizations, [12]. The uplift formula implies $m = 2g/3$.

The relevant part of the four-form flux for flux quantization is the component on the internal four-sphere,

$$F_{(4)} = \frac{\lambda \mu_0^{1/3}}{g^3 \Delta} \frac{U}{\Delta} \frac{\mu_1 \mu_2}{\mu_0} d\mu_1 \wedge d\mu_2 \wedge \sigma_1 \wedge \sigma_2 + \dots \,. \tag{3.32}$$

We impose quantization conditions on the fluxes,

$$\left(2\pi l_s\right) F_{(0)} = n_0 \in \mathbb{Z} \,, \qquad \frac{1}{\left(2\pi l_s\right)^3} \int_{\tilde{S}^4} F_{(4)} = N \in \mathbb{Z} \,, \tag{3.33}$$

where l_s is the string length. For the solutions, these imply that

$$g^8 = \frac{1}{(2\pi l_s)^8} \frac{18\pi^6}{N^3 n_0}\,, \qquad \lambda^8 = \frac{8\pi^2}{9Nn_0^3}\,, \tag{3.34}$$

where we have $n_0 = 8 - N_f$ and N_f is the number of D8-branes. These results are identical to the case of minimal $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions in [12].

For the metric of the form in the string frame,

$$ds^2_{\text{s.f.}} = e^{2A} \left(ds^2_{AdS_2} + ds^2_{M_8} \right)\,, \tag{3.35}$$

the Bekenstein-Hawking entropy is by, [34, 35], and in [12],

$$S_{\text{BH}} = \frac{1}{4G_N^{(2)}} = \frac{8\pi^2}{(2\pi l_s)^8} \int e^{8A-2\Phi} \text{vol}_{M_8}\,. \tag{3.36}$$

For the solutions, we obtain the Bekenstein-Hawking entropy to be

$$S_{\text{BH}} = \frac{1}{(2\pi l_s)^8} \frac{9\,(3\pi\lambda)^4\, k_8^{1/2}}{20 g^8 k_4} 4\pi\kappa\,(1-\mathfrak{g})\, A_h = \frac{1}{(2\pi l_s)^8} \frac{\sqrt{3}\,(3\pi\lambda)^4}{20 g^8} 4\pi\kappa\,(1-\mathfrak{g})\, A_h\,, \tag{3.37}$$

where the area of the horizon of black hole, multi-charged $AdS_2 \times \mathbb{\Sigma}$, in (3.7) is

$$A_h = \frac{1}{2}\,(y_2 - y_1)\,\Delta z\,, \tag{3.38}$$

and y_1 and y_2 are two relevant roots of $P(y)$. The free energy of 5d USp$(2N)$ gauge theory on $S^3 \times \Sigma_{\mathfrak{g}}$ is given by, [12, 41, 42],

$$\begin{aligned} \mathcal{F}_{S^3\times\Sigma_{\mathfrak{g}}} &= \frac{16\pi^3}{(2\pi\, l_s)^8} \int e^{8A-2\Phi} \text{vol}_{M_6} \\ &= \frac{16\pi\kappa\,(1-\mathfrak{g})\, N^{5/2}\, (\mathtt{z}^2-\kappa^2)^{3/2} \left(\sqrt{\kappa^2+8\mathtt{z}^2} - \kappa \right)}{5\sqrt{8-N_f} \left(\kappa\sqrt{\kappa^2+8\mathtt{z}^2} - \kappa^2 + 4\mathtt{z}^2 \right)^{3/2}}\,. \end{aligned} \tag{3.39}$$

By comparing (3.39) with (3.36), we find the Bekenstein-Hawking entropy to be

$$\begin{aligned} S_{\text{BH}} &= \frac{1}{2\pi} \mathcal{F}_{S^3\times\Sigma_{\mathfrak{g}}} A_h \\ &= \frac{8\kappa\,(1-\mathfrak{g})\, N^{5/2}\, (\mathtt{z}^2-\kappa^2)^{3/2} \left(\sqrt{\kappa^2+8\mathtt{z}^2} - \kappa \right)}{5\sqrt{8-N_f} \left(\kappa\sqrt{\kappa^2+8\mathtt{z}^2} - \kappa^2 + 4\mathtt{z}^2 \right)^{3/2}} A_h\,, \end{aligned} \tag{3.40}$$

and, for $\kappa = -1$ and $\mathtt{z} = 1$, (3.19), we obtain[2]

$$S_{\text{BH}} = \left(\frac{3}{8} \right)^{3/2} \frac{32\,(\mathfrak{g}-1)\, N^{5/2}}{5\sqrt{8-N_f}} A_h\,. \tag{3.41}$$

