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Bootstrapping the effect of the twist operator in symmetric orbifold CFTs

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Abstract: We study the 2D symmetric orbifold CFT of two copies of free bosons. The twist operator can join the two separated copies in the untwisted sector into a joined copy in the twisted sector. Starting with a state with any number of quanta in the untwisted sector, the state in the twisted sector obtained by the action of the twist operator can be computed by using the covering map method. We develop a new method to compute the effect of a twist operator by using the Bogoliubov ansatz and conformal symmetry. This may lead to more efficient tools to compute correlation functions involving twist operators.

Keywords: Field Theories in Lower Dimensions, Scale and Conformal Symmetries

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Contents

1 Introduction

Symmetric orbifold CFTs are widely used in AdS_3/CFT_2 as boundary theories to understand the bulk physics [1–8]. Consider a seed CFT with target space M. The symmetric orbifold CFT has the target space

$$M^N/S_N \tag{1.1}$$

where N is the number of copies of the seed CFT and S_N is the permutation group of N elements.

Due to the S_N orbifold, there exist twist operators. If we circle the insertion of a twist operator, different copies of the seed CFT permute into each other. The traditional way to compute correlation functions of twist operators is to map the 2-d base space to the covering space [9–13]. On this covering space, the ramification caused by the twist operator in the base space is resolved, such that the target space becomes a single copy of M. The correlation functions on the base space can be computed from a combination of the correlation functions in the covering space and a Liouville factor which takes into account the covering map.

The effect of a twist operator can be studied by using the covering map [14–18]. To be specific, take two copies of the seed CFT of a free boson, such that $M = \mathbb{R}$. Suppose around $z = 0$ that the state is in the untwisted sector as shown in figure 1. Around this point, there are two separate copies of the seed CFT. Let us put a twist operator σ_2 at z_0, which is the unique twist operator for two copies. This twist operator generates a branch cut from z_0 to infinity. The two separate copies join into a single doubly wound copy for $|z| > |z_0|$. If an initial state is given at $z = 0$ for the two separate copies, what is the state after the twist operator, e.g. the state at infinity? The result of this question tells us all the three-point functions involving a twist operator σ_2.

It has been observed that the effect of a twist operator is in the form of a Bogoliubov transformation between the modes before and after the twist operator. From the covering space method point of view, the map from the base space to the covering space leads to this linear transformation of modes. The effect of a twist operator is encoded in this linear transformation. This can be derived from the covering map or by matching the modes just before and after the twist [19, 20]. In the covering map method, it seems that the details of the map are necessary. In the latter method, the covering map is not needed but in practice not all effects can be obtained since it requires one to invert an infinite-dimensional matrix.

In this paper, we will develop a method that does not involve the covering map and can obtain the effects completely. To do that, we will use the 'weak' Bogoliubov ansatz[1] and conformal symmetry

$$\text{Weak Bogoliubov ansatz} + \text{Conformal symmetry} \Rightarrow \text{Effect of a twist operator} \tag{1.2}$$

[1]'Weak' means that we don't require some relations among the coefficients in the original Bogoliubov ansatz. For more details, see section 4. These relations come out naturally. As a result, this method does not involve the inversion of infinite-dimensional matrices.

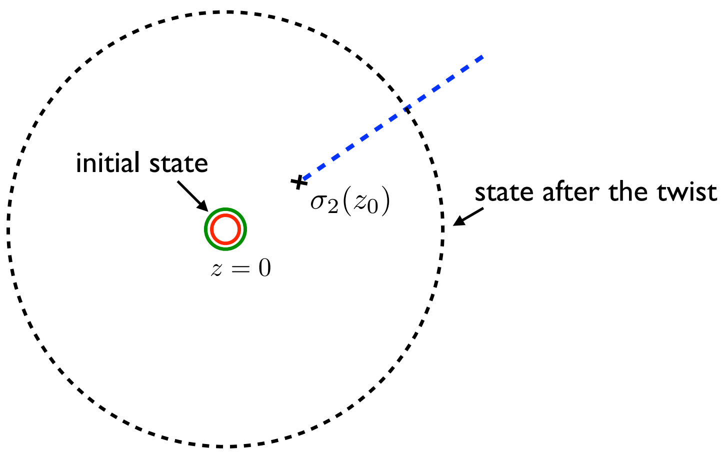

Figure 1. The effect of a twist operator. The red and green circles represent the states living on two separate copies located before the twist operator. The twist operator produces a branch cut from z_0 to infinity. The dashed circle represents the state living on the joined copy with two sheets.

For earlier work in this direction see [21] which incorporates the use of the covering map and conformal symmetry. There have also been works to compute correlation functions of twist operators using conformal symmetry, see e.g. [7, 22–24]. For recent work, see [25].

The plan of the paper is as follows: in section 2 we outline the orbifold CFT of one boson. In section 3 we describe the effect of the twist operator. In section 4 we discuss the Bogoliubov transformation. In section 5 we bootstrap the effect of the twist operator using the weak Bogoliubov ansatz and conformal symmetry. In section 6 we discuss our results and future work.

2 Orbifold CFT of one boson

Symmetric orbifold CFTs are obtained by orbifolding N copies of a seed CFT by the permutation group S_N, which results in the target space

$$M^N/S_N \tag{2.1}$$

where M is the target space of the seed CFT. In this paper, we will consider the simplest case where the seed CFT is a free boson with target space $M = \mathbb{R}$. The base space is the complex z plane.

The N copies of the free boson are labeled by $X^{(i)}$ with $i = 1, \dots, N$. In the untwisted sector, the fields have the boundary condition

$$X^{(i)} \to X^{(i)}, \qquad z \to ze^{2\pi i} \tag{2.2}$$

There also exist twist sectors where the N copies can join into many linked copies in all possible ways. For example, a k-wound linked copy has the boundary condition

$$X^{(1)} \to X^{(2)} \to \dots \to X^{(k)} \to X^{(1)}, \qquad z \to ze^{2\pi i} \tag{2.3}$$

It is convenient to define a single field X living on the k-wound copy. On the i-th segment of the k-wound copy, the field X equals to $X^{(i)}$, such that the field X has the boundary condition

$$X \to X, \qquad z \to z e^{2\pi k i} \tag{2.4}$$

Notice that the field X is multi-valued in the base space and should be thought of as a single-valued field living on a Riemann surface with k sheets.

In radial quantization, the modes of the holomorphic part in the untwisted sector are defined as

$$\alpha_n^{(i)} = \frac{1}{2\pi} \oint_{C_0} dz\, z^n \partial X^{(i)}(z) \tag{2.5}$$

where C_0 is a contour centered around $z = 0$. The n is an integer as required by the boundary condition (2.2). The commutation relation is

$$[\alpha_m^{(i)}, \alpha_n^{(j)}] = m\delta^{ij}\delta_{m+n,0} \tag{2.6}$$

We also define

$$\alpha_n^{(i)\dagger} = \alpha_{-n}^{(i)} \tag{2.7}$$

The vacuum $|0\rangle^{(i)}$ of copy i is defined by the condition

$$\alpha_n^{(i)}|0\rangle^{(i)} = 0, \qquad n \geq 0 \tag{2.8}$$

The Virasoro generators can be expanded in terms of a sum over bilinears of the modes

$$L_m = \frac{1}{2} \sum_i \sum_n \alpha_n^{(i)} \alpha_{m-n}^{(i)} \tag{2.9}$$

with implicit normal-ordering. Using the commutation relation (2.6), we have

$$[L_m, \alpha_n^{(i)}] = -n\alpha_{m+n}^{(i)} \tag{2.10}$$

For the field X living on the k-wound copy, the modes are defined as

$$\alpha_n = \frac{1}{2\pi} \oint_{C_0^{(2\pi k)}} dz\, z^n \partial X(z) \tag{2.11}$$

where the contour of the integral $C_0^{(2\pi k)}$ is again centered around $z = 0$ but now from angle 0 to $2\pi k$. The boundary condition for the field X requires that $n = m/k$ where m is an integer. The commutation relation is given by

$$[\alpha_m, \alpha_n] = km\delta_{m+n,0} \tag{2.12}$$

We also define

$$\alpha_n^\dagger = \alpha_{-n} \tag{2.13}$$

The vacuum $|0^k\rangle$ of the k-wound copy is defined by the condition

$$\alpha_n|0^k\rangle = 0, \qquad n \geq 0 \tag{2.14}$$

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The Virasoro generators can be expanded in terms of a sum over bilinears of modes

$$L_m = \frac{1}{2k} \sum_n \alpha_n \alpha_{m-n} \tag{2.15}$$

again with implicit normal-ordering. Using the commutation relation (2.12), we have

$$[L_m, \alpha_n] = -n\alpha_{m+n} \tag{2.16}$$

The k-wound copy can be produced by applying the twist operator σ_k to the untwisted sector. The twist operator σ_k has dimension [10]

$$h(\sigma_k) = \frac{c}{24}\left(k - \frac{1}{k}\right) \tag{2.17}$$

For a single free boson, $c = 1$.

3 The effect of a twist operator

In this section, we will briefly review the effect of a twist operator. In this paper, we restrict ourselves to the simplest case where there are only two copies of the seed CFT, such that $N = 2$. Suppose at $z = 0$ an initial state in the untwisted sector is given by

$$\alpha^{(i_1)}_{-n_1}\alpha^{(i_2)}_{-n_2}\dots\alpha^{(i_m)}_{-n_m}|0\rangle^{(1)}|0\rangle^{(2)} \tag{3.1}$$

where $n_k > 0$ and $i_k = 1, 2$ is the copy label. Let us apply the twist operator σ_2 at z_0. The question is to find out the state ϕ at $|z| > |z_0|$ which is after the twist operator. The state ϕ is defined as

$$|\phi\rangle = \sigma_2(z_0)\alpha^{(i_1)}_{-n_1}\alpha^{(i_2)}_{-n_2}\dots\alpha^{(i_k)}_{-n_k}|0\rangle^{(1)}|0\rangle^{(2)} \tag{3.2}$$

which lives on a doubly wound copy since the twist operator has joined the two singly wound copies in the initial state. This question has been addressed completely by the covering map method, which will be reproduced for a single boson in appendix A. The effect of a twist operator can be summarized by the following three basic rules:

(i) Contraction: two modes $\alpha^{(i)}_{-m}$ and $\alpha^{(j)}_{-n}$ in the initial state (3.1) can 'Wick contract', giving a number

$$C^{ij}[m, n] \equiv C[\alpha^{(i)}_{-m}\alpha^{(j)}_{-n}] \tag{3.3}$$

For the process of Wick contraction, we consider all possible pairs of modes. For each such pair, we get a term where the pair contracts to the above number, and a term where the pair does not contract but will pass through the twist as shown in step (ii) below.

