

All-loop-orders relation between Regge limits of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity four-point amplitudes

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ABSTRACT: We examine in detail the structure of the Regge limit of the (nonplanar) $\mathcal{N} = 4$ SYM four-point amplitude. We begin by developing a basis of color factors C_{ik} suitable for the Regge limit of the amplitude at any loop order, and then calculate explicitly the coefficients of the amplitude in that basis through three-loop order using the Regge limit of the full amplitude previously calculated by Henn and Mistlberger. We compute these coefficients exactly at one loop, through $\mathcal{O}(\epsilon^2)$ at two loops, and through $\mathcal{O}(\epsilon^0)$ at three loops, verifying that the IR-divergent pieces are consistent with (the Regge limit of) the expected infrared divergence structure, including a contribution from the three-loop correction to the dipole formula. We also verify consistency with the IR-finite NLL and NNLL predictions of Caron-Huot et al. Finally we use these results to motivate the conjecture of an all-orders relation between one of the coefficients and the Regge limit of the $\mathcal{N} = 8$ supergravity four-point amplitude.

KEYWORDS: Extended Supersymmetry, Scattering Amplitudes, Supergravity Models, Supersymmetric Gauge Theory

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Contents

1	Introduction	1
2	Group-theory constraints on four-point amplitudes	4
3	Regge limit of the $\mathcal{N} = 4$ four-point amplitude	7
4	Regge basis of color factors	9
4.1	One-loop basis	11
4.2	Two-loop basis	12
4.3	Three-loop basis	13
4.4	Higher-loop basis	14
5	IR-divergence structure of the $\mathcal{N} = 4$ SYM amplitude	15
5.1	Reduced amplitude	17
5.2	One loop	18
5.3	Two loops	19
5.4	Three loops	20
5.5	All loop results	22
6	SYM/supergravity relation in the Regge limit	23
7	Conclusions	25

1 Introduction

Maximally supersymmetric Yang-Mills and supergravity theories have garnered especial interest ever since it was shown that they emerge from the low energy limit of superstring theory, which allowed the first one-loop calculation of four-point amplitudes in each [1]. In the decades following, higher-loop $\mathcal{N} = 4$ SYM four-point amplitudes were calculated [2–8] in terms of planar and nonplanar scalar integrals through the use of generalized unitarity [9, 10]. The infrared divergences of these massless integrals were dimensionally regulated in $D = 4 - 2\epsilon$ dimensions, and Laurent expansions in ϵ of the planar two- and three-loop integrals [11, 12] were used to obtain explicit expressions for planar (leading in the $1/N$ expansion) $\mathcal{N} = 4$ SYM four-point amplitudes [13, 14]. Using these results, together with the well-studied structure of infrared divergences of gauge theories [15, 16], Bern, Dixon, and Smirnov proposed their celebrated all-loop-orders ansatz for (maximally-helicity-violating) planar n -point amplitudes [14], whose validity for $n = 4$ and 5 follows from dual conformal invariance of the planar theory [17, 18]. Progress on the evaluation of nonplanar two- and three-loop integrals has taken longer [19–22], but these results have allowed explicit

expressions for the full (nonplanar) $\mathcal{N} = 4$ SYM four-point amplitude at two [23, 24] and, more recently, three [25] loops. These expressions were verified to be consistent with the expected (nonplanar) infrared divergence structure through three loops [26–35].

Higher-loop four-point amplitudes of $\mathcal{N} = 8$ supergravity, a theory believed to be ultraviolet finite to a high loop order (see ref. [36] and references therein), have also been calculated in terms of planar and nonplanar integrals [3, 4, 7, 37]. Again using Laurent expansions of the planar and nonplanar integrals, explicit expressions for the integrated $\mathcal{N} = 8$ supergravity four-point amplitudes have been obtained at two [38–40] and, more recently, three [41] loops.

Relations between supergravity and subleading-color SYM amplitudes and their IR divergences were explored in refs. [23, 24, 40, 42–44]. Exact relations between $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity four-point amplitudes were established at one- and two-loop levels [23], but attempts at finding relations at three loops and beyond were generally unsuccessful.

Scattering amplitudes often exhibit dramatic simplification in the high-energy (or Regge) limit $s \gg -t$. The known structure of infrared divergences of gauge theory amplitudes [14–16, 26–35] simplifies at high energies, giving rise to leading and subleading logarithmic behavior [45–48]. Moreover, an effective Hamiltonian approach based on Balitsky-Fadin-Kuraev-Lipatov theory was used to compute IR-finite large logarithmic behavior (NLL and NNLL) in the Regge limit [49–52]. The Regge limit of the $\mathcal{N} = 4$ SYM four-point amplitude was examined in refs. [25, 50, 53–58].

The high-energy limit of gravity amplitudes is in some ways even simpler than gauge theory amplitudes [59–67]. The high-energy behavior of $\mathcal{N} = 8$ supergravity amplitudes has been studied in refs. [40, 41, 68–74], and recently, eikonal exponentiation in impact parameter space has been used to obtain an exact all-loop-orders expression for the high-energy limit of the $\mathcal{N} = 8$ supergravity four-point amplitude [75, 76].

In this paper, we examine in detail the structure of the Regge limit of the (nonplanar) $\mathcal{N} = 4$ SYM four-point amplitude. We begin by developing a basis of color factors C_{ik} suitable for the Regge limit of the amplitude at any loop order, and then calculate explicitly the coefficients of the amplitude in that basis through three-loop order using the Regge limit of the full amplitude previously calculated by Henn and Mistlberger [25]. We compute these coefficients exactly at one loop, through $\mathcal{O}(\epsilon^2)$ at two loops, and through $\mathcal{O}(\epsilon^0)$ at three loops, verifying that the IR-divergent pieces are consistent with (the Regge limit of) the expected infrared divergence structure as determined in refs. [14–16, 26–35, 45–50], including a contribution from the three-loop correction to the dipole formula [35, 50]. We also verify consistency¹ with the IR-finite NLL and NNLL predictions of Caron-Huot et al. [49, 50]. Finally we use these results to motivate the conjecture of an all-orders relation between one of the coefficients and the Regge limit of the $\mathcal{N} = 8$ supergravity four-point amplitude.

Color-ordered n -gluon amplitudes (i.e. the coefficients of the amplitude in a trace basis) of an $SU(N)$ gauge theory are not independent but obey relations derived from

¹The authors of ref. [50] also reported consistency of their results with ref. [25].

group theory [1, 77–82]. For four-gluon amplitudes, there are $3\ell - 1$ independent color-ordered amplitudes at $\ell \geq 2$ loops (with two at tree level, and three at one loop). In the Regge limit, each of these is reduced by one. Inspired by refs. [46, 48–50], we define a set of color factors C_{ik} by

$$\begin{aligned}
 C_{00} &= \tilde{f}^{a_4 a_1 e} \tilde{f}^{a_2 a_3 e}, \\
 C_{ii} &= (\mathbf{T}_{s-u})^i C_{00}, & i \geq 1 \\
 C_{i1} &= [\mathbf{T}_t^2, \dots, [\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] \dots] C_{00}, & i \geq 2 \\
 C_{i,i-1} &= [\mathbf{T}_{s-u}^2, \dots, [\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] \dots] C_{00}, & i \geq 2
 \end{aligned} \tag{1.1}$$

where \mathbf{T}_t^2 and \mathbf{T}_{s-u}^2 are operators (cf. eq. (4.7)) that act on the tree-level t -channel color factor C_{00} to generate higher-loop color factors. Each color factor C_{ik} in eq. (1.1) contains exactly i operators \mathbf{T}_t^2 or \mathbf{T}_{s-u}^2 with k factors of \mathbf{T}_{s-u}^2 . These color factors have well-defined signature (either + or –) under the crossing symmetry operation that exchanges external legs 2 and 3 (i.e., $s \leftrightarrow u$). As we will see in section 5, this basis is particularly well-suited to describe the IR divergences of the amplitude in the Regge limit.

Having defined the Regge basis of color factors, we write the $\mathcal{N} = 4$ SYM four-point amplitude in the Regge limit as

$$\mathcal{A} = A_1^{(0)} \sum_{\ell=0}^{\infty} \tilde{a}^\ell \sum_{i=0}^{\ell} \sum_k B_{ik}^{(\ell)} N^{\ell-i} C_{ik} \tag{1.2}$$

where \tilde{a} is the loop expansion parameter, and the range of k is

$$k = \begin{cases} 0, & \text{when } i = 0, \\ 1, & \text{when } i = 1, \\ 1, 2, & \text{when } i = 2, \\ 1, i - 1, i, & \text{when } i \geq 3. \end{cases} \tag{1.3}$$

By using the decomposition of the color factors C_{ik} into a trace basis, the coefficients $B_{ik}^{(\ell)}$ can be expressed as linear combinations of color-ordered amplitudes. By Bose symmetry of the full amplitude, the coefficients $B_{ik}^{(\ell)}$ will have the same signature under crossing symmetry as their associated color factors C_{ik} , which has implications [50] for the reality properties of $B_{ik}^{(\ell)}$, as we will see in this paper.

By examining the Regge limit of the structure of IR divergences for $\mathcal{N} = 4$ SYM theory, we determine that the $B_{ik}^{(\ell)}$ are polynomials of order $\ell - k$ in $L \equiv \log |s/t| - \frac{1}{2}i\pi$. Thus the leading logarithmic (LL) behavior of the amplitude at ℓ loops is entirely contained in the coefficient $B_{00}^{(\ell)}$, with the other coefficients contributing to subleading logarithmic behavior.

The full amplitude \mathcal{A} can be factored into an infrared-divergent prefactor times an infrared-finite hard function \mathcal{H} , whose coefficients in the Regge basis of color factors are denoted $h_{ik}^{(\ell)}$, analogous to eq. (1.2). Using the Regge limit of the $\mathcal{N} = 4$ SYM color-ordered amplitudes given in ref. [25], we compute the IR-finite coefficients $h_{ik}^{(\ell)}$ exactly at one loop, through $\mathcal{O}(\epsilon^2)$ at two loops, and through $\mathcal{O}(\epsilon^0)$ at three loops. The NLL and NNLL contributions to $h_{ik}^{(\ell)}$ are verified to agree with the predictions of refs. [49, 50].

Finally, using our results through three loops, we observe that one of the coefficients, $B_{\ell\ell}^{(\ell)}$, which contains no $\log|s/t|$ dependence, is equal (up to a sign) to the Regge limit of the ℓ -loop $\mathcal{N} = 8$ supergravity four-point amplitude. At one and two loops, this equality follows from the exact relations presented in ref. [23]. Moreover, this relation is consistent with the leading IR divergences of the amplitudes for all ℓ . We therefore conjecture that this SYM-supergravity relation holds to all loop orders in the Regge limit.

This paper is structured as follows. We review the group theory constraints on four-gluon amplitudes in section 2, and the Regge limits of the tree-level and one-loop $\mathcal{N} = 4$ SYM amplitudes in section 3. In section 4, we develop a basis of color factors C_{ik} suitable for the Regge limit of the $\mathcal{N} = 4$ SYM amplitude at any loop order. We then obtain expressions (through three loops) for the coefficients $B_{ik}^{(\ell)}$ in this basis as linear combinations of color-ordered amplitudes (and for all loops for $B_{\ell\ell}^{(\ell)}$). In section 5, after reviewing the structure of (the Regge limit of) the infrared divergences of the $\mathcal{N} = 4$ SYM amplitude through three loops, we compute the infrared-finite hard function using the results of ref. [25]. In section 6, we review the Regge limit of the $\mathcal{N} = 8$ supergravity four-graviton amplitude, and conjecture its equivalence to $B_{\ell\ell}^{(\ell)}$. In section 7, we offer some concluding remarks.