Although formally the Bekenstein-Hawking entropy is in the identical expression of the one for minimal $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions in [12], note that the black holes that give the area, A_h, are different: it was minimal $AdS_2 \times \mathbb{\Sigma}$ in [12], but now it is multi-charged $AdS_2 \times \mathbb{\Sigma}$, [14]. We refer [15] for the explicit expression of A_h for the multi-charged solutions.

[2]We would like to thank Hyojoong Kim for comments on this limit.

JHEP02(2024)205

4 Gravitational blocks

In this section, we briefly review the off-shell quantities from gluing gravitational blocks, [43], and show that extremization of off-shell quantity correctly reproduces the Bekenstein-Hawking entropy, central charge, and free energy, depending on the dimensionality, [12]. Then apply the gravitational block calculations to the solutions we constructed in the previous sections.

Depending on the dimensionality, the Bekenstein-Hawking entropy, central charge, and free energy are obtained by extremizing the off-shell quantity, [12],

$$F_d^{\pm}\left(\Delta_i,\epsilon;\mathfrak{n}_i,n_+,n_-,\sigma\right)=\frac{1}{\epsilon}\Big(\mathcal{F}_d\left(\Delta_i^+\right)\pm\mathcal{F}_d\left(\Delta_i^-\right)\Big)\,, \tag{4.1}$$

where $\mathcal{F}_d$ are the gravitational blocks, [43]. We also define quantities,

$$\Delta_i^{\pm}\equiv\varphi_i\pm\mathfrak{n}_i\epsilon\,, \tag{4.2}$$

and

$$\varphi_i\equiv\Delta_i+\frac{r_i}{2}\frac{n_+-\sigma n_-}{n_+n_-}\epsilon\,, \tag{4.3}$$

where $\sigma=+1$ and $\sigma=-1$ for twist and anti-twist solutions, respectively. The expressions of gravitational blocks are

$$\mathcal{F}_3=b_3\left(\Delta_1\Delta_2\Delta_3\Delta_4\right)^{1/2}\,,\quad\mathcal{F}_4=b_4\left(\Delta_1\Delta_2\Delta_3\right)\,,\quad\mathcal{F}_5=b_5\left(\Delta_1\Delta_2\right)^{3/2}\,,\quad\mathcal{F}_6=b_6\left(\Delta_1\Delta_2\right)^2\,, \tag{4.4}$$

and the constants, b_d, will be given later. The relative sign for gluing gravitational blocks in (4.1) is $-\sigma$ for $d=3,5$ and $-$ for $d=4,6$. The twist conditions on the magnetic flux through the spindle, $\mathfrak{n}_i$, is given by

$$\sum_{i=1}^{\mathfrak{d}}\mathfrak{n}_i=\frac{n_++\sigma n_-}{n_+n_-}\,, \tag{4.5}$$

where n_+ and n_- are the orbifold numbers of spindle and $\mathfrak{d}$ is the rank of global symmetry group of dual field theory, i.e., $\mathfrak{d}=4$ for $d=3$, $\mathfrak{d}=3$ for $d=4$, and $\mathfrak{d}=2$ for $d=5,6$. The constants are constrained by

$$\sum_{i=1}^{\mathfrak{d}}r_i=2\,, \tag{4.6}$$

and they parametrize the ambiguities of defining the flavor symmetries. The U(1) R-symmetry flux gives

