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Carrollian structure of the null boundary solution space

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Abstract: We study pure D dimensional Einstein gravity in spacetimes with a generic null boundary. We focus on the symplectic form of the solution phase space which comprises a $2D$ dimensional boundary part and a $2(D(D-3)/2+1)$ dimensional bulk part. The symplectic form is the sum of the bulk and boundary parts, obtained through integration over a codimension 1 surface (null boundary) and a codimension 2 spatial section of it, respectively. Notably, while the total symplectic form is a closed 2-form over the solution phase space, neither the boundary nor the bulk symplectic forms are closed due to the symplectic flux of the bulk modes passing through the boundary. Furthermore, we demonstrate that the $D(D-3)/2+1$ dimensional Lagrangian submanifold of the bulk part of the solution phase space has a Carrollian structure, with the metric on the $D(D-3)/2$ dimensional part being the Wheeler-DeWitt metric, and the Carrollian kernel vector corresponding to the outgoing Robinson-Trautman gravitational wave solution.

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1 Introduction

Question of field theories on spacetimes with boundaries appears in various areas of physics. In particular, the boundary can be a codimension 1 null or timelike surface. It is well known that consistency of the theory besides the usual bulk modes requires introduction of boundary degrees of freedom (dof). These dof reside on the boundary while interacting with the bulk modes and among themselves. Furthermore, in theories with local (gauge) symmetries, it is known that the boundary dof can be labeled and studied by extending the notion of gauge invariance in the presence of the boundary. Explicitly, residual gauge symmetries and the corresponding surface charges [1–13] provide a systematic framework to formulate boundary dof.

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In this work, our primary emphasis lies in D dimensional pure Einstein gravity in presence of a null boundary. This problem is motivated by questions within the realm of black hole physics, where the horizon plays the role of the null boundary from the viewpoint of non-free-fall, fiducial, observers situated "outside" the horizon. The same problem arises when exploring gravity in an asymptotic flat spacetime, where the null boundary is asymptotic future (or past) null infinity. While our main focus will be on the former, the latter has been subject of intense study in the recent years, particularly in connection with memory effects and gravitational wave observations [14–18].

In the analysis of asymptotic or boundary symmetries, the conventional approach typically commences with specifying/prescribing the falloff behavior of the fields near the boundary, as demonstrated in, for example, [1, 6, 7, 9, 15, 19–37] and [38–40]. However, in general one may formulate the problem in a different way: one can start with constructing the set of all solutions to the theory which accommodate presence of the boundary. For the case of a gravity on a spacetime with a null boundary (horizon), this program was outlined in [41] and worked through in [39, 42–46], see also [47–50] for related work.

In particular, a complete null boundary solution space for D dimensional Einstein gravity has been constructed in [44]. We place the null boundary at $r = 0$ and construct the solution space by solving the Einstein equations perturbatively around $r = 0$. This solution space, which we will briefly review in section 2, is characterized by $D + D(D-3)/2$ functions over the $D-1$ dimensional null boundary $\mathcal{N}$.

Before we delve further, let us introduce a terminology that proves particularly useful in the context of light-front field theory formulation and when dealing with a null boundary. As depicted in figure 1, the null boundary $\mathcal{N}$ is spanned by v (parametrizing the null direction) and $D-2$ dimensional spatial coordinates x^A. In light-front field theories, $\mathcal{N}$ plays the role of partial Cauchy surface where the dynamics means evolution in the light-cone (null) time direction, v, and the conjugate momentum is related to the derivative of the fields w.r.t. v. If $\Phi(r; v, x^A)$ denotes a generic field, the solution space is spanned by $\Phi(r = 0; v, x^A)$ and the momentum conjugate to Φ, which is proportional to $\partial_v\Phi$, does not carry an independent information. Consequently, unlike the usual spatial (partial) Cauchy surfaces, in the Hamiltonian formulation of light-front field theories, half of Hamilton's equations, which define conjugate momenta, simply represent relationships among Cauchy data. Noting this peculiar feature, it proves useful to distinguish between the "solution space" and "solution phase space" in the light-front field theories and when formulating the theory with a null boundary. If the latter space is $2N$ dimensional, the former is N dimensional and is obtained by imposing half of Hamilton's equations on the fields. In essence, the equations defining conjugate momenta may be regarded as (second class) constraints on the system, and the solution space serves as the "reduced phase space" of the theory. For further details, refer to appendix A. A notable feature of the light-front solution space is that in addition to its symplectic structure it possesses a metric structure. In simple field theories, such as the one discussed in appendix A, the metric on the field space also appears in the symplectic form. Consequently, the symplectic form on the solution space contains information about the metric on the solution space.

In our analysis, the primary focus is on the symplectic form of the theory. We begin with the symplectic form on the solution phase space, and by imposing the boundary equations of motion and definition of conjugate light-cone momenta, we derive the on-shell symplectic

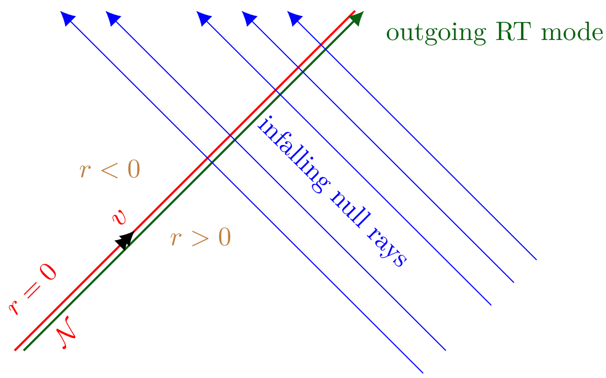

Figure 1. Depiction of a null boundary at $r = 0$, spanned by v and x^A (perpendicular to the plane) coordinates. Our null boundary solution space encompasses all solutions to pure D dimensional Einstein gravity in the region $r \geq 0$. In our notation, the infalling gravitational waves correspond to $\mathcal{N}_{AB}$ modes and the RT mode denoting the Robinson-Trautman gravitational wave solution [51, 52]. In addition to the "bulk modes" (RT and infalling modes) we also have boundary modes residing on $\mathcal{N}$. These modes are specified by D functions at an arbitrary given $v = v_b$.

form, which pertains to the symplectic form over the solution space. This on-shell symplectic form consists of two parts, a boundary part given by integrals over constant v slice at $\mathcal{N}$ and the bulk part determined by codimension 1 integrals over $\mathcal{N}$. The on-shell boundary part comprises D modes and in the absence of bulk modes (when there are no gravitational waves passing through the boundary), it is closed and invertible. In such cases, one can invert the symplectic form, calculate Poisson brackets over the solution space (which are Dirac brackets over the solution phase space), and obtain the algebra of surface charges. In a suitable slicing of the solution space, this is a direct sum of Heisenberg and Diff$(D-2)$ algebras [44–46]. When we turn on bulk modes, neither the boundary nor bulk parts of the on-shell symplectic forms are individually closed or invertible. This feature is anticipated due to the flux of bulk modes passing through the boundary.

The symplectic form on a sector of solution phase space (referred to as the off-shell symplectic form) and the corresponding Poisson/Dirac brackets, were recently studied in [53]. In this work, we narrow our focus on the bulk part of the on-shell symplectic form. To streamline our analysis, we consider the co-rotating case, reducing the number of boundary modes to two. Interestingly, one of the modes, specifically the volume form over the codimension 2 spacelike surface spanned by x^A, appears both as a boundary mode and a bulk mode. Consequently, we have $D(D-3)/2+1$ bulk modes, which is one more than the anticipated number of gravitational wave polarizations. Our analysis reveals that this space, at any given point on $\mathcal{N}$, possesses a $(D-1)(D-2)/2$ dimensional Carrollian structure. The metric on this Carrollian geometry is the Wheeler-DeWitt metric, with the kernel vector aligning with the mode shared by both the boundary and bulk parts.

Organization of the paper. In section 2, we provide an overview of the construction of the null boundary solution space, which also establish the conventions and notations used in this work. In section 3, we introduce the off-shell symplectic form and the basic Poisson brackets on the solution phase space. In section 4, we delve into the analysis of the on-shell symplectic form and compute Poisson brackets over the solution space. In section 5, we explore the geometry of bulk solution space and demonstrate that it exhibits a Carrollian structure. In section 6, we study three distinctive sectors within the solution space. Section 7 is devoted to providing an outlook and concluding remarks. Within various appendices, we have compiled essential background information to enhance the self-contained nature of this paper, as well as in-depth calculations. In appendix A, we review some basic features of light-front scalar field theory, including its symplectic form and on-shell Poisson brackets. We establish the equivalence between on-shell Poisson brackets and Dirac brackets on the off-shell solution phase space. In appendix B, we review basics of Carrollian geometry. Appendix C discusses boundary symmetry generators and corresponding field variations. In appendix D, we discuss analysis of boundary symmetry charges and their algebra. In appendix E, we provide details of computation of on-shell symplectic form. In appendix F, we provide details of inverting on-shell bulk symplectic form and extracting the corresponding Poisson brackets. In appendix G we present specification of the general results of the main text to the physically significant $D = 4$ case.

2 Einstein gravity on a spacetime with a null boundary, a quick review

Considering a null surface at $r = 0$, we define r as a coordinate to measures deviation from the given null surface. We denote the advanced time by v and the transverse coordinates by x^A, where $A = 1, \cdots, D-2$. The line-element in the Gaussian null-type coordinate system takes the form

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -V\,\mathrm{d}v^2 + 2\eta\,\mathrm{d}v\,\mathrm{d}r + q_{AB}\left(\mathrm{d}x^A + U^A\,\mathrm{d}v\right)\left(\mathrm{d}x^B + U^B\,\mathrm{d}v\right), \tag{2.1}$$

where $x^\mu = \{v, r, x^A\}$.

Expanding around null surface. Here, V, q_{AB}, U^A represent unknown functions that have r dependence, while η does not depend on r. We *presume* these functions are smooth and amenable to Taylor expansion around $r = 0$. The metric on transverse surface, q_{AB} can be expanded as

$$q_{AB} = \Omega_{AB} + 2\,r\,\eta\left(\lambda_{AB} + \frac{\lambda}{D-2}\Omega_{AB}\right) + \mathcal{O}(r^2)\,. \tag{2.2}$$

In our notation λ_{AB} is traceless. We also use the notation $\Omega := \sqrt{\det\Omega_{AB}}$. We can also expand U^A and V as

$$U^A = \mathcal{U}^A - r\,\eta\,\Omega^{-1}\mathcal{J}^A + \mathcal{O}(r^2)\,, \tag{2.3a}$$

$$V = -\eta\left(\Gamma - \frac{2}{D-2}\frac{\mathcal{D}_v\Omega}{\Omega} + \frac{\mathcal{D}_v\eta}{\eta}\right) r + \mathcal{O}(r^2)\,, \tag{2.3b}$$

where $\mathcal{D}_v = \partial_v - \mathcal{L}_\mathcal{U}$, with $\mathcal{L}_\mathcal{U}$ is the Lie derivative along $\mathcal{U}$.

Null geometric quantities. One may also define the induced metric using two null vector fields,

$$l_\mu \, \mathrm{d}x^\mu = -\frac{1}{2} V \, \mathrm{d}v + \eta \, \mathrm{d}r \,, \qquad l^\mu \partial_\mu = \partial_v - U^A \partial_A + \frac{V}{2\eta} \partial_r \,, \tag{2.4a}$$

$$n_\mu \, \mathrm{d}x^\mu = -\, \mathrm{d}v \,, \qquad n^\mu \partial_\mu = -\frac{1}{\eta} \partial_r \,, \tag{2.4b}$$

where l^μ and n^μ are outward-pointing and inward-pointing, respectively, and they are normalized such that $l \cdot n = -1$. The induced metric on the transverse surface can be expressed as $q_{\mu\nu} = g_{\mu\nu} + 2\, l_{(\mu} n_{\nu)}$. The deviation tensor associated with the vector field l^μ is

$$B_{\mu\nu} := \left(q_\mu^\alpha q_\nu^\beta \nabla_\beta l_\alpha \right)_{r=0} = \frac{1}{D-2} \theta \, q_{\mu\nu} + N_{\mu\nu} \,, \tag{2.5}$$

where

$$\theta = \frac{\mathcal{D}_v \Omega}{\Omega} \,, \qquad N_{AB} = \frac{1}{2} \mathcal{D}_v \Omega_{AB} - \frac{1}{D-2} \frac{\mathcal{D}_v \Omega}{\Omega} \Omega_{AB} \,, \tag{2.6}$$

are respectively expansion and shear/news tensors. For later convenience, we introduce the following notations:

$$\mathcal{P} := \ln \left(\eta \, \theta^{-2} \right) , \tag{2.7a}$$

$$\gamma_{AB} := \Omega^{-2/(D-2)} \Omega_{AB} \,, \qquad \det \gamma_{AB} = 1 \,, \tag{2.7b}$$

$$\mathcal{N}_{AB} := \frac{1}{2} \mathcal{D}_v \gamma_{AB}, \qquad N_{AB} = \Omega^{2/(D-2)} \mathcal{N}_{AB} \,. \tag{2.7c}$$

We also assume that the unimodular metric γ_{AB} is invertible and denote its inverse as γ^{AB}, i.e. $\gamma_{AB} \gamma^{BC} = \delta_A{}^C$. As the notation suggests, these are scalar, vector, and tensor with respect to the codimension 2 diffeomorphisms along a constant v slice of the null hypersurface located at $r = 0$. These quantities carry A, B-type indices and we will use γ_{AB} or γ^{AB} to lower or raise indices on other quantities.

Equations of motion and solution space. To construct the solution space, we need to impose Einstein's equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, where Λ represents the cosmological constant and $G_{\mu\nu}$ is the Einstein tensor. We can solve Einstein's equations in an order by order manner in powers of r. Among these equations, there are two conspicuous equations, namely the Raychaudhuri and Damour equations. Once we expand metric functions and parameters around $r = 0$, we acquire: [44, 45]

- Two pairs of scalars, (Γ, Ω), $(\theta, \mathcal{P})$;
- A pair of vectors, $(\mathcal{U}^A, \mathcal{J}_A)$;
- A pair of two-tensors, $(\mathcal{N}_{AB}, \gamma^{AB})$;

These pairs are subject to equations of motion (constraints),[1]

$$-\partial_v\Omega+\vartheta\approx 0\,,\tag{2.8a}$$

$$-\partial_v\mathcal{P}+\varpi+2\,\theta^{-1}\mathcal{N}^2\approx 0\,,\tag{2.8b}$$

$$-\partial_v\mathcal{S}_A+\partial_B(\mathcal{U}^B\mathcal{S}_A)+\mathcal{S}_B\partial_A\mathcal{U}^B+2\,\Omega\,\partial_A(\theta^{-1}\mathcal{N}^2)-2\nabla^B(\Omega\,\mathcal{N}_{AB})\approx 0\,,\tag{2.8c}$$

$$-\partial_v\gamma_{AB}+\nabla_A\mathcal{U}_B+\nabla_B\mathcal{U}_A-\frac{2}{D-2}\nabla_C\mathcal{U}^C\gamma_{AB}+2\,\mathcal{N}_{AB}\approx 0\,,\tag{2.8d}$$

where ∇_A denotes the covariant derivative w.r.t. unimodular metric γ_{AB} and new quantities are defined as follows:

$$\mathcal{S}_A:=\mathcal{J}_A+\Omega\partial_A\mathcal{P}\,,\qquad \varpi:=\Gamma+\mathcal{U}^A\partial_A\mathcal{P},\qquad \vartheta:=\Omega\theta+\partial_A(\Omega\mathcal{U}^A)\,,\qquad \mathcal{N}^2:=\mathcal{N}_{AB}\mathcal{N}^{AB}\,.\tag{2.9}$$

$\mathcal{S}_A$ is the "spin superrotations" part (or superspin, for short) of the angular momentum (superrotations). It is derived by subtracting the orbital superrotations $-\Omega\partial_A\mathcal{P}$ from the total superrotations $\mathcal{J}_A$.

Equations (2.8a) and (2.8d) can be regarded as two equations of motion that specify the expansion and news tensor. The Damour (2.8c) specifies angular velocity aspect $\mathcal{U}^\mathcal{A}$ and the Raychaudhuri equation (2.8b) can be seen as an equation for ϖ. Therefore, the solution phase space for a null boundary is entirely characterized by two scalars $\Omega,\mathcal{P}$, one vector $\mathcal{S}_A$, and one tensor mode γ_{AB} (along with their canonical conjugates).

3 Off-shell null boundary symplectic form

For later convenience, we adopt the following notation:

$$\oint_{\mathcal{N}_b}=\int \mathrm{d}^{D-2}x\,,\qquad \int_{\mathcal{N}}=\int \mathrm{d}v\int \mathrm{d}^{D-2}x\,,\tag{3.1}$$

where $\mathcal{N}_b$ denotes a constant $v=v_b$ slice on the null boundary. For simplicity, we also assume that the codimension 2 transverse surface $\mathcal{N}_b$ is compact.