2 Group-theory constraints on four-point amplitudes

In this section, we briefly review the group-theory constraints on the ℓ -loop color-ordered four-gluon amplitudes in an $SU(N)$ gauge theory that were derived in ref. [80]. These will play an important role in the following sections when we define a new basis of color factors for four-gluon amplitudes in the Regge limit.

The color-ordered amplitudes of a gauge theory are the coefficients of the full amplitude in a basis using traces of generators T^a in the fundamental representation of the gauge group.² Color-ordered amplitudes have the advantage of being individually gauge-invariant. Four-gluon amplitudes of $SU(N)$ gauge theories can be expressed in terms of a six-dimensional basis $c_{[\lambda]}$ of single and double traces [78]

$$\begin{aligned} c_{[1]} &= \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \text{Tr}(T^{a_1} T^{a_4} T^{a_3} T^{a_2}), & c_{[4]} &= \text{Tr}(T^{a_1} T^{a_3}) \text{Tr}(T^{a_2} T^{a_4}), \\ c_{[2]} &= \text{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) + \text{Tr}(T^{a_1} T^{a_3} T^{a_4} T^{a_2}), & c_{[5]} &= \text{Tr}(T^{a_1} T^{a_4}) \text{Tr}(T^{a_2} T^{a_3}), \\ c_{[3]} &= \text{Tr}(T^{a_1} T^{a_4} T^{a_2} T^{a_3}) + \text{Tr}(T^{a_1} T^{a_3} T^{a_2} T^{a_4}), & c_{[6]} &= \text{Tr}(T^{a_1} T^{a_2}) \text{Tr}(T^{a_3} T^{a_4}). \end{aligned} \quad (2.1)$$

All other possible trace terms vanish in $SU(N)$ since $\text{Tr}(T^a) = 0$. The ℓ -loop color-ordered amplitudes can be further decomposed [2] in powers of N as

$$\mathcal{A}^{(\ell)} = \sum_{\lambda=1}^3 \left(\sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} N^{\ell-2k} A_{\lambda}^{(\ell,2k)} \right) c_{[\lambda]} + \sum_{\lambda=4}^6 \left(\sum_{k=0}^{\lfloor \frac{\ell-1}{2} \rfloor} N^{\ell-2k-1} A_{\lambda}^{(\ell,2k+1)} \right) c_{[\lambda]} \quad (2.2)$$

where $A_{\lambda}^{(\ell,0)}$ are leading-order-in- N (planar) amplitudes, and $A_{\lambda}^{(\ell,k)}$, $k = 1, \dots, \ell$, are subleading-order, yielding $(3\ell + 3)$ color-ordered amplitudes at ℓ loops. The $1/N$ expansion

²Our conventions are $\text{Tr}(T^a T^b) = \delta^{ab}$ so that $[T^a, T^b] = i\sqrt{2} f^{abc} T^c$ and $f^{abc} = (-i/\sqrt{2}) \text{Tr}([T^a, T^b] T^c)$.

in eq. (2.2) suggests an enlargement of the trace basis to an extended $(3\ell + 3)$ -dimensional trace basis $t_\lambda^{(\ell)}$, defined by

$$\begin{aligned} t_{1+6k}^{(\ell)} &= N^{\ell-2k} c_{[1]}, & t_{4+6k}^{(\ell)} &= N^{\ell-2k-1} c_{[4]}, \\ t_{2+6k}^{(\ell)} &= N^{\ell-2k} c_{[2]}, & t_{5+6k}^{(\ell)} &= N^{\ell-2k-1} c_{[5]}, \\ t_{3+6k}^{(\ell)} &= N^{\ell-2k} c_{[3]}, & t_{6+6k}^{(\ell)} &= N^{\ell-2k-1} c_{[6]}, \end{aligned} \tag{2.3}$$

in terms of which eq. (2.2) becomes

$$\mathcal{A}^{(\ell)} = \sum_{\lambda=1}^{3\ell+3} A_\lambda^{(\ell)} t_\lambda^{(\ell)}, \quad \text{where} \quad A_{\lambda+6k}^{(\ell)} = \begin{cases} A_\lambda^{(\ell,2k)}, & \lambda = 1, 2, 3, \\ A_\lambda^{(\ell,2k+1)}, & \lambda = 4, 5, 6. \end{cases} \tag{2.4}$$

The $(3\ell + 3)$ components $A_\lambda^{(\ell)}$ are not independent but are related by various group-theory constraints. These constraints can be conveniently expressed [80] in terms of a set of null vectors in the space spanned by $t_\lambda^{(\ell)}$. Each ℓ -loop null vector $r_\lambda^{(\ell)}$ gives rise to a linear constraint on the components $A_\lambda^{(\ell)}$, namely

$$0 = \sum_{\lambda=1}^{3\ell+3} r_\lambda^{(\ell)} A_\lambda^{(\ell)}. \tag{2.5}$$

At tree level, the single null vector

$$r^{(0)} = (1, 1, 1) \tag{2.6}$$

corresponds to the constraint [1, 77]

$$0 = A_1^{(0)} + A_2^{(0)} + A_3^{(0)}. \tag{2.7}$$

At one loop, there are three null vectors

$$r^{(1)} = \begin{cases} (6, 6, 6, -1, -1, -1), \\ (0, 0, 0, 1, -2, 1), \\ (0, 0, 0, 1, 0, -1). \end{cases} \tag{2.8}$$

The corresponding constraints imply that the three one-loop subleading-color amplitudes are equal and proportional to the sum of leading-color amplitudes [78]

$$A_4^{(1)} = A_5^{(1)} = A_6^{(1)} = 2(A_1^{(1)} + A_2^{(1)} + A_3^{(1)}). \tag{2.9}$$

There are exactly four null vectors for every loop level two and above [80]. The four null vectors at two loops are

$$r^{(2)} = \begin{cases} (6, 6, 6, -1, -1, -1, 0, 0, 0), \\ (0, 0, 0, 1, -2, 1, 1, -2, 1), \\ (0, 0, 0, 1, 0, -1, 1, 0, -1), \\ (0, 0, 0, 0, 0, 0, 1, 1, 1), \end{cases} \tag{2.10}$$

which give rise to four two-loop group-theory relations [79]. Consequently, the two-loop amplitude may be written in terms of five independent components $A_1^{(2)}, A_2^{(2)}, A_3^{(2)}, A_7^{(2)}, A_9^{(2)}$, with the four remaining color-ordered amplitudes given by

$$\begin{aligned} A_4^{(2)} &= 2 \left(A_1^{(2)} + A_2^{(2)} + A_3^{(2)} \right) - A_7^{(2)}, \\ A_5^{(2)} &= 2 \left(A_1^{(2)} + A_2^{(2)} + A_3^{(2)} \right) + A_7^{(2)} + A_9^{(2)}, \\ A_6^{(2)} &= 2 \left(A_1^{(2)} + A_2^{(2)} + A_3^{(2)} \right) - A_9^{(2)}, \\ A_8^{(2)} &= -A_7^{(2)} - A_9^{(2)}. \end{aligned} \tag{2.11}$$

The four null vectors at three loops are

$$r^{(3)} = \begin{cases} (6, 6, 6, -1, -1, -1, 2, 2, 2, 0, 0, 0), \\ (0, 0, 0, 0, 0, 0, 6, 6, 6, -1, -1, -1), \\ (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -2, 1), \\ (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1). \end{cases} \tag{2.12}$$

As a result, the three-loop amplitude can be written in terms of eight independent components $A_1^{(3)}, A_2^{(3)}, A_3^{(3)}, A_4^{(3)}, A_6^{(3)}, A_7^{(3)}, A_8^{(3)}, A_9^{(3)}$, with the four remaining color-ordered amplitudes given by

$$\begin{aligned} A_5^{(3)} &= 6 \left(A_1^{(3)} + A_2^{(3)} + A_3^{(3)} \right) - A_4^{(3)} - A_6^{(3)} + 2 \left(A_7^{(3)} + A_8^{(3)} + A_9^{(3)} \right), \\ A_{10}^{(3)} &= 2 \left(A_7^{(3)} + A_8^{(3)} + A_9^{(3)} \right), \\ A_{11}^{(3)} &= 2 \left(A_7^{(3)} + A_8^{(3)} + A_9^{(3)} \right), \\ A_{12}^{(3)} &= 2 \left(A_7^{(3)} + A_8^{(3)} + A_9^{(3)} \right). \end{aligned} \tag{2.13}$$

For all even loop levels beyond two ($\ell = 2m + 2$), the four null vectors are given by prepending $6m$ zeros to each of the two-loop null vectors (2.10), which give rise to the constraints

$$\begin{aligned} A_{3\ell-2}^{(\ell)} &= 2 \left(A_{3\ell-5}^{(\ell)} + A_{3\ell-4}^{(\ell)} + A_{3\ell-3}^{(\ell)} \right) - A_{3\ell+1}^{(\ell)}, \\ A_{3\ell-1}^{(\ell)} &= 2 \left(A_{3\ell-5}^{(\ell)} + A_{3\ell-4}^{(\ell)} + A_{3\ell-3}^{(\ell)} \right) + A_{3\ell+1}^{(\ell)} + A_{3\ell+3}^{(\ell)}, \\ A_{3\ell}^{(\ell)} &= 2 \left(A_{3\ell-5}^{(\ell)} + A_{3\ell-4}^{(\ell)} + A_{3\ell-3}^{(\ell)} \right) - A_{3\ell+3}^{(\ell)}, & \text{for even } \ell \geq 2 \\ A_{3\ell+2}^{(\ell)} &= -A_{3\ell+1}^{(\ell)} - A_{3\ell+3}^{(\ell)}. \end{aligned} \tag{2.14}$$

For all odd loop levels beyond three ($\ell = 2m + 3$), the four null vectors are given by prepending $6m$ zeros to each of the three-loop null vectors (2.12) giving rise to the constraints

$$\begin{aligned} A_{3\ell-4}^{(\ell)} &= 6 \left(A_{3\ell-8}^{(\ell)} + A_{3\ell-7}^{(\ell)} + A_{3\ell-6}^{(\ell)} \right) - A_{3\ell-5}^{(\ell)} - A_{3\ell-3}^{(\ell)} + 2 \left(A_{3\ell-2}^{(\ell)} + A_{3\ell-1}^{(\ell)} + A_{3\ell}^{(\ell)} \right), \\ A_{3\ell+1}^{(\ell)} &= 2 \left(A_{3\ell-2}^{(\ell)} + A_{3\ell-1}^{(\ell)} + A_{3\ell}^{(\ell)} \right), \\ A_{3\ell+2}^{(\ell)} &= 2 \left(A_{3\ell-2}^{(\ell)} + A_{3\ell-1}^{(\ell)} + A_{3\ell}^{(\ell)} \right), & \text{for odd } \ell \geq 3 \\ A_{3\ell+3}^{(\ell)} &= 2 \left(A_{3\ell-2}^{(\ell)} + A_{3\ell-1}^{(\ell)} + A_{3\ell}^{(\ell)} \right). \end{aligned} \tag{2.15}$$

Using eq. (2.4), these constraints can be rewritten in terms of the color-ordered amplitudes $A_\lambda^{(\ell,k)}$, in which case it becomes evident that they involve only the two or three most-subleading-color amplitudes (i.e., subleading in the $1/N$ expansion).

3 Regge limit of the $\mathcal{N} = 4$ four-point amplitude

In this section, we briefly review the Regge limit of the $\mathcal{N} = 4$ four-point amplitude at tree and one-loop level to establish conventions and prepare for the generalization to higher loops in subsequent sections.

