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Explorations in scalar fermion theories: β-functions, supersymmetry and fixed points

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Abstract: Results for β-functions and anomalous dimensions in general scalar fermion theories are presented to three loops. Various constraints on the individual coefficients for each diagram following from supersymmetry are analysed. The results are used to discuss potential fixed points in the ε-expansion for scalar fermion theories, with arbitrary numbers of scalar fields, and where there are just two scalar couplings and one Yukawa coupling. For different examples the fixed points follow a similar pattern as the numbers of fermions is varied. For diagrams with subdivergences there are extensive consistency constraints arising from the existence of a perturbative a-function and these are analysed in detail. Further arbitrary scheme variations which preserve the form of β functions and anomalous dimensions in terms of 1PI diagrams are also discussed. The existence of linear and quadratic scheme invariants is demonstrated and the consistency condition are shown to be expressible in terms of these invariants.

Keywords: Conformal and W Symmetry, Global Symmetries, Renormalization Group, Supersymmetry and Duality

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1 Introduction

Whatever the role of supersymmetry in the phenomenological description of the world at accessible energies there is no doubt that supersymmetric quantum field theories in various dimensions have enhanced our understanding of quantum field theories more generally. This is especially true non perturbatively where the duality between different theories was first developed and the existence of conformal fixed points in three and higher dimensions is

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much better understood. However there are also constraints at the perturbative level where supersymmetric non-renormalisation theorems have implications for more general β-functions and related quantities when they are reduced to the supersymmetric case.

In this paper we explore these, and other, constraints for general scalar fermion theories in four space-time dimensions at up to three loops. For pure scalar theories, this is hardly state of the art as general three loop results have been known for more than 30 years [1], and higher orders are available [2–5]. Nevertheless, the corresponding expressions for general scalar fermion theories, allowing for arbitrary Yukawa couplings, have only been obtained quite recently [6–8]. While field anomalous dimensions and Yukawa β-functions were obtained, these depended on results already found for a variety of special cases. The general β- and γ-functions are expressed in terms of contractions of generalised coupling tensors with each term corresponding to a specific allowed Feynman diagram at each loop order. The associated results for the quartic scalar β-function at three loops have not previously been fully determined [6]. Closing this gap would also represent a stepping stone towards complete three-loop renormalisation group equations of any renormalisable QFT, which is now feasible after recent advances in general four-loop gauge and three-loop Yukawa results [7–9]. Without gauge interactions, each term corresponds to a one particle irreducible (1PI) diagram, whose numbers increase rapidly with each loop order. With the quartic scalar coupling in standard regularisation schemes all one vertex reducible diagrams (or *snail diagrams*) can be omitted so that the necessary diagrams are one vertex irreducible (1VI). The unknown coefficients may be partially fixed with direct calculations, e.g. [10–16] in our case. However, their number can be greatly reduced and literature results cross-checked by applying constraints arising from special cases such as supersymmetry, which is the exercise undertaken here. We are then able to fully determine the three loop beta function for the quartic scalar couplings in general scalar fermion theories.

To carry out our analysis for general four-dimensional renormalisable scalar fermion theories, it is natural to consider a basis with n_s real scalars ϕ^a and essentially n_f pseudo-real Majorana fermions ψ where the couplings are just a symmetric 4 index real tensor λ^{abcd} and a Yukawa coupling y^a which is a symmetric $n_f \times n_f$ real matrix in the non spinorial fermion indices (which are here suppressed) [6, 9]. The discussion in subsequent sections then concerns the beta functions ${\beta_\lambda}^{abcd}$, ${\beta_y}^a$ as well as associated anomalous dimensions ${\gamma_\phi}^{ab}$, and γ_ψ. These quantities completely determine β-functions when superrenormalisable couplings, corresponding to operators with dimension three or less, are introduced, if background field methods are used and ${\beta_\lambda}^{abcd}$ is extended to $\beta_V(\phi)$ for an arbitrary quartic scalar potential V. Equivalently by applying the so called dummy field technique [17–19]. The results here encompass those for Dirac fermions, the corresponding reduction is described later.

For each fermion loop graph which leads to a trace over products of the Yukawa coupling matrices y^a then with our conventions the numerical coefficient for each such trace for a four-dimensional four-component Majorana spinor Ψ should have an additional factor 2 times the results quoted here. Such four-dimensional Majorana spinors reduce in three dimensions to two two-component real spinors which belong to inequivalent representations of the three-dimensional Dirac algebra.

General theories of course can be restricted by imposing symmetries. With complex fields, n_s even, then we may take $y^a \to (y^i, \bar{y}_i)$ with $y^i, \bar{y}_i$ not necessarily square matrices but related by hermitian conjugation. Imposing a U(1) symmetry where both scalar and fermion fields carry a charge, so that all lines in any diagram are directed, the number of diagrams is significantly reduced (for the three loop 1PI Yukawa vertex diagrams from 52 to 12) [20].

As a special case the U(1) symmetric theory encompasses the Wess-Zumino theory with $\mathcal{N} = 1$ supersymmetry and four supercharges [21]. In a superspace formalism, for the renormalisable theory, there are complex chiral superfields Φ_i, $\bar{\Phi}^i$, with an overall U(1) symmetry and the general couplings are given by a symmetric 3 index tensor Y^{ijk} and its conjugate $\bar{Y}_{ijk}$, which determine the scalar quartic couplings. There are then very strong non-renormalisation theorems [22, 23] which ensure that the β-functions $\beta_Y{}^{ijk}$, $\beta_{\bar{Y}}{}_{ijk}$ are determined just in terms of the anomalous dimensions γ_Φ, γ. Moreover, dedicated literature for such supersymmetric QFT's is available to high orders [24–29]. This yields conditions on the β-functions and anomalous dimensions for an arbitrary scalar fermion theory but these do not significantly reduce the number of independent terms [6].

In three dimensions there are scalar-fermion theories with just two supercharges [30]. In a superfield formalism the theory is described in terms of a real superfield Φ^a and for current interest there are just real cubic couplings given by the symmetric three index tensor Y^{abc}. Such theories can emerge at fixed points under RG flow [31–34] and may be relevant for fixed point exponents in some condensed matter systems. For a single scalar field and a $\mathbb{Z}_2$ symmetry this is a supersymmetric version of the 3d Ising model. For several scalar fields then extending the theory away from three dimensions it is possible to set up an epsilon expansion determining potential fixed points and their associated critical exponents [35]. The 3d supersymmetric Ising model has been explored using the bootstrap [36–40], with extensions to several fields in [39]. Of course extending supersymmetric theories away from their natural integer dimension, even just in a perturbative expansion, is potentially fraught with problems. Various Ward identities necessary for supersymmetry are no longer valid. These relate contributions with different numbers of fermion loops and depend on Fierz identities. However these problems do not arise at low loop order, up to three loops in our case. A discussion of the four dimensional $\mathcal{N} = 1$ supersymmetry algebra extended away from four dimensions using a form of dimensional reduction was given in [41]. The minimal three dimensional supersymmetric scalar-fermion theory would define an apparent four dimensional theory with $\mathcal{N} = \frac{1}{2}$ supersymmetry.[1] These theories of course do not exist as well defined Lorentz invariant unitary theories though there exists the possibility of considering such a theory away from three dimensions where the full d-dimensional Lorentz symmetry is broken. Dimensional regularisation breaks supersymmetry but for $\mathcal{N} = 1$ theories such anomalous contributions breaking supersymmetric Ward identities should be removable by an appropriate redefinition of the couplings, or essentially a change of scheme. At one, two or three loops it is sufficient just to ensure that fermion traces are appropriately normalised. We defer further discussion to the conclusion.

In section 8 we make use of the three loop results to discuss possible fixed points in the ε-expansion for fermion scalar theories. We consider generalisations of the Gross-Neveu, Nambu Jona-Lasinio and Heisenberg theories which have $n_s = 1, 2, 3$ scalar fields and have $O(n_s)$ symmetry. For $n_s \geq 4$ there are theories with reduced $H \subset O(n_s)$ symmetry which have two scalar couplings and one Yukawa coupling. For a consistent RG flow with the

[1]This is different from the $\mathcal{N} = \frac{1}{2}$ supersymmetry discussed in [42, 43] which involve non anti-commuting θ's or $\bar{\theta}'$s. The renormalisation of these theories was considered in [44–50].

reduced set of couplings it is necessary to impose completeness relations on the matrices defining the Yukawa couplings. For square matrices we identify six different examples where these are satisfied. In each case the numbers of fermions n_f can be arbitrarily large. For vanishing Yukawa coupling these theories generally have just the $O(n_s)$ and Gaussian fixed points though one example is equivalent to the scalar theory with hypertetrahedral symmetry where there are two further fixed points with $\mathcal{S}_{n_s+1} \times \mathbb{Z}_2$ symmetry. Assuming just lowest order β-functions the Yukawa β-function does not contain the scalar couplings and is easily solved. For the scalar couplings there are then relations between the fixed points for small and large n_f. A similar pattern emerges in each example. Even if there are four fixed points when $n_f = 0$ these reduce to two except for very tiny or very large n_f. Generally there are two fixed points for low and large n_f and for intermediate n_f either 0 or 4. These do not necessarily lead to scalar potentials which are bounded below, the Gaussian fixed point becomes unstable when $n_f > 0$, but there is a stable potential for large n_f related to the Gaussian fixed point as $n_f \to 0$. We also consider an example where the Yukawa matrices are not square, corresponding to chiral fermions, and where there is a $\mathrm{U}(r) \times \mathrm{U}(s)$ symmetry and $n_s = rs$. The purely scalar theory may have four fixed points for suitable r, s but with a non zero Yukawa coupling there is a similar pattern.

Further constraints relating the coefficients for the contributions of various diagrams to β-functions and anomalous dimensions can be obtained from applying a perturbative version of the a-theorem [1, 9, 20, 51, 52]. In general this relates certain combinations of β and γ-function coefficients at a particular loop order to lower order contributions. In the present context this provides relations for the coefficients of the three loop Yukawa β-function and also γ_ϕ and γ_ψ. Such conditions were analysed at length by Poole and Thomsen [9], including also gauge couplings. We present their results here without any explicit evaluation of lower order one and two loop contributions so that the structure of the conditions is more apparent. We also consider the restriction to U(1) symmetry where results are more tractable.

The outline of the paper is as follows: In the next four sections we list the diagrams for the scalar and fermion anomalous dimensions and the Yukawa and quartic scalar β-functions for the general scalar fermion theory at up to three loops. We also give the values for the corresponding coefficients, 143 at three loops, which are all consistent with the various relations obtained later. Of course at one and two loops results have been known for a long time, we list the coefficients diagram by diagram. Our conventions match those in [9] and our numerical results at one and two loops agree precisely once they are multiplied by the required factor to ensure overall symmetry. Similarly the three loop results for the Yukawa β-function and also the anomalous dimensions agree exactly with [8]. In the case of the quartic scalar β-function the relations obtained here are used to provide complete results for all terms appearing in the general expansion. For simplicity the Yukawa couplings are rescaled by 4π and the scalar quartic couplings by $16\pi^2$. These coefficients all correspond to what would be obtained in a $\overline{MS}$ scheme although no explicit calculation is undertaken here.

The results are simplified in section 6 where a U(1) symmetry is imposed which significantly reduces the number of terms present in the expansions of the general β-functions and anomalous dimensions. The U(1) restriction contains as a special case $\mathcal{N} = 1$ supersymmetry and the various necessary linear constraints are derived in section 7. We there

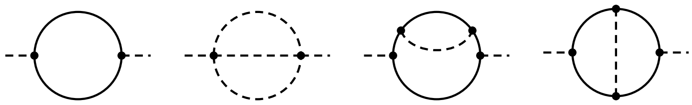

Figure 1. One and two loop diagrams giving contributions to the scalar field anomalous dimensions, containing Yukawa and quartic scalar couplings. Fermion lines are solid, scalar lines are dashed.

also consider also the example of what is here termed $\mathcal{N} = \frac{1}{2}$ supersymmetry where there are a significant number of linear constraints which are all satisfied by the explicit results listed earlier in sections 2, 3, 4, 5.

Besides supersymmetry conditions there are also relations for the various coefficients derived from the existence of a perturbative a-function. We list the conditions for the general scalar-fermion theory which are all derived from [9]. For the two loop anomalous dimensions and the Yukawa β-function there are 4 relations whereas at three loops there are 42. At three loops it is necessary to also allow for 5 possible antisymmetric contributions to the anomalous dimensions and 4 relations for these are obtained.

Section 8 contains our discussion of scalar fermion fixed points. For multiple scalars we show there are theories which can be restricted to a single Yukawa coupling and two quartic scalar couplings. As special cases these include the well known renormalisable Gross-Neveu and Nambu Jona-Lasinio theories.

In general results for individual coefficients corresponding to particular diagrams are scheme dependent. In section 10 scheme variations which preserve the structure in terms of contributions from 1PI diagrams are considered. Coefficients corresponding to primitive diagrams, which have no subdivergences, are individually invariant but this of course not true in general. We demonstrate how scheme invariants can be formed and applied in detail to the three loop Yukawa β-function. These can be linear or higher order in the coefficients. The scheme invariance of the a-function relations is also verified.

Some further details are considered in various appendices. Appendix A describes a basis for Majorana fermions relevant for reduction to three dimensions and their possible extensions away from $d = 3$ with broken Lorentz invariance. In appendix B we outline some tensorial calculations relevant for the fixed point discussion. Some figures elucidating how the fixed points in scalar fermion theories vary with differing numbers of fermions are given in appendix C. In appendix D we describe the derivation of the a-function relations at two and three loops after restricting to U(1) symmetry, there is then one relation at two loops and 12 at three. Finally in appendix E we discuss some general features of scheme changes which preserve the perturbative structure in terms of contributions corresponding to 1PI diagrams.

2 Scalar anomalous dimension

The one and two loop 1PI and 1VI diagrams relevant for ${\gamma_\phi}^{ab} = {\gamma_\phi}^{ba}$ are just shown in figure 1, while at three loops are shown in figure 2.

The corresponding expansions are then

$$\begin{aligned}
\gamma_\phi^{(1)ab} &= \gamma_{\phi 1}\, \mathrm{tr}(y^{ab})\,, \\
\gamma_\phi^{(2)ab} &= \gamma_{\phi 2a}\, \lambda^{acde}\lambda^{bcde} + \gamma_{\phi 2b}\, \mathrm{tr}(y^{abcc}) + \gamma_{\phi 2c}\, \mathrm{tr}(y^{acbc})\,,
\end{aligned}$$

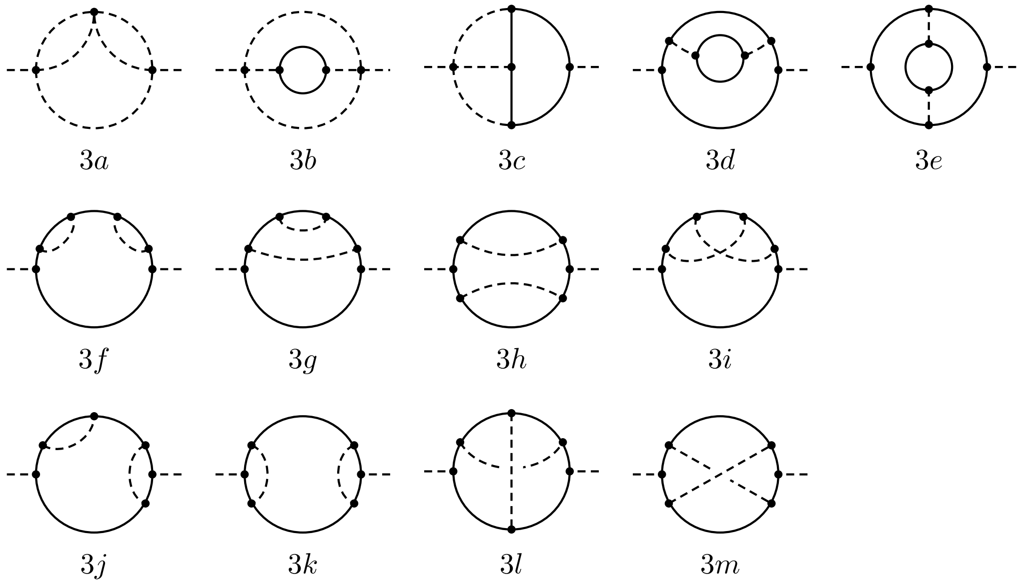

Figure 2. Three loop diagrams giving contributions to the scalar field anomalous dimensions, containing Yukawa and quartic scalar couplings.

$$
\begin{aligned}
\gamma_\phi^{(3)ab} = {} & \gamma_{\phi 3a}\, \lambda^{acde}\lambda^{defg}\lambda^{bcfg} + \gamma_{\phi 3b}\, \lambda^{acde}\lambda^{bcdf}\, \mathrm{tr}(y^{ef}) + \gamma_{\phi 3c}\, \mathcal{S}_2\, \lambda^{acde}\, \mathrm{tr}(y^{bcde}) \\
& + \gamma_{\phi 3d}\, \mathrm{tr}(y^{abcd})\, \mathrm{tr}(y^{cd}) + \gamma_{\phi 3e}\, \mathrm{tr}(y^{acbd})\, \mathrm{tr}(y^{cd}) + \gamma_{\phi 3f}\, \mathrm{tr}(y^{abccdd}) \\
& + \gamma_{\phi 3g}\, \mathrm{tr}(y^{abcddc}) + \gamma_{\phi 3h}\, \mathrm{tr}(y^{accbdd}) + \gamma_{\phi 3i}\, \mathrm{tr}(y^{abcdcd}) + \gamma_{\phi 3j}\, \mathcal{S}_2\, \mathrm{tr}(y^{acbcdd}) \\
& + \gamma_{\phi 3k}\, \mathrm{tr}(y^{acdbdc}) + \gamma_{\phi 3l}\, \mathrm{tr}(y^{acbdcd}) + \gamma_{\phi 3m}\, \mathrm{tr}(y^{acdbcd}) \,,
\end{aligned}
\tag{2.1}
$$

employing the abbreviation $y^{abcd\ldots} = y^a y^b y^c y^d \ldots$ and where $\mathcal{S}_2$ denotes the sum over two terms necessary to ensure symmetry for $a \leftrightarrow b$ so that $\gamma_\phi^{ab} = \gamma_\phi^{ba}$ for the three loop expressions. The normalisations of the traces correspond to the fermions having two components, as would be appropriate in three dimensions. For four dimensional Majorana fermions $y_a \to \left(\begin{smallmatrix} y_a & 0 \\ 0 & y_a \end{smallmatrix}\right)$ so that

$$
\mathrm{tr}(y^{a_1} y^{a_2} \ldots y^{a_p})\big|_{\text{Majorana}} = \big(1 + (-1)^p\big)\, \mathrm{tr}(y^{a_1 \cdots a_p}) \,. \tag{2.2}
$$

The coefficients for the trace corresponding to a fermion loop then has an additional factor two and there are only an even number of Yukawa couplings on any loop.

With this notation the results of calculation for the individual coefficients in the general fermion scalar theory are [6–8, 53]

$$
\begin{aligned}
& \gamma_{\phi 1} = \tfrac{1}{2} \,, && \gamma_{\phi 2a} = \tfrac{1}{12} \,, && \gamma_{\phi 2b} = -\tfrac{3}{4} \,, && \gamma_{\phi 2c} = -\tfrac{1}{2} \,, && \\
& \gamma_{\phi 3a} = -\tfrac{1}{16} \,, && \gamma_{\phi 3b} = -\tfrac{5}{32} \,, && \gamma_{\phi 3c} = \tfrac{5}{8} \,, && \gamma_{\phi 3d} = 1 \,, && \gamma_{\phi 3e} = \tfrac{9}{16} \,, \\
& \gamma_{\phi 3f} = -\tfrac{3}{16} \,, && \gamma_{\phi 3g} = \tfrac{5}{16} \,, && \gamma_{\phi 3h} = \tfrac{1}{32} \,, && \gamma_{\phi 3i} = -\tfrac{3}{8} \,, && \\
& \gamma_{\phi 3j} = \tfrac{7}{16} \,, && \gamma_{\phi 3k} = \tfrac{7}{4} \,, && \gamma_{\phi 3l} = -\tfrac{3}{4} \,, && \gamma_{\phi 3m} = \tfrac{3}{2}\zeta_3 - 1 \,. &&
\end{aligned}
\tag{2.3}
$$

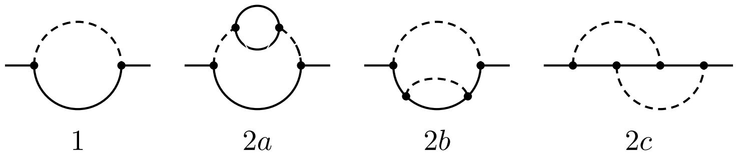

Figure 3. One and two loop diagrams giving contributions to the fermion field anomalous dimensions, containing Yukawa and quartic scalar couplings.

At three loop order there is the further possibility of 1PI antisymmetric contributions to the anomalous dimension [14, 53–55] which take the form [9]

$$\upsilon_\phi^{(3)ab} = \upsilon_{\phi 3c}\, \mathcal{A}_2\, \lambda^{acde}\, \mathrm{tr}(y^{bcde}) + \upsilon_{\phi 3j}\, \mathcal{A}_2\, \mathrm{tr}(y^{acbcdd})\,, \tag{2.4}$$

where now $\mathcal{A}_2\, \mathrm{tr}(y^{acbcdd}) = \mathrm{tr}(y^{acbcdd}) - \mathrm{tr}(y^{bcacdd})$. Such terms can usually be neglected but they play a role in finding fixed points with vanishing energy momentum tensor trace. In this context the results [8] are then

$$\upsilon_{\phi 3c} = -\tfrac{5}{8}\,, \qquad \upsilon_{\phi 3j} = -\tfrac{3}{4}\,. \tag{2.5}$$

3 Fermion anomalous dimension

For $\gamma_\psi = \gamma_\psi{}^T$, at one and two loops the 1PI, 1VI diagrams are displayed in figure 3 and at three loops there are 16 1PI diagrams shown in figure 4.

Corresponding to figures 3 and 4 the contributions have the general form

$$\begin{aligned}
\gamma_\psi^{(1)} &= \gamma_{\psi 1}\, y^{aa}\,, \\
\gamma_\psi^{(2)} &= \gamma_{\psi 2a}\, y^{ab}\, \mathrm{tr}(y^{ab}) + \gamma_{\psi 2b}\, y^{abba} + \gamma_{\psi 2c}\, y^{abab}\,, \\
\gamma_\psi^{(3)} &= \gamma_{\psi 3a}\, \lambda^{acde}\lambda^{bcde}\, y^{ab} + \gamma_{\psi 3b}\, \lambda^{abcd}\, y^{abcd} + \gamma_{\psi 3c}\, y^{ab}\, \mathrm{tr}(y^{ac})\, \mathrm{tr}(y^{bc}) \\
&\quad + \big(\gamma_{\psi 3d}\, y^{accb} + \gamma_{\psi 3e}\, y^{cabc} + \gamma_{\psi 3f}\, \mathcal{S}_2\, y^{acbc}\big)\, \mathrm{tr}(y^{ab}) + \gamma_{\psi 3g}\, y^{ab}\, \mathrm{tr}(y^{abcc}) \\
&\quad + \gamma_{\psi 3h}\, y^{ab}\, \mathrm{tr}(y^{acbc}) + \gamma_{\psi 3i}\, y^{abbcca} + \gamma_{\psi 3j}\, y^{abccba} + \gamma_{\psi 3k}\, y^{abacbc} + \gamma_{\psi 3l}\, y^{abcbca} \\
&\quad + \gamma_{\psi 3m}\, \mathcal{S}_2\, y^{abcbac} + \gamma_{\psi 3n}\, \mathcal{S}_2\, y^{abbcac} + \gamma_{\psi 3o}\, y^{abccab} + \gamma_{\psi 3p}\, y^{abcabc}\,,
\end{aligned} \tag{3.1}$$

where here $\mathcal{S}_2\, y^{acbc} = y^{acbc} + y^{cbca}$, $\mathcal{S}_2\, y^{abcbac} = y^{abcbac} + y^{cabcba}$ and similarly as necessary for the symmetry $\gamma_\psi^{(3)} = \gamma_\psi^{(3)T}$. In this case the coefficients in a $\overline{MS}$ scheme are then [6–8, 53]

$$\begin{array}{llll}
\gamma_{\psi 1} = \tfrac{1}{2}\,, & \gamma_{\psi 2a} = -\tfrac{3}{8}\,, & \gamma_{\psi 2b} = -\tfrac{1}{8}\,, & \gamma_{\psi 2c} = 0\,, \\
\gamma_{\psi 3a} = -\tfrac{11}{96}\,, & \gamma_{\psi 3b} = 1\,, & \gamma_{\psi 3c} = -\tfrac{3}{32}\,, & \gamma_{\psi 3d} = -\tfrac{1}{8}\,, \\
\gamma_{\psi 3e} = \tfrac{9}{32}\,, & \gamma_{\psi 3f} = -\tfrac{3}{32}\,, & \gamma_{\psi 3g} = 1\,, & \gamma_{\psi 3h} = \tfrac{1}{2}\,, \\
\gamma_{\psi 3i} = -\tfrac{5}{32}\,, & \gamma_{\psi 3j} = \tfrac{1}{16}\,, & \gamma_{\psi 3k} = \tfrac{1}{2}\,, & \gamma_{\psi 3l} = -\tfrac{5}{16}\,, \\
\gamma_{\psi 3m} = 0\,, & \gamma_{\psi 3n} = \tfrac{3}{32}\,, & \gamma_{\psi 3o} = \tfrac{1}{4}\,, & \gamma_{\psi 3p} = \tfrac{3}{2}\zeta_3 - 1\,.
\end{array} \tag{3.2}$$

For potential antisymmetric contributions

$$\upsilon_\psi^{(3)} = \upsilon_{\psi 3f}\, \mathcal{A}_2\, y^{acbc}\, \mathrm{tr}(y^{ab}) + \upsilon_{\psi 3m}\, \mathcal{A}_2\, y^{abcbac} + \gamma_{\psi 3n}\, \mathcal{A}_2\, y^{abbcac}\,, \tag{3.3}$$

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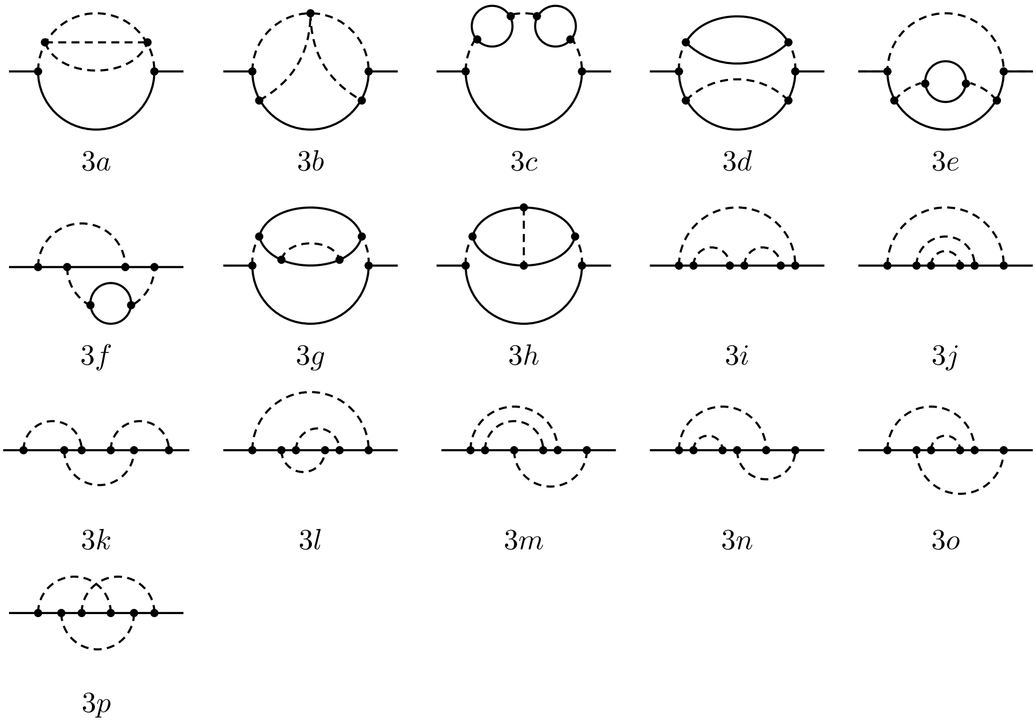

Figure 4. Three-loop diagrams giving contributions to the fermion field anomalous dimensions, containing Yukawa and quartic scalar couplings.

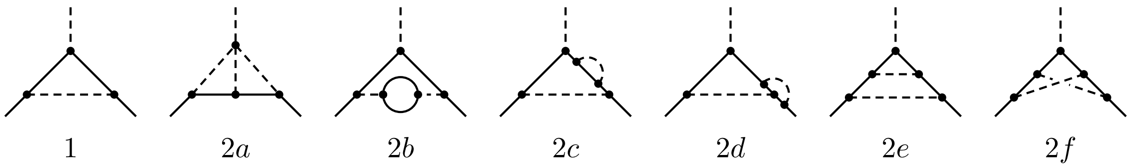

Figure 5. One and two loop Yukawa vertex diagrams.

with $\mathcal{A}_2\, y^{acbc}\,\mathrm{tr}(y^{ab}) = (y^{acbc} - y^{cacb})\,\mathrm{tr}(y^{ab})$, $\mathcal{A}_2\, y^{abcbac} = y^{abcbac} - y^{acbabc}$. In this case [8]

$$\upsilon_{\psi 3f} = \tfrac{7}{16}\,, \qquad \upsilon_{\psi 3m} = -\tfrac{3}{8}\,, \qquad \upsilon_{\psi 3n} = -\tfrac{5}{16}\,. \tag{3.4}$$

4 Yukawa couplings

The Yukawa coupling β-function can be decomposed as

$$\beta_y^a = \beta_y^{aT} = \tilde{\beta}_y^a + \gamma_\phi^{ab}\, y^b + \gamma_\psi\, y^a + y^a\, \gamma_\psi\,, \tag{4.1}$$

where $\tilde{\beta}_y^a$ is determined solely by the contributions of 1PI diagrams.

At one and two loops the relevant 1PI diagrams are shown in figure 5, where

$$\begin{aligned}
\tilde{\beta}_y^{(1)a} &= \beta_{y1}\, y^{bab}\,, \\
\tilde{\beta}_y^{(2)a} &= \beta_{y2a}\,\lambda^{abcd} y^{bcd} + \beta_{y2b}\, y^{bac}\,\mathrm{tr}(y^{bc}) + \beta_{y2c}\,\mathcal{S}_2\, y^{baccb} + \beta_{y2d}\,\mathcal{S}_2\, y^{bacbc} \\
&\quad + \beta_{y2e}\, y^{bcacb} + \beta_{y2f}\, y^{bcabc}\,,
\end{aligned} \tag{4.2}$$

with $\mathcal{S}_2\, y^{baccb} = y^{baccb} + y^{bccab}$, as necessary for symmetry. Old results, with our conventions, give

$$\beta_{y1} = 2\,, \quad \beta_{y2a} = -2\,, \quad \beta_{y2b} = \beta_{y2c} = -1\,, \quad \beta_{y2d} = 0\,, \quad \beta_{y2e} = -2\,, \quad \beta_{y2f} = 2\,. \tag{4.3}$$

At three loops there are 52 distinct diagrams so we use the alphabet twice over as labels (figure 6). Joining the external lines to a single vertex the resulting vacuum diagrams can be either planar or non planar. In the above list $3c, 3s, 3x, 3\tilde{g}, 3\tilde{o}, 3\tilde{p}, 3\tilde{r}, 3\tilde{s}, 3\tilde{u}, 3\tilde{v}, 3\tilde{w}, 3\tilde{x}, 3\tilde{y}$ are non planar.

With this diagrammatic decomposition the three loop Yukawa β-function is expanded as

$$\begin{aligned}
\tilde{\beta}_y^{(3)a} = {} & \beta_{y3a}\, \lambda^{bdef}\lambda^{cdef}\, y^{bac} + \lambda^{abef}\lambda^{efcd}\big(\beta_{y3b}\, \mathcal{S}_2\, y^{bcd} + \beta_{y3c}\, y^{cbd}\big) \\
& + \lambda^{acde}\, \mathrm{tr}(y^{eb})\, \big(\beta_{y3d}\, y^{cbd} + \beta_{y3e}\, \mathcal{S}_2\, y^{bcd}\big) + \lambda^{bcde}\, \big(\beta_{y3f}\, y^{bcade} + \beta_{y3g}\, \mathcal{S}_2\, y^{bacde}\big) \\
& + \lambda^{abcd}\big(\beta_{y3h}\, y^{ebcde} + \beta_{y3i}\, y^{beced} + \beta_{y3j}\, \mathcal{S}_2\, y^{beecd} + \beta_{y3k}\, \mathcal{S}_2\, y^{ebecd} + \beta_{y3l}\, \mathcal{S}_2\, y^{ebced}\big) \\
& + \beta_{y3m}\, y^{bac}\, \mathrm{tr}(y^{bd})\, \mathrm{tr}(y^{cd}) \\
& + \big(\beta_{y3n}\, y^{dbacd} + \beta_{y3o}\, y^{bdadc} + \beta_{y3p}\, \mathcal{S}_2\, y^{baddc} + \beta_{y3q}\, \mathcal{S}_2\, y^{dabdc}\big)\, \mathrm{tr}(y^{bc}) \\
& + \big(\beta_{y3r}\, \mathcal{S}_2\, y^{badcd} + \beta_{y3s}\, \mathcal{S}_2\, y^{dbadc} + \beta_{y3t}\, \mathcal{S}_2\, y^{dabcd}\big)\, \mathrm{tr}(y^{bc}) \\
& + y^{bac}\big(\beta_{y3u}\, \mathrm{tr}(y^{bcdd}) + \beta_{y3v}\, \mathrm{tr}(y^{bdcd})\big) + \big(\beta_{y3w}\, y^{bcd} + \beta_{y3x}\, \mathcal{S}_2\, y^{bdc}\big)\, \mathrm{tr}(y^{abcd}) \\
& + \beta_{y3y}\, y^{bccaddb} + \beta_{y3z}\, \mathcal{S}_2\, y^{baccddb} + \beta_{y3\tilde{a}}\, \mathcal{S}_2\, y^{bacddcb} + \beta_{y3\tilde{b}}\, \mathcal{S}_2\, y^{bcacddb} \\
& + \beta_{y3\tilde{c}}\, \mathcal{S}_2\, y^{bcaddcb} + \beta_{y3\tilde{d}}\, \mathcal{S}_2\, y^{bcadcdb} + \beta_{y3\tilde{e}}\, \mathcal{S}_2\, y^{bacdcdb} + \beta_{y3\tilde{f}}\, y^{bcdadcb} \\
& + \beta_{y3\tilde{g}}\, y^{bcdacdb} + \beta_{y3\tilde{h}}\, \mathcal{S}_2\, y^{bcbaddc} + \beta_{y3\tilde{\imath}}\, \mathcal{S}_2\, y^{baccdbd} + \beta_{y3\tilde{\jmath}}\, \mathcal{S}_2\, y^{bacddbc} \\
& + \beta_{y3\tilde{k}}\, \mathcal{S}_2\, y^{bacbddc} + \beta_{y3\tilde{l}}\, \mathcal{S}_2\, y^{bacbdcd} + \beta_{y3\tilde{m}}\, \mathcal{S}_2\, y^{bacdbdc} + \beta_{y3\tilde{n}}\, \mathcal{S}_2\, y^{badcdbc} \\
& + \beta_{y3\tilde{o}}\, \mathcal{S}_2\, y^{bcabddc} + \beta_{y3\tilde{p}}\, \mathcal{S}_2\, y^{bcaddbc} + \beta_{y3\tilde{q}}\, \mathcal{S}_2\, y^{bcacdbd} + \beta_{y3\tilde{r}}\, \mathcal{S}_2\, y^{bcadbdc} \\
& + \beta_{y3\tilde{s}}\, \mathcal{S}_2\, y^{bcabdcd} + \beta_{y3\tilde{t}}\, y^{bcbadcd} + \beta_{y3\tilde{u}}\, y^{bcdadbc} + \beta_{y3\tilde{v}}\, \mathcal{S}_2\, y^{bacdbcd} \\
& + \beta_{y3\tilde{w}}\, \mathcal{S}_2\, y^{bcdacbd} + \beta_{y3\tilde{x}}\, \mathcal{S}_2\, y^{bcadbcd} + \beta_{y3\tilde{y}}\, y^{bcdabcd} + \beta_{y3\tilde{z}}\, \mathcal{S}_2\, y^{bcadcbd}\,,
\end{aligned} \tag{4.4}$$

with [6–8, 53]

$$\begin{aligned}
&\beta_{y3a} = -\tfrac{3}{8}\,, && \beta_{y3b} = \tfrac{1}{2}\,, && \beta_{y3c} = \tfrac{3}{2}\,, && \beta_{y3d} = \tfrac{3}{2}\,, && \beta_{y3e} = \tfrac{1}{2}\,, \\
&\beta_{y3f} = 2\,, && \beta_{y3g} = 3\,, && \beta_{y3h} = 5\,, && \beta_{y3i} = 3\,, && \beta_{y3j} = \tfrac{1}{2}\,, \\
&\beta_{y3k} = -1\,, && \beta_{y3l} = 2\,, && \beta_{y3m} = -\tfrac{1}{2}\,, && \beta_{y3n} = 2\,, && \beta_{y3o} = -1\,, \\
&\beta_{y3p} = -\tfrac{1}{2}\,, && \beta_{y3q} = \tfrac{3}{2}\,, && \beta_{y3r} = -\tfrac{3}{2}\,, && \beta_{y3s} = -\tfrac{1}{2}\,, && \beta_{y3t} = \tfrac{25}{16}\,, \\
&\beta_{y3u} = \tfrac{25}{8}\,, && \beta_{y3v} = \tfrac{5}{4}\,, && \beta_{y3w} = 0\,, && \beta_{y3x} = 2(3\zeta_3 - 2)\,, && \beta_{y3y} = -\tfrac{1}{2}\,, \\
&\beta_{y3z} = -\tfrac{1}{2}\,, && \beta_{y3\tilde{a}} = \tfrac{7}{16}\,, && \beta_{y3\tilde{b}} = -1\,, && \beta_{y3\tilde{c}} = 2\,, && \beta_{y3\tilde{d}} = -1\,, \\
&\beta_{y3\tilde{e}} = -1\,, && \beta_{y3\tilde{f}} = 4\,, && \beta_{y3\tilde{g}} = 6\zeta_3 - 5\,, && \beta_{y3\tilde{h}} = -\tfrac{3}{2}\,, && \beta_{y3\tilde{\imath}} = -\tfrac{3}{2}\,, \\
&\beta_{y3\tilde{\jmath}} = \tfrac{1}{2}\,, && \beta_{y3\tilde{k}} = \tfrac{3}{2}\,, && \beta_{y3\tilde{l}} = 2\,, && \beta_{y3\tilde{m}} = 1\,, && \beta_{y3\tilde{n}} = -2 \\
&\beta_{y3\tilde{o}} = -\tfrac{1}{2}\,, && \beta_{y3\tilde{p}} = -\tfrac{3}{2}\,, && \beta_{y3\tilde{q}} = -3\,, && \beta_{y3\tilde{r}} = 3(2\zeta_3 - 1)\,, && \beta_{y3\tilde{s}} = 1\,, \\
&\beta_{y3\tilde{t}} = -2\,, && \beta_{y3\tilde{u}} = -3\,, && \beta_{y3\tilde{v}} = 3(2\zeta_3 - 1)\,, && \beta_{y3\tilde{w}} = 2(3\zeta_3 - 1)\,, && \beta_{y3\tilde{x}} = 2(3\zeta_3 - 1)\,, \\
&\beta_{y3\tilde{y}} = 2\,, && \beta_{y3\tilde{z}} = -4\,.
\end{aligned} \tag{4.5}$$

Of the above two and three loop diagrams $2a$, $2f$, $3f$, $3l$, $3\tilde{w}$, $3\tilde{x}$, $3\tilde{y}$, $3\tilde{z}$ do not have subdivergences and are primitive.

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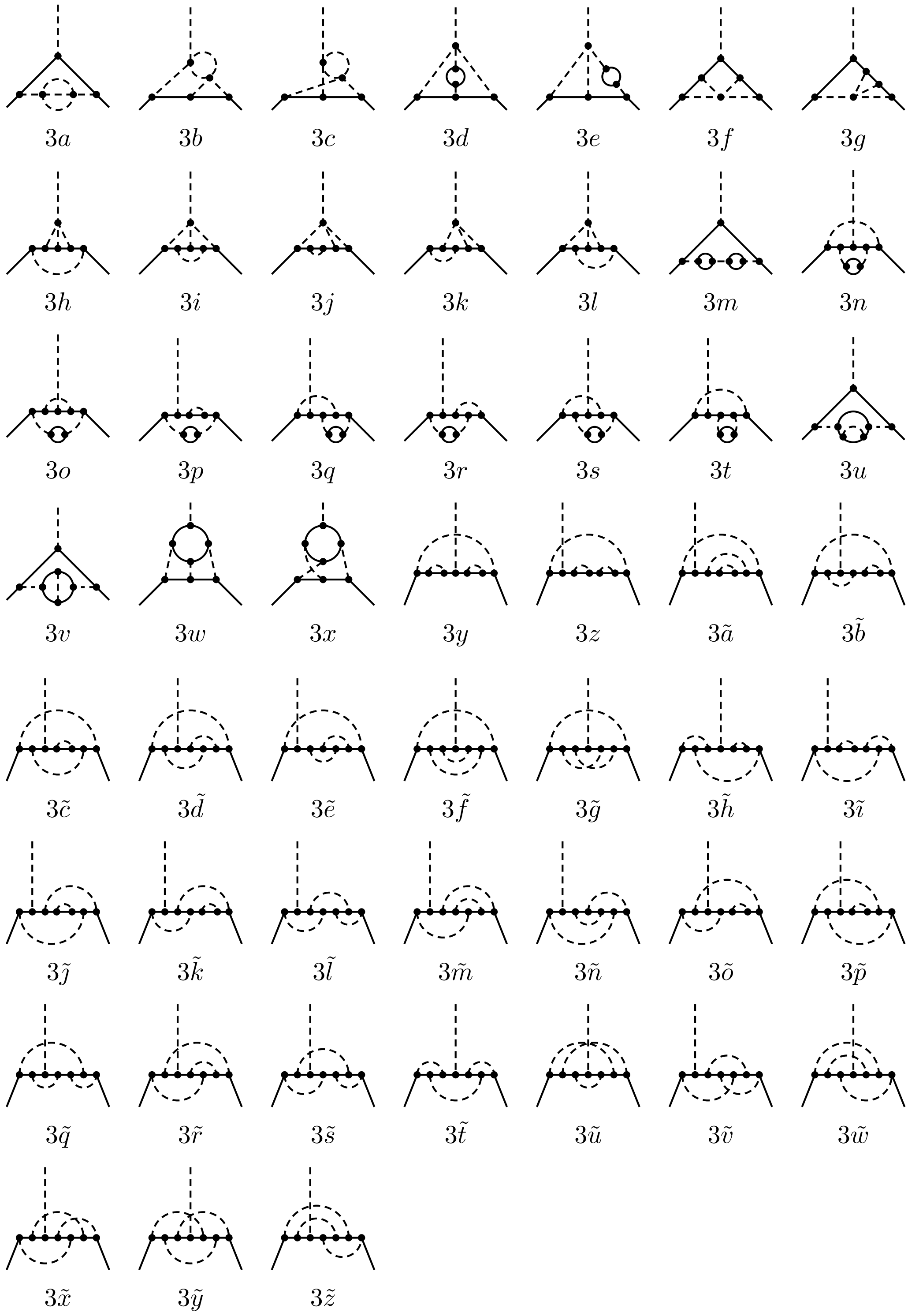

Figure 6. Three loop Yukawa vertex diagrams.

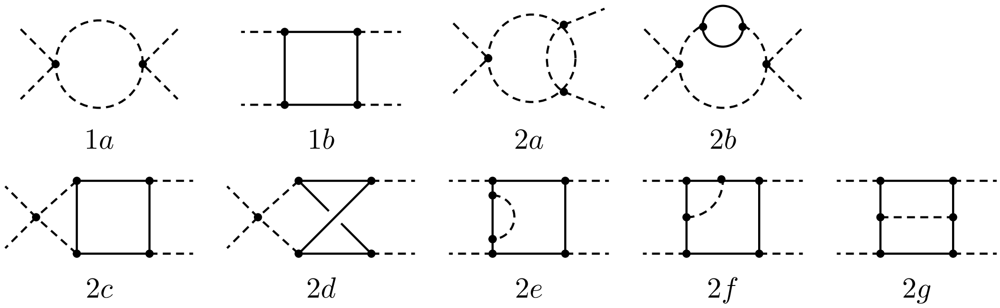

Figure 7. One and two loop diagrams relevant for the scalar quartic β-function.

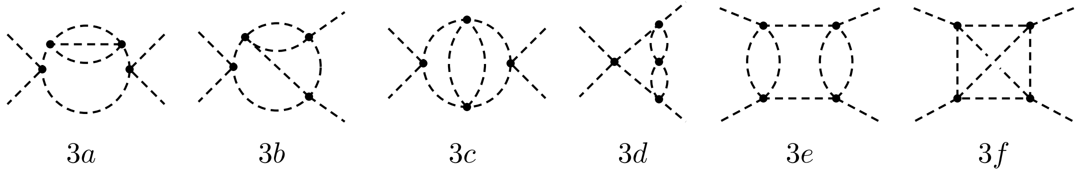

Figure 8. Three loop diagrams involving the quartic scalar coupling contributions to the scalar β-function.

5 Scalar quartic couplings

The scalar quartic coupling is a symmetric 4 index tensor λ^{abcd} and the β-function has a similar decomposition as for the Yukawa coupling in (4.1)

$$\beta_\lambda^{abcd} = \tilde{\beta}_\lambda^{abcd} + \gamma_\phi^{ae}\lambda^{ebcd} + \gamma_\phi^{be}\lambda^{aecd} + \gamma_\phi^{ce}\lambda^{abed} + \gamma_\phi^{de}\lambda^{abce} = \tilde{\beta}_\lambda^{abcd} + \mathcal{S}_4\,\gamma_\phi^{ae}\lambda^{ebcd} \tag{5.1}$$

with $\tilde{\beta}_\lambda^{abcd}$ given in terms of 1PI diagrams and $\mathcal{S}_4$ here denoting the sum over the four terms, each term with unit weight, necessary to obtain a fully symmetric result.