Off-shell symplectic form. The standard Lee-Wald off-shell symplectic form can be read from the Einstein-Hilbert action as [45]

$$\begin{aligned}\boldsymbol{\Omega}_{\text{LW}}[\delta\Phi,\delta\Phi;g]&=\frac{1}{16\pi G}\int_{\mathcal{N}}\left[\,\delta\mathcal{U}^A\wedge\delta\mathcal{J}_A-\delta\Gamma\wedge\delta\Omega+\delta(\Omega\,\theta)\wedge\delta\mathcal{P}+\delta\mathcal{N}_{AB}\wedge\delta(\Omega\,\gamma^{AB})\,\right]\\&=\frac{1}{16\pi G}\int_{\mathcal{N}}\left[\,\delta\mathcal{U}^A\wedge\delta\mathcal{S}_A+\delta\Omega\wedge\delta\varpi+\delta\vartheta\wedge\delta\mathcal{P}+\delta\mathcal{N}_{AB}\wedge\delta(\Omega\,\gamma^{AB})\,\right].\end{aligned}\tag{3.2}$$

where the above is subject to second-class constraints,

$$C_1=\det\gamma-1=0,\qquad C_2:=\gamma^{AB}\mathcal{N}_{AB}=0.\tag{3.3}$$

[1] In writing the Raychaudhuri (2.8b) and Damour (2.8c) equations, we assume that $\theta\neq 0$ (as we have divided these equations by θ to obtain these forms). In the special case, where $\theta=0$, the Raychaudhuri equation leads $\mathcal{N}_{AB}=0$ and the Damour equation takes the form $\mathcal{D}_v\mathcal{J}_A+\Omega\partial_A\Gamma=0$ [44].

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In our adopted coordinate system, r is taken to be the affine parameter along null geodesics generated by the null vector field n^μ. Hence, it bears a resemblance to the usual time coordinate, particularly when we are consider the Hamiltonian formulation based on the ADM decomposition. Strictly speaking, the bulk evolution is orchestrated by r. Consequently, the Lee-Wald symplectic form (3.2) can be viewed as a symplectic form written on a constant affine parameter. Advanced time is one of the coordinates on the specified hypersurface.

By inverting the above symplectic form, one can read nonvanishing "off-shell" Poisson brackets

$$\Big\{\mathcal{U}^A(v_1,\mathbf{x}_1),\mathcal{S}_B(v_2,\mathbf{x}_2)\Big\}_{\text{PB}} = 16\pi G\,\delta^A_B\,\delta(v_1-v_2)\,\delta^{D-2}(\mathbf{x}_1-\mathbf{x}_2)\,, \tag{3.4a}$$

$$\{\Omega(v_1,\mathbf{x}_1),\varpi(v_2,\mathbf{x}_2)\}_{\text{PB}} = 16\pi G\,\delta(v_1-v_2)\,\delta^{D-2}(\mathbf{x}_1-\mathbf{x}_2)\,, \tag{3.4b}$$

$$\{\vartheta(v_1,\mathbf{x}_1),\mathcal{P}(v_2,\mathbf{x}_2)\}_{\text{PB}} = 16\pi G\,\delta(v_1-v_2)\,\delta^{D-2}(\mathbf{x}_1-\mathbf{x}_2)\,, \tag{3.4c}$$

$$\Big\{\mathcal{N}_{AB}(v_1,\mathbf{x}_1),\Omega(v_2,\mathbf{x}_2)\,\gamma^{CD}(v_2,\mathbf{x}_2)\Big\}_{\text{DB}} = 16\pi G\,P^{CD}_{AB}(v_2,\mathbf{x}_2)\,\delta(v_1-v_2)\,\delta^{D-2}(\mathbf{x}_1-\mathbf{x}_2)\,. \tag{3.4d}$$

where

$$P^{CD}_{AB}(v) := \delta^C_{(A}\delta^D_{B)} - \frac{1}{D-2}\gamma_{AB}(v)\gamma^{CD}(v)\,, \tag{3.5}$$

is a projection operator, projecting tensors to their trace-less parts. In the expression above, $\delta^{D-2}(\mathbf{x}_1-\mathbf{x}_2)$ is merely a product of $D-2$ delta-functions and does not involve Ω. Additionally, (3.4d) is the Dirac bracket, which is subjected to the constraints (3.3), see appendix F.1 for more details of the calculation. P^{CD}_{AB} is symmetric w.r.t. A,B and C,D indices and it is traceless, $P^{CD}_{AB}\gamma_{CD} = P^{CD}_{AB}\gamma^{AB} = 0$. It is worth noting that (3.4d) can also be expressed as

$$\{\mathcal{N}_{AB}(x_1),\gamma_{CD}(x_2)\}_{\text{DB}} = -\frac{16\pi G}{\Omega(x_2)}\mathcal{G}_{ABCD}(x_2)\;\delta(v_1-v_2)\,\delta^{D-2}(\mathbf{x}_1-\mathbf{x}_2)\,, \tag{3.6}$$

where

$$\mathcal{G}_{ABCD} := \frac{1}{2}\left(\gamma_{AC}\gamma_{BD} + \gamma_{AD}\gamma_{BC} - \frac{2}{D-2}\gamma_{AB}\gamma_{CD}\right)\,, \tag{3.7}$$

is the Wheeler-DeWitt (WdW) metric [54]. This exhibits the following symmetry structures and trace properties,

$$\begin{gathered}\mathcal{G}_{ABCD} = \mathcal{G}_{BACD} = \mathcal{G}_{ABDC} = \mathcal{G}_{CDAB}\,,\\ \gamma^{AB}\mathcal{G}_{ABCD} = \gamma^{CD}\mathcal{G}_{ABCD} = 0\,, \qquad \gamma^{AC}\mathcal{G}_{ABCD} = \frac{D(D-3)}{2(D-2)}\gamma_{BD}\,.\end{gathered} \tag{3.8}$$

Note that the above Poisson brackets are still subject to the "constraint equations" (2.8). We will impose these constraints in the next section.

4 On-shell symplectic form and Dirac brackets

One can view the equations of motion (2.8) as (second-class) constraints and compute Dirac brackets. Alternatively, these equations can be directly inserted into the symplectic form (3.2) to compute the "on-shell symplectic form". In the conventional constrained system terminology, this process involves solving the constraints and going to the reduced phase space. We have shown in appendix A that, for a null-front field theory, these two methods yield the same result. In the other words, the Poisson brackets derived from the "on-shell" symplectic form defined on the reduced phase space (solution space) and the Dirac brackets using half of the Hamilton's equations as constraints (over the solution phase space) are equivalent.

In our particular case, on the reduced phase space, we have one canonical pair of scalars, one vector, and one tensor mode. We opt to solve (2.8b) for ϖ (while keeping $\mathcal{P}$), solve (2.8a) for ϑ (while keeping Ω), solve (2.8c) for $\mathcal{U}^A$ (while keeping the spin superrotations $\mathcal{S}_A$) and ultimately solve (2.8d) to determine $\mathcal{N}^{AB}$ (while keeping γ_{AB}). Explicitly,

$$\vartheta = \partial_v \Omega \,, \qquad \varpi = \partial_v \mathcal{P} - \frac{2}{\theta}\mathcal{N}^2 \,, \qquad \mathcal{N}_{AB} = \frac{1}{2}\mathcal{D}_v \gamma_{AB} \,, \tag{4.1a}$$

$$\mathcal{U}^A = \bar{\mathcal{U}}^A(\mathcal{S}_A) + (\mathcal{O}^{-1})^{AB} \left(\partial_v \mathcal{S}_B - 2\mathcal{X}_B\right) \,, \quad \mathcal{O}_{AB} := \mathcal{S}_A \partial_B + \mathcal{S}_B \partial_A + \partial_B \mathcal{S}_A \,, \tag{4.1b}$$

where $\mathcal{O}_{AB}\bar{\mathcal{U}}^B = 0$, $\mathcal{X}_A = \partial_A(\theta^{-1}\mathcal{N}^2) - \nabla^B(\Omega \mathcal{N}_{AB})$ and $(\mathcal{O}^{-1})^{AB}$ is the inverse of $\mathcal{O}_{AB}$ which has a linear functional dependence on $\mathcal{S}_A$.

4.1 Vanishing news $\mathcal{N}_{AB} = 0$ case

For this case, by plugging (4.1) into the symplectic form, we obtain [45]

$$\begin{aligned} \boldsymbol{\Omega}_{\text{LW}} &= \frac{1}{16\pi G}\int_{\mathcal{N}} \partial_v\left(\delta\Omega \wedge \delta\mathcal{P} + \mathcal{F}^{AB}\delta\mathcal{S}_A \wedge \delta\mathcal{S}_B\right) \\ &= \frac{1}{16\pi G}\oint_{\mathcal{N}_b} \left[\delta\Omega \wedge \delta\mathcal{P} + \mathcal{F}^{AB}\delta\mathcal{S}_A \wedge \delta\mathcal{S}_B\right] , \end{aligned} \tag{4.2}$$

where $\mathcal{F}^{AB} = (\{\mathcal{S}_A, \mathcal{S}_B\})^{-1}$ (see below). It is evident that in the absence of $\mathcal{N}_{AB}$ modes, the on-shell symplectic form simplifies into a boundary term represented as an integral over $\mathcal{N}_b$ [46]. That is, when $\mathcal{N}_{AB} = 0$ the system is solely described by boundary dof. The symplectic form (4.2) is a closed 2-form and is invertible. The inverse symplectic form yields Poisson brackets,

$$\begin{aligned} &\{\Omega(v_b, \mathbf{x}), \mathcal{P}(v_b, \mathbf{y})\}_{\text{DB}} = 16\pi G\ \delta^{D-2}(\mathbf{x} - \mathbf{y}) \,, \\ &\{\Omega(v_b, \mathbf{x}), \Omega(v_b, \mathbf{y})\}_{\text{DB}} = \{\mathcal{P}(v_b, \mathbf{x}), \mathcal{P}(v_b, \mathbf{y})\}_{\text{DB}} = 0 \,, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\{\Omega(v_b, \mathbf{x}), \mathcal{S}_A(v_b, \mathbf{y})\}_{\text{DB}} = \{\mathcal{P}(v_b, \mathbf{x}), \mathcal{S}_A(v_b, \mathbf{y})\}_{\text{DB}} = 0 \,, \\ &\{\mathcal{S}_A(v_b, \mathbf{x}), \mathcal{S}_B(v_b, \mathbf{y})\}_{\text{DB}} = 16\pi G \left(\mathcal{S}_A(v_b, \mathbf{y})\frac{\partial}{\partial x^B} - \mathcal{S}_B(v_b, \mathbf{x})\frac{\partial}{\partial y^A}\right)\delta^{D-2}(\mathbf{x} - \mathbf{y}) \,. \end{aligned} \tag{4.4}$$

The above is Heisenberg $\oplus$ Diff($\mathcal{N}_b$) algebra discussed in [45], see also [46, 55].

4.2 Nonvanishing $\mathcal{N}_{AB}$ and co-rotating case

To keep equations less cumbersome and to illustrate how turning on the news affects the symplectic form, we study the co-rotating case with $\mathcal{U}_A = 0$. In this case, the on-shell symplectic potential and symplectic form respectively take the form

$$16\pi G\ \boldsymbol{\Theta}_{\text{on-shell}} = \oint_{\mathcal{N}_b} \Omega\, \delta\mathcal{P} - \int_{\mathcal{N}} \Omega \mathcal{N}^{AB}\left[\delta\gamma_{AB} - 2\,\frac{\mathcal{N}_{AB}}{\Omega\theta}\,\delta\Omega\right] , \tag{4.5a}$$

$$16\pi G\ \boldsymbol{\Omega}_{\text{on-shell}} = \oint_{\mathcal{N}_b} \delta\Omega \wedge \delta\mathcal{P} + \int_{\mathcal{N}}\left[-\delta\Omega^2 \wedge \delta\left(\frac{\mathcal{N}^2}{\partial_v\Omega}\right) + \frac{1}{2}\partial_v\delta\gamma_{AB} \wedge \delta(\Omega\,\gamma^{AB})\right] . \tag{4.5b}$$

In the above, "on-shell" means imposing (2.8) with $\mathcal{U}^A = 0$, explicitly, this means

$$\theta = \frac{\partial_v \Omega}{\Omega}\,, \qquad \mathcal{N}_{AB} = \frac{1}{2}\partial_v \gamma_{AB}\,. \tag{4.6}$$

The Damour equation takes the form $\partial_v \mathcal{S}_A = 2\,\Omega\,\partial_A(\theta^{-1}\mathcal{N}^2) - 2\nabla^B(\Omega\,\mathcal{N}_{AB})$ which fully specifies the time dependence of the angular momentum aspect. Strictly speaking, the Damour equation specifies $\mathcal{S}_A$ up to codimension 2 functions, denoted as $\bar{\mathcal{S}}_A(v_b, \boldsymbol{x})$. These integration constants provide us with angular momentum charges. Therefore, the co-rotating reduced phase space is spanned by $\Omega\,, \mathcal{P}\,, \gamma_{AB}$.

The on-shell symplectic form consists of a codimension 1 integral over $\mathcal{N}$ and a codimension 2 part, integral over $\mathcal{N}_b$. However, there still remains an arbitrariness in separating them into boundary and bulk parts, as $\oint_{\mathcal{N}_b} X = \int_{\mathcal{N}} \partial_v X$. This arbitrariness may be removed upon some other (physical) requirements. For example, we may require that the boundary and bulk parts are closed 2-forms over the solution space:

$$\boldsymbol{\Omega}_{\text{on-shell}} = \boldsymbol{\Omega}^c_{\text{bdy}} + \boldsymbol{\Omega}^c_{\text{Bulk}}\,, \tag{4.7a}$$

$$\boldsymbol{\Omega}^c_{\text{bdy}} = \frac{1}{16\pi G}\oint_{\mathcal{N}_b} \delta\Omega \wedge \delta\mathcal{P}\,, \qquad \boldsymbol{\Omega}^c_{\text{Bulk}} = \frac{1}{16\pi G}\int_{\mathcal{N}}\left[-\delta\Omega^2 \wedge \delta\left(\frac{\mathcal{N}^2}{\partial_v\Omega}\right) + \frac{1}{2}\partial_v\delta\gamma_{AB}\wedge\delta(\Omega\,\gamma^{AB})\right]. \tag{4.7b}$$

As another way to separate the symplectic form into bulk and boundary parts is to stipulate that the bulk part follows the generic form outlined in appendix A, i.e. the bulk part takes the form $\int_{\mathcal{N}} \not{\delta} X \wedge \partial_v \not{\delta} X$, where $\not{\delta}$ is used to emphasis that $\not{\delta} X$ is not necessarily a closed 1-form on the phase space. After some manipulations, the details of which are given in appendix E, one arrives at the following expression:

$$\boldsymbol{\Omega}_{\text{on-shell}} = \boldsymbol{\Omega}_{\text{bdy}} + \boldsymbol{\Omega}_{\text{Bulk}}\,, \tag{4.8a}$$

$$\boldsymbol{\Omega}_{\text{bdy}} = \frac{1}{16\pi G}\oint_{\mathcal{N}_b} \delta\Omega \wedge \not{\delta}\hat{\mathcal{P}}\,, \qquad \boldsymbol{\Omega}_{\text{Bulk}} = \frac{1}{32\pi G}\int_{\mathcal{N}} \Omega\mathcal{G}^{ABCD}\, \not{\delta}\hat{\gamma}_{AB}\wedge\partial_v\not{\delta}\hat{\gamma}_{CD}\,, \tag{4.8b}$$

where non-closed 1-forms are defined as

$$\not{\delta}\hat{\mathcal{P}} := \delta\mathcal{P} - \frac{\mathcal{N}^{AB}}{\theta}\delta\gamma_{AB} = \delta\mathcal{P} + \frac{\Omega\partial_v\gamma^{AB}}{2\partial_v\Omega}\delta\gamma_{AB}\,, \tag{4.9a}$$

$$\not{\delta}\hat{\gamma}_{AB} := \delta\gamma_{AB} - \frac{2\mathcal{N}_{AB}}{\Omega\theta}\delta\Omega = \delta\gamma_{AB} - \frac{\partial_v\gamma_{AB}}{\partial_v\Omega}\delta\Omega\,, \tag{4.9b}$$

and

$$\mathcal{G}^{ABCD} := \frac{1}{2}\left(\gamma^{AC}\gamma^{BD} + \gamma^{AD}\gamma^{BC} - \frac{2}{D-2}\gamma^{AB}\gamma^{CD}\right), \tag{4.10}$$

is the inverse WdW metric (3.7) which possesses the same properties in (3.8) and

$$\mathcal{G}^{ABEF}\mathcal{G}_{EFCD} = P^{AB}_{CD}\,, \qquad P^{AB}_{CD}\mathcal{G}^{CDEF} = \mathcal{G}^{ABEF}\,, \qquad P^{AB}_{CD}\mathcal{G}_{ABEF} = \mathcal{G}_{CDEF}\,. \tag{4.11}$$

The relation between the two boundary/bulk separations mentioned above is as follows:

$$\boldsymbol{\Omega}_{\text{bdy}} = \boldsymbol{\Omega}^c_{\text{bdy}} + \mathbf{F}_{\text{bdy}}\,, \qquad \boldsymbol{\Omega}_{\text{Bulk}} = \boldsymbol{\Omega}^c_{\text{Bulk}} - \mathbf{F}_{\text{bdy}}\,, \tag{4.12}$$

where

$$\mathbf{F}_{\rm bdy} = -\frac{1}{16\pi G}\oint_{\mathcal{N}_b} \frac{\mathcal{N}^{AB}}{\theta}\delta\Omega \wedge \delta\gamma_{AB} = -\frac{1}{16\pi G}\oint_{\mathcal{N}_b} \frac{\mathcal{N}^{AB}}{\theta}\delta\Omega \wedge \not{\delta}\hat{\gamma}_{AB}\,, \tag{4.13}$$

represents the boundary symplectic flux due to the passage of the bulk flux. $\mathbf{\Omega}^{c}_{\rm bdy}$ as given in (4.8b) is closed and invertible over $\mathcal{N}_b$, yielding the boundary Poisson brackets (4.3). On the other hand, $\mathbf{\Omega}^{c}_{\rm Bulk}$, while closed, is not invertible over $\mathcal{N}$. We will discuss this further in the next sections.

