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Higgs boson production and quark scattering amplitudes at high energy through the next-to-next-to-leading power in quark mass

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Abstract: We study the amplitudes of the quark scattering by an external electromagnetic field and of the light quark mediated Higgs boson production via gluon fusion in the high-energy limit. The asymptotic behavior of the quark form factors is obtained in the double-logarithmic approximation to all orders in strong coupling constant through $\mathcal{O}(m_q^3)$ in the small quark mass expansion and the asymptotic formula is given in a closed analytic form. In the case of the two-gluon Higgs boson form factor we obtain a complete analytic result for the three-loop $\mathcal{O}(m_q^3)$ double-logarithmic term while the all-order analysis is performed in the large-N_c limit of QCD and for the abelian gauge group. An estimate of the high-order high-power light quark mass effect in the Higgs boson production and decay is given.

Keywords: Higgs Physics, Perturbative QCD, Resummation

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1 Introduction

Quantum corrections are known to significantly alter the high-energy properties of the gauge theory scattering amplitudes. The asymptotic behavior of the amplitudes which are not suppressed by the ratio of a characteristic infrared scale to the process energy is governed by the "Sudakov" radiative corrections enhanced by the second power of the large logarithm of the scale ratio per each power of the coupling constant. Sudakov logarithms exponentiate and result in a strong universal suppression of the scattering amplitudes in the limit when all the kinematic invariants of the process are large [1–9]. The structure of the power suppressed logarithmically enhanced contributions is by far more complex and the corresponding renormalization group analysis poses a serious challenge to the modern effective field theory. One of the important problems in this category is the analysis of the scattering amplitudes involving massive particle in the limit of small mass or high energy. The mass effects on the leading-power contributions have been extensively studied in the context of the high-order electroweak and QED radiative corrections [10–20]. The next-to-leading power contributions for a number of key processes in QED and QCD have been analysed in the leading (double) [21–30] and the next-to-leading logarithmic approximation [31–33].[1]

In the processes with massive fermions already at the next-to-leading power the origin of the logarithmic corrections and the asymptotic behavior of the amplitudes drastically differ from the leading-power Sudakov case. The double-logarithmic terms in this case are related to the effect of the eikonal (color) charge nonconservation in the process with soft fermion exchange and result in asymptotic exponential enhancement for a wide class of

[1] The next-to-leading power logarithmic contributions corrections have also been recently discussed in many different incarnations [34–47].

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amplitudes and in a breakdown of a formal power counting [23, 27, 28]. Thus, it is of a primary theoretical interest to get insight into the asymptotic behavior of the next-to-next-to-leading power contributions and determine whether any qualitatively new phenomenon appears in this order. The renormalization group analysis has not yet been extended beyond the next-to-leading power for any kind of power corrections to the high-energy processes. In this paper we present for the first time such an analysis of the simplest but fundamental and phenomenologically important amplitudes of the quark scattering in an external electromagnetic field and of the light quark mediated Higgs boson production in gluon fusion. The results of the analysis are used to get a quantitative estimate of the accuracy of the fixed-order calculations [48, 49] and the calculations based on the small-mass expansion [50, 51] of the light quark contribution to the Higgs boson production and decays.

The paper is organized as follows. In the next section we discuss the scattering of a massive quark by an external electromagnetic field in the limit of large momentum transfer, recall the main features of the double-logarithmic result for the next-to-leading power contribution and extend the analysis to the $\mathcal{O}(m_q^3)$ amplitude. In section 3 we discuss the amplitude of the Higgs boson production at $\mathcal{O}(m_q^3)$, derive the analytic result for the three-loop double-logarithmic term and extend it to all orders in the large-N_c limit and in the case of the abelian gauge group. Section 4 is our summary.

2 Quark scattering by electromagnetic field

The amplitude $\mathcal{F}$ of a quark scattering in an external field can be parameterized in the standard way by the Dirac and Pauli form factors

$$\mathcal{F} = e_q\bar{\psi}(p_1)\left(\gamma_\mu F_1 + \frac{i\sigma_{\mu\nu}q^\nu}{2m_q}F_2\right)\psi(p_2)\,, \tag{2.1}$$

where e_q is the quark charge. For on-shell quark $p_1^2 = p_2^2 = m_q^2$ and the large Euclidean momentum transfer $Q^2 = -(p_2-p_1)^2$ when the ratio $\rho \equiv m_q^2/Q^2$ is positive and small the form factors can be expanded in an asymptotic series

$$F_i = Z_q^2\sum_{n=0}^{\infty}\rho^n F_i^{(n)}\,, \tag{2.2}$$

where the universal Sudakov factor for the external on-shell quark lines which incorporates all the infrared divergencies of the amplitude. In dimensional regularization with $d = 4-2\varepsilon$ in the double-logarithmic approximation it reads

$$Z_q^2 = \exp\left[-C_F\left(\frac{\alpha_s}{2\pi}\frac{\ln\rho}{\varepsilon} + x\right)\right], \tag{2.3}$$

where $x = \frac{\alpha_s}{4\pi}\ln^2\rho$ is the double-logarithmic variable and $C_F = (N_c^2-1)/(2N_c)$ for the $\mathrm{SU}(N_c)$ color group. The infrared finite coefficients $F_i^{(n)}$ in a given order of perturbation theory depends on ρ only logarithmically, and in the double-logarithmic approximation are functions of x. Due to factorization of Sudakov logarithms into Z_q^2 these coefficients include only non-Sudakov double logarithms and the leading-power Dirac form factor with

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the logarithmic accuracy is just $F_1^{(0)} = 1$. At the same time the Pauli form factor describe the scattering with a flip of the quark chirality and therefore has to vanish in the high-energy or small-mass limit i.e. $F_2^{(0)} = 0$.

The next-to-leading power double-logarithmic contribution to the Dirac form factor is generated by the soft quark pair exchange and starts with two loops. The corresponding coefficient reads [27, 28]

$$F_1^{(1)} = \frac{C_F(C_A - 2C_F)}{6} x^2 f(-z) \,, \tag{2.4}$$

where $z = (C_A - C_F)x$, $C_A = N_c$ and the function f has the following integral representation

$$f(z) = 12 \int_0^1 \mathrm{d}\eta_1 \int_{\eta_1}^1 \mathrm{d}\eta_2 \int_0^{1-\eta_2} \mathrm{d}\xi_2 \int_{\xi_2}^{1-\eta_1} \mathrm{d}\xi_1 \, e^{2z\eta_1(\xi_1-\xi_2)} e^{2z\xi_2(\eta_2-\eta_1)} \,. \tag{2.5}$$

Due to the $1/m_q$ factor in the definition of F_2 the coefficient $F_2^{(1)}$ corresponds to the $\mathcal{O}(m_q)$ scattering amplitude. However, as we will see in the double-logarithmic approximation $F_2^{(1)} = 0$ and the Pauli form factor starts to contribute at $\mathcal{O}(m_q^3)$, i.e. at the next-to-next-to-leading power in small-mass expansion.