$$\frac{1}{2\pi}\int_\Sigma dA_R=\frac{n_++\sigma n_-}{n_+n_-}\,, \tag{4.7}$$

and the fugacities of dual field theories are normalized by

$$\sum_{i=1}^{\mathfrak{d}}\Delta_i=2\,. \tag{4.8}$$

The off-shell quantity can be written by

$$F_d^{\pm}\left(\varphi_i,\epsilon;\mathfrak{n}\right)=\frac{1}{\epsilon}\Big(\mathcal{F}_d\left(\varphi_i+\mathfrak{n}_i\epsilon\right)\pm\mathcal{F}_d\left(\varphi_i-\mathfrak{n}_i\epsilon\right)\Big)\,, \tag{4.9}$$

where the variables satisfy the constraint,

$$\sum_{i=1}^{\mathfrak{d}}\varphi_i-\frac{n_+-\sigma n_-}{n_+n_-}\epsilon=2\,, \tag{4.10}$$

which originates from (4.6) and (4.8).

4.1 M5-branes wrapped on $\mathbb{\Sigma}\times\Sigma_{\mathfrak{g}}$

For the $AdS_3\times\mathbb{\Sigma}\times\Sigma_{\mathfrak{g}}$ solutions, there is standard topological twist on $\Sigma_{\mathfrak{g}}$ for the magnetic charges, $\mathfrak{s}_i$, and anti-twist on $\mathbb{\Sigma}$ for $\mathfrak{n}_i$. Then the off-shell central charge is given by

$$\begin{aligned}S(\varphi_i,\epsilon_1,\epsilon_2;\mathfrak{n}_i,\mathfrak{s}_i)=-\frac{1}{4\epsilon_1\epsilon_2}\Big[&\mathcal{F}_6(\varphi_i+\mathfrak{n}_i\epsilon_1+\mathfrak{s}_i\epsilon_2)-\mathcal{F}_6(\varphi_i-\mathfrak{n}_i\epsilon_1+\mathfrak{s}_i\epsilon_2)\\&-\mathcal{F}_6(\varphi_i+\mathfrak{n}_i\epsilon_1-\mathfrak{s}_i\epsilon_2)+\mathcal{F}_6(\varphi_i-\mathfrak{n}_i\epsilon_1-\mathfrak{s}_i\epsilon_2)\Big]\,,\end{aligned} \tag{4.11}$$

with the constraints,

$$\mathfrak{n}_1+2\mathfrak{n}_2=\frac{n_+-n_-}{n_+n_-}\,,\qquad\mathfrak{s}_1+\mathfrak{s}_2=2(1-\mathfrak{g})\,,\qquad\varphi_1+2\varphi_2-\frac{n_++n_-}{n_+n_-}\epsilon_1=2\,. \tag{4.12}$$

For the calculations, we employ

$$b_4=-\frac{3}{2}N^2\,,\qquad b_6=-N^3\,. \tag{4.13}$$

Extremizing it with respect to ϵ_2 gives $\epsilon_2=0$ and renaming $\epsilon_1\mapsto\epsilon$, we find the off-shell central charge expressed by

$$\begin{aligned}S(\varphi_i,\epsilon;\mathfrak{n}_i,\mathfrak{s}_i)&=2N^3\mathfrak{s}_1\left(\mathfrak{n}_1\varphi_2\varphi_3+\varphi_1\mathfrak{n}_2\varphi_3+\varphi_1\varphi_2\mathfrak{n}_3+\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3\epsilon^2\right)|_{3\mapsto2}\\&\quad+2N^3\mathfrak{s}_2\left(\mathfrak{n}_1\varphi_2\varphi_3+\varphi_1\mathfrak{n}_2\varphi_3+\varphi_1\varphi_2\mathfrak{n}_3+\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3\epsilon^2\right)|_{3\mapsto1}\\&=2N^3\mathfrak{s}_1\left(-\frac{1}{3N^2}F_4^-\right)\Big|_{3\mapsto2}+2N^3\mathfrak{s}_2\left(-\frac{1}{3N^2}F_4^-\right)\Big|_{3\mapsto1}\,.\end{aligned} \tag{4.14}$$