Further comments. Given the above on-shell symplectic forms, some comments are in order:

I. The appearance of non-exact solution space one-forms $\not{\delta}\hat{\mathcal{P}}$ and $\not{\delta}\hat{\gamma}_{AB}$ is the hallmark of the fact that neither of the boundary nor bulk parts of the phase space are individually closed.

II. While the boundary and bulk terms are not closed, the combined bulk+boundary system is closed, $\delta\mathbf{\Omega}_{\text{on-shell}} = 0$.

III. The closure of the bulk+boundary symplectic form is necessary, but not a sufficient condition for invertibility of the symplectic form. Upon closer inspection, it becomes apparent that $\mathbf{\Omega}_{\text{on-shell}}$ is not invertible over the solution space on $\mathcal{N}$.

IV. The non-closedness of individual bulk and boundary parts is a manifestation of the non-integrability of charges associated with symmetry generators, as discussed in appendix C. See also appendix D for the analysis of surface charges and their algebra.

V. Non-closed parts of $\not{\delta}\hat{\mathcal{P}}$ and $\not{\delta}\hat{\gamma}_{AB}$ are proportional to the news $\mathcal{N}_{AB}$. When $\not{\delta}\hat{\gamma}_{AB} = 0$, it indicates a degeneracy direction of the bulk symplectic form $\mathbf{\Omega}_{\rm Bulk}$. Nonetheless, note that $\mathbf{\Omega}_{\rm Bulk}$ is invertible in the (codimension 1) subspace spanned by γ_{AB}. In the next section, we will show that the bulk part of the on-shell solution space is indeed a Carrollian geometry. By inverting the invertible parts of the bulk and boundary symplectic forms, namely

$$\mathbf{\Omega}^{\rm I}_{\rm Bulk} = \frac{1}{32\pi G}\int_{\mathcal{N}} \Omega\, \mathcal{G}^{ABCD}\ \delta\gamma_{AB}\wedge\partial_v\delta\gamma_{CD}\,, \qquad \mathbf{\Omega}^{\rm I}_{\rm bdy} = \frac{1}{16\pi G}\oint_{\mathcal{N}_b} \delta\Omega\wedge\delta\mathcal{P}\,, \tag{4.14}$$

over the null boundary $\mathcal{N}$ and $\mathcal{N}_b$ respectively yields (see appendix F.2 for more details)

$$\{\Omega(v_b,\mathbf{x}),\mathcal{P}(v_b,\mathbf{x}')\} = 16\pi G\,\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,, \tag{4.15a}$$

$$\{\Omega(v_b,\mathbf{x}),\gamma_{AB}(v,\mathbf{x}')\} = 0\,, \tag{4.15b}$$

$$\{\mathcal{P}(v_b,\mathbf{x}),\gamma_{AB}(v,\mathbf{x}')\} = 0\,, \tag{4.15c}$$

$$\{\gamma_{AB}(v,\mathbf{x}),\gamma_{CD}(v',\mathbf{x}')\} = \frac{16\pi G}{\sqrt{\Omega(x)\Omega(x')}}\,\mho^{\rm S}_{ABCD}(v,v';\mathbf{x},\mathbf{x}')\,H(v-v')\ \delta^{D-2}(\mathbf{x}-\mathbf{x}')\,, \tag{4.15d}$$

where $\mho^{\rm S}_{ABCD}(v,v';\mathbf{x},\mathbf{x}')$ is given by (F.33).

Notice that to obtain (4.15d) we assume $D > 3$. For $D = 3$ pure Einstein gravity, we do not have any bulk modes, explicitly γ_{AB}. In $D = 3$ case the solution space is solely governed by the boundary modes. See [42] for a detailed analysis.

5 Bulk part of solution phase space is a Carrollian geometry

Recalling the analysis of appendix A, in particular, (A.17), the on-shell symplectic form is expected to carry information about the metric over the field space. Let us focus on the bulk part of the symplectic form

$$\boldsymbol{\Omega}_{\text{Bulk}} = \frac{1}{32\pi G} \int_{\mathcal{N}} \int_{\mathcal{N}}' \delta\varphi^{\mathbb{I}}(x)\, \boldsymbol{\Omega}_{\mathbb{IJ}}[x;x'] \wedge \delta\varphi^{\mathbb{J}}(x')\,, \tag{5.1}$$

where $\varphi^{\mathbb{I}} = \{\Omega\,, \gamma_{AB}\}$ denotes the bulk phase space variables and $\boldsymbol{\Omega}_{\mathbb{IJ}}[x;x']$ is a $(D-1)(D-2)/2$ dimensional antisymmetric matrix. Recalling (4.8b) we can write it as

$$\boldsymbol{\Omega}_{\mathbb{IJ}}[x;x'] = \boldsymbol{\mathcal{G}}_{\mathbb{IJ}}[x;x']\, \partial_v \delta(v-v')\, \delta^{D-2}(\mathbf{x}-\mathbf{x}')\,, \tag{5.2}$$

where $\boldsymbol{\mathcal{G}}_{\mathbb{IJ}}[x;x'] = \boldsymbol{\mathcal{G}}_{\mathbb{JI}}[x';x]$ and

$$\boldsymbol{\mathcal{G}}_{\mathbb{IJ}}[x;x'] = \begin{pmatrix} \boldsymbol{\mathcal{G}}_{\Omega\Omega}[x;x'] & (\boldsymbol{\mathcal{G}}_{\Omega\gamma})^{AB}[x;x'] \\ (\boldsymbol{\mathcal{G}}_{\gamma\Omega})^{AB}[x;x'] & (\boldsymbol{\mathcal{G}}_{\gamma\gamma})^{ABCD}[x;x'] \end{pmatrix}\,, \tag{5.3}$$

with

$$\boldsymbol{\mathcal{G}}_{\Omega\Omega}[x;x'] = 4\, (\boldsymbol{\mathcal{G}}_{\gamma\gamma})^{ABCD}[x;x'] \frac{\mathcal{N}_{AB}(x)}{\Omega(x)\theta(x)} \frac{\mathcal{N}_{CD}(x')}{\Omega(x')\theta(x')}\,, \tag{5.4a}$$

$$(\boldsymbol{\mathcal{G}}_{\Omega\gamma})^{AB}[x;x'] = -2\, (\boldsymbol{\mathcal{G}}_{\gamma\gamma})^{ABCD}[x;x'] \frac{\mathcal{N}_{CD}(x)}{\Omega(x)\theta(x)}\,, \tag{5.4b}$$

$$(\boldsymbol{\mathcal{G}}_{\gamma\gamma})^{ABCD}[x;x'] = \sqrt{\Omega(x)\Omega(x')}\, \mathcal{G}^{ABCD}[x;x']\,, \tag{5.4c}$$

where $\mathcal{G}^{ABCD}[x;x']$ is the *point-split WdW metric*:

$$\begin{aligned} \mathcal{G}^{ABCD}[x;x'] = \frac{1}{4}\Big[&\gamma^{AC}(x)\gamma^{BD}(x') + \gamma^{AD}(x)\gamma^{BC}(x') + \gamma^{AC}(x')\gamma^{BD}(x) \\ &+ \gamma^{AD}(x')\gamma^{BC}(x) - \frac{4}{D-2}\gamma^{AB}(x)\gamma^{CD}(x')\Big]\,. \end{aligned} \tag{5.5}$$

such that $\mathcal{G}^{ABCD}[x;x] = \mathcal{G}^{ABCD}(x)$ is the traceless WdW metric given in (4.10).

The symmetric matrix $\boldsymbol{\mathcal{G}}_{\mathbb{IJ}}$ is the metric over the bulk part of the Lagrangian submanifold of the solution phase space. Explicitly, we may define "line element" over this space as:

$$\begin{aligned} \delta\mathbb{S}^2 := \int_{\mathcal{N}}\int_{\mathcal{N}}' \boldsymbol{\mathcal{G}}_{\mathbb{IJ}}[x;x']\delta\varphi^{\mathbb{I}}(x)\,\delta\varphi^{\mathbb{J}}(x') &= \int_{\mathcal{N}}\int_{\mathcal{N}}' \sqrt{\Omega(x)\Omega(x')}\,\mathcal{G}^{ABCD}[x;x'] \not{\delta}\hat{\gamma}_{AB}(x) \not{\delta}\hat{\gamma}_{CD}(x') \\ = \int_{\mathcal{N}}\int_{\mathcal{N}}' \sqrt{\Omega(x)\Omega(x')}\,\mathcal{G}^{ABCD}[x;x'] &\left(\delta\gamma_{AB}(x) - \frac{\partial_v\gamma_{AB}(x)}{\partial_v\Omega(x)}\delta\Omega(x)\right)\left(\delta\gamma_{CD}(x') - \frac{\partial_{v'}\gamma_{CD}(x')}{\partial_{v'}\Omega(x')}\delta\Omega(x')\right). \end{aligned} \tag{5.6}$$

This manifold is describing a Carrollian geometry (see appendix B for more discussions), as it has a kernel vector

$$K^{\mathbb{I}}[x]\,\frac{\delta}{\delta\varphi^{\mathbb{I}}} = \frac{\delta}{\delta\Omega} + \frac{2\mathcal{N}_{AB}}{\Omega\theta}\,\frac{\delta}{\delta\gamma_{AB}}\,, \tag{5.7}$$

such that $\boldsymbol{\mathcal{G}}_{\mathbb{IJ}}[x;x']K^{\mathbb{J}}[x']=0$ and the $D(D-3)/2$ dimensional metric $\sqrt{\Omega(x)\Omega(x')}\,\mathcal{G}^{ABCD}[x;x']$. The integral curves of this kernel vector on the phase space are given by

$$\gamma_{AB}(x)=\gamma_{AB}(\Omega(x),x^C)+\tilde{\gamma}_{AB}(v_b,x^C)\,. \tag{5.8}$$

We can also introduce an extra structure and define the Ehresmann connection

$$E_{\mathbb{I}}\,\delta\varphi^{\mathbb{I}}=\delta\Omega+X^{AB}\not{\delta}\hat{\gamma}_{AB}\,, \tag{5.9}$$

such that $K^{\mathbb{I}}[x]E_{\mathbb{I}}[x]=1$. One can choose X^{AB} to be zero, without loss of generality. This Carrollian structure sheds further light on the bulk part of the symplectic form that is non-invertible.

6 Various sub-sectors of the co-rotating solution space

In this section, we explore the various sub-sectors of the solution space. The first notable sector is characterized by the vanishing news $\mathcal{N}_{AB}=0$ within the solution space. In this case, boundary surface charges become integrable. In an appropriate slicing of the solution space, this yields a Heisenberg plus *Diff*$(\mathcal{N}_b)$ algebra (4.3), (4.4). This sector has been extensively studied in previous works [44, 45, 56] and we will not repeat it here. In the rest of this section, we discuss other interesting subsectors.

6.1 Outgoing Robinson-Trautman gravitational wave sector

From (4.9) we infer that when $\not{\delta}\hat{\mathcal{P}}=0$ or $\not{\delta}\hat{\gamma}_{AB}=0$, respectively the boundary or bulk directions of the symplectic form vanish. When $\not{\delta}\hat{\gamma}_{AB}=0$,

$$\mathcal{N}_{AB}=\frac{1}{2}\frac{\partial\Omega}{\partial v}\frac{\delta\gamma_{AB}}{\delta\Omega}\qquad\Rightarrow\qquad\gamma_{AB}(x)=\gamma_{AB}(\Omega(x),\mathbf{x})+\tilde{\gamma}_{AB}(v_b,\mathbf{x})\,. \tag{6.1}$$

That is, $\not{\delta}\hat{\gamma}_{AB}=0$ along the integral curves of the kernel vector on the Carrollian solution space. The above describes the (non-rotating) Robinson-Trautman "spherical gravitational waves" solutions [51, 52]. One can readily check that $\mathbf{F}_{\text{bdy}}$ vanishes and hence $\boldsymbol{\Omega}_{\text{on-shell}}=\boldsymbol{\Omega}^{\text{c}}_{\text{bdy}}$.[2] This yields Poisson brackets (4.3).

The Robison-Trautman solutions are particular gravitational waves in several respects. They are entirely characterized by the scalar (density) mode Ω defined over the null surface $\mathcal{N}$ as opposed to the traceless-symmetric modes $\mathcal{N}_{AB}$. These solutions do not contribute to the bulk or boundary symplectic forms as well as the boundary symplectic flux, given in (4.12). In our analysis Ω serves a dual role: it acts as a boundary mode, associated with the entropy density in the null surface thermodynamics description [45] and as a bulk mode, parameterizing the Robinson-Trautman solution.

6.2 Non-expanding null boundaries

In this subsection, we consider the non-expanding null surfaces $\theta=0$ which, using boundary equations of motion, yields $\Omega=\Omega(v_b,\mathbf{x})$. In this case, the Raychaudhuri equation leads to

[2] In this sector $\not{\delta}\hat{\mathcal{P}}=\delta\mathcal{P}-\frac{2\mathcal{N}^2}{\Omega\theta^2}\delta\Omega=\delta\mathcal{P}+\delta\mathcal{F}(\Omega)$, where $\delta\mathcal{F}/\delta\Omega=\frac{1}{2}\frac{\delta\gamma_{AB}}{\delta\Omega}\frac{\delta\gamma^{AB}}{\delta\Omega}\Omega$.

$\mathcal{N}_{AB} = 0$ [44, 45]. The Damour equation simplifies to $\mathcal{D}_v\mathcal{J}_A + \Omega\partial_A\Gamma = 0$. For the "co-rotating" ($\mathcal{U}^A = 0$) case we are considering here, this equation reduces to $\partial_v\mathcal{J}_A = -\Omega(v_b, \mathbf{x})\partial_A\Gamma$, which fixes the v-dependence of $\mathcal{J}_A$ in terms of $\Gamma = \Gamma(v, \mathbf{x})$. The on-shell symplectic form in the non-expanding case becomes

$$\mathbf{\Omega}_{\text{NE}}[\delta\Omega, \delta\Gamma] = \frac{1}{16\pi G}\int_{\mathcal{N}} \delta\Omega(v_b, \mathbf{x}) \wedge \delta\Gamma(v, \mathbf{x})\,. \tag{6.2}$$

If $\Gamma(v, \mathbf{x}) = \partial_v\rho(v, \mathbf{x})$, then $\mathcal{J}_A = -\Omega(v_b, \mathbf{x})\partial_A\rho + \bar{\mathcal{J}}_A(v_b, \mathbf{x})$ and the on-shell symplectic form takes the form

$$\mathbf{\Omega}_{\text{NE}}[\delta\Omega, \delta\rho] = \frac{1}{16\pi G}\oint_{\mathcal{N}_b} \delta\Omega(v_b, \mathbf{x}) \wedge \delta\rho(v_b, \mathbf{x})\,. \tag{6.3}$$

This yields the on-shell Poisson brackets

$$\{\Omega(v_b,\mathbf{x}),\Omega(v_b,\mathbf{y})\} = 0\,, \quad \{\Omega(v_b,\mathbf{x}),\rho(v_b,\mathbf{y})\} = 16\pi G\delta^{D-2}(\mathbf{x}-\mathbf{y})\,, \quad \{\rho(v_b,\mathbf{x}),\rho(v_b,\mathbf{y})\} = 0\,. \tag{6.4}$$

So, we have a quantum mechanical system at $\mathcal{N}_b$ for the non-expanding non-rotating case. Strictly speaking, we have turned off all of the gravitational waves, including outgoing Robinson-Trautman mode, and hence Ω becomes solely a boundary mode.