The leading-order contribution to the Pauli form factor is given by the one-loop vertex diagram and can be written as follows

$$[F_2]_{1-\mathrm{loop}} = \frac{C_F \alpha_s}{\pi} \frac{1}{1+4\rho} I_1 \,, \tag{2.6}$$

where the scalar integral over the virtual gluon momentum

$$I_1 = -i \int \frac{\mathrm{d}^4 l}{\pi^2} \frac{(p_1 l) + (p_2 l)}{l^2 \left((p_1 - l)^2 - m_q^2\right) \left((p_2 - l)^2 - m_q^2\right)} \tag{2.7}$$

corresponds to a single insertion of the loop momentum in the numerator of a quark propagator, figure 1(a). At the same time the terms without the loop momentum do not provide the relevant Lorentz structure. The logarithmically enhanced corrections to the on-shell (or almost on-shell) amplitudes in the high-energy limit are universally associated with the emission of the virtual particles which are soft and/or collinear to the large external momenta. For the one-loop Pauli form factor the virtual gluon momentum in the numerator cancels one of the eikonal propagators and makes the integrand not sufficiently singular to develop the double-logarithmic contribution in the leading order in ρ. Hence the integral generates only a single soft logarithm

$$I_1 = -\ln \rho + \ldots \,, \tag{2.8}$$

where the ellipsis stands for the power-suppressed terms. Since the higher-order Sudakov corrections factor out we get $F_2^{(1)} = 0$ in the double-logarithmic approximation and

$$F_2^{(1)} = -\frac{C_F \alpha_s}{\pi} \ln \rho \tag{2.9}$$

in the next-to-leading logarithmic approximation to all orders of perturbative expansion. The absence of the double-logarithmic contribution in the leading-power Pauli form factor

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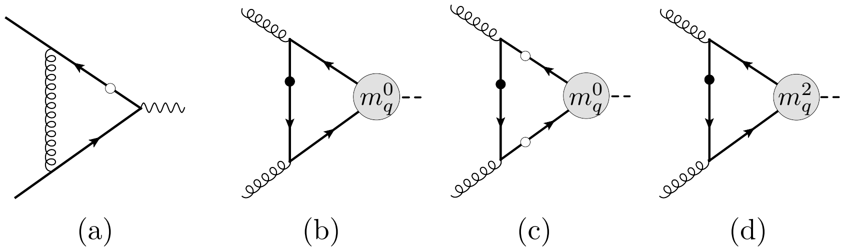

Figure 1. (a) the leading-order one-loop Feynman diagram for the $\mathcal{O}(m_q)$ logarithmic contribution to the quark Pauli form factor. The Feynman diagrams representing (b) $\mathcal{O}(m_q)$ and (c), (d) $\mathcal{O}(m_q^3)$ double-logarithmic contributions to the ggH amplitude. The black (empty) circle represents the mass (loop momentum) insertion. The gray blobs correspond to the double-logarithmic off-shell scalar form factor at $\mathcal{O}(m_q^0)$ and $\mathcal{O}(m_q^2)$. Symmetric diagrams and the diagrams with the opposite direction of the closed quark line are not shown.

can be easily understood. Indeed, in a physical gauge the collinear logarithms are generated by the self-energy corrections to the on-shell external lines [2] while a new Lorentz structure in the quark coupling to the external electromagnetic field can only result from a vertex correction. We, however are interested in the $\mathcal{O}(\rho^2)$ double logarithms contributing to $F_2^{(2)}$. The analysis of such terms is more complicated since the formal expansion of the integrand in eq. (2.7) results in more singular integrals, which may have double-logarithmic scaling. A systematic way to study the mass-suppressed double logarithms has been suggested in [23] and discussed in detail in [25]. It is based on the expansion by regions approach [52–54] which gives the coefficients of the small-mass expansion in terms of the singular homogeneous integrals. The coefficient of the double-logarithmic term can be read of the highest singularity of these integrals. This singularity in turn can be obtained by the classical method of Sudakov [1] which on its own is blind to the power corrections. As it has been shown in [23] the exchange by *massless* soft gauge boson does not produce double logarithms in the first order in ρ. It can be directly checked for the one-loop integral which gives

$$\left[F_2^{(2)}\right]_{1-\text{loop}} = \mathcal{O}(\ln\rho)\,, \tag{2.10}$$

and the result extends to an arbitrary number of the soft gluon exchanges (see [25] for detailed discussion). We would like to emphasize that the above statement is valid only for the massless soft particles. A presence of a mass in the soft propagator does lead to the double-logarithmic corrections at $\mathcal{O}(\rho)$ as we will see in the next section.

Thus the soft gluon exchanges do not contribute to $F_2^{(2)}$ in the double-logarithmic approximation. At the same time as in the case of the Dirac form factor at $\mathcal{O}(m_q^2)$ [23] starting with two loops the double-logarithmic contribution to this coefficient is generated through the soft virtual quark pair exchange, figure 2(a), where the mass suppression factor comes from the numerators of the soft propagators, i.e. is associated with the chirality flip. This makes the soft quark propagators sufficiently singular to produce the double-logarithmic contribution. In figure 2(a) the large external momenta flow through the edges of the diagram which for the soft loop momenta form the eikonal lines. Thus the corresponding momentum configuration is the opposite of the standard Sudakov case with

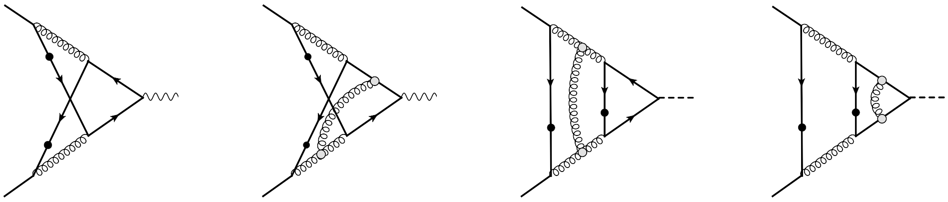

Figure 2. (a) the leading two-loop Feynman diagram for the $\mathcal{O}(m_q^3)$ double-logarithmic correction to the quark Pauli form factor. The diagrams with an effective soft gluon exchange which incorporate the non-Sudakov double-logarithmic corrections (b) to the $\mathcal{O}(m_q^3)$ Pauli form factor and (c), (d) to the $\mathcal{O}(m_q^2)$ scalar form factor of a quark. The effective vertices (gray circles) are defined in the text.

soft gauge bosons and eikonal fermions. The Pauli form factor structure requires an additional chirality flip on an eikonal quark line, which is provided either by the mass term of the propagator or by an external momentum $\not{p}_i$ acting on the corresponding on-shell quark field. Such terms cancel in the diagrams with the soft gluon exchange but give a contribution which is not suppressed by the soft momentum in the diagram figure 2(a), where due to the topology of the fermion flow the operators $\not{p}_i$ should be commuted with the photon vertex before the equation of motion for the initial and final quark states can be applied. Let us consider the origin of the relevant Lorentz structure in more detail. Due to chirality conservation the eikonal gauge bosons must have transversal polarization. By using the property $\gamma_i\gamma_j\gamma_i\gamma_j = 0$ of the Dirac matrices in two-dimensional transversal space we find that after neglecting the virtual momenta the entire contribution to the Pauli form factor is generated by the photon vertex $\not{p}_2\gamma_\mu\not{p}_1$ part of the Dirac chain where one of the external momenta is converted into m_q by the equation of motion. The evaluation of the corresponding two-loop double-logarithmic integral is discussed in detail in [23, 25] and gives

$$\left[F_2^{(2)}\right]_{2-\text{loop}} = \frac{2}{3}C_F(C_A - 2C_F)x^2 + \dots\,, \tag{2.11}$$

where the ellipsis stands for the subleading logarithms, which agrees with the expansion of the exact result [55]. The higher-order double-logarithmic corrections are generated by the multiple exchanges of the leading-power soft gluons with light-cone polarization.[2] The emission of a soft gluon with polarization α off the eikonal quark is trivial in the spinor space due to the identity

$$\left(\not{p}_i + m_q\right)\gamma^\alpha\left(\not{p}_i + m_q\right) = 2p_i^\alpha\left(\not{p}_i + m_q\right). \tag{2.12}$$

Thus the higher-order double-logarithmic corrections do not affect the spinor part of the amplitude and the Lorentz structure relevant for the Pauli form factor is generated by the same vertex part of the Dirac chain as in two loops. As a consequence they are identical to the double-logarithmic corrections to the coefficient $F_1^{(1)}$ stemming from the same diagram figure 2(a). After factoring out the Sudakov corrections to the external quark lines, the

[2] A covariant gauge is implied.