(ii) Propagation: any modes left after the contraction will pass through the twist and become modes after the twist operator, which are modes on the doubly wound copy.

$$\alpha^{(i)}_{-n} \quad\longrightarrow\quad \sum_{p>0} f_i[-n, -p]\alpha_{-p}, \qquad i = 1, 2 \tag{3.4}$$

where α_{-p} is a mode on the doubly wound copy and $p = m/2$ where m is a positive integer. In section 5.1, we will show that the nontrivial f_i's are

$$\begin{aligned} &f_i[-n,-n] = 1/2 \\ &f_i[-n,-p] \neq 0, \qquad \text{when} \quad p \neq n \quad \text{and} \quad p \text{ is a positive half integer} \end{aligned} \tag{3.5}$$

(iii) Pair creation: after the previous two steps, the modes in the initial state have either been contracted or passed through the twist. We are left with the twist operator acting on the untwisted vacuum

$$|\chi\rangle \equiv \sigma_2(z_0)|0\rangle^{(1)}|0\rangle^{(2)} = \exp\left(\sum_{m,n>0}\gamma_{mn}\alpha_{-m}\alpha_{-n}\right)|0^2\rangle \tag{3.6}$$

where the dimension of the twist operator σ_2 is (2.17)

$$h = h(\sigma_2) = 1/16 \tag{3.7}$$

which takes into account the difference of dimensions between the vacuum of doubly wound copy $|0^2\rangle$ and the vacuum of two singly wound copies $|0\rangle^{(1)}|0\rangle^{(2)}$. In section 5.1, we will show that

$$\gamma_{mn} \neq 0 \qquad \text{only if } m, n \text{ are positive half integers} \tag{3.8}$$

The above rules are shown in figure 2. To better understand these rules, let us consider an example with one initial mode

$$\sigma_2(z_0)\alpha^{(i)}_{-n}|0\rangle^{(1)}|0\rangle^{(2)} = \sum_{p>0} f_i[-n,-p]\alpha_{-p}\exp\left(\sum_{m,n>0}\gamma_{mn}\alpha_{-m}\alpha_{-n}\right)|0^2\rangle \tag{3.9}$$

We first use the propagation rule (3.4) to pass the initial mode through the twist. Then the twist operator acts on the untwisted vacuum to produce pairs using the rule (3.6). Let us now consider an example of two initial modes

$$\begin{aligned} \sigma_2(z_0)\alpha^{(i)}_{-n_1}\alpha^{(j)}_{-n_2}|0\rangle^{(1)}|0\rangle^{(2)} = &\left(\sum_{p_1>0} f_i[-n_1,-p_1]\alpha_{-p_1}\sum_{p_2>0} f_j[-n_2,-p_2]\alpha_{-p_2} + C^{ij}[n_1,n_2]\right) \\ &\times\exp\left(\sum_{m,n>0}\gamma_{mn}\alpha_{-m}\alpha_{-n}\right)|0^2\rangle \end{aligned} \tag{3.10}$$

The first term in the parentheses comes from the propagation of the two initial modes while the second term is from the contraction. The exponent in the last line comes from the pair creation.

If the effect of an operator satisfies the above three rules but with independent and undetermined coefficients f_i, C^{ij}, and γ, we call it the weak Bogoliubov form. As will be explained in the next section, in the 'normal' Bogoliubov transformation these coefficients are not independent of each other.

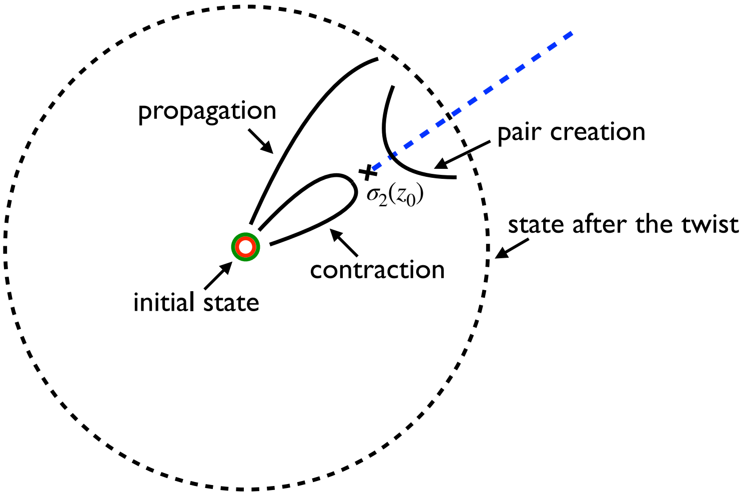

Figure 2. The three basic rules to compute the effect of a twist operator.

4 Bogoliubov transformation

The Bogoliubov transformation is a linear transformation that mixes creation and annihilation operators

$$\begin{aligned}\hat{a} &= \alpha\,\hat{b} + \beta\,\hat{b}^\dagger \\ \hat{a}^\dagger &= \alpha^*\,\hat{b}^\dagger + \beta^*\,\hat{b}\end{aligned} \tag{4.1}$$

where $|\alpha|^2 - |\beta|^2 = 1$ to make both sets of operators canonical

$$[\hat{a}, \hat{a}^\dagger] = 1, \qquad [\hat{b}, \hat{b}^\dagger] = 1 \tag{4.2}$$

The 'a' vacuum and 'b' vacuum are defined by the conditions

$$\hat{a}|0\rangle_a = 0, \qquad \hat{b}|0\rangle_b = 0 \tag{4.3}$$

Since the Bogoliubov transformation mixes the creation and annihilation operators, the 'a' vacuum is no longer the 'b' vacuum

$$\hat{a}|0\rangle_a = (\alpha\,\hat{b} + \beta\,\hat{b}^\dagger)|0\rangle_a = 0 \quad \longrightarrow \quad |0\rangle_a = e^{\frac{1}{2}\gamma\,\hat{b}^\dagger\hat{b}^\dagger}|0\rangle_b \tag{4.4}$$

where

$$\gamma = -\frac{\beta}{\alpha} \tag{4.5}$$

Consider an example with two initial modes on the 'a' vacuum

$$\begin{aligned}\hat{a}^\dagger\hat{a}^\dagger|0\rangle_a &= (\alpha^*\,\hat{b}^\dagger + \beta^*\,\hat{b})(\alpha^*\,\hat{b}^\dagger + \beta^*\,\hat{b})e^{\frac{1}{2}\gamma\,\hat{b}^\dagger\hat{b}^\dagger}|0\rangle_b \\ &= \Big[(\alpha^* + \beta^*\,\gamma)\hat{b}^\dagger(\alpha^* + \beta^*\,\gamma)\hat{b}^\dagger + \beta^*\alpha^*\Big]\,e^{\frac{1}{2}\gamma\,\hat{b}^\dagger\hat{b}^\dagger}|0\rangle_b\end{aligned} \tag{4.6}$$

It is similar to the effect of a twist operator with two initial modes (3.10). The first term in the bracket comes from the propagation of the two initial modes while the second term is from their contraction. The exponent in the last line comes from the pair creation. Similarly, we can compute the case with multiple initial modes

$$\hat{a}^\dagger \ldots \hat{a}^\dagger|0\rangle_a \tag{4.7}$$

using the following three rules that are similar to the rules in section 3.

(i) Contraction:

$$C[\hat{a}^\dagger, \hat{a}^\dagger] = \beta^* \alpha^* \tag{4.8}$$

(ii) Propagation:

$$\hat{a}^\dagger \to (\alpha^* + \beta^* \gamma)\hat{b}^\dagger \tag{4.9}$$

(iii) Pair creation:

$$\hat{a}|0\rangle_a = e^{\frac{1}{2}\gamma \hat{b}^\dagger \hat{b}^\dagger}|0\rangle_b \tag{4.10}$$

The coefficients in these expressions are not independent of each other. They are related through (4.5). We will call the above three rules with the constraint (4.5) the 'normal' Bogoliubov ansatz and the same rules without the constraint the 'weak' Bogoliubov ansatz.

In [19], it has been shown that the effect of a twist operator explained in section 3 satisfies the 'normal' Bogoliubov ansatz with the generalization: the creation and annihilation operators are generalized to infinite-dimensional vectors, i.e. $\hat{a}$ becomes $\hat{a}_i$; the coefficients α, β, γ are generalized to infinite-dimensional matrices, i.e. α becomes α_{ij}. In [19], by matching the modes before and after the twist, they obtained the matrices α and β. However, there is no systematic way to invert the infinite-dimensional matrices α to obtain γ through the constraint (4.5). In this paper, we will take the weak Bogoliubov ansatz which is without the constraint. We will show that it is enough to determine all the coefficients without inverting any infinite-dimensional matrices. The constraint (4.5) is satisfied automatically.

5 Bootstrapping the effect of a twist operator

In this section, we will take the weak Bogoliubov ansatz and apply the Virasoro generators to obtain recursion relations for all the coefficients in the ansatz. The solutions to these recursion relations match the results from the covering map method.