6.3 Decoupling of bulk modes from boundary modes

As discussed, Ω is a special degree of freedom with a dual role- it appears both as a boundary and a bulk mode, and it couples the boundary and bulk parts of the symplectic form. We can identify a special subsector of the solution space, by imposing the $\delta\Omega = 0$ condition. As it is evident from (4.8b) that when $\delta\Omega = 0$ the boundary symplectic form vanishes, and the bulk part becomes both closed and invertible. Specifically, in this sector

$$\mathbf{\Omega}_{\text{on-shell}} = \frac{1}{2}\int_{\mathcal{N}} \Omega\,\mathcal{G}^{ABCD}\;\delta\gamma_{AB} \wedge \partial_v\delta\gamma_{CD}\,. \tag{6.5}$$

This symplectic form is invertible and yields Poisson brackets (4.15d) (see appendix F for more details).

7 Discussion and outlook

Building upon the results presented in [44–46], where the null boundary solution (phase) space for D dimension was constructed, and surface charges analyzed, we focused more on the symplectic structure on the solution space and the associated Poisson brackets. The motivation behind this study is paving the way for the "quantization" of the null boundary solution space. This endeavor is aimed at addressing fundamental questions in black hole physics, particularly the microstates, and the information problem. The null boundary solution space is intricately structured, comprising both boundary and bulk parts that interact through the symplectic flux (4.13). For stationary black holes, we deal with the solution space restricted to $\theta = 0$, cf. section 6.2. In this case, the symplectic flux vanishes and the boundary system is described by $\Omega(v_b, \mathbf{x}), \rho(v_b, \mathbf{x})$ that form a Heisenberg algebra in which $16\pi G$ plays the role of $\hbar$. In this system, Ω is the entropy density, while ρ is its canonical/thermodynamical conjugate, playing the role of temperature. Assuming that $\mathcal{N}_b$ is compact with finite volume, upon applying semiclassical quantization, i.e. promoting the solution space to a Hilbert space, Ω and ρ to operators while substituting Poisson bracket (6.4) with commutators, we get quantization of the "horizon area" and/or variations in temperature. Further investigation into this semiclassical quantization could provide valuable insights.

Both the bulk and boundary systems are open as their symplectic 2-forms are nonclosed. However, when combined as the bulk+boundary system, the symplectic form becomes closed. Nonetheless, the symplectic form of the whole system is not invertible, indicating that the Poisson brackets in the invertible subsectors do not carry the whole information in the system. This non-invertibility has two sources: the existence of symplectic flux (4.13) and the Carrollian structure of the bulk solution space. This Carrollian structure is in the solution space and should not be mistaken with the Carrollian geometry of the null boundary $\mathcal{N}$. The former is infinite-dimensional, dealing with functions over $\mathcal{N}$, and occurs in the $(D-1)(D-2)/2$ dimensional matrices at any given point on $\mathcal{N}$. The null direction on the Carrollian solution space corresponds to the class of Robinson-Trautman (RT) gravitational waves [51, 52]. The existence of Carrollian solution space will have interesting and important ramifications to questions regarding black holes, which need to be thoroughly explored. Here we just mention two such ramifications. Our analysis clearly states that RT solution contributes neither to the boundary flux nor to the null boundary symplectic form. This implies that Hawking radiation in a pure gravity theory would be mainly dominated by the modes that are not of the RT type. Hence, it naively would not be relevant to the resolution of the information problem. Additionally, the appearance of the RT solution as the null direction in the solution space Carrollian geometry implies that we should label these geometries by charges that may be computed from our symplectic form. We hope to return to this point in an upcoming publication.

In addition to our previous line of research, our analysis here is partially motivated by the recent paper [53]. In comparison to our *on-shell* symplectic form, that paper focuses on *partially off-shell* symplectic form and inverts it to read off-shell Poisson brackets. In our terminology, 'off-shell' is when none of the null boundary constraints, (2.8), are imposed; 'partially off-shell' when only the expansion and news constraints, (2.8a) and (2.8d), are imposed and 'on-shell' is when all null boundary constraints, (2.8), are imposed. Similar to our analysis in section 4.2, [53] also considers the non-rotating case (which more precisely should be called co-rotating). Dealing with null boundaries and formulation of gravity on null-fronts, as illustrated in appendix A, it is physically meaningful (and necessary) to consider on-shell symplectic form and work with solution space. Furthermore, the authors of [53] have excluded the $\mathcal{P}$ mode in their analysis, whereas our analysis includes this mode along with its corresponding chemical potential in the boundary symplectic form.

To state our main results, namely on-shell symplectic form (4.8) and the Carrollian nature of the null boundary solution space (cf. section 5), we focused on the co-rotating sector in which the "super-spin" part $\mathcal{S}_A$ (and its off-shell canonical conjugate $\mathcal{U}^A$) have been turned off. We expect both of these results to extend to the most general solution space where $\mathcal{S}_A$ is also included. Explicitly, we expect an additional term in $\boldsymbol{\Omega}_{\text{bdy}}$ (4.8) which contains $\oint\!\!\!\!/\,\mathcal{S}_A \wedge \oint\!\!\!\!/\,\mathcal{S}_B$ where $\oint\!\!\!\!/\,\mathcal{S}_A$ is nonclosed 1-form and the nonclosed part is proportional to the divergence of the news, the (Bondi) angular momentum aspect. We also expect the bulk part of solution space to still remains Carrollian.

In our analysis, we did not focus on the surface charges and only briefly mentioned them in appendix D. These charges and their variations may be used to define horizon memory effect [44] and null surface thermodynamics [45]. Our symplectic form analysis here and the notion of symplectic flux shed some further light on both of these effects. In particular, it would be instructive to study if there is a horizon memory effect involving RT solutions and the null direction on the Carrollian solution space. Our analysis here on null boundary $\mathcal{N}$, with slight modifications, can be extended to asymptotic null infinity. It would be interesting to study the memory effect involving RT solutions in that setting.

Finally, we would like to comment on the possible relation between the two Carrollian geometries we have in our setting: Carrollian geometry on the $D-1$ dimensional null boundary and the one on the bulk solution space. One may argue that the Carrollian structure of the solution space is a consequence of studying the solution space in the presence of a null boundary. A careful inspection of the details of the construction of the null boundary solution space (see [44]) reveals that Ω being along the kernel vector in the bulk solution space stems from the fact that in metric (2.1) $V=0$ at $\mathcal{N}$. This latter is of course equivalent to dealing with a null boundary. If this intuition is correct, this means that replacing the null boundary with a timelike boundary, e.g. replacing horizon with the stretched horizon, should also lift the null direction in the solution space. In other words, the Carrollian structure of the solution space is a consequence of dealing with a null boundary and in a timelike boundary case we do not expect to see a Carrollian solution space. Explicitly, our intuition is that $V_0 = V(r=0)$ in the solution space is expected to play the role of speed of light in usual Lorentzian geometry, such that in $V_0 \to 0$ limit the solution space becomes Carrollian. It would be instructive to establish this point.

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A Symplectic analysis of massless scalar theory on the light front

Let us consider flat spacetime in the light-cone coordinate system with the line element

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}\,, \tag{A.1}$$

where u and v are null coordinates, and $\mathbf{x}^A$ are coordinates on a constant (u,v) $(D-2)$-surface. The action for a system of massless scalar fields $\phi_i, i=1,2,\cdots,N$, can be written as:

$$S=\int \mathrm{d}u\, L[\phi_i], \qquad L[\phi_i]=\int_{u=cte.} \mathrm{d}v\, \mathrm{d}^{D-2}x\, \mathcal{L}, \qquad \mathcal{L}=G^{ij}\left[\partial_v\phi_i\,\partial_u\phi_j-\frac{1}{2}\left(\partial_\mathbf{x}\phi_i\cdot\partial_\mathbf{x}\phi_j\right)\right], \tag{A.2}$$

where $L[\phi_i]$ is the Lagrangian, $\mathcal{L}$ is the Lagrangian density, and G^{ij} is the metric over the field space. While for physically interesting cases G^{ij} is either a function of x and/or fields ϕ_i, for simplicity and as an illustrative case, we take G^{ij} to be a constant matrix. In our case, we assume the field space metric G^{ij} to be invertible and $G^{ij}G_{jk} = \delta^i{}_k$. For the case of our interest, the off-shell or on-shell bulk symplectic forms, the metric on the field space is indeed field-dependent. The analysis of this appendix will be completed by those in appendix F.

We take u to be the light-cone time from the bulk viewpoint. The conjugate light-cone momentum and the canonical Hamiltonian read

$$\pi^i = \frac{\partial \mathcal{L}}{\partial(\partial_u \phi_i)} = G^{ij}\partial_v \phi_j \,, \tag{A.3a}$$

$$\mathcal{H}_c = \frac{1}{2} G^{ij} \partial_{\mathbf{x}} \phi_i \cdot \partial_{\mathbf{x}} \phi_j \,. \tag{A.3b}$$

As it has been discussed in [57, 58], we shall consider (A.3a) a primary constraint, i.e.

$$\chi^i := \pi^i - G^{ij}\partial_v \phi_j \approx 0 \,. \tag{A.4}$$

Hence we can rewrite the action as

$$S[\phi_i, \pi^i; \lambda_i] = \int \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}^{D-2}x \left[\partial_u \phi_i \, \pi^i - \mathcal{H}_T \right] \,, \qquad \mathcal{H}_T = \mathcal{H}_c + \lambda_i \, \chi^i \,, \tag{A.5}$$

where $\mathcal{H}_T$ is the total Hamiltonian density, which on a constant u surface, say $u = 0$, is given as

$$H_T := \int_{u=0} \mathrm{d}v \, \mathrm{d}^{D-2}x \; \mathcal{H}_T = \int_{u=0} \mathrm{d}v \, \mathrm{d}^{D-2}x \left[\frac{1}{2} G^{ij} \partial_{\mathbf{x}} \phi_i \cdot \partial_{\mathbf{x}} \phi_j + \lambda_i \, \chi^i \right] \,, \tag{A.6}$$

and λ_i are Lagrangian multipliers. Hence the Hamilton equations read as

$$\partial_u \phi_i = \frac{\delta H_T}{\delta \pi^i} = \lambda_i \,, \tag{A.7a}$$

$$\partial_u \pi^i = -\frac{\delta H_T}{\delta \phi_i} = G^{ij} \left(\partial^2_{\mathbf{x}} \phi_j - \partial_v \lambda_j \right) \,. \tag{A.7b}$$

Before proceeding further, we want to emphasize the differences between the light-front analysis and the usual case where the Cauchy data (solution phase space data) are given on spatial constant time slices. In the light-front case, the "Cauchy" (boundary) data are given as $\phi_i^{(0)}(v, \mathbf{x}) = \phi_i(u = 0; v, \mathbf{x})$, and the momentum conjugate is $\pi_i = \partial_v \phi_i$. Therefore, the relation between the momenta and their conjugate fields at the null boundary $u = 0$ provides a relation among the "Cauchy" data. This is in contrast to the spatial boundary case, where momenta and their conjugate fields at the spatial boundary are independent variables on the solution phase space. In the spatial boundary case, (A.4) represent constraints on the solution phase space. These constraints, as we discuss below, are second-class constraints.

The symplectic potential can be inferred by taking the variation of the action. The symplectic potential on a constant u slice at the null boundary, for example, $u = 0$, is given by:

$$\mathbf{\Theta} = \int_{u=0} \mathrm{d}v \, \mathrm{d}^{D-2}x \, \phi_i \, \delta\pi^i \,, \tag{A.8}$$

and hence the symplectic form can be expressed as:

$$\mathbf{\Omega} = \int_{u=0} \mathrm{d}v \, \mathrm{d}^{D-2}x \, \delta\phi_i \wedge \delta\pi^i \,. \tag{A.9}$$

Now we can immediately infer the Poisson bracket as follows:

$$\begin{aligned}
&\{\phi_i(v,\mathbf{x}),\phi_j(v',\mathbf{x}')\}_{\rm PB}=0\,,\\
&\{\phi_i(v,\mathbf{x}),\pi^j(v',\mathbf{x}')\}_{\rm PB}=\delta_i{}^j\;\delta(v-v')\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,,\\
&\{\pi^i(v,\mathbf{x}),\pi^j(v',\mathbf{x}')\}_{\rm PB}=0\,.
\end{aligned}\tag{A.10}$$

Defining the smeared constraints as $X=\int \mathrm{d}v\,\mathrm{d}^{D-2}x\;f_i\chi^i$, where $f_i(v,\mathbf{x})$ are test functions that vanish at the boundary, one finds the Poisson bracket for χ^i as:

$$\{\chi^i(v,\mathbf{x}),\chi^j(v',\mathbf{x}')\}_{\rm PB}=-2G^{ij}\;\partial_v\delta(v-v')\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,.\tag{A.11}$$

The consistency condition leads to nothing new

$$\partial_u\chi^i=\{\chi^i,H_T\}_{\rm PB}\approx 0\,.\tag{A.12}$$

Dirac bracket. To compute the Dirac bracket, we need to inverse the Poisson brackets of the second-class constraints, namely $\Delta^{ij}(v,\mathbf{x};v',\mathbf{x}')=\{\chi^i(v,\mathbf{x}),\chi^j(v',\mathbf{x}')\}_{\rm PB}$,

$$(\Delta^{-1})_{ij}(v,\mathbf{x};v',\mathbf{x}')=\frac{1}{4}G_{ij}\;H(v-v')\;\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,,\tag{A.13}$$

where

$$H(v-v'):=\begin{cases}\frac{(v-v')}{|v-v'|} & v\neq v'\\ 0 & v=v'\end{cases}\tag{A.14}$$

is the Heaviside step function. Note that with the above definition $\partial_v H(v-v')=2\delta(v-v')$.

From this, one can determine the Dirac bracket of two arbitrary functions in the phase space, f and g, as follows:

$$\begin{aligned}
&\{f(v,\mathbf{x}),g(v',\mathbf{x}')\}_{\rm DB}=\{f(v,\mathbf{x}),g(v',\mathbf{x}')\}_{\rm PB}\\
&-\int \mathrm{d}\tilde{v}\,\mathrm{d}^{D-2}\tilde{x}\int \mathrm{d}\bar{v}\,\mathrm{d}^{D-2}\bar{x}\{f(v,\mathbf{x}),\chi^i(\tilde{v},\tilde{\mathbf{x}})\}_{\rm PB}(\Delta^{-1})_{ij}(\tilde{v},\tilde{\mathbf{x}};\bar{v},\bar{\mathbf{x}})\{\chi^j(\bar{v},\bar{\mathbf{x}}),g(v',\mathbf{x}')\}_{\rm PB}\,.
\end{aligned}\tag{A.15}$$

The Dirac bracket of two scalar fields is then given as

$$\{\phi_i(v,\mathbf{x}),\phi_j(v',\mathbf{x}')\}_{\rm DB}=\frac{1}{4}G_{ij}\;H(v-v')\;\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,.\tag{A.16}$$

On-shell symplectic form and Poisson brackets on the reduced phase space. One can directly obtain the bracket (A.16) from equation (A.8) without using the Dirac bracket procedure. To do so, we note that by imposing the $\chi^i\approx 0$ constraints, which are essentially half of Hamilton's equations, and replacing the momenta with the derivatives of the fields, we arrive at the "on-shell" symplectic form:

$$\begin{aligned}
\mathbf{\Omega}_{\text{on-shell}}&=\int \mathrm{d}v\,\mathrm{d}^{D-2}x\,G^{ij}\,\delta\phi_i\wedge\partial_v\delta\phi_j\\
&=\frac{1}{2}\int \mathrm{d}v\,\mathrm{d}^{D-2}x\int \mathrm{d}v'\,\mathrm{d}^{D-2}x'\;\delta\phi_i(v,\mathbf{x})\mathbf{\Omega}^{ij}(v,\mathbf{x};v',\mathbf{x}')\wedge\delta\phi_j(v',\mathbf{x}')\,,
\end{aligned}\tag{A.17}$$

where

$$\boldsymbol{\Omega}^{ij}(v,\mathbf{x};v',\mathbf{x}') = -\boldsymbol{\Omega}^{ji}(v',\mathbf{x}';v,\mathbf{x}) = 2G^{ij}\ \partial_v\delta(v-v')\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,. \tag{A.18}$$

In the terminology of typical constrained systems, $\boldsymbol{\Omega}_{\text{on-shell}}$ represents the symplectic form over the reduced phase space obtained after solving and imposing constraints. One can invert the above symplectic potential over this reduced phase space to find:

$$(\boldsymbol{\Omega}^{-1})_{ij}(v,\mathbf{x};v',\mathbf{x}') = \frac{1}{4}G_{ij}\ H(v-v')\ \delta^{D-2}(\mathbf{x}-\mathbf{x}')\,. \tag{A.19}$$

It is evident that the symplectic form above directly yields the bracket (A.16).