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remaining non-Sudakov double logarithms are described by the diagram figure 2(b) [27, 28]. The effective soft gluon exchange in this diagram exponentiate and its coupling is obtained from the standard QCD expression by replacing the quadratic Casimir operator C_A of the adjoint representation (gluon color charge) with the difference $C_A - C_F$ which reflects the nonconservation of the color charge along the eikonal lines in the process with soft quark exchange, which is the physical origin of the non-Sudakov corrections.[3] The result for the $F_2^{(2)}$ then reads

$$F_2^{(2)} = \frac{2}{3} C_F (C_A - 2C_F) x^2 f(-z) \tag{2.13}$$

with the function $f(z)$ given by eq. (2.5). The Taylor expansion of this function

$$f(z) = 1 + \frac{z}{5} + \frac{11}{420} z^2 + \frac{z^3}{378} + \dots \tag{2.14}$$

gives the loop-by-loop double-logarithmic approximation of the Pauli form factor. The three-loop pure gluon contribution to the massive quark form factors is known so far only in the large-N_c planar approximation [56]. The nonplanar diagram figure 2(b), however, has a subleading color factor $C_A - 2C_F$. Thus the leading logarithmic power-suppressed contribution vanishes in the large-N_c limit through $\mathcal{O}(m_q^3)$ and cannot be explicitly verified against this result. At the same time the leading-color analysis [56] confirms the absence of the three-loop mass-suppressed leading logarithms in agreement with eq. (2.4).

In the limit of large momentum transfer the asymptotic behavior of the form factor crucially depends on the gauge group. The variable z is negative in QED and positive in QCD. The relevant asymptotic expressions at $z \to \infty$ are respectively

$$f(z) \sim 6 \left[\ln\left(\frac{z}{2}\right) + \gamma_E \right] \left(\frac{2\pi e^z}{z^5} \right)^{1/2} \tag{2.15}$$

and

$$f(-z) \sim \left[(\ln(2z) + \gamma_E)^2 - \frac{\pi^2}{2} \right] \frac{3}{z^2}, \tag{2.16}$$

where $\gamma_E = 0.577215\dots$ is the Euler constant. The details of the derivation of the above asymptotic formulae are given in appendix A.[4] Thus in QED the Pauli form factor at $\mathcal{O}(m_q^3)$ has the leading asymptotic behavior given by the exponential factor $e^{x/2}$. At the same time in QCD it scales with the double-logarithmic variable as $\ln^2 x$.

3 Higgs boson production in gluon fusion

A quark loop mediated ggH amplitude can be written as follows

$$\mathcal{M}_{ggH}^q = T_F \frac{\alpha_s}{\pi} \frac{y_q m_q}{m_H^2} \left(p_1^\mu p_2^\nu - g^{\mu\nu}(p_1 p_2) \right) A_\nu^a(p_1) A_\mu^a(p_2) H M_{ggH}^q, \tag{3.1}$$

[3]The sensitivity to the eikonal color charge variation is characteristic to the soft emission. The single logarithms resulting from the collinear emission do not have this property [31].

[4]In [27] the coefficients of the leading powers of logarithms in eqs. (2.15, 2.16) have been estimated numerically with a rather low accuracy of the fit.

where y_q is the quark Yukawa coupling, m_H is the Higgs boson mass, $p_i^2 = 0$, $(p_1p_2) = -m_H^2/2$, the gauge condition $\partial^\mu A_\mu^a = 0$ is implied and one can choose the transversal polarization of the gluon fields. In the heavy quark limit $m_q \gg m_H$ the scalar amplitude approaches the value $M^q_{\gamma\gamma H} = -2/(3\rho)$, where now $\rho = m_q^2/m_H^2$ is a Minkowskian parameter. In the opposite limit of light quark $m_q \ll m_H$ it can be expanded in an asymptotic series

$$M^q_{ggH} = Z_g^2 \sum_{n=0}^{\infty} \rho^n M^{(n)}_{ggH}\,, \tag{3.2}$$

where the coefficients $M^{(n)}_{ggH}$ are finite and

$$Z_g^2 = \exp\left[-\frac{C_A s^{-\varepsilon}}{\varepsilon^2}\frac{\alpha_s}{2\pi}\right] \tag{3.3}$$

with $s = m_H^2$ is the universal Sudakov factor for the external on-shell gluon lines which incorporates all the infrared divergencies of the amplitude. Note that as in the case of Pauli form factor the amplitude is loop generated and in the high-energy (small-mass) limit is suppressed by the quark mass due to chirality flip at the Higgs boson vertex.

The leading-order one-loop scalar amplitude reduces to

$$\left[M^q_{ggH}\right]_{1-\mathrm{loop}} = 2J_1 + 8J_2\,, \tag{3.4}$$

where

$$J_1 = i\int \frac{\mathrm{d}^4 l}{\pi^2}\frac{2(p_1p_2)}{(l^2-m_q^2)\left((p_1-l)^2-m_q^2\right)\left((p_2-l)^2-m_q^2\right)} \tag{3.5}$$

and

$$J_2 = i\int \frac{\mathrm{d}^4 l}{\pi^2}\frac{l^2-4(lp_1)(lp_2)/(p_1p_2)}{(l^2-m_q^2)\left((p_1-l)^2-m_q^2\right)\left((p_2-l)^2-m_q^2\right)} \tag{3.6}$$

are the scalar integrals corresponding to no and double insertion of the loop (soft quark) momentum $\not{l}$ in the numerators of the quark propagators, respectively (cf. the diagrams in figure 1(b) and figure 1(c) with the leading-order Higgs boson vertex). The integral J_1 is responsible for the leading double-logarithmic contribution to the $\mathcal{O}(m_q)$ coefficient $M^{(0)}_{ggH}$. For the further analysis it is instructive to recall the evaluation of this contribution (see e.g. [28]). With the double-logarithmic accuracy the propagators of the soft and eikonal quarks can be approximated as follows

$$\frac{1}{l^2-m_q^2} \approx -i\pi\delta(l^2-m_q^2)\,, \tag{3.7}$$

$$\frac{1}{(p_i-l)^2-m_q^2} \approx -\frac{1}{2(p_il)}\,. \tag{3.8}$$

Then the standard Sudakov parametrization of the soft quark momentum $l = up_1+vp_2+l_\perp$ is introduced, where the first two terms correspond to the light-cone components and the last term corresponds to the transversal component in the plane orthogonal to the gluon momenta. For $|u|,|v| \gtrsim 1$ the eikonal approximation in eq. (3.8) breaks down and the quadratic dependence of the quark propagators on the virtual momentum is restored cutting off the logarithmic integral. Thus the logarithmic scaling of the integrand requires $|u|, |v| < 1$ and the additional kinematical constraints $|uv| > \rho$ has to be imposed to ensure that the soft quark propagator in eq. (3.7) can go on the mass shell. After integrating eq. (3.5) over $l_\perp$ with the double-logarithmic accuracy we get