5.1 Global modes

In this section we will show (3.5) and (3.8). Let us start with

$$\begin{aligned}
\alpha_n\sigma_2(z_0) - \sigma_2(z_0)(\alpha_n^{(1)} + \alpha_n^{(2)}) &= \frac{1}{2\pi}\oint_{C_{z_0}^{(4\pi)}} dz\; z^n \partial X(z)\sigma_2(z_0)\\
&= z_0^n \frac{1}{2\pi}\oint_{C_{z_0}^{(4\pi)}} dz\; \left(1 + (z-z_0)/z_0\right)^n \partial X(z)\sigma_2(z_0)\\
&= z_0^n(\alpha_0^{(z_0)} + n\alpha_1^{(z_0)}/z_0 + \ldots)\sigma_2(z_0)
\end{aligned} \tag{5.1}$$

where n is an integer. To obtain the first equality, we join the two contours, one before and one after the twist operator, into a single contour centered around the twist operator. We

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note that modes with superscript (z_0), i.e. $\alpha_0^{(z_0)}$, are the modes centered around z_0, while modes without it, i.e. α_n and $\alpha_n^{(i)}$, are the modes centered around $z = 0$. Since the twist operator σ_2 is defined as the lowest dimension operator that produces the twisted sector, we have

$$\alpha_n^{(z_0)}\sigma_2(z_0) = 0, \quad n \geq 0 \tag{5.2}$$

Thus (5.1) becomes

$$\alpha_n\sigma_2(z_0) - \sigma_2(z_0)(\alpha_n^{(1)} + \alpha_n^{(2)}) = 0 \tag{5.3}$$

Consider the following state where m and n are positive integers

$$\begin{aligned}\langle 0^2|\alpha_m\sigma_2(z_0)\alpha_{-n}^{(i)}|0\rangle^{(1)}|0\rangle^{(2)} &= \langle 0^2|\sigma_2(z_0)(\alpha_m^{(1)} + \alpha_m^{(2)})\alpha_{-n}^{(i)}|0\rangle^{(1)}|0\rangle^{(2)} \\ &= \delta_{mn}n\end{aligned} \tag{5.4}$$

which is nonzero only if m and n are equal. From $\langle 0^2|\alpha_n\alpha_{-n}|0^2\rangle = 2n$, we find

$$\sigma_2(z_0)\alpha_{-n}^{(i)}|0\rangle^{(1)}|0\rangle^{(2)} = \left(\frac{1}{2}\alpha_{-n} + \text{half integer modes}\right)|0^2\rangle \tag{5.5}$$

which is (3.5). To show (3.8), consider the following state where m is a positive integer

$$\alpha_m|\chi\rangle = \alpha_m\sigma_2(z_0)|0\rangle^{(1)}|0\rangle^{(2)} = \sigma_2(z_0)(\alpha_m^{(1)} + \alpha_m^{(2)})|0\rangle^{(1)}|0\rangle^{(2)} = 0 \tag{5.6}$$

Thus the state from the pair creation χ can not have any excitations of integer modes, which is stated in (3.8).

5.2 Pair creation

In this subsection, we will derive the pair creation coefficients γ_{mn}.

5.2.1 Relations from L_{-1}

Starting with

$$L_{-1}|0\rangle^{(1)}|0\rangle^{(2)} = 0 \tag{5.7}$$

where we have used (2.8) and (2.9), we have

$$\begin{aligned}0 &= \sigma_2(z_0)L_{-1}|0\rangle^{(1)}|0\rangle^{(2)} \\ &= (L_{-1}\sigma_2(z_0) - [L_{-1}, \sigma_2(z_0)])|0\rangle^{(1)}|0\rangle^{(2)}\end{aligned} \tag{5.8}$$

Let's compute the commutator. We have

$$[L_{-1}, \sigma_2(z_0)] = \oint_{C_{z_0}^{(4\pi)}} \frac{dz}{2\pi i}T(z)\sigma_2(z_0) = L_{-1}^{(z_0)}\sigma_2(z_0) = \partial\sigma_2(z_0) \tag{5.9}$$

The mode $L_{-1}^{(z_0)}$ with superscript (z_0) is the mode centered around z_0, while the mode without it, i.e. L_{-1}, is the mode centered around $z = 0$. Therefore (5.8) becomes

$$\begin{aligned}0 &= (L_{-1} - \partial)\sigma_2(z_0)|0\rangle^{(1)}|0\rangle^{(2)} \\ &= (L_{-1} - \partial)\exp\left(\sum_{m,n\geq 0}\gamma_{m+1/2,n+1/2}\,\alpha_{-(m+1/2)}\alpha_{-(n+1/2)}\right)|0^2\rangle\end{aligned} \tag{5.10}$$

Here we have relabeled $m \to m + 1/2$ and $n \to n + 1/2$ in (3.6) where m and n are now integers. Our aim is to find relations which will help us solve for $\gamma_{m+1/2,n+1/2}$. There are four cases we will look at it.

$m, n > 0$. The first case is $m, n > 0$. We will look for ways to obtain terms propotional to

$$\alpha_{-(m+1/2)}\alpha_{-(n+1/2)}|0^2\rangle \tag{5.11}$$

There are three ways to get this. From (5.10) we obtain

$$\begin{aligned}0 = \gamma_{m-1/2,n+1/2}\big[L_{-1},\alpha_{-(m-1/2)}\big]\alpha_{-(n+1/2)} + \gamma_{m+1/2,n-1/2}\alpha_{-(m+1/2)}\big[L_{-1},\alpha_{-(n+1/2)}\big]\\ - \partial\gamma_{m+1/2,n+1/2}\alpha_{-(m+1/2)}\alpha_{-(n+1/2)}\end{aligned} \tag{5.12}$$

Using (2.10) we find the relation

$$\gamma_{m-1/2,n+1/2}(m-1/2) + \gamma_{m+1/2,n-1/2}(n-1/2) = \partial\gamma_{m+1/2,n+1/2} \tag{5.13}$$

$m = 0, n > 0$. Next we consider the case where $m = 0, n > 0$. We need terms proportional to

$$\alpha_{-1/2}\alpha_{-(n+1/2)}|0^2\rangle \tag{5.14}$$

Looking at (5.10) we get

$$0 = \gamma_{1/2,n-1/2}\alpha_{-1/2}[L_{-1},\alpha_{-(n-1/2)}] - \partial\gamma_{1/2,n+1/2}\alpha_{-1/2}\alpha_{-(n+1/2)} \tag{5.15}$$

Again using (2.10) we obtain the relation

$$\gamma_{1/2,n-1/2}(n-1/2) = \partial\gamma_{1/2,n+1/2} \tag{5.16}$$

$m > 0, n = 0$. Similar to the previous case, for $m > 0, n = 0$ we obtain the relation

$$\gamma_{m-1/2,1/2}(m-1/2) = \partial\gamma_{m+1/2,1/2} \tag{5.17}$$

$m = n = 0$. For $m = n = 0$ we need terms proportional to

$$\alpha_{-1/2}\alpha_{-1/2}|0^2\rangle \tag{5.18}$$

For this we use (2.15) with $k = 2$ where

$$L_{-1}|0^2\rangle = \frac{1}{4}\alpha_{-1/2}\alpha_{-1/2}|0^2\rangle \tag{5.19}$$

Using this in (5.10) we find the relation

$$\frac{1}{4} = \partial\gamma_{1/2,1/2} \tag{5.20}$$

5.2.2 The solution

To find the initial condition for these differential equations, we consider

$$\begin{aligned}|0^2\rangle &= \sigma_2(z_0 = 0)|0\rangle^{(1)}|0\rangle^{(2)}\\ &= \exp\left(\sum_{m,n\geq 0}\gamma_{m+1/2,n+1/2}(z_0 = 0)\,\alpha_{-(m+1/2)}\alpha_{-(n+1/2)}\right)|0^2\rangle\end{aligned} \tag{5.21}$$

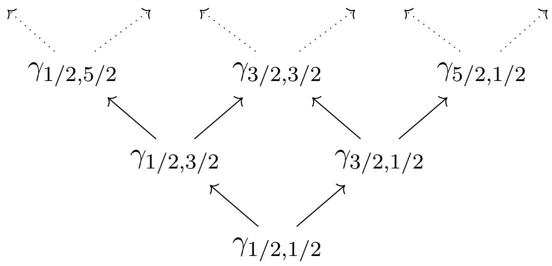
The first line comes from the definition of the twist operator σ_2 which is the lowest dimension operator that changes the untwisted sector to the twisted sector. Thus we have

$$\gamma_{m+1/2,n+1/2}(z_0 = 0) = 0 \tag{5.22}$$

Using these initial conditions, we can solve the differential equations. The solution to (5.20) is

$$\gamma_{1/2,1/2} = \frac{1}{4} z_0 \tag{5.23}$$

We can find all other $\gamma_{m+1/2,n+1/2}$'s by using the relations (5.13), (5.16), and (5.17). You can think of the $\gamma_{m+1/2,n+1/2}$'s as forming an inverted triangle with integer lattice spacing with $\gamma_{1/2,1/2}$ as the bottom lattice point. The relation (5.16) moves you along the right edge, (5.17) moves you along the left edge and (5.13) moves you within the interior.



(5.24)

The solution is

$$\gamma_{m+1/2,n+1/2} = \frac{z_0^{m+n+1}\Gamma[\frac{3}{2}+m]\Gamma[\frac{3}{2}+n]}{(2m+1)(2n+1)(1+m+n)\pi\Gamma[m+1]\Gamma[n+1]} \tag{5.25}$$

where m and n are non-negative integers.

5.3 Propagation

Here we derive the propagation coefficients $f_i[-n,-p]$ which correspond to a mode passing through the twist.