The tree-level four-gluon amplitude is

$$\mathcal{A}^{(0)} = \sum_{\lambda=1}^3 A_\lambda^{(0)} t_\lambda^{(0)} \tag{3.1}$$

with color-ordered amplitudes³

$$A_1^{(0)} = -g^2 \mu^{2\epsilon} \frac{4K}{st}, \quad A_2^{(0)} = -g^2 \mu^{2\epsilon} \frac{4K}{su}, \quad A_3^{(0)} = -g^2 \mu^{2\epsilon} \frac{4K}{tu}. \tag{3.2}$$

The polarization-dependent factor K is defined in eq. (7.4.42) of ref. [83] and the Mandelstam variables are

$$s = (k_1 + k_2)^2, \quad t = (k_1 + k_4)^2, \quad u = (k_1 + k_3)^2. \tag{3.3}$$

The components in eq. (3.2) manifestly obey eq. (2.7) by virtue of massless momentum conservation $s + t + u = 0$.

In the Regge limit $s \gg -t$ (so that $u \approx -s$), one has $A_2^{(0)} \ll A_1^{(0)}$ and $A_3^{(0)} \approx -A_1^{(0)}$ so that

$$\mathcal{A}^{(0)} \xrightarrow{s \gg -t} A_1^{(0)} C_{00} \tag{3.4}$$

where we define

$$C_{00} \equiv t_1^{(0)} - t_3^{(0)}. \tag{3.5}$$

It is straightforward (using the conventions of footnote 2) to show that C_{00} corresponds to the tree-level t -channel color factor

$$C_{00} = \tilde{f}^{a_4 a_1 e} \tilde{f}^{a_2 a_3 e} \tag{3.6}$$

where $\tilde{f}^{abc} \equiv i\sqrt{2}f^{abc}$.

We digress slightly to discuss the behavior of the tree-level amplitude under crossing symmetry which exchanges external legs 2 and 3. From their definitions, we can see that the trace basis color factors (2.3) obey

$$t_{1+3j}^{(\ell)} \leftrightarrow t_{3+3j}^{(\ell)}, \quad t_{2+3j}^{(\ell)} \rightarrow t_{2+3j}^{(\ell)} \tag{3.7}$$

³Anticipating the need to dimensionally regularize loop amplitudes, we take the spacetime dimension as $D = 4 - 2\epsilon$. Choosing g to be a dimensionless parameter, the coupling constant in D dimensions is $g_D = g\mu^\epsilon$, where μ is a renormalization scale.

under $a_2 \leftrightarrow a_3$ and thus the color factor (3.5) is odd: $C_{00} \rightarrow -C_{00}$. In the Regge limit, $u \approx -s$, so the Mandelstam variables transform under $k_2 \leftrightarrow k_3$ as

$$s \rightarrow -s, \quad t \rightarrow t \quad (3.8)$$

while the factor K is invariant under all permutations of external legs. Thus the tree-level color-ordered amplitude $A_1^{(0)}$ is also odd: $A_1^{(0)} \rightarrow -A_1^{(0)}$. The full tree-level amplitude (3.4), being the product of two odd objects, is thus invariant, as expected from Bose symmetry.

Next, we turn to the one-loop $\mathcal{N} = 4$ SYM four-gluon amplitude [3]

$$\mathcal{A}^{(1)} = -ig^4 \mu^{2\epsilon} (4K) \left[\mathcal{I}^{(1)}(s, t) c_{1234}^{(1)} + \mathcal{I}^{(1)}(u, s) c_{1243}^{(1)} + \mathcal{I}^{(1)}(t, u) c_{1324}^{(1)} \right] \quad (3.9)$$

where [84]

$$\begin{aligned} \mathcal{I}^{(1)}(s, t) &= \mu^{2\epsilon} \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2 (p - k_1)^2 (p - k_1 - k_2)^2 (p + k_4)^2} \\ &= \frac{i}{8\pi^2} \frac{(4\pi e^{-\gamma})^\epsilon r(\epsilon)}{st\epsilon^2} \left[\left(\frac{\mu^2}{-s} \right)^\epsilon F\left(\epsilon, 1 + \frac{s}{t}\right) + \left(\frac{\mu^2}{-t} \right)^\epsilon F\left(\epsilon, 1 + \frac{t}{s}\right) \right] \end{aligned} \quad (3.10)$$

with

$$\begin{aligned} F(\epsilon, z) &\equiv {}_2F_1(1, -\epsilon, 1 - \epsilon, z), \\ r(\epsilon) &\equiv e^{\gamma\epsilon} \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} = 1 - \frac{1}{2}\zeta_2\epsilon^2 - \frac{7}{3}\zeta_3\epsilon^3 - \frac{47}{16}\zeta_4\epsilon^4 + \dots \end{aligned} \quad (3.11)$$

The one-loop color factors $c_{1234}^{(1)} \equiv \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{ca_3d} \tilde{f}^{da_4e}$, etc. can be written in the trace basis as

$$\begin{aligned} c_{1234}^{(1)} &= t_1^{(1)} + 2 \left(t_4^{(1)} + t_5^{(1)} + t_6^{(1)} \right), \\ c_{1243}^{(1)} &= t_2^{(1)} + 2 \left(t_4^{(1)} + t_5^{(1)} + t_6^{(1)} \right), \\ c_{1324}^{(1)} &= t_3^{(1)} + 2 \left(t_4^{(1)} + t_5^{(1)} + t_6^{(1)} \right). \end{aligned} \quad (3.12)$$

Inserting these into eq. (3.9), we obtain one-loop color-ordered amplitudes

$$\begin{aligned} A_1^{(1)} &= A_1^{(0)} \tilde{a} \frac{1}{\epsilon^2} \left[- \left(e^{i\pi x} \right)^\epsilon F\left(\epsilon, 1 - \frac{1}{x}\right) - F(\epsilon, 1 - x) \right], \\ A_2^{(1)} &= A_1^{(0)} \tilde{a} \frac{1}{\epsilon^2} \left(\frac{x}{1-x} \right) \left[- \left(e^{i\pi x} \right)^\epsilon F\left(\epsilon, \frac{-x}{1-x}\right) - \left(\frac{x}{1-x} \right)^\epsilon F(\epsilon, x) \right], \\ A_3^{(1)} &= A_1^{(0)} \tilde{a} \frac{1}{\epsilon^2} \left(\frac{1}{1-x} \right) \left[F\left(\epsilon, \frac{1}{1-x}\right) + \left(\frac{x}{1-x} \right)^\epsilon F\left(\epsilon, \frac{1}{x}\right) \right], \\ A_4^{(1)} &= A_5^{(1)} = A_6^{(1)} = 2 \left(A_1^{(1)} + A_2^{(1)} + A_3^{(1)} \right), \end{aligned} \quad (3.13)$$

where we define the dimensionless parameters

$$\begin{aligned} \tilde{a} &\equiv \frac{g^2}{8\pi^2} \frac{\Gamma^2(1 - \epsilon)\Gamma(1 + \epsilon)}{\Gamma(1 - 2\epsilon)} \left(\frac{4\pi\mu^2}{-t} \right)^\epsilon, \\ x &\equiv \frac{-t}{s}. \end{aligned} \quad (3.14)$$

We analytically continue to positive s using $-s = e^{-i\pi}s$, so that $(-t)/(-s) = e^{i\pi}x$.

To take the Regge limit, one expands the hypergeometric functions in x , keeping only the $\mathcal{O}(x^0)$ term

$$\begin{aligned}
 A_1^{(1)} &= A_1^{(0)}\tilde{a} \left[-\frac{2}{\epsilon^2} - \frac{\log x + i\pi}{\epsilon} + f_1(\epsilon) \right] + \mathcal{O}(x), \\
 A_2^{(1)} &= \mathcal{O}(x), \\
 A_3^{(1)} &= A_1^{(0)}\tilde{a} \left[\frac{2}{\epsilon^2} + \frac{\log x}{\epsilon} - f_1(\epsilon) \right] + \mathcal{O}(x), \\
 A_4^{(1)} &= A_5^{(1)} = A_6^{(1)} = A_1^{(0)}\tilde{a} \left[\frac{-2\pi i}{\epsilon} \right] + \mathcal{O}(x)
 \end{aligned}
 \tag{3.15}$$

where

$$\begin{aligned}
 f_1(\epsilon) &\equiv \frac{2}{\epsilon^2} - \psi(-\epsilon) - \pi \cot(\pi\epsilon) - \gamma \\
 &= \sum_{m=0}^{\infty} [2 + (-1)^m] \zeta_{m+2} \epsilon^m.
 \end{aligned}
 \tag{3.16}$$

Two- and higher-loop amplitudes for $\mathcal{N} = 4$ four-gluon amplitudes are known in terms of linear combinations of planar and nonplanar integrals [2–8]. The two-loop planar integrals were evaluated in Laurent expansions through $\mathcal{O}(\epsilon^2)$, and the three-loop planar integrals through $\mathcal{O}(\epsilon^0)$, in ref. [14]. More recently, the Laurent expansions of the corresponding nonplanar integrals have been evaluated to the same accuracy [19–22]. Using these, the color-ordered four-point amplitudes through three loops have been evaluated in ref. [25] with results given in ancillary files. We have used these results to derive the Regge limits of these amplitudes. As at tree level and one loop, the color-ordered amplitude $A_2^{(\ell)}$ is suppressed relative to the other color-ordered amplitudes, and this holds to all orders.⁴ Once we have developed a suitable basis of higher-loop color factors in section 4, and discussed the infrared divergent structure of the higher-loop amplitudes in section 5, we will be able to present our results in a compact way.

4 Regge basis of color factors

As noted in section 2, the ℓ -loop color-ordered amplitudes $A_\lambda^{(\ell)}$, coefficients in the extended trace basis $t_\lambda^{(\ell)}$ with $\lambda = 1, \dots, 3\ell + 3$, are not independent due to group-theory constraints. Moreover, $A_2^{(\ell)}$ is suppressed in the Regge limit relative to the other color-ordered amplitudes, so taking into account the group-theory constraints, there remain *in the Regge limit* one independent amplitude at tree level, two independent amplitudes at one loop, and $3\ell - 2$ independent amplitudes for $\ell \geq 2$.

In this section, we develop an alternative basis C_{ik} of color factors that will be useful for characterizing independent amplitudes in the Regge limit. In particular, the color factors C_{ik} are chosen to have definite signature under the exchange of external legs 2 and 3. We

⁴This is because $A_2^{(\ell)}$ is leading in the large- N (planar) limit, and the BDS ansatz [14] expresses the all-orders planar amplitude in terms of the tree-level and one-loop amplitude.

then express the coefficients $B_{ik}^{(\ell)}$ of the ℓ -loop amplitude in this basis as linear combinations of the color-ordered amplitudes $A_\lambda^{(\ell)}$.

First, let's count the number of independent even and odd signature amplitudes. As noted in section 3, the exchange of legs 2 and 3 acts on the $3\ell + 3$ elements of the extended trace basis (2.3) according to eq. (3.7), so one can form linear combinations with $2\ell + 2$ even elements $t_{1+3j}^{(\ell)} + t_{3+3j}^{(\ell)}$ and $t_{2+3j}^{(\ell)}$, and $\ell + 1$ odd elements $t_{1+3j}^{(\ell)} - t_{3+3j}^{(\ell)}$. The null vectors that correspond to group-theory constraints can also be chosen to have definite signature. The tree-level null vector (2.6) is even, there are two even and one odd one-loop null vectors (2.8), and for two loops and above, there are three even and one odd null vectors, cf. eqs. (2.10) and (2.12). In the Regge limit, the coefficient of $t_2^{(\ell)}$ is suppressed, eliminating one more even element. The upshot is that there remain *in the Regge limit* one odd independent amplitude at tree level, one odd and one even independent amplitude at one loop, and ℓ odd and $2\ell - 2$ even independent amplitudes for $\ell \geq 2$. Our choice of basis must reflect this.

The Regge basis of color factors C_{ik} will be built from operators acting on the odd tree-level color factor C_{00} given by eq. (3.5). In the color-space formalism introduced in refs. [15, 85, 86], the color operator \mathbf{T}_i acts on a color factor by inserting a generator in the adjoint representation of $SU(N)$ on the i th external leg, specifically $(\mathbf{T}_i^a)^{bc} = \frac{1}{\sqrt{2}} \tilde{f}^{bac}$, while acting as the identity on all the other external legs. Color conservation implies that

$$\sum_{i=1}^4 \mathbf{T}_i = 0 \tag{4.1}$$

when acting on a four-point amplitude [15].