$$J_1 \approx \int_\rho^1 \frac{\mathrm{d}v}{v} \int_{\rho/v}^1 \frac{\mathrm{d}u}{u} = \ln^2\rho \int_0^1 \mathrm{d}\xi \int_0^{1-\xi} \mathrm{d}\eta = \frac{\ln^2\rho}{2}\,, \tag{3.9}$$

where the normalized logarithmic variables $\eta = \ln v/\ln\rho$ and $\xi = \ln u/\ln\rho$ are introduced.[5] As in the case of the soft gluon exchange the expansion of J_1 to $\mathcal{O}(\rho)$ does not result in a double-logarithmic contribution. However, in the given kinematics this can be seen immediately since $p_i^2 = 0$. Indeed, the expansion of the eikonal quark propagators then reads

$$\frac{1}{(p_i - l)^2 - m_q^2} = -\frac{1}{2(p_i l)}\left(1 + \frac{l^2 - m_q^2}{2(p_i l)} + \ldots\right) \tag{3.10}$$

so that all the subleading terms cancel soft quark propagator and have no double-logarithmic scaling. Thus we get

$$J_1 = \frac{1}{2}\Big(\ln^2\rho + \mathcal{O}(\rho\ln\rho)\Big) + \ldots\,, \tag{3.11}$$

which in double-logarithmic approximation gives

$$\Big[M_{ggH}^{(0)}\Big]_{1-\mathrm{loop}} = \ln^2\rho \tag{3.12}$$

and does not contribute to $M_{ggH}^{(1)}$.

The case of the integral eq. (3.6) is less trivial. It remains finite at $m_q \to 0$ but its expansion to $\mathcal{O}(\rho)$ does produce a double-logarithmic contribution. The second term in the numerator of eq. (3.6) cancels both eikonal quark propagators up to the terms proportional to $l^2 - m_q^2$ and can be omitted. In the first term for the on-shell soft quark we can replace l^2 with m_q^2 and get the same double-logarithmic integral as for J_1 up to an overall factor $-\rho$, which gives

$$J_2 = -\frac{1}{2}\left(1 + \rho\ln^2\rho + \ldots\right)\,, \tag{3.13}$$

and

$$\Big[M_{ggH}^{(1)}\Big]_{1-\mathrm{loop}} = -4\ln^2\rho\,. \tag{3.14}$$

Let us consider the higher-order double-logarithmic contributions due to dressing of the one-loop diagram with multiple leading-power gluon exchanges. After factoring out the external line Sudakov corrections into Z_g^2 the remaining non-Sudakov mass logarithms are determined by the effective soft gluon corrections to the Higgs boson vertex which can be obtained from the double-logarithmic result for the off-shell quark form factor given in the appendix B. At $\mathcal{O}(m_q)$ only the leading-power term should be kept in eq. (B.1)

[5] The contributions of the positive and negative Sudakov parameters are symmetric so we rewrite the total integral in terms of positive u and v.

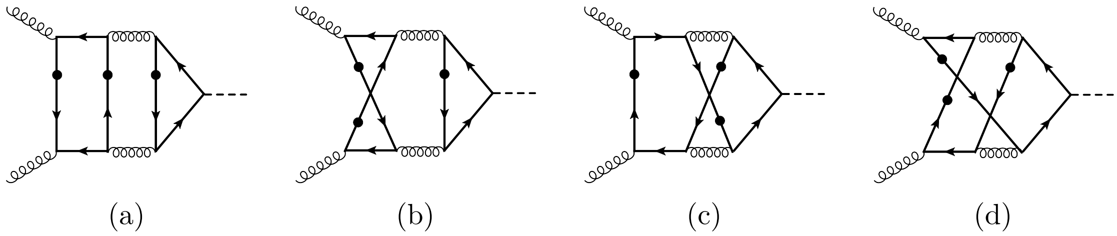

Figure 3. The three-loop Feynman diagrams for the Higgs boson two-photon decay amplitude with triple soft quark exchange.

and included into the one-loop integrand of eq. (3.9), see figure 1(b). To account for the variation of the color charge along the eikonal lines in the process of soft quark emission the color weight in the Sudakov factor eq. (B.2) for the off-shell quark lines should be changed from C_F to $C_F - C_A$ [27, 28]. This gives

$$M^{(0)}_{ggH} = \ln^2\!\rho\, g(z)\,, \tag{3.15}$$

where

$$g(z) = 2\int_0^1 \mathrm{d}\xi \int_0^{1-\xi} \mathrm{d}\eta e^{2z\eta\xi} = {}_2F_2\,(1,1;3/2,2;z/2) \tag{3.16}$$

is the generalized hypergeometric function with the Taylor expansion

$$g(z) = 2\sum_0^\infty \frac{n!}{(2n+2)!}(2z)^n\,. \tag{3.17}$$

At $\mathcal{O}(m_q^3)$ the loop momentum insertion do not affect the structure of the leading-power soft gluon contribution which is represented by the diagram in figure 1(c). As it was discussed above, the logarithmic integral over the soft quark momentum is identical to the $\mathcal{O}(m_q)$ term and the corresponding contribution to $M^{(1)}_{ggH}$ is given by $-4M^{(0)}_{ggH}$, cf. eqs. (3.12) and (3.14). In principle in this order of the small-mass expansion one has to consider the leading power correction to the off-shell form factor itself. However, as it has been pointed out the soft gluons do not generate the double-logarithmic $\mathcal{O}(\rho)$ contribution. At the same time the power corrections in the off-shell quark momenta Δ_i/m_H^2 to eq. (B.1) cancel an eikonal quark propagator up to the terms proportional to $l^2 - m_q^2$ and also can be omitted.

Starting with three loops the diagrams with triple soft quark exchange, figures 3(a–d), may contribute to $\mathcal{O}(m_q^3)$ amplitude. The double-logarithmic part of the diagrams figures 3(b–d) vanishes after taking the spinor trace over the closed quark loop. At the same time the diagram figure 3(a) includes a two-loop subdiagram corresponding to the double-logarithmic off-shell scalar form factor eq. (B.1), i.e. has the structure of figure 1(d). To get the corresponding corrections to the amplitude the next-to-leading power term $Z_q^2(\eta,\xi)F_S^{(1)}(\eta,\xi)$ should be included into the one-loop integrand of eq. (3.9) and as before in the factor $Z_q^2(\eta,\xi)$ the color weight C_F should be changed to $C_F - C_A$. In this way we get the double-logarithmic corrections to the coefficient $M^{(1)}_{ggH}$