5.3.1 Relations from L_0

Here we use the generator L_0 to derive a relation for $f_1[-1,-p]$. We begin with

$$\begin{aligned}\sigma_2(z_0)\alpha_{-1}^{(1)}|0\rangle^{(1)}|0\rangle^{(2)} &= \sigma_2(z_0)L_0\alpha_{-1}^{(1)}|0\rangle^{(1)}|0\rangle^{(2)}\\ &= (L_0\sigma_2(z_0) - [L_0,\sigma_2(z_0)])\alpha_{-1}^{(1)}|0\rangle^{(1)}|0\rangle^{(2)}\end{aligned} \tag{5.26}$$

where the initial mode is on the first copy. Let us compute the commutator. We have

$$\begin{aligned}[L_0,\sigma_2(z_0)] &= \oint_{C_{z_0}^{(4\pi)}} \frac{dz}{2\pi i}\, z\, T(z)\sigma_2(z_0)\\ &= \oint_{C_{z_0}^{(4\pi)}} \frac{dz}{2\pi i}\, (z_0 + z - z_0)\, T(z)\sigma_2(z_0)\\ &= (z_0 L_{-1}^{(z_0)} + L_0^{(z_0)})\sigma_2(z_0)\\ &= (z_0\partial + h)\sigma_2(z_0)\end{aligned} \tag{5.27}$$

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Therefore

$$\sigma_2(z_0)\alpha_{-1}^{(1)}|0\rangle^{(1)}|0\rangle^{(2)} = (L_0 - (z_0\partial + h))\sigma_2(z_0)\alpha_{-1}^{(1)}|0\rangle^{(1)}|0\rangle^{(2)} \tag{5.28}$$

which gives

$$\sum_{p>0} f_1[-1,-p]\alpha_{-p}|\chi\rangle = (L_0 - (z_0\partial + h))\sum_{p>0} f_1[-1,-p]\alpha_{-p}|\chi\rangle \tag{5.29}$$

where χ is state (3.6) from pair creation. Taking the term proportional to $\alpha_{-p}|0^2\rangle$ with p being a positive half integer, we have

$$f_1[-1,-p]\alpha_{-p}|0^2\rangle = (L_0 - (z_0\partial + h))f_1[-1,-p]\alpha_{-p}|0^2\rangle \tag{5.30}$$

which gives

$$f_1[-1,-p] = (p - z_0\partial)f_1[-1,-p] \tag{5.31}$$

The solution is

$$f_1[-1,-p] \propto z_0^{p-1} \tag{5.32}$$

where p is a positive half integer.

5.3.2 Relations from L_1

Here we use L_1 to determine the proportional coefficients in (5.32). Since

$$\alpha_0^{(1)}|0\rangle^{(1)} = 0 \tag{5.33}$$

we begin with the following relation

$$0 = \sigma_2(z_0)L_1\alpha_{-1}^{(1)}|0\rangle^{(1)}|0\rangle^{(2)} \tag{5.34}$$

Bringing L_1 through the twist we obtain

$$0 = (L_1\sigma_2(z_0) - [L_1, \sigma_2(z_0)])\alpha_{-1}^{(1)}|0\rangle^{(1)}|0\rangle^{(2)} \tag{5.35}$$

Let us compute the commutator. We have

$$\begin{aligned}
[L_1, \sigma_2(z_0)] &= \oint_{C_{z_0}^{(4\pi)}} \frac{dz}{2\pi i}\, z^2\, T(z)\sigma_2(z_0) \\
&= \oint_{C_{z_0}^{(4\pi)}} \frac{dz}{2\pi i}\, (z_0 + z - z_0)^2 T(z)\sigma_2(z_0) \\
&= (z_0^2 L_{-1}^{(z_0)} + 2z_0 L_0^{(z_0)} + L_1^{(z_0)})\sigma_2(z_0) \\
&= z_0(z_0\partial + 2h)\sigma_2(z_0)
\end{aligned} \tag{5.36}$$

Inserting this into (5.35) yields

$$\begin{aligned}
0 &= (L_1 - z_0(z_0\partial + 2h))\sigma_2(z_0)\alpha_{-1}^{(1)}|0\rangle^{(1)}|0\rangle^{(2)} \\
&= (L_1 - z_0(z_0\partial + 2h))\sum_{p>0} f_1[-1,-p]\alpha_{-p}\exp\left(\sum_{m,n\geq 0}\gamma_{m+1/2,n+1/2}\alpha_{-(m+1/2)}\alpha_{-(n+1/2)}\right)|0^2\rangle
\end{aligned} \tag{5.37}$$

We again match terms to obtain relations which one solves to find $f_1[-1,-p]$. We want to keep terms which are $\alpha_{-(m+1/2)}|0^2\rangle$ with $m>0$. The terms which contribute are

$$\begin{aligned}0=&2f_1[-1,-1/2]\gamma_{m+1/2,1/2}\alpha_{-(m+1/2)}[[L_1,\alpha_{-1/2}],\alpha_{-1/2}]\\&+f_1[-1,-(m+1/2)]\gamma_{1/2,1/2}\alpha_{-(m+1/2)}[[L_1,\alpha_{-1/2}],\alpha_{-1/2}]\\&+f_1[-1,-(m+3/2)][L_1,\alpha_{-(m+3/2)}]-z_0(z_0\partial+2h)f_1[-1,-(m+1/2)]\alpha_{-(m+1/2)}\end{aligned} \tag{5.38}$$

Using the commutation relation (2.16), this gives a recursion relation for $m>0$

$$\begin{aligned}&(m+3/2)f_1[-1,-(m+3/2)]\\&=\left[z_0(z_0\partial+2h)-\frac{1}{2}\gamma_{1/2,1/2}\right]f_1[-1,-(m+1/2)]-f_1[-1,-1/2]\gamma_{m+1/2,1/2}\end{aligned} \tag{5.39}$$

Now we need to find a relation for $f_1[-1,-3/2]$. In order to do so we will need terms which are proportional to $\alpha_{-1/2}|0^2\rangle$. From (5.37) we get

$$\begin{aligned}0=&3f_1[-1,-1/2]\gamma_{1/2,1/2}[[L_1,\alpha_{-1/2}],\alpha_{-1/2}]\alpha_{-1/2}\\&+f_1[-1,-3/2][L_1,\alpha_{-3/2}]-z_0(z_0\partial+2h)f_1[-1,-1/2]\alpha_{-1/2}\end{aligned} \tag{5.40}$$

where we have collected terms coming from appropriate commutators in order to leave only one $\alpha_{-1/2}$. Again using commutation relations we obtain the relation

$$\frac{3}{2}f_1[-1,-3/2]=z_0(z_0\partial+2h)f_1[-1,-1/2]-\frac{3}{2}f_1[-1,-1/2]\gamma_{1/2,1/2} \tag{5.41}$$

Therefore all $f_1[-1,-p]$ can be determined from $f_1[-1,-1/2]$.

5.3.3 The solution

The solution satisfying the recursion relations (5.39) and (5.41) is

$$f_1[-1,-p]=C\frac{\Gamma[p-1]}{\Gamma[p+1/2]}z_0^{p-1} \tag{5.42}$$

where p is a positive half integer. The constant C will be derived in (5.61)

$$C=\pm\frac{i}{4\sqrt{\pi}} \tag{5.43}$$

To obtain $f_2[-1,-p]$ for an excitation on copy 2, we change the location of the twist by $z_0\to z_0e^{2\pi i}$. It interchanges copy 1 and copy 2. Since p is a half integer, we have $z_0^{p-1}\to -z_0^{p-1}$. Therefore, we have

$$f_2[-1,-p]=-C\frac{\Gamma[p-1]}{\Gamma[p+1/2]}z_0^{p-1} \tag{5.44}$$

Thus the two possible signs of (5.43) correspond to two possible conventions of labeling copy 1 and copy 2. By applying L_{-1} repeatedly, we can compute $f_i[-n,-p]$ for $n>1$, which will be shown in appendix B. The final expressions are given by

$$\begin{aligned}
f_1[-n,-p] &= \frac{1}{2}\delta_{n,p}\\
f_1[-n,-p] &= \frac{iz_0^{p-n}\Gamma(\frac{1}{2}+n)\Gamma(p)}{\pi(2p-2n)\Gamma(n)\Gamma(p+\frac{1}{2})}, \quad n\neq p\\
f_2[-n,-p] &= \frac{1}{2}\delta_{n,p}\\
f_2[-n,-p] &= -f_1[-n,-p], \quad n\neq p
\end{aligned} \tag{5.45}$$

5.4 Contraction

Here we derive the expression for the contraction of two modes in the initial state under the effect of the twist.

5.4.1 Relations from L_1

Let's start with the following state with $n>1$

$$\begin{aligned}
\sigma_2(z_0)\alpha^{(i)}_{-1}\alpha^{(j)}_{-(n-1)}|0\rangle^{(1)}|0\rangle^{(2)} &= \frac{1}{n}\sigma_2(z_0)L_1\alpha^{(i)}_{-1}\alpha^{(j)}_{-n}|0\rangle^{(1)}|0\rangle^{(2)}\\
&= \frac{1}{n}[L_1 - z_0(z_0\partial+2h)]\sigma_2(z_0)\alpha^{(i)}_{-1}\alpha^{(j)}_{-n}|0\rangle^{(1)}|0\rangle^{(2)}
\end{aligned} \tag{5.46}$$

where we remind the reader that $i,j=1,2$ are copy labels. So we have the relation

$$[L_1 - z_0(z_0\partial+2h)]\sigma_2(z_0)\alpha^{(i)}_{-1}\alpha^{(j)}_{-n}|0\rangle^{(1)}|0\rangle^{(2)} = n\sigma_2(z_0)\alpha^{(i)}_{-1}\alpha^{(j)}_{-(n-1)}|0\rangle^{(1)}|0\rangle^{(2)} \tag{5.47}$$

which becomes

$$\begin{aligned}
&[L_1 - z_0(z_0\partial+2h)]\left(\sum_{p>0} f_i[-1,-p]\alpha_{-p}\sum_{p'>0} f_j[-n,-p']\alpha_{-p'} + C^{ij}[1,n]\right)|\chi\rangle\\
&= n\left(\sum_{p>0} f_i[-1,-p]\alpha_{-p}\sum_{p'>0} f_j[-(n-1),-p']\alpha_{-p'} + C^{ij}[1,n-1]\right)|\chi\rangle
\end{aligned} \tag{5.48}$$

We collect terms which carry no bosonic modes in order to easily isolate C^{ij}. Terms of this kind are given by

$$\begin{aligned}
&\big(f_i[-1,-1/2]f_j[-n,-1/2] + C^{ij}[1,n]\gamma_{1/2,1/2}\big)[[L_1,\alpha_{-1/2}],\alpha_{-1/2}]\\
&\quad - z_0(z_0\partial+2h)C^{ij}[1,n] = nC^{ij}[1,n-1]
\end{aligned} \tag{5.49}$$

Using the commutation relations yield for $n>1$

$$\frac{1}{2}f_i[-1,-1/2]f_j[-n,-1/2] + \frac{z_0}{8}C^{ij}[1,n] - z_0(z_0\partial+2h)C^{ij}[1,n] = nC^{ij}[1,n-1] \tag{5.50}$$

Setting $h=1/16$ (the dimension of the twist operator), the relation becomes

$$\frac{1}{2}f_i[-1,-1/2]f_j[-n,-1/2] - z_0^2\partial C^{ij}[1,n] = nC^{ij}[1,n-1] \tag{5.51}$$

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where $n > 1$. To find $C^{ij}[1,1]$, notice that in (5.47) if $n = 1$, the r.h.s. vanishes. Thus the r.h.s. of (5.51) vanishes if $n = 1$, which gives

$$\frac{1}{2} f_i[-1,-1/2] f_j[-1,-1/2] = z_0^2 \partial C^{ij}[1,1] \tag{5.52}$$

The solution is

$$C^{ij}[1,1] = -(-1)^{i+j} \pi C^2 z_0^{-2} \tag{5.53}$$

where we have used (5.42) and (5.44).