B A quick review of Carrollian geometry

Let us consider a Lorentizan d-dimensional manifold equipped with a degenerate metric g_{ab} of rank $(d-1)$. We can use coordinate $x^a = \{v, x^A\},\ A = 1, 2, \cdots, d-1$ on the given manifold. The generic line element in this case takes the form:

$$\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = g_{AB}(\mathrm{d}x^A + U^A\,\mathrm{d}v)(\mathrm{d}x^B + U^B\,\mathrm{d}v)\,. \tag{B.1}$$

To describe Carrollian geometry, we need to define the kernel of the metric

$$K^a\partial_a = \alpha\left(\partial_v - U^A\partial_A\right)\,, \qquad\qquad K^a g_{ab} = 0\,, \tag{B.2}$$

which is the Carrollian kernel vector and the Ehresmann connection

$$E_a\,\mathrm{d}x^a = \frac{1}{\alpha}\,\mathrm{d}v + E_A(\mathrm{d}x^A + U^A\,\mathrm{d}v)\,, \qquad\qquad K^a E_a = 1\,. \tag{B.3}$$

The metric (B.1) and (B.2) define a Carrollian geometry, which is naturally supplemented with the Ehresmann connection.

The minors of g_{ab} is proportional to the product K^aK^b [59, 60], i.e.

$$\mathfrak{g}^{ab} = \beta^2\,K^aK^b\,. \tag{B.4}$$

The determinant of g_{ab} vanishes, however, its determinant is replaced by the density β. While the metric is degenerate and hence non-invertible, we can still raise indices by exploiting the Ehresmann connection. We may define a symmetric $(2,0)$-type tensor h^{ab} such that:

$$h^{ac}g_{cb} = \delta^a_b - K^aE_b\,. \tag{B.5}$$

To fully determine h^{ab}, we would need to impose an additional condition $h^{ab}E_aE_b = 0$.

Let us set $\alpha = 1$, $E_A = 0$, for simplicity. Then the components of h^{ab} read

$$h^{vv} = 0\,, \qquad h^{vA} = 0\,, \qquad h^{AB} = g^{AB}\,. \tag{B.6}$$

This justifies the discussion on the invertable part of the bulk symplectic form in item V. in section 4.2.

C Null surface boundary symmetry generators

The following vector field represents the null surface symmetries that preserves the form of the metric and moves us into the solution space

$$\xi = T\,\partial_v + r\,(\mathcal{D}_v T - W)\,\partial_r + \Big(Y^A - \mathcal{U}^A T - r\eta\partial^A T\Big)\,\partial_A + \mathcal{O}(r^2)\,, \tag{C.1}$$

in which $T(v, x^A)$, $W(v, x^A)$, and $Y^A(v, x^B)$ are codimension 1 symmetry generators of the causal boundary. This vector field keeps $r = 0$ a null surface and generates the following variations over the solution phase space functions

$$\delta_\xi \eta = T\mathcal{D}_v\eta + 2\eta\mathcal{D}_v T - W\eta + Y^A\partial_A\eta\,, \tag{C.2a}$$

$$\delta_\xi \gamma_{AB} = T\mathcal{D}_v\gamma_{AB} + 2\nabla_{\langle A}Y_{B\rangle} \tag{C.2b}$$

$$\delta_\xi \mathcal{J}_A = T\mathcal{D}_v\mathcal{J}_A + \mathcal{L}_Y\mathcal{J}_A + \Omega\big[\partial_A W - \Gamma\partial_A T - 2N_{AB}\partial^B T\big]\,, \tag{C.2c}$$

$$\delta_\xi \Gamma = -\mathcal{D}_v(W - \Gamma T) + Y^A\partial_A\Gamma\,, \tag{C.2d}$$

$$\delta_\xi \mathcal{N}_{AB} = \mathcal{D}_v(T\mathcal{N}_{AB}) + \mathcal{L}_Y\mathcal{N}_{AB} - \frac{2\nabla_C Y^C}{D-2}\mathcal{N}_{AB}\,, \tag{C.2e}$$

and $\mathcal{L}_Y$ denotes the Lie derivative along Y^A.

Transformation of charge densities and their canonical conjugates. From the above, one may compute variations of $\varpi, \Omega; \vartheta, \mathcal{P}; \mathcal{U}^A, \mathcal{S}_A$,

$$\delta_\xi\varpi = \mathcal{D}_v(T\varpi) + Y^A\partial_A\varpi - \partial_v W + \mathcal{U}^A\partial_A T\partial_v\mathcal{P} + \partial_A\mathcal{P}\partial_v(Y^A - \mathcal{U}^A T)\,, \tag{C.3a}$$

$$\delta_\xi\Omega = T\mathcal{D}_v\Omega + \partial_A(\Omega Y^A)\,, \tag{C.3b}$$

$$\delta_\xi\vartheta = \partial_v[\vartheta T - T\partial_A(\Omega\mathcal{U}^A)] + \nabla_A(\vartheta Y^A + \Omega\partial_v Y^A) = \partial_v(\delta_\xi\Omega)\,, \tag{C.3c}$$

$$\delta_\xi\mathcal{P} \approx T\varpi + (Y^A - T\mathcal{U}^A)\partial_A\mathcal{P} - W + \frac{2T}{\theta}\mathcal{N}^2, \tag{C.3d}$$

$$\delta_\xi\mathcal{U}^A = \mathcal{D}_v Y^A\,, \tag{C.3e}$$

$$\delta_\xi\mathcal{S}_A \approx \nabla_B(Y^B\mathcal{S}_A) + \mathcal{S}_B\nabla_A Y^B + 2\Omega\nabla_A(T\theta^{-1}\mathcal{N}^2) - 2\nabla^B_A(T\Omega\mathcal{N}_{AB})\,, \tag{C.3f}$$

where $\gamma_{AB}, \mathcal{P}, \mathcal{N}_{AB}$ are defined in (2.7), (2.9) and

$$\mathcal{D}_v\mathcal{N}_{AB} := \partial_v\mathcal{N}_{AB} - \mathcal{L}_{\mathcal{U}}\mathcal{N}_{AB} + \frac{2}{D-2}\mathcal{N}_{AB}\,\nabla_C\mathcal{U}^C\,. \tag{C.4}$$

Note that the variation of the superspin does not depend on W (compare it with that of (C.2c)). $\delta_\xi\mathcal{P}$ and $\delta_\xi\mathcal{S}_A$ are written "on-shell" using the Raychaudhuri (2.8b) and Damour (2.8c) equations. For completeness, we also present variations $\not{\delta}_\xi\hat{\mathcal{P}}, \not{\delta}_\xi\hat{\gamma}_{AB}$ defined in (4.9)

$$\not{\delta}_\xi\hat{\mathcal{P}} \approx -W + \varpi T + (Y^A - T\mathcal{U}^A)\partial_A\mathcal{P} - 2\theta^{-1}\mathcal{N}^A_B\nabla_A Y^B\,, \tag{C.5a}$$

$$\not{\delta}_\xi\hat{\gamma}_{AB} = 2\nabla_{\langle A}Y_{B\rangle} - \frac{2\mathcal{N}_{AB}}{\Omega\theta}\nabla_C(\Omega Y^C)\,. \tag{C.5b}$$

Interestingly, note that $\not{\delta}_\xi\hat{\gamma}_{AB}$ has no T, W dependence. The first term in (C.5b) is variation of a $D-2$ dimensional symmetric 2 tensor and the term proportional to $\mathcal{N}^{AB}$ is stemming from the non-closedness of $\not{\delta}\hat{\gamma}$.

D Charge analysis

Given the symplectic form, one can define the surface charge variation as $\not{\delta} Q_\xi = \mathbf{\Omega}[\delta_\xi g, \delta g; g]$ which becomes a surface integral over co-dimension 2 surface on-shell [61, 62]. Surface charge variation associated with symmetry generator (C.1) is [44, 45]

$$\not{\delta} Q_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_b}\left[(W-\Gamma T)\,\delta\Omega + Y^A\delta\mathcal{J}_A + T\,\Omega\,\theta\,\not{\delta}\hat{\mathcal{P}}\right], \tag{D.1}$$

or in the Heisenberg-direct sum slicing, the charge variation can be split into the integrable part

$$\tilde{Q}^{\mathrm{I}}_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_b}\left(\tilde{W}\Omega + Y^A\mathcal{S}_A + \tilde{T}\mathcal{P}\right), \tag{D.2}$$

and the flux

$$\mathbf{F}_\xi(\delta g) = -\frac{1}{16\pi G}\int_{\mathcal{N}_b}\left[\tilde{T} - \partial_C(\Omega Y^C)\right]\theta^{-1}\mathcal{N}^{AB}\delta\gamma_{AB}\,, \tag{D.3}$$

where [44]

$$\tilde{W} = W - \Gamma T - Y^A\partial_A\mathcal{P}, \qquad \tilde{T} = \Omega\theta T + \partial_A(\Omega Y^A)\,. \tag{D.4}$$

We note that $\delta_\xi\Omega = \tilde{T}$ and hence when $Y^A = 0$, $\mathbf{F}_\xi(\delta g) = \mathbf{F}_{\text{bdy}}|_{\delta\Omega=\delta_\xi\Omega}$, where $\mathbf{F}_{\text{bdy}}$ is the boundary symplectic flux defined in (4.13). This clarifies the relation between the charge flux and the symplectic flux.[3]

Now, by using the adjusted bracket proposed by Barnich and Troessaert [7], the charge algebra can be read as follows

$$\{\Omega(v,\mathbf{x}),\Omega(v,\mathbf{x}')\} = \{\mathcal{P}(v,\mathbf{x}),\mathcal{P}(v,\mathbf{x}')\} = 0\,, \tag{D.5a}$$

$$\{\Omega(v,\mathbf{x}),\mathcal{P}(v,\mathbf{x}')\} = 16\pi G\delta^{D-2}\left(\mathbf{x}-\mathbf{x}'\right), \tag{D.5b}$$

$$\{\mathcal{S}_A(v,\mathbf{x}),\Omega(v,\mathbf{x}')\} = \{\mathcal{S}_A(v,\mathbf{x}),\mathcal{P}(v,\mathbf{x}')\} = 0\,, \tag{D.5c}$$

$$\{\mathcal{S}_A(v,\mathbf{x}),\mathcal{S}_B(v,\mathbf{x}')\} = 16\pi G\left(\mathcal{S}_A(v,\mathbf{x}')\partial_B - \mathcal{S}_B(v,\mathbf{x})\partial'_A\right)\delta^{D-2}\left(\mathbf{x}-\mathbf{x}'\right). \tag{D.5d}$$

Note that the above are equal-v charge brackets. This v may be taken to be the arbitrary value v_b. Thus, the above is the same as (4.3), (4.4).

E Details of the on-shell symplectic form computations

In this appendix, we provide a detailed of derivation of (4.8), starting from (4.5b). To this end, we note that (4.5b) may be written as:

$$\begin{aligned}16\pi G\;\mathbf{\Omega}_{\text{on-shell}} = &\oint_{\mathcal{N}_b}(\delta\Omega\wedge\delta\mathcal{P}) + \frac{1}{2}\int_{\mathcal{N}}\Omega\mathcal{G}^{ABCD}\bigg[\frac{4}{\Omega^2\theta^2}\mathcal{N}_{AB}\mathcal{N}_{CD}\delta\Omega\wedge\partial_v\delta\Omega + \delta\gamma_{AB}\wedge\partial_v\delta\gamma_{CD}\\ &-\frac{4}{\Omega\theta}\mathcal{N}_{AB}\,\delta\Omega\wedge\partial_v\delta\gamma_{CD} + \frac{1}{(D-3)\Omega}\gamma_{AB}\,\delta\Omega\wedge\partial_v\delta\gamma_{CD} + \frac{8}{\Omega\theta}\mathcal{N}_{AE}\mathcal{N}^E_B\,\delta\Omega\wedge\delta\gamma_{CD}\bigg].\end{aligned} \tag{E.1}$$

---

[3] Note that the inner product of the vector field induced by ξ and the symplectic flux (4.13) is as follows:

$$\begin{aligned}I_\xi\mathbf{F}_{\text{bdy}} &= -\frac{1}{16\pi G}\oint_{\mathcal{N}_b}\frac{\mathcal{N}^{AB}}{\theta}(\delta_\xi\Omega\delta\gamma_{AB} - \delta\Omega\delta_\xi\gamma_{AB})\\ &= -\frac{1}{16\pi G}\int_{\mathcal{N}_b}\left[\tilde{T}\frac{\mathcal{N}^{AB}}{\theta}\not{\delta}\hat{\gamma}_{AB} - 2\delta\Omega\left(\frac{\mathcal{N}^{AB}}{\theta}\nabla_A Y_B - \frac{\mathcal{N}^2}{\theta^2}\nabla_C(\Omega Y^C)\right)\right].\end{aligned}$$

Now, we concentrate on the bulk term, which is the integral over $\mathcal{N}$ in (E.1), and establish its relationship with $\mathbf{\Omega}_{\text{Bulk}}$ from (4.8). Let us start from (4.8b),

$$16\pi G\,\mathbf{\Omega}_{\text{Bulk}}=\frac{1}{2}\int_{\mathcal{N}}\Omega\mathcal{G}^{ABCD}\not{\delta}\hat{\gamma}_{AB}\wedge\partial_v\not{\delta}\hat{\gamma}_{CD}=\mathbf{\Omega}_1+\mathbf{\Omega}_2+\mathbf{\Omega}_3+\mathbf{\Omega}_4\,, \tag{E.2}$$

where using (4.9b) we have

$$\begin{aligned}
\mathbf{\Omega}_1&=\frac{1}{2}\int_{\mathcal{N}}\Omega\mathcal{G}^{ABCD}\,\delta\gamma_{AB}\wedge\partial_v\delta\gamma_{CD}\,,\\
\mathbf{\Omega}_4&=\frac{4}{2}\int_{\mathcal{N}}\Omega\mathcal{G}^{ABCD}\left(\frac{\mathcal{N}_{AB}}{\Omega\,\theta}\,\delta\Omega\right)\wedge\partial_v\left(\frac{\mathcal{N}_{CD}}{\Omega\,\theta}\,\delta\Omega\right)=\int_{\mathcal{N}}\frac{2\mathcal{N}^2}{\Omega\,\theta^2}\delta\Omega\wedge\partial_v\delta\Omega\,,\\
\mathbf{\Omega}_2&=-\int_{\mathcal{N}}\Omega\mathcal{G}^{ABCD}\,\delta\gamma_{AB}\wedge\partial_v\left(\frac{\mathcal{N}_{CD}}{\Omega\,\theta}\,\delta\Omega\right)\,,\\
\mathbf{\Omega}_3&=-\int_{\mathcal{N}}\Omega\mathcal{G}^{ABCD}\left(\frac{\mathcal{N}_{AB}}{\Omega\,\theta}\,\delta\Omega\right)\wedge\partial_v\delta\gamma_{CD}\,.
\end{aligned} \tag{E.3}$$

Straightforward manipulations yield

$$\begin{aligned}
\mathbf{\Omega}_2&=-\oint_{\mathcal{N}_b}\Omega\mathcal{G}^{ABCD}\,\delta\gamma_{AB}\wedge\frac{\mathcal{N}_{CD}}{\Omega\theta}\,\delta\Omega+\int_{\mathcal{N}}\partial_v\left(\Omega\mathcal{G}^{ABCD}\,\delta\gamma_{AB}\right)\wedge\left(\frac{\mathcal{N}_{CD}}{\Omega\theta}\,\delta\Omega\right)\\
&=-\oint_{\mathcal{N}_b}\frac{\mathcal{N}^{AB}}{\theta}\delta\gamma_{AB}\wedge\delta\Omega+\int_{\mathcal{N}}\left\{\mathcal{N}^{AB}\delta\gamma_{AB}\wedge\delta\Omega+\frac{\mathcal{N}_{CD}}{\theta}\left[\partial_v(\mathcal{G}^{ABCD})\,\delta\gamma_{AB}+\delta\partial_v\gamma_{CD}\right]\wedge\delta\Omega\right\},\\
\mathbf{\Omega}_3&=-\int_{\mathcal{N}}\Omega\mathcal{G}^{ABCD}\left(\frac{\mathcal{N}_{AB}}{\Omega\theta}\,\delta\Omega\right)\wedge\partial_v\delta\gamma_{CD}=\int_{\mathcal{N}}\frac{\mathcal{N}^{AB}}{\theta}\delta\partial_v\gamma_{AB}\wedge\delta\Omega.
\end{aligned} \tag{E.4}$$

Recalling the definition of the Wheeler-DeWitt metric (4.10) yields

$$\int_{\mathcal{N}}\frac{\mathcal{N}_{CD}}{\theta}\,\partial_v(\mathcal{G}^{ABCD})\,\delta\gamma_{AB}\wedge\delta\Omega=-\int_{\mathcal{N}}\frac{4}{\theta}\mathcal{N}^{AC}\mathcal{N}_C{}^B\delta\gamma_{AB}\wedge\delta\Omega\,. \tag{E.5}$$

Therefore, we get

$$\begin{aligned}
\mathbf{\Omega}_2+\mathbf{\Omega}_3=&-\oint_{\mathcal{N}_b}\frac{\mathcal{N}^{AB}}{\theta}\delta\gamma_{AB}\wedge\delta\Omega\\
&+\int_{\mathcal{N}}\left\{\mathcal{N}^{AB}\delta\gamma_{AB}\wedge\delta\Omega-\frac{4}{\theta}\mathcal{N}^{AC}\mathcal{N}_C{}^B\delta\gamma_{AB}\wedge\delta\Omega+2\frac{\mathcal{N}^{AB}}{\theta}\delta\partial_v\gamma_{AB}\wedge\delta\Omega\right\}.
\end{aligned} \tag{E.6}$$

Using definition of $\mathcal{N}_{AB}$, we also note that

$$\int_{\mathcal{N}}\gamma^{AB}\,\delta\Omega\wedge\partial_v\delta\gamma_{AB}=-\int_{\mathcal{N}}\partial_v\gamma^{AB}\,\delta\Omega\wedge\delta\gamma_{AB}=+2\int_{\mathcal{N}}\mathcal{N}^{AB}\,\delta\Omega\wedge\delta\gamma_{AB}\,. \tag{E.7}$$

Putting all the above together and summing $\mathbf{\Omega}_1+\mathbf{\Omega}_2+\mathbf{\Omega}_3+\mathbf{\Omega}_4$ yields the desired result.