$$\ln^2\!\rho\,\frac{T_F C_F}{45}x^2 h(z)\,, \tag{3.18}$$

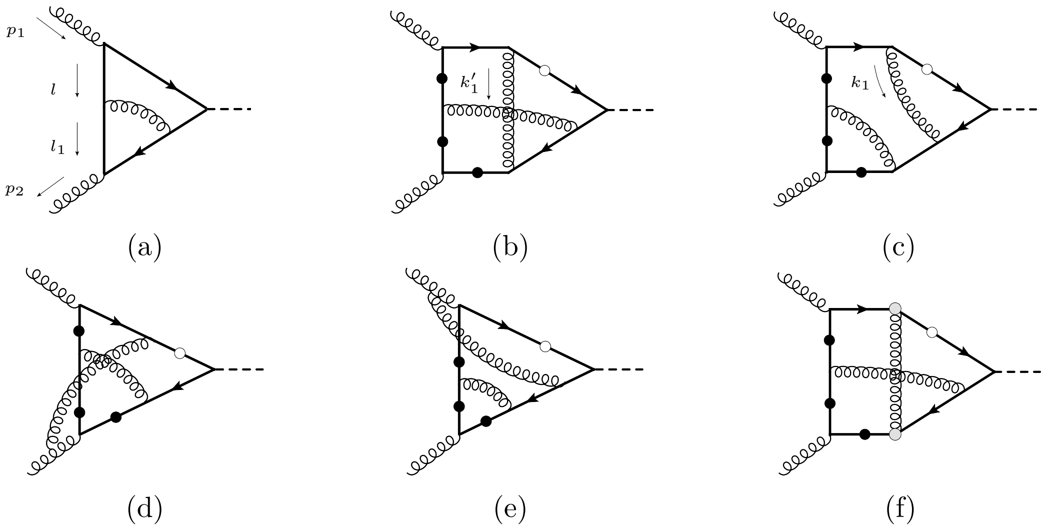

Figure 4. The Feynman diagrams representing (a) the two-loop and (b)–(e) the three-loop corrections to the ggH amplitude with an additional eikonal gluon emitted by the soft quark. (f) the Feynman diagram with the effective soft gluon exchange, which represents the total QCD three-loop non-Sudakov double-logarithmic correction associated with the eikonal gluon emission, eq. (3.20).

where the function $h(z)$ has the following integral representation

$$h(z) = 6! \int_0^1 \mathrm{d}\eta \int_0^{1-\eta} \mathrm{d}\xi \int_0^{\eta} \mathrm{d}\eta_2 \int_0^{\xi} \mathrm{d}\xi_2 \int_0^{\eta_2} \mathrm{d}\eta_1 \int_0^{\xi_2} \mathrm{d}\xi_1 \, e^{2z(\eta\xi - \eta_2\xi_2 + \eta_1\xi_1)} \,. \tag{3.19}$$

The coefficients of the Taylor series $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ can be computed for any given n corresponding to the $(n+3)$-loop double-logarithmic contribution. The first eight coefficients of the series are listed in table 1.

All the double-logarithmic contributions we have considered so far factored out into the (effective) corrections to the Higgs boson vertex. In three loops a new source of the double-logarithmic corrections opens up with an additional eikonal gluon connecting one of the eikonal and the soft quark lines. To trace the origin of this contributions let us consider first the two-loop diagram in figure 4(a). This diagram formally may have a double-logarithmic scaling. Indeed, for the virtual momentum l collinear to p_1 and for the soft momentum l_1 the gluon propagator becomes eikonal and proportional to $1/(l_1 l)$. Thus we get a standard scalar double-logarithmic integral over l_1 with external on-shell momenta p_2 and l which is proportional to the eikonal factor $1/(p_2 l)$. To get the double-logarithmic scaling of the integral over l it must be canceled since the same factor is already present in the second eikonal quark propagator. However, due to transversal polarization of the external gluons the relevant structure with the light-cone component of the momentum l does not appear in the tensor decomposition of the Feynman diagram and the double-logarithmic contribution of this type does not appear in two loops. At the same time the relevant tensor structures do appear in three loops. The corresponding abelian and nonabelian diagrams are given in figures 4(b,c) and (d,e), respectively. The details of the calculation of the abelian contribution are given in appendix C. Note that for the planar

n	1	2	3	4	5	6	7	8
$n^2 2^n n! h_n$	$\frac{3}{7}$	$\frac{8}{9}$	$\frac{90}{77}$	$\frac{59392}{45045}$	$\frac{5360}{3861}$	$\frac{7559936}{5360355}$	$\frac{583744}{415701}$	$\frac{2110652416}{1527701175}$
$n! j_n^{\mathrm{ab}}$	$\frac{17}{28}$	$\frac{83}{175}$	$\frac{241}{550}$	$\frac{47984}{105105}$	$\frac{3645}{7007}$	$\frac{97228}{153153}$	$\frac{772588}{944775}$	$\frac{19563776}{17782765}$

Table 1. The normalized coefficients of the Taylor series for the function $h(z)$ and $j^{\mathrm{ab}}(z)$ up to $n = 8$.

topology figure 4(c) due to a cancellation specific to three loops the double-logarithmic contribution vanishes. Moreover, in the diagram figure 4(e) the contribution of the soft gluon momentum coming from the three-gluon vertex is not included. Though such a term does produce a double-logarithmic contribution for a given diagram, it is proportional to the momentum of the on-shell soft gluon and after we cut the corresponding gluon line it vanishes in the sum of the diagrams by the Ward identity. After separating the infrared divergencies in the same way as it has been done for the functions $f(z)$ and $g(z)$, the remaining infrared finite double-logarithmic contribution is described by the diagram figure 4(f) with the effective soft gluon exchange. The corresponding Feynman integral is the same as for the abelian diagram in figure 4(b) computed in appendix C, which gives the following contribution to $M^{(1)}_{ggH}$

$$-\ln^2\rho \, \frac{(C_A - C_F)\,(C_A - 2C_F)}{9} x^2 \,. \tag{3.20}$$

The color structure of eq. (3.20) is quite peculiar. As it has been previously discussed the factor $C_A - C_F$ accounts for the eikonal color charge variation caused by a soft quark emission. The remaining factor $C_A - 2C_F$ reflects the change of the eikonal quark and antiquark state into color octet after the emission of the eikonal gluon.

The higher-order double-logarithmic corrections of this type are obtained by dressing the diagram in figure 4(a) with multiple soft gluons. This results in multiplication of eq. (3.20) by a function of the double-logarithmic variable $j(z) = 1 + \sum_{n=1}^{\infty} j_n z^n$. Thus the complete double-logarithmic approximation for the next-to-next-to-leading power coefficient can be written as follows

$$M^{(1)}_{ggH} = \ln^2\rho \left[-4g(z) + \left(\frac{T_F C_F}{45} h(z) - \frac{(C_A - C_F)(C_A - 2C_F)}{9} j(z) \right) x^2 \right] . \tag{3.21}$$

Calculation of the functions $j(z)$ requires a systematic factorization of the soft emissions with respect to the emission of the additional eikonal gluon. For QCD this is a rather complicated computational problem due to the soft interaction of the eikonal gluon, which starts to contribute in four loops. The full QCD analysis, however, goes beyond the scope of the present paper. Instead, we consider two complementary limits where such a complication is absent. First we discuss QCD with the large number of colors $N_c \to \infty$. In this case the color factor of the diagram figure 4(a) vanishes and the double logarithmic approximation is entirely determined by the function $g(z)$ where $z = N_c x/2$. In the opposite abelian limit $C_A = 0$ the gluon self-coupling is absent but the analysis of the factorization is nevertheless quite nontrivial, see appendix C. For $C_A = 0$ we get the following integral