We have started with the $d=6$ gravitational blocks, $\mathcal{F}_6$, and we observe the $d=4$ structure, F_4^-, naturally emerges. See section 5.2 of [12] for the calculations of $d=4$ gravitational blocks. From the $d=4$ point of view, the $\mathfrak{s}_1$ term of $S(\varphi_i,\epsilon;\mathfrak{n}_i,\mathfrak{s}_i)$ in (4.14) is the off-shell central charge for $\mathfrak{n}_1\neq\mathfrak{n}_2=\mathfrak{n}_3$ and the $\mathfrak{s}_2$ terms is for $\mathfrak{n}_1=\mathfrak{n}_3\neq\mathfrak{n}_2$. Thus, extremization gives disparate results for each term. However, for the solution, as we have

$$\mathfrak{s}_1=2\left(1-\mathfrak{g}\right)\,,\qquad\mathfrak{s}_2=0\,, \tag{4.15}$$

the solution chooses the $\mathfrak{s}_1$ term in the off-shell central charge. Extremizing this we find the values,

$$\epsilon^*=\frac{\frac{n_+-\sigma n_-}{n_+n_-}}{2\left(\frac{\sigma}{n_+n_-}-\left(\mathfrak{n}_1\mathfrak{n}_2+\mathfrak{n}_2\mathfrak{n}_3+\mathfrak{n}_3\mathfrak{n}_1\right)\right)}\Bigg|_{3\mapsto2}\,,\quad\varphi_2^*=\frac{\mathfrak{n}_2\left(\mathfrak{n}_2-\mathfrak{n}_3-\mathfrak{n}_1\right)}{2\left(\frac{\sigma}{n_+n_-}-\left(\mathfrak{n}_1\mathfrak{n}_2+\mathfrak{n}_2\mathfrak{n}_3+\mathfrak{n}_3\mathfrak{n}_1\right)\right)}\Bigg|_{3\mapsto2}\,. \tag{4.16}$$

JHEP02(2024)205

Then the off-shell central charge gives

$$S(\varphi_i^*, \epsilon^*; \mathfrak{n}_i) = 4N^3\,(\mathfrak{g}-1)\,\frac{\mathfrak{n}_1\mathfrak{n}_2\mathfrak{n}_3}{\frac{\sigma}{n_+n_-} - (\mathfrak{n}_1\mathfrak{n}_2 + \mathfrak{n}_2\mathfrak{n}_3 + \mathfrak{n}_3\mathfrak{n}_1)}\Bigg|_{3\mapsto 2}\,, \tag{4.17}$$

which precisely matches the holographic central charge from the supergravity solutions, (2.27), with $\sigma = -1$.

4.2 D4-branes wrapped on $\mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$

For the $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions, there is standard topological twist on $\Sigma_{\mathfrak{g}}$ for the magnetic charges, $\mathfrak{s}_i$, and anti-twist on $\mathbb{\Sigma}$ for $\mathfrak{n}_i$. Then the entropy function is given by

$$\begin{aligned} S(\varphi_i, \epsilon_1, \epsilon_2; \mathfrak{n}_i, \mathfrak{s}_i) = -\frac{1}{4\epsilon_1\epsilon_2}\Big[&\mathcal{F}_5(\varphi_i + \mathfrak{n}_i\epsilon_1 + \mathfrak{s}_i\epsilon_2) + \mathcal{F}_5(\varphi_i - \mathfrak{n}_i\epsilon_1 + \mathfrak{s}_i\epsilon_2) \\ &- \mathcal{F}_5(\varphi_i + \mathfrak{n}_i\epsilon_1 - \mathfrak{s}_i\epsilon_2) - \mathcal{F}_5(\varphi_i - \mathfrak{n}_i\epsilon_1 - \mathfrak{s}_i\epsilon_2)\Big]\,, \end{aligned} \tag{4.18}$$

with the constraints,

$$\mathfrak{n}_1 + 3\mathfrak{n}_2 = \frac{n_+ - n_-}{n_+n_-}\,, \qquad \mathfrak{s}_1 + \mathfrak{s}_2 = 2(1-\mathfrak{g})\,, \qquad \varphi_1 + 3\varphi_2 - \frac{n_+ + n_-}{n_+n_-}\epsilon_1 = 2\,. \tag{4.19}$$

For the calculations, we employ

$$b_3 = -\frac{\sqrt{2}\pi}{3}N^{3/2}\,, \qquad b_5 = -\frac{2^{5/2}\pi}{15}\frac{N^{5/2}}{\sqrt{8-N_f}}\,. \tag{4.20}$$