5.4.2 Relations from L_{-2}

Here we will derive the constant C which appears in (5.42), (5.44), and (5.53). Start with

$$\sigma_2(z_0) L_{-2} |0\rangle^{(1)} |0\rangle^{(2)} = \sigma_2(z_0) \frac{1}{2} (\alpha_{-1}^{(1)} \alpha_{-1}^{(1)} + \alpha_{-1}^{(2)} \alpha_{-1}^{(2)}) |0\rangle^{(1)} |0\rangle^{(2)} \tag{5.54}$$

Using the contraction (5.53), the term without any modes on the r.h.s. is

$$\frac{1}{2} (C^{11}[1,1] + C^{22}[1,1]) |0^2\rangle = -\pi C^2 z_0^{-2} |0^2\rangle \tag{5.55}$$

Let us compute the l.h.s. of (5.54) in another way. We have

$$\sigma_2(z_0) L_{-2} = L_{-2} \sigma_2(z_0) - [L_{-2}, \sigma_2(z_0)] \tag{5.56}$$

where

$$\begin{aligned}
[L_{-2}, \sigma_2(z_0)] &= \oint_{C_{z_0}^{(4\pi)}} \frac{dz}{2\pi i} z^{-1} T(z) \sigma_2(z_0) \\
&= z_0^{-1} \oint_{C_{z_0}^{(4\pi)}} \frac{dz}{2\pi i} \left(1 + \frac{z - z_0}{z_0}\right)^{-1} T(z) \sigma_2(z_0) \\
&= z_0^{-1} \oint_{C_{z_0}^{(4\pi)}} \frac{dz}{2\pi i} \left(1 - \frac{z - z_0}{z_0} + \dots\right) T(z) \sigma_2(z_0) \\
&= z_0^{-1} (L_{-1}^{(z_0)} - z_0^{-1} L_0^{(z_0)}) \sigma_2(z_0) \\
&= z_0^{-1} (\partial - z_0^{-1} h) \sigma_2(z_0)
\end{aligned} \tag{5.57}$$

Thus we have

$$\sigma_2(z_0) L_{-2} |0\rangle^{(1)} |0\rangle^{(2)} = [L_{-2} - z_0^{-1}(\partial - z_0^{-1} h)] |\chi\rangle \tag{5.58}$$

The term without any modes is

$$z_0^{-2} h |0^2\rangle = \frac{z_0^{-2}}{16} |0^2\rangle \tag{5.59}$$

where we have used $h = 1/16$. Comparing to (5.55), we obtain

$$-\pi C^2 = \frac{1}{16} \tag{5.60}$$

which gives

$$C = \pm \frac{i}{4\sqrt{\pi}} \tag{5.61}$$

As explained in section 5.3.3, the two signs correspond to the two different conventions of labeling the copies.

5.4.3 The solution

Using (5.61), the contraction (5.53) becomes

$$C^{ij}[1,1] = -(-1)^{i+j}\pi C^2 z_0^{-2} = (-1)^{i+j}\frac{1}{16}z_0^{-2} \tag{5.62}$$

Solving (5.51) recursively we find the following solution

$$C^{ij}[1,n] = (-1)^{i+j}\frac{z_0^{-(1+n)}\Gamma(\frac{1}{2}+n)}{4(1+n)\sqrt{\pi}\Gamma(n)} \tag{5.63}$$

The contractions $C^{ij}[n_1,n_2]$ with $n_1, n_2 > 0$ are computed in appendix C. The final expressions are

$$C^{ij}[n_1,n_2] = (-1)^{i+j}\frac{z_0^{-(n_1+n_2)}}{2(n_1+n_2)\pi}\frac{\Gamma(\frac{1}{2}+n_1)\Gamma(\frac{1}{2}+n_2)}{\Gamma(n_1)\Gamma(n_2)} \tag{5.64}$$

where $i, j = 1, 2$ are copy labels.

6 Discussion

The traditional way to compute the effect of a twist operator is by using the covering map method. In this paper, we have developed a new method using the Bogoliubov ansatz and conformal symmetry. The Bogoliubov ansatz includes three quantities which characterize the final state uniquely. These three quantities correspond to three effects produced by the twist operator. One effect is pair creation in which the twist, when acting on the vacuum in the initial state, produces pairs of modes in the final state. This effect is encoded in the coefficients γ_{mn} which are computed in (5.25). Another effect, which arises by applying the twist operator to a single mode in the initial state, is propagation. This effect is encoded in the functions $f_i[-n,-p]$ which are computed in (5.45). The third effect produced by the twist is the contraction of two modes in the initial state. This process is encoded in the functions $C^{ij}[n_1,n_2]$. These expressions were computed in (5.64). Each of these quantities agrees with the results from the covering map method in appendix A. Using the Bogoliubov ansatz along with conformal symmetry we have derived a new method for computing effects of the twist operator in orbifold CFTs.

Our results also answer an important question about the nature of the effect of a twist operator. From the general idea of the covering map, we know that the twist operator generates a Bogoliubov transformation. This raises the following question: if we know that the effect of a twist is a Bogoliubov transformation what else do we need to completely determine the coefficients in this transformation? From the covering map method, it seems that the answer is the covering map. The covering map is essential to determine the effect of a twist operator. However, our results show that with conformal symmetry we do not need other input to determine its effect. The nature of the effect of a twist operator is captured by the form of the Bogoliubov transformation and conformal symmetry.

In the study of the perturbative D1D5 CFT, twist operators and their effects carry a wide variety of applications since a class of marginal deformations of the theory contain

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twist operators [26–44]. The D1D5 CFT is realized by four free bosons and four free fermions. Our results, which are for one boson, can be simply generalized to four bosons where each boson has its own propagation, contraction and pair creation. These effects for each of the four bosons of the D1D5 CFT are the same as those of one free boson computed in this paper, thus yielding the same expressions for the three relevant quantities. The generalization to free fermions is straight forward but will contain some additional features. We will present these results in a future work. Furthermore, to compute higher order effects in the D1D5 CFT, which is relevant for holography, it is necessary to compute higher order twist correlation functions [45–49]. In the covering map method, these computations become more challenging due to the growing complexity of the covering maps themselves. The method developed in this paper provides tools to possibly compute these higher order effects.

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A Covering space method

In this appendix we review the covering space method which is traditionally used to compute the effect of a twist operator. We will compute the coefficients of the three rules in section 3. In [14, 15], these coefficients are computed for the D1D5 CFT with $\mathcal{N} = 4$ superconformal symmetry, which is realized by four free bosons and four free fermions. In this appendix, we will focus on the theory of one free boson. The results are similar to the boson in the supersymmetric case with some normalization differences.

A.1 Mode definitions

Let us first define the bosonic modes on the z-plane as in section 2. Imagine we place the twist operator at a point z_0. Before the twist which corresponds to the point, $|z| < |z_0|$, we have two singly wound copies. They are defined as follows

$$\alpha_m^{(i)} = \frac{1}{2\pi} \oint_{C_0} dz z^m \partial X^{(i)}(z), \quad i = 1, 2 \tag{A.1}$$

Here m is an integer. The commutation relations before the twist is given by

$$[\alpha_m^{(i)}, \alpha_n^{(j)}] = m\, \delta^{ij} \delta_{m+n,0} \tag{A.2}$$

After the twist where $|z| > |z_0|$, we have a doubly wound copy and the modes are defined as follows

$$\alpha_m = \frac{1}{2\pi} \oint_{C_0^{(4\pi)}} dz z^m \partial X(z) \tag{A.3}$$

Here m can be integer or half integer since these are modes defined on a doubly wound copy (we will indeed show that only half integer modes are nontrivially affected by the twist operator). The commutation relation after the twist is given by

$$[\alpha_m, \alpha_n] = 2m\delta_{m+n,0} \tag{A.4}$$

where the factor of two comes from the fact that the theory is defined on a doubly wound copy. Next we discuss the covering map.

A.2 Covering map

In order to compute these quantities we use the covering space method. In the z-plane, the effect of the twist introduces a branch cut. To resolve this branch cut, one can pass to the covering space t defined as

$$z = z_0 + t^2 \tag{A.5}$$

where

$$z_0 = a^2 \tag{A.6}$$

For $|z| > |z_0|$ we have a doubly wound copy. In the t-plane, $|z| \to \infty$ corresponds to $t \to \infty$. In the z-plane we have two single copies in the initial state located at the origin, $z = 0$. The location of these states gets mapped to two different points in the t-plane. In our convention, they are

$$\text{Copy 1}: \quad t = ia, \quad \text{Copy 2}: \quad t = -ia \tag{A.7}$$

One can also use the other convention where the copy labels are interchanged. The location of the twist operator is at $z_0 = a^2$. It maps to the location $t = 0$ in the t-plane. Here the covering map is of order 2 and is therefore the location where one crosses from one branch to the other. Now that we have analyzed the covering map let us compute the various ansatz quantities by mapping them to the t-plane. In the following sections we will compute the quantities, γ, f_i, C^{ij}, which determine the twisted state in the theory of one free boson.