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representation of the function $j(z)$

$$j^{\rm ab}(z) = 72\int_0^1 {\rm d}\eta \int_0^{1-\eta}{\rm d}\xi\int_0^{1-\xi}{\rm d}\eta_1\int_0^{1-\eta_1-\xi}{\rm d}\xi_1\, \eta\xi_1 e^{2z\eta(\xi+\xi_1)}$$
$$\times\left[1+\frac{e^{-2z\eta\xi}-1}{2}+\frac{e^{-2z\eta\xi}-1+2z\eta\xi}{4z\eta\xi_1}\right], \tag{3.22}$$

where in the abelian approximation the double-logarithmic variable reduces to $z=-C_Fx$. The first eight coefficients of the Taylor series for $j^{\rm ab}(z)$ are listed in table 1. The perturbative expansion of eq. (3.21) reads

$$M^{(1)}_{ggH} = \ln^2\rho\left[-4-\frac{2}{3}(C_A-C_F)x+\left(\frac{T_FC_F}{45}-\frac{14}{45}C_F^2+\frac{23}{45}C_FC_A-\frac{9}{45}C_A^2\right)x^2\right.$$
$$\left.+c_4x^3+\ldots\right], \tag{3.23}$$

where the four-loop coefficient is $c_4=-N_c^3/840$ in the large-N_c approximation and

$$c_4=-\frac{T_FC_F^2}{210}+\frac{13}{90}C_F^3 \tag{3.24}$$

in the abelian approximation. The series eq. (3.23) can be compared to the existing fixed-order results. The two-loop term agrees with the expansion of the exact analytic result [57]. The high-energy expansion of the three-loop ggH amplitude has been obtained numerically in ref. [48]. Eq. (3.23) corresponds to the following coefficient of the L_s^6/z^2 term in eq. (C.1) of [48]

$$\frac{1}{23040}\left(-T_FC_F+14C_F^2-23C_FC_A+9C_A^2\right), \tag{3.25}$$

which agrees with its numerical value 0.0005738811728. The result eq. (3.21) for the gluon fusion amplitude can be transformed into the one for the amplitude of the Higgs boson two-photon decay by changing the color charge of the external lines from C_A to zero. This results in the replacement $C_A-C_F\to -C_F$ in the definition of the double-logarithmic variable z and in the coefficient of eq. (3.20). By adopting the notations similar to the gluon fusion case we get

$$M^{(1)}_{H\gamma\gamma}=\ln^2\rho\left[-4+\frac{2}{3}C_Fx+\left(\frac{T_FC_F}{45}-\frac{14}{45}C_F^2+\frac{C_FC_A}{9}\right)x^2+\ldots\right]. \tag{3.26}$$

The three-loop term can be compared to the numerical result for the high-energy expansion of the amplitude given in ref. [49]. It corresponds to the coefficient

$$-\frac{1}{3840}\left(T_FC_F-14C_F^2+5C_FC_A\right) \tag{3.27}$$

of the L_s^6/z^2 term in eq. (C.1) and agrees with its numerical value 0.001099537037. The agreement holds for the contributions of the individual color factors [58].

Let us consider the all-order asymptotic behavior of the $\mathcal{O}(m_q^3)$ amplitude in the high-energy (small-mass) limit. In the large-N_c approximation it reads

$$M^{(1)}_{ggH}=-4\ln^2\rho\, g\left(\frac{N_cx}{2}\right), \tag{3.28}$$

where

$$g(z) \sim \left(\frac{2\pi e^z}{z^3}\right)^{\frac{1}{2}} \tag{3.29}$$

at $z \to \infty$, i.e. the amplitude is exponentially enhanced. Note that the limit $N_c \to \infty$ is taken first and in general may not commute with the kinematical limit $z \to \infty$. In the abelian approximation the relevant asymptotic behavior of the functions in eq. (3.21) at $z \to -\infty$ reads

$$g(z) \sim -\frac{\ln(-2z) + \gamma_E}{z}, \quad h(z) = \mathcal{O}(1/z^3), \quad j^{\rm ab}(z) \sim \frac{9}{2z^2}\,. \tag{3.30}$$

Thus the coefficient asymptotically approaches the value $M^{(1)}_{ggH} = -\ln^2\rho$, i.e. the double logarithmic corrections effectively reduce the leading-order coefficient by factor four.

Now we can estimate the effect of the high-order $\mathcal{O}(m_q^3)$ terms for the physical values of the parameters. The relative correction to the $\mathcal{O}(m_q)$ amplitude is given by the factor

$$1 + \rho\left[-4 + \left(\frac{T_F C_F}{45}h(z) - \frac{(C_A - C_F)(C_A - 2C_F)}{9}j(z)\right)\frac{x^2}{g(z)}\right]. \tag{3.31}$$

In the large-N_c approximation eq. (3.31) reduces to $1 - 4\rho$ with $\rho \approx 1.6 \cdot 10^{-3}$, which amounts of approximately 0.64% negative correction to the $\mathcal{O}(m_q)$ contribution. It does not depend on x and is the same for the gluon and photon external lines. Hence it gives a universal all-order estimate of the next-to-next-to-leading power corrections both for the production and decay amplitudes.

4 Summary

We have studied the high-energy asymptotic behavior of the electromagnetic quark scattering and the light quark loop mediated Higgs boson production in the third order of the small quark mass expansion. To our knowledge this is the first example of the renormalization group analysis of the next-to-next-to-leading power amplitudes.

For the $\mathcal{O}(m_q^3)$ quark scattering the asymptotic behavior is determined by the double-logarithmic corrections to the Pauli form factor with the structure similar to the Dirac and scalar form factors at $\mathcal{O}(m_q^2)$. These non-Sudakov double logarithms result from the eikonal color charge nonconservation in the process with the exchange of the soft virtual quark pair. They are described by a universal function which shows exponential growth for the large values of the double-logarithmic variable in QED and a logarithmic scaling in QCD. We present for the first time the complete analytic asymptotic result for this function, eqs. (2.15), (2.16).

The double-logarithmic corrections to the $\mathcal{O}(m_q^3)$ Higgs boson production and decay amplitudes are induced by single and triple soft quark exchanges. This is the first example where the mass suppression of the double-logarithmic contribution is not entirely associated with the chirality flip on a fermion line. Starting with three loops a new source of the double-logarithmic corrections opens up with an emission of an additional virtual eikonal gluon by the soft quark. Our analytic result agrees with the previous numerical evaluation of the three-loop QCD corrections to the Higgs boson production [48] and two-photon decay [49]. Beyond three loops the all-order double-logarithmic asymptotic behavior of the amplitudes has been derived in two complementary approximations. In the large-N_c limit, which is supposed to catch the qualitative behavior of real QCD, the structure of the double-logarithmic corrections significantly simplifies and becomes identical to the one of the leading $\mathcal{O}(m_q)$ contribution, which is exponentially enhanced for the large values of the double-logarithmic variable. The opposite abelian limit $C_A = 0$, though less phenomenologically relevant, reveals a more complex structure of the double-logarithmic contributions and represents the general case for the mass-suppressed amplitudes at the next-to-next-to-leading power.

We have also presented a quantitative estimate of the accuracy of the high-order calculations based on the small-mass expansion for the Higgs boson production and decays mediated by the bottom quark loop, which may become relevant with the permanently increasing accuracy of the QCD predictions [59]. On the basis of the double-logarithmic analysis we conclude that neglecting the terms suppressed by the mass ratio m_b^2/m_H^2 in such a calculation introduces a relative error at the scale of one percent in every order of the perturbative expansion. Our result can also be generalized to estimate the high-order subleading top quark mass effects on the double Higgs boson production in the high-energy limit [60, 61], where the role of the next-to-next-to-leading power terms could be significant.