F Derivation of bulk off-shell and on-shell Poisson brackets

In this appendix, we first provide the details of the derivation of (3.6), and then we will demonstrate how to compute on-shell Poisson brackets of the bulk modes in the sector where the bulk symplectic form is invertible.

F.1 Off-shell bulk Poisson brackets, (3.6)

The symplectic form leads to off-shell Poisson brackets

$$\Big\{\mathcal{U}^A(x_1), \mathcal{S}_B(x_2)\Big\}_{\text{PB}} = 16\pi G\, \delta^A_B\, \delta^{D-1}(x_1 - x_2)\,, \tag{F.1a}$$

$$\{\Omega(x_1), \varpi(x_2)\}_{\text{PB}} = 16\pi G\, \delta^{D-1}(x_1 - x_2)\,, \tag{F.1b}$$

$$\{\vartheta(x_1), \mathcal{P}(x_2)\}_{\text{PB}} = 16\pi G\, \delta^{D-1}(x_1 - x_2)\,, \tag{F.1c}$$

$$\Big\{\mathcal{N}_{AB}(x_1), \Omega(x_2)\, \gamma^{CD}(x_2)\Big\}_{\text{PB}} = 16\pi G\, \delta^C_{(A}\delta^D_{B)}\, \delta^{D-1}(x_1 - x_2)\,. \tag{F.1d}$$

Poisson brackets among the following second-class constraints,

$$C_1 := \det\gamma - 1 \approx 0, \qquad C_2 := \gamma^{AB}\mathcal{N}_{AB} \approx 0, \tag{F.2}$$

read as

$$\{C_1(x_1), C_1(x_2)\} = 0\,, \tag{F.3a}$$

$$\{C_1(x_1), C_2(x_2)\} = \frac{16\pi G\,(D-2)}{\Omega(x_1)}\, \det\gamma(x_1)\, \delta^{D-1}(x_1 - x_2)\,, \tag{F.3b}$$

$$\{C_2(x_1), C_2(x_2)\} = 0\,, \tag{F.3c}$$

where the following equation was used

$$\frac{\delta \det\gamma}{\delta\gamma^{AB}} = -\det\gamma\, \gamma_{AB} \tag{F.4}$$

The above can be written as a matrix

$$\Delta_{ij}(x_1, x_2) \approx \frac{16\pi G\,(D-2)}{\Omega(x_1)}\, \delta^{D-1}(x_1 - x_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{F.5}$$

where $i, j = 1, 2$. We may use the equation

$$\int_{\mathcal{N}}^{\prime\prime} \Delta_{ik}(x, x'')(\Delta^{-1})^{kj}(x'', x') = \delta_i^j\, \delta^{D-1}(x - x') \tag{F.6}$$

to define the inverse of $\Delta_{ij}(x_1, x_2)$ and hence

$$(\Delta^{-1})^{ij}(x_1, x_2) = \frac{\Omega(x_1)}{16\pi G\,(D-2)}\, \delta^{D-1}(x_1 - x_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{F.7}$$

$$\begin{aligned}
\Big\{\mathcal{N}_{AB}(x), \gamma^{CD}(x')\Big\}_{\text{DB}} &= \Big\{\mathcal{N}_{AB}(x), \gamma^{CD}(x')\Big\}_{\text{PB}} \\
&\quad - \int_{\mathcal{N}}^{\prime\prime}\int_{\mathcal{N}}^{\prime\prime\prime} \{\mathcal{N}_{AB}(x), C_i(x'')\}_{\text{PB}} (\Delta^{-1})^{ij}(x''; x''') \{C_j(x'''), \gamma^{CD}(x')\}_{\text{PB}} \\
&= \frac{16\pi G}{\Omega(x)}\, \delta^C_{(A}\delta^D_{B)}\, \delta^{D-1}(x - x') \\
&\quad + \int_{\mathcal{N}}^{\prime\prime} \frac{\Omega(x'')}{16\pi G\,(D-2)} \{\mathcal{N}_{AB}(x), C_1(x'')\}_{\text{PB}} \{C_2(x''), \gamma^{CD}(x')\}_{\text{PB}} \\
&= \frac{16\pi G}{\Omega(x)} \left[\delta^C_{(A}\delta^D_{B)} - \frac{1}{(D-2)}\gamma_{AB}(x)\gamma^{CD}(x)\right] \delta^{D-1}(x - x')
\end{aligned} \tag{F.8}$$

then

$$\left\{\mathcal{N}_{AB}(x),\gamma^{CD}(x')\right\}_{\rm DB}=\frac{16\pi G}{\Omega(x)}\left(\delta^C_{(A}\delta^D_{B)}-\frac{1}{D-2}\gamma_{AB}\gamma^{CD}\right)\delta^{D-1}(x-x')\,. \tag{F.9}$$

Using the identity

$$\begin{aligned}\left\{\mathcal{N}_{AB}(x),\gamma^{CD}(x')\right\}_{\rm DB}&=\left\{\mathcal{N}_{AB}(x),\gamma^{CE}(x')\gamma^{DF}(x')\gamma_{EF}(x')\right\}_{\rm DB}\\&=\left\{\mathcal{N}_{AB}(x),\gamma_{EF}(x')\right\}_{\rm DB}\gamma^{CE}(x')\gamma^{DF}(x')\\&\quad+\left\{\mathcal{N}_{AB}(x),\gamma^{CE}(x')\right\}_{\rm DB}\gamma^{DF}(x')\gamma_{EF}(x')\\&\quad+\left\{\mathcal{N}_{AB}(x),\gamma^{DF}(x')\right\}_{\rm DB}\gamma^{CE}(x')\gamma_{EF}(x')\end{aligned} \tag{F.10}$$

one can readily check (3.6).

F.2 On-shell bulk Poisson brackets

In the example of scalar fields discussed in appendix A, the metric on the field space G^{ij} was considered to be field-independent. In our case, however, the invertible part of the bulk symplectic form is spanned by γ_{AB}, with $\Omega(x)$ being a fixed function. Therefore

$$\begin{aligned}\mathbf{\Omega}^{\rm I}_{\rm Bulk}&=\frac{1}{32\pi G}\int_{\mathcal{N}}\delta^A_{(C}\delta^B_{D)}\Omega\,\partial_v\delta\gamma_{AB}\wedge\delta\gamma^{CD}\\&:=\int_{\mathcal{N}}\int'_{\mathcal{N}}\mathbf{\Omega}^{ABCD}(x,x')\delta\gamma_{AB}(x)\wedge\delta\gamma_{CD}(x')\,,\end{aligned} \tag{F.11}$$

where using $\delta\gamma^{CD}=-\gamma^{CE}\gamma^{DF}\delta\gamma_{EF}$, we obtain

$$\mathbf{\Omega}^{ABCD}[x;x']=\frac{\sqrt{\Omega(x)\Omega(x')}}{64\pi G}\mathcal{G}^{ABCD}[x;x'](\partial_v\delta(v-v')-\partial_{v'}\delta(v'-v))\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,, \tag{F.12}$$

where $\mathcal{G}^{ABCD}[x;x']$ is the *point-split WdW metric* (5.5) which satisfies the following equations

$$\gamma_{AB}(x)\mathcal{G}^{ABCD}[x;x']=\mathcal{G}^{ABCD}[x;x']\gamma_{CD}(x')=0\,, \tag{F.13a}$$

$$\mathcal{G}^{ABCD}[x;x']=\mathcal{G}^{CDAB}[x';x]=\mathcal{G}^{BACD}[x;x']=\mathcal{G}^{ABDC}[x;x']\,. \tag{F.13b}$$

Let us denote the inverted antisymmetric bivector by $\mho_{ABCD}$,

$$\mho^{\rm I}_{\rm Bulk}:=\int_{\mathcal{N}}\int'_{\mathcal{N}}\mho_{ABCD}[x;x']\frac{\delta}{\delta\gamma_{AB}(x)}\vee\frac{\delta}{\delta\gamma_{CD}(x')}\,, \tag{F.14}$$

such that $\mho_{ABCD}[x;x']$ is trace-free, that is

$$\gamma^{AB}(x)\mho_{ABCD}[x;x']=\mho_{ABCD}[x;x']\gamma^{CD}(x')=0\,, \tag{F.15}$$

and it is defined via[4]

$$\int''_{\mathcal{N}}\mathbf{\Omega}^{ABEF}[x;x'']\mho_{EFCD}[x'';x']=P^{AB}_{CD}\ \delta^{D-1}(x-x')\,. \tag{F.16}$$

Then by definition, the on-shell Poisson bracket is

$$\{\gamma_{AB}(x_1),\gamma_{CD}(x_2)\}=\mho_{ABCD}[x_1;x_2] \tag{F.17}$$

[4]One can verify that $\int''_{\mathcal{N}}\mho_{CDEF}[x;x'']\mathbf{\Omega}^{EFAB}[x'';x']=P^{AB}_{CD}\ \delta^{D-1}(x-x')$ is equivalent to (F.16).

Similarly to the analysis of appendix F.1, when working with trace-free $\mho_{ABCD}[x;x']$, it is important to note that the constraints (3.3) are already implemented. This is because $C_2 \propto \partial_v C_1$ and hence C_2 is already satisfied "on-shell" if C_1 holds and C_1 is guaranteed by the traceless property of $\mho_{ABCD}[x;x']$. To solve for $\mho_{ABCD}(x_1;x_2)$, recalling analysis in appendix A and inspired by the point-split WdW metric (5.5), one can immediately observe the following form

$$\mho_{ABCD}[x;x'] = \frac{16\pi G}{\sqrt{\Omega(x)\Omega(x')}}\, \mho^{\rm S}_{ABCD}[x;x']\; H(v-v')\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,. \tag{F.18}$$

By substituting the above expression in (F.16), we have

$$\begin{aligned}
2\,P^{AB}_{CD}\;\delta(v-v') = &\int {\rm d}v'' \sqrt{\frac{\Omega(v)}{\Omega(v')}}\,\partial_{v''}\left[\mathcal{G}^{ABEF}[v;v'']\,\mho^{\rm S}_{EFCD}[v'';v']\; H(v''-v')\right]\delta(v-v'')\\
&= \int {\rm d}v''\Big[2\,\mathcal{G}^{ABEF}_{\rm TF}(v)\,\mho^{\rm S}_{EFCD}[v;v]\,\delta(v-v')\,\delta(v-v'')\\
&\qquad + \sqrt{\frac{\Omega(v)}{\Omega(v')}}\left(\partial_v\mathcal{G}^{ABEF}[v'';v]\,\mho^{\rm S}_{EFCD}[v;v'] + \mathcal{G}^{ABEF}_{\rm TF}(v)\,\partial_v\mho^{\rm S}_{EFCD}[v;v']\right)\\
&\qquad \times H(v-v')\,\delta(v-v'')\Big]\\
&= 2\,\mathcal{G}^{ABEF}_{\rm TF}(v)\,\mho^{\rm S}_{EFCD}[v;v]\delta(v-v')\\
&\quad + \sqrt{\frac{\Omega(v)}{\Omega(v')}}\left[\mathcal{G}^{ABEF}_{\rm TF}(v)\,\partial_v\mho^{\rm S}_{EFCD}[v;v'] + \mho^{\rm S}_{EFCD}[v;v']\boldsymbol{\mathcal{A}}^{ABEF}(v)\right] H(v-v')\,,
\end{aligned} \tag{F.19}$$

where

$$\begin{aligned}
\boldsymbol{\mathcal{A}}^{ABCD}(v) &= \int {\rm d}v''\partial_v\mathcal{G}^{ABCD}[v'';v]\,\delta(v-v'')\\
&= \frac{1}{2}\,\partial_v\mathcal{G}^{ABCD} + \frac{1}{D-2}\left(\mathcal{N}^{CD}(v)\gamma^{AB}(v) - \mathcal{N}^{AB}(v)\gamma^{CD}(v)\right)\\
&= -\frac{1}{2}\Big[\gamma^{AC}(v)\mathcal{N}^{BD}(v) + \gamma^{AD}(v)\mathcal{N}^{BC}(v) + \mathcal{N}^{AC}(v)\gamma^{BD}(v) + \mathcal{N}^{AD}(v)\gamma^{BC}(v)\\
&\qquad - \frac{4}{D-2}\gamma^{AB}(v)\mathcal{N}^{CD}(v)\Big]\,.
\end{aligned} \tag{F.20}$$

Note that $\gamma_{AB}(v)\boldsymbol{\mathcal{A}}^{ABCD}(v) = 0, \boldsymbol{\mathcal{A}}^{ABCD}(v)\gamma_{CD}(v) = -2\mathcal{N}^{AB}$.

Eq. (F.19) yields the following equations:

Continuity condition at $v = v'$. From the coefficient of the delta function in (F.19) we can read the following algebraic equation:

$$\mathcal{G}^{ABEF}(v)\,\mho^{\rm S}_{EFCD}[v;v] = P^{AB}_{CD}(v)\,, \tag{F.21}$$

which yields

$$\mho^{\rm S}_{ABCD}[v;v] = \mathcal{G}_{ABCD}(v)\,. \tag{F.22}$$

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Differential equation for $v \neq v'$. From the coefficient of the Heaviside function in (F.19) we can read the following first-order differential equation:

$$\mathcal{G}^{ABEF}(v)\,\partial_v \mho^{\rm S}_{EFCD}[v;v'] + \boldsymbol{\mathcal{A}}^{ABEF}(v)\mho^{\rm S}_{EFCD}[v;v'] = 0\,. \tag{F.23}$$

This equation may be written as

$$\partial_v\left(\mathcal{G}^{ABEF}(v)\,\mho^{\rm S}_{EFCD}[v;v']\right) + \left(\boldsymbol{\mathcal{A}}^{ABEF}(v) - \partial_v\mathcal{G}^{ABEF}(v)\right)\mho^{\rm S}_{EFCD}[v;v'] = 0\,. \tag{F.24}$$

Now by using the explicit expression of $\boldsymbol{\mathcal{A}}^{ABEF}(v)$ in equation (F.20), we get

$$\partial_v\left(\mathcal{G}^{ABEF}(v)\,\mho^{\rm S}_{EFCD}[v;v']\right) - \left(\frac{1}{2}\partial_v\mathcal{G}^{ABEF}(v) - \frac{1}{D-2}\mathcal{N}^{EF}(v)\gamma^{AB}(v)\right)\mho^{\rm S}_{EFCD}[v;v'] = 0\,, \tag{F.25}$$

where we used $\gamma^{EF}(v)\mho^{\rm S}_{EFCD}[v;v'] = 0$, leading to

$$\partial_v \boldsymbol{\mathcal{X}}^{AB}{}_{CD}[v;v'] - \boldsymbol{\mathcal{B}}^{AB}{}_{EF}(v)\mathcal{X}^{EF}{}_{CD}[v;v'] = 0\,, \tag{F.26}$$

with[5]

$$\boldsymbol{\mathcal{X}}^{AB}{}_{CD}[v;v'] := \mathcal{G}^{ABEF}(v)\,\mho^{\rm S}_{EFCD}[v;v']\,, \qquad \boldsymbol{\mathcal{B}}^{AB}{}_{CD}(v) = -2\mathcal{N}^{(A}_{(C}\delta^{B)}_{D)}\,. \tag{F.27}$$

We note that $\boldsymbol{\mathcal{B}}^{AB}{}_{CD}(v)\gamma^{CD}(v) = \partial_v\gamma^{AB}, \gamma_{AB}(v)\boldsymbol{\mathcal{B}}^{AB}{}_{CD}(v) = -\partial_v\gamma_{CD}$.