At one and two loops the relevant diagrams (figure 7) are so that

$$\begin{aligned}
\tilde{\beta}_\lambda^{(1)abcd} &= \beta_{\lambda 1a}\,\mathcal{S}_3\,\lambda^{abef}\lambda^{cdef} + \beta_{\lambda 1b}\,\mathcal{S}_3\,\mathrm{tr}(y^{abcd})\,, \\
\tilde{\beta}_\lambda^{(2)abcd} &= \beta_{\lambda 2a}\,\mathcal{S}_6\,\lambda^{abef}\lambda^{cfgh}\lambda^{degh} + \beta_{\lambda 2b}\,\mathcal{S}_3\,\lambda^{abef}\lambda^{cdeg}\,\mathrm{tr}(y^{fg}) \\
&\quad + \beta_{\lambda 2c}\,\mathcal{S}_6\,\lambda^{abef}\mathrm{tr}(y^{efcd}) + \beta_{\lambda 2d}\,\mathcal{S}_6\,\lambda^{abef}\mathrm{tr}(y^{ecfd}) \\
&\quad + \beta_{\lambda 2e}\,\mathcal{S}_{12}\,\mathrm{tr}(y^{aeebcd}) + \beta_{\lambda 2f}\,\mathcal{S}_{12}\,\mathrm{tr}(y^{aebcde}) + \beta_{\lambda 2g}\,\mathcal{S}_6\,\mathrm{tr}(y^{eabecd})\,,
\end{aligned} \tag{5.2}$$

with $\mathcal{S}_n$ denoting the sum over n terms necessary to achieve symmetrisation over all permutations of a, b, c, d. Historic results, with our conventions, give

$$\begin{array}{lllll}
\beta_{\lambda 1a} = 1\,, & \beta_{\lambda 1b} = -4\,, & & \beta_{\lambda 2a} = -1\,, & \beta_{\lambda 2b} = -1\,, \\
\beta_{\lambda 2c} = 0\,, & \beta_{\lambda 2d} = 2\,, & \beta_{\lambda 2e} = 2\,, & \beta_{\lambda 2f} = 4\,, & \beta_{\lambda 2g} = 4\,.
\end{array} \tag{5.3}$$

At three loops there are $\mathrm{O}(\lambda^4, \lambda^3 y^2, \lambda^2, y^4, \lambda y^6, y^8)$ contributions to the scalar quartic β-function. For our discussion it is convenient to isolate related sets of diagrams out of a total of 62. The $\mathrm{O}(\lambda^4)$ purely scalar contribution corresponds to the diagrams shown

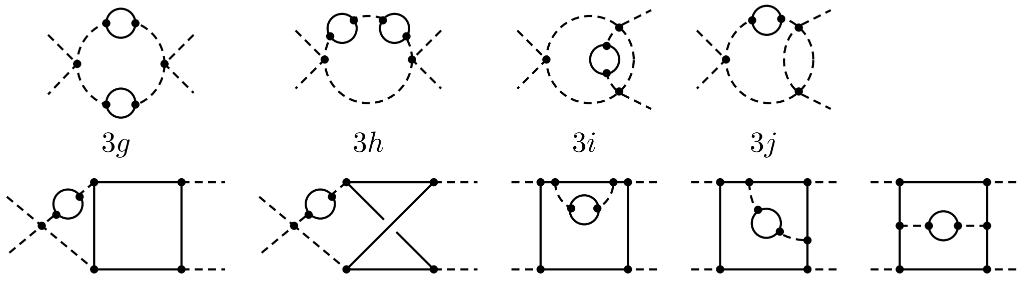

Figure 9. Three loop diagrams involving fermion bubble contributions to the scalar β-function.

in figure 8, so that

$$\begin{aligned}\tilde\beta^{(3)abcd}_{\lambda A} = {} & \beta_{\lambda 3a}\,\mathcal{S}_3\,\lambda^{abef}\lambda^{ehij}\lambda^{ghij}\lambda^{cdfg} + \beta_{\lambda 3b}\,\mathcal{S}_{12}\,\lambda^{abef}\lambda^{cegh}\lambda^{fgij}\lambda^{dhij} \\ & + \beta_{\lambda 3c}\,\mathcal{S}_3\,\lambda^{abef}\lambda^{egij}\lambda^{fhij}\lambda^{cdgh} + \beta_{\lambda 3d}\,\mathcal{S}_6\,\lambda^{abef}\lambda^{cegh}\lambda^{ghij}\lambda^{dfij} \\ & + \beta_{\lambda 3e}\,\mathcal{S}_6\,\lambda^{afge}\lambda^{bfgh}\lambda^{ceij}\lambda^{dhij} + \beta_{\lambda 3f}\,\lambda^{aefg}\lambda^{behi}\lambda^{cfhj}\lambda^{dgij}\,. \end{aligned} \tag{5.4}$$

For the purely scalar case the general three loop coefficients have long been known:

$$\beta_{\lambda 3a} = -\tfrac{3}{8}\,, \qquad \beta_{\lambda 3b} = 2\,, \qquad \beta_{\lambda 3c} = \tfrac{1}{2}\,, \qquad \beta_{\lambda 3d} = -\tfrac{1}{2}\,, \qquad \beta_{\lambda 3e} = -\tfrac{1}{2}\,, \qquad \beta_{\lambda 3f} = 12\zeta_3\,. \tag{5.5}$$

Diagrams involving two or one insertions of fermion bubbles into internal scalar propagator lines in one or two loop diagrams are just depicted in figure 9. The corresponding contributions are then

$$\begin{aligned}\tilde\beta^{abcd}_{\lambda B} = {} & \beta_{\lambda 3g}\,\mathcal{S}_3\,\lambda^{abef}\lambda^{cdgh}\,\mathrm{tr}(y^{eg})\,\mathrm{tr}(y^{fh}) + \beta_{\lambda 3h}\,\mathcal{S}_3\,\lambda^{abef}\lambda^{cdeg}\,\mathrm{tr}(y^{fh})\,\mathrm{tr}(y^{hg}) \\ & + \beta_{\lambda 3i}\,\mathcal{S}_6\,\lambda^{abef}\lambda^{cegh}\lambda^{dfgi}\,\mathrm{tr}(y^{hi}) + \beta_{\lambda 3j}\,\mathcal{S}_{12}\,\lambda^{abef}\lambda^{cegh}\lambda^{dghi}\,\mathrm{tr}(y^{fi}) \\ & + \beta_{\lambda 3k}\,\mathcal{S}_{12}\,\lambda^{abef}\mathrm{tr}(y^{fg})\,\mathrm{tr}(y^{gecd}) + \beta_{\lambda 3l}\,\mathcal{S}_6\,\lambda^{abef}\mathrm{tr}(y^{fg})\,\mathrm{tr}(y^{gced}) \\ & + \mathcal{S}_{12}\big(\beta_{\lambda 3m}\,\mathrm{tr}(y^{abcdef}) + \beta_{\lambda 3n}\,\mathrm{tr}(y^{abcedf})\big)\mathrm{tr}(y^{ef}) \\ & + \beta_{\lambda 3o}\,\mathcal{S}_6\,\mathrm{tr}(y^{abecdf})\,\mathrm{tr}(y^{ef})\,, \end{aligned} \tag{5.6}$$

with

$$\begin{aligned}& \beta_{\lambda 3g} = -\tfrac{1}{4}\,, \qquad \beta_{\lambda 3h} = -\tfrac{1}{2}\,, \qquad \beta_{\lambda 3i} = 2\,, \qquad \beta_{\lambda 3j} = -\tfrac{1}{2}\,, \qquad \beta_{\lambda 3k} = 3\,, \quad \beta_{\lambda 3l} = 2\,, \\ & \beta_{\lambda 3m} = -\tfrac{25}{8}\,, \qquad \beta_{\lambda 3n} = -4\,, \qquad \beta_{\lambda 3o} = -3\,. \end{aligned} \tag{5.7}$$

There are further $\mathrm{O}(\lambda^2)$ diagrams which are depicted in figure 10, which give

$$\begin{aligned}\tilde\beta^{abcd}_{\lambda C} = {} & \mathcal{S}_3\,\lambda^{abef}\lambda^{cdeg}\big(\beta_{\lambda 3p}\,\mathrm{tr}(y^{fghh}) + \beta_{\lambda 3q}\,\mathrm{tr}(y^{fhgh})\big) \\ & + \mathcal{S}_{12}\,\lambda^{abef}\lambda^{cegh}\big(\beta_{\lambda 3r}\,\mathrm{tr}(y^{dfgh}) + \beta_{\lambda 3s}\,\mathrm{tr}(y^{dgfh})\big) \\ & + \mathcal{S}_3\,\lambda^{abef}\lambda^{cdgh}\big(\beta_{\lambda 3t}\,\mathrm{tr}(y^{efgh}) + \beta_{\lambda 3u}\,\mathrm{tr}(y^{egfh})\big) \\ & + \mathcal{S}_{12}\,\lambda^{aefg}\lambda^{befh}\big(\beta_{\lambda 3v}\,\mathrm{tr}(y^{cdgh}) + \beta_{\lambda 3w}\,\mathrm{tr}(y^{cgdh})\big)\,. \end{aligned} \tag{5.8}$$

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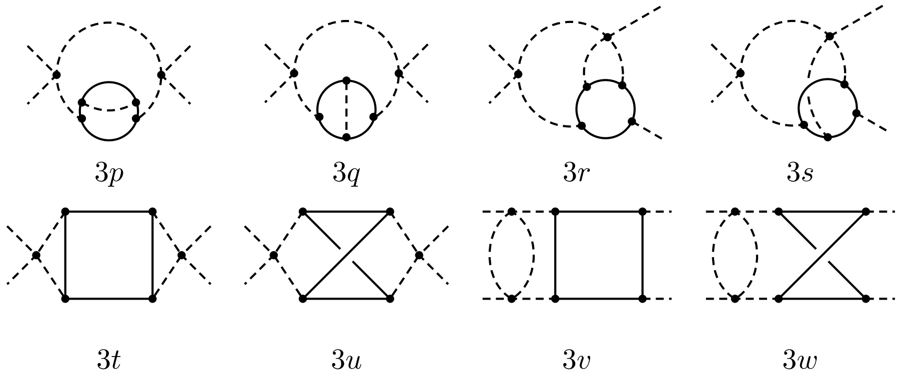


Figure 10. Three-loop $O(\lambda^2)$ diagrams contributing to the scalar β-function.

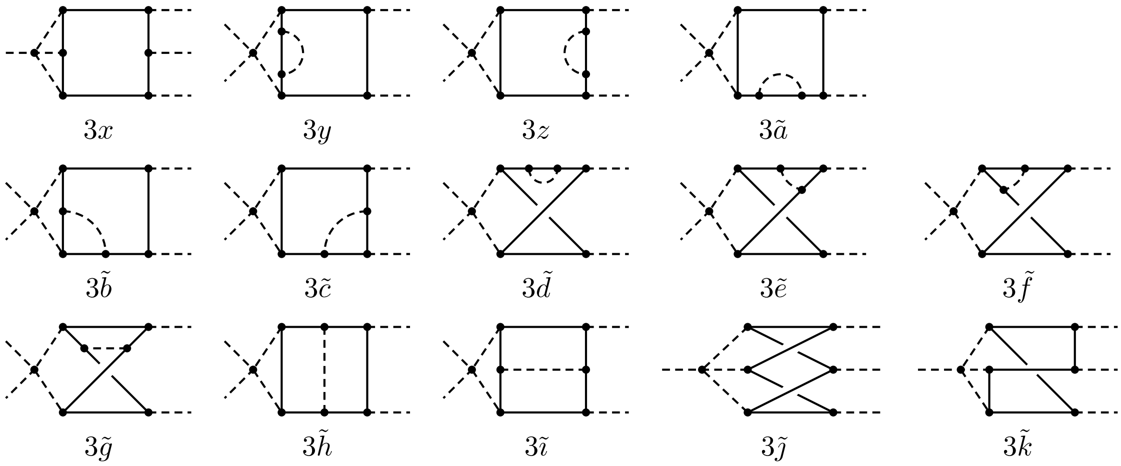


Figure 11. Three-loop diagrams containing one scalar vertex and one fermion loop.

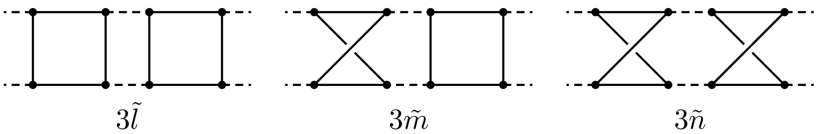


Figure 12. Double fermion loop diagrams without scalar vertex.

Diagrams with a single scalar vertex are shown in figure 11. These correspond to

$$\begin{aligned}\tilde{\beta}^{abcd}_{\lambda D} = \;& \beta_{\lambda 3x}\, \mathcal{S}_{12}\, \lambda^{aefg}\, \mathrm{tr}(y^{bcdefg}) + \mathcal{S}_6\, \lambda^{abef}\big(\beta_{\lambda 3y}\, \mathrm{tr}(y^{cdeggf}) + \beta_{\lambda 3z}\, \mathrm{tr}(y^{cggdef})\big) \\ & + \mathcal{S}_{12}\, \lambda^{abef}\big(\beta_{\lambda 3\tilde{a}}\, \mathrm{tr}(y^{cdggef}) + \beta_{\lambda 3\tilde{b}}\, \mathrm{tr}(y^{cdgegf}) + \beta_{\lambda 3\tilde{c}}\, \mathrm{tr}(y^{efgcgd})\big) \\ & + \mathcal{S}_{12}\, \lambda^{abef}\,\big(\beta_{\lambda 3\tilde{d}}\, \mathrm{tr}(y^{cedggf}) + \beta_{\lambda 3\tilde{e}}\, \mathrm{tr}(y^{cegdgf})\big) + \beta_{\lambda 3\tilde{f}}\, \mathcal{S}_6\, \lambda^{abef}\, \mathrm{tr}(y^{cgegdf}) \\ & + \mathcal{S}_6\, \lambda^{abef}\big(\beta_{\lambda 3\tilde{g}}\, \mathrm{tr}(y^{cegdfg}) + \beta_{\lambda 3\tilde{h}}\, \mathrm{tr}(y^{cdgefg}) + \beta_{\lambda 3\tilde{\imath}}\, \mathrm{tr}(y^{egfcgd})\big) \\ & + \beta_{\lambda 3\tilde{\jmath}}\, \mathcal{S}_4\, \lambda^{aefg}\, \mathrm{tr}(y^{becfdg}) + \beta_{\lambda 3\tilde{k}}\, \mathcal{S}_{24}\, \lambda^{aefg}\, \mathrm{tr}(y^{bcedfg})\,. \end{aligned} \tag{5.9}$$

The remaining diagrams have no quartic scalar vertex. Those which involve two fermion loops are just depicted in figure 12:

$$\tilde{\beta}^{abcd}_{\lambda E} = \mathcal{S}_6\big(\beta_{\lambda 3\tilde{l}}\, \mathrm{tr}(y^{abef}) + \beta_{\lambda 3\tilde{m}}\, \mathrm{tr}(y^{aebf})\big)\, \mathrm{tr}(y^{cdef}) + \beta_{\lambda 3\tilde{n}}\, \mathcal{S}_3\, \mathrm{tr}(y^{aebf})\, \mathrm{tr}(y^{cedf})\,. \tag{5.10}$$

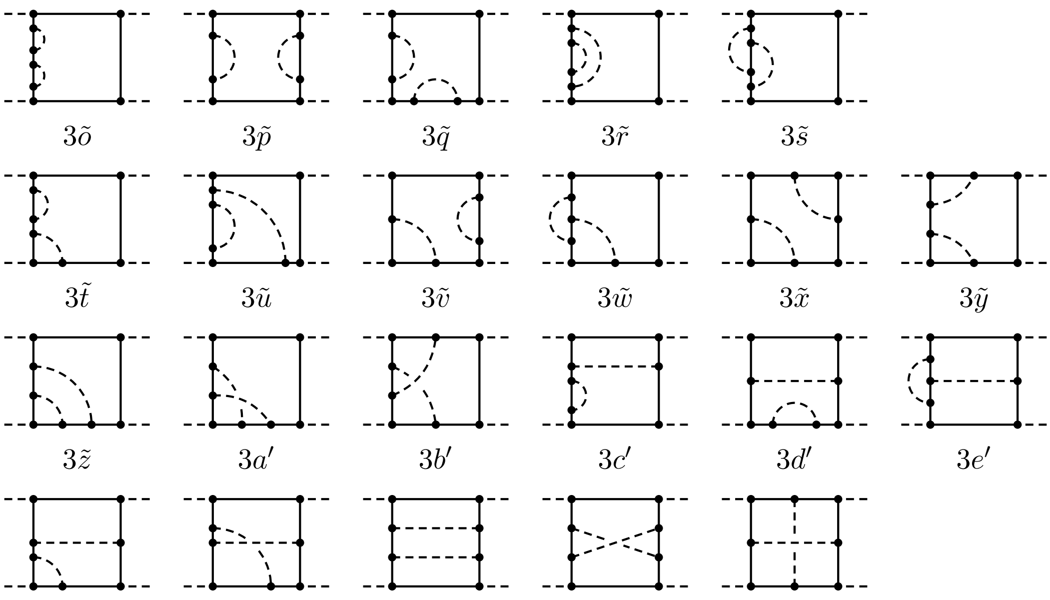


Figure 13. Three-loop diagrams containing only Yukawa couplings with a single fermion loop.

The final set of diagrams, shown in figure 13, is then

$$\begin{aligned}\tilde{\beta}_{\lambda F}^{abcd} = \; & \beta_{\lambda 3\tilde{o}}\, \mathcal{S}_{12}\, \mathrm{tr}(y^{abcdeeff}) + \beta_{\lambda 3\tilde{p}}\, \mathcal{S}_{6}\, \mathrm{tr}(y^{abeecdff}) + \beta_{\lambda 3\tilde{q}}\, \mathcal{S}_{12}\, \mathrm{tr}(y^{aeebffcd}) \\ & + \mathcal{S}_{12}\left(\beta_{\lambda 3\tilde{r}}\; \mathrm{tr}(y^{abcdeffe}) + \beta_{\lambda 3\tilde{s}}\; \mathrm{tr}(y^{abcdefef})\right) \\ & + \mathcal{S}_{24}\left(\beta_{\lambda 3\tilde{t}}\; \mathrm{tr}(y^{affebecd}) + \beta_{\lambda 3\tilde{u}}\; \mathrm{tr}(y^{aeffbecd}) + \beta_{\lambda 3\tilde{v}}\; \mathrm{tr}(y^{aebecffd})\right) \\ & + \beta_{\lambda 3\tilde{w}}\; \mathcal{S}_{24}\, \mathrm{tr}(y^{aefebfcd}) + \beta_{\lambda 3\tilde{x}}\; \mathcal{S}_{6}\, \mathrm{tr}(y^{aebecfdf}) + \beta_{\lambda 3\tilde{y}}\; \mathcal{S}_{12}\, \mathrm{tr}(y^{abecefdf}) \\ & + \mathcal{S}_{12}\left(\beta_{\lambda 3\tilde{z}}\; \mathrm{tr}(y^{aefbfecd}) + \beta_{\lambda 3a'}\; \mathrm{tr}(y^{aefbefcd}) + \beta_{\lambda 3b'}\; \mathrm{tr}(y^{abecfedf})\right) \\ & + \beta_{\lambda 3c'}\; \mathcal{S}_{24}\, \mathrm{tr}(y^{affebced}) + \mathcal{S}_{12}\left(\beta_{\lambda 3d'}\; tr(y^{abecffde}) + \beta_{\lambda 3e'}\; \mathrm{tr}(y^{aefebcfd})\right) \\ & + \mathcal{S}_{24}\left(\beta_{\lambda 3f'}\; \mathrm{tr}(y^{aefbfced}) + \beta_{\lambda 3g'}\; \mathrm{tr}(y^{aefbecfd})\right) \\ & + \mathcal{S}_{6}\left(\beta_{\lambda 3h'}\; \mathrm{tr}(y^{aefbcfed}) + \beta_{\lambda 3i'}\; \mathrm{tr}(y^{aefbcefd})\right) + \beta_{\lambda 3j'}\; \mathcal{S}_{3}\, \mathrm{tr}(y^{aebfcedf})\,. \qquad (5.11)\end{aligned}$$

Of the quartic scalar diagrams $2g, 3f, 3\tilde{h}, 3\tilde{\imath}, 3\tilde{\jmath}, 3\tilde{k}, 3b', 3g', 3h', 3i', 3j'$ are primitive. The diagrams $3f, 3l, 3s, 3u, 3w, 3\tilde{d}, 3\tilde{e}, 3\tilde{f}, 3\tilde{g}, 3\tilde{\jmath}, 3\tilde{k}, 3a', 3b', 3g', 3i', 3j'$ are non planar.

As explained subsequently the 62 individual coefficients can be determined so that

$$\begin{aligned}
&\beta_{\lambda 3a} = -\tfrac{3}{8}\,, && \beta_{\lambda 3b} = 2\,, && \beta_{\lambda 3c} = \tfrac{1}{2}\,, && \beta_{\lambda 3d} = -\tfrac{1}{2}\,, \\
&\beta_{\lambda 3e} = -\tfrac{1}{2}\,, && \beta_{\lambda 3f} = 12\zeta_3\,, && \beta_{\lambda 3g} = -\tfrac{1}{4}\,, && \beta_{\lambda 3h} = -\tfrac{1}{2}\,, \\
&\beta_{\lambda 3i} = 2\,, && \beta_{\lambda 3j} = -\tfrac{1}{2}\,, && \beta_{\lambda 3k} = 3\,, && \beta_{\lambda 3l} = 2\,, \\
&\beta_{\lambda 3m} = -\tfrac{25}{8}\,, && \beta_{\lambda 3n} = -4\,, && \beta_{\lambda 3o} = -3\,, && \beta_{\lambda 3p} = \tfrac{25}{8}\,, \\
&\beta_{\lambda 3q} = \tfrac{5}{4}\,, && \beta_{\lambda 3r} = 2\,, && \beta_{\lambda 3s} = 2(3\zeta_3 - 2)\,, && \beta_{\lambda 3t} = 2\,, \\
&\beta_{\lambda 3u} = 3\zeta_3 - 1\,, && \beta_{\lambda 3v} = 3\,, && \beta_{\lambda 3w} = 1\,, && \beta_{\lambda 3x} = -10\,, \\
&\beta_{\lambda 3y} = -1\,, && \beta_{\lambda 3z} = -3\,, && \beta_{\lambda 3\tilde{a}} = -3\,, && \beta_{\lambda 3\tilde{b}} = -2\,, \\
&\beta_{\lambda 3\tilde{c}} = -6\,, && \beta_{\lambda 3\tilde{d}} = -4\,, && \beta_{\lambda 3\tilde{e}} = -4\,, && \beta_{\lambda 3\tilde{f}} = 4(3\zeta_3 - 2)\,, \\
&\beta_{\lambda 3\tilde{g}} = 2(6\zeta_3 - 5)\,, && \beta_{\lambda 3\tilde{h}} = -2\,, && \beta_{\lambda 3\tilde{\imath}} = -10\,, && \beta_{\lambda 3\tilde{\jmath}} = -24\zeta_3\,,
\end{aligned}$$

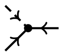
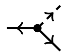
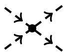
$$
\begin{array}{llll}
\beta_{\lambda 3\tilde{k}} = -12\zeta_3\,, & \beta_{\lambda 3\tilde{l}} = -8\,, & \beta_{\lambda 3\tilde{m}} = -12\,, & \beta_{\lambda 3\tilde{n}} = -4 \\
\beta_{\lambda 3\tilde{o}} = 1\,, & \beta_{\lambda 3\tilde{p}} = 1\,, & \beta_{\lambda 3\tilde{q}} = 1\,, & \beta_{\lambda 3\tilde{r}} = -\tfrac{7}{8}\,, \\
\beta_{\lambda 3\tilde{s}} = 2\,, & \beta_{\lambda 3\tilde{t}} = 2\,, & \beta_{\lambda 3\tilde{u}} = -4\,, & \beta_{\lambda 3\tilde{v}} = 2\,, \\
\beta_{\lambda 3\tilde{w}} = 2\,, & \beta_{\lambda 3\tilde{x}} = 4\,, & \beta_{\lambda 3\tilde{y}} = 4\,, & \beta_{y 3\tilde{z}} = -8\,, \\
\beta_{\lambda 3a'} = -2(6\zeta_3 - 5)\,, & \beta_{\lambda 3b'} = 0\,, & \beta_{\lambda 3c'} = -1\,, & \beta_{\lambda 3d'} = -1\,, \\
\beta_{\lambda 3e'} = -6\,, & \beta_{\lambda 3f'} = -2\,, & \beta_{\lambda 3g'} = -12\zeta_3\,, & \beta_{\lambda 3h'} = -4\,, \\
\beta_{\lambda 3i'} = -12\zeta_3\,, & \beta_{\lambda 3j'} = -24\zeta_3\,. & &
\end{array}
\tag{5.12}
$$

This completes the expressions given in [6], which have been obtained using $\mathcal{N} = 1$ SUSY relations as well as explicit literature results for the SM [10–14] and Gross-Neveu type models [15, 16]. In this paper $\mathcal{N} = \frac{1}{2}$ SUSY conditions are also considered, which are not sufficient to obtain (5.12), but overcomplete the conditions in [6] without inconsistencies. Hence, literature results [15, 16] are cross-checked by the SUSY relations and explicit SM computations.

6 Reduction to U(1) symmetry

For complex fields with a U(1) symmetry the number of diagrams is significantly reduced.[2] This restriction is achieved by taking $\phi^a = (\varphi_i, \bar{\varphi}^i)$, so that $\phi^a \phi'^a = \varphi_i\, \bar{\varphi}'^i + \bar{\varphi}^i\, \varphi'_i$, and

$$
\tfrac{1}{24}\,\lambda^{abcd}\,\phi^a\phi^b\phi^c\phi^d \to \tfrac{1}{4}\,\lambda_{ij}{}^{kl}\,\bar{\varphi}^i\bar{\varphi}^j\varphi_k\varphi_l\,, \quad \phi^a y^a \to \varphi_i\, y^i\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) + \bar{\varphi}^i\,\bar{y}_i\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right). \tag{6.1}
$$

The scalar and fermion lines on each diagram then have arrows with the basic vertices for the Yukawa couplings y^i, $\bar{y}_i$ represented by  ,  and for the scalar quartic coupling $\lambda_{ij}{}^{kl}$ by  . The triangle graphs present for real couplings are no longer allowed. With this prescription then for traces over the Yukawa couplings

$$
\mathrm{tr}(y^{a_1a_2a_3\ldots a_{2n}}) \to \begin{cases} \frac{1}{2}\big(\mathrm{tr}(\bar{y}_{i_1}y^{i_2}\bar{y}_{i_3}\ldots y^{i_{2n}}) + \mathrm{tr}(y^{i_{2n}}\ldots y^{i_2}\bar{y}_{i_1})\big)\,, \\ \frac{1}{2}\big(\mathrm{tr}(y^{i_1}\bar{y}_{i_2}y^{i_3}\ldots \bar{y}_{i_{2n}}) + \mathrm{tr}(\bar{y}_{i_{2n}}\ldots \bar{y}_{i_2}y^{i_1})\big)\,. \end{cases} \tag{6.2}
$$

In general, for the anomalous dimensions

$$
\gamma_\phi{}^{ab} = (\gamma_{\varphi\, i}{}^j, \gamma_{\varphi\, j}{}^i)\,, \qquad \gamma_\psi = \begin{pmatrix} \gamma_\psi & 0 \\ 0 & \bar{\gamma}_\psi \end{pmatrix}, \tag{6.3}
$$

and for the β-functions from (4.1) and (5.1)

$$
\begin{aligned}
\beta_y{}^i = \tilde{\beta}_y{}^i + \gamma_\psi\, y^i + y^i\,\bar{\gamma}_\psi + y^j\gamma_{\varphi\, j}{}^i\,, \qquad \beta_{\bar{y}i} = \tilde{\beta}_{\bar{y}i} + \bar{\gamma}_\psi\,\bar{y}_i + \bar{y}_i\,\gamma_\psi + \gamma_{\varphi\, i}{}^j\,\bar{y}_j\,, \\
\beta_{\lambda\, ij}{}^{kl} = \tilde{\beta}_{\lambda\, ij}{}^{kl} + \gamma_{\varphi\, i}{}^m\lambda_{mj}{}^{kl} + \gamma_{\varphi\, i}{}^m\lambda_{im}{}^{kl} + \lambda_j{}^{ml}\,\gamma_{\varphi\, m}{}^k + \lambda_j{}^{km}\,\gamma_{\varphi\, m}{}^l\,,
\end{aligned} \tag{6.4}
$$

where $\gamma_\psi \to \bar{\gamma}_\psi$ $\beta_y{}^i \to \beta_{\bar{y}i}$ by taking $y^i \leftrightarrow \bar{y}_i$, $\lambda_{ij}{}^{kl} \to \lambda_{kl}{}^{ij}$ in each contribution. For $\bar{y}_i = (y^i)^\dagger$, $\lambda_{kl}{}^{ij} = (\lambda_{ij}{}^{kl})^*$ then $\gamma_\psi = \gamma_\psi{}^\dagger$ and $\gamma_{\varphi\, j}{}^i = (\gamma_{\varphi\, i}{}^j)^*$. At one loop

$$
\begin{aligned}
\gamma_{\varphi\, i}{}^{j(1)} = \gamma_{\phi 1}\,\mathrm{tr}(\bar{y}_i\, y^j)\,, \qquad \gamma_\psi{}^{(1)} = \gamma_{\psi 1}\, y^i\bar{y}_i\,, \qquad \tilde{\beta}_y{}^{i(1)} = 0\,, \\
\tilde{\beta}_{\lambda\, ij}{}^{kl(1)} = \beta_{\lambda 1a}(\lambda_{ij}{}^{mn}\lambda_{mn}{}^{kl} + 2\,\mathcal{S}_2\,\lambda_{im}{}^{kn}\lambda_{jn}{}^{ml}) + \tfrac{1}{2}\,\beta_{\lambda 1b}\,\mathcal{S}_2\,\mathrm{tr}(\bar{y}_i\, y^k\bar{y}_j\, y^l)\,,
\end{aligned} \tag{6.5}
$$

[2]This example was considered in [20].

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where $\mathcal{S}_2 X_{ij}{}^{kl} = X_{ij}{}^{kl} + X_{ji}{}^{kl} = X_{ij}{}^{kl} + X_{ij}{}^{lk}$. At two loops

$$
\begin{aligned}
\gamma_{\varphi\,i}{}^{j\,(2)} &= 3\,\gamma_{\phi 2a}\,\lambda_{ik}{}^{mn}\lambda_{mn}{}^{kj} + \tfrac{1}{2}\,\gamma_{\phi 2b}\left(\mathrm{tr}(\bar y_i\, y^k \bar y_k\, y^j) + \mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^k)\right),\\
\gamma_\psi{}^{(2)} &= \gamma_{\psi 2a}\, y^i \bar y_j\, \mathrm{tr}(\bar y_i\, y^j) + \gamma_{\psi 2b}\, y^i \bar y_j\, y^j \bar y_i\,,\\
\tilde\beta_y{}^{i\,(2)} &= \beta_{y2a}\,\lambda_{jk}{}^{li}\, y^j \bar y_l\, y^k + \beta_{y2f}\, y^j \bar y_k\, y^i \bar y_j\, y^k\,,\\
\tilde\beta_{\lambda\, ij}{}^{kl\,(2)} &= \beta_{\lambda 2a}\big(2\,\lambda_{ij}{}^{mn}\lambda_{mp}{}^{qk}\lambda_{nq}{}^{pl} + 2\,\lambda_{ip}{}^{mq}\lambda_{jq}{}^{np}\lambda_{mn}{}^{kl} + \mathcal{S}_4\,\lambda_{ip}{}^{qk}\lambda_{jq}{}^{mn}\lambda_{mn}{}^{pl}\\
&\qquad\quad + 2\,\mathcal{S}_4\,\lambda_{im}{}^{pn}\lambda_{jn}{}^{qk}\lambda_{pq}{}^{ml}\big)\\
&\quad + \beta_{\lambda 2b}\big(\lambda_{ij}{}^{mp}\lambda_{mq}{}^{kl} + \mathcal{S}_4\,\lambda_{im}{}^{kp}\lambda_{jq}{}^{lm}\big)\,\mathrm{tr}(\bar y_p\, y^q)\\
&\quad + \tfrac{1}{2}\,\beta_{\lambda 2c}\,\mathcal{S}_4\,\lambda_{im}{}^{kn}\big(\mathrm{tr}(\bar y_j\, y^l \bar y_n\, y^m) + \mathrm{tr}(\bar y_j\, y^m \bar y_n\, y^l)\big)\\
&\quad + \beta_{\lambda 2d}\left(\lambda_{ij}{}^{mn}\,\mathrm{tr}(\bar y_m\, y^k \bar y_n\, y^l) + \lambda_{mn}{}^{kl}\,\mathrm{tr}(\bar y_i\, y^m \bar y_j\, y^n)\right)\\
&\quad + \tfrac{1}{2}\,\beta_{\lambda 2e}\,\mathcal{S}_4\left(\mathrm{tr}(\bar y_m\, y^k \bar y_i\, y^l \bar y_j\, y^m) + \mathrm{tr}(\bar y_m\, y^m \bar y_i\, y^k \bar y_j\, y^l)\right)\\
&\quad + \tfrac{1}{2}\,\beta_{\lambda 2g}\,\mathcal{S}_4\,\mathrm{tr}(\bar y_i\, y^k \bar y_m\, y^l \bar y_j\, y^m)\,,
\end{aligned}
\tag{6.6}
$$

with $\mathcal{S}_4 X_{ij}{}^{kl} = X_{ij}{}^{kl} + X_{ji}{}^{kl} + X_{ij}{}^{lk} + X_{ji}{}^{lk}$.

At three loops the results here reduce to 8 contributions for γ_ϕ

$$
\begin{aligned}
\gamma_{\varphi\,i}{}^{j\,(3)} = {}& \gamma_{\phi 3a}\left(\lambda_{ik}{}^{mn}\lambda_{mn}{}^{pq}\lambda_{pq}{}^{kj} + 4\,\lambda_{ik}{}^{mn}\lambda_{ml}{}^{kp}\lambda_{np}{}^{lj}\right)\\
& + \gamma_{\phi 3b}\left(\lambda_{ik}{}^{mn}\lambda_{mn}{}^{lj} + 2\,\lambda_{im}{}^{ln}\lambda_{kn}{}^{mj}\right)\mathrm{tr}(\bar y_l\, y^k)\\
& + \gamma_{\phi 3c}\left(\lambda_{ik}{}^{lm}\,\mathrm{tr}(\bar y_l\, y^k\, \bar y_m\, y^j) + \mathrm{tr}(\bar y_i\, y^l \bar y_k\, y^m)\,\lambda_{lm}{}^{kj}\right)\\
& + \tfrac{1}{2}\,\gamma_{\phi 3d}\left(\mathrm{tr}(\bar y_i\, y^k \bar y_l\, y^j) + \mathrm{tr}(\bar y_i\, y^j \bar y_l\, y^k)\right)\mathrm{tr}(\bar y_k\, y^l)\\
& + \tfrac{1}{2}\,\gamma_{\phi 3f}\left(\mathrm{tr}(\bar y_i\, y^k \bar y_k\, y^l\, \bar y_l\, y^j) + \mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^k \bar y_l\, y^l)\right)\\
& + \tfrac{1}{2}\,\gamma_{\phi 3g}\left(\mathrm{tr}(\bar y_i\, y^k \bar y_l\, y^l\, \bar y_k\, y^j) + \mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^l \bar y_l\, y^k)\right)\\
& + \gamma_{\phi 3h}\,\mathrm{tr}(\bar y_k\, y^k \bar y_i\, y^l\, \bar y_l\, y^j) + \gamma_{\phi 3m}\,\mathrm{tr}(\bar y_k\, y^l \bar y_i\, y^k \bar y_l\, y^j)\,,
\end{aligned}
\tag{6.7}
$$

and 9 for γ_ψ

$$
\begin{aligned}
\gamma_\psi{}^{(3)} = {}& 3\,\gamma_{\psi 3a}\, y^i \bar y_j\,\lambda_{im}{}^{kl}\lambda_{kl}{}^{mj} + \gamma_{\psi 3b}\, y^i \bar y_k\, y^j \bar y_l\,\lambda_{ij}{}^{kl}\\
& + \gamma_{\psi 3c}\, y^i \bar y_j\,\mathrm{tr}(y^j \bar y_k)\,\mathrm{tr}(y^k \bar y_i) + \left(\gamma_{\psi 3d}\, y^i \bar y_k\, y^k \bar y_j + \gamma_{\psi 3e}\, y^k \bar y_j\, y^i \bar y_k\right)\mathrm{tr}(y^j \bar y_i)\\
& + \tfrac{1}{2}\,\gamma_{\psi 3g}\, y^i \bar y_j\left(\mathrm{tr}(y^j \bar y_k\, y^k \bar y_i) + \mathrm{tr}(y^k \bar y_k\, y^j \bar y_i)\right)\\
& + \gamma_{\psi 3i}\, y^i \bar y_j\, y^j \bar y_k\, y^k \bar y_i + \gamma_{\psi 3j}\, y^i \bar y_j\, y^k \bar y_k\, y^j \bar y_i + \gamma_{\psi 3p}\, y^i \bar y_j\, y^k \bar y_i\, y^j \bar y_k\,,
\end{aligned}
\tag{6.8}
$$

and 12 for $\tilde\beta_y$

$$
\begin{aligned}
\tilde\beta_y{}^{i\,(3)} = {}& 2\,\beta_{y3b}\, y^j \bar y_k\, y^l\big(\lambda_{jm}{}^{ni}\lambda_{nl}{}^{km} + \lambda_{jm}{}^{kn}\lambda_{nl}{}^{mi}\big) + \beta_{y3c}\, y^j \bar y_k\, y^l\,\lambda_{jl}{}^{mn}\lambda_{mn}{}^{ki}\\
& + \beta_{y3d}\, y^j \bar y_m\, y^k \lambda_{jk}{}^{li}\,\mathrm{tr}(y^m \bar y_l) + \beta_{y3e}\left(y^k \bar y_l\, y^m + y^m \bar y_l\, y^k\right)\lambda_{jk}{}^{li}\,\mathrm{tr}(y^j \bar y_m)\\
& + \beta_{y3f}\, y^k \bar y_m\, y^i \bar y_n\, y^l\,\lambda_{kl}{}^{mn} + \beta_{y3j}\left(y^k \bar y_m\, y^m \bar y_j\, y^l + y^k \bar y_j\, y^m \bar y_m y^l\right)\lambda_{kl}{}^{ji}\\
& + \beta_{y3l}\left(y^m \bar y_j\, y^k \bar y_m\, y^l + y^k \bar y_m\, y^l \bar y_j\, y^m\right)\lambda_{kl}{}^{ji}\\
& + \beta_{y3s}\left(y^j \bar y_l\, y^i \bar y_k\, y^l + y^l \bar y_k\, y^i \bar y_l\, y^j\right)\mathrm{tr}(y^k \bar y_j) + \tfrac{1}{2}\,\beta_{y3w}\left(y^j \bar y_k\, y^l + y^l \bar y_k\, y^j\right)\mathrm{tr}(\bar y_j\, y^k \bar y_l\, y^i)\\
& + \beta_{y3\tilde o}\left(y^k \bar y_j\, y^j \bar y_l\, y^i \bar y_k\, y^l + y^k \bar y_l\, y^i \bar y_k\, y^j \bar y_j\, y^l\right)\\
& + \beta_{y3\tilde p}\left(y^k \bar y_l\, y^j \bar y_j\, y^i \bar y_k\, y^l + y^k \bar y_l\, y^i \bar y_j\, y^j \bar y_k\, y^l\right)\\
& + \beta_{y3\tilde z}\left(y^j \bar y_k\, y^l \bar y_j\, y^i \bar y_l\, y^k + y^k \bar y_l\, y^i \bar y_j\, y^l \bar y_k\, y^j\right).
\end{aligned}
\tag{6.9}
$$

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For the scalar quartic β-function at three loops the 1PI contributions are restricted to 43 diagrams as $3n$, $3q$, $3\tilde{b}$, $3\tilde{c}$, $3\tilde{e}$, $3\tilde{f}$, $3\tilde{s}$, $3\tilde{t}$, $3\tilde{u}$, $3\tilde{v}$, $3\tilde{w}$, $3\tilde{x}$, $3\tilde{y}$, $3\tilde{z}$, $3e'$, $3f'$, $3g'$, $3i'$, $3j'$ are no longer present. There remain 7 primitive 3 loop diagrams.

There is one possible antisymmetric term at three loops

$$\upsilon_{\varphi\, i}{}^{j(3)} = \upsilon_{\phi 3c}\, \big(\lambda_{ik}{}^{lm}\, \mathrm{tr}(\bar{y}_l\, y^k \bar{y}_m\, y^j) - \mathrm{tr}(\bar{y}_i\, y^l \bar{y}_k\, y^m)\, \lambda_{lm}{}^{kj}\big)\,. \tag{6.10}$$

7 Supersymmetry relations

Supersymmetry of course relates bosons and fermions. Imposing symmetry on the scalar fermion theory leads to linear relations between the anomalous dimension and β-function coefficients which we describe below.

7.1 $\mathcal{N} = 1$ supersymmetry

The Wess Zumino theory for scalars and fermions is a special case which can be obtained by restricting the couplings of the theory with U(1) symmetry so that

$$y^i \to Y^{ijk} = Y^{(ijk)}\,, \qquad \bar{y}_i \to \bar{Y}_{ijk} = \bar{Y}_{(ijk)}\,, \qquad \lambda_{ij}{}^{kl} \to \bar{Y}_{ijm} Y^{mkl}\,. \tag{7.1}$$

The usual non renormalisation theorems require

$$\gamma_\psi{}^i{}_j = \gamma_{\varphi j}{}^i = \gamma^i{}_j\,, \qquad \tilde{\beta}_Y{}^{ijk} = 0\,, \qquad \tilde{\beta}_{\lambda\, ij}{}^{kl} = 2\, \bar{Y}_{ijm}\, \gamma^m{}_n Y^{nkl}\,. \tag{7.2}$$

At one loop this just imposes

$$\gamma_1^S = \gamma_{\psi 1} = \gamma_{\phi 1}\,, \qquad 4\,\beta_{\lambda 1a} + \beta_{\lambda 1b} = 0\,, \qquad \beta_{\lambda 1a} = 2\,\gamma_1^S\,. \tag{7.3}$$

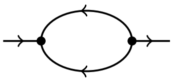
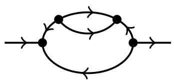
At two loops the necessary conditions are

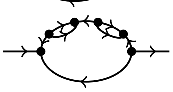
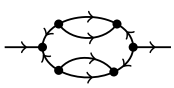
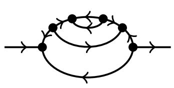
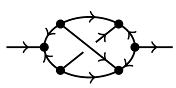
$$\begin{aligned}
&\gamma_2^S = \gamma_{\psi 2a} + \gamma_{\psi 2b} = 3\,\gamma_{\phi 2a} + \gamma_{\phi 2b}\,, \qquad \beta_{y2a} + \beta_{y2f} = 0\,,\\
&2\,\beta_{\lambda 2a} + \beta_{\lambda 2d} = 0\,, \quad 4\,\beta_{\lambda 2a} + 2\,\beta_{\lambda 2c} + \beta_{\lambda 2g} = 0\,, \quad \beta_{\lambda 2a} + \beta_{\lambda 2b} + \beta_{\lambda 2e} = 0,\\
&\beta_{\lambda 2b} = 2\,\gamma_2^S\,.
\end{aligned} \tag{7.4}$$

At three loops the conditions on the anomalous dimensions and Yukawa couplings are then

$$\begin{aligned}
&\gamma_{3A}^S = \gamma_{\psi 3c} + \gamma_{\psi 3i} = \gamma_{\phi 3a} + \gamma_{\phi 3f}\,, \qquad \gamma_{3B}^S = \gamma_{\psi 3d} = \gamma_{\phi 3b} + \gamma_{\phi 3h}\,,\\
&\gamma_{3C}^S = 3\,\gamma_{\psi 3a} + \gamma_{\psi 3e} + \gamma_{\psi 3g} + \gamma_{\psi 3j} = 2\,\gamma_{\phi 3b} + \gamma_{\phi 3d} + \gamma_{\phi 3g}\,,\\
&\gamma_{3D}^S = \gamma_{\psi 3b} + \gamma_{\psi 3p} = 4\,\gamma_{\phi 3a} + 2\,\gamma_{\phi 3c} + \gamma_{\phi 3m}\,,\\
&4\,\beta_{y3b} + \beta_{y3f} + 2\,\beta_{y3l} + \beta_{y3w} + 2\,\beta_{y3\tilde{z}} = 0\,,\\
&\beta_{y3e} + \beta_{y3j} + \beta_{y3s} + \beta_{y3\tilde{o}} = 0\,, \qquad \beta_{y3c} + \beta_{y3d} + 2\,\beta_{y3\tilde{p}} = 0\,,
\end{aligned} \tag{7.5}$$

so that

$\gamma_\phi^{S(1)} = \gamma_1^S$  , $\qquad \gamma_\phi^{S(2)} = \gamma_2^S$  ,

$\gamma_\phi^{S(3)} = \gamma_{3A}^S$  $+\ \gamma_{3B}^S$  $+\ \gamma_{3C}^S$  $+\ \gamma_{3D}^S$  , (7.6) with

$$\gamma_1^S = \tfrac{1}{2}\,, \qquad \gamma_2^S = -\tfrac{1}{2}\,, \qquad \gamma_{3A}^S = -\tfrac{1}{4}\,, \qquad \gamma_{3B}^S = -\tfrac{1}{8}\,, \qquad \gamma_{3C}^S = 1\,, \qquad \gamma_{3D}^S = \tfrac{3}{2}\zeta_3\,. \tag{7.7}$$

The contribution (6.10) to υ vanishes on reduction to supersymmetry as in (7.1).

The constraints arising from (7.2) at three loops leads to 4 conditions relating $\tilde{\beta}_\lambda$ to the anomalous dimensions

$$\beta_{\lambda 3h} = 2\,\gamma_{3A}^S\,, \qquad \beta_{\lambda 3g} = 2\,\gamma_{3B}^S\,, \qquad 3\,\beta_{\lambda 3a} + \beta_{\lambda 3p} = 2\,\gamma_{3C}^S\,, \qquad 2\,\beta_{\lambda 3c} + \beta_{\lambda 3u} = 2\,\gamma_{3D}^S\,, \tag{7.8}$$

and 14 linear homogeneous relations for $\tilde{\beta}_\lambda$

$$\begin{aligned}
&3\,\beta_{\lambda 3a} + \beta_{\lambda 3i} + \beta_{\lambda 3m} + \beta_{\lambda 3p} + \beta_{\lambda 3\tilde{r}} = 0\,,\\
&4\,\beta_{\lambda 3b} + 4\,\beta_{\lambda 3d} + 2\,\beta_{\lambda 3r} + \beta_{\lambda 3\tilde{\imath}} = 0\,, \qquad \beta_{\lambda 3b} + 2\,\beta_{\lambda 3j} + \beta_{\lambda 3k} + \beta_{\lambda 3\tilde{a}} + \beta_{\lambda 3c'} = 0\,,\\
&4\,\beta_{\lambda 3c} + 4\,\beta_{\lambda 3e} + 2\,\beta_{\lambda 3t} + 4\,\beta_{\lambda 3v} + 2\,\beta_{\lambda 3\tilde{h}} + \beta_{\lambda 3\tilde{l}} + \beta_{\lambda 3h'} = 0\,,\\
&2\,\beta_{\lambda 3c} + 2\,\beta_{\lambda 3i} + \beta_{\lambda 3o} + 2\,\beta_{\lambda 3y} = 0\,, \qquad \beta_{\lambda 3d} + \beta_{\lambda 3h} + \beta_{\lambda 3\tilde{o}} = 0\,, \qquad 2\,\beta_{\lambda 3j} + \beta_{\lambda 3\tilde{q}} = 0\,,\\
&\beta_{\lambda 3e} + 2\,\beta_{\lambda 3g} + \beta_{\lambda 3\tilde{p}} = 0\,, \qquad 2\,\beta_{\lambda 3e} + \beta_{\lambda 3i} + \beta_{\lambda 3v} + \beta_{\lambda 3z} + \beta_{\lambda 3d'} = 0\,,\\
&\beta_{\lambda 3b} + \beta_{\lambda 3i} + \beta_{\lambda 3\tilde{d}} = 0\,, \qquad 8\,\beta_{\lambda 3e} + 8\,\beta_{\lambda 3w} + \beta_{\lambda 3\tilde{n}} = 0\,, \qquad 4\,\beta_{\lambda 3j} + \beta_{\lambda 3l} = 0\,,\\
&4\,\beta_{\lambda 3b} + 2\,\beta_{\lambda 3s} + \beta_{\lambda 3x} + \beta_{\lambda 3a'} = 0\,,\\
&4\,\beta_{\lambda 3b} + 4\,\beta_{\lambda 3d} + \beta_{\lambda 3f} + 2\,\beta_{\lambda 3r} + \beta_{\lambda 3\tilde{g}} + 2\,\beta_{\lambda 3\tilde{k}} + \beta_{\lambda 3b'} = 0\,.
\end{aligned} \tag{7.9}$$

The last 5 relations involve contributions arising from non planar diagrams.

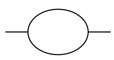
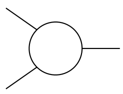
7.2 $\mathcal{N} = \frac{1}{2}$ supersymmetry

This is a special case of the general scalar fermion theory where of course the number of real scalars matches the number of fermions with $y^a \to Y^{abc}$ a symmetric real tensor and also $\lambda^{abcd} \to \mathcal{S}_3\, Y^{abe} Y^{cde}$. For this theory ϕ, ψ can be combined as a real superfield Φ and in a perturbative expansion the diagrams reduce to those of a simple ϕ^3 theory. Under renormalisation as a four dimensional theory there is a β-function $\beta_Y{}^{abc}$ and anomalous dimension $\gamma_\Phi{}^{ab}$. For this case, unlike for $\mathcal{N} = 1$ supersymmetry, there are non trivial divergent vertex graphs so that β_Y and γ_Φ are independent although the scalar β-function β_λ is determined in terms of these.

At lowest one loop order this gives

$$\gamma_{\Phi 1} = \gamma_{\phi 1} = \gamma_{\psi 1}\,, \qquad \beta_{Y1} = 2\,\beta_{\lambda 1a} = \beta_{y1} = 4\,\gamma_{\phi 1}\,, \qquad 4\,\beta_{\lambda 1a} + \beta_{\lambda 1b} = 0\,, \tag{7.10}$$

where

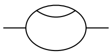
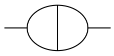
$$\gamma_\Phi{}^{(1)} = \gamma_{\Phi 1}$$  , $$\tilde{\beta}_Y{}^{(1)} = \beta_{Y1}$$  . (7.11)

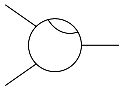
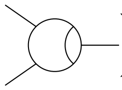
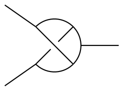
The equality $\gamma_{\phi 1} = \gamma_{\psi 1}$ is a reflection of the choice of normalisation of fermion traces in the main body of results. Each fermion trace gives the contribution of a two component real fermion propagating round the loop.