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A Evaluation of the function $f(\pm z)$ in the limit $z \to \infty$

It is more convenient to use an alternative integral representation of $f(z)$

$$f(z) = 24 \int_0^1 \mathrm{d}\eta_2 \int_0^{1-\eta_2} \mathrm{d}\xi_2\, e^{2z\eta_2\xi_2} \int_0^{\eta_2} \mathrm{d}\eta_1 \int_0^{\xi_2} \mathrm{d}\xi_1\, e^{-2z\eta_1\xi_1}\,, \tag{A.1}$$

which is equivalent to eq. (2.5). Then, the integration over ξ_1 and η_1 can be done explicitly with the result

$$\frac{1}{2z}\left(-\mathrm{Ei}(-2z\eta_2\xi_2) + \ln(2z\eta_2\xi_2) + \gamma_E\right). \tag{A.2}$$

For the remaining integrals we introduce new variables $\xi_2 = y\lambda^{1/2}$, $\eta_2 = \lambda^{1/2}/y$ with the Jacobian $|J| = y$ and the integration limits

$$\frac{1-(1-4\lambda)^{1/2}}{2\lambda^{1/2}} < y < \frac{1+(1-4\lambda)^{1/2}}{2\lambda^{1/2}}\,, \qquad 0 < \lambda < 1/4\,. \tag{A.3}$$

The logarithmic integral over y gives

$$\ln\left(\frac{1+(1-4\lambda)^{1/2}}{1-(1-4\lambda)^{1/2}}\right). \tag{A.4}$$

The further analysis depends on the sign of the argument of $f(z)$. For $z \to \infty$ the factor $e^{2z\eta_2\xi_2} = e^{2z\lambda}$ provides exponential enhancement and the integral over λ is saturated with the region in vicinity of the maximal value $\lambda = 1/4$. Thus eq. (A.2) can be approximated as follows

$$\frac{1}{2z}\left(\ln(z/2)+\gamma_E\right), \tag{A.5}$$

and eq. (A.4) reduces to $2(1-4\lambda)^{1/2}$. The asymptotic expansion of the resulting integral is straightforward

$$\int_0^{\frac{1}{4}} 2(1-4\lambda)^{\frac{1}{2}} e^{2z\lambda} \mathrm{d}\lambda \sim \left(\frac{\pi e^z}{2z^3}\right)^{\frac{1}{2}}, \tag{A.6}$$

which gives eq. (2.15).

For the negative value of the argument $f(-z)$ at $z \to \infty$ the asymptotic expansion of the resulting integral is more involved but can be performed by the standard techniques

$$\int_0^{\frac{1}{4}} \ln\left(\frac{1+(1-4\lambda)^{1/2}}{1-(1-4\lambda)^{1/2}}\right)\left(\mathrm{Ei}(2z\lambda)-\ln(2z\lambda)-\gamma_E\right)e^{-2z\lambda}\mathrm{d}\lambda \sim \frac{\left(\ln(2z)+\gamma_E\right)^2-\pi^2/2}{4z}, \tag{A.7}$$

which gives eq. (2.16)

B Off-shell scalar form factor of massive quark

We consider the scalar form factor of a quark with the off-shell external momenta $\Delta_i = (p_i - l)^2 - m_q^2 \neq 0$ and Minkowskian momentum transfer $(p_2 - p_1)^2 = m_H^2$. In the double-logarithmic approximation we can set $l^2 = m_q^2$ so that $\Delta_1 = -2(p_1 l) = v m_H^2$, $\Delta_2 = -2(p_2 l) = u m_H^2$, and consider the case $|\Delta_i| \gg m_q^2$. The form factor can be simultaneously expanded in $\rho = m_q^2/m_H^2$ and Δ_i/m_H^2 as follows

$$F_S(\eta,\xi) = Z_q^2(\eta,\xi)\sum_{n=0}^{\infty}\rho^n F_S^{(n)}(\eta,\xi) + \mathcal{O}(\Delta_i/m_H^2), \tag{B.1}$$

where $Z_q^2(\eta,\xi)$, $F_S^{(n)}(\eta,\xi)$ are the functions of the logarithmic variables η, ξ. Though formally $|\Delta_i| \gg m_q^2$ we are not interested in the terms vanishing for $\Delta_i = 0$ since they do not produce the double-logarithmic corrections to the Higgs boson decay amplitude. As in eq. (2.2) the coefficient

$$Z_q^2(\eta,\xi) = e^{-2C_F x\eta\xi} \tag{B.2}$$

represents the usual Sudakov factor now computed for the off-shell quarks. Thus in the double-logarithmic approximation the leading-power coefficient is just $F_S^{(0)}(\eta,\xi) = 1$. The next-to-leading power coefficient $F_S^{(1)}(\eta,\xi)$ gets the double-logarithmic contribution from the diagrams with the soft quark pair exchange, figures 2(c,d), and has been evaluated for the on-shell quarks in [27, 28]. This analysis can be extended to the off-shell case in a straightforward way by changing the infrared cutoff from m_q^2 to Δ_i. The result reads

$$F_S^{(1)}(\eta,\xi) = 8C_F T_F x^2 \int_0^\eta \mathrm{d}\eta_2 \int_0^\xi \mathrm{d}\xi_2 \int_0^{\eta_2} \mathrm{d}\eta_1 \int_0^{\xi_2} \mathrm{d}\xi_1 \, e^{-2z\eta_2\xi_2} e^{2z\eta_1\xi_1} \,, \tag{B.3}$$

where the exponential factors corresponding to the diagrams figure 2(c) and figure 2(d) are given separately. The only difference of the above equation with respect to eqs. (12,13) of [27] is in the integration limits over the logarithmic Sudakov variables η_i, ξ_i corresponding to each loop momenta. Since we perform the calculation for $m_q = 0$ there is no correlation between the η_i and ξ_i variables unlike eq. (3.9) and the logarithmic integration intervals in eq. (B.3) are given just by ordering these variables along the eikonal lines $\xi > \xi_2 > \xi_1$, $\eta > \eta_2 > \eta_1$.

C Evaluation of the function $j(z)$ in abelian approximation

Let us begin with the calculation of the leading three-loop term. In the abelian approximation only two diagrams in figures 4(b,c) may have double-logarithmic scaling. We consider the nonplanar topology first. Defining the loop momenta l and l_1 as in figure 4(a) we introduce the following Sudakov parametrization $l_1 = u_1 l + v_1 p_2 + l_{1\perp}$, $k_1' = r_1' p_1 + w_1' p_2 + k_{1\perp}'$ and assume that in the light-cone coordinates $p_1 = p_1^-$, $p_2 = p_2^+$. Then the on-shell condition for the soft quark propagators requires $uv > \rho$, $u_1 v_1 > \rho/u$, and the logarithmic scaling of the integrals over the Sudakov parameters imposes the conditions $v < w_1'$, $uu_1 < r_1' < u$. The double-logarithmic scalar integral over l_1 results in the factor $1/(p_2 l) = 1/(p_2^+ l^-)$ which has the same structure as the lower eikonal quark propagator

$$S(p_2 - k_1' - l) = -\frac{\gamma^-}{l^-} + \ldots \,, \tag{C.1}$$

where we used the relation $k_1'^- \ll l^-$ valid in the logarithmic integration region. To get the logarithmic integral over l^- one of the $1/l^-$ factors must be cancelled. By taking into account that the real (virtual) gluons have transversal (light-cone) polarization we find that the only relevant tensor structure is given by the l^- term in the numerator of the upper eikonal quark propagator

$$S(p_1 - k_1' - l) = -\frac{\gamma^+}{k_1'^+}\left(1 - \frac{k_1'^- + l^-}{p_1^-}\right) + \ldots \,, \tag{C.2}$$

as it is indicated in figure 4(b). The integral over r_1' and w_i' within the logarithmic limits specified above results in the standard one-loop Sudakov correction factor $2z\eta\xi_1$, where $z = -C_F x$ and we introduce the logarithmic variables $\eta_1 = \ln v_1/\ln\rho$ and $\xi_1 = \ln u_1/\ln\rho$ in the same way as for the loop momentum l. Then the $C_A = 0$ abelian double-logarithmic contribution of the diagram figure 4(b) to the coefficient $M_{ggH}^{(1)}$ reads

$$-8\ln^2\!\rho\, z^2 \int_0^1 \mathrm{d}\eta \int_0^{1-\eta} \mathrm{d}\xi \int_0^{1-\xi} \mathrm{d}\eta_1 \int_0^{1-\eta_1-\xi} \mathrm{d}\xi_1 \, \eta\xi_1 = -\left(\frac{\ln\!\rho\, z}{3}\right)^2 . \tag{C.3}$$