Extremizing it with respect to ϵ_2 gives $\epsilon_2 = 0$ and renaming $\epsilon_1 \mapsto \epsilon$, we find the entropy function expressed by

$$\begin{aligned} S(\varphi_i, \epsilon; \mathfrak{n}_i, \mathfrak{s}_i) = \frac{c}{\epsilon}\Bigg[&\mathfrak{s}_1\left(\sqrt{(\varphi_1 + \mathfrak{n}_1\epsilon)\,(\varphi_2 + \mathfrak{n}_2\epsilon)^3} + \sqrt{(\varphi_1 - \mathfrak{n}_1\epsilon)\,(\varphi_2 - \mathfrak{n}_2\epsilon)^3}\right) \\ &+\mathfrak{s}_2\left(\sqrt{(\varphi_1 + \mathfrak{n}_1\epsilon)^3\,(\varphi_2 + \mathfrak{n}_2\epsilon)} + \sqrt{(\varphi_1 - \mathfrak{n}_1\epsilon)^3\,(\varphi_2 - \mathfrak{n}_2\epsilon)}\right)\Bigg]\,, \end{aligned} \tag{4.21}$$

where we have

$$c \equiv \frac{\sqrt{2}\pi}{5}\frac{N^{5/2}}{\sqrt{8-N_f}}\,. \tag{4.22}$$

We have started with the $d = 5$ gravitational blocks, $\mathcal{F}_5$, and we observe the $d = 3$ structure naturally emerges. See section 5.1 of [12] for the calculations of $d = 3$ gravitational blocks. From the $d = 3$ point of view, the $\mathfrak{s}_1$ term of $S(\varphi_i, \epsilon; \mathfrak{n}_i, \mathfrak{s}_i)$ in (4.21) is the entropy function for $\mathfrak{n}_1 \neq \mathfrak{n}_2 = \mathfrak{n}_3 = \mathfrak{n}_4$ and the $\mathfrak{s}_2$ terms is for $\mathfrak{n}_1 = \mathfrak{n}_2 = \mathfrak{n}_3 \neq \mathfrak{n}_4$. Thus, extremization gives disparate results for each term. However, for the solution, as we have

$$\mathfrak{s}_1 = 2\,(1-\mathfrak{g})\,, \qquad \mathfrak{s}_2 = 0\,, \tag{4.23}$$

the solution chooses the $\mathfrak{s}_1$ term in the entropy function. However, in this case, the algebraic equations appearing in the extremization procedure are quite complicated and we do not pursue it further here.

JHEP02(2024)205

5 Conclusions

In this work, we have constructed multi-charged $AdS_3 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ and $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions from M5-branes and D4-branes. We have uplifted the solutions to eleven-dimensional and massive type IIA supergravity, respectively. We have also studied their spindle properties and calculated the holographic central charge and the Bekenstein-Hawking entropy, respectively.

Although we have only considered the $AdS_{2,3} \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions for spindle, $\mathbb{\Sigma}$, the local form of our solutions naturally allows solutions for disk, $\mathbb{\Sigma}$, by different global completion. However, the $AdS_3 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solution for disk, $\mathbb{\Sigma}$, was already constructed and studied in [21]. Thus, it would be interesting to analyze the $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions for disk, $\mathbb{\Sigma}$, from the solutions we have constructed.

Unlike the minimal $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions in [12] where $\mathtt{z}$ is a free parameter, only $\mathtt{z} = 1$ is allowed for our multi-charged $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions, (3.19). We would like to understand why it is required to fix the parameter for the solutions and if there are more general multi-charged solutions with additional parameters.

The solutions we have obtained could be seen as generalizations of $AdS_3 \times \Sigma_{\mathfrak{g}_1} \times \Sigma_{\mathfrak{g}_2}$ solutions in [44] and $AdS_2 \times \Sigma_{\mathfrak{g}_1} \times \Sigma_{\mathfrak{g}_2}$ solutions in [45–47]. In particular, via the AdS/CFT correspondence, [48], the Bekenstein-Hawking entropy of $AdS_2 \times \Sigma_{\mathfrak{g}_1} \times \Sigma_{\mathfrak{g}_2}$ solutions was microscopically counted by the topologically twisted index of 5d USp($2N$) gauge theories, [42, 49]. It would be most interesting to derive the Bekenstein-Hawking entropy of the $AdS_2 \times \mathbb{\Sigma} \times \Sigma_{\mathfrak{g}}$ solutions from the field theory calculations.