A.3 Pair creation

In this subsection we compute the Bogoliubov coefficient γ by using the covering map (A.5). We start with the state (3.6). Let us first compute the amplitude

$$\langle 0^2|\alpha_m\alpha_n\sigma_2(z_0)|0\rangle^{(1)}|0\rangle^{(2)} = \langle 0^2|\alpha_m\alpha_n|\chi\rangle \tag{A.8}$$

Using the expression (3.6) we obtain the following

$$\gamma_{mn} = \frac{1}{8mn}\langle 0^2|\alpha_m\alpha_n\sigma_2(z_0)|0\rangle^{(1)}|0\rangle^{(2)} \tag{A.9}$$

In order to remove the twist we map our result to the t-plane. This gives the relation

$$\gamma_{mn} = \frac{1}{8mn}\,{}_t\langle 0|\alpha'_m\alpha'_n|0\rangle_t \tag{A.10}$$

The primes denote modes which have been mapped to the t-plane and the subscript t denotes t-plane states. The state $|0\rangle_t$ is the vacuum at the origin and the conjugate state ${}_t\langle 0|$ is the vacuum at $t = \infty$. We see that the insertion of the twist is removed since it's action is encoded in the covering map. The problem is then simplified to computing a wick contraction between terms within the t-plane. To do so we must define modes which are natural to the t-plane. For γ_{mn} we will only need modes defined after the twist but for the other coefficients we'll need modes before the twist. We therefore record both types below.

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Modes before the twist. Modes natural to the t-plane which correspond to initial states are given by

$$\begin{aligned}\text{Copy 1:} \quad \tilde{\alpha}_m^{t\to ia} &= \frac{1}{2\pi}\oint_{C_{ia}} dt(t-ia)^m \partial X(t) \\ \text{Copy 2:} \quad \tilde{\alpha}_m^{t\to -ia} &= \frac{1}{2\pi}\oint_{C_{-ia}} dt(t+ia)^m \partial X(t)\end{aligned} \tag{A.11}$$

Here m is an integer. We have the commutation relations

$$\begin{aligned}[\tilde{\alpha}_m^{t\to ia}, \tilde{\alpha}_n^{t\to ia}] &= m\delta_{m+n,0} \\ [\tilde{\alpha}_m^{t\to -ia}, \tilde{\alpha}_n^{t\to -ia}] &= m\delta_{m+n,0}\end{aligned} \tag{A.12}$$

Modes after the twist. Modes natural to the t-plane corresponding to the image of the location of final states in the z-plane are given by

$$\tilde{\alpha}_m^{t\to\infty} = \frac{1}{2\pi}\oint_{C_\infty} dt\, t^m \partial X(t) \tag{A.13}$$

Here m is an integer. Also we have the commutation relation

$$[\tilde{\alpha}_m^{t\to\infty}, \tilde{\alpha}_n^{t\to\infty}] = m\delta_{m+n,0} \tag{A.14}$$

Modes defined after the twist in the z-plane map to

$$\alpha_m \to \alpha'_m = \frac{1}{2\pi}\oint_{C_\infty} dt(z_0+t^2)^m \partial X(t) \tag{A.15}$$

Let's expand this mode around $t=\infty$. The integrand is

$$(z_0+t^2)^m = t^{2m}(1+z_0 t^{-2})^m = \sum_{k\geq 0} {}^mC_k\, z_0^k\, t^{2m-2k} \tag{A.16}$$

Inserting this expansion into (A.15) and using (A.13) gives

$$\alpha'_m = \sum_{k\geq 0} {}^mC_k\, z_0^k\, \tilde{\alpha}_{2m-2k}^{t\to\infty} \tag{A.17}$$

Inserting this into (A.10) gives

$$\gamma_{mn} = \frac{1}{8mn}\sum_{k,k'\geq 0} {}^mC_k\, {}^nC_{k'}\, z_0^{k+k'} [\tilde{\alpha}_{2m-2k}^{t\to\infty}, \tilde{\alpha}_{2n-2k'}^{t\to\infty}] \tag{A.18}$$

In order to have a nonzero contraction we require

$$m > k \text{ and } n < k' \tag{A.19}$$

Using commutation relation (A.14) gives

$$\gamma_{mn} = \frac{1}{4mn}\sum_{k=0}^{\lceil m-1\rceil} (m-k)\, {}^mC_k\, {}^nC_{m+n-k}\, z_0^{m+n} \tag{A.20}$$

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For $m, n \in \mathbb{Z}$, γ_{mn} is zero. For the case where m, n are half-integers we take

$$\begin{aligned} m &= m' + 1/2 \\ n &= n' + 1/2 \end{aligned} \tag{A.21}$$

and we obtain

$$\gamma_{m'+1/2,n'+1/2} = \frac{z_0^{m'+n'+1}\Gamma(\frac{3}{2}+m')\Gamma(\frac{3}{2}+n')}{(2m'+1)(2n'+1)(1+m'+n')\pi\Gamma(1+m')\Gamma(1+n')} \tag{A.22}$$

where $m', n' \in \mathbb{Z}_+$.

A.4 Propagation

In this subsection we compute $f_i[-n, -p]$ which describes the propagation of a mode through the twist operator where $i = 1, 2$ is the copy label of the initial mode. Let us start with the state

$$\sigma_2(z_0)\alpha_{-n}^{(i)}|0\rangle^{(1)}|0\rangle^{(2)} = \sum_{p>0} f_i[-n, -p]\alpha_{-p}|\chi\rangle \tag{A.23}$$

The propagation f_i can be computed from

$$f_i[-n, -p] = \frac{1}{2p}\langle 0^2|\alpha_p\sigma_2(z_0)\alpha_{-n}^{(i)}|0\rangle^{(1)}|0\rangle^{(2)} \tag{A.24}$$

where we have used the commutation relation (A.4).

Again, mapping to the t-plane gives

$$f_i[-n, -p] = \frac{1}{2p}{}_t\langle 0|\alpha'_p\alpha'^{(i)}_{-n}|0\rangle_t \tag{A.25}$$

In order to compute the Wick contraction we will first expand the mode $\alpha'^{(i)}_{-n}$ around it's t-plane image corresponding to a mode in the initial state. We will then expand these modes at $t = \infty$ enabling us to perform the Wick contraction with the mode in the final state. We first write the mode $\alpha'^{(1)}_{-n}$ which maps to the image $t = ia$

$$\alpha'^{(1)}_{-n} = \frac{1}{2\pi}\oint_{C_{ia}} dt(a^2+t^2)^{-n}\partial X(t) \tag{A.26}$$

We expand the integrand around $t = ia$ as follows

$$\begin{aligned} (a^2+t^2)^{-n} &= (t-ia)^{-n}(2ia+t-ia)^{-n} \\ &= \sum_{k\geq 0} {}^{-n}C_k\,(2ia)^{-(n+k)}(t-ia)^{k-n} \end{aligned} \tag{A.27}$$

Inserting this expansion into (A.26) and using the definition in (A.11) gives

$$\alpha'^{(1)}_{-n} = \sum_{k\geq 0} {}^{-n}C_k\,(2ia)^{-(n+k)}\tilde{\alpha}^{t\to ia}_{k-n} \tag{A.28}$$

We note that for copy 2 we simply take $a \to -a$ which gives the expansion

$$\alpha'^{(2)}_{-n} = \sum_{k \geq 0} {}^{-n}C_k (-2ia)^{-(n+k)} \tilde{\alpha}^{t \to -ia}_{k-n} \tag{A.29}$$

If there is no other operator which maps to the inside of the contour, the requirement to obtain a nonzero result imposed by acting $\alpha^{t \to ia}_{k-n}$ on the vacuum defined at $t = ia$,

$$\alpha^{t \to ia}_{k-n} |0\rangle \neq 0 \tag{A.30}$$

is that

$$k - n < 0 \quad \Longrightarrow \quad k < n \tag{A.31}$$

This gives

$$\alpha'^{(1)}_{-n} = \sum_{k=0}^{n-1} {}^{-n}C_k \, (2ia)^{-(n+k)} \tilde{\alpha}^{t \to ia}_{k-n} \tag{A.32}$$

and similarly for copy 2

$$\alpha'^{(2)}_{-n} = \sum_{k=0}^{n-1} {}^{-n}C_k \, (-2ia)^{-(n+k)} \tilde{\alpha}^{t \to -ia}_{k-n} \tag{A.33}$$

These modes obey the commutation relations (A.12). Next we expand the mode sitting at $t = ia$ to $t = \infty$. To do so we first write the mode on the r.h.s. of (A.32)

$$\tilde{\alpha}^{t \to ia}_{k-n} = \frac{1}{2\pi} \oint_{C_{ia}} dt (t - ia)^{k-n} \partial X(t) \tag{A.34}$$

Let's expand the integrand at $t = \infty$

$$\begin{aligned} (t - ia)^{k-n} &= t^{k-n} (1 - iat^{-1})^{k-n} \\ &= \sum_{k' \geq 0} {}^{k-n}C_{k'} (-ia)^{k'} t^{k-n-k'} \end{aligned} \tag{A.35}$$

Inserting (A.35) into (A.34) and using (A.13) yields

$$\tilde{\alpha}^{t \to ia}_{k-n} = \sum_{k' \geq 0} {}^{k-n}C_{k'} (-ia)^{k'} \tilde{\alpha}^{t \to \infty}_{k-n-k'} \tag{A.36}$$

Inserting this into (A.32) gives

$$\alpha'^{(1)}_{-n} = \sum_{k=0}^{n-1} \sum_{k' \geq 0} {}^{-n}C_k \, {}^{k-n}C_{k'} \, (2ia)^{-(n+k)} (-ia)^{k'} \tilde{\alpha}^{t \to \infty}_{k-n-k'} \tag{A.37}$$

Again, if there is no other operator which maps to the inside of the contour, to obtain a nonzero result we require that

$$\tilde{\alpha}^{t \to \infty}_{k-n-k'} |0\rangle_t \neq 0 \tag{A.38}$$

which implies that

$$q = k - n - k' < 0 \implies k' = k - n - q \tag{A.39}$$

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Since $k' \geq 0$ we find that

$$k - n - q \geq 0 \implies k \geq n + q \tag{A.40}$$

For the sum over k we have the following ranges

$$\begin{aligned} n+q \leq 0: &\qquad 0 \leq k \leq n-1 \\ n+q \geq 0: &\qquad n+q \leq k \leq n-1 \end{aligned} \tag{A.41}$$