At two loops equality of γ_ϕ, γ_ψ and symmetry of β_y requires

$$\gamma_{\Phi 2A} = 3\,\gamma_{\phi 2a} + \gamma_{\phi 2b} = \gamma_{\psi 2a} + \gamma_{\psi 2b}\,, \qquad \gamma_{\Phi 2B} = 6\,\gamma_{\phi 2a} + \gamma_{\phi 2c} = \gamma_{\psi 2c}\,,$$
$$\beta_{Y2A} = \beta_{y2b} = \beta_{y2c}\,, \quad \beta_{Y2B} = \beta_{y2a} + \beta_{y2d} = \beta_{y2e}\,, \quad \beta_{Y2C} = \beta_{y2a} + \beta_{y2f}\,, \tag{7.12}$$

where

$\gamma_\Phi{}^{(2)} = \gamma_{\Phi 2A}$  $+\, \gamma_{\Phi 2B}$  ,

$\tilde\beta_Y{}^{(2)} = \mathcal{S}_3\Big(\beta_{Y2A}$  $+\, \beta_{Y2B}$  $\Big) + \beta_{Y2C}$  . (7.13)

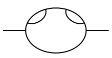
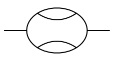
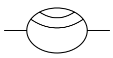
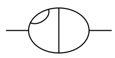
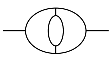
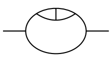
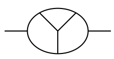
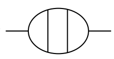
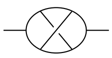
Determining β_λ in terms of β_Y and γ_ϕ imposes the restrictions

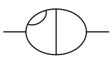
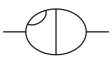
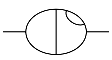
$$\beta_{\lambda 2a} = 2\,\gamma_{\Phi 2A} = \beta_{Y2A} = \tfrac{1}{2}\,\beta_{Y2B}\,, \qquad 0 = \gamma_{\Phi 2B} = \beta_{Y2C}\,,$$
$$4\,\beta_{\lambda 2a} = 4\,\beta_{\lambda 2b} = -2\,\beta_{\lambda 2d} = -2\,\beta_{\lambda 2e} = -\beta_{\lambda 2f} = -\beta_{\lambda 2g}\,, \qquad \beta_{\lambda 2c} = 0\,, \tag{7.14}$$

which implies further constraints on the Yukawa β-functions

$$\beta_{y2a} = -\beta_{y2f} = 2\,\beta_{y2b} - \beta_{y2d}\,, \qquad \beta_{y2b} = 2(\gamma_{\psi 2a} + \gamma_{\psi 2b})\,. \tag{7.15}$$

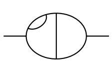
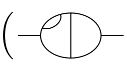
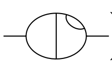
At three loops there are 9 propagator diagrams

$\gamma_\Phi{}^{(3)} = \gamma_{\Phi 3A}$  $+\,\gamma_{\Phi 3B}$  $+\,\gamma_{\Phi 3C}$  $+\,\gamma_{\Phi 3D}\,\mathcal{S}_2$ 
$+\,\gamma_{\Phi 3E}$  $+\,\gamma_{\Phi 3F}$  $+\,\gamma_{\Phi 3G}$  $+\,\gamma_{\Phi 3H}$ 
$+\,\gamma_{\Phi 3I}$  , (7.16)

with $\mathcal{S}_2$  $=$  $+$ . Reducing general results requires

$$\begin{aligned}
\gamma_{\Phi 3A} &= \gamma_{\phi 3a} + \gamma_{\phi 3f} = \gamma_{\psi 3c} + \gamma_{\psi 3i}\,,\\
\gamma_{\Phi 3B} &= \gamma_{\phi 3b} + \gamma_{\phi 3h} = \gamma_{\psi 3d}\,,\\
\gamma_{\Phi 3C} &= 2\,\gamma_{\phi 3b} + \gamma_{\phi 3d} + \gamma_{\phi 3g} = 3\,\gamma_{\psi 3a} + \gamma_{\psi 3e} + \gamma_{\psi 3g} + \gamma_{\psi 3j}\,,\\
\gamma_{\Phi 3D} &= 2\,\gamma_{\phi 3a} + 2\gamma_{\phi 3b} + \gamma_{\phi 3j} = \gamma_{\psi 3f} + \gamma_{\psi 3n}\,,\\
\gamma_{\Phi 3E} &= 2\,\gamma_{\phi 3b} + \gamma_{\phi 3e} = \gamma_{\psi 3o}\,,\\
\gamma_{\Phi 3F} &= 2\,\gamma_{\phi 3a} + \gamma_{\phi 3i} = 6\,\gamma_{\psi 3a} + \gamma_{\psi 3h} + \gamma_{\psi 3l}\,,\\
\gamma_{\Phi 3G} &= 12\,\gamma_{\phi 3a} + 4\gamma_{\phi 3c} + \gamma_{\phi 3l} = \gamma_{\psi 3b} + 2\,\gamma_{\psi 3m}\,,\\
\gamma_{\Phi 3H} &= 4\,\gamma_{\phi 3a} + \gamma_{\phi 3k} = \gamma_{\psi 3b} + \gamma_{\psi 3k}\,,\\
\gamma_{\Phi 3I} &= 4\,\gamma_{\phi 3a} + 2\gamma_{\phi 3c} + \gamma_{\phi 3m} = \gamma_{\psi 3b} + \gamma_{\psi 3p}\,.
\end{aligned} \tag{7.17}$$

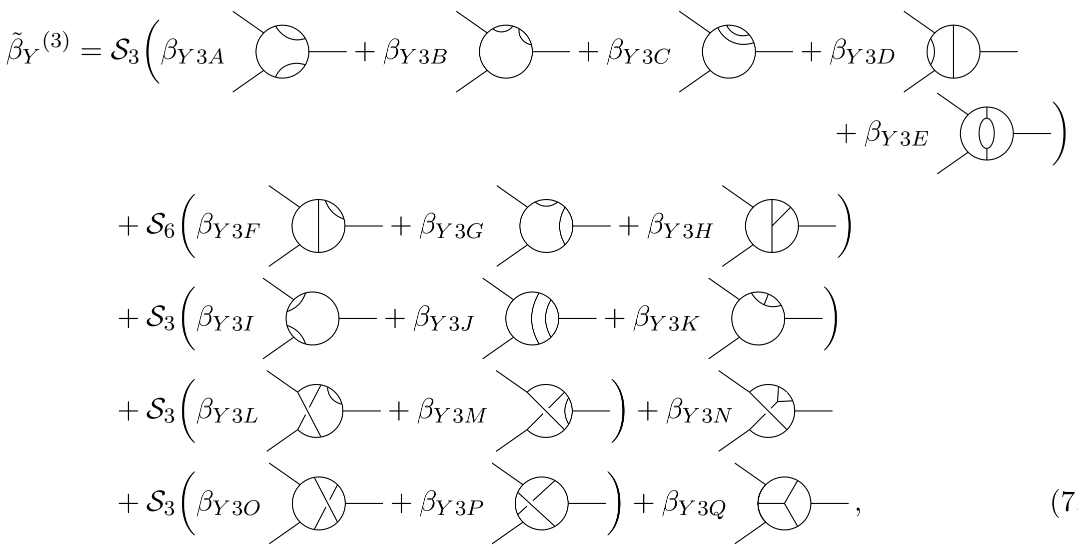
For an antisymmetric contribution

$\upsilon_\Phi{}^{(3)} = \upsilon_{3D}\,\mathcal{A}_2$  $= \upsilon_{\Phi 3D}\Big($  $-$  $\Big)$, (7.18)

where from (2.4) and (3.3)

$$\upsilon_{3D} = \upsilon_{\phi 3j} = -\upsilon_{\psi 3f} + \upsilon_{\psi 3n}\,. \tag{7.19}$$

At three loops there are 17 1PI contributions to the symmetric β-function which are expressible diagrammatically as



(7.20)

with $\mathcal{S}_6$ denoting the sum over six inequivalent permutations. Imposing symmetry on the general Yukawa β-function in this case requires 18 relations

$$
\begin{aligned}
&\beta_{y3p} - \beta_{y3y} = 0\,, \qquad \beta_{y3m} - \beta_{y3z} = 0\,, \qquad 6\,\beta_{y3a} + \beta_{y3v} - \beta_{y3\tilde{e}} = 0\,,\\
&\beta_{y3d} - \beta_{y3n} + \beta_{y3\tilde{j}} = 0\,, \qquad \beta_{y3e} - \beta_{y3o} + \beta_{y3\tilde{h}} = 0\,, \qquad \beta_{y3e} + \beta_{y3q} - \beta_{y3\tilde{c}} = 0\,,\\
&\beta_{y3j} + \beta_{y3r} - \beta_{y3\tilde{b}} = 0\,,\\
&3\,\beta_{y3a} - \beta_{y3t} + \beta_{y3u} - \beta_{y3\tilde{a}} = 0\,, \qquad \beta_{y3b} - \beta_{y3j} - \beta_{y3r} + \beta_{y3\tilde{\imath}} = 0\,,\\
&\beta_{y3g} + \beta_{y3k} - \beta_{y3\tilde{f}} + \beta_{y3\tilde{l}} = 0\,, \qquad \beta_{y3g} - \beta_{y3h} - \beta_{y3\tilde{g}} + \beta_{y3\tilde{v}} = 0\,,\\
&\beta_{y3h} - \beta_{y3i} + \beta_{y3\tilde{d}} - \beta_{y3\tilde{m}} = 0\,, \qquad 2\,\beta_{y3b} - \beta_{y3f} - 2\,\beta_{y3k} + \beta_{y3\tilde{q}} - \beta_{y3\tilde{t}} = 0\,,\\
&2\,\beta_{y3b} + \beta_{y3g} - \beta_{y3h} + \beta_{y3l} - \beta_{y3\tilde{d}} + \beta_{y3\tilde{n}} = 0\,,\\
&2\,\beta_{y3e} - \beta_{y3j} + 2\,\beta_{y3q} - \beta_{y3\tilde{c}} - \beta_{y3\tilde{k}} = 0\,,\\
&2\,\beta_{y3c} - \beta_{y3k} - \beta_{y3\tilde{s}} + \beta_{y3\tilde{u}} = 0\,, \qquad \beta_{y3g} - \beta_{y3h} - \beta_{y3\tilde{g}} + \beta_{y3\tilde{v}} = 0\,,\\
&\beta_{y3c} + \beta_{y3d} - \beta_{y3e} - \beta_{y3j} - \beta_{y3s} - \beta_{y3\tilde{o}} + 2\,\beta_{y3\tilde{p}} = 0\,,\\
&2\,\beta_{y3b} + 2\,\beta_{y3c} - \beta_{y3f} + \beta_{y3l} + \beta_{y3x} + \beta_{y3\tilde{w}} - 2\,\beta_{y3\tilde{x}} - \beta_{y3\tilde{y}} = 0\,.
\end{aligned}
\tag{7.21}
$$

Subject to (7.21)

$$
\begin{aligned}
&\beta_{Y3A} = \beta_{y3p}\,, \quad \beta_{Y3B} = \beta_{y3m}\,, \quad \beta_{Y3C} = 3\,\beta_{y3a} + \beta_{y3u}\,, \quad \beta_{Y3D} = \beta_{y3o}\,, \quad \beta_{Y3E} = \beta_{y3n}\,,\\
&\beta_{Y3F} = \beta_{y3e} + \beta_{y3q}\,, \quad \beta_{Y3G} = \beta_{y3j} + \beta_{y3r}\,, \quad \beta_{Y3H} = \beta_{y3h} + \beta_{y3\tilde{d}}\,,\\
&\beta_{Y3I} = 2\,\beta_{y3b} + \beta_{y3\tilde{q}}\,, \quad \beta_{Y3J} = \beta_{y3\tilde{f}}\,, \quad \beta_{Y3K} = 6\,\beta_{y3a} + \beta_{y3v}\,,\\
&\beta_{Y3L} = \beta_{y3e} + \beta_{y3j} + \beta_{y3s} + \beta_{y3\tilde{o}}\,, \qquad \beta_{Y3M} = \beta_{y3k} + \beta_{y3\tilde{s}}\,,\\
&\beta_{Y3N} = 2\,\beta_{y3c} + \beta_{y3i} + 2\,\beta_{y3\tilde{r}}\,, \qquad \beta_{Y3O} = \beta_{y3h} + \beta_{y3\tilde{g}}\,,\\
&\beta_{Y3P} = \beta_{y3f} + 2\,\beta_{y3\tilde{x}} + \beta_{y3\tilde{y}}\,, \quad \beta_{Y3Q} = 4\,\beta_{y3b} + \beta_{y3f} + 2\,\beta_{y3l} + \beta_{y3w} + 2\,\beta_{y3\tilde{z}}\,.
\end{aligned}
\tag{7.22}
$$

Explicit results for this $\mathcal{N}=\frac{1}{2}$ theory in the $\overline{MS}$ scheme are then

$$\begin{aligned}
&\gamma_{\Phi 1}=\tfrac{1}{2}\,, \quad &&\gamma_{\Phi 2A}=-\tfrac{1}{2}\,, \quad &&\gamma_{\Phi 2B}=0\,, \quad &&\gamma_{\Phi 3A}=-\tfrac{1}{4}\,, \quad &&\gamma_{\Phi 3B}=-\tfrac{1}{8}\,, \quad &&\gamma_{\Phi 3C}=1\,, \\
&\gamma_{\Phi 3D}=0\,, \quad &&\gamma_{\Phi 3E}=\tfrac{1}{4}\,, \quad &&\gamma_{\Phi 3F}=-\tfrac{1}{2}\,, \quad &&\gamma_{\Phi 3G}=1\,, \quad &&\gamma_{\Phi 3H}=\tfrac{3}{2}\,, \quad &&\gamma_{\Phi 3I}=\tfrac{3}{2}\zeta_3\,,
\end{aligned} \tag{7.23}$$

with $\upsilon_{\Phi 3D}=-\frac{3}{4}$ and

$$\begin{aligned}
&\beta_{Y1}=2\,, \quad &&\beta_{Y2A}=-1\,, \quad &&\beta_{Y2B}=-2\,, \quad &&\beta_{Y2C}=0\,, \\
&\beta_{3A}=-\tfrac{1}{2}\,, \quad &&\beta_{Y3B}=-\tfrac{1}{2}\,, \quad &&\beta_{Y3C}=2\,, \quad &&\beta_{Y3D}=-1\,, \\
&\beta_{Y3E}=2\,, \quad &&\beta_{Y3F}=2\,, \quad &&\beta_{Y3G}=-1\,, \quad &&\beta_{3H}=4\,, \\
&\beta_{Y3I}=-2\,, \quad &&\beta_{Y3J}=4\,, \quad &&\beta_{Y3K}=-1\,, \\
&\beta_{Y3N}=12\zeta_3\,, \quad &&\beta_{Y3O}=6\zeta_3\,, \quad &&\beta_{3P}=12\zeta_3\,, \quad &&\beta_{Y3L}=\beta_{Y3M}=\beta_{Y3Q}=0\,.
\end{aligned} \tag{7.24}$$

These results can be obtained directly from superspace calculations [56].

Reducing the three loop scalar β-function to the $\mathcal{N}=\frac{1}{2}$ theory requires large numbers of relations. For the anomalous dimension and the symmetric β function

$$\begin{aligned}
&\beta_{\lambda 3h}=2\,\gamma_{\Phi 3A}\,, \quad \beta_{\lambda 3g}=2\,\gamma_{\Phi 3B}\,, \quad 3\,\beta_{\lambda 3a}+\beta_{\lambda 3p}=2\,\gamma_{\Phi 3C}\,, \quad 0=\gamma_{\Phi 3D}\,, \\
&\beta_{\lambda 3c}=2\,\gamma_{\Phi 3E}=\tfrac{1}{2}\,\gamma_{\Phi 3G}\,, \quad 6\,\beta_{\lambda 3a}+\beta_{\lambda 3q}=2\,\gamma_{\Phi 3F}\,, \\
&2\,\beta_{\lambda 3c}+\beta_{\lambda 3t}=2\,\gamma_{\Phi 3H}\,, \quad 2\,\beta_{\lambda 3c}+\beta_{\lambda 3u}=2\,\gamma_{\Phi 3I}\,, \\
&\beta_{\lambda 3d}=2\,\beta_{\lambda 3g}=\beta_{\lambda 3h}=\beta_{\lambda 3j}=\tfrac{1}{2}(\beta_{\lambda 3i}+\beta_{\lambda 3z})=\tfrac{1}{2}(\beta_{\lambda 3b}+\beta_{\lambda 3\tilde{a}})=\tfrac{1}{4}(2\,\beta_{\lambda 3b}+\beta_{\lambda 3\tilde{c}}) \\
&\ =\tfrac{1}{2}(6\,\beta_{\lambda 3a}+\beta_{\lambda 3q})=\beta_{Y3A}=\beta_{3B}=\tfrac{1}{2}\,\beta_{Y3D}=\tfrac{1}{2}\,\beta_{Y3G}=\tfrac{1}{4}\,\beta_{Y3I}=\tfrac{1}{2}\,\beta_{Y3K}\ \ , \\
&3\,\beta_{\lambda 3a}+\beta_{\lambda 3p}=\beta_{\lambda 3b}=\beta_{\lambda 3i}=2\,\beta_{\lambda 3j}+\beta_{\lambda 3k}=2\,\beta_{\lambda 3c}+\beta_{\lambda 3t}+\tfrac{1}{2}\,\beta_{\lambda 3\tilde{h}} \\
&\ =\beta_{Y3C}=\beta_{Y3E}=\beta_{Y3F}=\tfrac{1}{2}\,\beta_{Y3H}=\tfrac{1}{2}\,\beta_{Y3J}\,, \\
&\qquad\qquad\qquad 2\,\beta_{\lambda 3c}+\beta_{\lambda 3y}=4\,\beta_{\lambda 3c}+\beta_{\lambda 3\tilde{b}}=4\,\beta_{\lambda 3d}+\beta_{\lambda 3r}=0\,, \\
&\beta_{\lambda 3b}+\beta_{\lambda 3i}+\beta_{\lambda 3\tilde{d}}=4\,\beta_{\lambda 3j}+\beta_{\lambda 3l}=\beta_{Y3L}\,, \qquad 0=2\,\beta_{\lambda 3b}+\beta_{\lambda 3\tilde{e}}=\beta_{Y3M}\,, \\
&4\,\beta_{\lambda 3b}+\beta_{\lambda 3\tilde{f}}=\beta_{Y3N}\,, \\
&2\,\beta_{\lambda 3b}+\beta_{\lambda 3s}=4\,\beta_{\lambda 3c}+2\,\beta_{\lambda 3u}=2\,\beta_{\lambda 3b}+2\,\beta_{\lambda 3d}+\beta_{\lambda 3r}+\tfrac{1}{2}\,\beta_{\lambda 3\tilde{g}}=\beta_{Y3O}=\tfrac{1}{2}\,\beta_{Y3P}\,, \\
&4\,\beta_{\lambda 3b}+4\,\beta_{\lambda 3d}+2\,\beta_{\lambda 3r}+\beta_{\lambda 3\tilde{\imath}}=\beta_{Y3Q}\,.
\end{aligned} \tag{7.25}$$

There are here 18 linear relations on the 3 loop β_λ coefficients. There are also 33 additional consistency equations. For those involving contributions from planar diagrams

$$\begin{aligned}
&4\,\beta_{\lambda 3b}+\beta_{\lambda 3\tilde{z}}=0\,, \quad \beta_{\lambda 3b}+\beta_{\lambda 3i}+\beta_{\lambda 3\tilde{u}}=0\,, \quad 2\,\beta_{\lambda 3i}+\beta_{\lambda 3n}=0\,, \quad 2\,\beta_{\lambda 3j}+\beta_{\lambda 3\tilde{q}}=0\,, \\
&2\,\beta_{\lambda 3d}+2\,\beta_{\lambda 3j}+\beta_{\lambda 3\tilde{t}}=0\,, \quad 4\,\beta_{\lambda 3d}+4\,\beta_{\lambda 3e}+\beta_{\lambda 3\tilde{y}}=0\,, \quad \beta_{\lambda 3d}+\beta_{\lambda 3h}+\beta_{\lambda 3\tilde{o}}=0\,, \\
&8\,\beta_{\lambda 3e}+\beta_{\lambda 3\tilde{x}}=0\,, \quad 2\,\beta_{\lambda 3e}+2\,\beta_{\lambda 3j}+\beta_{\lambda 3\tilde{v}}=0\,, \quad \beta_{\lambda 3e}+2\,\beta_{\lambda 3g}+\beta_{\lambda 3\tilde{p}}=0\,, \\
&6\,\beta_{\lambda 3a}+2\,\beta_{\lambda 3d}+\beta_{\lambda 3q}+\beta_{\lambda 3\tilde{s}}=0\,, \quad 3\,\beta_{\lambda 3a}+\beta_{\lambda 3i}+\beta_{\lambda 3m}+\beta_{\lambda 3p}+\beta_{\lambda 3\tilde{r}}=0\,, \\
&4\,\beta_{\lambda 3b}+4\,\beta_{\lambda 3d}+2\,\beta_{\lambda 3r}+\beta_{\lambda 3\tilde{\imath}}=0\,, \quad 4\,\beta_{\lambda 3b}+4\,\beta_{\lambda 3c}+2\,\beta_{\lambda 3\tilde{b}}+\beta_{\lambda 3e'}=0\,,
\end{aligned}$$

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$$4\,\beta_{\lambda 3b}+4\,\beta_{\lambda 3d}+\beta_{\lambda 3r}+\beta_{\lambda 3x}+\beta_{\lambda 3\tilde{w}}=0\,,\qquad 2\,\beta_{\lambda 3b}+4\,\beta_{\lambda 3e}+2\,\beta_{\lambda 3v}+\beta_{\lambda 3\tilde{c}}+\beta_{\lambda 3f'}=0\,,$$
$$\beta_{\lambda 3b}+2\,\beta_{\lambda 3j}+\beta_{\lambda 3k}+\beta_{\lambda 3\tilde{a}}+\beta_{\lambda 3c'}=0\,,\qquad 2\,\beta_{\lambda 3c}+2\,\beta_{\lambda 3i}+\beta_{\lambda 3o}+2\,\beta_{\lambda 3y}=0\,,$$
$$2\,\beta_{\lambda 3e}+\beta_{\lambda 3i}+\beta_{\lambda 3v}+\beta_{\lambda 3z}+\beta_{\lambda 3d'}=0\,,\qquad 4\,\beta_{\lambda 3v}+\beta_{\lambda 3\tilde{m}}=0\,,$$
$$4\,\beta_{\lambda 3c}+4\,\beta_{\lambda 3e}+2\,\beta_{\lambda 3t}+4\,\beta_{\lambda 3v}+2\,\beta_{\lambda 3\tilde{h}}+\beta_{\lambda 3\tilde{l}}+\beta_{\lambda 3h'}=0\,. \tag{7.26}$$

For the relations which involve contributions from the non planar diagrams for the quartic β-function,

$$\beta_{\lambda 3b}+\beta_{\lambda 3i}+\beta_{\lambda 3\tilde{d}}=0\,,\qquad 2\,\beta_{\lambda 3b}+\beta_{\lambda 3\tilde{e}}=0\,,\qquad 4\,\beta_{\lambda 3j}+\beta_{\lambda 3l}=0\,,$$
$$2\,\beta_{\lambda 3e}+\beta_{\lambda 3w}=0\,,\qquad 8\,\beta_{\lambda 3e}+8\,\beta_{\lambda 3w}+\beta_{\lambda 3\tilde{n}}=0\,,$$
$$2\,\beta_{\lambda 3f}+\beta_{\lambda 3\tilde{\jmath}}=0\,,\qquad 4\,\beta_{\lambda 3b}+\beta_{\lambda 3f}+\beta_{\lambda 3\tilde{f}}+2\,\beta_{\lambda 3\tilde{k}}=0\,,$$
$$4\,\beta_{\lambda 3b}+2\,\beta_{\lambda 3s}+\beta_{\lambda 3x}+\beta_{\lambda 3a'}=0\,,$$
$$4\,\beta_{\lambda 3b}+4\,\beta_{\lambda 3d}+\beta_{\lambda 3f}+2\,\beta_{\lambda 3r}+\beta_{\lambda 3\tilde{g}}+2\,\beta_{\lambda 3\tilde{k}}+\beta_{\lambda 3b'}=0\,,$$
$$4\,\beta_{\lambda 3b}+\beta_{\lambda 3f}+2\,\beta_{\lambda 3s}+\beta_{\lambda 3\tilde{k}}+\beta_{\lambda 3g'}=0\,,$$
$$8\,\beta_{\lambda 3c}+\beta_{\lambda 3f}+4\,\beta_{\lambda 3u}+2\,\beta_{\lambda 3i'}=0\,,\qquad 2\,\beta_{\lambda 3f}+\beta_{\lambda 3j'}=0\,. \tag{7.27}$$

The 14 homogeneous relations in (7.9) are contained in (7.26), (7.27). Combining (7.26), (7.27) with (7.25) would apparently generate 51 conditions but 2 are redundant. Two of the conditions in (7.26) imply $\beta_{Y3L}=\beta_{Y3Q}=0$ and the relations in (7.25) $0=2\,\beta_{\lambda 3b}+\beta_{\lambda 3\tilde{e}}$ and $\beta_{\lambda 3b}+\beta_{\lambda 3i}+\beta_{\lambda 3\tilde{d}}=4\,\beta_{\lambda 3j}+\beta_{\lambda 3l}$ can be omitted since they are all zero in (7.26). There remain 49 independent equations.

The conditions $\beta_{Y3A}=\beta_{3B}=\frac{1}{2}\,\beta_{Y3D}=\frac{1}{2}\,\beta_{Y3G}=\frac{1}{4}\,\beta_{Y3I}=\frac{1}{2}\,\beta_{Y3K}$, $\beta_{3C}=\beta_{Y3E}=\beta_{Y3F}=\frac{1}{2}\,\beta_{Y3H}=\frac{1}{2}\,\beta_{Y3J}$, $\beta_{Y3O}=\frac{1}{2}\,\beta_{Y3P}$ and $\beta_{3L}=\beta_{Y3M}=\beta_{Y3Q}=0$ impose 13 further relations on the Yukawa β-functions from (7.22).

8 Special cases and fixed points

To analyse the RG flow in scalar fermion theories, and potentially find fixed points in an $\varepsilon=4-d$ expansion, it is generally necessary to restrict to cases where the RG flow is constrained to a small number of couplings. Here we describe various examples where symmetries are imposed so that the RG flow is reduced to two scalar couplings and one Yukawa coupling. Of course with minimal subtraction ε only appears at zeroth order in a loop expansion so that various perturbative results listed here can easily be used in the hunt for fixed points. Possible fixed points are first determined by using the one loop contributions to the β-functions and are described in this section. Corresponding two and three formulae which give $\varepsilon^2,\varepsilon^3$ contributions are obtained by restriction of the general results and are presented in supplementary material attached to this paper.

At one loop the results obtained here for the general case give

$$\beta_y{}^{(1)a}=2\,y^b y^a y^b+\tfrac{1}{2}(y^b y^b\,y^a+y^a\,y^b y^b)+\tfrac{1}{2}\,y^b\,\mathrm{tr}(y^b y^a)\,,$$
$$\beta_\lambda{}^{(1)abcd}=3\,\lambda^{ef(ab}\lambda^{cd)ef}+2\,\lambda^{e(abc}\,\mathrm{tr}(y^{d)}y^e)-12\,\mathrm{tr}(y^{(a}y^b y^c y^{d)})\,,$$
$$\gamma_\phi{}^{(1)ab}=\tfrac{1}{2}\mathrm{tr}(y^a y^b)\,,\qquad \gamma_\psi{}^{(1)}=\tfrac{1}{2}\,y^a y^a\,, \tag{8.1}$$

with $\{y^a\}$ symmetric and real.

For n_s real scalars and n_f pseudo real Majorana fermions the reduction to three couplings is achieved by assuming

$$y^a \to y\, t^a\, \mathbb{1}_m\,, \quad \lambda^{abcd} = \lambda(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) + g\, h^{abcd}\,, \tag{8.2}$$

with $\{t^a\}$ a set of real traceless $n \times n$ symmetric matrices and m essentially arbitrary. We assume[3]

$$\operatorname{tr}(t^{(a}t^b t^c t^{d)}) = n\,\alpha\, h^{abcd}\,. \tag{8.3}$$

In general the symmetry group is given by

$$R^{-1} y^a\, R = \tilde{R}^{ab} y^b \quad \text{for} \quad R \in H_f \subset O(n_f)\,, \quad [\tilde{R}^{ab}] \in H_s \subset O(n_s)\,, \tag{8.4}$$

and then h^{abcd} is an H_s invariant $O(n_s)$ symmetric tensor. Assuming (8.2) $H_f \simeq H \times O(m)$ with $H \subset O(n)$. For simplicity we take t^a to be traceless and $H_s \simeq H/\mathbb{Z}_2$ with H simple.

At one loop a consistent RG flow is achieved by requiring for the Yukawa β-function the conditions

$$t^a t^a = n_s\,\alpha\, \mathbb{1}_n\,, \qquad \operatorname{tr}(t^a t^b) = n\,\alpha\,\delta^{ab}\,, \qquad t^b t^a t^b = \beta\, t^a\,, \tag{8.5}$$

where $n_f = n\,m$ and $\alpha > 0$ depends on a choice of scale for $\{t^a\}$. For the scalar coupling it is also necessary that

$$h^{ef(ab}h^{cd)ef} = A(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) + B\, h^{abcd}\,. \tag{8.6}$$

The tensor h^{abcd} may be further decomposed as

$$h^{abcd} = d^{abcd} + r\big(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\big)\,, \qquad r = \frac{2\,n_s\,\alpha + \beta}{3(n_s+2)}\,, \tag{8.7}$$

for d^{abcd} symmetric and traceless and (8.2) is alternatively expressed as

$$\lambda^{abcd} = \hat{\lambda}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) + g\, d^{abcd}\,, \qquad \hat{\lambda} = \lambda + r\, g\,. \tag{8.8}$$

With this definition (8.6) is equivalent to

$$d^{ef(ab}d^{cd)ef} = a\,\delta^{(ab}\delta^{cd)} + b\, d^{abcd}\,, \tag{8.9}$$

where $b = B - 4\,r$, $a = 3(A + rB) - (n_s+8)r^2$.

At higher loops the necessary constraints are such that the β-functions are reduced to β_y, β_λ, β_g and the anomalous dimension matrices have the form $\gamma_\phi\,\delta^{ab}$, $\gamma_\psi\,\mathbb{1}_m \times \mathbb{1}_n$.

[3]The corresponding scalar potential should be bounded below. The constraints on λ, g may be determined by the inequalities for any hermitian traceless $n \times n$ t

$$k_+ \operatorname{tr}(t^2)^2 \geq \operatorname{tr}(t^4) \geq k_- \operatorname{tr}(t^2)^2\,, \qquad k_+ = \frac{(n-1)^3+1}{n^2(n-1)}\,, \quad k_- = \begin{cases} \frac{1}{n}\,, & n \text{ even} \\ \frac{n^2+3}{n(n^2-1)}\,, & n \text{ odd} \end{cases}.$$

For the potential to be bounded below it is possible for either g or λ to be negative so long as $\lambda > 0$, $3\,\lambda + k_+ n\alpha\, g > 0$ or $g > 0$, $3\,\lambda + k_- n\alpha\, g > 0$.

Using

$$h^{abcc} = \tfrac{1}{3}(2\,n_s\,\alpha + \beta)\,\delta^{ab}\,, \tag{8.10}$$

the lowest order results (8.1) are consistent with this form and give,

$$\begin{aligned}
\beta_y{}^{(1)} &= (n_s\,\alpha + 2\,\beta + \tilde n_f)y^3\,,\\
\beta_\lambda{}^{(1)} &= (n_s+8)\lambda^2 + \tfrac{2}{3}(2\,n_s\,\alpha+\beta)\lambda\,g + 4\,\tilde n_f\,\lambda\,y^2 + 3\,A\,g^2\,,\\
\beta_g{}^{(1)} &= 12\,\lambda\,g + 3\,B\,g^2 + 4\,\tilde n_f\,g\,y^2 - 24\,\tilde n_f\,y^4\,,\\
\gamma_\phi{}^{(1)} &= \tilde n_f\,y^2\,, \qquad \gamma_\psi{}^{(1)} = \tfrac{1}{2}\,n_s\,\alpha\,y^2\,, \qquad \tilde n_f = \tfrac{1}{2}\,\alpha\,n_f\,.
\end{aligned} \tag{8.11}$$

Alternatively

$$\hat\lambda = \lambda + r\,g\,, \qquad \beta_{\hat\lambda}{}^{(1)} = (n_s+8)\hat\lambda^2 + a\,g^2 + 4\,\tilde n_f\,\hat\lambda\,y^2 - 24\,\tilde n_f\,r\,y^4\,,$$
$$\beta_g{}^{(1)} = 12\,\hat\lambda\,g + 3\,b\,g^2 + 4\,\tilde n_f\,g\,y^2 - 24\,\tilde n_f\,y^4\,. \tag{8.12}$$

For quadratic scalar operators then at lowest order the anomalous dimension for the singlet $\sigma = \phi^2$ and the corresponding matrix for $\rho^{ab} = \phi^a\phi^b - \frac{1}{n_s}\delta^{ab}\phi^2$ are just

$$\begin{aligned}
\gamma_\sigma{}^{(1)} &= (n_s+2)\hat\lambda + 2\,\tilde n_f\,y^2\,,\\
\gamma_\rho{}^{(1)ab,cd} &= 2(\hat\lambda + \tilde n_f\,y^2)\big(\tfrac{1}{2}(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc} - \tfrac{1}{n_s}\delta^{ab}\delta^{cd}\big) + g\,d^{abcd}\,.
\end{aligned} \tag{8.13}$$

At higher orders, besides the symmetric traceless tensor d^{abcd}, it is necessary to take into account the mixed symmetry tensor w^{abcd} defined by

$$2\,\mathrm{tr}(t^{(a}t^{b)}t^{(c}t^{d)}) - \mathrm{tr}(t^a t^c t^b t^d) - \mathrm{tr}(t^a t^d t^b t^c) = n\,\alpha\big(w^{abcd} - s(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc} - 2\,\delta^{ab}\delta^{cd})\big)\,, \tag{8.14}$$

which with

$$s = \frac{n_s\,\alpha - \beta}{n_s - 1}\,, \tag{8.15}$$

satisfies $w^{abcd} = w^{(ab)(cd)} = w^{cdab}$, $w^{a(bcd)} = 0$ and is traceless on contraction of any pair of indices. This contributes to $\gamma_\rho{}^{ab,cd}$ at two and higher loops. The anomalous dimensions are then dictated by the eigenvalues of d^{abcd} and w^{abcd} as $\frac{1}{2}(n-1)(n+2) \times \frac{1}{2}(n-1)(n+2)$ symmetric matrices. There are discussed in appendix B. In general there are three eigenvalues as symmetric traceless tensors decompose into components belonging to representation spaces of the reduced symmetry group H.

If $a = 0$ then in (8.7) $d^{abcd} = 0$ and the g coupling is redundant. The scalar β-function at one loop is given just by $\beta_{\hat\lambda}{}^{(1)}$ in (8.12) with $g = 0$. This restriction necessarily holds for $n = 2, 3$ since, for any traceless t, $\mathrm{tr}(t^4) = \frac{1}{2}\mathrm{tr}(t^2)^2$. This translates into the condition $\frac{1}{2}(n_s+2)n\,\alpha = 2\,n_s\,\alpha + \beta$. In this case $H = O(n_s)$ and there are just two anomalous dimensions for quadratic scalars γ_σ, γ_ρ with

$$\gamma_\sigma{}^{(1)} = (n_s+2)\hat\lambda + 2\,\tilde n_f\,y^2\,, \qquad \gamma_\rho{}^{(1)} = 2(\hat\lambda + \tilde n_f\,y^2)\,. \tag{8.16}$$

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Two extreme examples of matrices satisfying (8.5) are given by

1. *Symmetric.* $t^a \to s^a$ where $\{s^a\}$ are a basis for symmetric traceless $n \times n$ real matrices with $n \geq 2$ satisfying the completeness condition $(s^a)_{\alpha\beta}(s^a)_{\gamma\delta} = \delta_{\alpha\gamma}\,\delta_{\beta\delta} + \delta_{\alpha\delta}\,\delta_{\beta\gamma} - \frac{2}{n}\,\delta_{\alpha\beta}\,\delta_{\gamma\delta}$, and $\mathrm{tr}(s^a s^b) = 2\,\delta^{ab}$. $\{s^a\}$ are the generators corresponding to the coset $Sl(n,\mathbb{R})/\mathrm{SO}(n)$.

2. *Diagonal.* t^a are traceless diagonal $n \times n$ real matrices with $n \geq 2$. A basis is obtained by taking $(t^a)_{\alpha\beta} = e^a_\alpha\,\delta_{\alpha\beta}$ where e^a_α, $\alpha = 1, \dots, n$ form the n vertices of a $n-1$ dimensional hypertetrahedron and satisfy $\sum_\alpha e^a_\alpha = 0$, $\sum_\alpha e^a_\alpha e^b_\alpha = 2\,\delta^{ab}$ with $e^a_\alpha e^a_\beta = 2\,\delta_{\alpha\beta} - \frac{2}{n}$. In this example the tensor w^{abcd} in (8.14) vanishes.

For these cases we have

y^a	n_s	n_f	α	β	A	B	H
1.	$\frac{1}{2}(n-1)(n+2)$	$n\,m$	$\frac{2}{n}$	$\frac{1}{n}(n-2)$	$\frac{2}{9n^2}(n^2+6)$	$\frac{1}{9n}(2n^2+9n-36)$	$O(n)$
2.	$n-1$	$n\,m$	$\frac{2}{n}$	$\frac{2}{n}(n-1)$	$\frac{4}{3n^2}$	$\frac{2}{n}(n-2)$	$\mathcal{S}_n$

(8.17)

For purely scalar theories these examples were described long ago in [57].

In general defining

$$\begin{aligned} S_{\alpha\beta\gamma\delta} &= (t^a)_{(\alpha\beta}(t^a)_{\gamma\delta)} - \tfrac{2\,n_s\,\alpha}{3(n+2)}\big(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\big)\,, \\ W_{\alpha\beta\gamma\delta} &= (t^a)_{\alpha\beta}(t^a)_{\gamma\delta} - \tfrac{1}{2}\big((t^a)_{\alpha\gamma}(t^a)_{\beta\delta} + (t^a)_{\alpha\delta}(t^a)_{\beta\gamma}\big) \\ &\qquad + \tfrac{n_s\,\alpha}{n-1}\big(\delta_{\alpha\beta}\delta_{\gamma\delta} - \tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\big)\,, \end{aligned} \tag{8.18}$$

then positivity of $S_{\alpha\beta\gamma\delta}S_{\alpha\beta\gamma\delta}$ and $W_{\alpha\beta\gamma\delta}W_{\alpha\beta\gamma\delta}$ give the bounds

$$\big(\tfrac{2\,n_s}{n+2} - \tfrac{1}{2}\,n\big)\alpha \leq \beta \leq \big(n - \tfrac{n_s}{n-1}\big)\alpha\,, \tag{8.19}$$

which entails $n_s \leq \frac{1}{2}(n-1)(n+2)$. This is of course saturated in case 1 and the upper bound on β is saturated in case 2. With $O^{ab} = t^a\,t^b - \frac{1}{n_s}\,\delta^{ab}\,t^c t^c$ then since $|\mathrm{tr}(O^{ab}\,O^{ab})| \leq \mathrm{tr}(O^{ba}\,O^{ab})$ we must have also the bounds

$$-(n_s - 2)\alpha \leq \beta \leq n_s\,\alpha\,. \tag{8.20}$$

General results for fermion scalar theories can be restricted to n_s real scalars and n_f Dirac fermions by taking

$$y^a \to \begin{pmatrix} 0 & y\,t^a \\ \bar{y}\,\bar{t}^a & 0 \end{pmatrix} \mathbb{1}_m\,, \qquad \bar{t}^a = (t^a)^\dagger\,, \tag{8.21}$$

with $\{t^a\}$ $n \times n$ matrices so that $n_f = n\,m$. Assuming the Yukawa interaction satisfies

$$U^{-1} y^a\,U = \tilde{R}^{ab} y^b \quad \text{for} \quad U \in H_f \subset \mathrm{U}(n_f) \times \mathrm{U}(n_f)\,, \quad [\tilde{R}^{ab}] \in H_s \subset O(n_s)\,, \tag{8.22}$$

to preserve the form (8.21) $U = \left(\begin{smallmatrix} U_- & 0 \\ 0 & U_+ \end{smallmatrix}\right)$ with $U_\pm \in H \times \mathrm{U}(m)$ so that the symmetry groups become $H_f = H \times H \times \mathrm{U}(m)$ with $H_s = H/\mathrm{U}(1)$. Here we require as previously that $\{t^a\}$ are traceless so that it is necessary to take $U_\pm = U$.

For each fermion trace the reduction of general results is obtained by taking

$$\begin{aligned}\mathrm{tr}(y^{a_1}y^{a_2}\dots y^{a_{2p}}) \to m\,(y\bar y)^n\big(&\mathrm{tr}(t^{a_1}\bar t^{a_2}\dots \bar t^{a_{2p}}) + \mathrm{tr}(\bar t^{a_1}t^{a_2}\dots t^{a_{2p}}) \\ &+ \mathrm{tr}(\bar t^{a_{2p}}\dots \bar t^{a_2}t^{a_1}) + \mathrm{tr}(t^{a_{2p}}\dots t^{a_2}\bar t^{a_1})\big)\,, \\ \mathrm{tr}(y^{a_1}y^{a_2}\dots y^{a_{2p+1}}) \to 0\,. \end{aligned} \tag{8.23}$$

The identities in (8.5) and also (8.3) become

$$t^a\bar t^a = n_s\,\alpha\,\mathbb{1}_n\,,\quad \mathrm{tr}(t^a\bar t^b) = n\,\alpha\,\delta^{ab}\,,\quad t^b\bar t^a t^b = \beta\, t^a\,,\quad \mathrm{tr}\big(t^{(a}\bar t^b t^c \bar t^{d)}\big) = n\,\alpha\, h^{abcd}\,. \tag{8.24}$$

As before $\alpha > 0$ but the bounds in (8.19) are no longer valid although, as previously with $O^{ab} = t^a\,\bar t^b - \frac{1}{n_s}\,\delta^{ab}\,t^c\,\bar t^c$, (8.20) remains. The results in (8.1) and (8.11) then remain valid after taking $n_f \to 4\,n_f$.

Various examples of matrices $\{t^a\}$ satisfying (8.24) are obtained from the generators in the fundamental representation of classical Lie groups

3, Unitary. t^a, $\bar t^a \to \lambda^a$ where $\{\lambda^a\}$ are hermitian traceless $n\times n$ matrices, $n \geq 2$, forming generators for SU(n), satisfying the completeness condition $(\lambda^a)_\alpha{}^\beta(\lambda^a)_\gamma{}^\delta = 2(\delta_\alpha{}^\delta\delta_\gamma{}^\beta - \frac{1}{n}\,\delta_\alpha{}^\beta\delta_\gamma{}^\delta)$, and $\mathrm{tr}(\lambda^a\lambda^b) = 2\,\delta^{ab}$.

4, Antisymmetric. t^a, $-\bar t^a \to a^a$ where $\{a^a\}$ are antisymmetric $n\times n$ real matrices, $n \geq 2$, forming generators for SO(n), satisfying the completeness condition $(a^a)_{\alpha\beta}(a^a)_{\gamma\delta} = \delta_{\alpha\gamma}\,\delta_{\beta\delta} - \delta_{\alpha\delta}\,\delta_{\beta\gamma}$, and $\mathrm{tr}(a^a a^b) = -2\,\delta^{ab}$.

5, Symplectic. t^a, $\bar t^a \to \sigma^a$ where $\{\sigma^a\}$ are hermitian $n\times n$ matrices, $n = 2p$, $p \geq 1$, which are generators of Sp(n) so that for $J_{\alpha\beta}, (J^{-1})^{\alpha\beta}$ antisymmetric matrices then $J^{-1}\sigma^a J = -(\sigma^a)^T$ or $(\sigma^a J)^T = \sigma^a J$. The assumed completeness relation is then $(\sigma^a)_\alpha{}^\beta(\sigma^a)_\gamma{}^\delta = \delta_\alpha{}^\delta\delta_\gamma{}^\beta + J_{\alpha\gamma}\,(J^{-1})^{\beta\delta}$ and $\mathrm{tr}(\sigma^a\sigma^b) = 2\,\delta^{ab}$.

6, Symplectic′. t^a, $\bar t^a \to \tilde\sigma^a$ where $\{\tilde\sigma^a\}$ are hermitian traceless $n\times n$ matrices, $n = 2p$, $p \geq 2$, corresponding to generators belonging to the coset SU(n)/Sp(n). With J, J^{-1} antisymmetric matrices as in case 4 $J^{-1}\tilde\sigma^a J = (\tilde\sigma^a)^T$ or $(\tilde\sigma^a J)^T = -\tilde\sigma^a J$ and the completeness relation becomes $(\tilde\sigma^a)_\alpha{}^\beta(\tilde\sigma^a)_\gamma{}^\delta = \delta_\alpha{}^\delta\delta_\gamma{}^\beta - J_{\alpha\gamma}\,(J^{-1})^{\beta\delta} - \frac{2}{n}\,\delta_\alpha{}^\beta\delta_\gamma{}^\delta$ and $\mathrm{tr}(\tilde\sigma^a\tilde\sigma^b) = 2\,\delta^{ab}$.

For the different cases we have

y^a	n_s	n_f	α	β	A	B	H
3.	n^2-1	nm	$\frac{2}{n}$	$-\frac{2}{n}$	$\frac{4}{9n^2}(n^2+3)$	$\frac{4}{9n}(n^2-9)$	SU(n)
4.	$\frac{1}{2}n(n-1)$	nm	$\frac{2}{n}$	1	$\frac{2}{9}$	$\frac{1}{9}(2n-1)$	SO(n)
5.	$\frac{1}{2}n(n+1)$	nm	$\frac{2}{n}$	-1	$\frac{2}{9}$	$\frac{1}{9}(2n+1)$	Sp(n)
6.	$\frac{1}{2}(n-2)(n+1)$	nm	$\frac{2}{n}$	$-\frac{1}{n}(n+2)$	$\frac{2}{9n^2}(n^2+6)$	$\frac{1}{9n}(2n^2-9n-36)$	Sp(n)

(8.25)

For case 4 and $n = 2$, H reduces to $\mathbb{Z}_2$.

Beyond one loop there are further conditions necessary on t^a, $\bar t^a$ for each primitive diagram (which are those with no subdivergences). At two loops it is sufficient to require

$$\begin{aligned} &t^b\,\bar t^c\,t^a\,\bar t^b\,t^c = \gamma\,t^a\,, \\ &\mathrm{tr}\big(t^e\,\bar t^{(a}\,t^b\,\bar t^{|e|}\,t^c\,\bar t^{d)}\big) = n\alpha\big(\delta\,h^{abcd} + \tfrac{1}{3}\alpha(n_s\alpha - \delta)(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})\big)\,. \end{aligned} \tag{8.26}$$

In general

$$\gamma = n_s\,\alpha(2\alpha+\delta) - (2\alpha-\beta)\delta - \beta^2\,. \tag{8.27}$$

For the different examples considered here results for a, b in (8.9) and also γ, δ are then

y^a	a	b	γ	δ
1.	$\frac{(n-3)(n-2)(n+1)(n+4)(n+6)}{6\,n(n_s+2)^2}$	$\frac{2n^4+11n^3-71n^2-90n+72}{18\,n(n_s+2)}$	$\frac{1}{n^2}\big(n^3+3n^2-4n+4\big)$	$1-\frac{2}{n}$
2.	$\frac{8(n-3)(n-2)}{n(n_s+2)^2}$	$2\,\frac{n^2-5n+2}{n(n_s+2)}$	$\frac{4}{n^2}(n-1)^2$	$2-\frac{2}{n}$
3.	$\frac{4\,(n_s-3)(n_s-8)}{3(n_s+2)^2}$	$\frac{4(n^4-20n^2+9)}{9\,n(n_s+2)}$	$\frac{4}{n^2}(n^2+1)$	$-\frac{2}{n}$
4.	$\frac{8\,(n_s-1)(n_s-3)}{3(n_s+2)^2}$	$\frac{(n-5)(n+4)(2n-1)}{9(n_s+2)}$	$3-n$	-1
5.	$\frac{2(n_s-1)(n_s-3)}{3(n_s+2)^2}$	$\frac{(n-4)(n+5)(2n+1)}{18(n_s+2)}$	$3+n$	1
6.	$\frac{(n-6)(n-4)(n-1)(n+2)(n+3)}{6\,n(n_s+2)^2}$	$\frac{2n^4-11n^3-71n^2+90n+72}{18\,n(n_s+2)}$	$\frac{1}{n^2}\big(-n^3+3n^2+4n+4\big)$	$-1-\frac{2}{n}$

(8.28)

The results for $d^{abcd}d^{abcd} = \frac{1}{2}n_s(n_s+2)a$ correspond in cases 3,4,5 to the evaluation of the quartic Casimir for $\mathrm{SU}(n)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(n)$ [58]. In general for $a > 0$ it is necessary to restrict $n > 3$ except for case 6 when $n > 6$ is required.