Applying the operator $\mathbf{T}_i \cdot \mathbf{T}_j \equiv \sum_a \mathbf{T}_i^a \mathbf{T}_j^a$ to a given color factor acts to attach a rung between external legs i and j . For example, the operators $\mathbf{T}_1 \cdot \mathbf{T}_2$ and $\mathbf{T}_1 \cdot \mathbf{T}_3$ convert the tree-level t -channel color factor C_{00} to a one-loop box and crossed box color factor, respectively, but C_{00} is an eigenstate of $\mathbf{T}_1 \cdot \mathbf{T}_4$, since $\tilde{f}^{dae} \tilde{f}^{ebg} \tilde{f}^{gcd} = N \tilde{f}^{abc}$. More precisely,

$$\mathbf{T}_1 \cdot \mathbf{T}_2 C_{00} = -\frac{1}{2} c_{1234}^{(1)}, \quad \mathbf{T}_1 \cdot \mathbf{T}_3 C_{00} = \frac{1}{2} c_{1342}^{(1)}, \quad \mathbf{T}_1 \cdot \mathbf{T}_4 C_{00} = -\frac{1}{2} N C_{00}. \tag{4.2}$$

Also

$$(\mathbf{T}_i^2)^{bc} = \frac{1}{2} \tilde{f}^{bad} \tilde{f}^{dac} = N \delta^{bc} \tag{4.3}$$

for any i , so that eq. (4.1) implies

$$\mathbf{T}_3 \cdot \mathbf{T}_4 = \mathbf{T}_1 \cdot \mathbf{T}_2, \quad \mathbf{T}_2 \cdot \mathbf{T}_4 = \mathbf{T}_1 \cdot \mathbf{T}_3, \quad \mathbf{T}_2 \cdot \mathbf{T}_3 = \mathbf{T}_1 \cdot \mathbf{T}_4. \tag{4.4}$$

Next, following refs. [45, 46, 87], we define color operators associated with color flow in each channel

$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2, \quad \mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_3, \quad \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_4 \tag{4.5}$$

with color conservation (4.1) implying $\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = 4N$. Finally we define [50]

$$\mathbf{T}_{s-u}^2 \equiv \frac{1}{2} (\mathbf{T}_s^2 - \mathbf{T}_u^2). \tag{4.6}$$

A key observation is that the operators

$$\begin{aligned}\mathbf{T}_t^2 &= 2N + 2\mathbf{T}_1 \cdot \mathbf{T}_4 = -2(\mathbf{T}_1 \cdot \mathbf{T}_2 + \mathbf{T}_1 \cdot \mathbf{T}_3), \\ \mathbf{T}_{s-u}^2 &= \mathbf{T}_1 \cdot \mathbf{T}_2 - \mathbf{T}_1 \cdot \mathbf{T}_3\end{aligned}\tag{4.7}$$

are even and odd, respectively, under the exchange of external legs 2 and 3 (which implies $s \leftrightarrow u$ and $t \rightarrow t$).

4.1 One-loop basis

At one loop, we have one independent odd amplitude and one independent even amplitude. Recalling that both C_{00} and \mathbf{T}_{s-u}^2 are odd under $2 \leftrightarrow 3$, and defining

$$C_{11} \equiv \mathbf{T}_{s-u}^2 C_{00}\tag{4.8}$$

we form a one-loop basis of color factors $\{NC_{00}, C_{11}\}$ whose elements have signature $\{-, +\}$. Their components in the one-loop trace basis $t_\lambda^{(1)}$

$$\begin{aligned}NC_{00} &= (1, 0, -1, 0, 0, 0), \\ C_{11} &= (-\frac{1}{2}, 0, -\frac{1}{2}, -2, -2, -2),\end{aligned}\tag{4.9}$$

can be obtained by using eqs. (3.12) and (4.2). Expressing the amplitude in this one-loop basis

$$\mathcal{A}^{(1)} = A_1^{(0)} \tilde{a} \left[B_{00}^{(1)} NC_{00} + B_{11}^{(1)} C_{11} \right]\tag{4.10}$$

we obtain

$$\begin{aligned}A_1^{(0)} \tilde{a} B_{00}^{(1)} &= \frac{1}{2} \left(A_1^{(1)} - A_3^{(1)} \right), \\ A_1^{(0)} \tilde{a} B_{11}^{(1)} &= - \left(A_1^{(1)} + A_3^{(1)} \right).\end{aligned}\tag{4.11}$$

The coefficients $B_{00}^{(1)}$ and $B_{11}^{(1)}$ inherit the same signature under crossing symmetry (which takes $A_1^{(1)} \leftrightarrow A_3^{(1)}$) as their associated color factors, leaving the whole amplitude Bose symmetric.

Using eq. (3.15) in eq. (4.11), we obtain

$$\begin{aligned}B_{00}^{(1)} &= -\frac{2}{\epsilon^2} + \frac{L}{\epsilon} + f_1(\epsilon), \\ B_{11}^{(1)} &= \frac{i\pi}{\epsilon}\end{aligned}\tag{4.12}$$

where we define [50]

$$L \equiv -\log x - \frac{1}{2}i\pi.\tag{4.13}$$

Note that L is the Regge limit of the even-signature combination of logarithms

$$-\frac{1}{2} \left[\log \left(\frac{-t}{-s} \right) + \log \left(\frac{-t}{-u} \right) \right] = -\frac{1}{2} \left[\log \left(e^{i\pi} x \right) + \log \left(\frac{x}{1-x} \right) \right] \xrightarrow{x \rightarrow 0} L.$$

As noted in ref. [50], when expressed in terms of the natural variable L , the coefficients of odd-signature color factors are real, whereas the coefficients of even-signature color factors are imaginary. We will see this at higher loops as well.

4.2 Two-loop basis

At two loops, we have two independent odd amplitudes and two independent even amplitudes. Thus, to the previous two color factors, we add two more

$$\begin{aligned} C_{21} &= [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] C_{00}, \\ C_{22} &= (\mathbf{T}_{s-u}^2)^2 C_{00}, \end{aligned} \tag{4.14}$$

to form a two-loop basis of color factors $\{N^2 C_{00}, NC_{11}, C_{21}, C_{22}\}$ whose elements have signature $\{-, +, +, -\}$. In order to obtain explicit expressions for these color factors, it is convenient to express the operators \mathbf{T}_t^2 and \mathbf{T}_{s-u}^2 as matrices⁵

$$\mathbf{T}_t^2 = \begin{pmatrix} N & 0 & 0 & 0 & 0 & -1 \\ 0 & 2N & 0 & 1 & 0 & 1 \\ 0 & 0 & N & -1 & 0 & 0 \\ 0 & 2 & 0 & 2N & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2N \end{pmatrix}, \quad \mathbf{T}_{s-u}^2 = \begin{pmatrix} -\frac{N}{2} & 0 & 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{N}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 2 & N & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & -N \end{pmatrix} \tag{4.15}$$

in the original six-dimensional trace basis (2.1). Using these, we can obtain the components of the elements of the two-loop basis in the extended trace basis $t_\lambda^{(2)}$

$$\begin{aligned} N^2 C_{00} &= (1, 0, -1, 0, 0, 0, 0, 0, 0), \\ NC_{11} &= (-\frac{1}{2}, 0, -\frac{1}{2}, -2, -2, -2, 0, 0, 0), \\ C_{21} &= (0, 0, 0, -2, 4, -2, 2, -4, 2), \\ C_{22} &= (\frac{1}{4}, 0, -\frac{1}{4}, -3, 0, 3, 3, 0, -3). \end{aligned} \tag{4.16}$$

Expressing the amplitude in this two-loop basis

$$\mathcal{A}^{(2)} = A_1^{(0)} \tilde{a}^2 [B_{00}^{(2)} N^2 C_{00} + B_{11}^{(2)} NC_{11} + B_{21}^{(2)} C_{21} + B_{22}^{(2)} C_{22}] \tag{4.17}$$

we obtain⁶

$$\begin{aligned} A_1^{(0)} \tilde{a}^2 B_{00}^{(2)} &= \frac{1}{2} (A_1^{(2)} - A_3^{(2)}) - \frac{1}{24} (A_7^{(2)} - A_9^{(2)}), \\ A_1^{(0)} \tilde{a}^2 B_{11}^{(2)} &= - (A_1^{(2)} + A_3^{(2)}), \\ A_1^{(0)} \tilde{a}^2 B_{21}^{(2)} &= \frac{1}{4} (A_7^{(2)} + A_9^{(2)}), \\ A_1^{(0)} \tilde{a}^2 B_{22}^{(2)} &= \frac{1}{6} (A_7^{(2)} - A_9^{(2)}). \end{aligned} \tag{4.18}$$

⁵The matrices given here, which also appear in appendix C of ref. [50], are the transpose of those in ref. [80]. This is because here we take these matrices to act to the right on the color factors, whereas in ref. [80] the matrices acted to the left.

⁶It is easiest to obtain these results (and similarly at higher loops) by expressing $t_\lambda^{(2)}$ in terms of an enlarged basis consisting of $\{N^2 C_{00}, NC_{11}, C_{21}, C_{22}\}$ together with $t_2^{(2)}$ and the two-loop null vectors (2.10). The coefficients of $t_2^{(2)}$ and the null vectors will automatically vanish by virtue of $A_2^{(2)} = 0$ and the group theory constraints (2.11).

As we saw previously, the coefficients $B_{ik}^{(2)}$ inherit the same signature under crossing symmetry (which takes $A_1^{(2)} \leftrightarrow A_3^{(2)}$, and $A_7^{(2)} \leftrightarrow A_9^{(2)}$) as their associated color factors, leaving the whole amplitude Bose symmetric. We will use eq. (4.18) in section 5 to compute the Laurent expansions of $B_{ik}^{(2)}$ through $\mathcal{O}(\epsilon^2)$.