The mode in (A.37) can thus be written as

$$\alpha'^{(1)}_{-n} = \sum_{q\leq -1} \left(\sum_{k=\max[0,n+q]}^{n-1} {}^{-n}C_k \, {}^{k-n}C_{k-n-q} (2ia)^{-(n+k)} (-ia)^{k-n-q} \right) \tilde{\alpha}_q^{t\to\infty} \tag{A.42}$$

Performing the sum in parenthesis gives

$$\alpha'^{(1)}_{-n} = \sum_{q\leq -1} \frac{(-1)^n i^{-q} a^{-2n-q} \Gamma(-\frac{q}{2})}{2\Gamma(n)\Gamma(1-(n+\frac{q}{2}))} \tilde{\alpha}_q^{t\to\infty} \tag{A.43}$$

Inserting (A.17) and (A.43) into (A.25) for copy 1 gives

$$\begin{aligned} f_1[-n,-p] &= \frac{1}{2p} \, {}_t\langle 0|\alpha'_p \alpha'^{(1)}_{-n}|0\rangle_t \\ &= \frac{1}{2p} \sum_{q\leq -1} \frac{(-1)^n i^{-q} a^{-2n-q} \Gamma(-\frac{q}{2})}{2\Gamma(n)\Gamma(1-(n+\frac{q}{2}))} \sum_{j\geq 0} {}^pC_j z_0^j [\tilde{\alpha}^{t\to\infty}_{2p-2j}, \tilde{\alpha}_q^{t\to\infty}] \\ &= \frac{1}{2p} \sum_{q\leq -1} \frac{(-1)^n i^{-q} a^{-2n-q} \Gamma(-\frac{q}{2})}{2\Gamma(n)\Gamma(1-(n+\frac{q}{2}))} \sum_{j\geq 0} {}^pC_j z_0^j (2p-2j) \delta_{2p-2j+q,0} \end{aligned} \tag{A.44}$$

The delta function constraint gives

$$\begin{aligned} 0 = 2p - 2j + q \quad &\implies \quad j = p + \frac{q}{2} \\ j \geq 0 \implies p + \frac{q}{2} \geq 0 &\implies q \geq -2p \end{aligned} \tag{A.45}$$

Making these substitutions give

$$f_1[-n,-p] = \frac{a^{2p-2n}}{2p} \sum_{q=-2p}^{-1} \frac{(-1)^n i^{-q} \Gamma(-\frac{q}{2})(-\frac{q}{2})}{\Gamma(n)\Gamma(1-(n+\frac{q}{2}))} \, {}^pC_{p+\frac{q}{2}} \tag{A.46}$$

Since the index j must be an integer (A.45) indicates that if p is an integer then q is required to be an even integer and if p is half integer then q is required to be an odd integer. We first consider the case where p is an integer and q is an even integer.

$$p = p', \qquad q = 2q', \qquad p', q' \in \mathbb{Z} \tag{A.47}$$

Our result becomes

$$f_1[-n,-p'] = \frac{a^{2p'-2n}}{2p'} \sum_{q'=-p'}^{-1} \frac{(-1)^n i^{-2q'} \Gamma(-q'+1)}{\Gamma(n)\Gamma(1-(n+q'))} \, {}^{p'}C_{p'+q'} = \frac{1}{2}\delta_{n,p'} \tag{A.48}$$

Now consider when p is half integer which also requires q to be odd

$$p = p' - \frac{1}{2}, \qquad q = 2q' + 1 \tag{A.49}$$

Inserting this into the above and simplifying gives

$$\begin{aligned} f_1\left[-n, -p' + \frac{1}{2}\right] &= -ia^{2p'-2n-1} \sum_{q'=-p'}^{-1} \frac{(-1)^{n+q'}\Gamma(-\frac{1}{2}+p')}{2\Gamma(n)\Gamma(\frac{1}{2}-(n+q'))\Gamma(1+q'+p')} \\ &= \frac{ia^{2p'-2n-1}\Gamma(\frac{1}{2}+n)\Gamma(-\frac{1}{2}+p')}{\pi(-1-2n+2p')\Gamma(n)\Gamma(p')} \end{aligned} \tag{A.50}$$

which is

$$f_1[-n, -p] = \frac{ia^{2p-2n}\Gamma(\frac{1}{2}+n)\Gamma(p)}{\pi(2p-2n)\Gamma(n)\Gamma(p+\frac{1}{2})} \tag{A.51}$$

For copy 2 we simply take $a \to -a$. For integer p the result is the same as (A.48)

$$f_2[-n, -p] = \frac{1}{2}\delta_{n,p} \tag{A.52}$$

for half integer p the result changes by a minus sign

$$f_2[-n, -p] = -\frac{ia^{2p-2n}\Gamma(\frac{1}{2}+n)\Gamma(p)}{\pi(2p-2n)\Gamma(n)\Gamma(p+\frac{1}{2})} \tag{A.53}$$

A.5 Contraction

Here we compute the contraction terms C^{ij} using the covering map. We start with two modes on copy 1 in the initial state and compute the following amplitude

$$C^{11}[n_1, n_2] = \langle 0^2|\sigma_2(z_0)\alpha^{(1)}_{-n_1}\alpha^{(1)}_{-n_2}|0\rangle^{(1)}|0\rangle^{(2)} \tag{A.54}$$

Mapping to the t-plane using (A.5) gives

$$C^{11}[n_1, n_2] = {}_t\langle 0|\alpha'^{(1)}_{-n_1}\alpha'^{(1)}_{-n_2}|0\rangle_t \tag{A.55}$$

where the primed modes are defined in (A.32) and (A.33). The contour of the mode labeled by n_1 is mapped to the outside of the contour of the mode labeled by n_2. To obtain a contraction, the inside contour should give negative modes and the outside contour should give positive modes. Our expression becomes

$$C^{11}[n_1, n_2] = \sum_{j>n_1}\sum_{k=0}^{n_2-1} {}^{-n_1}C_j \, {}^{-n_2}C_k (2ia)^{-(n_1+j)}(2ia)^{-(n_2+k)} {}_t\langle 0|\tilde{\alpha}^{t\to ia}_{j-n_1}\,\tilde{\alpha}^{t\to ia}_{k-n_2}|0\rangle_t \tag{A.56}$$

Using commutation relations (A.12) we find that

$$C^{11}[n_1, n_2] = -\sum_{k=0}^{n_2-1} {}^{-n_1}C_{n_1+n_2-k} \, {}^{-n_2}C_k (2ia)^{-(2n_1+n_2-k)}(2ia)^{-(n_2+k)}(k-n_2) \tag{A.57}$$

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Performing the sum gives

$$C^{11}[n_1,n_2] = \frac{a^{-2(n_1+n_2)}}{2(n_1+n_2)\pi}\frac{\Gamma(\frac{1}{2}+n_1)\Gamma(\frac{1}{2}+n_2)}{\Gamma(n_1)\Gamma(n_2)} \tag{A.58}$$

Notice that this expression is symmetric between n_1 and n_2. By switching $a \to -a$, we find

$$C^{22}[n_1,n_2] = C^{11}[n_1,n_2] \tag{A.59}$$

Let's compute the contraction C^{12}. We start with

$$\begin{aligned} C^{12}[n_1,n_2] =& {}_t\langle 0|\alpha'^{(1)}_{-n_1}\alpha'^{(2)}_{-n_2}|0\rangle_t \\ =& \sum_{j=0}^{n_1-1}\sum_{k=0}^{n_2-1} {}^{-n_1}C_j {}^{-n_2}C_k (2ia)^{-(n_1+j)}(2ia)^{-(n_2+k)} {}_t\langle 0|\tilde{\alpha}^{t\to ia}_{j-n_1}\tilde{\alpha}^{t\to -ia}_{k-n_2}|0\rangle_t \end{aligned} \tag{A.60}$$

On the t-plane, the contour of copy 1 is located at $t=ia$ and the contour of copy 2 is located at $t=-ia$. Both contours should be left with negative modes. To compute the expectation value, we expand the contour of the mode located at $t=ia$ around the point $t=-ia$. To do this we start with a mode defined at $t=ia$

$$\tilde{\alpha}^{t\to ia}_m = \frac{1}{2\pi}\oint_{C_{ia}} dt(t-ia)^m\partial X(t) \tag{A.61}$$

We expand the integrand in the following way

$$(t-ia)^m = (-2ia+t+ia)^m = \sum_{p\geq 0} {}^mC_p(-2ia)^{m-p}(t+ia)^p \tag{A.62}$$

Inserting this into (A.61) gives

$$\tilde{\alpha}^{t\to ia}_m = (-1)\sum_{p\geq 0} {}^mC_p(-2ia)^{m-p}\tilde{\alpha}^{t\to -ia}_p \tag{A.63}$$

where we have used the modes defined around $t=-ia$ in (A.11). Using commutation relations (A.12), the expectation value of a mode at $t=ia$ and a mode at $t=-ia$ becomes

$${}_t\langle 0|\tilde{\alpha}^{t\to ia}_m\,\tilde{\alpha}^{t\to -ia}_n|0\rangle_t = (-1)\sum_{p\geq 0} {}^mC_p(-2ia)^{m-p}p\delta_{p+n,0} = {}^mC_{-n}(-2ia)^{m+n}n \tag{A.64}$$

Using this expression in (A.60) gives

$$C^{12}[n_1,n_2] = (2a)^{-2(n_1+n_2)}\sum_{j=0}^{n_1-1}\sum_{k=0}^{n_2-1} {}^{-n_1}C_j {}^{-n_2}C_k {}^{j-n_1}C_{n_2-k}(-1)^{j+k}(k-n_2) \tag{A.65}$$

Performing the sums give

$$C^{12}[n_1,n_2] = -\frac{a^{-2(n_1+n_2)}}{2(n_1+n_2)\pi}\frac{\Gamma(\frac{1}{2}+n_1)\Gamma(\frac{1}{2}+n_2)}{\Gamma(n_1)\Gamma(n_2)} \tag{A.66}$$

Notice that this expression is symmetric between n_1 and n_2. By switching $a\to -a$, we find

$$C^{12}[n_1,n_2] = C^{21}[n_1,n_2] \tag{A.67}$$

Therefore, all the contractions can be written as

$$C^{ij}[n_1,n_2] = (-1)^{i+j}\frac{a^{-2(n_1+n_2)}}{2(n_1+n_2)\pi}\frac{\Gamma(\frac{1}{2}+n_1)\Gamma(\frac{1}{2}+n_2)}{\Gamma(n_1)\Gamma(n_2)} \tag{A.68}$$

where $i,j=1,2$ are copy labels.