8.1 Further algebraic relations

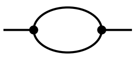
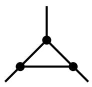
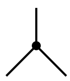
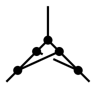
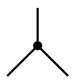
For the characterisation of the different possibilities we may further define for cases 1,2,3,6 additional invariant tensors

$$\mathrm{tr}\big(t^{(a}t^bt^{c)}\big) = n\,\alpha\,d^{abc}\,,$$  $= n\alpha$  , (8.29)

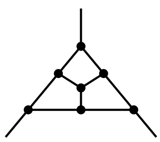
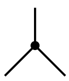
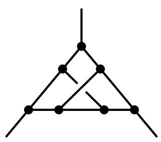
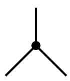
where d^{abc} is symmetric and traceless, These three index d-tensors are constrained by the one and two loop identities

$$d^{acd}d^{bcd} = \alpha_d\,\delta^{ab}\,, \quad d^{ade}d^{bef}d^{cfd} = \beta_d\,d^{abc}\,, \quad d^{dbf}d^{efg}d^{agh}d^{dhi}d^{eic} = \gamma_d\,d^{abc}\,, \tag{8.30}$$

or diagrammatically

 $= \alpha_d$,  $= \beta_d$  ,  $= \gamma_d$  , (8.31)

and at three loops there are two primitive diagrams and it is then necessary that

 $= \delta_d$  ,  $= \epsilon_d$  . (8.32)

More general versions of these equations with more than one d-tensor were discussed for various n in [35].

For the particular cases considered here

	α_d	β_d	γ_d
1.	$\frac{1}{2n}(n-2)(n+4)$	$\frac{1}{4n}(n^2+4n-24)$	$\frac{1}{8n^2}(n-4)(3n^2+4n-80)$
2.	$\frac{2}{n}(n-2)$	$\frac{2}{n}(n-3)$	$\frac{4}{n^2}(n^2-6n+10)$
3.	$\frac{1}{n}(n^2-4)$	$\frac{1}{2n}(n^2-12)$	$-\frac{4}{n^2}(n^2-10)$
6.	$\frac{1}{2n}(n-4)(n+2)$	$\frac{1}{4n}(n^2-4n-24)$	$-\frac{1}{8n^2}(n+4)(3n^2-4n-80)$,

(8.33)

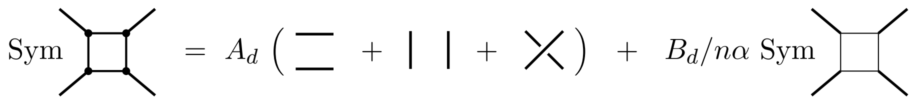
and

	δ_d	ϵ_d
1.	$\frac{1}{n^3}\big(\frac{1}{64}n^6+\frac{9}{64}n^5+\frac{1}{8}n^4-\frac{11}{2}n^3+2n^2+116n-256\big)$	$\frac{1}{n^3}\big(\frac{5}{64}n^5+\frac{7}{16}n^4-\frac{19}{4}n^3-n^2+116n-264\big)$
2.	$\frac{8}{n^3}(n^3-9n^2+29n-32)$	$\frac{8}{n^3}(n^3-9n^2+29n-33)$
3.	$\frac{1}{8n^3}(n^2-8)(n^4-8n^2+256)$	$-\frac{1}{2n^3}(n^4-68n^2+528)$
6.	$\frac{1}{n^3}\big(\frac{1}{64}n^6-\frac{9}{64}n^5+\frac{1}{8}n^4+\frac{11}{2}n^3+2n^2-116n-256\big)$	$-\frac{1}{n^3}\big(\frac{5}{64}n^5-\frac{7}{16}n^4-\frac{19}{4}n^3+n^2+116n+264\big)$.

(8.34)

The results for cases 2 and 3 were given previously in [59, 60] and [61].

For further applications



(8.35)

where $(n_s+2)A_d + \frac{1}{3}(2n_s\,\alpha+\beta)B_d = \frac{1}{3}(2\alpha_d+\beta_d)\alpha_d$. For the different cases we have

	A_d	B_d	β_1
1.	$\frac{1}{12n^2}(3n^2+32)$	$\frac{1}{8n}(n^2+8n-64)$	$\frac{1}{2}(5n^2+14n-72)$
2.	$\frac{8}{3n^2}$	$\frac{2}{n}(n-4)$	$2(7n-18)$
3.	$\frac{1}{6n^2}(3n^2+16)$	$\frac{1}{4n}(n^2-32)$	$5n^2-36$
6.	$\frac{1}{12n^2}(3n^2+32)$	$\frac{1}{8n}(n^2-8n-64)$	$\frac{1}{2}(5n^2-14n-72)$

. (8.36)

The tensors d^{abc} defined as in (8.29) may be used to form symmetric traceless Yukawa couplings so that the number of fermions $n_f = m\,n_s$. In this case (8.11) becomes

$$
\begin{aligned}
&{\beta_y}^{(1)} = (\alpha_d + 2\,\beta_d + \tfrac{1}{2}\,\alpha_d\,m)y^3\,,\\
&{\beta_\lambda}^{(1)} = (n_s+8)\lambda^2 + \tfrac{2}{3}(2\,n_s\,\alpha+\beta)\lambda\,g + 2\,\alpha_d\,m\,\lambda\,y^2 + 3\,A\,g^2 - 12\,A_d\,m\,y^4\,,\\
&{\beta_g}^{(1)} = 12\,\lambda\,g + 3\,B\,g^2 + 2\,\alpha_d\,m\,g\,y^2 - 12\,B_d\,m\,y^4\\
&{\gamma_\phi}^{(1)} = \tfrac{1}{2}\,\alpha_d\,m\,y^2\,, \qquad {\gamma_\psi}^{(1)} = \tfrac{1}{2}\,\alpha_d\,y^2\,.
\end{aligned} \tag{8.37}
$$

As was discussed in detail in [62] for $m=1$, $n_s = n_f$ there is a reduction to a single component $\mathcal{N} = \frac{1}{2}$ supersymmetric theory with the couplings constrained by

$$
\lambda = \tfrac{2}{n}\,y^2\,, \qquad g = 3\,y^2\,. \tag{8.38}
$$

The results in (8.37) with this restriction all correspond to

$$\beta_{y^2}{}^{(1)} = \tfrac{1}{n}\, \beta_1\, y^4 \,, \tag{8.39}$$

with formulae for β_1 given in (8.36). The reduction depends on non trivial relations between A, A_d, B, B_d which are satisfied in each of the cases listed. Except for the last case $\beta_1 > 0$, and there are fixed points in an ε-expansion, for $n \geq 3$. For the $\mathcal{N} = \frac{1}{2}$ supersymmetric theory then taking $Y^{abc} \to y\, d^{abc}$ there is a single coupling theory with a β-function and γ_Φ, with three loop coefficients given in (7.23), (7.24), expanded as

$$\begin{aligned} \beta_y = {} & (\tfrac{3}{2}\, \alpha_d + 2\, \beta_d) y^3 - 3(\tfrac{1}{2}\, \alpha_d{}^2 + \alpha_d\, \beta_d + 2\, \beta_d{}^2) y^5 \\ & + (\tfrac{15}{8}\, \alpha_d{}^3 + \tfrac{9}{4}\, \alpha_d{}^2 \beta_d + \tfrac{27}{2}\, \alpha_d\, \beta_d{}^2 + 30\, \beta_d{}^3) y^7 \\ & + ((\tfrac{9}{2}\, \alpha_d + 30\, \beta_d)\gamma_d + 36\, \epsilon_d)\zeta_3\, y^7 + \mathrm{O}(y^9) \,, \\ \gamma_\Phi = {} & \tfrac{1}{2}\, \alpha_d\, y^2 - \tfrac{1}{2}\, \alpha_d{}^2 y^4 + (\tfrac{5}{8}\, \alpha_d{}^3 - \tfrac{1}{4}\, \alpha_d{}^2 \beta_d + \tfrac{5}{2}\, \alpha_d\, \beta_d{}^2 + \tfrac{3}{2}\, \alpha_d\, \gamma_d\, \zeta_3) y^6 + \mathrm{O}(y^8) \,. \end{aligned} \tag{8.40}$$

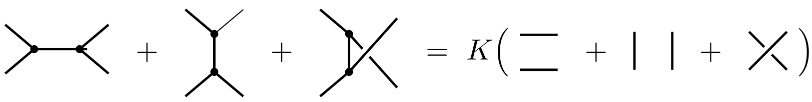
Of course the lowest order term corresponds to (8.37). Furthermore

$$\begin{aligned} \gamma_{\Phi^2} = {} & 2\, \alpha_d\, y^2 - (5\, \alpha_d{}^2 + 4\, \alpha_d \beta_d) y^4 \\ & + (10\, \alpha_d{}^3 + 7\, \alpha_d{}^2 \beta_d + 22\, \alpha_d\, \beta_d{}^2 + 6(\alpha_d{}^2 \beta_d + 6\, \alpha_d\, \beta_d{}^2 + 4\, \alpha_d\, \gamma_d)\zeta_3) y^6 \\ & + 2\, \gamma_\Phi + \mathrm{O}(y^8) \,. \end{aligned} \tag{8.41}$$

The bounds (8.19) becomes $-\frac{n_s - 2}{2(n_s+2)}\, \alpha_d \leq \beta_d \leq \frac{n_s-2}{n_s-1}\, \alpha_d$. The upper bound for β_d corresponds to the vanishing of $W_{\alpha\beta\gamma\delta}$ in (8.18) and holds exactly for case 2 in (8.33) for any n. The lower bound for β_d corresponds to the vanishing of $S_{\alpha\beta\gamma\delta}$ in (8.18) which becomes in this case the condition

$$d^{abe} d^{cde} + d^{ade} d^{bce} + d^{ace} d^{dbe} = K\big(\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} + \delta^{ac}\delta^{db}\big) \,, \qquad K = \tfrac{2\, \alpha_d}{n_s + 2} \,, \tag{8.42}$$

or diagrammatically


(8.43)

This was analysed in [63] and related to the F_4 family of Lie groups.

A uniform treatment is obtained by considering hermitian traceless $n \times n$ matrices $\{e_a\}$, $a = 1, \dots, n_s$, satisfying

$$\tfrac{1}{2}(e_a e_b + e_b e_a) = \tfrac{1}{n}\, \delta_{ab}\, \mathbb{1}_n + d_{abc}\, e_c \,, \qquad d_{abc} = d_{(abc)} \,, \quad d_{aac} = 0 \,. \tag{8.44}$$

Of course this implies $e_a e_a = \frac{1}{n} n_s\, \mathbb{1}_n$, $\mathrm{tr}(e_a e_b) = \delta_{ab}$. The algebra defined by (8.44) is equivalent to the result that hermitian real, complex and quaternionic matrices form a special real Jordan algebra. Furthermore 3×3 hermitian octonionic matrices also form an exceptional real Jordan, or Albert, algebra, with F_4 as the automorphism group. For 3×3 hermitian traceless matrices

$$x = x_a e_a \,, \qquad x^3 - \tfrac{1}{2}\, \mathrm{tr}(x^2)\, x - \tfrac{1}{3}\, \mathrm{tr}(x^3)\, \mathbb{1}_3 = 0 \,. \tag{8.45}$$

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This is just the Cayley Hamilton theorem for real or complex matrices in these cases. The result also extends to hermitian traceless quaternionic and even octonionic matrices [64] where the trace is just the sum of the real diagonal elements. (8.45) is equivalent to (8.42) with just $K = \frac{1}{6}$. For each case it is straightforward to check

$$\begin{array}{c|cccc} & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{O} \\ \hline n_s & 5 & 8 & 14 & 26 \end{array} \quad . \tag{8.46}$$

Defining $\tilde{\alpha}_d$, $\tilde{\beta}_d$, $\tilde{\gamma}_d$, $\tilde{\delta}_d$, $\tilde{\epsilon}_d$ just as in (8.31) and (8.32) with normalisation dictated by (8.44) then applying (8.43) to each diagram gives the relations

$$\begin{aligned} & 2\,\tilde{\beta}_d + \tilde{\alpha}_d = \tfrac{1}{3}\,, \quad \tilde{\gamma}_d + 2\,\tilde{\beta}_d{}^2 = \tfrac{1}{6}(\tilde{\beta}_d + 2\,\tilde{\alpha}_d)\,, \quad 2\,\tilde{\epsilon}_d + \tilde{\beta}_d\,\tilde{\gamma}_d = \tfrac{1}{6}\,\tilde{\beta}_d(2\,\tilde{\beta}_d + \tilde{\alpha}_d)\,, \\ & \tilde{\delta}_d + \tilde{\epsilon}_d + \tilde{\beta}_d{}^3 = \tfrac{1}{6}(\tilde{\gamma}_d + \tilde{\beta}_d{}^2 + \tilde{\alpha}_d{}^2)\,. \end{aligned} \tag{8.47}$$

Since (8.42) directly determines $\tilde{\alpha}_d$ the results are then

$\tilde{\alpha}_d$	$\tilde{\beta}_d$	$\tilde{\gamma}_d$	$\tilde{\delta}_d$	$\tilde{\epsilon}_d$
$\frac{1}{12}(n_s+2)$	$-\frac{1}{24}(n_s-2)$	$-\frac{1}{2^5.3^2}(n_s{}^2-10\,n_s-16)$	$\frac{1}{2^8.3^3}(n_s{}^3-3\,n_s{}^2+80\,n_s+100)$	$-\frac{1}{2^9.3^3}\,n_s(n_s-2)(n_s-10)$

(8.48)

For $n_s = 5, 8, 14$ these results are identical to the corresponding results obtained above apart from a change of normalisation. That the only solutions of (8.42) are given by (8.46) was demonstrated in [63] as a consequence of various bounds following from (8.42).

8.2 Fixed points

Extending to $4-\varepsilon$ dimensions the interactions become marginally relevant and the β-functions in a $\overline{MS}$ scheme take the form

$$\hat{\beta}_y{}^a = -\tfrac{1}{2}\,\varepsilon\, y^a + \beta_y{}^a\,, \qquad \hat{\beta}_\lambda{}^{abcd} = -\varepsilon\,\lambda^{abcd} + \beta_\lambda{}^{abcd}\,. \tag{8.49}$$

There are then fixed points which can be analysed in terms of an ε expansion.

In the restricted theories described previously we consider first the case where there are just two couplings y, λ. Within the examples discussed here this coresponds to requiring $a = 0$ as given in (8.28) where the various possiblities arise for $n_s = 1, 2, 3, 5, 8, 14$. At lowest order from (8.11) and (8.37), with $\tilde{n}_f = \frac{1}{2}\alpha\, n_f$ from (8.11),

$$y_*{}^2 = \tfrac{1}{2}\, Y\,\varepsilon\,, \qquad \lambda_{*\pm} = \frac{1}{2(n_s+8)}\big(1 - 2\,Y\,\tilde{n}_f \pm \sqrt{Z}\,\big)\varepsilon\,, \tag{8.50}$$

for

$$Y = \frac{1}{n_s\,\alpha + 2\,\beta + \tilde{n}_f}\,, \qquad Z = 1 + \frac{12}{n_s+2}\,\big(n_s(n_s+10)\alpha + 4\,\beta\big)\, Y^2\,\tilde{n}_f\,. \tag{8.51}$$

For n_f Dirac fermions the results for fixed points remain unchanged except $y^2 \to y\bar{y}$ and it is necessary to take in (8.50) and (8.51) $n_f \to 4n_f$. For $Z = 0$ there is a bifurcation point. This requires β to be sufficiently negative but with just the lower bound in (8.20) this is impossible and for non zero n_f $Z > 1$ so that $\lambda_{*-} < 0$ which leads to an unstable scalar potential. For

$n_f = 0$ the fixed points are just $\lambda_{*+} = \varepsilon/(n_s+8)$, $\lambda_{*-} = 0$ which reproduce the fixed points for the purely scalar $O(n_s)$ theory. From (8.50) and (8.51) to leading order for large $\tilde{n}_f$

$$y_*^{\,2} \sim \frac{1}{2\,\tilde{n}_f}\,\varepsilon\,, \qquad \lambda_{*+} \sim 6\,r\,\frac{1}{\tilde{n}_f}\,\varepsilon\,, \qquad \lambda_{*-} \sim -\frac{1}{n_s+8}\,\varepsilon\,. \tag{8.52}$$

Since $\lambda_{*-} < 0$ this case is seemingly not relevant.

The anomalous dimensions at the fixed point to lowest order are then just

$$\gamma_\phi = \tfrac{1}{2}\,\tilde{n}_f\,Y\,\varepsilon\,, \qquad \gamma_\psi = \tfrac{1}{4}\,n_s\,\alpha\,Y\,\varepsilon\,. \tag{8.53}$$

For just two couplings the stability matrix at a fixed point becomes

$$M = \begin{pmatrix} \partial_\lambda \hat{\beta}_\lambda & \partial_\lambda \beta_y \\ \partial_y \beta_\lambda & \partial_y \hat{\beta}_y \end{pmatrix}\bigg|_{\lambda=\lambda_*, y=y_*}\,. \tag{8.54}$$

At lowest order since $\partial_\lambda \beta_y = 0$ the eigenvalues obtained from (8.1) and (8.11) for the fixed points corresponding to $\lambda_{*\pm}$ are then given by

$$\kappa_\pm = (1, \pm\sqrt{Z}\,)\varepsilon\,. \tag{8.55}$$

The theories described in (8.17) and (8.24) reduce to the case where there are just two couplings λ, y when $n = 2, 3$. Theories corresponding to $n = 2$ are well known. For case 2 in (8.17) and case 3 in (8.24) there is just a single scalar and both correspond to the Gross Neveu model, a fermionic generalisation of the Ising model. For case 1 in (8.17) then $n_s = 2$ when $n = 2$ and this is a renormalisable form of Nambu Jona-Lasinio model which has complex scalars and extends the XY model. Case 3 and case 5 are identical for $n = 2$, reflecting $\mathrm{SU}(2) \simeq \mathrm{Sp}(2)$ and this extends the Heisenberg theory for $n_s = 3$. Examples corresponding to taking $n = 3$ do not seem to have been considered previously. The lowest order results taking $\tilde{n}_f \to \frac{1}{2}N$, are then given in terms of

results for	n	n_s	y_*	$\lambda_{*\pm}$
Antisymmetric	2	1	$\frac{1}{N+6}$	$\frac{6-N\pm\sqrt{N^2+132N+36}}{18(N+6)}$
	3	3	$\frac{1}{N+8}$	$\frac{8-N\pm\sqrt{N^2+160N+64}}{22(N+8)}$
Symmetric	2	2	$\frac{1}{N+4}$	$\frac{4-N\pm\sqrt{N^2+152N+16}}{20(N+4)}$
	3	5	$\frac{1}{N+8}$	$\frac{8-N\pm\sqrt{N^2+192N+64}}{26(N+8)}$
Diagonal	3	2	$\frac{1}{N+8}$	$\frac{8-N\pm\sqrt{N^2+144N+64}}{20(N+8)}$
Unitary	2	3	$\frac{1}{N+2}$	$\frac{2-N\pm\sqrt{N^2+172N+4}}{22(N+2)}$
	3	8	$\frac{1}{N+8}$	$\frac{8-N\pm\sqrt{N^2+240N+64}}{32(N+8)}$
Symplectic′	4	5	$\frac{1}{N-1}$	$\frac{-1-N\pm\sqrt{N^2+106N+1}}{26(N-1)}$
	6	14	$\frac{1}{N+4}$	$\frac{4-N\pm\sqrt{N^2+168N+16}}{44(N+4)}$

. (8.56)

Of course the $n = 2$ results are in accord in special cases with lowest order results already in the literature [15, 16, 33, 65–67].[4] Results in [15] extend to four loops. There is an extensive literature considering large n_f [33]. For the XY theory and $\tilde{n}_f = 1$ then, as is well known [31–33], there is a supersymmetric fixed point with, at lowest order $\lambda_{*+} = y_*^2 = \frac{1}{6}\,\varepsilon$, $\gamma_{\phi *} = \gamma_{\psi *} = \frac{1}{6}\,\varepsilon$, $\gamma_{\sigma *} = \varepsilon$, $\gamma_{\rho *} = \frac{2}{3}\,\varepsilon$ and stability matrix eigenvalues $(1,3)\varepsilon$. For the Ising case and $\tilde{n}_f = \frac{1}{2}$ there is an apparent $\mathcal{N} = 1$ supersymmetric fixed point [31–34] with $\lambda_{*+} = y_*^2 = \frac{1}{7}\,\varepsilon$, $\gamma_{\phi *} = \gamma_{\psi *} = \frac{1}{14}\,\varepsilon$, $\gamma_{\sigma *} = \frac{4}{7}\,\varepsilon$ and stability matrix eigenvalues $(1, \frac{13}{7})\varepsilon$.

For three non zero couplings results are more involved. At lowest order from (8.11) fixed points are determined by solving

$$\begin{aligned}
0 =& \; -\tfrac{1}{2}\,\varepsilon\, y + (n_s\,\alpha + 2\,\beta + \tilde{n}_f)y^3\,, \\
0 =& \; (-\varepsilon + 4\,\tilde{n}_f\, y^2)\lambda + (n_s+8)\lambda^2 + \tfrac{2}{3}(2\,n_s\,\alpha + \beta)\lambda\, g + 3\,A\, g^2\,, \\
0 =& \; (-\varepsilon + 4\,\tilde{n}_f\, y^2) g + 12\,\lambda\, g + 3\,B\, g^2 - 24\,\tilde{n}_f\, y^4\,,
\end{aligned} \tag{8.57}$$

for the various choices of α, β and n_s. The example of hermitian y^a was considered in [68] though our results differ in one term in β_λ. A large N analysis for Yukawa couplings given in terms of Lie algebra generators was discussed in [69]. For the purely scalar theories obtained when $n_f = 0$ there are no fixed points with both λ, g non zero for the theories discussed here for allowed n except in case 4 when $n = 4$ and in case 2 for arbitrary n. This latter case corresponds to the hypertetrahedral theory discussed in [70] and more recently in [71]. Besides the Gaussian fixed point with vanishing couplings there is of course always the Heisenberg fixed point with, at lowest order,

$$\lambda_{*H} = \frac{1}{n_s+8}\,\varepsilon\,, \quad g_{H*} = 0\,, \quad (\kappa_{1H}, \kappa_{2H}) = \Big(1,\, \tfrac{4-n_s}{n_s+8}\Big)\varepsilon\,, \tag{8.58}$$

which clearly becomes unstable, in this approximation when $n_s > 4$.

For a non zero Yukawa coupling it is trivial to solve (8.57) to determine y_*^2 to $\mathrm{O}(\varepsilon)$. Furthermore λ appears only linearly in the β_g equation so that the fixed point equations reduce to just finding the roots of a quartic polynomial $f(g)$. In consequence there are generically four possible roots though of course these may be complex. Once y_*^2 is eliminated the equations (8.57) have a crucial symmetry under

$$\tilde{n}_f \to \frac{(n_s\,\alpha + 2\,\beta)^2}{\tilde{n}_f}\,, \qquad \varepsilon \to -\varepsilon\,. \tag{8.59}$$

This relates solutions for large and small $\tilde{n}_f$. Under this transformation it is easy to verify from (8.50) and (8.51) that, since $Y \to \tilde{n}_f Y/(n_s\,\alpha + 2\beta)$, $\lambda_{*+} \leftrightarrow \lambda_{*-}$.

When $\tilde{n}_f$ is small the Yukawa interaction is weakly relevant. The Gaussian fixed point is perturbed to give

$$g_* \sim -\frac{6}{(n_s\,\alpha + 2\beta)^2}\,\tilde{n}_f\,\varepsilon + \frac{108\,B}{(n_s\,\alpha + 2\beta)^4}\,\tilde{n}_f^{\,2}\,\varepsilon\,, \qquad \lambda_* \sim \frac{108\,A}{(n_s\,\alpha + 2\beta)^4}\,\tilde{n}_f^{\,2}\,\varepsilon\,, \tag{8.60}$$

[4] The results in [15] correspond to those described here by taking $\lambda \to \lambda/4$, $y \to 2y^2$, $n \to \frac{1}{2}\tilde{n}_f$ and $\gamma_\phi/2 \to \gamma_\phi$, $\gamma_\psi/2 \to \gamma_\psi$, $\beta_y/4 \to y\beta_y$, $4\,\beta_\lambda \to \beta_\lambda$. The results in [33] also relate to those here by taking $g_2/(4\pi)^2 \to 3\lambda$, $g_1/4\pi \to y$, $N \to 2\tilde{n}_f$ while $(4\pi)^2\beta_{g_2}/3 \to \beta_\lambda$, $4\pi\beta_{g_1} \to \beta_y$.

and starting from the $O(n_s)$ symmetric fixed point

$$g_* \sim \frac{6(n_s+8)}{(n_s-4)(n_s\,\alpha+2\beta)^2}\,\tilde{n}_f\,\varepsilon\,, \qquad \lambda_* \sim \frac{1}{n_s+8}\,\varepsilon + \frac{6\big(n_s(n_s+12)\alpha+8\beta\big)}{(n_s-4)(n_s+8)(n_s\,\alpha+2\beta)^4}\,\tilde{n}_f\,\varepsilon\,. \tag{8.61}$$

If there are additional fixed point solutions for $\tilde{n}_f = 0$, as in the special cases described above, these disappear for very tiny non zero $\tilde{n}_f$.

When $\tilde{n}_f$ is large a corresponding pattern related by (8.59) emerges. For λ, g both of O(1) as $n_f \to \infty$, then since $4\,\tilde{n}_f\,y_*^2 \to 2\varepsilon$, it is easy to see that the fixed point equations are, to leading order, of the same form as the purely scalar theory but with $\varepsilon \to -\varepsilon$. For $\varepsilon > 0$ this leads in general to a scalar potential which is not bounded below and in any event there are no solutions except in the cases described above. For large n_f there are solutions with both g, λ small which have the form

$$y_*{}^2 \sim \frac{1}{2\,\tilde{n}_f}\,\varepsilon\,, \qquad g_* \sim 6\,\frac{1}{\tilde{n}_f}\,\varepsilon - 108\,B\,\frac{1}{\tilde{n}_f{}^2}\,\varepsilon\,, \qquad \lambda_* \sim -108\,A\,\frac{1}{\tilde{n}_f{}^2}\,\varepsilon\,, \tag{8.62}$$

where the scalar potential is bounded below, and also for $\lambda = \mathrm{O}(1)$

$$g_* \sim 6\,\frac{n_s+8}{n_s-4}\frac{1}{\tilde{n}_f}\,\varepsilon\,, \qquad \lambda_* \sim -\frac{1}{n_s+8}\,\varepsilon\,. \tag{8.63}$$

The second solution leads to instabilities so only (8.62) remains as a valid possibility.

For intermediate $\tilde{n}_f$ the possible fixed points depend on n. For lowish $n \lesssim 7$ the number of solutions drops to zero as $\tilde{n}_f$ increases and then goes back to two (for case 2 this happens if $n < 5$ and for case 6 if $n \le 14$). For higher n the number of solutions jumps from 2 to 4 with increasing $\tilde{n}_f$ and then reverts to two which match on to (8.62) and (8.63) for large $\tilde{n}_f$ (for case 2 and case 4 if $n = 4$ there are four solutions for very large $\tilde{n}_f$ and very tiny $\tilde{n}_f$ as the purely scalar theories have fixed point in these cases). The critical n dividing the two cases is determined by $d(n_c) = 0$ for $d(n) = \frac{4}{9}(2n_s\alpha+\beta)^2 - 12(n_s+8)A$. For $d(n) < 0$ as happens for $n < n_c$ then $h(\lambda, g) = (n_s+8)\lambda^2 + \frac{2}{3}(2n_s\alpha+\beta)\lambda g + 3A\,g^2 > 0$. For $4\,\tilde{n}_f\,y_*^2 = \varepsilon$, or $n_f = n_s\alpha + 2\beta$, the lowest order fixed point equations require $h(\lambda_*, g_*) = 0$. This ensures that there can be no fixed point solutions for a finite region $n \lesssim n_c$. Conversely for $n \gtrsim n_c$ $h(\lambda, g)$ is no longer positive definite and there are solutions with g non zero. In consequence for $n \approx n_c$ and $\tilde{n}_f \approx n_{f,c} = (n_s\alpha + 2\beta)|_{n=n_c}$ there are either 0 or 4 solutions For the symmetric case $n_c = 6$ and $\tilde{n}_{f,c} = 8$.[5] For the other cases the results for $(n_c,\, n_{f,c})$ are then 2. $(4.37, 2.31)$, 3. $(6.58, 12.24)$, 4. $(7.37, 8.37)$, 5. $(6.37, 5.37)$, 6. $(14.11, 10.69)$. The jumps are associated with bifurcation points which correspond to there being two coincident roots of the polynomial $f(g)$, or that its discriminant vanishes. The boundaries of the regions where there are jumps from 2 to 0 or 2 to 4 correspond to $\tilde{n}_f$ linked by (8.59) though the fixed point couplings have opposite signs. At the fixed points in general the couplings do not give potentials which are bounded below except for one which matches (8.62) when $\tilde{n}_f$ is large. The positivity condition remains satisfied as $\tilde{n}_f$ is reduced until just above the upper bifurcation point.

Diagrams showing the structure of fixed point solutions outlined above are presented in appendix C.

[5] For $n = 6$, $\tilde{n}_f = 8$ the fixed point solutions become $y_*{}^2 = \frac{1}{32}\varepsilon$, $\lambda_* = \pm\frac{1}{24}\varepsilon$, $g_* = \mp\frac{1}{4}\varepsilon$. In neither case are the conditions for a stable potential satisfied.

For the stability matrix eigenvalues the absence of g, λ contributions to the Yukawa β-function at lowest order ensures that one eigenvalue is ε for any $n, \tilde{n}_f$ and the remaining eigenvalues are obtained from

$$M = \begin{pmatrix} \partial_\lambda \hat{\beta}_\lambda & \partial_g \beta_\lambda \\ \partial_\lambda \beta_g & \partial_g \hat{\beta}_g \end{pmatrix} \Big|_{\lambda=\lambda_*, g=g_*, y=y_*} . \tag{8.64}$$

For $\tilde{n}_f$ small from (8.60)

$$\kappa_\pm \sim -\varepsilon - \Big(9B + n_s\alpha - \beta \pm \sqrt{(9B - 2\,n_s\alpha - \beta)^2 + 648A}\Big) \frac{2\,\tilde{n}_f}{(n_s\,\alpha + 2\,\beta)^2}\,\varepsilon\,. \tag{8.65}$$

Starting from (8.61) the eigenvalues are

$$\kappa_1 \sim -\frac{n_s - 4}{n_s + 8}\,\varepsilon + \mathrm{O}(\tilde{n}_f)\,, \qquad \kappa_2 \sim \varepsilon + \frac{6(n_s(n_s+10)\alpha + 4\beta)}{(n_s+2)(n_s\alpha + 2\,\beta)^2}\,\tilde{n}_f\,\varepsilon\,. \tag{8.66}$$

Otherwise for large $\tilde{n}_f$ from the fixed point (8.62)

$$\kappa_\pm \sim \varepsilon + \Big(9B + n_s\alpha - \beta \pm \sqrt{(9B - 2\,n_s\alpha - \beta)^2 + 648A}\Big) \frac{2}{\tilde{n}_f}\,\varepsilon\,. \tag{8.67}$$

For the supersymmetric case using (8.40) there are possible fixed points in the ε expansion if $4\beta_d + 3\alpha_d > 0$. From the lower bound on β_d this is satisfied whenever $n_s > 2$.

8.3 U(1) symmetry

A similar reduction is possible for complex scalar fields where there is at least an overall U(1) symmetry. In this case we consider chiral fermions ψ and χ of opposite chirality which need not be equal in number. Writing

$$y^i = y\,\mathbb{1}_m t^i\,, \qquad \bar{y}_i = \bar{y}\,\mathbb{1}_m \bar{t}_i\,, \qquad \bar{t}_i = (t^i)^\dagger\,, \tag{8.68}$$

then t^i, $i = 1, \dots, n$, need not be a square matrix but is assumed to be $r \times s$. In this case we assume

$$U_-^{-1} t^i\, U_+ = R^i{}_j\, t^j\,, \quad U_- \in \mathrm{U}(r)\,,\; U_+ \in \mathrm{U}(s)\,, \quad [R^i{}_j] \in H \subset \mathrm{U}(n)\,. \tag{8.69}$$

Corresponding to the previous discussion we assume, with a choice of normalisation for t^i,

$$t^i\,\bar{t}_i = s\,\mathbb{1}_r\,, \qquad \bar{t}_i\,t^i = r\,\mathbb{1}_s\,, \qquad \mathrm{tr}(t_i\,\bar{t}^j) = rs/n\,\delta_i{}^j\,. \tag{8.70}$$

Defining

$$\mathrm{tr}\big(\bar{t}_{(i}\,t^k\,\bar{t}_{j)}\,t^l\big) = h_{ij}{}^{kl}\,, \qquad h_{ik}{}^{jk} = \tfrac{1}{2}\,rs(r+s)/n\,\delta_i{}^j\,, \tag{8.71}$$

the scalar coupling is assumed to have the form[6]

$$\lambda_{ij}{}^{kl} = \lambda(\delta_i{}^k\delta_j{}^l + \delta_i{}^l\delta_j{}^k) + g\,h_{ij}{}^{kl}\,. \tag{8.72}$$

[6]The necessary conditions for a positive potential can be obtained from

$$\big(\mathrm{tr}(\bar{t}\,t)\big)^2 \big/ \min(r,s) \leq \mathrm{tr}(\bar{t}\,t\,\bar{t}\,t) \leq \big(\mathrm{tr}(\bar{t}\,t)\big)^2\,,$$

where t is a $r \times s$ complex matrix and $\bar{t}$ its hermitian conjugate.

As above we assume conditions are imposed such that β-functions determining the RG flow are reduced to β_y, β_λ, β_g with $\beta_{\bar y} = (\beta_y)^*$ and the anomalous dimension matrices become $\gamma_\varphi \mathbb{1}_n, \gamma_\psi \mathbb{1}_r, \gamma_\chi \mathbb{1}_s$ with $\gamma_\varphi = \gamma_{\bar\varphi}$. Quadratic operators are decomposed as

$$\varphi_i \bar\varphi^j = \sigma\, \delta_i{}^j + \rho_i{}^j \,, \quad \rho_i{}^i = 0 \,, \quad \varphi_i \varphi_j = (\varphi^2)_{ij} \,, \tag{8.73}$$

and the corresponding anomalous dimension matrices are then of the general form

$$\begin{aligned} &\gamma_\sigma\, \delta_i{}^j \,, \qquad \gamma_\rho{}_i{}^j{}_k{}^l = \gamma_\rho \left(\delta_i{}^l \delta_k{}^j - \tfrac{1}{n} \delta_i{}^j \delta_k{}^l\right) + \gamma'_\rho\, d_{ik}{}^{jl} + \gamma''_\rho\, w_{ik}{}^{jl} \,, \\ &\gamma_{\varphi^2}{}_{ij}{}^{kl} = \gamma_{\varphi^2} \tfrac{1}{2} (\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k) + \gamma'_{\varphi^2}\, d_{ij}{}^{kl} \,, \end{aligned} \tag{8.74}$$

where

$$\begin{aligned} h_{ij}{}^{kl} &= q(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k) + d_{ij}{}^{kl} \,, \qquad (n+1)q = \tfrac{1}{2}\, rs(r+s)/n \,, \\ \mathrm{tr}(\bar t_{[i}\, t^k\, \bar t_{j]}\, t^l) &= p(\delta_i{}^k \delta_j{}^l - \delta_i{}^l \delta_j{}^k) + w_{ij}{}^{kl} \,, \qquad (n-1)p = \tfrac{1}{2}\, rs(r-s)/n \,, \end{aligned} \tag{8.75}$$

so that both $d_{ij}{}^{kl}$ and $w_{ij}{}^{kl}$ are zero on contraction of up and down indices. Instead of (8.72)

$$\lambda_{ij}{}^{kl} = \hat\lambda(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k) + g\, d_{ij}{}^{kl} \,, \qquad \hat\lambda = \lambda + q\, g \,. \tag{8.76}$$

The eigenvalues and corresponding degeneracies for the d and w tensors as they appear in (8.74) are given in appendix B.

At one loop it is sufficient to impose the conditions

$$\begin{aligned} h_{ij}{}^{mn} h_{mn}{}^{kl} &= \tilde A(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k) + \tilde B\, h_{ij}{}^{kl} \,, \\ h_{m(i}{}^{n(k} h_{j)n}{}^{l)m} &= \tilde A'(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k) + \tilde B'\, h_{ij}{}^{kl} \,. \end{aligned} \tag{8.77}$$

Hence

$$\begin{aligned} \beta_y{}^{(1)} &= \tfrac{1}{2}(r + s + m\, rs/n) y^2 \bar y \,, \\ \beta_\lambda{}^{(1)} &= 2(n+4)\lambda^2 + 4(n+1) q\, \lambda\, g + (\tilde A + 4\tilde A') g^2 + 2\, m\, rs/n\, \lambda\, y\bar y \,, \\ \beta_g{}^{(1)} &= 12\, \lambda\, g + (\tilde B + 4\tilde B') g^2 + 2\, m\, rs/n\, g\, y\bar y - 4\, m\, (y\bar y)^2 \,. \end{aligned} \tag{8.78}$$

A necessary condition is

$$h_{ik}{}^{mn} h_{mn}{}^{kj} = (n+1)(\tilde A + q\, \tilde B) \delta_i{}^j = \left(2(n+1)(\tilde A' + q\, \tilde B') - (n+1)^2 q^2\right) \delta_i{}^j \,. \tag{8.79}$$

Defining

$$\tilde a = \tilde A + q\, \tilde B - 2q^2 \,, \qquad d_{ik}{}^{mn} d_{mn}{}^{kj} = (n+1) \tilde a\, \delta_i{}^j \,, \tag{8.80}$$

then the one loop scalar β-function can be alternatively expressed as

$$\beta_{\hat\lambda}{}^{(1)} = 2(n+4)\, \hat\lambda^2 + 3\, \tilde a\, g^2 + m\, rs/n\, \hat\lambda\, y\bar y - 4\, m\, q\, (y\bar y)^2 \,. \tag{8.81}$$

For $\tilde a = 0$ $d_{ij}{}^{kl} = 0$ and $h_{ij}{}^{kl} = q(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k)$ so is no longer independent and the g coupling is redundant.

This framework encompasses a variety of theories discussed in the literature. As an illustration we may consider $\{t^a\}$ to be a basis for $r \times s$ complex matrices where $n = rs$ and which satisfy (8.70). The scalar field symmetry group is $H = \mathrm{U}(r) \times \mathrm{U}(s)/\mathrm{U}(1)$. Positivity of the potential holds if

$$\lambda > 0\,, \quad 2\,\lambda + g > 0 \quad \text{or} \quad g > 0\,, \quad 2\,\lambda + g/\min(r,s) > 0\,. \tag{8.82}$$

This example requires (8.77)

$$\tilde{A} = \tfrac{1}{2}\,, \qquad \tilde{B} = 0\,, \qquad \tilde{A}' = \tfrac{1}{4}\,, \qquad \tilde{B}' = \tfrac{1}{4}(r+s)\,. \tag{8.83}$$

In this case from (8.80) and (8.75)

$$\tilde{a} = \tfrac{(r^2-1)(s^2-1)}{2(n+1)^2}\,, \qquad q = \tfrac{r+s}{2(n+1)}\,. \tag{8.84}$$

For the purely scalar theory, without fermions, results for β-functions have been obtained to five loops in [72] and more recently to six loops in [5, 73] and a bootstrap analysis has been undertaken in [74].

The lowest order β-functions and anomalous dimensions are then, with $n = rs$,

$$\begin{aligned}
&\beta_y{}^{(1)} = \big(\tfrac{1}{2}(r+s+m)y\bar{y}\big)\,y\,, \quad \beta_\lambda{}^{(1)} = 2(n+4)\lambda^2 + 2(r+s)\lambda\,g + \tfrac{3}{2}\,g^2 + 2m\,\lambda\,y\bar{y}\,,\\
&\beta_g{}^{(1)} = 12\,g\,\lambda + (r+s)\,g^2 + 2m\,g\,y\bar{y} - 4m\,(y\bar{y})^2\,,\\
&\gamma_\varphi{}^{(1)} = \tfrac{1}{2}m\,y\bar{y}\,, \qquad \gamma_\psi{}^{(1)} = \tfrac{1}{2}r\,y\bar{y}\,, \qquad \gamma_\chi{}^{(1)} = \tfrac{1}{2}s\,y\bar{y}\,,
\end{aligned} \tag{8.85}$$

and in (8.74)

$$\begin{aligned}
&\gamma_\sigma{}^{(1)} = 2(n+1)\,\hat{\lambda} + m\,y\bar{y}\,, \qquad \gamma_\rho{}^{(1)} = \gamma_{\varphi^2}{}^{(1)} = 2\,\hat{\lambda} + m\,y\bar{y}\,,\\
&\gamma'_{\varphi^2}{}^{(1)} = \gamma'_\rho{}^{(1)} = 2\,g\,, \qquad \gamma''_\rho{}^{(1)} = 0\,.
\end{aligned} \tag{8.86}$$

Setting $s = 1$, $r = n$, so that $\tilde{a} = 0$ and the d-tensor vanishes, the β-function reduces to

$$\beta_{\hat{\lambda}}{}^{(1)} = 2(n+4)\,\hat{\lambda}^2 + 2m\,\hat{\lambda}\,y\bar{y} - 2m\,(y\bar{y})^2\,. \tag{8.87}$$

For $n = 1$ this coincides with the Nambu Jona-Lasinio extended XY model, with U(1) symmetry when $m = 1$, so long as $y\bar{y} \to 2\,y^2$.

At higher orders further relations corresponding to primitive diagrams are necessary. The primitive Yukawa diagrams $2a$, $2f$ and scalar diagram $2g$ correspond to

$$t^j\bar{t}_k t^l\,d_{jl}{}^{ki} = (n+1)\tilde{a}\,t^i\,, \quad t^j\bar{t}_k t^i\bar{t}_j t^k = t^i\,, \quad \mathrm{tr}\big(\bar{t}_{(i}t^m\bar{t}_{j)}\,t^{(k}\bar{t}_m t^{l)}\big) = \tfrac{1}{2}\big(\delta_i{}^k\delta_j{}^l + \delta_i{}^l\delta_j{}^k\big)\,. \tag{8.88}$$

The two loop 1PI contributions to the Yukawa and scalar β-functions are, with $\tilde{a}$, q as in (8.84), then

$$\begin{aligned}
\tilde{\beta}_y{}^{(2)} = {}& \big(-2(r+s)\,\hat{\lambda}\,y\bar{y} - 2(n+1)\,\tilde{a}\,g\,y\bar{y} + 2\,(y\bar{y})^2\big)\,y\,,\\
\beta_{\hat{\lambda}}{}^{(2)} = {}& -4(5n+11)\hat{\lambda}^3 - 6(n+7)\,\tilde{a}\,g^2\hat{\lambda} - 4(n-5)q\,\tilde{a}\,g^3 - 2(n+4)m\,\hat{\lambda}^2\,y\bar{y} - 3\,\tilde{a}\,m\,g^2\,y\bar{y}\\
&+ 8\,q\,m\,\hat{\lambda}\,(y\bar{y})^2 + 4\,\tilde{a}\,m\,g\,(y\bar{y})^2 + 4\big((r+s)q+1\big)m\,(y\bar{y})^3\,,\\
\tilde{\beta}_g{}^{(2)} = {}& -12(n+7)\,g\,\hat{\lambda}^2 - 2(n-5)q\,g^2(12\,\hat{\lambda} + m\,y\bar{y}) - 2\big(n+4-18(n-1)q^2\big)g^3\\
&- 12\,m\,g\,\hat{\lambda}\,y\bar{y} + 8\,m\,\hat{\lambda}\,(y\bar{y})^2 - 8\,q\,m\,g\,(y\bar{y})^2 + 4(r+s)m\,(y\bar{y})^3\,.
\end{aligned} \tag{8.89}$$

At two loops the anomalous dimensions are given by

$$\begin{aligned}
\gamma_\varphi{}^{(2)} &= \tfrac{1}{2}(n+1)(\hat\lambda^2 + \tfrac{1}{2}\,\tilde a\, g^2) - \tfrac{3}{8}\, m\,(r+s)\,(y\bar y)^2\,,\\
\gamma_\psi{}^{(2)} &= -\,\tfrac{1}{8}\, r(s+3m)\,(y\bar y)^2\,, \qquad \gamma_\chi{}^{(2)} = -\tfrac{1}{8}\, s(r+3m)\,(y\bar y)^2\,,\\
\gamma_\sigma{}^{(2)} &= -\,6(n+1)(\hat\lambda^2 + \tfrac{1}{2}\,\tilde a\, g^2) - 2(n+1)\, m\,\hat\lambda\, y\bar y + 2\,\gamma_\varphi{}^{(2)}\,,\\
\gamma_{\varphi^2}{}^{(2)} &= -\,2(n+3)\,\hat\lambda^2 - 2\,\tilde a\, g^2 - 2\, m\,\hat\lambda\, y\bar y + 4\, q\, m\,(y\bar y)^2 + 2\,\gamma_\varphi{}^{(2)}\,,\\
\gamma'_{\varphi^2}{}^{(2)} &= -\,8\,\hat\lambda\, g - 2(n-3)q\, g^2 - 2\, m\, g\, y\bar y + 4\, m\,(y\bar y)^2\,,\\
\gamma_\rho{}^{(2)} &= -\,2(n+3)\,\hat\lambda^2 - \tfrac{n-3}{n-1}\,\tilde a\, g^2 - 2\, m\,\hat\lambda\, y\bar y + 2\,\gamma_\varphi{}^{(2)}\,,\\
\gamma'_\rho{}^{(2)} &= -\,8\,\hat\lambda\, g - (n-7)q\, g^2 - 2\, m\, g\, y\bar y\,, \qquad \gamma''_\rho{}^{(2)} = -\tfrac{1}{2}\,(r-s)\, g^2\,.
\end{aligned} \tag{8.90}$$

At three loops further relations are necessary. Corresponding to $3\tilde z$

$$t^j\bar t_k t^l \bar t_j t^i \bar t_l t^k + t^k \bar t_l t^i \bar t_j t^l \bar t_k t^j = (r+s)\, t^i\,. \tag{8.91}$$

For just the quartic scalar coupling at three loops then corresponding to $3f$ there is the relation

$$\begin{aligned}
&d_{m(i}{}^{nr}\, d_{j)r}{}^{pq}\, d_{ps}{}^{m(k}\, d_{nq}{}^{l)s} = \mathcal{A}(\delta_i{}^k\delta_j{}^l + \delta_i{}^l\delta_j{}^k) + \tilde c\, d_{ij}{}^{kl}\,,\\
&\mathcal{A} = \tfrac{1}{8}(n+4-18\,q^2)\,\tilde a\,, \qquad \tilde c = -\tfrac{1}{8}\,(5\,n+29)\,q + 2(5\,n-1)\,q^3\,.
\end{aligned} \tag{8.92}$$

For the purely scalar theory obtained by setting the Yukawa couplings to zero there are non trivial fixed points which to lowest order have the form, for $r, s > 1$,

$$g_{*\pm} = \frac{n-2}{D_{rs} \pm 3\sqrt{R_{rs}}}\,\varepsilon\,, \qquad \hat\lambda_{*\pm} = \frac{D_{rs} \pm (n+1)\sqrt{R_{rs}}}{4(n+1)(D_{rs} \pm 3\sqrt{R_{rs}})}\,\varepsilon\,, \tag{8.93}$$

for

$$D_{rs} = (n-5)(r+s)\,, \qquad R_{rs} = r^2 + s^2 - 10\, rs + 24\,. \tag{8.94}$$

Since $R_{rs} = (r - n_+(s))(r - n_-(s))$ for $n_\pm(s) = 5s \pm 2\sqrt{6}\sqrt{s^2-1}$ then $R_{rs} \geq 0$ and there are real fixed points if $r \geq n_+(s)$ or $r \leq n_-(s)$. For $r > s$ only the first case is relevant. The corresponding stability matrix eigenvalues at the fixed points in (8.93) are then

$$(\kappa_{1\pm},\, \kappa_{2\pm}) = \Big(1, \pm\tfrac{(n-2)\sqrt{R_{rs}}}{D_{rs}\pm 3\sqrt{R_{rs}}}\Big)\varepsilon\,. \tag{8.95}$$

Integer solutions for the bifurcation points when $R_{rs} = 0$ can be obtained iteratively, for $r > s$, by taking $r_i = 10\, r_{i-1} - s_{i-1},\ s_i = r_{i-1}$ starting from $r_1 = 5,\ s_1 = 1$. This scalar theory with $H = \mathrm{U}(r) \times \mathrm{U}(s)/\mathrm{U}(1)$ symmetry is an obvious extension of the bifundamental theory with real scalars and $O(m) \times O(n)/\mathbb{Z}_2$ symmetry discussed recently in [71, 75] which contains earlier citations. Defining the invariants

$$||\lambda||^2 = \lambda_{ij}{}^{kl}\lambda_{kl}{}^{ij} = 2n(n+1)(\hat\lambda^2 + \tfrac{1}{2}\,\tilde a\, g^2)\,, \qquad |\lambda_{*\pm}| = \lambda_{ij}{}^{ij} = n(n+1)\,\hat\lambda\,, \tag{8.96}$$

then at the fixed point (8.93)

$$||\lambda_{*\pm}||^2 = \tfrac{1}{24}\, n(\varepsilon^2 - \kappa_{2\pm}{}^2)\,, \qquad |\lambda_{*\pm}| = \tfrac{1}{4}\, n(\varepsilon + \kappa_{2\pm})\,. \tag{8.97}$$

Clearly $||\lambda_{*\pm}||^2 \leq \frac{1}{24}\, n\, \varepsilon^2$ in accord, of course, with the bound obtained by Hogervorst and Toldo [76] extending the results in [75]. For any n there is also the Heisenberg fixed point with $O(2n)$ symmetry where

$$\hat{\lambda}_{*H} = \frac{1}{2(n+4)}\,\varepsilon \qquad g_{*H} = 0\,, \tag{8.98}$$

and

$$(\kappa_{1H}, \kappa_{2H}) = \Big(1, \tfrac{2-n}{n+4}\Big)\varepsilon\,, \quad ||\lambda_{*H}||^2 = \tfrac{n(n+1)}{2(n+4)^2}\,\varepsilon^2\,, \quad |\lambda_{*H}| = \tfrac{n(n+1)}{2(n+4)}\,\varepsilon\,. \tag{8.99}$$

For $n = 2$ these results coincide with (8.93), (8.95) and (8.97).