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A The equations of motion

A.1 U(1)^2-gauged supergravity in seven dimensions

We present the equations of motion derived from the Lagrangian in (2.1),

$$\begin{aligned} R_{\mu\nu} = &\, 6\partial_\mu\lambda_1\partial_\nu\lambda_1 + 6\partial_\mu\lambda_2\partial_\nu\lambda_2 + 8\partial_{(\mu}\lambda_1\partial_{\nu)}\lambda_2 + \frac{1}{5}g_{\mu\nu}V \\ &+ \frac{1}{2}e^{-4\lambda_1}\left(F^{12}_{\mu\rho}F^{12\rho}_{\nu} - \frac{1}{10}g_{\mu\nu}F^{12}_{\rho\sigma}F^{12\rho\sigma}\right) + \frac{1}{2}e^{-4\lambda_2}\left(F^{34}_{\mu\rho}F^{34\rho}_{\nu} - \frac{1}{10}g_{\mu\nu}F^{34}_{\rho\sigma}F^{34\rho\sigma}\right) \\ &+ \frac{1}{4}e^{-4\lambda_1-4\lambda_2}\left(S^5_{\mu\rho\sigma}S^{5\rho\sigma}_{\nu} - \frac{2}{15}g_{\mu\nu}S^5_{\rho\sigma\delta}S^{5\rho\sigma\delta}\right), \end{aligned} \tag{A.1}$$

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\left(3\lambda_1+2\lambda_2\right)\right) + \frac{1}{4}e^{-4\lambda_1}F^{12}_{\mu\nu}F^{12\mu\nu} + \frac{1}{12}e^{-4\lambda_1-4\lambda_2}S^5_{\mu\nu\rho}S^{5\mu\nu\rho} - \frac{g^2}{4}\frac{\partial V}{\partial\lambda_1} = 0\,, \\ &\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\left(2\lambda_1+3\lambda_2\right)\right) + \frac{1}{4}e^{-4\lambda_2}F^{34}_{\mu\nu}F^{34\mu\nu} + \frac{1}{12}e^{-4\lambda_1-4\lambda_2}S^5_{\mu\nu\rho}S^{5\mu\nu\rho} - \frac{g^2}{4}\frac{\partial V}{\partial\lambda_2} = 0\,, \end{aligned} \tag{A.2}$$

$$\begin{aligned} d\left(e^{-4\lambda_1} * F^{12}\right) + e^{-4\lambda_1-4\lambda_2} * S^5 \wedge F^{34} &= 0\,, \\ d\left(e^{-4\lambda_2} * F^{34}\right) + e^{-4\lambda_1-4\lambda_2} * S^5 \wedge F^{12} &= 0\,, \\ dS^5 - g e^{-4\lambda_1-4\lambda_2} * S^5 - F^{12}\wedge F^{34} &= 0\,. \end{aligned} \tag{A.3}$$

A.2 Matter coupled $F(4)$ gauged supergravity

We present the equations of motion derived from the action in (3.4),

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}\sum_{i=1}^{2}\partial_\mu\varphi_i\partial_\nu\varphi_i - \frac{1}{4}V g_{\mu\nu} - \frac{1}{2}\sum_{i=1}^{2} X_i^{-2}\left(F_{i\mu\rho}F_{i\nu}{}^{\rho} - \frac{1}{8}g_{\mu\nu}F_{i\rho\sigma}F_i{}^{\rho\sigma}\right) \\ &-\frac{m^2}{4}\left(X_1X_2\right)^{-1}\left(B_{\mu\rho}B_\nu{}^{\rho} - \frac{1}{8}g_{\mu\nu}B_{\rho\sigma}B^{\rho\sigma}\right) - \frac{1}{16}\left(X_1X_2\right)^2\left(H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} - \frac{1}{6}g_{\mu\nu}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}\right) = 0\,, \end{aligned} \tag{A.4}$$