4.3 Three-loop basis

At three loops, we have three independent odd amplitudes and four independent even amplitudes. To the previous four color factors, we add three more

$$\begin{aligned} C_{31} &= [\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] C_{00}, \\ C_{32} &= [\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] C_{00}, \\ C_{33} &= (\mathbf{T}_{s-u})^3 C_{00}, \end{aligned} \tag{4.19}$$

to form a three-loop basis of color factors $\{N^3 C_{00}, N^2 C_{11}, N C_{21}, N C_{22}, C_{31}, C_{32}, C_{33}\}$ whose elements have signature $\{-, +, +, -, +, -, +\}$. The elements of the three-loop basis have components in the extended trace basis $t_\lambda^{(3)}$ given by

$$\begin{aligned} N^3 C_{00} &= (1, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ N^2 C_{11} &= (-\frac{1}{2}, 0, -\frac{1}{2}, -2, -2, -2, 0, 0, 0, 0, 0, 0), \\ N C_{21} &= (0, 0, 0, -2, 4, -2, 2, -4, 2, 0, 0, 0), \\ N C_{22} &= (\frac{1}{4}, 0, -\frac{1}{4}, -3, 0, 3, 3, 0, -3, 0, 0, 0), \\ C_{31} &= (0, 0, 0, -2, -4, -2, 2, -8, 2, -8, -8, -8), \\ C_{32} &= (0, 0, 0, -1, 0, 1, -5, 0, 5, 0, 0, 0), \\ C_{33} &= (-\frac{1}{8}, 0, -\frac{1}{8}, -\frac{7}{2}, -\frac{1}{2}, -\frac{7}{2}, -3, 3, -3, -6, -6, -6). \end{aligned} \tag{4.20}$$

Expressing the amplitude in this three-loop basis

$$\mathcal{A}^{(3)} = A_1^{(0)} \tilde{a}^3 \left[B_{00}^{(3)} N^3 C_{00} + B_{11}^{(3)} N^2 C_{11} + B_{21}^{(3)} N C_{21} + B_{22}^{(3)} N C_{22} + B_{31}^{(3)} C_{31} + B_{32}^{(3)} C_{32} + B_{33}^{(3)} C_{33} \right] \tag{4.21}$$

we may derive

$$\begin{aligned} A_1^{(0)} \tilde{a}^3 B_{00}^{(3)} &= \frac{1}{2} (A_1^{(3)} - A_3^{(3)}) + \frac{5}{144} (A_4^{(3)} - A_6^{(3)}) - \frac{1}{144} (A_7^{(3)} - A_9^{(3)}), \\ A_1^{(0)} \tilde{a}^3 B_{11}^{(3)} &= -\frac{13}{12} (A_1^{(3)} + A_3^{(3)}) + \frac{1}{48} (A_4^{(3)} + A_6^{(3)}) + \frac{1}{48} (A_7^{(3)} + A_9^{(3)}), \\ A_1^{(0)} \tilde{a}^3 B_{21}^{(3)} &= \frac{3}{4} (A_1^{(3)} + A_3^{(3)}) - \frac{3}{16} (A_4^{(3)} + A_6^{(3)}) + \frac{5}{16} (A_7^{(3)} + A_9^{(3)}) + \frac{1}{4} A_8^{(3)}, \\ A_1^{(0)} \tilde{a}^3 B_{22}^{(3)} &= -\frac{5}{36} (A_4^{(3)} - A_6^{(3)}) + \frac{1}{36} (A_7^{(3)} - A_9^{(3)}), \\ A_1^{(0)} \tilde{a}^3 B_{31}^{(3)} &= -\frac{1}{4} (A_1^{(3)} + A_3^{(3)}) + \frac{1}{16} (A_4^{(3)} + A_6^{(3)}) - \frac{3}{16} (A_7^{(3)} + A_9^{(3)}) - \frac{1}{4} A_8^{(3)}, \\ A_1^{(0)} \tilde{a}^3 B_{32}^{(3)} &= -\frac{1}{12} (A_4^{(3)} - A_6^{(3)}) - \frac{1}{12} (A_7^{(3)} - A_9^{(3)}), \\ A_1^{(0)} \tilde{a}^3 B_{33}^{(3)} &= \frac{1}{3} (A_1^{(3)} + A_3^{(3)}) - \frac{1}{12} (A_4^{(3)} + A_6^{(3)}) - \frac{1}{12} (A_7^{(3)} + A_9^{(3)}). \end{aligned} \tag{4.22}$$

We will use these expressions in section 5 to compute the Laurent expansions of $B_{ik}^{(3)}$ through $\mathcal{O}(\epsilon^0)$.

4.4 Higher-loop basis

For each additional loop, the number of independent odd amplitudes increases by one and the number of independent even amplitudes increases by two. Thus to obtain the ℓ -loop basis for independent amplitudes we add to the $(\ell - 1)$ -loop basis three additional color factors

$$\begin{aligned} C_{\ell 1} &= [\mathbf{T}_t^2, \dots, [\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] \dots] C_{00}, \\ C_{\ell, \ell-1} &= [\mathbf{T}_{s-u}^2, \dots, [\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] \dots] C_{00}, \\ C_{\ell \ell} &= (\mathbf{T}_{s-u})^\ell C_{00}, \end{aligned} \tag{4.23}$$

where each color factor contains exactly ℓ operators \mathbf{T}^2 . The operator $C_{\ell k}$ contains k factors of \mathbf{T}_{s-u}^2 and thus has odd signature if k is even, and even signature if k is odd. Thus $C_{\ell 1}$ has even signature, and $C_{\ell, \ell-1}$ and $C_{\ell \ell}$ have opposite signatures, as required. This choice of basis is motivated by the Regge limit of the structure of IR divergences as studied in refs. [46, 48–50].

The ℓ -loop amplitude can then be expressed in this basis as

$$\mathcal{A}^{(\ell)} = A_1^{(0)} \tilde{a}^\ell \sum_{i=0}^{\ell} \sum_k B_{ik}^{(\ell)} N^{\ell-i} C_{ik} \tag{4.24}$$

where the range of k is

$$k = \begin{cases} 0, & \text{when } i = 0, \\ 1, & \text{when } i = 1, \\ 1, 2, & \text{when } i = 2, \\ 1, i - 1, i, & \text{when } i \geq 3. \end{cases} \tag{4.25}$$

Using eq. (4.15), the ℓ -loop Regge color factors $C_{\ell k}$ can be expressed in terms of $t_\lambda^{(\ell)}$, and the coefficients $B_{ik}^{(\ell)}$ obtained as linear combinations of the ℓ -loop color-ordered amplitudes $A_\lambda^{(\ell)}$.

The expressions for $B_{ik}^{(\ell)}$ grow increasingly complicated at higher loops, but the general expression for one of them, $B_{\ell \ell}^{(\ell)}$, can be easily guessed from explicit results (obtained through $\ell = 9$). The expressions differ depending on whether ℓ is even or odd. In the former case,

$$\begin{aligned} A_1^{(0)} \tilde{a}^\ell B_{\ell \ell}^{(\ell)} &= \frac{1}{2 \cdot 3^{\ell/2}} \left(A_{3\ell+1}^{(\ell)} - A_{3\ell+3}^{(\ell)} \right) \\ &= \frac{1}{2 \cdot 3^{\ell/2}} \left(A_1^{(\ell, \ell)} - A_3^{(\ell, \ell)} \right), \quad \text{for even } \ell. \end{aligned} \tag{4.26}$$

In the latter case, we have

$$\begin{aligned} A_1^{(0)} \tilde{a} B_{11}^{(1)} &= - \left(A_1^{(1)} + A_3^{(1)} \right) \\ &= - \left(A_1^{(1,0)} + A_3^{(1,0)} \right) \end{aligned} \tag{4.27}$$

for $\ell = 1$, and for odd $\ell > 1$, we have

$$\begin{aligned}
 A_1^{(0)} \tilde{a}^\ell B_{\ell\ell}^{(\ell)} &= \frac{1}{3^{(\ell-1)/2}} \left[\left(A_{3\ell-8}^{(\ell)} + A_{3\ell-6}^{(\ell)} \right) - \frac{1}{4} \left(A_{3\ell-5}^{(\ell)} + A_{3\ell-3}^{(\ell)} + A_{3\ell-2}^{(\ell)} + A_{3\ell}^{(\ell)} \right) \right] \\
 &= \frac{1}{3^{(\ell-1)/2}} \left[\left(A_1^{(\ell, \ell-3)} + A_3^{(\ell, \ell-3)} \right) - \frac{1}{4} \left(A_4^{(\ell, \ell-2)} + A_6^{(\ell, \ell-2)} + A_1^{(\ell, \ell-1)} + A_3^{(\ell, \ell-1)} \right) \right],
 \end{aligned} \tag{4.28}$$

for odd $\ell > 1$

where we have expressed these in terms of color-ordered amplitudes in both the $(3\ell + 3)$ -dimensional extended trace basis (2.2) and also the original six-dimensional trace basis (2.4).

In refs. [23, 56], we showed that the leading IR divergence of the color-ordered amplitude $A^{(\ell, k)}$ is $1/\epsilon^{2\ell-k}$, with planar amplitudes $A_\lambda^{(\ell, 0)}$ having the most severe $1/\epsilon^{2\ell}$ divergences, and the most-subleading-color amplitudes $A_\lambda^{(\ell, \ell)}$ having at most a $1/\epsilon^\ell$ divergence. At the end of section 5, we will show that $B_{\ell\ell}^{(\ell)}$ also has at most a $1/\epsilon^\ell$ divergence. For even ℓ , this is manifest from eq. (4.26), where $B_{\ell\ell}^{(\ell)}$ is expressed in terms of color-ordered amplitudes that are most-subleading in the $1/N$ expansion. For odd ℓ , however, eq. (4.28) shows that this is not the case, so the $1/\epsilon^\ell$ behavior of $B_{\ell\ell}^{(\ell)}$ requires some intricate cancellations of the more severe IR divergences appearing in $A_\lambda^{(\ell, \ell-3)}$, $A_\lambda^{(\ell, \ell-2)}$, and $A_\lambda^{(\ell, \ell-1)}$.

5 IR-divergence structure of the $\mathcal{N} = 4$ SYM amplitude

In this section we first briefly review the known structure of infrared divergences of the $\mathcal{N} = 4$ SYM four-point amplitude through three loops [14–16, 26–35, 45–50], focusing in particular on the Regge limit. We then take the Regge limit of known results for the four-point amplitude at one, two, and three loops [25] to confirm the expected IR divergences and to extract the IR-finite part of the amplitude in this limit, writing the result in terms of the Regge basis of color factors introduced in section 4. Finally we compare these to results obtained via an effective Hamiltonian approach based on Balitsky-Fadin-Kuraev-Lipatov theory in refs. [49, 50].

The amplitude may be factored into jet, soft, and hard functions [16, 26, 27]

$$\mathcal{A} \left(\frac{s_{ij}}{\mu^2} \right) = J \left(\frac{Q^2}{\mu^2} \right) \mathbf{S} \left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2} \right) \mathcal{H} \left(\frac{s_{ij}}{Q^2}, \frac{Q^2}{\mu^2} \right) \tag{5.1}$$

where the factors J and \mathbf{S} characterize the long-distance IR-divergent behavior, and \mathcal{H} , which is IR-finite, characterizes the short-distance behavior. Here $s_{ij} = (k_i + k_j)^2$, μ is the renormalization scale, and Q is an arbitrary factorization scale that serves to separate the long- and short-distance behavior. Since we are interested in the Regge limit $s \gg -t$, we choose the factorization scale as $Q^2 = -t$ in this paper.

Because $\mathcal{N} = 4$ SYM theory is conformally invariant, the jet function may be explicitly evaluated as [14]

$$J \left(\frac{-t}{\mu^2} \right) = \exp \left[- \sum_{\ell=1}^{\infty} \tilde{a}^\ell N^\ell \left(\frac{\gamma^{(\ell)}}{2(\ell\epsilon)^2} + \frac{\mathcal{G}_0^{(\ell)}}{\ell\epsilon} \right) \right] \tag{5.2}$$

where $\gamma^{(\ell)}$ and $\mathcal{G}_0^{(\ell)}$ are the coefficients of the cusp and collinear anomalous dimensions

$$\gamma^{(\ell)} = \{4, -4\zeta_2, 22\zeta_4, \dots\}, \quad \mathcal{G}_0^{(\ell)} = \left\{0, -\zeta_3, 4\zeta_5 + \frac{10}{3}\zeta_2\zeta_3, \dots\right\}. \quad (5.3)$$

The soft function \mathbf{S} depends on the coefficients $\mathbf{\Gamma}^{(\ell)}$ of the anomalous dimension matrix [16]. The one-loop anomalous dimension matrix is [26, 27]

$$\begin{aligned} \mathbf{\Gamma}^{(1)} &= \frac{1}{N} \sum_{i=1}^4 \sum_{j \neq i}^4 \mathbf{T}_i \cdot \mathbf{T}_j \log \left(\frac{Q^2}{-s_{ij}} \right) \\ &= \frac{4}{N} \left[\log \left(\frac{-t}{e^{-i\pi} s} \right) \mathbf{T}_1 \cdot \mathbf{T}_2 + \log \left(\frac{-t}{-u} \right) \mathbf{T}_1 \cdot \mathbf{T}_3 \right] \end{aligned} \quad (5.4)$$

which simplifies in the Regge limit to

$$\begin{aligned} \mathbf{\Gamma}^{(1)} &= \frac{4}{N} \left[\log \left(e^{i\pi} x \right) \mathbf{T}_1 \cdot \mathbf{T}_2 + \log(x) \mathbf{T}_1 \cdot \mathbf{T}_3 \right] \\ &= \frac{2}{N} \left[L\mathbf{T}_t^2 + i\pi\mathbf{T}_{s-u}^2 \right] \end{aligned} \quad (5.5)$$

where in the last line we used eqs. (4.7) and (4.13). On the assumption that the matrices $\mathbf{\Gamma}^{(\ell)}$ commute with one another, one may explicitly evaluate \mathbf{S} for $\mathcal{N} = 4$ SYM theory as⁷ [56]

$$\mathbf{S} = \exp \left[\sum_{\ell=1}^{\infty} \tilde{a}^\ell N^\ell \frac{\mathbf{\Gamma}^{(\ell)}}{2\ell\epsilon} \right]. \quad (5.6)$$