For non zero Yukawa couplings the one loop β function requires

$$(y\bar{y})_* = \varepsilon/(r+s+m)\,. \tag{8.100}$$

After using (8.100) the equations determining λ_*, g_* are then invariant under

$$m \to \frac{(r+s)^2}{m}\,, \qquad \varepsilon \to -\varepsilon\,, \tag{8.101}$$

relating results for large and small m. At $m = m_c$ for $m_c = r + s$ there are no solutions if $2(n+4)\lambda^2 + 2(r+s)\lambda g + \frac{3}{2}g^2 > 0$ or

$$2n \leq r^2 + s^2 < n + 12\,. \tag{8.102}$$

This is rather restrictive. Taking $r, s \geq 2$ and $r \leq s$ the only possibilities for (r,s) are just $(2,2)$, $(2,3)$, $(3,3)$, $(2,4)$ (in the last case $r^2 + s^2 = n + 12$). In these cases where $R_{rs} < 0$ there are two fixed points for small and large m but none over some interval centred on $r + s$, the interval shrinks to zero in the $(2,4)$ case where the bound in (8.102) is saturated. Otherwise for $R_{rs} < 0$ there are again two fixed points for small and large m but four over a region centred on $r + s$. If $R_{rs} \geq 0$ there are four fixed points for any m.

At large m there is a fixed point which is the counterpart of the Gaussian fixed point for small m and gives rise to a positive semi-definite potential and positive stability matrix eigenvalues

$$g_* \sim \frac{4}{m}\,\varepsilon - (r+s)\frac{16}{m^2}\,\varepsilon\,, \quad \lambda_* \sim -\frac{24}{m^2}\,\varepsilon\,, \quad \kappa_\pm \sim \varepsilon + \big(r+s\pm 4\big)\frac{6}{m}\,\varepsilon\,. \tag{8.103}$$

For $s = 1$ and $r = n$ there is just one scalar coupling and at lowest order

$$\hat{\lambda}_{*\pm} = \frac{1}{4(n+4)}\Big(-\frac{m-n-1}{m+n+1} \pm \sqrt{\tilde{Z}}\Big)\varepsilon\,, \quad \tilde{Z} = \frac{(m-n-1)^2 + 16(n+4)m}{(m+n+1)^2}\,. \tag{8.104}$$

For $n = 1$ this is identical to the XY case as given by (8.50) and (8.51).

9 Consistency relations

The existence of an a-function requires consistency relations between the coefficients for individual non primitive graphs appearing in the β-functions and the anomalous dimensions. The basic equation has the form, for couplings $\{g^I\}$ [20],

$$\partial_I A = T_{IJ}\, B^J\,, \tag{9.1}$$

where A is constructed from 1PI and 1VI vacuum diagrams, T_{IJ} also from 1PI, 1VI vacuum diagrams with two vertices identified and

$$B^I = \beta^I - (\upsilon g)^I \,, \tag{9.2}$$

with $\upsilon(g)$ corresponding to an element of the Lie algebra of the maximal symmetry group of the Lagrangian kinetic term. In general for a vanishing trace of the energy momentum tensor and hence conformal symmetry the requirement is that $B^I = 0$. In (9.1) T_{IJ} need not be symmetric, although any antisymmetric part has further constraints. In (9.2) $\upsilon(g)$ is necessarily present starting at 3 loops. There is a freedom in (9.1) where

$$A \sim A + g_{IJ} B^I B^J \,, \qquad T_{IJ} \sim T_{IJ} + \mathcal{L}_B \, g_{IJ} + \partial_I (g_{JK} B^K) - \partial_J (g_{IK} B^K) \,, \tag{9.3}$$

for arbitrary symmetric g_{IJ}. This does not preserve the symmetry of T_{IJ}.

In the present context the lowest order contribution to T_{IJ} is first present at two loops for the Yukawa couplings, T_{yy}, and at three for the scalar quartic couplings, $T_{\lambda\lambda}$. In consequence (9.1) provides potential relations between the Yukawa β-functions and fermion, scalar anomalous dimensions at ℓ loops and the scalar coupling β-function at $\ell - 1$ loops where A involves $\ell + 2$ loop diagrams. Eliminating A and T_{IJ} ensures that the relations contain non linear contributions involving the β-functions and anomalous dimensions at lower loop order. The elimination of any particular contribution to A is possible when the relevant diagram is not vertex transitive. If the diagram has n inequivalent vertices then (9.1) leads to n independent equations in this case. For $\ell = 2, 3$, and including also arbitrary gauge couplings, the possible relations were exhaustively analysed by Poole and Thomsen [9]. At this order the conditions relate contributions to the β-functions and anomalous dimensions which have insertions of one loop triangles and one loop bubbles.

For $\ell = 2$ there are 11 5-loop vacuum diagrams for A (3 are vertex transitive) and 9 possible 3 loop T_{IJ}, all of which are symmetric, and (9.1) gives rise to 21 equations. Nevertheless there are 4 conditions on the individual β-function coefficients which reduce to the vanishing of

$$\begin{aligned}
&B_1 = \gamma_{\phi 1} \, \beta_{y2a} - 3 \, \beta_{\lambda 1b} \, \gamma_{\phi 2a} \,, \qquad B_2 = 2 \, \gamma_{\phi 1} \, \beta_{y2c} + 2 \, \gamma_{\psi 1} \, \gamma_{\phi 2c} - \beta_{y1} \, \gamma_{\phi 2b} \,, \\
&B_3 = {\beta_{y1}}^2 \, \gamma_{\psi 2b} - \beta_{y1} \gamma_{\psi 1} \, (\beta_{y2c} + \gamma_{\psi 2c}) - {\gamma_{\psi 1}}^2 \, (\beta_{y2d} - \beta_{y2e}) \,, \\
&B_4 = {\beta_{y1}}^2 \, \gamma_{\psi 2a} - \beta_{y1} (\gamma_{\psi 1} \, \beta_{y2b} + \gamma_{\phi 1} \, \gamma_{\psi 2c}) + \gamma_{\phi 1} \gamma_{\psi 1} (\beta_{y2d} - \beta_{y2e}) \,,
\end{aligned} \tag{9.4}$$

or inserting one loop results this gives the conditions

$$\begin{gathered}
\beta_{y2a} = -24 \, \gamma_{\phi 2a} \,, \quad \beta_{y2c} = 2 \, \gamma_{\phi 2b} - \gamma_{\phi 2c} \,, \quad 4 \, \beta_{y2c} + \beta_{y2d} - \beta_{y2e} = 16 \, \gamma_{\psi 2a} - 4 \, \gamma_{\psi 2c} \,, \\
\beta_{y2b} + \beta_{y2c} = 4 \, (\gamma_{\psi 2a} + \gamma_{\psi 2b}) - 2 \, \gamma_{\psi 2c} \,.
\end{gathered} \tag{9.5}$$

The non planar β_{y2f} is not present since the associated vacuum graph obtained by joining the external lines is vertex transitive.

For $\ell = 3$ there are, for a general renormalisable fermion scalar theory 49 5 loop diagrams for A (6 are vertex transitive) and for T_{yy} there are 33 distinct contributions for T_{yy} which are symmetric and 20 with no symmetry. (9.1) then generates 152 equations which reduce to

42 conditions on the β-function, anomalous dimension coefficients. We consider first relations for non planar contributions to the β-function and anomalous dimension where there are 7 relations due to the vanishing of

$$
\begin{aligned}
&C_1 = \beta_{y3\tilde{w}} - \beta_{y3\tilde{x}}\,, \qquad C_2 = \beta_{y1}\big(\beta_{y3\tilde{o}} - \beta_{y3\tilde{p}}\big) - \gamma_{\psi 1}\big(\beta_{y3\tilde{s}} - \beta_{y3\tilde{u}}\big)\,,\\
&C_3 = \beta_{y1}\big(\beta_{y3\tilde{g}} - \beta_{y3\tilde{v}}\big) + \big(\beta_{y2d} - \beta_{y2e}\big)\beta_{y2f}\,,\\
&C_4 = \beta_{y1}\,\gamma_{\psi 3p} + \gamma_{\psi 1}\big(\beta_{y3\tilde{s}} - \beta_{y3\tilde{v}}\big) - \gamma_{\psi 2c}\,\beta_{y2f}\,,\\
&C_5 = \beta_{y1}\,\beta_{y3\tilde{o}} + \gamma_{\psi 1}\big(\beta_{y3\tilde{g}} - \beta_{y3\tilde{r}}\big) - \beta_{y2c}\,\beta_{y2f}\,,\\
&C_6 = \beta_{y1}\,\gamma_{\phi 3m} - \gamma_{\phi 1}\big(\beta_{y3\tilde{g}} - \beta_{y3\tilde{u}}\big) - \gamma_{\phi 2c}\,\beta_{y2f}\,,\\
&C_7 = \beta_{y1}\,\beta_{y3s} - \gamma_{\phi 1}\big(\beta_{y3\tilde{g}} + \beta_{y3\tilde{r}} + \beta_{y3\tilde{s}} - \beta_{y3\tilde{u}} - 2\,\beta_{y3\tilde{v}}\big) - \beta_{y2b}\,\beta_{y2f}\,.
\end{aligned}
\tag{9.6}
$$

For the planar three loop contributions there are 35 conditions in total. For those corresponding to contributions involving λ there are 14 corresponding to the vanishing of

$$
\begin{aligned}
D_1 &= \gamma_{\phi 1}\,\beta_{y3f} - 3\,\gamma_{\phi 2a}\,\beta_{\lambda 2g}\,, \qquad D_2 = \gamma_{\phi 1}\,\beta_{y3l} - 3\,\gamma_{\phi 2a}\,\beta_{\lambda 2g}\,,\\
D_3 &= \gamma_{\psi 1}\,\beta_{y3e} + \gamma_{\phi 1}\,\gamma_{\psi 3b} - \beta_{y2a}\,\gamma_{\psi 2a}\,,\\
D_4 &= \gamma_{\psi 1}(\beta_{y3g} - \beta_{y3k}) - \beta_{y1}\,\gamma_{\psi 3b} + \beta_{y2a}\,\gamma_{\psi 2c}\,,\\
D_5 &= 2\,\gamma_{\phi 1}(\beta_{y3b} - \beta_{y3c}) - 3\,\gamma_{\phi 2a}(\beta_{\lambda 2c} - 2\,\beta_{\lambda 2d})\,,\\
D_6 &= 2(\gamma_{\phi 1}\,\gamma_{\phi 3a} - \beta_{\lambda 1a}\,\gamma_{\phi 3b}) + 3\,\gamma_{\phi 2a}\,\beta_{\lambda 2b}\,,\\
D_7 &= \beta_{y1}(\beta_{y3g} - \beta_{y3h}) - \beta_{y2a}(\beta_{y2d} - \beta_{y2e})\,,\\
D_8 &= \beta_{y1}\,\beta_{y3d} + \gamma_{\phi 1}(2\,\beta_{y3g} - \beta_{y3h} - \beta_{y3i}) - \beta_{y2a}\,\beta_{y2b}\,,\\
D_9 &= \beta_{y1}\,\beta_{y3j} + \gamma_{\psi 1}(\beta_{y3h} - \beta_{y3i}) - \beta_{y2a}\,\beta_{y2c}\,,\\
D_{10} &= 2\,\beta_{y1}\,\gamma_{\phi 3c} - 2\,\gamma_{\phi 1}\,\beta_{y3h} + 3\,\beta_{\lambda 1b}\,\beta_{y3a} - \beta_{y2a}(\beta_{y2b} + 2\,\gamma_{\phi 2c}) + 6\,\gamma_{\phi 2a}\,\beta_{\lambda 2f}\,,\\
D_{11} &= \gamma_{\phi 1}(\gamma_{\psi 1}\,\beta_{y3a} - \beta_{y1}\,\gamma_{\psi 3a}) - \gamma_{\phi 2a}(\gamma_{\psi 1}\,\beta_{y2b} - \beta_{y1}\,\gamma_{\psi 2a})\,,\\
D_{12} &= \gamma_{\phi 1}\,\gamma_{\psi 1}(\beta_{y3h} + \beta_{y3k}) + 3\,\gamma_{\phi 2a}(\beta_{y1}\,\beta_{\lambda 2e} - \beta_{\lambda 1b}\,\beta_{y2c} - 2\,\gamma_{\psi 1}\,\beta_{\lambda 2f})\,,\\
D_{13} &= 2\,\gamma_{\psi 1}(\gamma_{\phi 1}\,\beta_{y3e} - \beta_{\lambda 1b}\,\gamma_{\phi 3b} + 2\,\gamma_{\phi 1}\,\gamma_{\phi 3c}) - \gamma_{\phi 1}(\gamma_{\phi 2b}\,\beta_{y2a} - 6\,\gamma_{\phi 2a}\,\beta_{\lambda 2e})\,,\\
D_{14} &= \beta_{y1}(\gamma_{\phi 1}\,\beta_{y3c} + 2\,\beta_{\lambda 1a}\,\gamma_{\phi 3c} - \beta_{\lambda 1b}\,\gamma_{\phi 3a}) - \gamma_{\phi 1}\,\beta_{\lambda 1a}\,\beta_{y3h}\\
&\qquad - 3\,\beta_{y1}\,\gamma_{\phi 2a}\,\beta_{\lambda 2d} - \beta_{\lambda 1a}(\beta_{y2a}\,\gamma_{\phi 2c} - 3\,\gamma_{\phi 2a}\,\beta_{\lambda 2f})\,.
\end{aligned}
\tag{9.7}
$$

Of the remaining 21 there is one relation which has total loop order 4 involving the vanishing of

$$
D_{15} = \beta_{y1}(\beta_{y3\tilde{\jmath}} - \beta_{y3\tilde{k}}) + \gamma_{\psi 1}(\beta_{y3\tilde{l}} - \beta_{y3\tilde{n}})\,,
\tag{9.8}
$$

and otherwise we have, for total loop order 5, 15 relations enforcing the vanishing of

$$
\begin{aligned}
D_{16} &= {\beta_{y1}}^2\gamma_{\phi 3d} - \beta_{y1}\,\gamma_{\phi 1}(2\,\beta_{y3t} + \gamma_{\phi 3i}) + 2\,{\gamma_{\phi 1}}^2\,\beta_{y3\tilde{e}} - 2\,\gamma_{\phi 2c}(\beta_{y1}\,\gamma_{\psi 2a} - \gamma_{\phi 1}\,\gamma_{\psi 2c})\,,\\
D_{17} &= {\beta_{y1}}^2\gamma_{\phi 3e} - \beta_{y1}\,\gamma_{\phi 1}(\beta_{y3n} - \beta_{y3o} + \gamma_{\phi 3l}) + 2\,{\gamma_{\phi 1}}^2(\beta_{y3\tilde{d}} - \beta_{y3\tilde{q}})\\
&\qquad - \gamma_{\phi 2c}(\beta_{y1}\,\beta_{y2b} - 2\,\gamma_{\phi 1}\,\beta_{y2d})\,,\\
D_{18} &= {\beta_{y1}}^2\gamma_{\phi 3g} - 2\,\beta_{y1}\,\gamma_{\phi 1}\,\beta_{y3\tilde{a}} - \beta_{y1}\,\gamma_{\psi 1}\,\gamma_{\phi 3i} + 2\,\gamma_{\phi 1}\,\gamma_{\psi 1}\,\beta_{y3\tilde{e}}\\
&\qquad - 2\,\gamma_{\phi 2c}(\beta_{y1}\,\gamma_{\psi 2b} - \gamma_{\psi 1}\,\gamma_{\psi 2c})\,,\\
D_{19} &= {\beta_{y1}}^2\gamma_{\psi 3e} - \beta_{y1}(\gamma_{\phi 1}\gamma_{\psi 3l} + \gamma_{\psi 1}\,\beta_{y3t}) + \gamma_{\phi 1}\,\gamma_{\psi 1}\,\beta_{y3\tilde{e}}\\
&\qquad - \gamma_{\psi 1}\,\gamma_{\psi 2a}(\beta_{y2d} - \beta_{y2e}) + \gamma_{\psi 2c}(\gamma_{\psi 1}\,\beta_{y2b} - 2\,\beta_{y1}\,\gamma_{\psi 2a} + 2\,\gamma_{\phi 1}\,\gamma_{\psi 2c})\,,
\end{aligned}
$$

$$
\begin{aligned}
D_{20} &= {\beta_{y1}}^2\gamma_{\psi 3g} - \beta_{y1}\,\gamma_{\psi 1}(\beta_{y3u} + 2\,\gamma_{\psi 3h}) + 2\,{\gamma_{\psi 1}}^2\,\beta_{y3v} + 2\,\beta_{y2c}\big(\gamma_{\psi 1}\beta_{y2b} - \beta_{y1}\,\gamma_{\psi 2a}\big)\,, \\
D_{21} &= {\beta_{y1}}^2\gamma_{\psi 3j} - \beta_{y1}\,\gamma_{\psi 1}(\beta_{y3\tilde a} + \gamma_{\psi 3l}) + {\gamma_{\psi 1}}^2\,\beta_{y3\tilde e} - \gamma_{\psi 1}\big((\beta_{y2d} - \beta_{y2e})\gamma_{\psi 2b} + \beta_{y2c}\,\gamma_{\psi 2c}\big)\,, \\
D_{22} &= {\beta_{y1}}^2\gamma_{\psi 3o} + \beta_{y1}\,\gamma_{\psi 1}(\beta_{y3\tilde h} - \beta_{y3\tilde\jmath} - 2\,\gamma_{\psi 3m}) + {\gamma_{\psi 1}}^2\big(\beta_{y3\tilde m} + \beta_{y3\tilde n} - \beta_{y3\tilde q} - \beta_{y3\tilde t}\big) \\
&\quad + \gamma_{\psi 2c}\big(\gamma_{\psi 1}(\beta_{y2d} + \beta_{y2e}) - \beta_{y1}\,\beta_{y2c}\big)\,, \\
D_{23} &= {\beta_{y1}}^2\beta_{y3p} - \beta_{y1}\,\gamma_{\phi 1}(\beta_{y3\tilde h} + \beta_{y3\tilde\imath}) + \beta_{y1}\,\gamma_{\psi 1}(\beta_{y3n} - \beta_{y3o}) \\
&\quad - \gamma_{\phi 1}\,\gamma_{\psi 1}\big(2\,\beta_{y3\tilde d} + \beta_{y3\tilde l} - \beta_{y3\tilde n} - 2\,\beta_{y3\tilde q}\big) - \beta_{y2c}\big(\beta_{y1}\beta_{y2b} - 2\,\gamma_{\phi 1}\,\beta_{y2d}\big)\,, \\
D_{24} &= {\beta_{y1}}^2(\beta_{y3n} - \beta_{y3q}) - \beta_{y1}\,\gamma_{\phi 1}(2\,\beta_{y3\tilde d} - \beta_{y3\tilde l} - \beta_{y3\tilde n}) \\
&\quad + (\beta_{y2d} - \beta_{y2e})\big(\beta_{y1}\,\beta_{y2b} - 2\,\gamma_{\phi 1}\,\beta_{y2d}\big)\,, \\
D_{25} &= {\beta_{y1}}^2\beta_{y3y} - \beta_{y1}\,\gamma_{\psi 1}\big(2\,\beta_{y3\tilde b} + 2\,\beta_{y3\tilde h} - \beta_{y3\tilde\imath} - \beta_{y3\tilde\jmath}\big) \\
&\quad + {\gamma_{\psi 1}}^2\big(2\,\beta_{y3\tilde d} - \beta_{y3\tilde l} - \beta_{y3\tilde m} - 2\,\beta_{y3\tilde n} + \beta_{y3\tilde q} + \beta_{y3\tilde t}\big) \\
&\quad - \beta_{y2c}\big(\beta_{y1}\,\beta_{y2c} + \gamma_{\psi 1}(3\,\beta_{y2d} - \beta_{y2e})\big) + (3\,\beta_{y2d} + \beta_{y2e})(\beta_{y1}\,\gamma_{\psi 2b} - \gamma_{\psi 1}\,\gamma_{\psi 2c})\,, \\
D_{26} &= {\beta_{y1}}^2(\beta_{y3\tilde c} - \beta_{y3\tilde\jmath}) - \beta_{y1}\,\gamma_{\psi 1}(\beta_{y3\tilde d} + \beta_{y3\tilde f} - \beta_{y3\tilde m} - \beta_{y3\tilde n}) \\
&\quad + (\beta_{y2d} - \beta_{y2e})\big(\beta_{y1}\,\beta_{y2c} - \gamma_{\psi 1}(\beta_{y2d} + \beta_{y2e})\big)\,, \\
D_{27} &= {\beta_{y1}}^2(\beta_{y3u} + 2\,\gamma_{\phi 3j}) - 2\,\beta_{y1}\,\gamma_{\psi 1}(\beta_{y3v} + 2\,\gamma_{\phi 3k} + \gamma_{\phi 3l}) - 4\,\beta_{y1}\,\gamma_{\phi 1}\,\beta_{y3\tilde\jmath} \\
&\quad - 4\,\gamma_{\phi 1}\,\gamma_{\psi 1}\big(\beta_{y3\tilde f} - \beta_{y3\tilde m} - \beta_{y3\tilde n} + \beta_{y3\tilde q}\big) \\
&\quad - 2\,\beta_{y2c}\big(\beta_{y1}(\beta_{y2b} + 2\,\gamma_{\phi 2c}) - 2\,\gamma_{\phi 1}\,\beta_{y2d}\big) \\
&\quad + 4(\beta_{y2d} + \beta_{y2e})\big(\beta_{y1}\,\gamma_{\psi 2a} - \gamma_{\psi 1}\,\beta_{y2b} - \gamma_{\phi 1}\,\gamma_{\psi 2c} + \gamma_{\psi 1}\,\gamma_{\phi 2c}\big)\,, \\
D_{28} &= {\beta_{y1}}^2(\beta_{y3\tilde a} + 2\,\gamma_{\psi 3n}) - \beta_{y1}\,\gamma_{\psi 1}(\beta_{y3\tilde e} + 2\,\beta_{y3\tilde\jmath} + 2\,\gamma_{\psi 3k} + 2\,\gamma_{\psi 3m}) \\
&\quad - 2\,{\gamma_{\psi 1}}^2\big(\beta_{y3\tilde l} - \beta_{y3\tilde m} - \beta_{y3\tilde n} + \beta_{y3\tilde t}\big) \\
&\quad + \beta_{y1}\big(- {\beta_{y2c}}^2 + 2\,\beta_{y2d}\,\gamma_{\psi 2b} - 2\,\beta_{y2c}\,\gamma_{\psi 2c}\big) \\
&\quad + \gamma_{\psi 1}\big(- \beta_{y2c}(\beta_{y2d} - \beta_{y2e}) + 2\,\beta_{y2e}\,\gamma_{\psi 2c}\big)\,, \\
D_{29} &= {\beta_{y1}}^2(\beta_{y3t} + 2\,\gamma_{\psi 3f}) - 2\,\beta_{y1}\,\gamma_{\psi 1}\,\beta_{y3n} - \beta_{y1}\,\gamma_{\phi 1}\,(\beta_{y3\tilde e} + 2\,\gamma_{\psi 3k} + 2\,\gamma_{\psi 3m}) \\
&\quad + 2\,\gamma_{\psi 1}\gamma_{\phi 1}(2\,\beta_{y3\tilde d} - \beta_{y3\tilde l} - \beta_{y3\tilde q}) \\
&\quad - \beta_{y1}\big(\beta_{y2b}(\beta_{y2c} + 2\,\gamma_{\psi 2c}) + 2\,\beta_{y2d}\,\gamma_{\psi 2a}\big) + 2\,\gamma_{\psi 1}\,\beta_{y2b}(\beta_{y2d} + \beta_{y2e}) \\
&\quad + \gamma_{\phi 1}\big(6\,\gamma_{\psi 2c}\,\beta_{y2d} + \beta_{y2c}(\beta_{y2d} - \beta_{y2e})\big)\,, \\
D_{30} &= \beta_{y1}\,\gamma_{\phi 1}(\beta_{y3\tilde b} + \beta_{y3\tilde\jmath}) - \beta_{y1}\,\gamma_{\psi 1}(\beta_{y3n} + \beta_{y3r}) \\
&\quad + \gamma_{\phi 1}\,\gamma_{\psi 1}\big(\beta_{y3\tilde d} - \beta_{y3\tilde f} + \beta_{y3\tilde l} - \beta_{y3\tilde n} - \beta_{y3\tilde q} + \beta_{y3\tilde t}\big) \\
&\quad + (\beta_{y2d} + \beta_{y2e})\big(2\,\beta_{y1}\,\gamma_{\psi 2a} - \gamma_{\psi 1}\,\beta_{y2b} - \gamma_{\phi 1}(\beta_{y2c} + 2\,\gamma_{\psi 2c})\big)\,,
\end{aligned}
\tag{9.9}
$$

and finally 5 with loop order six

$$
\begin{aligned}
D_{31} &= {\beta_{y1}}^3\,\gamma_{\psi 3c} + {\beta_{y1}}^2\big(\gamma_{\phi 1}(\beta_{y3t} - \gamma_{\psi 3h}) - \gamma_{\psi 1}\,\beta_{y3m}\big) \\
&\quad - \beta_{y1}\,\gamma_{\psi 1}\,\gamma_{\phi 1}(2\,\beta_{y3n} - \beta_{y3v}) + \beta_{y1}\,{\gamma_{\phi 1}}^2\big(\beta_{y3\tilde b} - \beta_{y3\tilde e} + \beta_{y3\tilde\jmath} - \gamma_{\psi 3k}\big) \\
&\quad + {\gamma_{\phi 1}}^2\,\gamma_{\psi 1}\big(3\,\beta_{y3\tilde d} - \beta_{y3\tilde m} - \beta_{y3\tilde n} - 2\,\beta_{y3\tilde q} + \beta_{y3\tilde t}\big) \\
&\quad + {\beta_{y1}}^2\,\gamma_{\psi 2a}\,\gamma_{\phi 2c} + \beta_{y1}\,\gamma_{\phi 1}\big(\gamma_{\psi 2a}\,\beta_{y2d} - \beta_{y2b}(\beta_{y2c} + \gamma_{\psi 2c})\big) - \beta_{y1}\,\gamma_{\psi 1}\,\gamma_{\phi 2c}\,\beta_{y2b} \\
&\quad + {\gamma_{\phi 1}}^2\big(\gamma_{\psi 2c}(\beta_{y2d} - \beta_{y2e}) - 2\,\beta_{y2c}\,\beta_{y2e}\big) + \gamma_{\phi 1}\,\gamma_{\psi 1}\,\beta_{y2b}\,\beta_{y2e}\,, \\
D_{32} &= {\beta_{y1}}^3\gamma_{\phi 3h} + {\beta_{y1}}^2\gamma_{\psi 1}\,\beta_{y3u} - \beta_{y1}\,\gamma_{\phi 1}\,\gamma_{\psi 1}\big(2\,\beta_{y3\tilde b} - \beta_{y3\tilde\imath} + \beta_{y3\tilde\jmath}\big) - 2\,\beta_{y1}\,{\gamma_{\psi 1}}^2(\beta_{y3v} + \gamma_{\phi 3k}) \\
&\quad + \gamma_{\phi 1}\,{\gamma_{\psi 1}}^2\big(2\,\beta_{y3\tilde d} - 2\,\beta_{y3\tilde f} - \beta_{y3\tilde l} + \beta_{y3\tilde m} - \beta_{y3\tilde q} + \beta_{y3\tilde t}\big) \\
&\quad - \beta_{y1}\,\gamma_{\phi 1}\,{\beta_{y2c}}^2 - \beta_{y1}\,\gamma_{\psi 1}\big(2\,\beta_{y2c}(\beta_{y2b} + \gamma_{\phi 2c}) + \gamma_{\psi 2a}(\beta_{y2d} - \beta_{y2e})\big) \\
&\quad + \gamma_{\phi 1}\,\gamma_{\psi 1}\big(\gamma_{\psi 2c}(\beta_{y2d} - \beta_{y2e}) + 2\,\beta_{y2c}\,\beta_{y2e}\big) \\
&\quad + {\gamma_{\psi 1}}^2\big(\beta_{y2b}(\beta_{y2d} - \beta_{y2e}) + 2\,\beta_{y2e}\,\gamma_{\phi 2c}\big)\,,
\end{aligned}
$$

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$$\begin{aligned}
D_{33} = {}& \beta_{y1}{}^3\,\gamma_{\psi 3i} + \beta_{y1}{}^2\gamma_{\psi 1}\,(\beta_{y3\tilde a} - \beta_{y3z} - \gamma_{\psi 3l}) + \beta_{y1}\,\gamma_{\psi 1}{}^2(\beta_{y3\tilde b} - \beta_{y3\tilde\jmath} - \gamma_{\psi 3k}) \\
& + \gamma_{\psi 1}{}^3(-\beta_{y3\tilde d} + \beta_{y3\tilde m} + \beta_{y3\tilde n} - \beta_{y3\tilde t}) \\
& - \beta_{y1}\,\gamma_{\psi 1}(\beta_{y2c}{}^2 + \beta_{y2c}\gamma_{2c} + \beta_{y2d}\,\gamma_{\psi 2b} - \gamma_{\psi 2c}{}^2) + \gamma_{\psi 1}{}^2(\beta_{y2c}\,\beta_{y2e} + 2\,\beta_{y2d}\,\gamma_{\psi 2c})\,, \\
D_{34} = {}& \beta_{y1}{}^3\,\gamma_{\psi 3d} + \beta_{y1}{}^2(\gamma_{\psi 1}\,\beta_{y3t} + \gamma_{\phi 1}\,\beta_{y3\tilde a}) - \beta_{y1}\,\gamma_{\phi 1}\,\gamma_{\psi 1}(2\,\beta_{y3\tilde e} + \beta_{y3\tilde\imath} + \beta_{y3\tilde\jmath} + 2\,\gamma_{\psi 3k}) \\
& - \gamma_{\phi 1}\,\gamma_{\psi 1}{}^2(3\,\beta_{y3\tilde l} - \beta_{y3\tilde m} - 2\,\beta_{y3\tilde n} - \beta_{y3\tilde q} + \beta_{y3\tilde t}) \\
& - \beta_{y1}\,\gamma_{\phi 1}\,\beta_{y2c}(\beta_{y2c} + \gamma_{\psi 2c}) - \beta_{y1}\,\gamma_{\psi 1}(\beta_{y2b}(2\,\beta_{y2c} + \gamma_{\psi 2c}) + \gamma_{\psi 2a}(3\,\beta_{y2d} - \beta_{y2e})) \\
& + \gamma_{\phi 1}\,\gamma_{\psi 1}(\beta_{y2c}(3\,\beta_{y2d} - \beta_{y2e}) + \gamma_{\psi 2c}(5\,\beta_{y2d} - \beta_{y2e})) + 2\,\gamma_{\psi 1}{}^2\,\beta_{y2b}\,\beta_{y2d}\,, \\
D_{35} = {}& \beta_{y1}{}^3\,\gamma_{\phi 3f} + \beta_{y1}{}^2(\gamma_{\psi 1}(\beta_{y3u} - \gamma_{\phi 3i}) - 2\,\gamma_{\phi 1}\,\beta_{y3z}) - 2\,\beta_{y1}\,\gamma_{\psi 1}{}^2(\beta_{y3v} + \gamma_{\phi 3k}) \\
& + 2\,\beta_{y1}\,\gamma_{\phi 1}\,\gamma_{\psi 1}(\beta_{y3\tilde e} + \beta_{y3\tilde\imath} - \beta_{y3\tilde\jmath}) - 2\,\gamma_{\phi 1}\,\gamma_{\psi 1}{}^2(\beta_{y3\tilde f} - \beta_{y3\tilde l} - \beta_{y3\tilde m} + \beta_{y3\tilde q}) \\
& + 2\,\beta_{y1}\,\gamma_{\psi 1}(\gamma_{\phi 2c}\,\gamma_{\psi 2c} + \gamma_{\psi 2a}(\beta_{y2d} + \beta_{y2e}) - \beta_{y2c}(\beta_{y2b} + \gamma_{\phi 2c})) \\
& - 2\,\gamma_{\phi 1}\,\gamma_{\psi 1}\,\gamma_{\psi 2c}(\beta_{y2d} + \beta_{y2e}) + 2\,\gamma_{\psi 1}{}^2(\gamma_{\phi 2c}\,\beta_{y2e} - \beta_{y2b}(\beta_{y2d} + \beta_{y2e}))\,.
\end{aligned} \tag{9.10}$$

To satisfy (9.1) with three loop β-functions it is necessary to include contributions to υ as in (2.4) and (3.3). The consistency relations require

$$\begin{aligned}
2\,\beta_{y1}\,\upsilon_{\phi 3c} &= -3\,\beta_{\lambda 1b}\,\beta_{y3a} + \beta_{y2a}\,\beta_{y2b}\,, \\
2\,\beta_{y1}\,\upsilon_{\phi 3j} &= -\beta_{y1}\,\beta_{y3u} + \gamma_{\psi 1}\,\beta_{y3v} + 2\,\beta_{y2b}\,\beta_{y2c}\,, \\
2\,\beta_{y1}(\beta_{y1}\,\upsilon_{\psi 3n} - \gamma_{\psi 1}\,\upsilon_{\psi 3m}) &= -\beta_{y1}(\beta_{y1}\,\beta_{y3\tilde a} - \gamma_{\psi 1}\,\beta_{y3\tilde e}) \\
&\quad + \gamma_{\psi 1}\,\beta_{y2c}(\beta_{y2d} - \beta_{y2e}) + \beta_{y1}\,\beta_{y2c}{}^2\,, \\
2\,\beta_{y1}(\beta_{y1}\,\upsilon_{\psi 3f} - \gamma_{\phi 1}\,\upsilon_{\psi 3m}) &= \beta_{y1}(\beta_{y1}\,\beta_{y3t} - \gamma_{\phi 1}\,\beta_{y3\tilde e}) \\
&\quad + \gamma_{\phi 1}\,\beta_{y2c}(\beta_{y2d} - \beta_{y2e}) - \beta_{y1}\,\beta_{y2b}\,\beta_{y2c}\,.
\end{aligned} \tag{9.11}$$

Together with (7.19) these suffice to determine the results in (2.5) and (3.4).

9.1 Supersymmetry reduction

For the reduction as described in subsection 7.2 the consistency relations reduce to just one at two loops given by the vanishing of

$$S_0 = \beta_{Y1}\,\gamma_{\Phi 2A} - 2\,\gamma_{\Phi 1}(\beta_{Y2A} + \gamma_{\Phi 2B})\,, \tag{9.12}$$

and at three loops for planar contributions there are 7 relations obtained by setting to zero

$$\begin{aligned}
S_1 &= \beta_{Y1}(\beta_{Y3E} - \beta_{Y3F}) - \gamma_{\Phi 1}(\beta_{Y3H} - \beta_{Y3J})\,, \\
S_2 &= \beta_{Y1}(2\,\beta_{Y3A} - \beta_{Y3C} - 2\,\gamma_{\Phi 3D} + 4\,\gamma_{\Phi 3E}) + 2\,\gamma_{\Phi 1}(\beta_{Y3F} - \beta_{Y3G} + \beta_{Y3K} - \gamma_{\Phi 3G} + 2\,\gamma_{\Phi 3H})\,, \\
S_3 &= \beta_{Y1}(\beta_{Y3A} + \gamma_{\Phi 3E}) - \gamma_{\Phi 1}(\beta_{Y3D} - \beta_{Y3E} + \beta_{Y3F} + \beta_{Y3G} + \gamma_{\Phi 3G}) \\
&\quad - \beta_{Y2A}(\beta_{Y2A} + \gamma_{\Phi 2B}) + \beta_{Y2B}\,\gamma_{\Phi 2A}\,, \\
S_4 &= \beta_{Y1}(\gamma_{\Phi 3A} + \gamma_{\Phi 3B}) + \gamma_{\Phi 1}(\beta_{Y3A} - 2\,\beta_{Y3B} + \beta_{Y3C} - 2\,\gamma_{\Phi 3D} + 2\,\gamma_{\Phi 3E} - \gamma_{\Phi 3F}) \\
&\quad - \gamma_{\Phi 2A}(2\,\beta_{Y2A} - \gamma_{\Phi 2B})\,, \\
S_5 &= \beta_{Y1}{}^2\,\beta_{Y3A} - \beta_{Y1}\gamma_{\Phi 1}(2\,\beta_{Y3D} - \beta_{Y3F} + \beta_{Y3G}) + 2\,\gamma_{\Phi 1}{}^2(\beta_{Y3I} - \beta_{Y3J}) \\
&\quad + \beta_{Y2A}(2\,\gamma_{\Phi 1}\beta_{Y2B} - \beta_{Y1}\beta_{Y2A})\,, \\
S_6 &= 2\,\beta_{Y1}{}^2\gamma_{\Phi 3B} - \beta_{Y1}\gamma_{\Phi 1}(2\,\beta_{Y3A} - \beta_{Y3C} + 2\,\gamma_{\Phi 3D}) + 2\,\gamma_{\Phi 1}{}^2(2\,\beta_{Y3D} - \beta_{Y3K} + \gamma_{\Phi 3G}) \\
&\quad - 2\,\gamma_{\Phi 1}(\beta_{Y2A}{}^2 + \beta_{Y2B}\,\gamma_{\Phi 2A})\,, \\
S_7 &= \beta_{Y1}{}^2\,\gamma_{\Phi 3C} - 2\,\beta_{Y1}\gamma_{\Phi 1}(\beta_{3C} + \gamma_{\Phi 3F}) + 4\,\gamma_{\Phi 1}{}^2\beta_{Y3K} - 4\,\gamma_{\Phi 1}\beta_{Y2A}\,\gamma_{\Phi 2B} = 0\,,
\end{aligned} \tag{9.13}$$

and for the non planar diagrams just 2

$$S_8 = \beta_{Y1}\beta_{Y3L} - \gamma_{\Phi 1}(\beta_{Y3N} - 2\,\beta_{Y3O}) - 2\,\beta_{Y2C}\,\beta_{Y2A}\,,$$
$$S_9 = \beta_{Y1}\gamma_{\Phi 3I} + \gamma_{\Phi 1}(\beta_{Y3M} - \beta_{Y3O}) - \beta_{Y2C}\,\gamma_{\Phi 2B}\,. \tag{9.14}$$

10 Scheme variations for scalar fermion theory to three loops

The coefficients appearing in the expansions of the β-functions and anomalous dimensions for a general scalar fermion theory are in general dependent on the choice of regularisation scheme. At ℓ loops possible scheme variations in ${\gamma_\phi}^{(\ell)}$, ${\gamma_\psi}^{(\ell)}$, ${\tilde\beta_y}^{(\ell)}$, ${\tilde\beta_\lambda}^{(\ell)}$ are determined in terms of arbitrary parameters related to the expansions of ${\gamma_\phi}^{(\ell-1)}$, ${\gamma_\psi}^{(\ell-1)}$, ${\tilde\beta_y}^{(\ell-1)}$, ${\tilde\beta_\lambda}^{(\ell-1)}$. This depends on preserving the form of the functions $\tilde\beta, \gamma$ in terms of contributions corresponding to 1PI and 1VI diagrams. Labelling the coefficients α in the expansion at ℓ loops by g, ℓ, r, where here $\mathrm{g} = \phi, \psi, y, \lambda$, the general forms of the variations for $\alpha_g \equiv \alpha_{\mathrm{g}\ell r}$, with $\alpha \to \gamma$ for $\mathrm{g} = \phi,\ \psi$ and $\alpha \to \beta$ for $\mathrm{g} \to y,\ \lambda$, are shown in appendix E to involve a sum over contributions

$$X_{g',g''} = -X_{g'',g'} = \alpha_{g'}\,\epsilon_{g''} - \epsilon_{g'}\,\alpha_{g''}\,, \qquad g' = \mathrm{g}'\,\ell' r'\,, \quad g'' = \mathrm{g}''\,\ell'' r''\,, \tag{10.1}$$

where $\{\epsilon_g\}$ are arbitrary parameters, so that

$$\delta\alpha_g = \sum_{\substack{g',g'' \\ \ell'+\ell''=\ell,\mathrm{g}''=\mathrm{g}}} {\mathcal{N}_g}^{g'g''}\,X_{g',g''}\,, \tag{10.2}$$

and ${\mathcal{N}_g}^{g'g''} = -{\mathcal{N}_g}^{g''g'}$ are integer coefficients. Of course one loop coefficients are scheme invariant and higher loops coefficients corresponding to primitive diagrams, which have a different topology and do not lead to integrals which have subdivergences, are also scheme invariant. Any e^g such that for a given ℓ

$$\sum_{g,\,\ell'+\ell''=\ell} e^g\,{\mathcal{N}_g}^{g'g''} = 0\,, \tag{10.3}$$

gives rise to a linear ℓ loop scheme invariant $\sum_g e^g\alpha_g$.

At two loops there are 10 possible X's but only 7 appear in scheme variations as $X_{\phi 1,\lambda 1b}$, $X_{\psi 1,\lambda 1a}$, $X_{y1,\lambda 1a}$, are not present, so we have

$$\begin{aligned}
&\delta\gamma_{\phi 2a} = \delta\gamma_{\psi 2b} = \delta\beta_{y2a} = \delta\beta_{y2d} = \delta\beta_{y2e} = \delta\beta_{y2f} = \delta\beta_{\lambda 2a} = \delta\beta_{\lambda 2g} = 0\,,\\
&\delta\gamma_{\phi 2b} = 4\,X_{\psi 1,\phi 1}\,, \quad \delta\gamma_{\phi 2c} = 2\,X_{y1,\phi 1}\,, \quad \delta\gamma_{\psi 2a} = 2\,X_{\phi 1,\psi 1}\,, \quad \delta\gamma_{\psi 2c} = 2\,X_{y1,\psi 1}\,,\\
&\delta\beta_{y2b} = 2\,X_{\phi 1,y1}\,, \quad \delta\beta_{y2c} = 2\,X_{\psi 1,y1}\,, \quad \delta\beta_{\lambda 2b} = 4\,X_{\phi 1,\lambda 1a}\,, \quad \delta\beta_{\lambda 2c} = 2\,X_{\lambda 1b,\lambda 1a}\,,\\
&\delta\beta_{\lambda 2d} = X_{\lambda 1b,\lambda 1a}\,, \quad \delta\beta_{\lambda 2e} = 2\,X_{\psi 1,\lambda 1b}\,, \quad \delta\beta_{\lambda 2f} = X_{y1,\lambda 1b}\,.
\end{aligned} \tag{10.4}$$

The cases $y2a$, $y2f$, $\lambda 2g$ correspond to primitive diagrams and so the variation is necessarily zero. Apart from those coefficients which are individually invariant there are four linear scheme invariants.

At three loops the results for scheme variations separate into different groups. There are six primitive three loop diagrams for the Yukawa β-function so that

$$\delta\beta_{y3f} = \delta\beta_{y3l} = \delta\beta_{y3\tilde w} = \delta\beta_{y3\tilde x} = \delta\beta_{y3\tilde y} = \delta\beta_{y3\tilde z} = 0\,, \tag{10.5}$$

and there are cases where the individual terms are scheme invariant

$$\delta\gamma_{\psi 3j} = \delta\beta_{y3\tilde{f}} = \delta\beta_{y3\tilde{l}} = \delta\beta_{y,3\tilde{n}} = 0\,. \tag{10.6}$$

For those diagrams containing the non planar subgraph corresponding to β_{y2f} the variations are

$$\begin{aligned}
&\delta\beta_{y3s} = 2\,X_{\phi 1,y2f}\,, \quad \delta\beta_{y3\tilde{o}} = \delta\beta_{y3\tilde{p}} = 2\,X_{\psi 1,y2f}\,,\\
&\delta\gamma_{\phi 3m} = 2\,X_{y2f,\phi 1}\,, \quad \delta\gamma_{\psi 3p} = 2\,X_{y2f,\psi 1}\,, && \text{(10.7a)}\\
&\delta\beta_{y3\tilde{g}} = \delta\beta_{y3\tilde{v}} = X_{y2f,y1}\,, \quad \delta\beta_{y3\tilde{r}} = \delta\beta_{y3\tilde{s}} = \delta\beta_{y3\tilde{u}} = X_{y1,y2f}\,. && \text{(10.7b)}
\end{aligned}$$

There are evidently 3 linear invariants from (10.7a) and 4 more from (10.7b). Otherwise for the variations of the anomalous dimension contributions arising from planar diagrams we have

$$\begin{aligned}
&\delta\gamma_{\phi 3a} = 6\,X_{\lambda 1a,\phi 2a}\,, \quad \delta\gamma_{\phi 3b} = 6\,X_{\phi 1,\phi 2a}\,, \quad \delta\gamma_{\phi 3c} = 3\,X_{\lambda 1b,\phi 2a} + X_{y2a,\phi 1}\,,\\
&\delta\gamma_{\phi 3d} = 2\,X_{\phi 1,\phi 2b} + 4\,X_{\psi 2a,\phi 1}\,, \quad \delta\gamma_{\phi 3f} = 4\,X_{\psi 1,\phi 2b}\,, \quad \delta\gamma_{\phi 3g} = 2\,X_{\psi 1,\phi 2b} + 4\,X_{\psi 2b,\phi 1}\,,\\
&\delta\gamma_{\phi 3h} = 2\,X_{\psi 1,\phi 2b}\,, && \text{(10.8a)}\\
&\delta\gamma_{\phi 3e} = 2\,X_{\phi 1,\phi 2c} + 2\,X_{y2b,\phi 1}\,, \quad \delta\gamma_{\phi 3i} = 4\,X_{\psi 2c,\phi 1} + 2\,X_{y1,\phi 2b}\,,\\
&\delta\gamma_{\phi 3j} = 4\,X_{\psi 1,\phi 2c} + X_{y1,\phi 2b} + 2\,X_{y2c,\phi 1}\,, \quad \delta\gamma_{\phi 3k} = 2\,X_{y1,\phi 2c} + 2\,X_{y2e,\phi 1}\,,\\
&\delta\gamma_{\phi 3l} = 2\,X_{y1,\phi 2c} + 4\,X_{y2d,\phi 1}\,, && \text{(10.8b)}
\end{aligned}$$

and

$$\begin{aligned}
&\delta\gamma_{\psi 3a} = 2\,X_{\phi 2a,\psi 1}\,, \quad \delta\gamma_{\psi 3b} = 2\,X_{y2a,\psi 1}\,, \quad \delta\gamma_{\psi 3c} = 4\,X_{\phi 1,\psi 2a}\,,\\
&\delta\gamma_{\psi 3d} = 2\,X_{\psi 1,\psi 2a} + 2\,X_{\phi 1,\psi 2b}\,, \quad \delta\gamma_{\psi 3e} = 2\,X_{\phi 1,\psi 2b} + 2\,X_{\psi 2a,\psi 1}\,,\\
&\delta\gamma_{\psi 3g} = 4\,X_{\psi 1,\psi 2a} + 2\,X_{\phi 2b,\psi 1}\,, \quad \delta\gamma_{\psi 3i} = 4\,X_{\psi 1,\psi 2b}\,, && \text{(10.9a)}\\
&\delta\gamma_{\psi 3f} = 2\,X_{\phi 1,\psi 2c} + X_{y1,\psi 2a} + X_{y2b,\psi 1}\,, \quad \delta\gamma_{\psi 3h} = 2\,X_{y1,\psi 2a} + 2\,X_{\phi 2c,\psi 1}\,,\\
&\delta\gamma_{\psi 3k} = 2\,X_{y1,\psi 2c} + 2\,X_{y2d,\psi 1}\,, \quad \delta\gamma_{\psi 3l} = 2\,X_{\psi 2c,\psi 1} + 2\,X_{y1,\psi 2b}\,,\\
&\delta\gamma_{\psi 3m} = X_{y1,\psi 2c} + X_{y2e,\psi 1} + X_{y2d,\psi 1}\,, \quad \delta\gamma_{\psi 3n} = 2\,X_{\psi 1,\psi 2c} + X_{y1,\psi 2b} + X_{y2c,\psi 1}\,,\\
&\delta\gamma_{\psi 3o} = 2\,X_{\psi 1,\psi 2c} + 2\,X_{y2c,\psi 1}\,. && \text{(10.9b)}
\end{aligned}$$

The remaining scheme variations are then

$$\begin{aligned}
&\delta\beta_{y3b} = \delta\beta_{y3c} = X_{\lambda 1a,y2a}\,, \quad \delta\beta_{y3d} = \delta\beta_{y3e} = 2\,X_{\phi 1,y2a}\,, \quad \delta\beta_{y3j} = 2\,X_{\psi 1,y2a}\,,\\
&\delta\beta_{y3w} = \delta\beta_{y3x} = X_{\lambda 1b,y2a}\,, && \text{(10.10a)}\\
&\delta\beta_{y3a} = 2\,X_{\phi 2a,y1}\,, \quad \delta\beta_{y3g} = \delta\beta_{y3h} = X_{y2a,y1}\,, \quad \delta\beta_{y3i} = \delta\beta_{y3k} = X_{y1,y2a}\,,\\
&\delta\beta_{y3m} = 4\,X_{\phi 1,y2b}\,, \quad \delta\beta_{y3n} = 2\,X_{\phi 1,y2e} + X_{y2b,y1}\,, \quad \delta\beta_{y3o} = 2\,X_{\phi 1,y2e} + X_{y1,y2b}\,,\\
&\delta\beta_{y3p} = 2\,X_{\phi 1,y2c} + 2\,X_{\psi 1,y2b}\,, \quad \delta\beta_{y3q} = 2\,X_{\phi 1,y2d} + X_{y2b,y1}\,, \quad \delta\beta_{y3r} = 2\,X_{\phi 1,y2d} + X_{y1,y2b}\,,\\
&\delta\beta_{y3t} = 2\,X_{\phi 1,y2c} + 2\,X_{\psi 2a,y1}\,, \quad \delta\beta_{y3u} = 4\,X_{\psi 1,y2b} + 2\,X_{\phi 2b,y1}\,,\\
&\delta\beta_{y3v} = 2\,X_{y1,y2b} + 2\,X_{\phi 2c,y1}\,, \quad \delta\beta_{y3y} = \delta\beta_{y3z} = 4\,X_{\psi 1,y2c}\,,\\
&\delta\beta_{y3\tilde{a}} = 2\,X_{\psi 1,y2c} + 2\,X_{\psi 2b,y1}\,, \quad \delta\beta_{y3\tilde{b}} = 2\,X_{\psi 1,y2e} + X_{y1,y2c}\,, \quad \delta\beta_{y3\tilde{c}} = 2\,X_{\psi 1,y2e} + X_{y2c,y1}\,,\\
&\delta\beta_{y3\tilde{d}} = X_{y1,y2e} + X_{y2d,y1}\,, \quad \delta\beta_{y3\tilde{e}} = 2\,X_{\psi 2c,y1} + 2\,X_{y1,y2c}\,,\\
&\delta\beta_{y3\tilde{h}} = \delta\beta_{y3\tilde{i}} = 2\,X_{\psi 1,y2d} + X_{y1,y2c}\,, \quad \delta\beta_{y3\tilde{j}} = \delta\beta_{y3\tilde{k}} = 2\,X_{\psi 1,y2d} + X_{y2c,y1}\,,\\
&\delta\beta_{y3\tilde{m}} = X_{y1,y2d} + X_{y2e,y1}\,, \quad \delta\beta_{y3\tilde{q}} = X_{y1,y2e} + X_{y1,y2d}\,, \quad \delta\beta_{y3\tilde{t}} = 2\,X_{y1,y2d}\,. && \text{(10.10b)}
\end{aligned}$$

Each variation is necessarily such that the number of fermion loops is conserved. The variations in (10.7a), (10.8a), (10.9a) and (10.10a), apart from that for β_{y3x}, correspond to the restriction to the U(1) invariant case. The 21 variations in (10.8a), (10.9a) and (10.10a) involve 14 different X's and there are 7 linear invariants. The 40 variations in (10.8b), (10.9b) and (10.10b) involve 23 different X's but there are 18 linear invariants since the equations are invariant under

$$\delta X_{\phi 1,\psi 2c} = \delta X_{\psi 1,y2b} = \rho\,, \qquad \delta X_{\phi 1,y2c} = \delta X_{\psi 1,\phi 2c} = \delta X_{y1,\psi 2a} = -\rho\,, \qquad \delta X_{y1,\phi 2b} = 2\rho\,. \tag{10.11}$$

Individual coefficients in the expansions of β or γ, besides those corresponding to primitive diagrams, are scheme invariant when the associated vertex or propagator subgraphs are a nested sequence all of the same form. Examples appear in (10.6). In this case $\gamma_{\psi 1}$, $\gamma_{\psi 2b}$, $\gamma_{\psi 3j}$ correspond to rainbow diagrams and β_{y1}, β_{y2e}, $\beta_{y3\tilde{f}}$ correspond to vertex ladder diagrams. For these cases there are exact all orders results [77, 78] obtained by solving quadratic equations

$$\begin{aligned} \gamma_\psi\big|_{\text{rainbow}} &= \sqrt{1+y^2} - 1 = \tfrac{1}{2}\, y^2 - \tfrac{1}{8}\, y^4 + \tfrac{1}{16}\, y^6 - \tfrac{5}{128}\, y^8 + \dots\,, \\ \beta_y/y\big|_{\text{ladder}} &= \sqrt{1+4\,y^2} - 1 = 2\, y^2 - 2\, y^4 + 4\, y^6 - 10\, y^8 + \dots\,. \end{aligned} \tag{10.12}$$

Further sequences of nested diagrams are also associated with β_{y1}, β_{y2d} and $\beta_{y3\tilde{l}}$ or $\beta_{y,3\tilde{n}}$ so these are necessarily scheme invariant.