We solve this equation with the following continuity condition (F.21), namely

$$\mathcal{X}^{AB}{}_{CD}[v;v] = P^{AB}_{CD}[v]\,. \tag{F.28}$$

To solve (F.26) we introduce the evolution matrix for $v \geq v'$

$$\boldsymbol{\mathcal{U}}^{AB}{}_{CD}[v;v'] = \mathbf{V}\exp\left[\int_{v'}^{v} \mathrm{d}\tilde{v}\;\boldsymbol{\mathcal{B}}(\tilde{v})\right]^{AB}_{\;\;CD}\,, \qquad \boldsymbol{\mathcal{U}}^{AB}{}_{CD}[v;v] = \delta^{(A}_{(C}\delta^{B)}_{D)} \tag{F.29}$$

where $\mathbf{V}$ denotes v-ordering and recalling the Dyson series, we have

$$\partial_v\boldsymbol{\mathcal{U}}^{AB}{}_{CD}[v;v'] = \boldsymbol{\mathcal{B}}^{AB}{}_{EF}(v)\boldsymbol{\mathcal{U}}^{EF}{}_{CD}[v;v'] \qquad v \geq v'\,. \tag{F.30}$$

So, imposing (F.28) to fix the integration constant, we learn

$$\mathcal{X}^{AB}{}_{CD}[v;v'] = \boldsymbol{\mathcal{U}}^{AB}{}_{EF}[v;v']P^{EF}_{CD}(v') \qquad v \geq v'\,. \tag{F.31}$$

Noting that $\mathcal{G}^{ABEF}(v)\mathcal{G}_{EFCD}(v) = P^{CD}_{AB}(v)$ and that $P^{EF}_{AB}(v)\mho^{\rm S}_{EFCD}[v;v'] = \mho^{\rm S}_{ABCD}[v;v']$, we have

$$\mho^{\rm S}_{ABCD}[v;v'] = \mathcal{G}_{ABEF}(v)\mathcal{X}^{EF}_{CD}[v;v'] = \mathcal{G}_{ABEF}(v)\boldsymbol{\mathcal{U}}^{EF}{}_{GH}[v;v']P^{GH}_{CD}(v')\,, \qquad v \geq v'\,. \tag{F.32}$$

The solution for $v' \geq v$ is then fixed through the symmetry requirement, $\mho^{\rm S}_{ABCD}[v;v'] = \mho^{\rm S}_{CDAB}[v';v]$. Explicitly,

$$\mho^{\rm S}_{ABCD}[v;v'] = \begin{cases} \mathcal{G}_{ABEF}(v)\,\boldsymbol{\mathcal{U}}^{EF}{}_{GH}[v;v']P^{GH}_{CD}(v')\,, & v \geq v' \\[2ex] \mathcal{G}_{CDEF}(v')\,\boldsymbol{\mathcal{U}}^{EF}{}_{GH}[v';v]P^{GH}_{AB}(v)\,, & v' \geq v \end{cases} \tag{F.33}$$

[5]Since $\gamma_{CD}(v)\mathcal{X}^{Cd}{}_{EF}[v;v'] = 0$, we have dropped the terms proportional to γ_{CD} in $\boldsymbol{\mathcal{B}}^{AB}{}_{CD}(v)$.

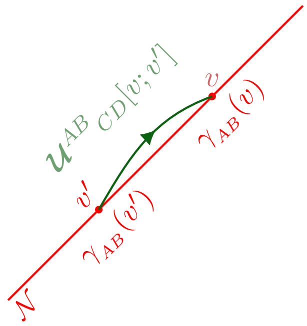

Figure 2. Depiction of two points $v' < v$ located at the same position transverse $\mathbf{x} = \mathbf{x}'$. A configuration of gravitational waves is evolved from v' to v using the evolution matrix $\boldsymbol{\mathcal{U}}^{AB}{}_{CD}[v;v']$. This evolution matrix do not change $\mathbf{x}$.

Discussion. Given the above result, which may look complicated, some clarifying comments are in order:

1. One may explicitly verify that

$$\gamma^{AB}(v)\mho^{\rm S}_{ABCD}[v;v']=0\,,\qquad \mho^{\rm S}_{ABCD}[v;v']\gamma_{CD}(v')=0\,,\qquad \mho^{\rm S}_{ABCD}[v;v']=\mho^{\rm S}_{CDAB}[v';v]\,, \tag{F.34}$$

and also

$$P^{AB}_{EF}(v)\mho^{\rm S}_{ABCD}[v;v'] = \mho^{\rm S}_{EFCD}[v;v']\,,\qquad P^{CD}_{EF}(v')\mho^{\rm S}_{ABCD}[v;v'] = \mho^{\rm S}_{ABEF}[v;v']\,. \tag{F.35}$$

2. As (F.19) shows $\mho^{\rm S}_{ABCD}[v;v']$ is a smooth function in v, v' and in particular $\mho^{\rm S}_{ABCD}[v;v] = \mathcal{G}_{ABCD}(v)$.

3. Recalling that $2\mathcal{N}_{AB}(v) = \partial_v\gamma_{AB}$, $\boldsymbol{\mathcal{U}}^{AB}{}_{CD}[v;v']$ evolves a given configuration of gravitational waves, parametrized in γ_{AB}, from an arbitrary boundary v' to v (for $v \geq v'$).

4. The propagator $\boldsymbol{\mathcal{U}}^{AB}{}_{CD}[v;v']$ only changes the v dependence of the gravitational waves and does not affect $\mathbf{x}$. This is expected as two points $(v,\mathbf{x})$ and $(v',\mathbf{x}')$ on $\mathcal{N}$ are causally connected only when $\mathbf{x} = \mathbf{x}'$.

5. The computation of bracket can be depicted as shown in figure 2. To find the bracket between the two generic points v, v' we need to start by evolving the one at $v' < v$ to the one at v and then compare the two. The bracket is obtained by noting that $\{\gamma_{AB}(v), \gamma_{CD}(v)\} = 0$ and using the definition of projector $P^{AB}_{CD}(v)$.

6. One also can verify that

$$\left\{\gamma_{AB}(x), \gamma^{CD}(x')\right\} = \frac{16\pi G}{\sqrt{\Omega(x)\Omega(x')}}\, \mho^{\rm S}_{ABEF}[x;x']\; \mathcal{G}^{EFCD}(x')\; H(v-v')\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,. \tag{F.36}$$

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7. Recalling that on-shell $\mathcal{N}_{AB} = \frac{1}{2}\partial_v \gamma_{AB}$ and using (F.17), one can compute $\{\mathcal{N}_{AB}(v), \gamma_{CD}(v')\}$ on-shell as follows:

$$
\begin{aligned}
&\{\mathcal{N}_{AB}(x), \gamma_{CD}(x')\} = \\
&\Bigg[\frac{16\pi G}{\Omega(x)}\mathcal{G}_{ABCD}(x)\delta(v-v') + 4\pi G\sqrt{\frac{\Omega(x)}{\Omega(x')}}\Bigg(\partial_v\left(\frac{1}{\Omega}\mathcal{G}_{ABEF}(x)\right) \\
&\quad -\frac{2}{(D-2)\Omega}\gamma_{AB}(x)\mathcal{N}_{EF}(x)\Bigg)\boldsymbol{\mathcal{U}}^{EF}{}_{KL}[v;v']P^{KL}_{CD}(v')H(v-v')\Bigg]\delta^{D-2}(\mathbf{x}-\mathbf{x}').
\end{aligned}
\tag{F.37}
$$

The $\delta(v-v')$ term in the above is what we had in the off-shell expression (3.6). The $H(v-v')$ terms, however, appear on-shell, as the evolution operator of the system and the WdW metric are both γ_{AB} dependent.

8. **Jacobi Identity.** We obtained the brackets through inverting the symplectic 2-form (F.11) in the $\delta\Omega = 0$ sector, which is clearly closed, as indicated by the first line in (F.11). The closedness of $\boldsymbol{\Omega}^{\text{I}}_{\text{Bulk}}$ implies the Jacobi identity for the bracket (F.17) obtained from inverting the symplectic form. Here we demonstrate this well-known statement for our specific case. Let us start from (F.16), and by taking derivatives w.r.t. to the "trace-free" $\hat{\gamma}_{KL}(y)$, i.e. $\frac{\delta\gamma_{EF}}{\delta\hat{\gamma}_{KL}(y)} = P^{KL}_{EF}$, we have:

$$
\int_{\mathcal{N}}^{x''}\frac{\delta\boldsymbol{\Omega}^{ABEF}[x;x'']}{\delta\hat{\gamma}_{KL}(y)}\mho_{EFCD}[x'';x'] + \boldsymbol{\Omega}^{ABEF}[x;x'']\frac{\delta\mho_{EFCD}[x'';x']}{\delta\hat{\gamma}_{KL}(y)} = 0 \tag{F.38}
$$

We may then use closedness relation,

$$
\frac{\delta\boldsymbol{\Omega}^{ABEF}[x;x'']}{\delta\hat{\gamma}_{KL}(y)} + \frac{\delta\boldsymbol{\Omega}^{KLAB}[y;x]}{\delta\hat{\gamma}_{EF}(x'')} + \frac{\delta\boldsymbol{\Omega}^{EFKL}[x'';y]}{\delta\hat{\gamma}_{AB}(x)} = 0\,, \tag{F.39}
$$

(F.16) and through straightforward algebraic manipulations using symmetry properties of $\mho_{ABCD}[x;x']$ we arrive at:

$$
\int_{\mathcal{N}}^{y}\mho_{KLEF}[y;x'']\,\frac{\delta\mho_{ABCD}[x;x']}{\delta\hat{\gamma}_{KL}(y)} + \mho_{KLCD}[y;x']\,\frac{\delta\mho_{EFAB}[x'';x]}{\delta\hat{\gamma}_{KL}(y)} + \mho_{KLAB}[y;x]\,\frac{\delta\mho_{CDEF}[x';x'']}{\delta\hat{\gamma}_{KL}(y)} = 0 \tag{F.40}
$$

which is immediately resulting in

$$
\begin{aligned}
&\{\{\gamma_{AB}(x_1),\gamma_{CD}(x_2)\},\gamma_{EF}(x_3)\} + \{\{\gamma_{EF}(x_3),\gamma_{AB}(x_1)\},\gamma_{CD}(x_2)\} \\
&+\{\{\gamma_{CD}(x_2),\gamma_{EF}(x_3)\},\gamma_{AB}(x_1)\} = 0
\end{aligned}
\tag{F.41}
$$

G On-shell symplectic form, $D = 4$ case

Here, we work out the physically interesting case of $D = 4$. The on-shell solution space is governed by four boundary modes $\Omega(v_b, \mathbf{x}), \mathcal{P}(v_b, \mathbf{x})$ and $\mathcal{S}_A(v_b, \mathbf{x})$ $(A = 1, 2)$ and three bulk modes $\Omega(v, \mathbf{x}), \gamma_{AB}(v, \mathbf{x})$. In this case, γ_{AB} has only two independent d.o.f which may be parametrized as a 2×2 matrix

$$
\gamma_{AB} = \begin{pmatrix} e^{\gamma}\cosh\phi & \sinh\phi \\ \sinh\phi & e^{-\gamma}\cosh\phi \end{pmatrix}, \qquad \gamma^{AB} = \begin{pmatrix} e^{-\gamma}\cosh\phi & -\sinh\phi \\ -\sinh\phi & e^{\gamma}\cosh\phi \end{pmatrix}. \tag{G.1}
$$

The news tensor readily can be read as follows

$$\mathcal{N}_{AB} = \frac{1}{2}\partial_v\gamma \begin{pmatrix} e^{\gamma}\cosh\phi & 0 \\ 0 & -e^{-\gamma}\cosh\phi \end{pmatrix} + \frac{1}{2}\partial_v\phi \begin{pmatrix} e^{\gamma}\sinh\phi & \cosh\phi \\ \cosh\phi & e^{-\gamma}\sinh\phi \end{pmatrix}, \tag{G.2}$$

and also with upper indices as

$$\mathcal{N}^{AB} = \frac{1}{2}\,\partial_v\gamma \begin{pmatrix} e^{-\gamma}\cosh\phi & 0 \\ 0 & -e^{\gamma}\cosh\phi \end{pmatrix} + \frac{1}{2}\partial_v\phi \begin{pmatrix} -e^{-\gamma}\sinh\phi & \cosh\phi \\ \cosh\phi & -e^{\gamma}\sinh\phi \end{pmatrix}, \tag{G.3}$$

resulting in

$$\mathcal{N}^2 = \frac{1}{2}(\partial_v\gamma)^2\cosh^2\phi + \frac{1}{2}(\partial_v\phi)^2\,. \tag{G.4}$$

WdW metric. The WdW metric,

$$\begin{aligned} \mathcal{G}^{ABCD} &= \frac{1}{2}(\gamma^{AC}\gamma^{BD} + \gamma^{AD}\gamma^{BC} - \gamma^{AB}\gamma^{CD})\,, \\ \mathcal{G}_{ABCD} &= \frac{1}{2}(\gamma_{AC}\gamma_{BD} + \gamma_{AD}\gamma_{BC} - \gamma_{AB}\gamma_{CD})\,, \end{aligned} \tag{G.5}$$

has the following components

$$\begin{aligned} &\mathcal{G}^{1111} = \frac{1}{2}e^{-2\gamma}\cosh^2\phi\,, && \mathcal{G}^{2222} = \frac{1}{2}e^{2\gamma}\cosh^2\phi\,, \\ &\mathcal{G}^{1122} = \mathcal{G}^{2211} = -\frac{1}{2}(1-\sinh^2\phi)\,, && \mathcal{G}^{1212} = \mathcal{G}^{2121} = \frac{1}{2}(1+\sinh^2\phi)\,, \\ &\mathcal{G}^{1112} = \mathcal{G}^{1211} = -\frac{1}{4}e^{-\gamma}\sinh 2\phi\,, && \mathcal{G}^{1222} = \mathcal{G}^{2212} = -\frac{1}{4}e^{\gamma}\sinh 2\phi\,. \end{aligned} \tag{G.6}$$

With the above, $\not{\delta}\hat{\gamma}_{AB}$ (4.9) is obtained as

$$\not{\delta}\hat{\gamma}_{AB} = \begin{pmatrix} e^{\gamma}(\cosh\phi\,\not{\delta}\hat{\gamma} + \sinh\phi\,\not{\delta}\hat{\phi}) & \cosh\phi\,\not{\delta}\hat{\phi} \\ \cosh\phi\,\not{\delta}\hat{\phi} & e^{-\gamma}(-\cosh\phi\,\not{\delta}\hat{\gamma} + \sinh\phi\,\not{\delta}\hat{\phi}) \end{pmatrix}, \tag{G.7}$$

where we introduced non-exact forms on solution space as follows

$$\begin{aligned} &\not{\delta}\hat{\mathcal{P}} = \delta\mathcal{P} - \Omega\frac{\partial_v\phi}{\partial_v\Omega}\delta\phi - \Omega\cosh^2\phi\frac{\partial_v\gamma}{\partial_v\Omega}\delta\gamma\,, \\ &\not{\delta}\hat{\gamma} = \delta\gamma - \frac{\partial_v\gamma}{\partial_v\Omega}\delta\Omega\,, \qquad \not{\delta}\hat{\phi} = \delta\phi - \frac{\partial_v\phi}{\partial_v\Omega}\delta\Omega\,. \end{aligned} \tag{G.8}$$

As a next step, we compute the symplectic potential (4.5a) and symplectic form (4.5b)

$$16\pi G\;\boldsymbol{\Theta}_{\text{on-shell}} = \oint_{\mathcal{N}_b}\Omega\,\delta\mathcal{P} - \int_{\mathcal{N}}\Omega\left[\partial_v\phi\,\not{\delta}\hat{\phi} + \cosh^2\phi\,\partial_v\gamma\,\not{\delta}\hat{\gamma}\right], \tag{G.9a}$$

$$16\pi G\;\boldsymbol{\Omega}_{\text{on-shell}} = \oint_{\mathcal{N}_b}\delta\Omega\wedge\delta\mathcal{P} + \int_{\mathcal{N}}\Omega\left[\sinh 2\phi\,\partial_v\gamma\,\not{\delta}\hat{\gamma}\wedge\not{\delta}\hat{\phi} + \cosh^2\phi\,\not{\delta}\hat{\gamma}\wedge\partial_v\not{\delta}\hat{\gamma} + \not{\delta}\hat{\phi}\wedge\partial_v\not{\delta}\hat{\phi}\right]. \tag{G.9b}$$

Neither the bulk nor the boundary parts are closed or invertible. The non-invertibility of the boundary part is due to its non-closedness, which is a consequence of having the boundary symplectic flux (4.13)

$$\mathbf{F}_{\text{bdy}} = \frac{1}{16\pi G}\oint_{\mathcal{N}_b}\frac{\Omega}{\partial_v\Omega}\left(\partial_v\phi\delta\phi - \cosh^2\phi\partial_v\gamma\delta\gamma\right)\wedge\delta\Omega\,. \tag{G.10}$$

Non-invertibility of the bulk piece is, however, due to both the presence of the flux (manifested in non-closed 1-forms $\oint X$) and the existence of a kernel vector in solution space (the Carrollian nature of the bulk solution space geometry).