Commutativity is guaranteed if we assume the anomalous dimension matrix is given by the dipole formula [29–34]

$$\mathbf{\Gamma}_{\text{dipole}}^{(\ell)} = \frac{1}{4} \gamma^{(\ell)} \mathbf{\Gamma}^{(1)} \quad (5.7)$$

which is valid through two loops [26, 27], but receives corrections at three [35] and four [49] loops. If the dipole formula were valid for all loops, then using eqs. (5.5) and (5.7) in eq. (5.6), one finds that the Regge limit of the soft function can be written in the compact form [45, 46]

$$\mathbf{S}_{\text{dipole}} = \exp \left[K \left(L\mathbf{T}_t^2 + i\pi\mathbf{T}_{s-u}^2 \right) \right] \quad (5.8)$$

where

$$K \equiv \sum_{\ell=1}^{\infty} \frac{N^{\ell-1} \gamma^{(\ell)}}{4\ell\epsilon} \tilde{a}^\ell = \frac{\tilde{a}}{\epsilon} \left(1 - \frac{1}{2} \zeta_2 N \tilde{a} + \frac{11}{6} \zeta_4 N^2 \tilde{a}^2 + \dots \right). \quad (5.9)$$

The three-loop correction to the dipole formula [35] persists in the Regge limit, and has the effect of modifying eq. (5.8) to [50]

$$\mathbf{S} = \exp \left[K \left(L\mathbf{T}_t^2 + i\pi\mathbf{T}_{s-u}^2 \right) + Q_\Delta^{(3)} \right] + \mathcal{O}(\tilde{a}^4) \quad (5.10)$$

⁷The expressions for eqs. (5.2) and (5.6) differ slightly from those in ref. [56] in that we are using \tilde{a} rather than $a = (g^2 N / 8\pi^2) (4\pi e^{-\gamma})^\epsilon$ as our loop expansion parameter. This only affects the form of the infrared-finite hard function \mathcal{H} .

where

$$\begin{aligned}
 Q_{\Delta}^{(3)} &= \frac{4}{3\epsilon} \tilde{a}^3 \Delta^{(3)} \\
 \Delta^{(3)} &= \frac{1}{4} i\pi (\zeta_3 L + 11\zeta_4) [\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] + \frac{1}{4} (\zeta_5 - 4\zeta_2\zeta_3) [\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] - \frac{1}{4} (\zeta_5 + 2\zeta_2\zeta_3) \mathbf{W} \\
 \mathbf{W} &= \frac{1}{2} \left\{ f^{abe} f^{cde} \left[\{\mathbf{T}_t^a, \mathbf{T}_t^d\} (\{\mathbf{T}_{s-u}^b, \mathbf{T}_{s-u}^c\} + \{\mathbf{T}_{s+u}^b, \mathbf{T}_{s+u}^c\}) \right. \right. \\
 &\quad \left. \left. + \{\mathbf{T}_{s-u}^a, \mathbf{T}_{s-u}^d\} \{\mathbf{T}_{s+u}^b, \mathbf{T}_{s+u}^c\} \right] - \frac{5}{8} N^2 \mathbf{T}_t^2 \right\}. \tag{5.11}
 \end{aligned}$$

Since the operators in the exponent of eq. (5.10) do not commute with one another, it is useful [46] to employ a variant of Campbell-Baker-Hausdorff known as the Zassenhaus formula

$$e^{K(X+Y)} = e^{KX} e^{KY} e^{-(1/2)K^2[X,Y]} e^{(1/6)K^3([X,[X,Y]]+2[Y,[X,Y]])} + \mathcal{O}(K^4) \tag{5.12}$$

to write

$$\begin{aligned}
 \mathbf{S} &= \exp \left[KL\mathbf{T}_t^2 \right] \exp \left[i\pi K\mathbf{T}_{s-u}^2 \right] \exp \left[-\frac{i\pi}{2} LK^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right] \\
 &\times \exp \left[\frac{i\pi}{6} K^3 L^2[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] - \frac{\pi^2}{3} K^3 L[\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] + \frac{4}{3\epsilon} \tilde{a}^3 \Delta^{(3)} \right] + \mathcal{O}(\tilde{a}^4) \tag{5.13}
 \end{aligned}$$

Putting all the pieces together, we write the Regge limit of the amplitude through $\mathcal{O}(\tilde{a}^3)$ as

$$\begin{aligned}
 \mathcal{A} &= \exp \left[-\sum_{\ell=1}^{\infty} \tilde{a}^{\ell} N^{\ell} \left(\frac{\gamma^{(\ell)}}{2(\ell\epsilon)^2} + \frac{\mathcal{G}_0^{(\ell)}}{\ell\epsilon} \right) \right] \exp \left[KL\mathbf{T}_t^2 \right] \exp \left[i\pi K\mathbf{T}_{s-u}^2 \right] \exp \left[-\frac{i\pi}{2} LK^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right] \\
 &\times \exp \left[\frac{i\pi}{6} K^3 L^2[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] - \frac{\pi^2}{3} K^3 L[\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] + \frac{4}{3\epsilon} \tilde{a}^3 \Delta^{(3)} \right] \mathcal{H} + \mathcal{O}(\tilde{a}^4) \tag{5.14}
 \end{aligned}$$

This equation exhibits all of the IR-divergent contributions to the amplitude through three loops, with the IR-finite part encoded in the hard function \mathcal{H} . To obtain \mathcal{H} , we need to compare eq. (5.14) with known expressions for the amplitude at one, two, and three loops. This we now proceed to do.

5.1 Reduced amplitude

Expanding eq. (5.14) in powers of the loop expansion parameter \tilde{a} , it is apparent that the first exponential term in eq. (5.14) is responsible for the most IR-divergent terms in the Laurent expansion at ℓ loops, starting with an $\mathcal{O}(1/\epsilon^{2\ell})$ term. It is also apparent that the second exponential term in eq. (5.14) is responsible for the leading log behavior, causing $\mathcal{A}^{(\ell)}$ to go as $\log^{\ell} |s/t|$ at ℓ loops, and leading to Reggeization [45, 46]. The amplitude, however, also has an intricate structure of subleading logarithms, to which the remaining terms in

eq. (5.14) contribute. To isolate this subleading logarithmic behavior, it is useful [49, 50] to define a reduced amplitude by factoring off the first two exponential terms:

$$\hat{\mathcal{A}} \equiv J^{-1} \exp \left[-KL\mathbf{T}_t^2 \right] \mathcal{A}. \quad (5.15)$$

The removal of J implies that the leading Laurent coefficient of $\hat{\mathcal{A}}$ at ℓ loops will be $1/\epsilon^\ell$ rather than $1/\epsilon^{2\ell}$. The removal of $e^{KL\mathbf{T}_t^2}$ implies that the leading logarithmic behavior of $\hat{\mathcal{A}}$ at ℓ loops will be $\log^{\ell-1} |s/t|$.

Note that the reduced amplitude (5.15) is almost, but not quite the same as, the reduced amplitude defined in refs. [49, 50], which multiplies \mathcal{A} by a factor $J^{-1} \exp[-\alpha_g L\mathbf{T}_t^2]$ rather than $J^{-1} \exp[-KL\mathbf{T}_t^2]$, where α_g is the Regge trajectory. The difference $\hat{\alpha}_g = \alpha_g - K$ vanishes at one-loop order (because we are expanding in \tilde{a} rather than a) and is IR-finite at two loops [50].

Using eq. (5.14) we may write the reduced amplitude (5.15) as

$$\begin{aligned} \hat{\mathcal{A}} = & \exp \left[i\pi K\mathbf{T}_{s-u}^2 \right] \exp \left[-\frac{1}{2}i\pi LK^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right] \\ & \times \exp \left[\frac{i\pi}{6}K^3L^2[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] - \frac{\pi^2}{3}K^3L[\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] + \frac{4}{3\epsilon}\tilde{a}^3\Delta^{(3)} \right] \mathcal{H} + \mathcal{O}(\tilde{a}^4). \end{aligned} \quad (5.16)$$

We then expand this in powers of \tilde{a} , using eq. (5.9), to obtain

$$\begin{aligned} \hat{\mathcal{A}} = & \mathcal{A}^{(0)} + \left[\tilde{a} \left(\frac{i\pi}{\epsilon} \mathbf{T}_{s-u}^2 \right) \mathcal{A}^{(0)} + \mathcal{H}^{(1)} \right] \\ & + \left[\tilde{a}^2 \left(-\frac{i\pi\zeta_2}{2\epsilon} N\mathbf{T}_{s-u}^2 - \frac{\pi^2}{2\epsilon^2} (\mathbf{T}_{s-u}^2)^2 - \frac{i\pi}{2\epsilon^2} L[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right) \mathcal{A}^{(0)} + \tilde{a} \left(\frac{i\pi}{\epsilon} \mathbf{T}_{s-u}^2 \right) \mathcal{H}^{(1)} + \mathcal{H}^{(2)} \right] \\ & + \left[\tilde{a}^3 \left(\frac{11i\pi\zeta_4}{6\epsilon} N^2\mathbf{T}_{s-u}^2 + \frac{\pi^2\zeta_2}{2\epsilon^2} N(\mathbf{T}_{s-u}^2)^2 - \frac{i\pi^3}{6\epsilon^3} (\mathbf{T}_{s-u}^2)^3 + \frac{i\pi\zeta_2}{2\epsilon^2} LN[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right. \right. \\ & \left. \left. + \frac{i\pi}{6\epsilon^3} L^2[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] - \frac{\pi^2}{3\epsilon^3} L[\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] + \frac{\pi^2}{2\epsilon^3} L\mathbf{T}_{s-u}^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] + \frac{4}{3\epsilon}\Delta^{(3)} \right) \mathcal{A}^{(0)} \right. \\ & \left. + \tilde{a}^2 \left(-\frac{i\pi\zeta_2}{2\epsilon} N\mathbf{T}_{s-u}^2 - \frac{\pi^2}{2\epsilon^2} (\mathbf{T}_{s-u}^2)^2 - \frac{i\pi}{2\epsilon^2} L[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \right) \mathcal{H}^{(1)} + \tilde{a} \left(\frac{i\pi}{\epsilon} \mathbf{T}_{s-u}^2 \right) \mathcal{H}^{(2)} + \mathcal{H}^{(3)} \right] \\ & + \mathcal{O}(\tilde{a}^4) \end{aligned} \quad (5.17)$$

where we denote the loop expansion of the hard function as

$$\mathcal{H} = \mathcal{A}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \mathcal{H}^{(3)} + \dots \quad (5.18)$$

We now consider the amplitude at each order in \tilde{a} .