For the restriction to the U(1) theory discussed in section 6, the scheme variations in (10.4), (10.8a), (10.9a), (10.10a) consistently restrict as they only involve the γ and β-function coefficients relevant in that case. The sequence of nested diagrams for $\gamma_\psi\big|_{\text{rainbow}}$ remains in this case. The scheme variations may be restricted to $\mathcal{N} = 1$ and $\mathcal{N} = \frac{1}{2}$ supersymmetry. They are consistent with the various constraints obtained earlier so long as all lower order conditions are imposed. For the latter case at two loops

$$\delta\gamma_{\Phi 2B} = 2\, X_{Y1,\Phi 1}\,, \quad \delta\beta_{Y2A} = 2\, X_{\Phi 1,Y1}\,, \quad \delta\gamma_{\Phi 2A} = \delta\beta_{Y2B} = \delta\beta_{Y2C} = 0\,, \tag{10.13}$$

and at three loops individual scheme invariant coefficients correspond to

$$\delta\gamma_{\Phi 3C} = \delta\beta_{Y3H} = \delta\beta_{Y3J} = \delta\beta_{Y3P} = \delta\beta_{Y3Q} = 0\,, \tag{10.14}$$

with β_{Y3P}, β_{Y3Q} arising from primitive diagrams while $\gamma_{\Phi 3C}$, along with $\gamma_{\Phi 1}$, $\gamma_{\Phi 2A}$, forms part of a sequence of nested rainbow diagrams. For planar contributions

$$\begin{aligned} &\delta\gamma_{\Phi 3A} = 2\,\delta\gamma_{\Phi 3B} = 4\, X_{\Phi 1,\Phi 2A}\,, \qquad \delta\gamma_{\Phi 3D} = 4\, X_{\Phi 1,\Phi 2B} + X_{Y1,\Phi 2A} + 2\, X_{Y2A,\Phi 1}\,, \\ &\delta\gamma_{\Phi 3E} = 2\, X_{\Phi 1,\Phi 2B} + 2\, X_{Y2A,\Phi 1}\,, \qquad \delta\gamma_{\Phi 3F} = 4\, X_{\Phi 2B,\Phi 1} + 2\, X_{Y1,\Phi 2A}\,, \\ &\delta\gamma_{\Phi 3G} = 2\, X_{Y1,\Phi 2B} + 4\, X_{Y2B,\Phi 1}\,, \qquad \delta\gamma_{\Phi 3H} = 2\, X_{Y1,\Phi 2B} + 2\, X_{Y2B,\Phi 1}\,, \\ &\delta\beta_{Y3A} = \delta\beta_{Y3B} = 4\, X_{\Phi 1,Y2A}\,, \qquad \delta\beta_{Y3C} = 4\, X_{\Phi 1,Y2A} + 2\, X_{\Phi 2A,Y1}\,, \\ &\delta\beta_{Y3D} = \delta\beta_{Y3G} = 2\, X_{\Phi 1,Y2B} + X_{Y1,Y2A}\,, \qquad \delta\beta_{Y3E} = \delta\beta_{Y3F} = 2\, X_{\Phi 1,Y2B} + X_{Y2A,Y1}\,, \\ &\delta\beta_{Y3I} = 2\, X_{Y1,Y2B}\,, \qquad \delta\beta_{Y3K} = 2\, X_{Y1,Y2A} + 2\, X_{\Phi 2B,Y1}\,, \end{aligned} \tag{10.15}$$

and for non planar

$$\begin{aligned} &\delta\gamma_{\Phi 3I} = 2\, X_{Y2C,\Phi 1}\,, \quad \delta\beta_{Y3L} = 4\, X_{\Phi 1,Y2C}\,, \\ &\delta\beta_{Y3M} = X_{Y1,Y2C}\,, \quad \delta\beta_{Y3N} = 2\, X_{Y1,Y2C}\,, \quad \delta\beta_{Y3O} = X_{Y2C,Y1}\,. \end{aligned} \tag{10.16}$$

From (10.13) there is one linear invariant, $\gamma_{\Phi 2B} + \beta_{Y2A}$. (10.15) contains 8 independent X's and 15 equations giving 7 linear invariants whereas in (10.16) 5 equations and 2 X's lead to 3 linear invariants.

These results may be used to verify the invariance of the consistency conditions obtained in section 8. The variations are either identically zero, using the antisymmetry of X, or lead to antisymmetrised products of three X's,

$$Y_{g_1,g_2,g_3} = \alpha_{g_1} X_{g_2,g_3} + \alpha_{g_3} X_{g_1,g_2} + \alpha_{g_2} X_{g_3,g_1} \,, \tag{10.17}$$

with $g_i = \mathrm{g}_i \ell_i r_i$. Given the definition of X in (10.1) necessarily $Y_{g_1,g_2,g_3} = 0$ and, from the antisymmetry of X_{g_1,g_2}, there is the identity

$$\alpha_{g_1} Y_{g_2,g_3,g_4} - \alpha_{g_2} Y_{g_3,g_4,g_1} + \alpha_{g_3} Y_{g_4,g_1,g_2} - \alpha_{g_4} Y_{g_1,g_2,g_3} = 0 \,, \tag{10.18}$$

and hence

$$\alpha_{g_1} Y_{g_2,g_3,g_4} - \alpha_{g_2} Y_{g_3,g_4,g_1} \not\supset X_{g_3,g_4} \,. \tag{10.19}$$

Apart from linear invariants there are also possible quadratic invariants

$$Q_\kappa = \kappa^{g_1 g_2} \alpha_{g_1} \alpha_{g_2} \,, \quad \kappa^{g_1 g_2} = \kappa^{g_2 g_1} \,, \tag{10.20}$$

if $\kappa^{g_1 g_2}$ is such that

$$\textstyle\sum_g \kappa^{g_1 g} \mathcal{N}_g{}^{g_2 g_3} = F^{g_1 g_2 g_3} = F^{[g_1 g_2 g_3]} \,, \tag{10.21}$$

as then $\delta Q_\kappa = \frac{2}{3} \sum_{g_1,g_2,g_3} F^{g_1 g_2 g_3} Y_{g_1,g_2,g_3}$. Higher order invariants are also possible as demonstrated later.

Applying this for (10.4) there are two quadratic invariants obtained from $Y_{\phi 1,\psi 1,y_1}$ and $Y_{\psi 1,y 1,\lambda 1b}$. However as a consequence of (10.19)

$$\begin{aligned}&\gamma_{\psi 1} Y_{\phi 1,\lambda 1a,\lambda 1b} - \beta_{\lambda 1a} Y_{\phi 1,\psi 1,\lambda 1b} \\ &= \gamma_{\psi 1}\big(\gamma_{\phi 1} X_{\lambda 1a,\lambda 1b} + \beta_{\lambda 1b} X_{\phi 1,\lambda 1a}\big) - \beta_{\lambda 1a}\big(\gamma_{\phi 1} X_{\psi 1,\lambda 1b} + \beta_{\lambda 1b} X_{\phi 1,\psi 1}\big) \,,\end{aligned} \tag{10.22}$$

leads to a further cubic invariant. At the next order from (10.7a), (10.7b) there are three possible Y's, $Y_{\phi 1,\psi 1,y 2f}$, $Y_{\phi 1,y 1,y 2f}$, $Y_{\psi 1,y 1,y 2f}$, which would lead to three potential quadratic invariants. Nevertheless these are not independent due to (10.18) and so there remain two quadratic invariants for the non planar coefficients. From (10.8a), (10.9a) and (10.10a) we can construct

$$\begin{aligned}&Y_{\phi 1,\psi 1,g} \,, && g = \phi 2a,\ \phi 2b,\ \psi 2a,\ \psi 2b,\ y 2a \,, \\ &Y_{\phi 1,\lambda 1a,g},\ Y_{\psi 1,\lambda 1b,g},\ Y_{\lambda 1a,\lambda 1b,g} \,, && g = \phi 2a,\ y 2a \,,\end{aligned} \tag{10.23}$$

which would appear to give 11 quadratic invariants. These are not independent due to the identity

$$\begin{aligned}&\gamma_{\psi 1}\big(\gamma_{\phi 1} Y_{\lambda 1a,\lambda 1b,g} + \beta_{\lambda 1b} Y_{\phi 1,\lambda 1a,g}\big) - \beta_{\lambda 1a}\big(\gamma_{\phi 1} Y_{\psi 1,\lambda 1b,g} + \beta_{\lambda 1b} Y_{\phi 1,\psi 1,g}\big) \\ &= \alpha_g\big(\gamma_{\psi 1} Y_{\phi 1,\lambda 1a,\lambda 1b} - \beta_{\lambda 1a} Y_{\phi 1,\psi 1,\lambda 1b}\big) \,, \quad g = \phi 2a, y 2a \,,\ \ \alpha_g = \gamma_{\phi 2a}, \beta_{y 2a} \,,\end{aligned} \tag{10.24}$$

so there remain 9 quadratic invariants when U(1) symmetry is imposed. In the general case there are additional invariants flowing from (10.8b), (10.9b) and (10.10b).

$$
\begin{aligned}
&Y_{\phi 1,\psi 1,g}\,,\ Y_{\phi 1,y1,g}\,,\ Y_{\psi 1,y1,g}\,, \qquad g=\phi 2c,\ \psi 2c,\ y2b,\ y2c,\ y2d, y2e\,, && (10.25a)\\
&Y_{\phi 1,y1,g}\,,\ Y_{\psi 1,y1,g}\,, \qquad g=\phi 2a,\ \phi 2b,\ \psi 2a,\ \psi 2b,\ y2a\,. && (10.25b)
\end{aligned}
$$

These are not independent due to (10.18). In (10.25a) we may then reduce to two sets of 6 and in (10.25b) to one set of 5. Possibilities are further restricted by requiring results are independent of ρ in (10.11) which leaves 16. There are then 11 independent Y's for which there is no constraint, a possible basis is given by

$$
\begin{aligned}
&Y_{\phi 1,y1,\phi 2a},\ Y_{\phi 1,y1,\phi 2c},\ Y_{\phi 1,y1,\psi 2b},\ Y_{\psi 1,y1,\psi 2c},\ Y_{\phi 1,y1,y2a},\ Y_{\phi 1,y1,y2b},\\
&Y_{\psi 1,y1,y2c},\ Y_{\phi 1,\psi 1,y2d},\ Y_{\phi 1,y1,y2d},\ Y_{\phi 1,\psi 1,y2e},\ Y_{\phi 1,y1,y2e}\,, \qquad (10.26)
\end{aligned}
$$

and 5 involving pairs of Y's formed from $Y_{\phi 1,\psi 1,g}$ or $Y_{\psi 1,y1,g}$, $g=\phi 2c,\, y2b$, $Y_{\phi 1,\psi 1,g}$ or $Y_{\phi 1,y1,g}$, $g=\psi 2c,\, y2c$, $Y_{\phi 1,y1,g}$ or $Y_{\psi 1,y1,g}$, $g=\psi 2a,\, \phi 2b$ which are ρ invariant. To achieve this (10.11) implies

$$
\begin{aligned}
&-\delta Y_{\phi 1,\psi 1,\phi 2c}=\delta Y_{\phi 1,\psi 1,y2b}=-\delta Y_{\phi 1,y1,\psi 2a}=\gamma_{\phi 1}\,\rho\,, \qquad \delta Y_{\phi 1,y1,\phi 2b}=2\,\gamma_{\phi 1}\,\rho\,,\\
&-\delta Y_{\phi 1,\psi 1,\psi 2c}=\delta Y_{\phi 1,\psi 1,y2c}=-\delta Y_{\psi 1,y1,\psi 2a}=\gamma_{\psi 1}\,\rho\,, \quad \delta Y_{\psi 1,y1,\phi 2b}=2\,\gamma_{\psi 1}\,\rho\,,\\
&-\delta Y_{\phi 1,y1,\psi 2c}=\delta Y_{\phi 1,y1,y2c}=\delta Y_{\psi 1,y1,\phi 2c}=-\delta Y_{\psi 1,y1,y2b}=\beta_{y1}\,\rho\,. \qquad (10.27)
\end{aligned}
$$

In (10.15) $Y_{\Phi 1,Y1,g}$, $g=\Phi 2A,\,\Phi 2B,\,Y2A,\,Y2B$, lead to four quadratic invariants and from (10.16) $Y_{\Phi 1,Y1,Y2C}$ to one more.

The various consistency conditions must be scheme invariant. We here check this by reducing their variations to sums of Y's which then show how they can be expressed in terms of quadratic invariants. At lowest order the variations of (9.4), using (10.4), are just

$$
\delta B_1=0\,, \qquad \delta B_2=4\,Y_{\phi 1,\psi 1,y1}\,, \qquad \delta B_3=0\,, \qquad \delta B_4=2\,\beta_{y1}\,Y_{\phi 1,\psi 1,y1}\,, \qquad (10.28)
$$

and for the non planar conditions (9.6) at the next order from (10.10a)

$$
\begin{aligned}
&\delta C_1=\delta C_2=\delta C_3=0\,,\\
&\delta C_4=-\delta C_5=2\,Y_{\psi 1,y1,y2f}\,, \quad \delta C_6=-\delta C_7=2\,Y_{\phi 1,y1,y2f}\,, \qquad (10.29)
\end{aligned}
$$

and for the variations of (9.7), (9.8), (9.9)

$$
\begin{aligned}
&\delta D_1=\delta D_2=\delta D_5=\delta D_7=0\,, \quad \delta D_3=-2\,Y_{\phi 1,\psi 1,y2a}\,, \quad \delta D_4=\delta D_9=-2\,Y_{\psi 1,y1,y2a}\,,\\
&\delta D_6=-\,12\,Y_{\phi 1,\phi 2a,\lambda 1a}\,, \quad \delta D_8=-2\,Y_{\phi 1,y1,y2a}\,, \quad \delta D_{10}=2\,Y_{\phi 1,y1,y2a}+6\,Y_{\phi 2a,y1,\lambda 1b}\,,\\
&\delta D_{11}=2\,\gamma_{\psi 1}Y_{\phi 1,\phi 2a,y1}-2\,\beta_{y1}Y_{\phi 1,\phi 2a,\psi 1}\,, \quad \delta D_{12}=-6\,\gamma_{\phi 2a}\,Y_{\psi 1,y1,\lambda 1b}\,,\\
&\delta D_{13}=12\,\gamma_{\phi 1}\,Y_{\phi 2a,\psi 1,\lambda 1b}-12\,\beta_{\lambda 1b}\,Y_{\phi 1,\phi 2a,\psi 1}+4\,B_1\,X_{\phi 1,\psi 1}\,,\\
&\delta D_{14}=\gamma_{\phi 1}\,Y_{y1,y2a,\lambda 1a}-3\,\gamma_{\phi 2a}\,Y_{y1,\lambda 1a,\lambda 1b}\\
&\qquad +6\,\beta_{y1}\,Y_{\phi 2a,\lambda 1a,\lambda 1b}-2\,\beta_{y1}\,Y_{\phi 1,y2a,\lambda 1a}+2\,\beta_{y2a}\,Y_{\phi 1,y1,\lambda 1a}-B_1\,X_{y1,\lambda 1a}\,,\\
&\delta D_{15}=0\,,
\end{aligned}
$$

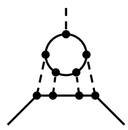
$$
\begin{aligned}
\delta D_{16} &= 4\,\gamma_{\phi 1}\big(Y_{\phi 1,y1,y2c}+Y_{\phi 1,\psi 2c,y1}\big)+2\,\beta_{y1}\big(Y_{\phi 1,\phi 2b,y1}-2\,Y_{\phi 1,\psi 2a,y1}\big)\\
&\quad -4\,\gamma_{\phi 2c}\,Y_{\phi 1,\psi 1,y1}-2\,B_2\,X_{\phi 1,y1}\,,\\
\delta D_{17} &= -4\,\gamma_{\phi 1}Y_{\phi 1,y1,y2d}+2\,\beta_{y1}\big(Y_{\phi 1,\phi 2c,y1}+Y_{\phi 1,y1,y2b}\big)\,,\\
\delta D_{18} &= 4\,\gamma_{\phi 1}Y_{\psi 1,y1,y2c}+4\,\gamma_{\psi 1}Y_{\phi 1,\psi 2c,y1}-2\,\beta_{y1}\big(2\,Y_{\phi 1,\psi 2b,y1}+Y_{\phi 2b,\psi 1,y1}\big)-2\,B_2\,X_{\psi 1,y1}\,,\\
\delta D_{19} &= 2\,\gamma_{\phi 1}\,Y_{\psi 1,\psi 2c,y1}+2\,\gamma_{\psi 1}\,Y_{\phi 1,y1,y2c}+2\,\beta_{y1}\,Y_{\phi 1,\psi 2b,y1}\\
&\quad -2\big(\gamma_{\psi 1}(\beta_{y2d}-\beta_{y2e})+2\,\beta_{y1}\,\gamma_{\psi 2c}\big)/\beta_{y1}\,Y_{\phi 1,\psi 1,y1}+2\big(B_3\,X_{\phi 1,y1}+B_4\,X_{\psi 1,y1}\big)/\beta_{y1}\,,\\
\delta D_{20} &= 4\,\gamma_{\psi 1}\big(Y_{\psi 1,y1,y2b}-Y_{\phi 2c,\psi 1,y1}\big)+2\,\beta_{y1}\big(2\,Y_{\psi 1,\psi 2a,y1}+Y_{\phi 2b,\psi 1,y1}\big)\\
&\quad -4\,\beta_{y2c}Y_{\phi 1,\psi 1,y1}+2\,B_2\,X_{\psi 1,y1}\,,\\
\delta D_{21} &= 2\,\gamma_{\psi 1}\big(Y_{\phi 1,\psi 2c,y1}+Y_{\psi 1,y1,y2c}\big)\,,\\
\delta D_{22} &= -2\,\gamma_{\psi 1}\big(Y_{\psi 1,y1,y2d}+Y_{\psi 1,y1,y2e}\big)+2\,\beta_{y1}\big(Y_{\psi 1,\psi 2c,y1}+Y_{\psi 1,y1,y2c}\big)\,,\\
\delta D_{23} &= 4\,\gamma_{\phi 1}Y_{\psi 1,y1,y2d}-2\,\beta_{y1}\big(Y_{\psi 1,y1,y2b}+Y_{\phi 1,y1,y2c}\big)\,,\\
\delta D_{24} &= 2\,\beta_{y1}\big(Y_{\phi 1,y1,y2d}-Y_{\phi 1,y1,y2e}\big)\,,\qquad \delta D_{25}=4\,\gamma_{\psi 1}Y_{\psi 1,y1,y2e}-4\,\beta_{y1}Y_{\psi 1,y1,y2c}\,,\\
\delta D_{26} &= 2\,\beta_{y1}\big(Y_{\psi 1,y1,y2d}-Y_{\psi 1,y1,y2e}\big)\,,\\
\delta D_{27} &= -8\,\beta_{y1}\big(Y_{\phi 1,\psi 1,y2d}+Y_{\phi 2c,\psi 1,y1}\big)-8\,\gamma_{\psi 1}Y_{\phi 1,y1,y2e}+4\,\beta_{y1}\big(Y_{\phi 1,y1,y2c}-Y_{\psi 1,y1,y2b}\big)\\
&\quad +8(2\,\beta_{y2d}+\beta_{y2e})\,Y_{\phi 1,\psi 1,y1}\,,\\
\delta D_{28} &= 4\,\beta_{y1}Y_{\psi 1,\psi 2c,y1}-2\,\gamma_{\psi 1}\big(Y_{\psi 1,y1,y2d}+Y_{\psi 1,y1,y2e}\big)\,,\\
\delta D_{29} &= 2\,\beta_{y1}\big(Y_{\psi 1,y1,y2b}-Y_{\phi 1,y1,y2c}+2\,Y_{\phi 1,\psi 2c,y1}\big)-2\,\gamma_{\phi 1}\big(3\,Y_{\psi 1,y1,y2d}+Y_{\psi 1,y1,y2e}\big)\\
&\quad +4\,\gamma_{\psi 1}\,Y_{\phi 1,y1,y2e}-4\,\beta_{y2d}\,Y_{\phi 1,\psi 1,y1}\,,\\
\delta D_{30} &= 2\,\beta_{y1}\big(Y_{\phi 1,\psi 1,y2d}+Y_{\phi 1,\phi 1,y2e}\big)+2\,(\beta_{y2d}+\beta_{y2e})\,Y_{\phi 1,\psi 1,y1}\,,\\
\delta D_{31} &= 4\,{\beta_{y1}}^2\,Y_{\psi 1,\psi 2b,y1}+2\,\beta_{y1}\gamma_{\psi 1}\big(Y_{\psi 1,\psi 2c,y1}+Y_{\psi 1,y1,y2c}\big)-2\,{\gamma_{\psi 1}}^2\,Y_{\psi 1,y1,y2e}+4\,B_3\,X_{\psi 1,y1}\,,\\
\delta D_{32} &= -2\,{\beta_{y1}}^2\,Y_{\phi 2b,\psi 1,y1}-4\,\beta_{y1}\gamma_{\psi 1}\big(Y_{\phi 1,\psi 1,y2e}+Y_{\psi 1,y1,y2b}\big)\\
&\quad -2\,\gamma_{\psi 1}\,(\beta_{y2d}-3\,\beta_{y2e})\,Y_{\phi 1,\psi 1,y1}-2\,B_2\,\beta_{y1}X_{\psi 1,y1}\,,\\
\delta D_{33} &= -4\,{\beta_{y1}}^2\,Y_{\phi 2b,\psi 1,y1}+4\,\beta_{y1}\gamma_{\psi 1}\big(Y_{\phi 1,\psi 2c,y1}-Y_{\psi 1,y1,y2b}\big)+8\,\beta_{y1}\gamma_{\phi 1}\,Y_{\psi 1,y1,y2c}\\
&\quad -4\,{\gamma_{\psi 1}}^2\,Y_{\phi 1,y1,y2e}+4\,\gamma_{\psi 1}\,(\beta_{y2d}+\beta_{y2e})\,Y_{\phi 1,\psi 1,y1}-4\,B_2\,\beta_{y1}\,X_{\psi 1,y1}\,,\\
\delta D_{34} &= 2\,{\beta_{y1}}^2\,\big(Y_{\phi 1,\psi 2b,y1}+Y_{\psi 1,\psi 2a,y1}\big)-2\,\beta_{y1}\gamma_{\phi 1}\,Y_{\psi 1,y1,y2c}-2\,\beta_{y1}\gamma_{\psi 1}\,Y_{\phi 1,y1,y2c}\\
&\quad -2\,\gamma_{\psi 1}(3\,\beta_{y2d}-\beta_{y2e})\,Y_{\phi 1,\psi 1,y1}+2\,B_3\,X_{\phi 1,y1}+2\,B_4\,X_{\psi 1,y1}\,,\\
\delta D_{35} &= 4\,{\beta_{y1}}^2\,Y_{\phi 1,\psi 2a,y1}-2\,\beta_{y1}\gamma_{\phi 1}\,\big(Y_{\phi 1,y1,y2c}+Y_{\phi 2c,\psi 1,y1}\big)+4\,\beta_{y1}\gamma_{\psi 1}\,Y_{\phi 1,y1,y2b}\\
&\quad -2{\gamma_{\phi 1}}^2\,\big(2\,Y_{\psi 1,y1,y2d}+Y_{\psi 1,y1,y2e}\big)+4\,\gamma_{\phi 1}\gamma_{\psi 1}\,Y_{\phi 1,y1,y2e}\\
&\quad +2\,\big(\gamma_{\phi 1}\,\beta_{y2d}+\beta_{y1}\,\gamma_{\phi 2c}\big)\,Y_{\phi 1,\psi 1,y1}+2\,B_4\,X_{\phi 1,y1}\,.
\end{aligned}
\tag{10.30}
$$

For the constraints obtained in the reduced $\mathcal{N}=\frac{1}{2}$ theory listed in sub section 9.1, corresponding to a Φ^3 interaction, then with (10.13), (10.15), (10.16),

$$
\begin{aligned}
&\delta S_0=\delta S_1=\delta S_2=0\,,\quad \delta S_3=-2\,Y_{\Phi 1,Y1,Y2A}\,,\quad \delta S_4=-6\,Y_{\Phi 1,Y1,\Phi 2A}\\
&\delta S_5=-4\,\beta_{Y1}\,Y_{\Phi 1,Y1,Y2A}+4\,\gamma_{\Phi 1}\,Y_{\Phi 1,Y1,Y2B}\,,\\
&\delta S_6=-4\,\beta_{Y1}\,Y_{\Phi 1,Y1,\Phi 2A}+8\,\gamma_{\Phi 1}\,Y_{\Phi 1,Y1,\Phi 2B}+4\,S_0\,X_{\Phi 1,Y1}\,,\\
&\delta S_7=8\,\gamma_{\Phi 1}\,\big(Y_{\Phi 1,Y1,Y2A}-Y_{\Phi 1,Y1,\Phi 2B}\big)\,,\\
&\delta S_8=-4\,Y_{\Phi 1,Y1,Y2C}\,,\quad \delta S_9=2\,Y_{\Phi 1,Y1,Y2C}\,.
\end{aligned}
\tag{10.31}
$$

11 Conclusion

The detailed results given here for β-functions and anomalous dimensions correspond to a $\overline{MS}$ regularisation scheme. As is well known attempting to extend supersymmetric theories away from their natural dimension is problematic and generally inconsistent and these issues affect any variant of dimensional regularisation [79–82]. For $\mathcal{N}=1$ supersymmetry and scalar fermion theories, without gauge fields, there are manifestly supersymmetric regularisation schemes and potential problems with $\overline{MS}$ arise only beyond three loops so long as the normalisation of fermion traces is chosen appropriately. These issues become significantly more severe for what we term $\mathcal{N}=\frac{1}{2}$ symmetry in this paper. Traces of three or more odd numbers of three dimensional Dirac gamma matrices are potentially non zero due to the appearance of the three dimensional antisymmetric symbol. This is not relevant for a fermion loop with three external scalars, due to momentum conservation, but such contributions are present if a fermion loop has five external scalar lines. Of course analogous problems with γ_5 are present with perturbative calculations using $\overline{MS}$ for chiral fermions. Such problems also arise in four dimensional chiral gauge theories for loops with two external vector lines and two external scalars and such loop diagrams contribute at four loops to 1PI contributions with two external vector lines and also to the Yukawa β-function [83]. In [83] it was shown how consistency with the a-function helps resolve some analogous γ_5 issues. In three dimensions similar potential problems arising for five vertex fermion loops as sub graphs occur at four loops in the Yukawa vertex renormalisation where the relevant diagrams are of the form



, (11.1)

together with various permutations of the internal vertices on the fermion loop. Such diagrams are primitive since there are no subdivergences when evaluated in four dimensions with some prescription for the contraction of two three dimensional ϵ symbols. A procedure for obtaining such contributions was described in [15]. However starting from four dimensional Dirac or Majorana fermions contributions related to (11.1) are absent [84]. The four dimensional fermion splits into two three dimensional fermions whose Yukawa couplings have the opposite sign, as shown here in appendix A.

In terms of the discussion of scheme changes and forming scheme invariants an alternative though equivalent approach is obtained within the framework of the Hopf algebra approach to Feynman diagrams [85–87]. The requirement of scheme invariance is identical with finding linear sums of graphs such that the Hopf algebra coproduct is cocommutative. A potentially interesting possibility is whether there is any extension of the Hopf algebraic approach to deriving consistency conditions, such as those considered here in section B.2, which might avoid some of the rather tortuous analysis required here and in [9].

The results obtained here suggest that there are potentially many interesting fixed points in scalar fermion theories once more than three scalar fields are allowed and the condition that there is just a single Yukawa coupling is relaxed. Finding a large n_f expansion for such theories may be tractable.

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12 Note added, fixed points with one scalar or one fermion field

Since this paper was finished Pannell and Stergiou [88] have investigated in detail possible fixed points in fermion scalar theories with low numbers of scalars and fermions. Many possiblities were discovered. Here we illustrate some results for either one scalar or one fermion which can be obtained quite easily.

For a single scalar field and n two component real fermions the couplings are just λ and y a real symmetric $n \times n$ matrix. For four dimensional Majorana fermions n should be even. At one loop order the β-functions in $4-\varepsilon$ dimensions reduce, with the usual rescaling to eliminate factors of 4π, to

$$\begin{aligned} \beta_y &= -\tfrac{1}{2}\,\varepsilon\, y + 3\, y^3 + \tfrac{1}{2}\, y\, \mathrm{tr}(y^2)\,, \\ \beta_\lambda &= -\,\varepsilon\,\lambda + 3\,\lambda^2 + 2\,\lambda\,\mathrm{tr}(y^2) - 12\,\mathrm{tr}(y^4)\,. \end{aligned} \tag{12.1}$$

To solve $\beta_y = 0,\ \beta_\lambda = 0$ we set $\varepsilon = 1$ and diagonalise y by an $O(n)$ transformation so that it has diagonal elements y_i, $i = 1, \dots, n$. The Yukawa β-function then gives

$$y_i = 6\, y_i^3 + y_i\, R\,, \quad R = \tilde{s} \sum_i y_i^2\,, \quad \Rightarrow \quad y_i^2 = \frac{1}{n+6}\,, \quad R = \frac{n}{n+6}\,. \tag{12.2}$$

Substituting in $\beta_\lambda = 0$ then gives

$$\lambda = \frac{1}{6(n+6)} \Big(6 - n \pm \sqrt{n^2 + 132\, n + 36} \Big)\,. \tag{12.3}$$

The stability matrix becomes

$$M = \begin{pmatrix} \partial_\lambda \beta_\lambda & \partial_\lambda \beta_{y_j} \\ \partial_{y_i} \beta_\lambda & \partial_{y_i} \beta_{y_j} \end{pmatrix}\,, \tag{12.4}$$

where y_i, λ are determined by (12.2), (12.3). Without any loss of generality we can take

$$y_i = \frac{1}{\sqrt{n+6}}\, s_i\,, \qquad s_i = \begin{cases} 1\,, & i = 1, \dots p, \\ -1, & i = p+1, \dots, n \end{cases}\,, \quad p = 0, \dots, n\,. \tag{12.5}$$

Then M becomes for the two possible solutions for λ in (12.3)

$$M = \frac{1}{n+6} \begin{pmatrix} \pm\sqrt{n^2 + 132\, n + 36} & 0 \\ * & 6\,\delta_{ij} + s_i s_j \end{pmatrix}\,. \tag{12.6}$$

The eigenvalues are then

$$\pm \tfrac{1}{n+6} \sqrt{n^2 + 132\, n + 36}\,, \quad 1\,, \quad \tfrac{6}{n+6} \text{ degeneracy } n-1\,, \tag{12.7}$$

where the eigenvectors for the last two cases can be given by

$$v_i = s_i \,, \quad v_1 = 1,\, v_i = \begin{cases} -1, & i = 2, \ldots, p \\ 1, & i = p+1, \ldots n \end{cases} \,, \; v_j = 0 \,, j \neq 1, i \,. \tag{12.8}$$

The fixed points corresponding to (12.5) are invariant under $O(p) \times O(n-p)$. Each point on the orbit corresponding to the coset $O(n)/O(p) \times O(n-p)$, of dimension $p(n-p)$, generated by the action of $O(n)$ defines an equivalent theory.

For one fermion and $n_s = m+1$ scalars then by an $O(n_s)$ rotation the Yukawa interaction can be considered to involve just one scalar σ while the remaining m scalars φ_a correspond to a purely scalar theory formed by quartic polynomial in φ together with interactions involving σ. If the maximal $O(m)$ symmetry is preserved then there are three couplings λ_1 corresponding the $O(m)$ invariant quartic $(\varphi^2)^2$, λ_2 for σ^4 and g for a $\varphi^2\sigma^2$ interaction. For $g = 0$ there are two decoupled theories. The resulting lowest order β-functions take the form

$$\begin{aligned} \beta_{\lambda_1} &= -\varepsilon\,\lambda_1 + (m+8)\lambda_1^2 + g^2 \,, \\ \beta_{\lambda_2} &= -\varepsilon\,\lambda_2 + 9\,\lambda_2^2 + m\,g^2 + 2\,y^2\lambda_2 - 4\,y^4 \,, \\ \beta_g &= -\varepsilon\,g + ((m+2)\lambda_1 + 3\,\lambda_2)g + 4\,g^2 + y^2 g \,. \end{aligned} \tag{12.9}$$

For $y = 0$ this is just a biconical theory, for $\lambda_1 = \lambda_2 = g$ there is an $O(m+1)$ symmetry. Other fixed points with g non zero are irrational and have two quadratic invariants, The Yukawa β-function gives at a fixed point $y^2 = \frac{1}{7}$. For $m = 4$, $\varepsilon = 1$ there is a rational solution $\lambda_1 = \frac{1}{21}$, $\lambda_2 = 0$, $g = \frac{1}{7}$, otherwise the $g \neq 0$ solutions are irrational. For the scalar invariants

$$S = 3m(m+2)\,\lambda_1^2 + 9\,\lambda_2^2 + m\,g^2 \,, \qquad a_0 = m(m+2)\,\lambda_1 + 3\,\lambda_2 + 2\,m\,g \,. \tag{12.10}$$

For the first few values of m these take the values, with $\varepsilon = 1$ and g, y non zero,

m	S				a_0			
0	$\frac{9}{49}$				$\frac{3}{7}$			
1	0.29635	0.23347			0.8081	0.6518		
2	0.42391	0.28682			1.2569	0.9066		
3	0.55581	0.34503			1.7530	1.1996		
4	0.68552	0.41049	0.66475	$\frac{32}{49}$	2.2841	1.5422	2.3582	$\frac{16}{7}$

(12.11)

Note that $\frac{32}{49} \approx 0.65306$. As a function of m, S is depicted in figure 14.
There are bifurcation points close to $m = 4, 6$ where two fixed points are created or annihilated.

At lowest order the stability matrix eigenvalues can be determined from the 3×3 matrix determined from $\beta_{\lambda_1}, \beta_{\lambda_2}, \beta_g$ analogously to (12.4). For $m = 1, 2$ the solution corresponding to the first case in (12.11) has three positive eigenvalues and is therefore RG stable. For $m = 4$ the two additional solutions each have a small eigenvalue of opposite sign. These both tend to zero as m approaches the bifurcation point, $m \approx 3.965$.

For $m \geq 2$ there are scalar theories with reduced symmetry which should lead to a ranged of additional fixed points with a Yukawa coupling to a single fermion.

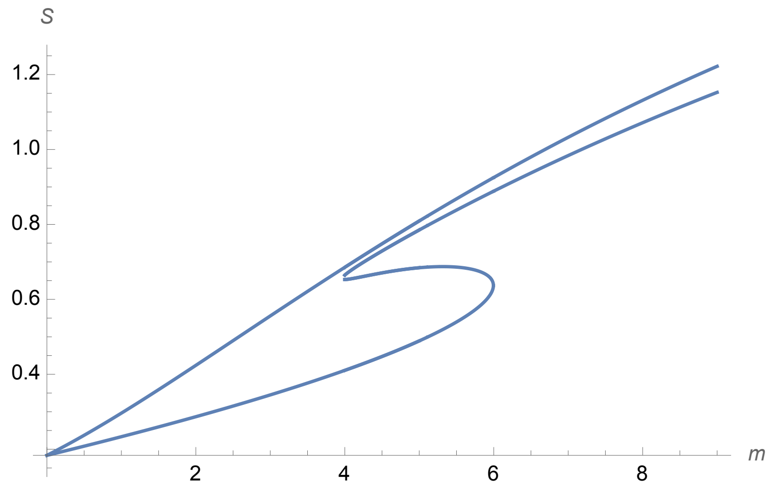

Figure 14. Scalar invariant S (12.10) as a function of the number of scalars m.

A Majorana fermions and their reduction

For a spinor field Ψ its conjugate $\bar{\Psi}$ is defined by

$$\bar{\Psi} = \Psi^\dagger A\,, \tag{A.1}$$

where A satisfies (the choice of both signs is a matter of convention, they are chosen here for later convenience)

$$A\,\gamma^\mu A^{-1} = -(\gamma^\mu)^\dagger\,, \qquad A^\dagger = -A\,, \tag{A.2}$$

with, for $d = 4$, the 4×4 Dirac matrices here defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\,\mathbb{1}\,, \qquad \eta^{\mu\nu} = \text{diag.}(-1,1,1,1)\,. \tag{A.3}$$

Taking $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, then $A\gamma_5 A^{-1} = -{\gamma_5}^\dagger$ and ${\gamma_5}^2 = \mathbb{1}$. Under a reflection in the x^1x^2 plane, charge conjugation and time reversal

$$\begin{aligned}
&\Psi \underset{\mathcal{R}_3}{\longrightarrow} \Psi_{r_3} = R\,\Psi\big|_{x^3\to -x^3}\,, && \bar{\Psi} \underset{\mathcal{R}_3}{\longrightarrow} \bar{\Psi}_{r_3} = \bar{\Psi}R^{-1}\big|_{x^3\to -x^3}\,,\\
&\Psi \underset{\mathcal{C}}{\longrightarrow} \Psi_c = C\,\bar{\Psi}^T\,, && \bar{\Psi} \underset{\mathcal{C}}{\longrightarrow} \bar{\Psi}_c = -\Psi^T C^{-1}\,,\\
&\Psi \underset{\mathcal{T}}{\longrightarrow} \Psi_t = T\,\Psi\big|_{x^0\to -x^0}\,, && \bar{\Psi} \underset{\mathcal{T}}{\longrightarrow} \bar{\Psi}_t = -\bar{\Psi}\,T^{-1}\big|_{x^0\to -x^0}\,,
\end{aligned} \tag{A.4}$$

where $\mathcal{T}$ is antilinear and

$$R^{-1}\gamma^\mu R = \begin{cases}\gamma^\mu, & \mu=0,1,2\\ -\gamma^3 & \mu=3\end{cases}, \qquad C^{-1}\gamma^\mu C = -(\gamma^\mu)^T, \qquad T^{-1}\gamma^{\mu *} T = \begin{cases}\gamma^\mu, & \mu=1,2,3\\ -\gamma^0 & \mu=0\end{cases}, \tag{A.5}$$

with $C^{-1}\gamma_5\,C = {\gamma_5}^T$, $T^{-1}{\gamma_5}^*\,T = \gamma_5$. In general

$$C^T = -C\,, \qquad A^* C^\dagger A = -C^{-1}\,, \qquad A^{-1}T^\dagger A = -T^{-1}\,, \qquad A^{-1}R^\dagger A = R^{-1}\,. \tag{A.6}$$

Using these results R, T can be given by

$$R = i\,\gamma^3\gamma_5\,, \qquad T = C^\dagger A\,\gamma^0\gamma_5 \quad \Rightarrow \quad TT^* = -\mathbb{1}\,, \tag{A.7}$$

and $\Psi \xrightarrow[\mathcal{R}_3{}^{2\prime}]{} R^2\Psi = \Psi$, $\Psi \xrightarrow[\mathcal{C}^{2\prime}]{} -CC^{-1T}\Psi = \Psi$, $\Psi \xrightarrow[\mathcal{T}^{2\prime}]{} TT^*\Psi = -\Psi$.

For a Majorana fermion

$$\Psi = \Psi_c\,, \tag{A.8}$$

and for present purposes we consider the Lagrangian

$$\mathcal{L}_M = -i\,\tfrac{1}{2}\bar\Psi\,\gamma\cdot\partial\,\Psi - i\,\tfrac{1}{2}\bar\Psi\,\mathcal{M}\,\Psi - i\,\tfrac{1}{2}\bar\Psi\,\mathcal{Y}^a\,\Psi\,\phi^a\,, \tag{A.9}$$

with both $\mathcal{M}$, $\mathcal{Y}^a$ real, symmetric and $[\mathcal{M},\gamma^\mu] = [\mathcal{Y}^a,\gamma^\mu] = 0$. With the conventions (A.1) and (A.2) ${\mathcal{L}_M}^\dagger = \mathcal{L}_M$.

For reduction to three dimensions a convenient basis is obtained by taking $\gamma^\mu \to \tilde\gamma^\mu$ with, adapting [84, 89, 90],

$$\tilde\gamma^\mu = \begin{pmatrix} \tilde\sigma^\mu & 0 \\ 0 & -\tilde\sigma^\mu \end{pmatrix}\,,\ \mu = 0,1,2\,, \qquad \tilde\sigma^\mu = (i\sigma_2, \sigma_3, -\sigma_1)\,, \qquad \tilde\gamma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,. \tag{A.10}$$

Here $\tilde\sigma^\mu$, $-\tilde\sigma^\mu$ correspond to the two inequivalent two dimensional irreducible representations for the $d = 3$ Dirac algebra. $\tilde\sigma^\mu i\sigma_2 = (-\mathbb{1}_2, \sigma_1, \sigma_3)$ form a basis for symmetric 2×2 matrices. For the $d = 4$ representation defined by (A.10)

$$\tau = \tilde\gamma^0\tilde\gamma^1\tilde\gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad \tilde\gamma_5 = i\tilde\gamma^0\tilde\gamma^1\tilde\gamma^2\tilde\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{A.11}$$

and

$$\begin{aligned} A = \tilde\gamma^0 &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}\,, & R &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \\ C = i\,\tilde\gamma^0\tilde\gamma^3\tilde\gamma_5 &= \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}\,, & T &= \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}\,. \end{aligned} \tag{A.12}$$

The representation in (A.10) can be related to the more commonplace chiral representation by

$$U\,\tilde\gamma^\mu\,U^{-1} = \begin{pmatrix} 0 & i\,\sigma^\mu \\ i\,\bar\sigma^\mu & 0 \end{pmatrix}\,, \quad U\,\tilde\gamma_5\,U^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,, \quad \sigma^\mu = (1, \boldsymbol{\sigma})\,, \quad \bar\sigma^\mu = (1, -\boldsymbol{\sigma})\,, \tag{A.13}$$

where

$$U = \frac{1}{2}\begin{pmatrix} \sigma_3 - \sigma_2 & \sigma_2 - \sigma_3 \\ \sigma_3\sigma_2 - 1 & \sigma_3\sigma_2 - 1 \end{pmatrix}\,, \quad U^{-1} = \frac{1}{2}\begin{pmatrix} \sigma_3 - \sigma_2 & \sigma_2\sigma_3 - 1 \\ \sigma_2 - \sigma_3 & \sigma_2\sigma_3 - 1 \end{pmatrix}\,. \tag{A.14}$$

With (A.10) the spinor field decomposes into two $d = 3$ two-component spinors as

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\,, \quad \bar\Psi = (\bar\psi_1, -\bar\psi_2)\,, \quad \bar\psi_a = {\psi_a}^\dagger i\sigma_2\,, \quad \psi_1 = {\psi_1}^*\,, \quad \psi_2 = -{\psi_2}^*\,. \tag{A.15}$$

Using the decomposition (A.15), and taking $\mathcal{M} \to \left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right)$, $\mathcal{Y}^a \to \begin{pmatrix} y^a & 0 \\ 0 & y^a \end{pmatrix}$, (A.9) becomes,

$$\begin{aligned}\mathcal{L}_M = & - i\,\tfrac{1}{2} \textstyle\sum_{a=1,2} \bar\psi_a\, \tilde\sigma \cdot \partial\, \psi_a - \tfrac{1}{2}(\bar\psi_1 \partial_3 \psi_2 + \bar\psi_2 \partial_3 \psi_1) \\ & - i\,\tfrac{1}{2}(\bar\psi_1 m \psi_1 - \bar\psi_2 m \psi_2) - i\,\tfrac{1}{2}(\bar\psi_1\, y^a \psi_1 - \bar\psi_2\, y^a \psi_2)\phi^a\,. \end{aligned} \tag{A.16}$$

For zero mass, $m = 0$, this has a $\mathbb{Z}_2$ symmetry where $\psi_1 \leftrightarrow \psi_2$, $\phi^a \to -\phi^a$. This ensures the cancellation of fermion loops with odd numbers of Yukawa vertices. For a symmetry $h^{-1} y^a h = y^a$, $h^{-1}\tilde\sigma^\mu h = \tilde\sigma^\mu$ with $h \in H_M \subset O(n_f)$ then in general there is a symmetry H_M but this extends to $(H_M \times H_M) \rtimes \mathbb{Z}_2$ when $d = 3$. For a single scalar this becomes $(O(n_f) \times O(n_f)) \rtimes \mathbb{Z}_2$ when $m = 0$.