B Propagation: higher modes

In this appendix we will derive the propagation $f_i[-n,-p]$ with $n>1$ from $f_i[-1,-p]$ by applying L_{-1} repeatedly. For copy 1 we begin with following expression

$$\sigma_2(z_0)\alpha^{(1)}_{-n}|0\rangle^{(1)}|0\rangle^{(2)} = \frac{1}{\Gamma(n)}\sigma_2(z_0)(L_{-1})^{n-1}\alpha^{(1)}_{-1}|0\rangle^{(1)}|0\rangle^{(2)} \tag{B.1}$$

Using relation (5.9) we obtain

$$\sigma_2(z_0)\alpha^{(1)}_{-n}|0\rangle^{(1)}|0\rangle^{(2)} = \frac{1}{\Gamma(n)}(L_{-1}-\partial)^{n-1}\sigma_2(z_0)\alpha^{(1)}_{-1}|0\rangle^{(1)}|0\rangle^{(2)} \tag{B.2}$$

Using (3.9) we pass the bosonic mode through the twist on both sides and only keep the terms with a single mode. This gives

$$\sum_{p>0} f_1[-n,-p]\alpha_{-p}|0^2\rangle = \frac{1}{\Gamma(n)}(L_{-1}-\partial)^{n-1}\sum_{p'>0} f_1[-1,-p']\alpha_{-p'}|0^2\rangle \tag{B.3}$$

Let's compute the r.h.s. We look at the term

$$(L_{-1}-\partial)^{n-1}f_1[-1,-p']\alpha_{-p'} = \sum_{k\geq 0}(L_{-1})^k\alpha_{-p'}\,{}^{n-1}C_k(-1)^{n-k-1}\partial^{n-k-1}f_1[-1,-p'] \tag{B.4}$$

We only keep terms where L_{-1} acts on $\alpha_{-p'}$ since this leaves us with just one mode. Only keeping terms which will leave us with one mode and using the expression in (5.42) and (5.44) we have

$$\begin{aligned}
&(L_{-1}-\partial)^{n-1}f_1[-1,-p']\alpha_{-p'}\\
&= f_1[-1,-p']_{z_0=1}\sum_{k=0}^{n-1}(L_{-1})^k\alpha_{-p'}\,{}^{n-1}C_k(-1)^{n-k-1}\partial^{n-k-1}z_0^{p'-1}\\
&= f_1[-1,-p']_{z_0=1}\sum_{k=0}^{n-1}\frac{\Gamma(p'+k)}{\Gamma(p')}\,{}^{n-1}C_k(-1)^{n-k-1}\frac{\Gamma(p')}{\Gamma(p'-(n-k-1))}z_0^{p'+k-n}\alpha_{-(p'+k)}+\dots\\
&= \sum_{k=0}^{n-1}f_1[-1,-p']_{z_0=1}\frac{\Gamma(p'+k)}{\Gamma(p'-(n-k-1))}\,{}^{n-1}C_k(-1)^{n-k-1}z_0^{p'+k-n}\alpha_{-(p'+k)}+\dots
\end{aligned} \tag{B.5}$$

Inserting this expression back into (B.3) we find

$$\begin{aligned}
\sum_{p>0} f_1[-n,-p]\alpha_{-p}|0^2\rangle &= \frac{1}{\Gamma(n)}\sum_{k=0}^{n-1}\sum_{p'>0}f_1[-1,-p']_{z_0=1}\frac{\Gamma(p'+k)}{\Gamma(p'-(n-k-1))}\,{}^{n-1}C_k(-1)^{n-k-1}\\
&\quad\times z_0^{p'+k-n}\alpha_{-(p'+k)}|0^2\rangle+\dots
\end{aligned} \tag{B.6}$$

In order to compare the terms on the left and right hand sides we take $p'+k=p$ which allows us to compare the following terms

$$\begin{aligned}
f_1[-n,-p] &= Cz_0^{p-n}\frac{\Gamma(p)}{\Gamma(n)\Gamma(p-(n-1))}\sum_{k=0}^{n-1}\frac{\Gamma(p-k-1)}{\Gamma(p-k+1/2)}\,{}^{n-1}C_k(-1)^{n-k-1}\\
&= Cz_0^{p-n}2(-1)^n\sqrt{\pi}\frac{\csc(\pi p)\Gamma(p)\Gamma(\frac{1}{2}+n)}{\Gamma(n)\Gamma(1-n+p)\Gamma(1+n-p)\Gamma(\frac{1}{2}+p)}
\end{aligned} \tag{B.7}$$

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Since p is a half integer, using the expression for C, (5.61), we obtain

$$f_1[-n,-p] = \frac{i z_0^{p-n}\Gamma(p)\Gamma(\frac{1}{2}+n)}{\pi(2p-2n)\Gamma(p+\frac{1}{2})\Gamma(n)} \tag{B.8}$$

where n is an integer and p is a half integer. To obtain $f_2[-n,-p]$ for an excitation on copy 2, we change the location of the twist by $z_0 \to z_0 e^{2\pi i}$. It interchanges copy 1 and copy 2. Since $p-n$ is a half integer, we have $z_0^{p-1} \to -z_0^{p-1}$, which gives

$$f_2[-n,-p] = -f_1[-n,-p] \tag{B.9}$$

Notice that if we take $z_0 = a^2$ this result agrees with the result from the covering map method (A.51) and (A.53).

C Contraction: higher modes

Here we derive $C^{ij}[n_1,n_2]$. We do this by looking at the following relation

$$\sigma_2(z_0)\alpha^{(i)}_{-n_1}\alpha^{(j)}_{-n_2}|0\rangle^{(1)}|0\rangle^{(2)} = \frac{1}{\Gamma(n_2)}\sigma_2(z_0)\alpha^{(i)}_{-n_1}(L_{-1})^{n_2-1}\alpha^{(j)}_{-1}|0\rangle^{(1)}|0\rangle^{(2)} \tag{C.1}$$

To compute the r.h.s., notice that

$$\alpha^{(i)}_{-n_1}L_{-1} = (L_{-1} - L_{-1}\circ)\alpha^{(i)}_{-n_1} \tag{C.2}$$

where

$$L_{-1}\circ O_{-n} = [L_{-1}, O_{-n}] \tag{C.3}$$

Thus, we have

$$\begin{aligned}\alpha^{(i)}_{-n_1}L_{-1}^{n_2-1} &= (L_{-1}-L_{-1}\circ)^{n_2-1}\alpha^{(i)}_{-n_1}\\ &= \sum_{k=0}^{n_2-1} {}^{n_2-1}C_k(-1)^{n_2-1-k}(L_{-1})^k(L_{-1}\circ)^{n_2-1-k}\alpha^{(i)}_{-n_1}\end{aligned} \tag{C.4}$$

where we have used the fact that $L_{-1}\circ L_{-1}=0$. Therefore, we obtain

$$\begin{aligned}\sigma_2(z_0)\alpha^{(i)}_{-n_1}\alpha^{(j)}_{-n_2}|0\rangle^{(1)}|0\rangle^{(2)} &= \frac{1}{\Gamma(n_2)}\sum_{k=0}^{n_2-1} {}^{n_2-1}C_k\frac{\Gamma(n_1+n_2-1-k)}{\Gamma(n_1)}(-1)^{n_2-1-k}\\ &\quad\times\sigma_2(z_0)(L_{-1})^k\alpha^{(i)}_{-(n_1+n_2-1-k)}\alpha^{(j)}_{-1}|0\rangle^{(1)}|0\rangle^{(2)}\end{aligned} \tag{C.5}$$

Using the relation (5.9) we have

$$\begin{aligned}\sigma_2(z_0)\alpha^{(i)}_{-n_1}\alpha^{(j)}_{-n_2}|0\rangle^{(1)}|0\rangle^{(2)} &= \frac{1}{\Gamma(n_1)\Gamma(n_2)}\sum_{k=0}^{n_2-1} {}^{n_2-1}C_k\Gamma(n_1+n_2-1-k)(-1)^{n_2-1-k}\\ &\quad\times(L_{-1}-\partial)^k\sigma_2(z_0)\alpha^{(i)}_{-(n_1+n_2-1-k)}\alpha^{(j)}_{-1}|0\rangle^{(1)}|0\rangle^{(2)}\end{aligned} \tag{C.6}$$

Keeping the terms which contain no modes on both the l.h.s. and r.h.s. we find

$$\begin{aligned} C^{ij}[n_1, n_2] = &\frac{1}{\Gamma(n_1)\Gamma(n_2)} \sum_{k=0}^{n_2-1} {}^{n_2-1}C_k \Gamma(n_1 + n_2 - 1 - k)(-1)^{n_2-1-k} \\ &\times (-\partial)^k C^{ij}[n_1 + n_2 - 1 - k, 1] \end{aligned} \tag{C.7}$$

Notice that $C^{ij}[1, n] = C^{ji}[n, 1]$ since the contraction is between two bosonic modes whose order can be changed. Using the expression in (5.63) we find that (C.7) becomes

$$C^{ij}[n_1, n_2] = \frac{(-1)^{i+j}(-1)^{n_2-1}}{4\sqrt{\pi}\Gamma(n_1)\Gamma(n_2)} \sum_{k=0}^{n_2-1} {}^{n_2-1}C_k \frac{\Gamma(-\frac{1}{2} + n_1 + n_2 - k)}{(n_1 + n_2 - k)} \partial^k z_0^{-(n_1+n_2-k)} \tag{C.8}$$

which gives

$$C^{ij}[n_1, n_2] = (-1)^{i+j} \frac{z_0^{-(n_1+n_2)}}{2(n_1 + n_2)\pi} \frac{\Gamma(\frac{1}{2} + n_1)\Gamma(\frac{1}{2} + n_2)}{\Gamma(n_1)\Gamma(n_2)} \tag{C.9}$$

where $i, j = 1, 2$ are copy labels.

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