5.2 One loop

The $\mathcal{O}(\tilde{a})$ term of eq. (5.17) is

$$\hat{\mathcal{A}}^{(1)} = \tilde{a} \left(\frac{i\pi}{\epsilon} \mathbf{T}_{s-u}^2 \right) \mathcal{A}^{(0)} + \mathcal{H}^{(1)}. \quad (5.19)$$

Recalling that $\mathcal{A}^{(0)} = A_1^{(0)} C_{00}$ and $C_{11} = \mathbf{T}_{s-u}^2 C_{00}$, this can be expressed in the Regge color factor basis as

$$\hat{\mathcal{A}}^{(1)} = A_1^{(0)} \tilde{a} \left[h_{00}^{(1)} N C_{00} + \left(\frac{i\pi}{\epsilon} + h_{11}^{(1)} \right) C_{11} \right] \quad (5.20)$$

where we define

$$\mathcal{H}^{(1)} = A_1^{(0)} \tilde{a} \left[h_{00}^{(1)} N C_{00} + h_{11}^{(1)} C_{11} \right]. \quad (5.21)$$

Using the exact (all orders in ϵ) one-loop amplitude from eqs. (4.10) and (4.12), one obtains the exact one-loop reduced amplitude

$$\hat{\mathcal{A}}^{(1)} = A_1^{(0)} \tilde{a} \left[f_1(\epsilon) N C_{00} + \frac{i\pi}{\epsilon} C_{11} \right]. \quad (5.22)$$

Comparing eqs. (5.20) and (5.22), we obtain the one-loop IR-finite contributions to all orders in ϵ :

$$\begin{aligned} h_{00}^{(1)} &= f_1(\epsilon) = \frac{1}{2}\pi^2 + \zeta_3\epsilon + \frac{1}{30}\pi^4\epsilon^2 + \zeta_5\epsilon^3 + \frac{1}{315}\pi^6\epsilon^4 + \dots, \\ h_{11}^{(1)} &= 0. \end{aligned} \quad (5.23)$$

5.3 Two loops

Next, we consider the $\mathcal{O}(\tilde{a}^2)$ term in eq. (5.17)

$$\hat{\mathcal{A}}^{(2)} = \tilde{a}^2 \left[i\pi \left(\frac{-\frac{1}{12}\pi^2 + h_{00}^{(1)}}{\epsilon} \right) N \mathbf{T}_{s-u}^2 + i\pi \left(-\frac{L}{2\epsilon^2} \right) [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] + \left(-\frac{\pi^2}{2\epsilon^2} \right) (\mathbf{T}_{s-u}^2)^2 \right] \mathcal{A}^{(0)} + \mathcal{H}^{(2)}. \quad (5.24)$$

All of the terms in this equation can be expressed in terms of the Regge color factor basis

$$\begin{aligned} \hat{\mathcal{A}}^{(2)} &= A_1^{(0)} \tilde{a}^2 \left[\hat{B}_{00}^{(2)} N^2 C_{00} + \hat{B}_{11}^{(2)} N C_{11} + \hat{B}_{21}^{(2)} C_{21} + \hat{B}_{22}^{(2)} C_{22} \right], \\ \mathcal{H}^{(2)} &= A_1^{(0)} \tilde{a}^2 \left[h_{00}^{(2)} N^2 C_{00} + h_{11}^{(2)} N C_{11} + h_{21}^{(2)} C_{21} + h_{22}^{(2)} C_{22} \right] \end{aligned} \quad (5.25)$$

where, using eq. (5.23), we have

$$\begin{aligned} \hat{B}_{00}^{(2)} &= h_{00}^{(2)}, \\ \hat{B}_{11}^{(2)} &= i\pi \left(\frac{\frac{5}{12}\pi^2}{\epsilon} + \zeta_3 + \frac{1}{30}\pi^4\epsilon + \zeta_5\epsilon^2 + \dots \right) + h_{11}^{(2)}, \\ \hat{B}_{21}^{(2)} &= i\pi \left(-\frac{L}{2\epsilon^2} \right) + h_{21}^{(2)}, \\ \hat{B}_{22}^{(2)} &= -\frac{\pi^2}{2\epsilon^2} + h_{22}^{(2)}. \end{aligned} \quad (5.26)$$

To determine the two-loop IR-finite contributions $h_{ik}^{(2)}$, we use the ancillary files of Henn and Mistlberger [25] to extract⁸ the Regge limit of the two-loop color-ordered amplitudes

⁸In ref. [25], s is taken to be negative. To convert their results to our conventions, we use the map $s_{HM} = u$, $t_{HM} = t$, $u_{HM} = s$.

$A_\lambda^{(2)}$ through $\mathcal{O}(\epsilon^2)$, then use eqs. (4.15) and (5.15) to derive the corresponding reduced amplitudes $\hat{A}_\lambda^{(2)}$, and finally use eq. (4.18) to determine $\hat{B}_{ik}^{(2)}$ to $\mathcal{O}(\epsilon^2)$. In doing so, we recover precisely the IR-divergent terms of eq. (5.26) together with the following IR-finite coefficients through $\mathcal{O}(\epsilon^2)$:

$$\begin{aligned}
 h_{00}^{(2)} &= -\frac{1}{2}\zeta_3 L + \frac{7}{360}\pi^4 - \left(\frac{1}{45}\pi^4 L + \frac{39}{2}\zeta_5 + \frac{5}{12}\pi^2\zeta_3 \right) \epsilon \\
 &\quad + \left(\frac{41}{2}\zeta_5 L + \frac{3}{2}\pi^2\zeta_3 L - \frac{47}{2}\zeta_3^2 - \frac{19}{168}\pi^6 \right) \epsilon^2 + \mathcal{O}(\epsilon^3), \\
 h_{11}^{(2)} &= i\pi \left[-\frac{1}{2}\zeta_3 - \frac{1}{45}\pi^4 \epsilon + \left(\frac{41}{2}\zeta_5 + \frac{3}{2}\pi^2\zeta_3 \right) \epsilon^2 \right] + \mathcal{O}(\epsilon^3), \\
 h_{21}^{(2)} &= i\pi \left[-3\zeta_3 - \left(9\zeta_3 L + \frac{13}{36}\pi^4 \right) \epsilon - \left(\frac{3}{20}\pi^4 L + 152\zeta_5 + \frac{28}{3}\pi^2\zeta_3 \right) \epsilon^2 \right] + \mathcal{O}(\epsilon^3), \\
 h_{22}^{(2)} &= 3\pi^2\zeta_3 \epsilon + \frac{1}{20}\pi^6 \epsilon^2 + \mathcal{O}(\epsilon^3).
 \end{aligned}
 \tag{5.27}$$

The coefficients of all the L -dependent terms of $h_{00}^{(2)}$ and $h_{21}^{(2)}$ are consistent⁹ with the NLL prediction (4.32) of ref. [50]. Also, $h_{22}^{(2)}$ is consistent with the NNLL prediction (4.33) of ref. [50], which in fact allows it to be computed to all orders in ϵ , viz.

$$h_{22}^{(2)} = \frac{\pi^2}{2\epsilon^2} \left[1 - \frac{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)\Gamma(1-3\epsilon)} \right].
 \tag{5.28}$$

5.4 Three loops

Finally we consider the $\mathcal{O}(\tilde{a}^3)$ term in eq. (5.17). Most of the terms can be immediately written in terms of the three-loop Regge color basis, with the exception of $\mathbf{T}_{s-u}^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]$ and also \mathbf{W} which appears in $\Delta^{(3)}$. Using eq. (4.15) and eq. (4.20), we may determine that

$$\mathbf{T}_{s-u}^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]C_{00} = -\frac{1}{12}N^3C_{00} + \frac{1}{3}N(\mathbf{T}_{s-u}^2)^2C_{00} + [\mathbf{T}_{s-u}^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]]C_{00}.
 \tag{5.29}$$

Using the matrix representation of \mathbf{W} given in eq. (C.8) of ref. [50], we find the components of $\mathbf{W}C_{00}$ in the extended three-loop trace basis

$$\mathbf{W}C_{00} = (0, 0, 0, -1, 0, 1, 13, 0, -13, 0, 0, 0).
 \tag{5.30}$$

Then using eq. (4.20) we can express it in terms of the three-loop Regge color factor basis

$$\mathbf{W}C_{00} = -\frac{1}{4}N^3C_{00} + N(\mathbf{T}_{s-u}^2)^2C_{00} - 2[\mathbf{T}_{s-u}^2[\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]]C_{00}.
 \tag{5.31}$$

⁹Because we are expanding in \tilde{a} rather than a , the values of the Regge trajectory coefficients $\hat{\alpha}_g^{(\ell)}$ differ from eqs. (D.1) and (D.2) of ref. [50]. In particular, $\hat{\alpha}_g^{(1)}$ vanishes in our case, as noted in that reference.

We thus obtain for the three-loop amplitude

$$\begin{aligned}
\hat{\mathcal{A}}^{(3)} = & \tilde{a}^3 \left[\left(-\frac{\pi^2}{24\epsilon^3} L - \frac{i\pi}{12\epsilon} h_{21}^{(2)} + \boxed{\frac{1}{12\epsilon} (\zeta_5 + 2\zeta_2 \zeta_3)} \right) N^3 \right. \\
& + \left(\frac{11i\pi\zeta_4}{6\epsilon} - \frac{i\pi\zeta_2}{2\epsilon} h_{00}^{(1)} + \frac{i\pi}{\epsilon} h_{00}^{(2)} \right) N^2 \mathbf{T}_{s-u}^2 + \left(\frac{i\pi\zeta_2}{2\epsilon^2} L - \frac{i\pi}{2\epsilon^2} L h_{00}^{(1)} \right) N [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2] \\
& + \left(\frac{\pi^2\zeta_2}{2\epsilon^2} - \frac{\pi^2}{2\epsilon^2} h_{00}^{(1)} + \frac{i\pi}{\epsilon} h_{11}^{(2)} + \frac{\pi^2}{6\epsilon^3} L + \frac{i\pi}{3\epsilon} h_{21}^{(2)} - \boxed{\frac{1}{3\epsilon} (\zeta_5 + 2\zeta_2 \zeta_3)} \right) N (\mathbf{T}_{s-u}^2)^2 \\
& + \left(\frac{i\pi}{6\epsilon^3} L^2 + \boxed{\frac{i\pi}{3\epsilon} (\zeta_3 L + 11\zeta_4)} \right) [\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] \\
& \left. + \left(\frac{\pi^2}{6\epsilon^3} L + \frac{i\pi}{\epsilon} h_{21}^{(2)} + \boxed{\frac{1}{\epsilon} \zeta_5} \right) [\mathbf{T}_{s-u}^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]] + \left(-\frac{i\pi^3}{6\epsilon^3} + \frac{i\pi}{\epsilon} h_{22}^{(2)} \right) (\mathbf{T}_{s-u}^2)^3 \right] \mathcal{A}^{(0)} + \mathcal{H}^{(3)}
\end{aligned} \tag{5.32}$$

where we have boxed all the terms that come from the three-loop dipole correction term $\Delta^{(3)}$. Expressing the amplitude in the three-loop Regge color factor basis

$$\begin{aligned}
\hat{\mathcal{A}}^{(3)} = & A_1^{(0)} \tilde{a}^3 \left[\hat{B}_{00}^{(3)} N^3 C_{00} + \hat{B}_{11}^{(3)} N^2 C_{11} + \hat{B}_{21}^{(3)} N C_{21} + \hat{B}_{22}^{(3)} N C_{22} + \hat{B}_{31}^{(3)} C_{31} + \hat{B}_{32}^{(3)} C_{32} + \hat{B}_{33}^{(3)} C_{33} \right], \\
\mathcal{H}^{(3)} = & A_1^{(0)} \tilde{a}^3 \left[h_{00}^{(3)} N^3 C_{00} + h_{11}^{(3)} N^2 C_{11} + h_{21}^{(3)} N C_{21} + h_{22}^{(3)} N C_{22} + h_{31}^{(3)} C_{31} + h_{32}^{(3)} C_{32} + h_{33}^{(3)} C_{33} \right]
\end{aligned} \tag{5.33}$$