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi_1\right) - \frac{\partial V}{\partial\varphi_1} - \frac{1}{2\sqrt{2}}X_1^{-2}F_{1\mu\nu}F_1{}^{\mu\nu} + \frac{1}{2\sqrt{2}}X_2^{-2}F_{2\mu\nu}F_2{}^{\mu\nu} = 0\,, \\ &\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi_2\right) - \frac{\partial V}{\partial\varphi_2} - \frac{1}{4\sqrt{2}}X_1^{-2}F_{1\mu\nu}F_1{}^{\mu\nu} - \frac{1}{4\sqrt{2}}X_2^{-2}F_{2\mu\nu}F_2{}^{\mu\nu} \\ &\qquad - \frac{m^2}{8\sqrt{2}}\left(X_1X_2\right)^{-1}B_{\mu\nu}B^{\mu\nu} + \frac{1}{24\sqrt{2}}\left(X_1X_2\right)^2 H_{\mu\nu\rho}H^{\mu\nu\rho} = 0\,, \end{aligned} \tag{A.5}$$

$$\begin{aligned} \mathcal{D}_\nu\left(X_1^{-2}F_1^{\nu\mu}\right) &= \frac{1}{24}\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma\tau\lambda}F_{2\nu\rho}H_{\sigma\tau\lambda}\,, \\ \mathcal{D}_\nu\left(X_2^{-2}F_2^{\nu\mu}\right) &= \frac{1}{24}\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma\tau\lambda}F_{1\nu\rho}H_{\sigma\tau\lambda}\,, \\ \mathcal{D}_\nu\left(\left(X_1X_2\right)^{-1}B^{\nu\mu}\right) &= \frac{1}{24}\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma\tau\lambda}B_{\nu\rho}H_{\sigma\tau\lambda}\,, \\ \mathcal{D}_\rho\left(\left(X_1X_2\right)^2 H^{\rho\nu\mu}\right) &= -\frac{1}{4}\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma\tau\lambda}\left(\frac{m^2}{2}B_{\rho\sigma}B_{\tau\lambda} + F_{i\rho\sigma}F_{i\tau\lambda}\right) - 2m^2\left(X_1X_2\right)^{-1}B^{\mu\nu}\,. \end{aligned} \tag{A.6}$$

B Consistent truncations of [1]

In this appendix, we briefly review the consistent truncation of seven-dimensional maximal gauged supergravity, [31], on a Riemann surface in [1] and explain the setup to uplift our solutions by employing the truncation ansatz.

The consistent truncation ansatz for the seven-dimensional metric on a Riemann surface, $\Sigma_\mathfrak{g}$, is given by

$$ds_7^2 = e^{-4\varphi}ds_5^2 + \frac{1}{g^2}e^{6\varphi}ds_{\Sigma_\mathfrak{g}}^2\,, \tag{B.1}$$

which introduces a scalar field, φ, in five dimensions. Also $g^2L_{AdS_5}^2 = 2^{4/3}$ for the gauge coupling, g, and the radius of asymptotic AdS_5, L_{AdS_5}. The SO(5) gauge fields are decomposed by SO(5) $\to$ SO(2) $\times$ SO(3),

$$
\begin{aligned}
A^{ab} &= \epsilon^{ab} A + \frac{1}{g}\omega^{ab}\,, \\
A_{a\alpha} &= -A^{\alpha a} = \psi^{1\alpha} e^a - \psi^{2\alpha}\epsilon^{ab} e^b\,, \\
A^{\alpha\beta} &= A^{\alpha\beta}\,,
\end{aligned}
\tag{B.2}
$$

where a, $b = 1, 2$, α, $\beta = 3, 4, 5$, $ds^2_{\Sigma_{\mathfrak{g}}} = e^a e^a$, and ω^{ab} is the spin connection on $\Sigma_{\mathfrak{g}}$. The ansatz introduces an SO(2) one-form, A, SO(3) one-forms, $A^{\alpha\beta}$, transforming in the $(\mathbf{1}, \mathbf{3})$ of SO(2) $\times$ SO(3), and six scalar fields, $\psi^{a\alpha} = (\psi^{1\alpha}, \psi^{2\alpha})$, transforming in the $(\mathbf{2}, \mathbf{3})$. The scalar fields are given by