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In the case of the planar diagram figure 4(c) the logarithmic intervals for the Sudakov parameters of the soft gluon momentum $k_1 = r_1 p_1 + w_1 p_2 + k_{1\perp}$ are $v < w_1$, $u < r_1$ and the integration over k_1 gives the factor $2z\eta\xi$. In contrast to the nonplanar case the required l^- term is generated in two different ways. Indeed, in the logarithmic integration region now $l^- \ll k_1{}^-$ and the denominator of the lower quark propagator can be expanded as follows

$$S(p_2 - k_1 - l) = -\frac{\gamma^-}{k_1{}^-}\left(1 - \frac{l^-}{k_1{}^-}\right) + \ldots\,, \tag{C.4}$$

while the upper quark propagator has the expansion similar to eq. (C.2)

$$S(p_1 - k_1 - l) = -\frac{\gamma^+}{k_1^+}\left(1 - \frac{k_1^- + l^-}{p_1^-}\right) + \ldots\,. \tag{C.5}$$

The product of l^-/k_1^- term in eq. (C.4) and k_1^-/p_1^- term in eq. (C.5) generates the relevant tensor structure which cancels the contribution from the product of the leading term in eq. (C.4) and l^-/p_1^- term in eq. (C.5). Thus the total double-logarithmic contribution of the planar three-loop diagram to the coefficient $M^{(1)}_{ggH}$ vanishes and eq. (C.3) with the equal contribution of the symmetric diagram determines the abelian part of eq. (3.20). The above cancellation, however, does not hold for the multiple soft gluon exchanges.

Let us now consider the case of n soft gluons. In the abelian approximation the double-logarithmic corrections are generated by the diagrams with m' leading-power exchanges of the topology figure 4(b) and $m = n - m'$ exchanges of the topology figure 4(c), with all possible permutations of n vertices along the upper quark line. Let k'_i and k_i be the momenta of the gluons from the first and the second group, respectively. Each of the n gluons contributes the term l^- from the numerator of the eikonal quark propagator, as in eqs. (C.2), (C.5). After the summation over all the permutations of the n vertices the integrals over the $k_i'^+$ and k_i^+ factorize. After the (redundant) summation over $m'!m!$ permutations of vertices along the lower quark line within each group the integrals over the $k_i'^-$ and $k_i{}^-$ also factorize. Thus the n-loop soft contribution can easily be evaluated with the result

$$\frac{n}{m'!m!}(2z\eta\xi_1)^{m'}(2z\eta\xi)^m\,. \tag{C.6}$$

The analysis of the l^- terms originating from the denominators of the eikonal quark propagators, as in eq. (C.4), is more subtle. These terms are generated by the soft gluons from the second group only and come from the expansion of the following expression

$$\frac{f_1 k_1^- + \ldots + f_m k_m^-}{(k_1^- + l^-)\ldots(k_1^- + \ldots + k_m^- + l^-)}\,, \tag{C.7}$$

where the gluon momenta are enumerated from the eikonal gluon to the Higgs boson vertex and f_i is the number of the eikonal propagators carrying the momentum k_i on the upper quark line for a given diagram. The numerator of eq. (C.7) can be rewritten as follows

$$\begin{aligned} &f_m(k_1^- + \ldots + k_m^- + l^-)\\ &+(f_{m-1} - f_m)(k_1^- + \ldots + k_{m-1}^- + l^-) + \ldots\\ &+(f_1 - f_2)(k_1^- + l^-) - f_1 l^-\,. \end{aligned} \tag{C.8}$$

Note that every term in eq. (C.8) except the last one cancels one of the eikonal quark propagators in eq. (C.7) removing the double-logarithmic scaling of the integrand. Thus we have to consider only the contribution of last term corresponding to the soft gluon emitted next to the eikonal gluon vertex and the total result is obtained by summing up the coefficients f_1 over the diagrams with all possible permutations of the remaining vertices. It is convenient to perform the double-logarithmic integration over k_i' and k_i first. Since in the logarithmic region the Sudakov parameters are ordered along the eikonal lines, for every diagram the n-fold integral over w_i' and w_i gives $\eta^n/n!$, the m'-fold integral over r_i' gives $\xi_1^{m'}/m'!$, and the m-fold integral over r_i gives $\xi^m/m!$. This combines into the common n-loop factor

$$\frac{(2z\eta\xi_1)^{m'}(2z\eta\xi)^m}{n!m'!m!}\,, \tag{C.9}$$

Since f_1 does not depend on the routing of the other loop momenta we can perform summation over the permutations within the groups of m' and $m-1$ remaining vertices on the lower quark line, which results in the factor $m'!(m-1)!$ for $m>0$. Now let j' and j be the numbers of the vertex with the soft momentum k_1 in a sequence of all n vertices and in a sequence of m vertices of the second group on the upper quark line, respectively, counted from the Higgs boson vertex. Then for a given diagram $f_1=j'$ and the sum over all the diagrams gives

$$\sum_{j=1}^{m}\sum_{j'=1}^{j+m'}\frac{(j'-1)!}{(j-1)!(j'-j)!}\frac{(n-j')!}{(m'+j-j')!(m-j)!}j'=\frac{n!}{m!m'!}\frac{(n+1)m}{2}\,, \tag{C.10}$$

where the combinatorial factor corresponds to the number of ways to arrange m' ordered vertices from the first group and $m-1$ ordered vertices from the second group for a given j' and j. Bringing all the factors together we get

$$-\frac{n+1}{2m'!m!}(2z\eta\xi_1)^{m'}(2z\eta\xi)^m\,, \tag{C.11}$$

which after adding the contribution eq. (C.6) gives the total result for $m>0$

$$\frac{m+m'-1}{2m'!m!}(2z\eta\xi_1)^{m'}(2z\eta\xi)^m\,. \tag{C.12}$$

The $m=0$ result can be obtained directly from eq. (C.6) and reads

$$\frac{(2z\eta\xi_1)^{m'}}{(m'-1)!}\,. \tag{C.13}$$

The dependence on m and m' in eq. (C.12) factorizes and the summation over the number of soft gluons in each group can be directly performed

$$\begin{aligned}&\sum_{m'=0}^{\infty}\frac{(2z\eta\xi_1)^{m'}}{m'!}\left[\sum_{m=0}^{\infty}\frac{m+m'-1}{2m!}(2z\eta\xi)^m+\frac{m'+1}{2}\right]\\&=\frac{e^{2z\eta(\xi+\xi_1)}}{2}\left[\left(e^{-2z\eta\xi}-1+2z\eta\xi\right)+(2z\eta\xi_1)\left(e^{-2z\eta\xi}+1\right)\right],\end{aligned} \tag{C.14}$$

where the second term in the first line provides the correct $m = 0$ contribution, eq. (C.13). After factoring out the leading soft gluon contribution $2z\eta\xi_1$ we obtain the integrand of eq. (3.22). Note that the contributions of the first and the second group of the soft gluons completely factorize and exponentiate in the final result.

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