Carrollian bulk solution space. Let us rewrite the bulk term in the symplectic form as

$$\mathbf{\Omega}_{\text{Bulk}} = \frac{1}{32\pi G}\int_{\mathcal{N}}\int'_{\mathcal{N}} \delta\varphi^{\mathbb{I}}(v,\mathbf{x})\,\mathbf{\Omega}_{\mathbb{IJ}}[v,\mathbf{x};v',\mathbf{x}']\wedge\delta\varphi^{\mathbb{J}}(v',\mathbf{x}')\,, \tag{G.11}$$

where $\varphi^{\mathbb{I}} = \{\Omega,\gamma,\phi\}$ and as in section 5 we define the bulk solution space metric as in (5.3)

$$\mathbf{\Omega}_{\mathbb{IJ}}[x;x'] = \boldsymbol{\mathcal{G}}_{\mathbb{IJ}}[x;x']\,\partial_v\delta(v-v')\delta^{D-2}(\mathbf{x}-\mathbf{x}')\,. \tag{G.12}$$

The 3×3 function-valued matrix $\boldsymbol{\mathcal{G}}_{\mathbb{IJ}}[x;x']$ leading to on-shell bulk solution space metric

$$\boldsymbol{\mathcal{G}}_{\mathbb{IJ}} = 2\Omega\begin{pmatrix} 2\frac{\mathcal{N}^2}{(\partial_v\Omega)^2} & -\frac{\partial_v\gamma}{\partial_v\Omega}\cosh^2\phi & -\frac{\partial_v\phi}{\partial_v\Omega}\\ -\frac{\partial_v\gamma}{\partial_v\Omega}\cosh^2\phi & \cosh^2\phi & 0\\ -\frac{\partial_v\phi}{\partial_v\Omega} & 0 & 1\end{pmatrix}\,, \tag{G.13}$$

where $\mathcal{N}^2$ is given in (G.4). More explicitly (cf. (5.6)),

$$\begin{aligned}\delta\mathbb{S}^2 = 2\int_{\mathcal{N}}\int'_{\mathcal{N}}\sqrt{\Omega(x)\Omega(x')}\,&\Big\{\cosh(\gamma(x)-\gamma(x'))\,\oint\hat{\phi}(x)\oint\hat{\phi}(x')\\ &+\left(\cosh\phi(x)\,\cosh\phi(x')-\cosh(\gamma(x)-\gamma(x'))\sinh\phi(x)\,\sinh\phi(x')\right)\,\oint\hat{\gamma}(x)\oint\hat{\gamma}(x')\\ &+\frac{1}{2}\sinh(\gamma(x)-\gamma(x'))\left(\sinh 2\phi(x')\,\oint\hat{\phi}(x)\oint\hat{\gamma}(x')-\sinh 2\phi(x)\,\oint\hat{\phi}(x')\oint\hat{\gamma}(x)\right)\Big\}\,.\end{aligned} \tag{G.14}$$

The above metric is clearly defining a Carrollian geometry, with a 2×2 "Carrollian metric"

$$\boldsymbol{\mathcal{G}}_{ij} = 2\Omega\begin{pmatrix}\cosh^2\phi & 0\\ 0 & 1\end{pmatrix}\,, \tag{G.15}$$

and the kernel vector $\mathbf{K}$ and the Ehresmann connection $\mathbf{E}$,

$$\mathbf{K} = \mathbf{K}^{\mathbb{I}}\frac{\delta}{\delta\phi^{\mathbb{I}}} = \partial_v\Omega\frac{\delta}{\delta\Omega}+\partial_v\phi\frac{\delta}{\delta\phi}+\partial_v\gamma\frac{\delta}{\delta\gamma}\,,\qquad \mathbf{E} = \mathbf{E}_{\mathbb{I}}\delta\phi^{\mathbb{I}} = \frac{1}{\partial_v\Omega}\delta\Omega+\frac{1}{\partial_v\gamma}\delta\gamma+\frac{1}{\partial_v\phi}\delta\phi\,. \tag{G.16}$$

As we see $\boldsymbol{\mathcal{G}}_{\mathbb{IJ}}\mathbf{K}^{\mathbb{J}} = 0$ and $\mathbf{K}^{\mathbb{I}}\mathbf{E}_{\mathbb{I}} = 1$.

4d bulk on-shell Poisson brackets. In our specific parametrization for γ_{AB} in terms of γ and ϕ, one can make the analysis of appendix F more explicit and find on-shell Poisson brackets of these variables. To this end, and a check of our previous analysis, we repeat them in this parametrization. One can simply read the integrable part of the symplectic form (4.14) (i.e. in the $\delta\Omega = 0$ sector)

$$\begin{aligned}\mathbf{\Omega}^{\text{I}}_{\text{Bulk}} &= \frac{1}{16\pi G}\int_{\mathcal{N}}\Omega\,\delta\mathcal{N}_{AB}\wedge\delta\gamma^{AB}\\ &= \frac{1}{16\pi G}\int_{\mathcal{N}}\Omega\left[\sinh 2\phi\,\partial_v\gamma\,\delta\gamma\wedge\delta\phi+\cosh^2\phi\,\delta\gamma\wedge\partial_v\delta\gamma+\delta\phi\wedge\partial_v\delta\phi\right]\,.\end{aligned} \tag{G.17}$$

As a check, one may immediately observe that $\mathbf{\Omega}^{\rm I}_{\rm Bulk}$ is closed in $\delta\Omega = 0$ sector, i.e. $\delta\mathbf{\Omega}^{\rm I}_{\rm Bulk} = 0$. This symplectic form may be written as

$$16\pi G\, \mathbf{\Omega}^{\rm I}_{\rm Bulk} = \frac{1}{2}\int_{\mathcal{N}}\int'_{\mathcal{N}} \delta\varphi^i(v,\mathbf{x})\, \mathbf{\Omega}_{ij}[v,\mathbf{x};v',\mathbf{x}'] \wedge \delta\varphi^j(v',\mathbf{x}')\,, \tag{G.18}$$

where $\varphi^i = \{\gamma,\phi\}$. By comparing the last two equations, one can simply find

$$\mathbf{\Omega}_{ij}[x;x'] = -\mathbf{A}_{ij}[x;x']\,\partial_{v'}\delta(v-v')\delta^2(\mathbf{x}-\mathbf{x}') + \mathbf{B}_{ij}[x;x']\,\delta(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,, \tag{G.19}$$

with

$$\mathbf{A}_{ij}[x;x'] = \hat{\mathbf{A}}_{ij}[x] + \hat{\mathbf{A}}_{ji}[x']\,, \qquad \mathbf{B}_{ij}[x;x'] = \hat{\mathbf{B}}_{ij}[x] - \hat{\mathbf{B}}_{ji}[x']\,, \tag{G.20}$$

where the explicit form of these matrices is given by

$$\hat{\mathbf{A}}_{ij}[x] = \Omega \begin{pmatrix} \cosh^2\phi & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad \hat{\mathbf{B}}_{ij}[x] = \Omega\, \sinh 2\phi\, \partial_v\gamma \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,. \tag{G.21}$$

The next step is to compute the inverse of the on-shell symplectic 2-form

$$\int''_{\mathcal{N}} \mathbf{\Omega}_{ik}[v,\mathbf{x};v'',\mathbf{x}'']\,\mho^{kj}[v'',\mathbf{x}'';v',\mathbf{x}'] = \delta_i^j\,\delta(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,. \tag{G.22}$$

This equation yields

$$-\int''_{\mathcal{N}} \mathbf{A}_{ik}[x;x'']\,\mho^{kj}[x'';x']\,\partial_{v''}\delta(v-v'') + \mathbf{B}_{ik}[x;x]\,\mho^{kj}[x;x'] = \delta_i^j\,\delta(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,. \tag{G.23}$$

One can simplify this equation further

$$\hat{\mathbf{A}}_{ik}[x]\partial_v\mho^{kj}[x;x'] + \partial_v\left(\hat{\mathbf{A}}_{ki}[x]\,\mho^{kj}[x;x']\right) + \mathbf{B}_{ik}[x]\,\mho^{kj}[x;x'] = \delta_i^j\,\delta(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,. \tag{G.24}$$

One can rewrite this equation as follows

$$\mathbf{A}_{ik}[x]\partial_v\mho^{kj}[x;x'] + \mathbf{C}_{ik}[x]\,\mho^{kj}[x;x'] = \delta_i^j\,\delta(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,, \tag{G.25}$$

where

$$\mathbf{A}_{ik}[x] := \mathbf{A}_{ik}[x;x]\,, \quad \mathbf{B}_{ik}[x] := \mathbf{B}_{ik}[x;x]\,, \quad \mathbf{C}_{ik}[x] := \mathbf{B}_{ik}[x] + \partial_v\hat{\mathbf{A}}_{ki}[x]\,. \tag{G.26}$$

Their explicit forms are given by

$$\mathbf{A}_{ij}[x] = 2\,\Omega \begin{pmatrix} \cosh^2\phi & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad \mathbf{C}_{ij}[x] = \Omega\, \sinh 2\phi \begin{pmatrix} \partial_v\phi & \partial_v\gamma \\ -\partial_v\gamma & 0 \end{pmatrix} + \frac{1}{2}\frac{\partial_v\Omega}{\Omega}\,\mathbf{A}_{ij}[x]\,, \tag{G.27}$$

To solve (G.25) we take the following ansatz

$$\mho^{ij}[x;x'] = \frac{1}{\sqrt{\Omega(x)\Omega(x')}}\,\mho^{ij}_{\rm S}[v,v';\mathbf{x}]H(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,. \tag{G.28}$$

To avoid cluttering and ease of notation, hereafter we will suppress the $\mathbf{x}$ dependence of our quantities e.g. $\mho_{\rm S}^{ij}[v,v';\mathbf{x}] = \mho_{\rm S}^{ij}[v,v']$. By using this ansatz (G.25) yields

$$\mho_{\rm S}^{ij}[v;v] = \frac{1}{2}\,\Omega\,\mathbf{A}^{ij}[v] = \frac{1}{4\,\cosh^2\phi}\begin{pmatrix}1 & 0\\ 0 & \cosh^2\phi\end{pmatrix}\,, \tag{G.29}$$

and

$$\partial_v \mho_{\rm S}^{ij}[v;v'] + \mathbf{D}^i{}_k\,\mho_{\rm S}^{kj}[v;v'] = 0\,, \tag{G.30}$$

where

$$\mathbf{D}^i{}_j = \mathbf{A}^{ik}\mathbf{C}_{kj} - \frac{1}{2}\frac{\partial_v\Omega}{\Omega}\delta^i_j = \tanh\phi\begin{pmatrix}\partial_v\phi & \partial_v\gamma\\ -\partial_v\gamma\,\cosh^2\phi & 0\end{pmatrix}\,. \tag{G.31}$$

Now let us consider

$$\mho_{\rm S}^{ij}[v;v'] = \begin{pmatrix}X(v,v') & Y(v,v')\\ Y(v',v) & Z(v,v')\end{pmatrix}\,, \tag{G.32}$$

where $X(v,v') = X(v',v)$ and $Z(v,v') = Z(v',v)$, then we need to solve four equations

$$\partial_v\left[\cosh\phi(v)\,X(v,v')\right] + Y(v',v)\,\sinh\phi(v)\,\partial_v\gamma(v) = 0\,, \tag{G.33a}$$

$$\partial_v Z(v,v') - \frac{1}{2}Y(v,v')\,\sinh 2\phi(v)\,\partial_v\gamma(v) = 0\,, \tag{G.33b}$$

$$\partial_v[\cosh\phi(x)\,Y(v,v')] + Z(v,v')\,\sinh\phi(v)\,\partial_v\gamma(v) = 0\,, \tag{G.33c}$$

$$\partial_v Y(v',v) - \frac{1}{2}X(v,v')\,\sinh 2\phi(v)\,\partial_v\gamma(v) = 0\,. \tag{G.33d}$$

To solve these equations, let us do a change of variables as

$$Z(v,v') = \frac{1}{4}\,\cos\chi(v,v')\,, \qquad Y(v,v') = \frac{1}{4}\,\frac{\sin\chi(v,v')}{\cosh\phi(v)}\,. \tag{G.34}$$

Using the above, equations (G.33b) and (G.33c) become a single equation

$$\partial_v\chi(v,v') + \sinh\phi(v)\partial_v\gamma(x) = 0\,, \tag{G.35}$$

which can be solved as follows

$$\chi(v,v') = -\chi(v',v) = -\int_{v'}^{v}\mathrm{d}\tilde{v}\,\sinh\phi(\tilde{v})\partial_{\tilde{v}}\gamma(\tilde{v})\,. \tag{G.36}$$

Next, we solve (G.33d) and find that

$$X(v,v') = \frac{\cos\chi(v,v')}{4\cosh\phi(v)\,\cosh\phi(v')}\,. \tag{G.37}$$

One can readily check that the initial condition (G.29) is satisfied. Now Dirac brackets read as

$$\{\gamma(v),\gamma(v')\} = \frac{8\pi G}{\sqrt{\Omega(x)\Omega(x')}}\frac{\cos\chi(v,v')}{\cosh\phi(x)\,\cosh\phi(x')}\,H(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,, \tag{G.38a}$$

$$\{\gamma(x),\phi(x')\} = \frac{8\pi G}{\sqrt{\Omega(x)\Omega(x')}}\frac{\sin\chi(v,v')}{\cosh\phi(x)}\,H(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,, \tag{G.38b}$$

$$\{\phi(x),\gamma(x')\} = -\frac{8\pi G}{\sqrt{\Omega(x)\Omega(x')}}\frac{\sin\chi(v,v')}{\cosh\phi(x')}\,H(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,, \tag{G.38c}$$

$$\{\phi(x),\phi(x')\} = \frac{8\pi G}{\sqrt{\Omega(x)\Omega(x')}}\,\cos\chi(v,v')\,H(v-v')\delta^2(\mathbf{x}-\mathbf{x}')\,, \tag{G.38d}$$

where $\chi(v,v')$ is given by (G.35). Therefore

$$\mho^{\rm S}_{1111}[x;x'] = \frac{1}{2}\sqrt{\frac{\gamma_{11}(x)}{\gamma_{22}(x)}}\sqrt{\frac{\gamma_{11}(x')}{\gamma_{22}(x')}}\,[(\gamma_{12}(x')-\gamma_{12}(x))\sin\chi(v,v')+(\gamma_{12}(x)\gamma_{12}(x')+1)\cos\chi(v,v')]\,, \tag{G.39a}$$

$$\mho^{\rm S}_{1112}[x;x'] = \frac{1}{2}\sqrt{\frac{\gamma_{11}(x)}{\gamma_{22}(x)}}\sqrt{\gamma_{11}(x')\gamma_{22}(x')}\,\,[\sin\chi(v,v')+\gamma_{12}(x)\cos\chi(v,v')]\,, \tag{G.39b}$$

$$\mho^{\rm S}_{1122}[x;x'] = \frac{1}{2}\sqrt{\frac{\gamma_{11}(x)}{\gamma_{22}(x)}}\sqrt{\frac{\gamma_{22}(x')}{\gamma_{11}(x')}}\,[(\gamma_{12}(x')+\gamma_{12}(x))\sin\chi(v,v')+(\gamma_{12}(x)\gamma_{12}(x')-1)\cos\chi(v,v')]\,, \tag{G.39c}$$

$$\mho^{\rm S}_{1212}[x;x'] = \frac{1}{2}\sqrt{\gamma_{11}(x)\gamma_{22}(x)}\sqrt{\gamma_{11}(x')\gamma_{22}(x')}\cos\chi(v,v')\,, \tag{G.39d}$$

$$\mho^{\rm S}_{2212}[x;x'] = \frac{1}{2}\sqrt{\frac{\gamma_{22}(x)}{\gamma_{11}(x)}}\sqrt{\gamma_{11}(x')\gamma_{22}(x')}\,\,(-\sin\chi(v,v')+\gamma_{12}(x)\cos\chi(v,v'))\,, \tag{G.39e}$$

$$\mho^{\rm S}_{2222}[x;x'] = \frac{1}{2}\sqrt{\frac{\gamma_{22}(x)}{\gamma_{11}(x)}}\sqrt{\frac{\gamma_{22}(x')}{\gamma_{11}(x')}}\,[(\gamma_{12}(x)-\gamma_{12}(x'))\sin\chi(v,v')+(1+\gamma_{12}(x)\gamma_{12}(x'))\cos\chi(v,v')]\,, \tag{G.39f}$$

where

$$\chi(v,v') = -\frac{1}{2}\int_{v'}^{v} \mathrm{d}\tilde v\;\gamma_{12}(\tilde v)\left(\frac{\partial_{\tilde v}\gamma_{11}(\tilde v)}{\gamma_{11}(\tilde v)} - \frac{\partial_{\tilde v}\gamma_{22}(\tilde v)}{\gamma_{22}(\tilde v)}\right)\,. \tag{G.40}$$

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