With conventions from (A.4)

$$\begin{aligned} &(\psi_{1r_3},\, \psi_{2r_3}) = (\psi_1, -\psi_2)\big|_{x^3 \to -x^3}\,, && (\bar\psi_{1r_3},\, \bar\psi_{2r_3}) = (\bar\psi_1, -\bar\psi_2)\big|_{x^3 \to -x^3}\,, \\ &(\psi_{1t},\, \psi_{2t}) = (i\sigma_2\psi_2, i\sigma_2\psi_1)\big|_{x^0 \to -x^0}\,, && (\bar\psi_{1t},\, \bar\psi_{2t}) = (-\bar\psi_2 i\sigma_2, -\bar\psi_1 i\sigma_2)\big|_{x^0 \to -x^0}\,. \end{aligned} \tag{A.17}$$

Thus (A.16) is invariant under $\mathcal{R}_3, \mathcal{T}$ so long as the time reversal $\mathcal{T}$ transformation is combined with $\psi_1 \leftrightarrow \psi_2$, $i\,\bar\psi_1\psi_1 \underset{\mathcal{T}}{\longrightarrow} -i\,\bar\psi_{2t}\psi_{2t}$. If $\mathcal{T}$ is combined with $\psi_1 \leftrightarrow \psi_2$ the sign of the mass term is reversed. However for a reflections $\mathcal{R}_1$ or $\mathcal{R}_2$, corresponding to $x^1 \to -x^1$ or $x^2 \to -x^2$, then instead $\bar\psi_1\psi_1 \leftrightarrow -\bar\psi_2\psi_2$ and the individual mass terms are not invariant by themselves under either $\mathcal{R}_1$ or $\mathcal{R}_2$, as expected for three dimensional spinors. The three dimensional theory with just one two component spinor cannot have mass terms which preserve $\mathcal{T}$ or $\mathcal{R}_1$ invariance [91].

If the Yukawa interaction is modified to

$$\mathcal{L}_Y = \tfrac{1}{2}\,\bar\Psi\,\tilde\gamma_3\tilde\gamma_5\,\mathcal{Y}^a\Psi\,\phi^a = -i\,\tfrac{1}{2}(\bar\psi_1\, y^a\psi_1 + \bar\psi_2\, y^a\psi_2)\phi^a\,, \tag{A.18}$$

it is then necessary to include the four loop diagrams corresponding to (11.1). the symmetry for $d = 3$ is enhanced to a subgroup of $O(2n_f)$, for $n_s = 1$ the symmetry is $O(2n_f)$. This prescription allows for fermion loops with odd numbers of Yukawa vertices but is not relevant up to three loops. By including contributions corresponding to diagrams of the form (A.15) it was implicitly followed in [15] in their four loop calculation. Nevertheless (A.18) breaks Lorentz invariance for $d \neq 3$ though $O(2,1)$ is preserved. Applying dimensional regularisation, $d = 4 - \varepsilon$, the one loop counterterms necessary when starting from (A.9) or equivalently (A.16) are, for a single scalar σ,

$$\mathcal{L}_{\rm ct}^{(1)} = \frac{y^2}{(4\pi)^2\varepsilon}\Big(i\,\tfrac{1}{2}\bar\Psi\,\gamma\cdot\partial\,\Psi - i\,m\,\bar\Psi\Psi + (\partial\sigma)^2 + 6\,m^2\sigma^2 - i\,y\,\bar\Psi\Psi\,\sigma + \tfrac{1}{3}\,y^2\,\sigma^4\Big)\,. \tag{A.19}$$

For the modified Yukawa interaction (A.18) the result becomes

$$\begin{aligned}\mathcal{L}_{\rm ct}^{(1)}\big|_{\rm modified} = \frac{y^2}{(4\pi)^2\varepsilon}\Big(& i\,\tfrac{1}{2}\bar\Psi\,\gamma\cdot\partial\,\Psi - i\,\bar\Psi\,\gamma_3\partial_3\,\Psi - i\,m\,\bar\Psi\Psi + \tfrac{2}{3}\big((\partial\sigma)^2 - (\partial_3\sigma)^2\big) + 4\,m^2\sigma^2 \\ & + \tfrac{1}{2}\,y\,\bar\Psi\gamma_3\gamma_5\Psi\,\sigma + \tfrac{1}{12}\,y^2\,\sigma^4\Big)\,. \end{aligned} \tag{A.20}$$

Besides breaking Lorentz invariance explicitly in the kinetic terms the counterterm has different coefficients for the Yukawa and the quartic scalar terms. For a consistent flow it would be necessary to allow for modified kinetic terms for the scalar and fermion fields so that the propagation velocity is different in the 3-direction from the 1,2 directions, bringing in two new parameters consistent with the breaking $O(3,1)$ to $O(2,1)$. Whether the ε-expansion can be applied in this case is unclear.

An alternative possibilty, yet to be explored, is to take $\mathcal{M} \to \left(\begin{smallmatrix} m & 0 \\ 0 & M \end{smallmatrix}\right)$ and require $\Lambda \gg M \gg m$ for Λ some cutoff. This breaks Lorentz invariance more softly and should lead to ψ_2 being decoupled so as to generate an effective theory for ψ_1. This non Lorentz invariant theory can potentially be extended to $\mathcal{N} = \frac{1}{2}$ supersymmetry away from $d = 3$.

B Algebra of d and w tensors

The tensors defined by (8.7) and (8.14) satisfy identities which allow determination of eigenvalues,

$$\begin{aligned} d^{abef}\, d^{cdef} &= \tfrac{1}{n_s-1}\, a\big(\tfrac{1}{2}n_s(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) - \delta^{ab}\delta^{cd}\big) + e\, w^{abcd} + b\, d^{abcd} \,, \\ d^{abef}\, w^{cdef} &= w^{abef}\, d^{cdef} = f\, w^{abcd} + h\, d^{abcd} \,, \\ w^{abef}\, w^{cdef} &= \tfrac{1}{n_s-1}\, a'\big(\tfrac{1}{2}n_s(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) - \delta^{ab}\delta^{cd}\big) + e'\, w^{abcd} + b'\, d^{abcd} \,, \end{aligned} \tag{B.1}$$

where a, b are as in (8.9). For consistency

$$e\, a' = h\, a \,, \qquad f\, a' = b'\, a \,, \qquad f\, h = b'\, e \,, \qquad \tfrac{n_s}{n_s-1}\, a = f^2 + e\, h - b\, f - e\, e' \,. \tag{B.2}$$

The relevant eigenvalue equations necessary for obtaining the anomalous dimensions ϕ^2 operators given general perturbative results are

$$d^{abcd}\, v^{cd} = \mu\, v^{ab} \,, \qquad w^{abcd}\, v^{cd} = \nu\, v^{ab} \,, \tag{B.3}$$

for symmetric traceless v^{ab}. (B.1) then requires

$$\mu^2 = e\,\nu + b\,\mu + \tfrac{n_s}{n_s-1}\, a \,, \qquad \mu\,\nu = f\,\nu + h\,\mu \,, \qquad \nu^2 = e'\,\nu + b'\,\mu + \tfrac{n_s}{n_s-1}\, a' \,. \tag{B.4}$$

As a consequence of (B.2) the last equation is redundant and then eliminating ν leads to a cubic equation for μ whose solutions determine ν. There are thus three possibilities μ_i, ν_i. The associated degeneracies are then determined by

$$\textstyle\sum_i d_i = \frac{1}{2}(n_s - 1)(n_s + 2) \,, \qquad \sum_i d_i\, \mu_i = \sum_i d_i\, \nu_i = 0 \,. \tag{B.5}$$

For the examples of interest here the coefficients appearing in (B.1), besides a, b which are listed in (8.28), are given by

y^a	e	f	h	a'	b'	e'
1.	$\frac{(n-3)(n+6)}{27n}$	$\frac{(n-3)(n+6)(n_s-1)}{9n(n_s+2)}$	$\frac{2n(n_s+2)}{3(n_s-1)}$	$\frac{3n(n-2)(n+1)(n+4)}{(n_s-1)(n_s+2)}$	$2n$	$\frac{n^3-5n^2+14n+24}{6(n_s-1)}$
3.	$\frac{2(n^2-9)}{27n}$	$\frac{2(n^2-9)(n_s-1)}{9n(n_s+2)}$	$\frac{4n(n_s+2)}{3(n_s-1)}$	$\frac{24n^2(n^2-4)}{(n_s-1)(n_s+2)}$	$4n$	$\frac{2n(n_s+11)}{3(n_s-1)}$
4.	$\frac{2(n+1)}{27}$	$\frac{(n-8)(n+1)^2}{9(n_s+2)}$	$\frac{8(n_s+2)}{3(n+1)}$	$\frac{24(n-3)(n^2-4)}{(n+1)(n_s+2)}$	$4(n-8)$	$\frac{4(n_s+11)}{3(n+1)}$
5.	$\frac{n-1}{27}$	$\frac{(n-1)^2(n+8)}{18(n_s+2)}$	$\frac{4(n_s+2)}{3(n-1)}$	$\frac{6(n^2-4)(n+3)}{(n-1)(n_s+2)}$	$2(n+8)$	$\frac{2(n_s+11)}{3(n-1)}$
6.	$\frac{(n-6)(n+3)}{27n}$	$\frac{(n-6)(n+3)(n_s-1)}{9n(n_s+2)}$	$\frac{2n(n_s+2)}{3(n_s-1)}$	$\frac{3n(n-4)(n-1)(n+2)}{(n_s-1)(n_s+2)}$	$2n$	$\frac{n^3+5n^2+14n-24}{6(n_s-1)}$

(B.6)

Solving (B.4) and (B.5) gives

y^a	μ_1	ν_1	d_1	μ_2	ν_2	d_2	μ_3	ν_3	d_3
1.	$\frac{(n-3)(n+1)(n+6)}{6(n_s+2)}$	$\frac{n^2(n+1)}{2(n_s-1)}$	$\frac{1}{2}(n-1)(n+2)$	$\frac{(n-3)(n^2-4)}{3n(n_s+2)}$	$-\frac{2(n^2-4)}{n_s-1}$	$\frac{1}{24}n(n-1)(n+1)(n+6)$	$-\frac{(n-1)(n+4)(n+6)}{6n(n_s+2)}$	$\frac{(n-1)(n+4)}{n_s-1}$	$\frac{1}{12}n(n-3)(n+1)(n+2)$
3.	$\frac{2n(n^2-9)}{3(n_s+2)}$	$\frac{2n^3}{n_s-1}$	n^2-1	$\frac{2(n-3)(n-2)(n+1)}{3n(n_s+2)}$	$-\frac{4(n-2)(n+1)}{n_s-1}$	$\frac{1}{4}n^2(n-1)(n+3)$	$-\frac{2(n-1)(n+2)(n+3)}{3n(n_s+2)}$	$\frac{4(n-1)(n+2)}{n_s-1}$	$\frac{1}{4}n^2(n-3)(n+1)$
4.	$\frac{n(n-3)(n+1)}{3(n_s+2)}$	$\frac{2n(n-3)}{n+1}$	$\frac{1}{2}(n-1)(n+2)$	$\frac{(n-1)(n-8)}{3(n_s+2)}$	$-\frac{4(n-1)}{n+1}$	$\frac{1}{12}n(n-3)(n+1)(n+2)$	$-\frac{2(n+1)(n+2)}{3(n_s+2)}$	$\frac{8(n+2)}{n+1}$	$\frac{1}{24}n(n-3)(n-2)(n-1)$
5.	$\frac{n(n-1)(n+3)}{6(n_s+2)}$	$\frac{n(n+3)}{n-1}$	$\frac{1}{2}(n-2)(n+1)$	$\frac{(n-2)(n-1)}{3(n_s+2)}$	$-\frac{4(n-2)}{n-1}$	$\frac{1}{24}n(n+1)(n+2)(n+3)$	$-\frac{(n+1)(n+8)}{6(n_s+2)}$	$\frac{2(n+1)}{n-1}$	$\frac{1}{12}n(n-2)(n-1)(n+3)$
6.	$\frac{(n-6)(n-1)(n+3)}{6(n_s+2)}$	$\frac{n^2(n-1)}{2(n_s-1)}$	$\frac{1}{2}(n-2)(n+1)$	$\frac{(n-6)(n-4)(n+1)}{6n(n_s+2)}$	$-\frac{(n-4)(n+1)}{n_s-1}$	$\frac{1}{12}n(n-2)(n-1)(n+3)$	$-\frac{(n^2-4)(n+3)}{3n(n_s+2)}$	$\frac{2(n^2-4)}{n_s-1}$	$\frac{1}{24}n(n-6)(n-1)(n+1)$

(B.7)

The results in (8.28) and (B.6) satisfy

$$\begin{aligned}
\{a,b,e,f,h,a',b',e'\}_1\big|_{n\to -n} &= \{a,-b,-e,-f,-h,a',-b',-e'\}_5\,,\\
\{a,b,e,f,h,a',b',e'\}_2\big|_{n\to -n} &= \{a,-b,-e,-f,-h,a',-b',-e'\}_2\,,\\
\{a,b,e,f,h,a',b',e'\}_3\big|_{n\to -n} &= \{4a,-2b,-2e,-2f,-2h,4a',-2b',-2e'\}_4\,,
\end{aligned} \tag{B.8}$$

which are a reflection of $SO(n) \simeq Sp(-n)$, $SU(n) \simeq SU(-n)$ [63]. There are corresponding relations for the eigenvalues and degeneracies in (B.6).

B.1 Results for U(1) case

Corresponding to subsection 8.3 a similar analysis can be applied. The basic equations relevant in (8.74) are

$$d_{ij}{}^{mn} d_{mn}{}^{kl} = \tilde a\,(\delta_i{}^k \delta_j{}^l + \delta_i{}^l \delta_j{}^k) + \tilde b\, d_{ij}{}^{kl}\,, \qquad \tilde a = \tfrac{1}{2}(1-4q^2)\,, \ \ \tilde b = -4\,q\,, \tag{B.9}$$

and, with $n = rs$,

$$\begin{aligned}
d_{im}{}^{jn} d_{nk}{}^{ml} &= \tfrac{1}{n-1}\,\tilde a\,(n\,\delta_i{}^l \delta_k{}^j - \delta_i{}^j \delta_k{}^l) + \hat b\, d_{ik}{}^{jl} + \hat e\, w_{ik}{}^{jl}\,,\\
d_{im}{}^{jn} w_{nk}{}^{ml} &= w_{im}{}^{jn} d_{nk}{}^{ml} = \hat h\, d_{ik}{}^{jl} + \hat f\, w_{ik}{}^{jl}\,,\\
w_{im}{}^{jn} w_{nk}{}^{ml} &= \tfrac{n+1}{(n-1)^2}\,\tilde a\,(n\,\delta_i{}^l \delta_k{}^j - \delta_i{}^j \delta_k{}^l) + \hat b'\, d_{ik}{}^{jl} + \hat e'\, w_{ik}{}^{jl}\,,
\end{aligned} \tag{B.10}$$

where

$$\begin{aligned}
&\hat b = \tfrac{n-3}{4(n+1)}(r+s)\,, \quad \hat e = \tfrac{1}{4}(r-s)\,, \quad \hat b' = \tfrac{1}{4}(r+s)\,, \quad \hat e' = \tfrac{n+3}{4(n+1)}(r-s)\,,\\
&\hat f = \tfrac{n-1}{4(n+1)}(r+s)\,, \qquad \hat h = \tfrac{n+1}{4(n-1)}(r-s)\,.
\end{aligned} \tag{B.11}$$

Eigenvalues and degeneracies are determined as before. From (B.9) there are two eigenvalues μ_1, μ_2 corresponding to eigenvectors $v_{ij} = v_{ji}$ which are given, with their associated degeneracies, by

μ_1	d_1	μ_2	d_2
$\frac{(r-1)(s-1)}{n+1}$	$\frac{1}{4}rs(r+1)(s+1)$	$-\frac{(r+1)(s+1)}{n+1}$	$\frac{1}{4}rs(r-1)(s-1)$

. (B.12)

From (B.10) with (B.11) there are three sets of eigenvalues μ_u, ν_u, $u = 1, 2, 3$, for eigenvectors $v_i{}^j$, $v_i{}^i = 0$ where $d_{il}{}^{jk} v_k{}^l = \mu\, v_i{}^j$, $w_{il}{}^{jk} v_k{}^l = \nu\, v_i{}^j$

$$
\begin{array}{cccccc}
\mu_1 & \nu_1 & \mu_2 & \nu_2 & \mu_3 & \nu_3 \\
\multicolumn{2}{c}{d_1} & \multicolumn{2}{c}{d_2} & \multicolumn{2}{c}{d_3} \\
\hline
\frac{s(r^2-1)}{2(n+1)} & \frac{s(r^2-1)}{2(n-1)} & \frac{r(s^2-1)}{2(n+1)} & -\frac{r(s^2-1)}{2(n-1)} & -\frac{r+s}{2(n+1)} & \frac{r-s}{2(n-1)} \\
\multicolumn{2}{c}{s^2-1} & \multicolumn{2}{c}{r^2-1} & \multicolumn{2}{c}{(r^2-1)(s^2-1)}
\end{array} . \tag{B.13}
$$

The degeneracies correspond to expected representations of $\mathrm{SU}(r) \times \mathrm{SU}(s)$. Related results are given in [74].

The results in (8.89) are obtained from

$$
\begin{aligned}
\tilde{\beta}_{\hat{\lambda}}^{(2)}\big|_{y=0} &= -4(n+11)\hat{\lambda}^3 - 6(n+7)\,\tilde{a}\,g^2\hat{\lambda} - 2\,(\tilde{b} + 4\,\hat{b})\,\tilde{a}\,g^3\,, \\
\tilde{\beta}_g^{(2)}\big|_{y=0} &= -12(n+7)\,g\,\hat{\lambda}^2 - 12(\tilde{b} + 4\,\hat{b})\,g^2\hat{\lambda} - 8\,\tfrac{n-2}{n-1}\,\tilde{a}\,g^3 - 8(\tilde{b}\,\hat{b} + \hat{b}^2 - \hat{e}\,\hat{h})\,g^3\,.
\end{aligned} \tag{B.14}
$$

C Figures

For some of the cases listed in (8.17) and (12.11) figure 15 displays the fixed point values of

$$
8\,||\lambda||^2/n_s\,, \quad |\lambda|/n_s\,, \qquad ||\lambda||^2 = \lambda^{abcd}\lambda^{abcd}\,, \quad |\lambda| = \lambda^{aabb}\,, \tag{C.1}
$$

respectively orange, blue, as functions of $\log m$ where m gives the number of fermions. The log plots exhibit the symmetry following from (8.59). For the purely scalar theory, when $m = 0$, $8\,||\lambda||^2/n_s \le 1$ [75] and when this is satisfied $|\lambda|/n_s = \frac{1}{2}$. These bounds are clearly violated for a non zero number of fermions.

At the $O(n_s)$ or Heisenberg fixed point

$$
||\lambda_H|| = \tfrac{3n_s(n_s+2)}{(n_s+8)^2}\,, \qquad |\lambda_H| = \tfrac{n_s(n_s+2)}{n_s+8}\,. \tag{C.2}
$$

Of course $\lambda^{aabb} < 0$ is indicative of an unstable potential.

Corresponding results for $\mathrm{U}(r) \times \mathrm{U}(s)$ fixed points are given in figure 16 where plots of $24||\lambda||^2/rs$ and $|\lambda|/rs$ are given for various representative r, s as functions of m determining the number of fermions. The intercepts at $m = 0$, and also for $m \to \infty$, are determined by results for the purely complex scalar $\mathrm{U}(r) \times \mathrm{U}(s)$ theory and are obtained from (8.95), (8.97), (8.99). There are two or four fixed points according to whether $R_{rs} < 0$ or $R_{rs} > 0$. For $r = 5$, $s = 49$ $R_{rs} = 0$ which is a bifurcation point and then $24||\lambda||^2/rs = 1$.

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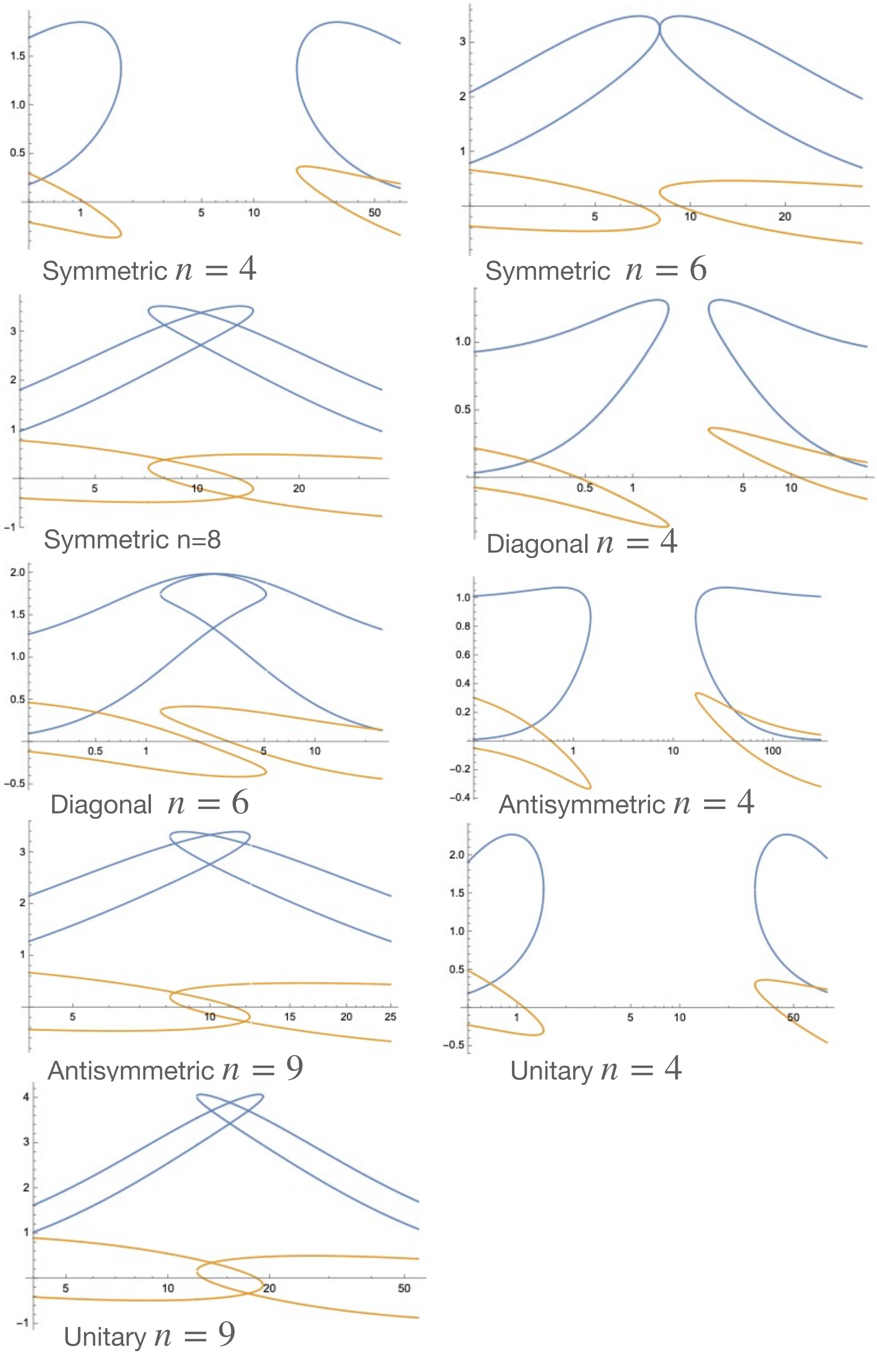

Figure 15. Fixed point values $|\lambda|/n_s$ (blue) and $8||\lambda||^2/n_s$ (orange) as a function of $\log m$.

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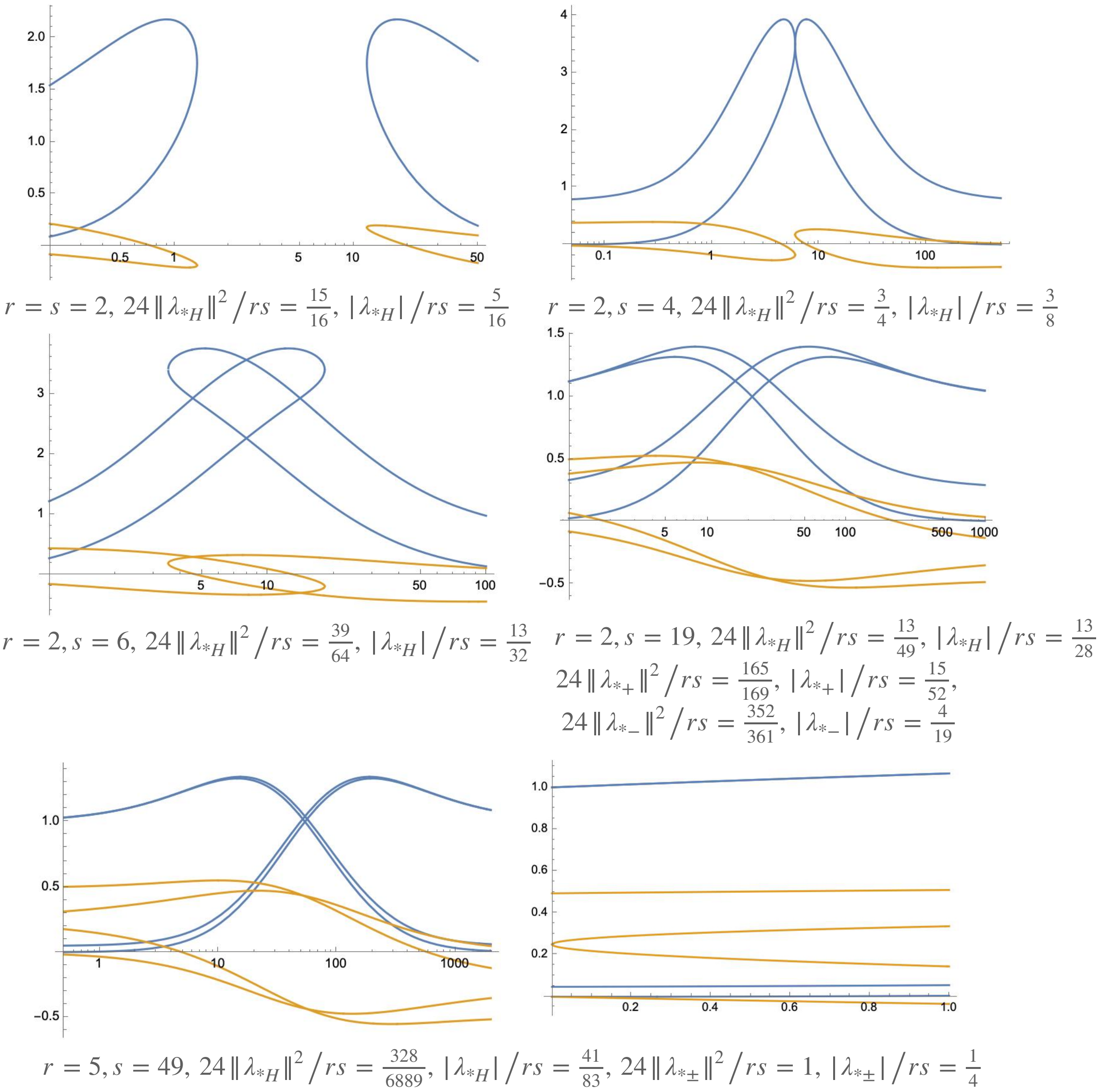

Figure 16. Fixed point values $|\lambda|/rs$ (blue) and $24||\lambda||^2/rs$ (orange) as a function of $\log m$ for the $U(r) \times U(s)$ theory.

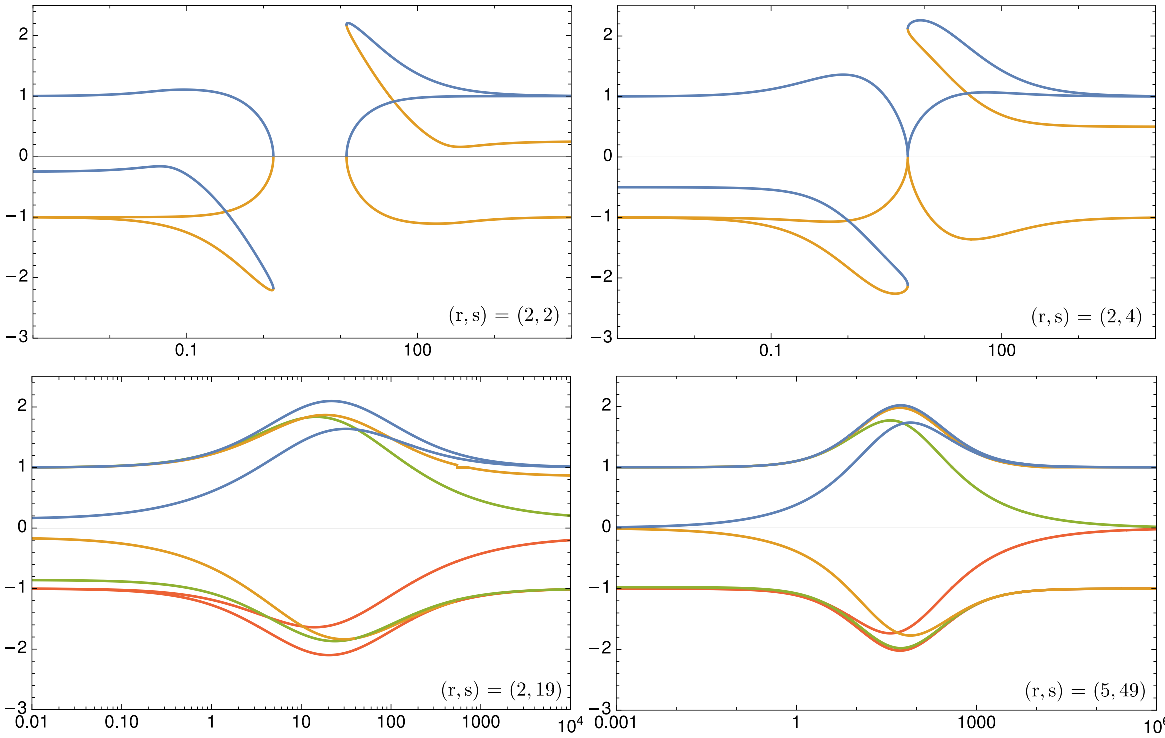

Figure 17. First coefficients of critical exponents as a function of m for indicated values of r and s. Different colour correspond to exponents of distinct fixed points.

For this case the eigenvalues κ of the stability matrix, taking $\varepsilon = 1$, as functions of m are given by figure 17.
The intercepts at $m = 0$ and $m \to \infty$ are given by (8.99) and, when $R_{rs} \geq 0$, (8.95). For $R_{rs} < 0$ there is, as $m \to 0$, one $\kappa \to 1$ and two $\kappa \to -1$, corresponding to the Gaussian fixed point. For $R_{rs} > 0$ there are two further cases with $\kappa \to 1$.

D U(1) scalar fermion theory consistency equations

The derivation of consistency relations for the three loop Yukawa couplings can be illustrated by restricting to case when U(1) symmetry is imposed. The number of couplings is significantly reduced but there remain non trivial relations which are a subset of the general case.[7]

At lowest order

$$T_{IJ}{}^{(2)}\, \mathrm{d}g^I \mathrm{d}'g^J = t_2\big(\mathrm{tr}(\mathrm{d}y^i\, \mathrm{d}'\bar{y}_i) + \mathrm{tr}(\mathrm{d}\bar{y}_i\, \mathrm{d}'y^i)\big)\,, \tag{D.1}$$

and

$$A^{(3)} = a_{3a}\,\big(\mathrm{tr}(\bar{y}_i\, y^i \bar{y}_j\, y^j) + \mathrm{tr}(y^i \bar{y}_i\, y^j \bar{y}_j)\big) + a_{3b}\,\mathrm{tr}(\bar{y}_i\, y^j)\,\mathrm{tr}(\bar{y}_j\, y^i)\,. \tag{D.2}$$

The consistency equations are then

$$a_{3a} = \tfrac{1}{2}\, t_2\, \gamma_{\psi 1}\,, \qquad a_{3b} = \tfrac{1}{2}\, t_2\, \gamma_{\phi 1}\,. \tag{D.3}$$

[7]The consistency relations in this case were discussed less completely in [20].

At the next order

$$
\begin{aligned}
{T_{IJ}}^{(3)}\, \mathrm{d}g^I \mathrm{d}'g^J\big|_{\mathrm{d}\lambda} &= t_{3\lambda}\, \mathrm{d}\lambda_{ij}{}^{kl}\, \mathrm{d}'\lambda_{kl}{}^{ij} \\
{T_{IJ}}^{(3)}\, \mathrm{d}g^I \mathrm{d}'g^J\big|_{\mathrm{d}\bar y} &= t_{3a_1}\, \big(\mathrm{tr}(\mathrm{d}\bar y_i\, \mathrm{d}'y^i\, \bar y_j\, y^j) + \mathrm{tr}(\mathrm{d}\bar y_i\, y^j \bar y_j\, \mathrm{d}'y^i)\big) \\
&\quad + t_{3a_2}\, \big(\mathrm{tr}(\mathrm{d}\bar y_i\, \mathrm{d}'y^j\, \bar y_j\, y^i) + \mathrm{tr}(\mathrm{d}\bar y_i\, y^i \bar y_j\, \mathrm{d}'y^j)\big) \\
&\quad + t_{3a_3}\, \big(\mathrm{tr}(\mathrm{d}\bar y_i\, y^i\, \mathrm{d}'\bar y_j\, y^j) + \mathrm{tr}(\mathrm{d}\bar y_i\, y^j\, \mathrm{d}'\bar y_j\, y^i)\big) \\
&\quad + t_{3b_1}\, \mathrm{tr}(\mathrm{d}\bar y_i\, \mathrm{d}'y^j)\, \mathrm{tr}(\bar y_j\, y^i) + t_{3b_2}\, \mathrm{tr}(\mathrm{d}\bar y_i\, y^j)\, \mathrm{tr}(\bar y_j\, \mathrm{d}'y^i) \\
&\quad + t_{3b_3}\, \mathrm{tr}(\mathrm{d}\bar y_i\, y^j)\, \mathrm{tr}(\mathrm{d}'\bar y_j\, y^i)\,,
\end{aligned} \tag{D.4}
$$

with ${T_{IJ}}^{(3)}\, \mathrm{d}g^I \mathrm{d}'g^J\big|_{\mathrm{d}y}$ obtained by conjugation and the notation indicates which 3 loop contribution in (D.2) each term corresponds to. At this order ${T_{IJ}}^{(3)}$ is symmetric. Four loop vacuum graphs give

$$
\begin{aligned}
A^{(4)} = {}& a_{4\lambda}\, (\lambda_{ij}{}^{kl}\lambda_{kl}{}^{mn}\lambda_{mn}{}^{ij} + 4\, \lambda_{ij}{}^{kl}\lambda_{km}{}^{\mathrm{in}}\lambda_{ln}{}^{jm}) \\
& + a_{4a}\, \lambda_{ij}{}^{kl}\, \mathrm{tr}(\bar y_l\, y^m)\, \lambda_{km}{}^{ij} + a_{4b}\, \lambda_{ij}{}^{kl}\, \mathrm{tr}(\bar y_k\, y^i \bar y_l\, y^j) \\
& + a_{4c}\, \mathrm{tr}(\bar y_i\, y^j)\, \mathrm{tr}(\bar y_j\, y^k)\, \mathrm{tr}(\bar y_k\, y^i) + a_{4d}\, \big(\mathrm{tr}(\bar y_i\, y^i\, \bar y_j\, y^j\, \bar y_k\, y^k) + \mathrm{tr}(y^i \bar y_i\, y^j \bar y_j\, y^k \bar y_k)\big) \\
& + a_{4e} \big(\mathrm{tr}(\bar y_i\, y^i\, \bar y_j\, y^k) + \mathrm{tr}(y^i \bar y_i\, y^k \bar y_j)\big)\, \mathrm{tr}(\bar y_k\, y^j) + a_{4f}\, \mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^k \bar y_j\, y^i) \\
& + a_{4g}\, \mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^i \bar y_j\, y^k)\,.
\end{aligned} \tag{D.5}
$$

In this case there are 11 relations corresponding to the number of inequivalent vertices in (D.5)

$$
\begin{aligned}
& a_{4\lambda} = \tfrac{1}{3}\, t_{3\lambda}\, \beta_{\lambda 1a}\,, \qquad a_{4a} = 3\, t_2\, \gamma_{\phi 2a} = 2\, t_{3\lambda}\, \gamma_{\phi 1}\,, \qquad a_{4b} = t_{3\lambda}\, \beta_{\lambda 1b} = \tfrac{1}{2}\, t_2\, \beta_{y2a}\,, \\
& a_{4c} = \tfrac{1}{3}(t_{3b_1} + t_{3b_2} + t_{3b_3})\, \gamma_{\phi 1}\,, \qquad a_{4d} = \tfrac{1}{3}(t_{3a_1} + t_{3a_2} + t_{3a_3})\, \gamma_{\psi 1}\,, \\
& a_{4e} = \tfrac{1}{2}\, t_2\, \gamma_{\phi 2b} + (t_{3b_2} + t_{3b_3})\, \gamma_{\psi 1} = t_2\, \gamma_{\psi 2a} + (t_{3a_2} + t_{3a_3})\, \gamma_{\phi 1} = t_{3a_1}\, \gamma_{\phi 1} + t_{3b_1}\, \gamma_{\psi 1}\,, \\
& a_{4f} = t_2\, \gamma_{\psi 2b} + (t_{3a_2} + t_{3a_3})\, \gamma_{\psi 1} = 2\, t_{3a_1}\, \gamma_{\psi 1}\,, \qquad a_{4g} = \tfrac{1}{3}\, t_2\, \beta_{y2f}\,.
\end{aligned} \tag{D.6}
$$

The $\mathrm{O}(\beta^2)$ freedom in A at this order corresponds just to the variations

$$
\delta a_{4c} = \epsilon_2\, {\gamma_{\phi 1}}^2\,, \quad \delta a_{4d} = \epsilon_2\, {\gamma_{\psi 1}}^2\,, \quad \delta a_{4e} = 2\, \epsilon_2\, \gamma_{\phi 1}\gamma_{\psi 1}\,, \quad \delta a_{4f} = 2\, \epsilon_2\, {\gamma_{\psi 1}}^2\,, \tag{D.7}
$$

which is compatible with (D.6) if

$$
\delta t_{3a_1} = \delta t_{3a_2} = \delta t_{3a3} = \epsilon_2\, \gamma_{\psi 1}\,, \qquad \delta t_{3b_1} = \delta t_{3b_2} = \delta t_{3b_3} = \epsilon_2\, \gamma_{\phi 1}\,. \tag{D.8}
$$

There is one non trivial relation at this order from the a_{4a}, a_{4b} equations requiring the vanishing of

$$
U_0 = \gamma_{\phi 1}\, \beta_{y2a} - 3\, \beta_{\lambda 1b}\, \gamma_{\phi 2a}\,, \tag{D.9}
$$

which is identical to B_1 in (9.4).

At the next order we initially focus on contributions involving λ

$$
\begin{aligned}
{T_{IJ}}^{(4)}\, \mathrm{d}g^I \mathrm{d}'g^J\big|_{\mathrm{d}\lambda} = {}& t_{4\lambda} (\mathrm{d}\lambda_{ij}{}^{kl}\, \mathrm{d}'\lambda_{kl}{}^{mn}\, \lambda_{mn}{}^{ij} + \mathrm{d}'\lambda_{ij}{}^{kl}\, \mathrm{d}\lambda_{kl}{}^{mn}\, \lambda_{mn}{}^{ij} + 8\, \mathrm{d}\lambda_{ij}{}^{kl}\, \mathrm{d}'\lambda_{km}{}^{\mathrm{in}}\, \lambda_{ln}{}^{jm}) \\
& + t_{4a_1} (\mathrm{d}\lambda_{ij}{}^{kl}\, \mathrm{d}'\lambda_{km}{}^{ij} + \mathrm{d}'\lambda_{ij}{}^{kl}\, \mathrm{d}\lambda_{km}{}^{ij})\, \mathrm{tr}(\bar y_l\, y^m) \\
& + t_{4a_2}\, \mathrm{d}\lambda_{ij}{}^{kl} (\lambda_{kl}{}^{im}\, \mathrm{tr}(\bar y_m\, \mathrm{d}'y^j) + \mathrm{tr}(\mathrm{d}'\bar y_l\, y^m)\lambda_{km}{}^{ij}) \\
& + t_{4a_3}\, \mathrm{d}\lambda_{ij}{}^{kl}\, (\lambda_{kl}{}^{im}\, \mathrm{tr}(\mathrm{d}'\bar y_m\, y^j) + \mathrm{tr}(\bar y_l\, \mathrm{d}'y^m)\lambda_{km}{}^{ij}) \\
& + t_{4b_1}\, \mathrm{d}\lambda_{ij}{}^{kl}\, (\mathrm{tr}(\bar y_k\, y^i \bar y_l\, \mathrm{d}'y^j) + \mathrm{tr}(\bar y_k\, y^i \mathrm{d}'\bar y_l\, y^j))\,,
\end{aligned} \tag{D.10}
$$

and

$$T_{IJ}^{(4)}\,\mathrm{d}g^I \mathrm{d}'g^J\big|_{\mathrm{d}\bar y} = t_{4a_4}\,\mathrm{d}'\lambda_{ij}{}^{kl}\,\mathrm{tr}(\mathrm{d}\bar y_l\, y^m)\,\lambda_{km}{}^{ij} + t_{4a_5}\,\mathrm{d}'\lambda_{ij}{}^{kl}\,\lambda_{kl}{}^{im}\,\mathrm{tr}(\mathrm{d}\bar y_m\, y^j)$$
$$+ t_{4a_6}\,\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}'y^j)\,\lambda_{jm}{}^{kl}\,\lambda_{kl}{}^{mi} + t_{4b_2}\,\mathrm{d}'\lambda_{ij}{}^{kl}\,\mathrm{tr}(\mathrm{d}\bar y_k\, y^i \bar y_l\, y^j)$$
$$+ t_{4b_3}\left(\mathrm{tr}(\mathrm{d}\bar y_i\,\mathrm{d}'y^k\,\bar y_j\, y^l) + \mathrm{tr}(\mathrm{d}\bar y_i\, y^k \bar y_j\,\mathrm{d}'y^l)\right)\lambda_{kl}{}^{ij}$$
$$+ t_{4b_4}\,\mathrm{tr}(\mathrm{d}\bar y_i\, y^k\,\mathrm{d}'\bar y_j\, y^l)\,\lambda_{kl}{}^{ij}\,, \tag{D.11}$$

with

$$A^{(5)}\big|_\lambda = a_{\lambda 5a}(\lambda_{ij}{}^{kl}\lambda_{kl}{}^{mn}\lambda_{mn}{}^{pq}\lambda_{pq}{}^{ij} + 8\,\lambda_{ij}{}^{kl}\lambda_{lm}{}^{mn}\lambda_{mn}{}^{pq}\lambda_{pq}{}^{ij})$$
$$+ a_{\lambda 5b}(\lambda_{ij}{}^{kl}\lambda_{km}{}^{jn}\lambda_{ln}{}^{pq}\lambda_{pq}{}^{im} + \lambda_{ij}{}^{kl}\lambda_{lm}{}^{\mathrm{in}}\lambda_{kp}{}^{mq}\lambda_{nq}{}^{jp}) + a_{\lambda 5c}\,\lambda_{ik}{}^{lm}\lambda_{lm}{}^{kj}\lambda_{jn}{}^{pq}\lambda_{pq}{}^{ni}$$
$$+ a_{5a}\left(\lambda_{ij}{}^{mn}\lambda_{mn}{}^{pq}\lambda_{pq}{}^{jk} + 4\,\lambda_{ij}{}^{mn}\lambda_{mp}{}^{jq}\lambda_{nq}{}^{pk}\right)\mathrm{tr}(\bar y_k\, y^i)$$
$$+ a_{5b}\left(\lambda_{ij}{}^{mn}\lambda_{mn}{}^{kl} + 2\,\lambda_{(i|m}{}^{nl}\lambda_{j)n}{}^{mk}\right)\mathrm{tr}(\bar y_k\, y^i)\,\mathrm{tr}(\bar y_l\, y^j)$$
$$+ a_{5c}\,\lambda_{ik}{}^{lm}\lambda_{lm}{}^{kj}\,\mathrm{tr}(\bar y_j\, y^n)\,\mathrm{tr}(\bar y_n\, y^i)$$
$$+ a_{5d}\,\lambda_{ik}{}^{lm}\lambda_{lm}{}^{kj}\left(\mathrm{tr}(\bar y_j\, y^i\,\bar y_n\, y^n) + \mathrm{tr}(\bar y_j\, y^n \bar y_n\, y^i)\right)$$
$$+ a_{5e}\,\lambda_{ij}{}^{mn}\lambda_{mn}{}^{kl}\,\mathrm{tr}(\bar y_k\, y^i\,\bar y_l\, y^j) + a_{5f}\,\lambda_{(i|m}{}^{(k|n}\lambda_{j)n}{}^{l)m}\,\mathrm{tr}(\bar y_k\, y^i\,\bar y_l\, y^j)$$
$$+ a_{5g}\,\lambda_{ij}{}^{kl}\left(\mathrm{tr}(\bar y_k\, y^m)\,\mathrm{tr}(\bar y_m\, y^i\,\bar y_l\, y^j) + \mathrm{tr}(\bar y_k\, y^i\,\bar y_l\, y^m)\,\mathrm{tr}(\bar y_m\, y^j)\right)$$
$$+ a_{5h}\,\lambda_{ij}{}^{kl}\left(\mathrm{tr}(\bar y_m\, y^m \bar y_k\, y^i\,\bar y_l\, y^j) + \mathrm{tr}(\bar y_m\, y^i\,\bar y_k\, y^j\,\bar y_l\, y^m)\right)$$
$$+ a_{5i}\,\lambda_{ij}{}^{kl}\,\mathrm{tr}(\bar y_k\, y^m \bar y_l\, y^i\,\bar y_m\, y^j)\,. \tag{D.12}$$

If T_{IJ} is symmetric only if

$$t_{4a_2} = t_{4a_4}\,, \qquad t_{4a_3} = t_{4a_5}\,, \qquad t_{4b_1} = t_{4b_2}\,. \tag{D.13}$$

The associated consistency equations deriving from $\mathrm{d}\lambda$ variations in $A^{(5)}$ are then

$$a_{\lambda 5a} = \tfrac{1}{2}\,t_{4\lambda}\,\beta_{\lambda 1a}\,, \quad a_{\lambda 5b} = 2\,t_{3\lambda}\,\beta_{\lambda 2a} + 4\,t_{4\lambda}\,\beta_{\lambda 1a}\,, \quad a_{\lambda 5c} = 3\,t_{3\lambda}\,\gamma_{\phi 2a}\,,$$
$$a_{5a} = t_{3\lambda}\,\beta_{\lambda 2b} + 4\,t_{4\lambda}\,\gamma_{\phi 1} = 2\,t_{4\lambda}\,\gamma_{\phi 1} + t_{4a_1}\,\beta_{\lambda 1a}\,, \qquad a_{5b} = t_{4a_1}\,\gamma_{\phi 1}\,,$$
$$a_{5c} = (t_{4a_1} + t_{4a_2} + t_{4a_3})\gamma_{\phi 1}\,, \qquad a_{5d} = t_{3\lambda}\,\gamma_{\phi 2b} + (t_{4a_2} + t_{4a_3})\gamma_{\psi 1}\,,$$
$$a_{5e} = t_{3\lambda}\,\beta_{\lambda 2d} + t_{4\lambda}\,\beta_{\lambda 1b}\,, \quad a_{5f} = 2\,t_{3\lambda}\,\beta_{\lambda 2c} + 4\,t_{4\lambda}\,\beta_{\lambda 1b}\,, \quad a_{5g} = t_{4a_1}\,\beta_{\lambda 1b} + t_{4b_1}\,\gamma_{\phi 1}\,,$$
$$a_{5h} = 2\,t_{3\lambda}\,\beta_{\lambda 2e} + 2\,t_{4b_1}\,\gamma_{\psi 1}\,, \qquad a_{5i} = 2\,t_{3\lambda}\,\beta_{\lambda 2g}\,, \tag{D.14}$$

and from $\mathrm{d}\bar y$ variations

$$a_{5a} = t_2\,\gamma_{\phi 3a} + (t_{4a_4} + t_{4a_5})\beta_{\lambda 1a}\,, \quad a_{5b} = \tfrac{1}{2}\,t_2\,\gamma_{\phi 3b} + \tfrac{1}{2}(t_{4a_4} + t_{4a_5})\gamma_{\phi 1}\,,$$
$$a_{5c} = 3\,t_{3b_2}\,\gamma_{\phi 2a} + (t_{4a_5} + t_{4a_6})\gamma_{\phi 1} = 3\,(t_{3b_1} + t_{3b_3})\,\gamma_{\phi 2a} + t_{4a_4}\,\gamma_{\phi 1}\,,$$
$$a_{5d} = 3\,t_2\,\gamma_{\psi 3a} + 3(t_{3a_2} + t_{3a_3})\gamma_{\phi 2a} = 3\,t_{3a_1}\,\gamma_{\phi 2a} + t_{4a_6}\,\gamma_{\psi 1}\,,$$
$$a_{5e} = \tfrac{1}{2}\,t_2\,\beta_{y3c} + \tfrac{1}{2}\,t_{4b_2}\,\beta_{\lambda 1a}\,, \qquad a_{5f} = 2\,t_2\,\beta_{y3b} + 2\,t_{4b_2}\,\beta_{\lambda 1a}\,,$$
$$a_{5g} = t_2\,\beta_{y3b} + (t_{4b_2} + t_{4b_3})\gamma_{\phi 1} = t_2\,\beta_{y3d} + (t_{4b_2} + t_{4b_4})\gamma_{\phi 1} = t_{3b_1}\,\beta_{y2a} + t_{4b_2}\,\gamma_{\phi 1}$$
$$= t_2(\gamma_{\phi 3c} - \upsilon_{\phi 3c}) + t_{3b_2}\,\beta_{y2a} + t_{4a_5}\,\beta_{\lambda 1b} = t_2(\gamma_{\phi 3c} + \upsilon_{\phi 3c}) + t_{3b_3}\,\beta_{y2a} + t_{4a_4}\,\beta_{\lambda 1b}\,,$$
$$a_{5h} = t_2\,\beta_{y3j} + (t_{4b_3} + t_{4b_4})\gamma_{\psi 1} = t_{3a_1}\,\beta_{y2a} + t_{4b_3}\,\gamma_{\psi 1} = t_2\,\gamma_{\psi 3b} + (t_{3a_2} + t_{3a_3})\beta_{y2a}\,,$$
$$a_{5i} = t_2\,\beta_{y3f} = t_2\,\beta_{y3l}\,. \tag{D.15}$$

For the λ dependent contributions to $A^{(5)}$ there is an additional $\mathrm{O}(\beta^2)$ freedom as well as that corresponding to (D.7)

$$\begin{aligned} &\delta a_{\lambda 5a} = \epsilon_{3\lambda}\, {\beta_{\lambda 1a}}^2\,, \qquad \delta a_{\lambda 5b} = 8\,\epsilon_{3\lambda}\, {\beta_{\lambda 1a}}^2\,, \qquad \delta a_{5a} = 8\,\epsilon_{3\lambda}\,\gamma_{\phi 1}\,\beta_{\lambda 1a}\,, \qquad \delta a_{5b} = 4\,\epsilon_{3\lambda}\, {\gamma_{\phi 1}}^2\,, \\ &\delta a_{5c} = 6\,\epsilon_2\,\gamma_{\phi 1}\gamma_{\phi 2a} + 4\,\epsilon_{3\lambda}\, {\gamma_{\phi 1}}^2\,, \qquad \delta a_{5d} = 6\,\epsilon_2\,\gamma_{\psi 1}\gamma_{\phi 2a}\,, \qquad \delta a_{5e} = 2\,\epsilon_{3\lambda}\,\beta_{\lambda 1a}\,\beta_{\lambda 1b}\,, \\ &\delta a_{5f} = 8\,\epsilon_{3\lambda}\,\beta_{\lambda 1a}\,\beta_{\lambda 1b}\,, \qquad \delta a_{5g} = \epsilon_2\,\gamma_{\phi 1}\,\beta_{y2a} + 4\,\epsilon_{3\lambda}\,\gamma_{\phi 1}\,\beta_{\lambda 1b}\,, \qquad \delta a_{5h} = 2\,\epsilon_2\,\gamma_{\psi 1}\,\beta_{y2a}\,. \end{aligned} \tag{D.16}$$

Correspondingly in addition to (D.8) we need to take

$$\begin{aligned} &\delta t_{4\lambda} = 2\,\epsilon_{3\lambda}\,\beta_{\lambda 1a}\,, \qquad \delta t_{4a_1} = \delta t_{4a_4} = \delta t_{4a_5} = 4\,\epsilon_{3\lambda}\,\gamma_{\phi 1}\,, \qquad \delta t_{4b_2} = 4\,\epsilon_{3\lambda}\,\beta_{\lambda 1b}\,, \\ &\delta t_{4a_2} = \delta t_{4a_3} = \delta t_{4a_6} = 3\,\epsilon_2\,\gamma_{\phi 2a}\,, \qquad \delta t_{4b_1} = \delta t_{4b_3} = \delta t_{4b_4} = \epsilon_2\,\beta_{y2a}\,, \end{aligned} \tag{D.17}$$

to ensure (D.14) and (D.15) are invariant. These variations do not preserve (D.13) as expected.