$$
T^{ab} = e^{-6\lambda}\delta^{ab}\,, \qquad T^{a\alpha} = 0\,, \qquad T^{\alpha\beta} = e^{4\lambda}\mathcal{T}^{\alpha\beta}\,,
\tag{B.3}
$$

which introduces a scalar field, λ, and five scalar fields in $\mathcal{T}^{\alpha\beta}$ which live on the coset manifold, SL(3)/SO(3). The three-form field is given by

$$
\begin{aligned}
S^a &= K^1_{(2)} \wedge e^a - \epsilon^{ab} K^2_{(2)} \wedge e^b\,, \\
S^\alpha &= h^\alpha_{(3)} + \chi^\alpha_{(1)} \wedge \text{vol}_{\Sigma_{\mathfrak{g}}}\,,
\end{aligned}
\tag{B.4}
$$

which introduces an SO(2) doublet of two-forms, $K^a_{(2)}$, three-forms, $h^\alpha_{(3)}$, and one-forms, $\chi^\alpha_{(1)}$.

To be particular, we consider a subtruncation of the general consistent truncations which reduces to SU(2) $\times$ U(1)-gauged $\mathcal{N} = 4$ supergravity in five dimensions, [29], which is presented in section 5.1 of [1]. In this case, we have the scalar fields to be

$$
\lambda = 3\varphi\,, \qquad \mathcal{T}_{\alpha\beta} = \delta_{\alpha\beta}\,, \qquad \psi^{a\alpha} = 0\,.
\tag{B.5}
$$

From the three-form field, we have a complex two-form field,

$$
\mathcal{C}_{(2)} = K^1_{(2)} + iK^2_{(2)}\,,
\tag{B.6}
$$

and a three-form field,

$$
*h^\alpha_{(3)} = \frac{1}{2} e^{-20\varphi}\epsilon_{\alpha\beta\gamma} F^{\beta\gamma}\,,
\tag{B.7}
$$

with $\chi^\alpha_{(1)} = 0$.

In order to match with the special case of U(1)^2-gauged supergravity in seven dimensions, (2.10), we further impose $A^{a\alpha}_{(1)} = 0$ and $\mathcal{C}_{(2)} = 0$. In U(1)^2-gauged supergravity in seven dimensions, the scalar fields of are given by

$$
T_{ij} = \text{diag}\left(e^{2\lambda_1}, e^{2\lambda_1}, e^{2\lambda_2}, e^{2\lambda_2}, e^{-4\lambda_1 - 4\lambda_2}\right).
\tag{B.8}
$$

By matching it with the consistent truncation ansatz,

$$
T_{ij} = \text{diag}\left(e^{-6\lambda}, e^{-6\lambda}, e^{4\lambda}, e^{4\lambda}, e^{4\lambda}\right),
\tag{B.9}
$$

we identify the scalar fields to be

$$
\lambda_1 = -3\lambda\,, \qquad \lambda_2 = 2\lambda\,.
\tag{B.10}
$$

The non-trivial three-form field, S^5, is given by $h^\alpha_{(3)}$ in (B.7).

Finally, we compare the actions of SU(2) × U(1)-gauged $\mathcal{N} = 4$ supergravity in five dimensions, [29], presented in (5.4) of [1] and in (2.1) with (3.1) of [6] to fix

$$X^{(1)} = 2^{-1/3}e^{10\varphi}\,, \qquad X^{(2)} = 2^{-1/3}e^{10\varphi}\,, \qquad X^{(3)} = 2^{2/3}e^{-20\varphi}\,. \tag{B.11}$$

With $X = X^{(1)} = X^{(2)}$, this determines the scalar fields to be

$$e^{-\frac{10}{9}\lambda_1} = 2^{1/3}X\,, \qquad e^{\frac{5}{3}\lambda_2} = 2^{1/3}X\,, \qquad e^{10\varphi} = 2^{1/3}X\,. \tag{B.12}$$

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