By eliminating the a_5's, t_4's from (D.14), and then the t_3's using (D.6) there remain 10 necessary consistency relations which become

$$\begin{aligned} U_1 &= \beta_{y3f} - \beta_{y3l}\,, \qquad U_2 = \gamma_{\phi 1}\,\beta_{y3f} - 3\,\gamma_{\phi 2a}\,\beta_{\lambda 2g}\,, \\ U_3 &= 2\,\gamma_{\phi 1}(\beta_{y3b} - \beta_{y3c}) - 3(\beta_{\lambda 2c} - 2\,\beta_{\lambda 2d})\,\gamma_{\phi 2a}\,, \\ U_4 &= 2\,(\gamma_{\phi 1}\,\gamma_{\phi 3a} - \beta_{\lambda 1a}\,\gamma_{\phi 3b}) + 3\,\beta_{\lambda 2b}\,\gamma_{\phi 2a}\,, \\ U_5 &= \gamma_{\psi 1}\,\beta_{y3e} + \gamma_{\phi 1}\,\gamma_{\psi 3b} - \beta_{y2a}\,\gamma_{\psi 2a}\,, \\ U_6 &= \gamma_{\psi 1}(\gamma_{\psi 1}\,\beta_{y3d} - \gamma_{\phi 1}\,\beta_{y3j}) - \beta_{y2a}(\gamma_{\psi 1}\,\gamma_{\psi 2a} - \gamma_{\phi 1}\,\gamma_{\psi 2b})\,, \\ U_7 &= 2\,\gamma_{\psi 1}\big(2\,\gamma_{\phi 1}\,\gamma_{\phi 3c} - \beta_{\lambda 1b}\,\gamma_{\phi 3b} + \gamma_{\phi 1}\,\beta_{y3e}\big) - \gamma_{\phi 1}\big(\beta_{y2a}\,\gamma_{\phi 2b} - 6\,\gamma_{\phi 2a}\,\beta_{\lambda 2e}\big)\,, \\ U_8 &= \gamma_{\psi 1}\big(\gamma_{\phi 1}\,\gamma_{\psi 3b} - 2\,\gamma_{\psi 1}\,\gamma_{\phi 3c} - 3\,\beta_{\lambda 1b}\,\gamma_{\psi 3a} + \beta_{y2a}(\gamma_{\phi 2b} + \gamma_{\psi 2a}) - 3\,\beta_{\lambda 2e}\,\gamma_{\phi 2a}\big) \\ &\quad - \gamma_{\phi 1}\,\beta_{y2a}\,\gamma_{\psi 2b}\,, \\ U_9 &= {\gamma_{\psi 1}}^2\big(2\,\gamma_{\phi 1}\,\beta_{y3b} - \beta_{\lambda 1a}\,\beta_{y3e} + \beta_{y2a}\,\beta_{\lambda 2b} - 3\,\gamma_{\phi 2a}\,\beta_{\lambda 2c}\big) \\ &\quad - \beta_{\lambda 1a}\big(\beta_{y2a}(\gamma_{\psi 1}\gamma_{\psi 2a} - \gamma_{\phi 1}\gamma_{\psi 2b}) + 3\,\gamma_{\psi 1}\gamma_{\phi 2a}\,\beta_{\lambda 2e}\big)\,, \end{aligned} \tag{D.18}$$

and

$$2\,\gamma_{\psi 1}\,\gamma_{\phi 2a}\,\upsilon_{\phi 3c} + \big(\gamma_{\phi 1}\,\gamma_{\psi 3a} - \gamma_{\phi 2a}\,\gamma_{\psi 2a}\big)\,\beta_{y2a} = 0\,. \tag{D.19}$$

This result for $\upsilon_{\phi 3c}$ is equivalent to the one in (9.11) using $B_1 = U_{11} = 0$. Substituting for one and two loop coefficients the relations reduce to

$$\begin{aligned} &\beta_{y3b} = \beta_{y3l} = 2\,, \quad \beta_{y3b} - \beta_{y3b} = 0\,, \quad \beta_{y3c} - \beta_{y3b} = \beta_{y3d} - \beta_{y3j} = 1\,, \\ &\beta_{y3b} + \gamma_{\psi 3b} = \tfrac{3}{2}\,, \quad \gamma_{\phi 3a} - 2\,\gamma_{\phi 3b} = \tfrac{1}{4}\,, \quad \gamma_{\phi 3b} + 3\,\gamma_{\psi 3a} = -\tfrac{1}{2}\,, \\ &2\,\gamma_{\phi 3c} - 24\,\gamma_{\psi 3a} - \gamma_{\psi 3b} = 3\,. \end{aligned} \tag{D.20}$$

There are 14 contributions to $A^{(5)}$ independent of λ which are

$$\begin{aligned}
A^{(5)}\big|_y = {} & a_{5j}\,\mathrm{tr}(\bar y_i\, y^j)\,\mathrm{tr}(\bar y_j\, y^k)\,\mathrm{tr}(\bar y_k\, y^l)\,\mathrm{tr}(\bar y_l\, y^i)\\
& + a_{5k}\,\big(\mathrm{tr}(\bar y_i\, y^k \bar y_j\, y^l) + \mathrm{tr}(\bar y_i\, y^l \bar y_j\, y^k)\big)\,\mathrm{tr}(\bar y_k\, y^i)\,\mathrm{tr}(\bar y_l\, y^j)\\
& + a_{5l}\,\big(\mathrm{tr}(\bar y_i\, y^k \bar y_j\, y^j) + \mathrm{tr}(\bar y_i\, y^j \bar y_j\, y^k)\big)\,\mathrm{tr}(\bar y_k\, y^l)\,\mathrm{tr}(\bar y_l\, y^i)\\
& + a_{5m}\,\big(\mathrm{tr}(\bar y_i\, y^l \bar y_j\, y^j \bar y_k\, y^k) + \mathrm{tr}(\bar y_i\, y^j \bar y_j\, y^k \bar y_k\, y^l)\big)\,\mathrm{tr}(\bar y_l\, y^i)\\
& + a_{5n}\,\big(\mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^k \bar y_j\, y^l) + \mathrm{tr}(\bar y_i\, y^l \bar y_j\, y^k \bar y_k\, y^j)\big)\,\mathrm{tr}(\bar y_l\, y^i)\\
& + a_{5o}\,\mathrm{tr}(\bar y_i\, y^j \bar y_j\, y^l \bar y_k\, y^k)\,\mathrm{tr}(\bar y_l\, y^i) + a_{5p}\,\mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^l \bar y_j\, y^k)\,\mathrm{tr}(\bar y_l\, y^i)\\
& + a_{5q}\,\big(\mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^l) + \mathrm{tr}(\bar y_i\, y^l \bar y_k\, y^j)\big)\,\mathrm{tr}(\bar y_j\, y^k \bar y_l\, y^i)\\
& + a_{5r}\,\big(\mathrm{tr}(\bar y_i\, y^i \bar y_j\, y^l) + \mathrm{tr}(\bar y_i\, y^k \bar y_j\, y^i)\big)\,\big(\mathrm{tr}(\bar y_k\, y^j \bar y_l\, y^l) + \mathrm{tr}(\bar y_l\, y^j \bar y_k\, y^l)\big)\\
& + a_{5s}\,\big(\mathrm{tr}(\bar y_i\, y^i \bar y_j\, y^j \bar y_k\, y^k \bar y_l\, y^l) + \mathrm{tr}(\bar y_i\, y^j \bar y_j\, y^k \bar y_k\, y^l \bar y_l\, y^i)\big)\\
& + a_{5t}\,\big(\mathrm{tr}(\bar y_i\, y^i \bar y_j\, y^j \bar y_k\, y^l \bar y_l\, y^k) + \mathrm{tr}(\bar y_i\, y^j \bar y_j\, y^k \bar y_l\, y^l \bar y_k\, y^i)\big)\\
& + a_{5u}\,\big(\mathrm{tr}(\bar y_i\, y^i \bar y_j\, y^k \bar y_l\, y^l \bar y_k\, y^j) + \mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^l \bar y_l\, y^k \bar y_j\, y^i)\big)\\
& + a_{5v}\,\big(\mathrm{tr}(\bar y_i\, y^i \bar y_j\, y^k \bar y_l\, y^j \bar y_k\, y^l) + \mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^l \bar y_j\, y^k \bar y_l\, y^i)\big)\\
& + a_{5w}\,\big(\mathrm{tr}(\bar y_i\, y^j \bar y_k\, y^i \bar y_l\, y^k \bar y_j\, y^l) + \mathrm{tr}(\bar y_j\, y^k \bar y_l\, y^i \bar y_k\, y^j \bar y_i\, y^l)\big)\,.
\end{aligned} \qquad \text{(D.21)}$$

The $\mathrm{O}(\beta^2)$ freedom in (D.7), (D.16) extends to

$$\begin{aligned}
\delta a_{5j} &= (\epsilon_{3b_1} + \epsilon_{3b_2} + 2\,\epsilon_{3b_3})\gamma_{\phi 1}{}^2\,, \quad \delta a_{5k} = 2\,\epsilon_2\,\gamma_{\phi 1}\,\gamma_{\psi 2a} + (\epsilon_{3a_2} + 2\,\epsilon_{3a_3})\gamma_{\phi 1}{}^2\,,\\
\delta a_{5l} &= \epsilon_2\,\gamma_{\phi 1}\,\gamma_{\phi 2b} + \epsilon_{3a_1}\,\gamma_{\phi 1}{}^2 + 2(\epsilon_{3b_1} + \epsilon_{3b_2} + 2\,\epsilon_{3b_3})\gamma_{\phi 1}\,\gamma_{\psi 1}\,,\\
\delta a_{5m} &= 2\,\epsilon_2\,\gamma_{\psi 1}\,\gamma_{\psi 2a} + 2(\epsilon_{3a_1} + \epsilon_{3a_2} + 2\,\epsilon_{3a_3})\gamma_{\phi 1}\,\gamma_{\psi 1} + \epsilon_{3b_1}\,\gamma_{\psi 1}{}^2\,,\\
\delta a_{5n} &= 2\,\epsilon_2\,(\gamma_{\phi 1}\,\gamma_{\psi 2b} + \gamma_{\psi 1}\,\gamma_{\psi 2b}) + 2(\epsilon_{3a_2} + 2\,\epsilon_{3a_3})\gamma_{\phi 1}\,\gamma_{\psi 1}\,,\\
\delta a_{5o} &= 4\,\epsilon_{3a_1}\,\gamma_{\phi 1}\,\gamma_{\psi 1} + 2\,\epsilon_{3b_1}\,\gamma_{\psi 1}{}^2\,, \quad \delta a_{5p} = 2\,\epsilon_2\,\gamma_{\phi 1}\,\beta_{y2f}\,, \quad \delta a_{5q} = \tfrac{1}{4}\,\epsilon_{3\lambda}\,\beta_{\lambda 1b}{}^2\,,\\
\delta a_{5r} &= 2\,\epsilon_2\,\gamma_{\psi 1}\,\gamma_{\phi 2b} + 2(\epsilon_{3b_2} + 2\,\epsilon_{3b_3})\,\gamma_{\psi 1}{}^2\,, \quad \delta a_{5s} = (\epsilon_{3a_1} + \epsilon_{3a_2} + 2\,\epsilon_{3a_3})\gamma_{\psi 1}{}^2\,,\\
\delta a_{5t} &= 2\,\epsilon_2\,\gamma_{\psi 1}\,\gamma_{\psi 2b} + (3\,\epsilon_{3a_1} + 2\,\epsilon_{3a_2} + 4\,\epsilon_{3a_3})\gamma_{\psi 1}{}^2\,,\\
\delta a_{5u} &= 2\,\epsilon_2\,\gamma_{\psi 1}\,\gamma_{\psi 2b} + (\epsilon_{3a_2} + 2\,\epsilon_{3a_3})\gamma_{\psi 1}{}^2\,, \quad \delta a_{5v} = 2\,\epsilon_2\,\gamma_{\psi 1}\,\beta_{y2f}\,.
\end{aligned} \qquad \text{(D.22)}$$

The contributions to the four loop T involving only the $y\bar y$ couplings can be obtained by determining inequivalent pairs of vertices in (D.5). There are 36 possible contributions. There are 36 equations resulting correspond to the number of inequivalent vertices in $A^{(5)}\big|_y$ as given in (D.21).

$$\begin{aligned}
a_{5j} &= \tfrac{1}{4}(t_{4c_1} + t_{4c_2} + t_{4c_3} + t_{4c_4} + t_{4c_5})\,\gamma_{\phi 1}\,,\\
a_{5k} &= \tfrac{1}{4}\,t_2\,\gamma_{\phi 3d} + \tfrac{1}{2}(t_{3b_2} + t_{3b_3})\,\gamma_{\psi 2a} + \tfrac{1}{2}\,(t_{4e_{11}} + t_{4e_{15}})\,\gamma_{\phi 1}\\
&= \tfrac{1}{2}\,t_{3b_1}\,\gamma_{\psi 2a} + \tfrac{1}{2}(t_{4e_1} + t_{4e_6} + t_{4e_9})\,\gamma_{\phi 1}\,,\\
a_{5l} &= t_2\,\gamma_{\psi 3c} + (t_{4e_2} + t_{4e_3} + t_{4e_4} + t_{4e_5})\gamma_{\phi 1} = t_{4c_1}\,\gamma_{\psi 1} + (t_{4e_7} + t_{4e_8} + t_{4e_{10}})\,\gamma_{\phi 1}\\
&= \tfrac{1}{2}(t_{3b_1} + t_{3b_3})\gamma_{\phi 2b} + (t_{4c_2} + t_{4c_4})\gamma_{\psi 1} + t_{4e_{12}}\,\gamma_{\phi 1}\\
&= \tfrac{1}{2}\,t_{3b_2}\,\gamma_{\phi 2b} + (t_{4c_3} + t_{4c_5})\gamma_{\psi 1} + (t_{4e_{13}} + t_{4e_{14}})\,\gamma_{\phi 1}\,,\\
a_{5m} &= \tfrac{1}{2}\,t_2\,\gamma_{\phi 3f} + (t_{4e_{11}} + t_{4e_{12}} + t_{4e_{14}} + t_{4e_{15}})\,\gamma_{\phi 1} = t_{4d_1}\,\gamma_{\phi 1} + (t_{4e_6} + t_{4e_7} + t_{4e_9})\,\gamma_{\psi 1}\\
&= t_{3a_2}\,\gamma_{\psi 2a} + (t_{4d_3} + t_{4d_5})\gamma_{\phi 1} + (t_{4e_4} + t_{4e_5})\,\gamma_{\psi 1}\\
&= (t_{3a_1} + t_{3a_3})\,\gamma_{\psi 2a} + (t_{4d_2} + t_{4d_4})\gamma_{\phi 1} + t_{4e_2}\,\gamma_{\psi 1}\,,
\end{aligned}$$

$$
\begin{aligned}
a_{5n} &= \tfrac{1}{2}\,t_2\,\gamma_{\phi 3g} + (t_{3b_2} + t_{3b_3})\gamma_{\psi 2b} + (t_{4e_{11}} + t_{4e_{15}})\,\gamma_{\psi 1} = t_{3a_1}\,\gamma_{\psi 2a} + t_{4e_1}\,\gamma_{\psi 1} + (t_{4f_7} + t_{4f_8})\,\gamma_{\phi 1} \\
&= t_2\,\gamma_{\psi 3e} + (t_{3a_2} + t_{3a_3})\gamma_{\psi 2a} + (t_{4f_2} + t_{4f_4})\gamma_{\phi 1} \\
&= t_{3b_1}\,\gamma_{\psi 2b} + (t_{4e_6} + t_{4e_9})\,\gamma_{\psi 1} + t_{4f_3}\,\gamma_{\phi 1}\,, \\
a_{5o} &= t_2\,\gamma_{\psi 3d} + (t_{4e_2} + t_{4e_4})\gamma_{\psi 1} + (t_{4f_1} + t_{4f_5})\,\gamma_{\phi 1} = 2\,t_{4e_7}\,\gamma_{\psi 1} + 2\,t_{4f_6}\,\gamma_{\phi 1} \\
&= t_2\,\gamma_{\psi 3h} + 2(t_{4e_{12}} + t_{4e_{14}})\gamma_{\psi 1}\,, \\
a_{5p} &= t_2\,\gamma_{\phi 3m} + (t_{3b_2} + t_{3b_3})\beta_{y2f} = t_2\,\beta_{y3s} + (t_{4g_1} + t_{4g_3})\gamma_{\phi 1} \\
a_{5q} &= \tfrac{1}{8}(t_2\,\beta_{y3w} + t_{4a_2}\,\beta_{\lambda 1b})\,, \\
a_{5r} &= \tfrac{1}{4}\,t_2\,\gamma_{\psi 3g} + \tfrac{1}{4}\,(t_{3a_2} + t_{3a_3})\,\gamma_{\phi 2b} + \tfrac{1}{2}(t_{4e_3} + t_{4e_5})\,\gamma_{\psi 1} \\
&= \tfrac{1}{4}\,t_{3a_1}\,\gamma_{\phi 2b} + \tfrac{1}{2}(t_{4e_8} + t_{4e_{10}} + t_{4e_{13}})\,\gamma_{\psi 1}\,, \\
a_{5s} &= \tfrac{1}{4}(t_{4d_1} + t_{4d_2} + t_{4d_3} + t_{4d_4} + t_{4d_5})\,\gamma_{\psi 1}\,, \\
a_{5t} &= t_2\,\gamma_{\psi 3i} + (t_{4f_1} + t_{4f_2} + t_{4f_4} + t_{4f_5})\,\gamma_{\psi 1} = (t_{4d_1} + 2\,t_{4f_6} + t_{4f_7} + t_{4f_8})\,\gamma_{\psi 1} \\
&= (t_{3a_1} + t_{3a_3})\,\gamma_{\psi 2b} + (t_{4d_2} + t_{4d_4} + t_{4f_1})\,\gamma_{\psi 1} \\
&= t_{3a_2}\,\gamma_{\psi 2b} + (t_{4d_3} + t_{4d_5} + t_{4f_3} + t_{4f_5})\,\gamma_{\psi 1}\,, \\
a_{5u} &= \tfrac{1}{2}\,t_2\,\gamma_{\psi 3j} + \tfrac{1}{2}(t_{3a_2} + t_{3a_3})\,\gamma_{\psi 2b} + \tfrac{1}{2}(t_{4f_2} + t_{4f_4})\,\gamma_{\psi 1} \\
&= \tfrac{1}{2}\,t_{3a_1}\,\gamma_{\psi 2b} + \tfrac{1}{2}(t_{4f_3} + t_{4f_7} + t_{4f_8})\,\gamma_{\psi 1}\,, \\
a_{5v} &= t_2\,\beta_{y3\tilde{o}} + (t_{4g_1} + t_{4g_3})\,\gamma_{\psi 1} = t_2\,\beta_{y3\tilde{p}} + (t_{4g_2} + t_{4g_3})\,\gamma_{\psi 1} \\
&= t_2\,\gamma_{\psi 3p} + (t_{3a_2} + t_{3a_3})\,\beta_{y2f} = t_{3a_1}\,\beta_{y2f} + t_{4g_1}\,\gamma_{\psi 1}\,, \\
a_{5w} &= \tfrac{1}{4}\,t_2\,\beta_{y3\tilde{z}}\,.
\end{aligned}
\tag{D.23}
$$

This leads to the non planar condition from the relations involving a_{5p}, a_{5v}

$$
U_{10} = 2\,\gamma_{\psi 1}(\beta_{y3s} - \gamma_{\phi 3m}) - 2\,\gamma_{\phi 1}(\beta_{y\tilde{o}} - \gamma_{\psi 3p}) + (\gamma_{\phi 2b} - 2\,\gamma_{\psi 2a})\,\beta_{y2f}\,, \tag{D.24}
$$

and just two other conditions

$$
\begin{aligned}
U_{11} &= \gamma_{\phi 1}(\gamma_{\psi 3j} - 2\,\gamma_{\psi 3i}) + \gamma_{\psi 1}(\gamma_{\psi 3d} + \gamma_{\psi 3e} + \gamma_{\phi 3f} - \gamma_{\phi 3g} - \gamma_{\phi 3h}) + (\gamma_{\phi 2b} - 2\,\gamma_{\psi 2a})\gamma_{\psi 2b}\,, \\
U_{12} &= 2\,{\gamma_{\phi 1}}^2\,\gamma_{\psi 1}\,\gamma_{\psi 3j} - \gamma_{\phi 1}\,{\gamma_{\psi 1}}^2(2\,\gamma_{\psi 3e} + \gamma_{\phi 3g}) + {\gamma_{\psi 1}}^3\gamma_{\phi 3d} \\
&\quad - (2\,\gamma_{\phi 1}\,\gamma_{\psi 2b} - \gamma_{\psi 1}\,\gamma_{\phi 2b})(\gamma_{\phi 1}\,\gamma_{\psi 2b} - \gamma_{\psi 1}\,\gamma_{\psi 2a})\,.
\end{aligned}
\tag{D.25}
$$

The results obtained in (D.9), (D.18), (D.24), (D.25) are a subset of those obtained in section B.2 in the general case. We list some scheme variations which are not immediately evident from previous results

$$
\begin{aligned}
\delta U_6 &= \,-\,2\,\gamma_{\psi 1}\,Y_{\phi 1,\psi 1,y2a}\,, \qquad \delta U_8 = -2\,\gamma_{\psi 1}\big(Y_{\phi 1,\psi 1,y2a} + 3\,Y_{\phi 2a,\psi 1,y1}\big)\,, \\
\delta U_9 &= 2\,\gamma_{\psi 1}\big(-\gamma_{\psi 1}\,Y_{\phi 1,y2a,\lambda 1a} - \beta_{y2a}\,Y_{\phi 1,\psi 1,\lambda 1a} + 3\,\gamma_{\phi 2a}\,Y_{\psi 1,\lambda 1a,\lambda 1b} + U_0\,X_{\psi 1,\lambda 1a}\big)\,, \\
\delta U_{10} &= \,-\,8\,Y_{\phi 1,\psi 1,y2f}\,, \qquad \delta U_{11} = -8\,Y_{\phi 1,\psi 1,\psi 2b}\,, \\
\delta U_{12} &= \,-\,2\,{\gamma_{\psi 1}}^2\big(2\,Y_{\phi 1,\psi 1,\psi 2a} + Y_{\phi 1,\phi 2b,\psi 1}\big)\,.
\end{aligned}
\tag{D.26}
$$

E Scheme variations in general

The coordinates in quantum field theories are the couplings. Physical results should be invariant under reparametrisations of the couplings or in this context changes of renormalisation scheme. Of course determining possible invariants is an exercise in differential geometry. Under changes of scale there is a RG flow in the space of couplings determined by a vector field, the β-function, and at any fixed point where the β-function vanishes the scale dimensions of operators are determined by the eigenvalues of the anomalous dimension matrix which, at a fixed point, is a two index tensor under reparametrisations.

In a perturbative context the possible reparametrisations of couplings are naturally restricted to preserve the form of the β-function in terms of contributions corresponding to 1PI diagrams which are superficially divergent. For an expansion in terms of diagrams with increasing loop order there a usually a restricted set of possible vertices $\{V_{\rm v}\}$, labelled by v and edges $\{E_{\rm e}\}$ labelled by e. The various possible e correspond to the different fields in the theory and each v to the different basic couplings. For $n_{\rm v}$ lines meeting at a particular vertex $V_{\rm v}$ there is then an associated coupling $(G^{\rm v})_{i_1\dots i_{n_{\rm v}}}$ where i_r an index associated with the a diagram line or edge, e, connected to the vertex v. In a diagram with a line e there is an associated two index link or propagator $(P^{\rm e})_{ij}$ with $i,j=1,\dots n_{\rm e}$. For each coupling $G^{\rm v}$ there is a symmetry group $G_{\rm v}\subset \mathcal{S}_{n_{\rm v}}$ generated by permutations of those lines corresponding to identical particles. For convenience we may consider a basis in which the couplings are real and $(P^{\rm e})_{ij}$ is symmetric, otherwise for complex couplings they form conjugate pairs. For simplicity we restrict to dimensionless couplings.

It is convenient to adopt a notation where for any set of $\{(R^{\rm e})_{ij}\}$ and $\{(\kappa^{\rm e})_{ij}\}$

$$\begin{aligned}(G^{\rm v}\circ R)_{i_1\dots i_{n_{\rm v}}} &= (G^{\rm v})_{j_1\dots j_{n_{\rm v}}}\,(R^1)_{j_1 i_1}\dots (R^{n_{\rm v}})_{j_{n_{\rm v}} i_{n_{\rm v}}}\,,\\ (G^{\rm v}\kappa)_{i_1\dots i_{n_{\rm v}}} &= \textstyle\sum_r (G^{\rm v})_{i_1\dots i_{r-1} j i_{r+1}\dots i_{n_{\rm v}}}\,(\kappa^r)_{j i_r}\,.\end{aligned} \tag{E.1}$$

Clearly $(G^{\rm v}\circ R)\circ R' = G^{\rm v}\circ RR'$, with $RR'=\{R_{\rm e}R'_{\rm e}\}$, $((G^{\rm v}\,\kappa)\,\kappa') - ((G^{\rm v}\,\kappa')\,\kappa) = (G^{\rm v}\,[\kappa,\kappa'])$. For an overall symmetry $\mathcal{G}$ then $G^{\rm v}\circ R = G^{\rm v}$, for each $R^{\rm e}$ belonging to the appropriate representation of $\mathcal{G}$, and $(P^{\rm e})_{kl}(R^{\rm e})_{ki}(R^{\rm e})_{lj} = (P^{\rm e})_{ij}$. For a vacuum diagram there is a corresponding amplitude formed by joining couplings for each vertex with appropriate propagators

$$A(G,P)=A(g)\,,\qquad g^{\rm v}=G^{\rm v}\circ P^{\frac12}\,,\qquad A(g)=A(g,\mathbb{1})\,, \tag{E.2}$$

with $G=\{G^{\rm v}\},\ P=\{P_{\rm e}\}$ and $P_{\rm e}{}^{\frac12}$ is required to be symmetric. In general we require

$$A(G\circ\mathcal{R},\mathcal{R}^T P\,\mathcal{R}) = A(G,P)\,,\quad A(g\circ\mathcal{R})=A(g)\,,\qquad \mathcal{R}=\{\mathcal{R}^{\rm e}\}\,,\quad \mathcal{R}^{\rm e}\mathcal{R}^{{\rm e}T}=\mathbb{1}^{\rm e}\,. \tag{E.3}$$

Reparametrisations of relevance here are generated by

$$\begin{aligned}\mathcal{D}_{v,w}(G,P) &= \sum_{\rm v} v^{\rm v}(G,P)\cdot\frac{\partial}{\partial G^{\rm v}} + 2\sum_{\rm e}(P^{\rm e}\,w^{\rm e}(G,P)P^{\rm e})\cdot\frac{\partial}{\partial P^{\rm e}}\\ &= \sum_{\rm v} v^{\rm v}(G,P)\cdot\frac{\partial}{\partial G^{\rm v}} - 2\sum_{\rm e} w^{\rm e}(G,P)\cdot\frac{\partial}{\partial P^{{\rm e}-1}}\,,\end{aligned} \tag{E.4}$$

where $v^{\rm v}(G,P)$, $(w^{\rm e}(G,P))_{ij}$ are determined in terms of sums of 1PI one and higher loop vertex and propagator graphs with vertices mapped to the appropriate $G^{\rm v}$ and similarly internal lines to $P^{\rm e}$. The 2 in (E.4) is introduced for later convenience. For a finite transformation $G^{\rm v}\to G^{\rm v}{}'$, $P^{\rm e}\to P^{\rm e}{}'$ then

$$G^{\rm v}{}'(G,P)=\exp\left(\mathcal{D}_{v,w}(G,P)\right)G^{\rm v}\,,\qquad P^{\rm e}{}'(G,P)=\exp\left(\mathcal{D}_{v,w}(G,P)\right)P^{\rm e}\,. \tag{E.5}$$

With this definition

$$G^{\mathrm{v}\prime}(G,P) = G^{\mathrm{v}} + f^{\mathrm{v}}(G,P)\,, \qquad P^{\mathrm{e}\prime}(G,P)^{-1} = P^{\mathrm{e}-1} - 2\,c^{\mathrm{e}}(G,P)\,, \tag{E.6}$$

where f^{v}, c^{e} can be expressed in terms of contributions from 1PI vertex and propagator graphs. Following from (E.2) then

$$f^{\mathrm{v}}(G,P) = f^{\mathrm{v}}(g)\circ P^{-\frac{1}{2}}\,, \qquad c^{\mathrm{e}}(G,P) = P^{\mathrm{e}-\frac{1}{2}}c^{\mathrm{e}}(g)P^{\mathrm{e}-\frac{1}{2}}\,, \tag{E.7}$$

and

$$g^{\mathrm{v}\prime}(G,P) = G^{\mathrm{v}\prime}(G,P)\circ P'(G,P)^{\frac{1}{2}} = g^{\mathrm{v}\prime}(g)\circ\mathcal{R}\,, \tag{E.8}$$

where

$$g^{\mathrm{v}\prime}(g) = \big(g^{\mathrm{v}} + f^{\mathrm{v}}(g)\big)\circ\big(\mathbb{1} - 2\,c(g)\big)^{-\frac{1}{2}}\,, \tag{E.9}$$

with

$$\mathcal{R}^{\mathrm{e}} = \big(\mathbb{1}^{\mathrm{e}} - 2\,c^{\mathrm{e}}(g)\big)^{\frac{1}{2}}\,P^{\mathrm{e}-\frac{1}{2}}\,P^{\mathrm{e}\prime}(G,P)^{\frac{1}{2}}\,. \tag{E.10}$$

From this definition

$$\mathcal{R}^{\mathrm{e}}\,\mathcal{R}^{\mathrm{e}T} = \mathbb{1}^{\mathrm{e}}\,. \tag{E.11}$$

The inverse of (E.8)

$$g^{\mathrm{v}}(g') = \big(g^{\mathrm{v}\prime} + f^{\mathrm{v}\prime}(g')\big)\circ\big(\mathbb{1} - 2\,c'(g')\big)^{-\frac{1}{2}}\,, \tag{E.12}$$

is obtained by taking

$$\begin{aligned} f^{\mathrm{v}\prime}(g') &= \,-\,f^{\mathrm{v}}(g)\circ\big(\mathbb{1} - c'(g')\big)^{\frac{1}{2}}\,, \\ c^{\mathrm{e}\prime}(g') &= \mathbb{1}^{\mathrm{e}} - \big(\mathbb{1}^{\mathrm{e}} - 2\,c^{\mathrm{e}}(g)\big)^{-1} = -\big(\mathbb{1}^{\mathrm{e}} - 2\,c^{\mathrm{e}}(g)\big)^{-\frac{1}{2}}\,c^{\mathrm{e}}(g)\,\big(\mathbb{1}^{\mathrm{e}} - c^{\mathrm{e}}(g)\big)^{-\frac{1}{2}}\,, \end{aligned} \tag{E.13}$$

from which it follows that $f^{\mathrm{v}\prime}(g')$, $c^{\mathrm{e}\prime}(g')$ are both expressible as expansions in g' in terms of contributions corresponding to 1PI diagrams. For f, c infinitesimal the generator of reparametrisations can be reduced, from (E.8), (E.11), to the form

$$\mathcal{D}_{f,c}(G,P) = \mathcal{D}_{f+(gc)}(g) + \mathcal{D}_{(g\omega)}(g)\,, \quad \mathcal{D}_h(g) = \sum_{\mathrm{v}} h^{\mathrm{v}}(g)\cdot\frac{\partial}{\partial g^{\mathrm{v}}}\,, \tag{E.14}$$

where

$$\mathcal{R}^{\mathrm{e}} = \mathbb{1}^{\mathrm{e}} + \omega^{\mathrm{e}}\,, \qquad \omega^{\mathrm{e}T} = -\omega^{\mathrm{e}}\,. \tag{E.15}$$

Under a reparametrisation

$$A'(G',P') = A(G,P) \qquad \Rightarrow \qquad A'(g') = A(g)\,, \tag{E.16}$$

is consistent with (E.2).

The essential RG functions $\tilde\beta^{\mathrm{v}}(G,P)$, $\gamma^{\mathrm{e}}(G,P)$ are formed from contributions corresponding to 1PI diagrams, with $\gamma^{\mathrm{e}}(G,P)$ symmetric. Corresponding to (E.3)

$$\tilde\beta^{\mathrm{v}}(G\circ\mathcal{R},\mathcal{R}^T P\,\mathcal{R}) = \tilde\beta^{\mathrm{v}}(G,P)\circ\mathcal{R}\,, \quad \gamma^{\mathrm{e}}(G\circ\mathcal{R},\mathcal{R}^T P\,\mathcal{R}) = \mathcal{R}^{\mathrm{e}T}\gamma^{\mathrm{e}}(G,P)\,\mathcal{R}^{\mathrm{e}}\,. \tag{E.17}$$

Reduction to the coupling g, determined as in (E.2), is achieved by taking

$$\tilde\beta^{\rm v}(G,P) = \tilde\beta^{\rm v}(g)\circ P^{-\frac12}\,, \qquad \gamma^{\rm e}(G,P) = P^{{\rm e}-\frac12}\gamma^{\rm e}(g)P^{{\rm e}-\frac12}\,, \tag{E.18}$$

and then $\tilde\beta^{\rm v}(g)$, $\gamma^{\rm e}(g)$ are expressible just in terms of 1PI contributions. RG flow is then generated by

$$\mathcal{D}_{\tilde\beta,\gamma}(G,P)\,, \tag{E.19}$$

where

$$\mathcal{D}_{\tilde\beta,\gamma}(G,P)\,A(g) = \mathcal{D}_\beta(g)\,A(g)\,, \qquad \beta^{\rm v}(g) = \tilde\beta^{\rm v}(g) + (g^{\rm v}\gamma(g))\,, \tag{E.20}$$

where we make use of

$$\mathcal{D}_{\tilde\beta,\,\gamma}(G,P)\,P^{{\rm e}\frac12} = P^{{\rm e}\frac12}(\gamma^{\rm e}+\omega^{\rm e}) = (\gamma^{\rm e}-\omega^{\rm e})P^{{\rm e}\frac12}\,, \quad \text{for some} \ \ \omega^{\rm e} = -\omega^{{\rm e}T}\,, \tag{E.21}$$

and

$$\mathcal{D}_{(g\omega)}(g)\,A(g) = 0\,. \tag{E.22}$$

In (E.20) the β-function $\beta^{\rm v}(g)$ then has the standard form in terms of 1PI contributions and from (E.3)

Under a reparametrisation as in (E.6)

$$\begin{aligned}
&\tilde\beta^{\rm v}{}'(G',P') = \mathcal{D}_{\tilde\beta,\gamma}(G,P)\,G^{\rm v}{}'(G,P)\,,\\
&\gamma^{\rm e}{}'(G',P') = \,-\,\mathcal{D}_{\tilde\beta,\gamma}(G,P)\,P^{\rm e}{}'(G,P)^{-1} = \gamma^{\rm e}(G,P) + \mathcal{D}_{\tilde\beta,\gamma}(G,P)\,c^{\rm e}(G,P)\,.
\end{aligned} \tag{E.23}$$

Defining

$$\tilde\beta^{\rm v}{}'(G',P')\circ P'^{\frac12} = \tilde\beta^{\rm v}{}'(g')\circ\mathcal{R}\,, \qquad P^{\rm e}{}'^{\frac12}\gamma^{\rm e}{}'(G',P')\circ P^{\rm e}{}'^{\frac12} = \mathcal{R}^{{\rm e}T}\gamma^{\rm e}{}'(g')\,\mathcal{R}^{\rm e}\,, \tag{E.24}$$

then reduces, with $\mathcal{R}$ given by (E.10), for $\tilde\beta(g)$, $\gamma(g)$ to

$$\begin{aligned}
&\tilde\beta^{\rm v}{}'(g') = \big(\tilde\beta^{\rm v}(g) + \mathcal{D}_\beta(g)\,f^{\rm v}(g) - (f_{\rm v}(g)\gamma(g))\big)\circ(\mathbb{1}-2\,c(g))^{-\frac12}\,,\\
&\gamma^{\rm e}{}'(g') = (\mathbb{1}-2\,c^{\rm e}(g))^{-\frac12}\big(\gamma^{\rm e}(g) + \mathcal{D}_\beta(g)\,c^{\rm e}(g) - \{\gamma^{\rm e}(g),c^{\rm e}(g)\}\big)(\mathbb{1}-2\,c^{\rm e}(g))^{-\frac12}\,.
\end{aligned} \tag{E.25}$$

To achieve (E.25) we make use of

$$\mathcal{D}_{(g\omega)}(g)\,f^{\rm v}(g) = (f^{\rm v}(g)\omega)\,, \qquad \mathcal{D}_{(g\omega)}(g)\,c^{\rm e}(g) = [c^{\rm e}(g),\omega]\,. \tag{E.26}$$

The expressions obtained in (E.25) ensure that $\tilde\beta^{\rm v}{}'(g')$, $\gamma^{\rm e}{}'(g')$ expanded in terms of g' as in (E.8) are expressible in terms of 1PI contributions and furthermore

$$\beta^{\rm v}{}'(g') = \tilde\beta^{\rm v}{}'(g') + (g^{\rm v}{}'\gamma(g')) = \mathcal{D}_\beta(g)\,g^{\rm v}{}' + (g^{\rm v}{}'\,\Omega)\,, \quad \Omega = -\Omega^T\,, \tag{E.27}$$

where

$$\begin{aligned}
&\Omega^{\rm e} = (\mathbb{1}-2\,c^{\rm e}(g))^{-\frac12}\big([c^{\rm e}(g),\gamma^{\rm e}(g)] + \hat\Omega^{\rm e}\big)(\mathbb{1}-2\,c^{\rm e}(g))^{-\frac12}\,,\\
&\mathcal{D}_\beta(g)(\mathbb{1}-2\,c^{\rm e}(g))^{-\frac12}\,(\mathbb{1}-2\,c^{\rm e}(g))^{\frac12} = (\mathbb{1}-2\,c^{\rm e}(g))^{-1}\big(\mathcal{D}_\beta(g)\,c^{\rm e}(g) - \hat\Omega^{\rm e}\big)\,.
\end{aligned} \tag{E.28}$$

From (E.23)

$$\mathcal{D}_{\tilde{\beta}',\gamma'}(G,P) = \exp\big(-\mathcal{D}_{v,w}(G,P)\big)\,\mathcal{D}_{\tilde{\beta},\gamma}(G,P)\exp\big(\mathcal{D}_{v,w}(G,P)\big)\,, \tag{E.29}$$

or

$$\tilde{\beta}^{\mathrm{v}\prime}(G,P) = \exp\big(-\mathcal{L}_{v,w}(G,P)\big)\tilde{\beta}^{\mathrm{v}}(G,P)\,, \quad \gamma^{\mathrm{e}\prime}(G,P) = \exp\big(-\mathcal{L}_{v,w}(G,P)\big)\gamma^{\mathrm{e}}(G,P)\,, \tag{E.30}$$

with, to lowest order,

$$\begin{aligned}\delta\tilde{\beta}^{\mathrm{v}}(G,P) &= -\mathcal{L}_{v,w}(G,P)\,\tilde{\beta}^{\mathrm{v}}(G,P) = \mathcal{D}_{\tilde{\beta},\gamma}(G,P)\,v^{\mathrm{v}}(G,P) - \mathcal{D}_{v,w}(G,P)\,\tilde{\beta}^{\mathrm{v}}(G,P)\,,\\ \delta\gamma^{\mathrm{e}}(G,P) &= -\mathcal{L}_{v,w}(G,P)\,\gamma^{\mathrm{e}}(G,P) = \mathcal{D}_{\tilde{\beta},\gamma}(G,P)\,w^{\mathrm{e}}(G,P) - \mathcal{D}_{v,w}(G,P)\,\gamma^{\mathrm{e}}(G,P)\,.\end{aligned} \tag{E.31}$$

For application here we set up a basis of 1PI vertex and propagator graphs so that

$$v^{\mathrm{v}}(G,P) = \sum_{\ell,r}\epsilon_{\mathrm{v}\ell r}\,\mathcal{S}_{p_{\mathrm{v}\ell r}}\,G^{\mathrm{v}\ell r}(G,P)\,, \qquad w^{\mathrm{e}}(G,P) = \sum_{\ell,r}\epsilon_{\mathrm{e}\ell r}\,\mathcal{S}_{p_{\mathrm{e}\ell r}}\,G^{\mathrm{e}\ell r}(G,P)\,. \tag{E.32}$$

In the expansion of $v^{\mathrm{v}}(G,P)$ the sum is over contributions $G^{\mathrm{v}\ell r}(G,P)$ corresponding to particular ℓ, $\ell = 1,2,\ldots,$ loop 1PI vertex graphs $\mathcal{G}^{\mathrm{v}\ell r}$, with the same external lines as v, and labelled by r. In each case $\mathcal{S}_{p_{\mathrm{v}\ell r}}$ denotes the sum over the $p_{\mathrm{v}\ell r}$ permutations of the external lines of $\mathcal{G}^{\mathrm{v}\ell r}$ necessary to ensure the symmetry under external line permutations satisfied by G^{v}. Similarly $G^{\mathrm{e}\ell r}(G,P)$ corresponds to a 1PI propagator graph $\mathcal{G}^{\mathrm{e}\ell r}$, the associated permutations over external lines are $\mathcal{S}_{p_{\mathrm{e}\ell r}}$ with $p_{\mathrm{e}\ell r} = 1,\,2$ according to whether $\mathcal{G}^{\mathrm{e}\ell r}$ is symmetric or not. Inserting vertex or propagator graphs generates an algebra which arises from

$$\begin{aligned}\mathcal{S}_{p_{\mathrm{v}'\ell' r'}}\,G^{\mathrm{v}'\ell' r'}(G,P)\cdot\frac{\partial}{\partial G}\,\mathcal{S}_{p_{\mathrm{g}\ell r}}\,G^{\mathrm{g}\ell r}(G,P) &= \sum_s N^{\mathrm{v}'\ell' r',\mathrm{g}\ell r}_{\mathrm{g}Ls}\,\mathcal{S}_{p_{\mathrm{g}Ls}}\,G^{\mathrm{g}Ls}(G,P)\,,\\ -\mathcal{S}_{p_{\mathrm{e}'\ell' r'}}\,G^{\mathrm{e}'\ell' r'}(G,P)\cdot\frac{\partial}{\partial P^{-1}}\,\mathcal{S}_{p_{\mathrm{g}\ell r}}\,G^{\mathrm{g}\ell r}(G,P) &= \sum_s N^{\mathrm{e}'\ell' r',\mathrm{g}\ell r}_{\mathrm{g}Ls}\,\mathcal{S}_{p_{\mathrm{g}Ls}}\,G^{\mathrm{g}Ls}(G,P)\,,\\ &\qquad \mathrm{g}\in\{\mathrm{v},\,\mathrm{e}\}\,,\quad L = \ell' + \ell\,.\end{aligned} \tag{E.33}$$

In general $N^{\mathrm{g}'\ell' r',\mathrm{g}\ell r}_{\mathrm{g}Ls}$ are integers.

Just as in (E.32) there is a similar expansion

$$\tilde{\beta}^{\mathrm{v}}(G,P) = \sum_{\ell,r}\beta_{\mathrm{v}\ell r}\,\mathcal{S}_{p_{\mathrm{v}\ell r}}\,G^{\mathrm{v}\ell r}(G,P)\,, \qquad \gamma^{\mathrm{e}}(G,P) = \sum_{\ell,r}\gamma_{\mathrm{e}\ell r}\,\mathcal{S}_{p_{\mathrm{e}\ell r}}\,G^{\mathrm{e}\ell r}(G,P)\,. \tag{E.34}$$

The results in (E.31) then ensure

$$\begin{aligned}&\delta\alpha_{\mathrm{g}Ls} = \sum_{\substack{\mathrm{g}'r'\ell',r\ell\\ \ell'+\ell=L}} n_{\mathrm{g}'}\,N^{\mathrm{g}'\ell' r',\mathrm{g}\ell r}_{\mathrm{g}Ls}\,X_{\mathrm{g}'\ell' r',\mathrm{g}\ell r}\,,\quad \alpha_{\mathrm{v}Ls} = \beta_{\mathrm{v}Ls}\,,\ \alpha_{\mathrm{e}Ls} = \gamma_{\mathrm{e}Ls}\,,\ n_{\mathrm{v}'} = 1,\ n_{\mathrm{e}'} = 2\,,\\ &X_{\mathrm{g}'\ell' r',\mathrm{g}\ell r} = \alpha_{\mathrm{g}'\ell' r'}\,\epsilon_{\mathrm{g}\ell r} - \epsilon_{\mathrm{g}'\ell' r'}\,\alpha_{\mathrm{g}\ell r}\,.\end{aligned} \tag{E.35}$$

This is then equivalent to (10.1) and (10.2) used in section 10.

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