and using eqs. (5.23) and (5.27) we obtain

$$\begin{aligned}
\hat{B}_{00}^{(3)} = & \left(-\frac{\pi^2}{24\epsilon^3} L + \frac{-\frac{2}{9}\pi^2\zeta_3 + \frac{1}{12}\zeta_5}{\epsilon} - \frac{3}{4}\pi^2\zeta_3 L - \frac{13}{432}\pi^6 \right) + h_{00}^{(3)}, \\
\hat{B}_{11}^{(3)} = & i\pi \left(\frac{-\frac{1}{540}\pi^4 - \frac{1}{2}\zeta_3 L}{\epsilon} - \frac{1}{45}\pi^4 L - \frac{39}{2}\zeta_5 - \frac{1}{2}\pi^2\zeta_3 \right) + h_{11}^{(3)}, \\
\hat{B}_{21}^{(3)} = & i\pi \left(-\frac{\pi^2}{6\epsilon^2} L - \frac{\zeta_3}{2\epsilon} L - \frac{1}{60}\pi^4 L \right) + h_{21}^{(3)}, \\
\hat{B}_{22}^{(3)} = & \left(\frac{\pi^2}{6\epsilon^3} L - \frac{\pi^4}{6\epsilon^2} + \frac{\frac{8}{9}\pi^2\zeta_3 - \frac{1}{3}\zeta_5}{\epsilon} + 3\pi^2\zeta_3 L + \frac{17}{135}\pi^6 \right) + h_{22}^{(3)}, \\
\hat{B}_{31}^{(3)} = & i\pi \left(\frac{1}{6\epsilon^3} L^2 + \frac{\frac{1}{3}\zeta_3 L + \frac{11}{270}\pi^4}{\epsilon} \right) + h_{31}^{(3)}, \\
\hat{B}_{32}^{(3)} = & \left(\frac{\pi^2}{6\epsilon^3} L + \frac{3\pi^2\zeta_3 + \zeta_5}{\epsilon} + 9\pi^2\zeta_3 L + \frac{13}{36}\pi^6 \right) + h_{32}^{(3)}, \\
\hat{B}_{33}^{(3)} = & i\pi \left(-\frac{\pi^2}{6\epsilon^3} + 3\pi^2\zeta_3 \right) + h_{33}^{(3)}.
\end{aligned} \tag{5.34}$$

Once again, to determine the IR-finite contributions $h_{ik}^{(3)}$, we use the ancillary files of Henn and Mistlberger [25] to extract the Regge limit of the three-loop color-ordered amplitudes $A_\lambda^{(3)}$ through $\mathcal{O}(\epsilon^0)$, then use eq. (5.15) to derive the corresponding reduced amplitudes $\hat{A}_\lambda^{(3)}$, and finally use eq. (4.22) to determine $\hat{B}_{ik}^{(3)}$ to $\mathcal{O}(\epsilon^0)$. In doing so, we

recover precisely the IR-divergent terms of eq. (5.34) together with the IR-finite coefficients at $\mathcal{O}(\epsilon^0)$:

$$\begin{aligned}
 h_{00}^{(3)} &= 2\zeta_5 L - \frac{11}{36}\pi^2\zeta_3 L - \frac{65\zeta_3^2}{36} - \frac{1397\pi^6}{102060} + \mathcal{O}(\epsilon), \\
 h_{11}^{(3)} &= i\pi \left(2\zeta_5 - \frac{5\pi^2\zeta_3}{36} \right) + \mathcal{O}(\epsilon), \\
 h_{21}^{(3)} &= i\pi \left(\frac{\pi^4}{60}L + 2\zeta_5 - \frac{13\pi^2\zeta_3}{6} \right) + \mathcal{O}(\epsilon), \\
 h_{22}^{(3)} &= \frac{4}{3}\pi^2\zeta_3 L - \frac{\zeta_3^2}{3} + \frac{565\pi^6}{6804} + \mathcal{O}(\epsilon), \\
 h_{31}^{(3)} &= i\pi \left(-\frac{11\zeta_3}{3}L^2 - \frac{29\pi^4}{90}L - \frac{65\zeta_5}{6} + \frac{5\pi^2\zeta_3}{12} \right) + \mathcal{O}(\epsilon), \\
 h_{32}^{(3)} &= \frac{10}{3}\pi^2\zeta_3 L + \zeta_3^2 + \frac{32\pi^6}{567} + \mathcal{O}(\epsilon), \\
 h_{33}^{(3)} &= i\pi \left(\frac{2\pi^2\zeta_3}{3} \right) + \mathcal{O}(\epsilon).
 \end{aligned}
 \tag{5.35}$$

The coefficient of L^2 of $h_{31}^{(3)}$ matches the NLL prediction (4.42) of ref. [50], and the coefficients of L of $h_{00}^{(3)}$, $h_{22}^{(3)}$, and $h_{32}^{(3)}$ match¹⁰ the NNLL prediction (4.44) of ref. [50].

5.5 All loop results

We wish to characterize the logarithmic dependence of each coefficient $B_{ik}^{(\ell)}$ in the expansion

$$\mathcal{A}^{(\ell)} = A_1^{(0)} \tilde{a}^\ell \sum_{i=0}^{\ell} \sum_k B_{ik}^{(\ell)} N^{\ell-i} C_{ik}.
 \tag{5.36}$$

In the IR-divergent prefactors in eq. (5.14), we observe that each power of L requires the presence of a factor of \mathbf{T}_t^2 . The Regge color factor C_{ik} contains $i - k$ factors of \mathbf{T}_t^2 , and $N^{\ell-i}$ can correspond to $\ell - i$ additional factors of \mathbf{T}_t^2 (since \mathbf{T}_t^2 acting on C_{00} produces a factor of N). Hence we anticipate that $B_{ik}^{(\ell)}$ can contain up to $\ell - k$ powers of L , that is

$$B_{ik}^{(\ell)} \sim L^{\ell-k} + \text{lower logarithmic terms}.
 \tag{5.37}$$

Thus $B_{00}^{(\ell)}$ alone contributes at leading log (LL) order, $B_{i1}^{(\ell)}$ starts at NLL, $B_{i2}^{(\ell)}$ at NNLL, etc. We cannot make this argument completely rigorous, because *a priori* we have no knowledge of the L dependence of the hard function \mathcal{H} , but the explicit results through three loops presented earlier are in accord with our expectations. (The L dependence of the reduced amplitude coefficients $\hat{B}_{ik}^{(\ell)}$ is sometimes milder than eq. (5.37) because we have stripped off some of the logarithms in eq. (5.15).)

By eq. (5.37), the coefficient $B_{\ell\ell}^{(\ell)}$ should have no logarithmic dependence whatsoever. In fact, the only term in eq. (5.14) that contributes to $B_{\ell\ell}^{(\ell)}$ is

$$\mathcal{A} \sim \exp \left[\frac{i\pi\tilde{a}}{\epsilon} \mathbf{T}_{s-u}^2 \right] \mathcal{H}
 \tag{5.38}$$

¹⁰See footnote 9. Also, $D_g^{(1)} = \frac{1}{4}Nr(\epsilon)f_1(\epsilon)$ since we are expanding in \tilde{a} rather than a .

hence

$$\mathcal{A}^{(\ell)} \sim \frac{1}{\ell!} \left(\frac{i\pi\tilde{a}}{\epsilon} \mathbf{T}_{s-u}^2 \right)^\ell \mathcal{A}^{(0)} + \text{contributions from } \mathcal{H}^{(2)}, \mathcal{H}^{(3)}, \dots \quad (5.39)$$

and thus

$$B_{\ell\ell}^{(\ell)} = \frac{1}{\ell!} \left(\frac{i\pi}{\epsilon} \right)^\ell + \mathcal{O}(\epsilon^{3-\ell}) \quad (5.40)$$

where the $\mathcal{O}(\epsilon^{3-\ell})$ corrections come from $\mathcal{H}^{(2)}, \mathcal{H}^{(3)}, \dots$. This is consistent with the explicit one-, two-, and three-loop results¹¹ (4.12), (5.26), (5.28), (5.34), and (5.35)

$$\begin{aligned} B_{11}^{(1)} &= \frac{i\pi}{\epsilon}, \\ B_{22}^{(2)} &= \left(-\frac{\pi^2}{2\epsilon^2} \right) \frac{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)\Gamma(1-3\epsilon)}, \\ B_{33}^{(3)} &= i\pi \left(-\frac{\pi^2}{6\epsilon^3} + \frac{11\pi^2\zeta_3}{3} \right) + \mathcal{O}(\epsilon). \end{aligned} \quad (5.41)$$

In section 6, we will observe that the coefficients $B_{\ell\ell}^{(\ell)}$ are closely related to the $\mathcal{N} = 8$ supergravity four-point amplitude in the Regge limit.

6 SYM/supergravity relation in the Regge limit

In this section, we review the Regge limit of the $\mathcal{N} = 8$ four-point amplitude. This will allow us to make an all-loop-orders conjecture between the $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity four-point amplitudes in the Regge limit.

The tree-level four-graviton amplitude is¹² [3]

$$\mathcal{M}^{(0)} = 8\pi G \mu^{2\epsilon} \frac{16K\tilde{K}}{stu}, \quad (6.1)$$

where K is defined as in eq. (3.2). The one-loop $\mathcal{N} = 8$ supergravity four-graviton amplitude is [3]

$$\begin{aligned} \mathcal{M}^{(1)} &= -i(8\pi G)^2 \mu^{2\epsilon} (16K\tilde{K}) \left[\mathcal{I}^{(1)}(s, t) + \mathcal{I}^{(1)}(u, s) + \mathcal{I}^{(1)}(t, u) \right] \\ &= \mathcal{M}^{(0)} \left(-8\pi i G stu \left[\mathcal{I}^{(1)}(s, t) + \mathcal{I}^{(1)}(u, s) + \mathcal{I}^{(1)}(t, u) \right] \right) \end{aligned} \quad (6.2)$$

which, using the expression of $\mathcal{I}^{(1)}$ given in eq. (3.10) becomes

$$\begin{aligned} \mathcal{M}^{(1)} &= \mathcal{M}^{(0)} \tilde{\eta} \frac{1}{\epsilon^2} \left[(x-1) \left(e^{i\pi} x \right)^\epsilon F \left(\epsilon, 1 - \frac{1}{x} \right) + (x-1) F \left(\epsilon, 1-x \right) \right. \\ &\quad \left. - x \left(e^{i\pi} x \right)^\epsilon F \left(\epsilon, \frac{-x}{1-x} \right) - x \left(\frac{x}{1-x} \right)^\epsilon F \left(\epsilon, x \right) \right. \\ &\quad \left. + F \left(\epsilon, \frac{1}{1-x} \right) + \left(\frac{x}{1-x} \right)^\epsilon F \left(\epsilon, \frac{1}{x} \right) \right] \end{aligned} \quad (6.3)$$

¹¹Note that $B_{\ell\ell}^{(\ell)} = \hat{B}_{\ell\ell}^{(\ell)}$ because the prefactors in eq. (5.15) cannot contribute to this coefficient.

¹²In this paper, we take G to be the four-dimensional Newton's constant, with $G_D = G\mu^{2\epsilon}$ its counterpart in $D = 4 - 2\epsilon$ dimensions.

where we define the dimensionless parameter¹³

$$\tilde{\eta} \equiv \frac{Gs}{\pi} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{4\pi\mu^2}{-t} \right)^\epsilon. \quad (6.4)$$

Writing the all-orders amplitude as

$$\mathcal{M} = \mathcal{M}^{(0)} \left[1 + \sum_{\ell=1}^{\infty} \tilde{\eta}^\ell M^{(\ell)} \right] \quad (6.5)$$

we see that eq. (6.3) simplifies dramatically in the Regge limit $x \rightarrow 0$ to

$$M^{(1)} = -\frac{i\pi}{\epsilon} + \mathcal{O}(x). \quad (6.6)$$

Using eikonal exponentiation in impact parameter space, the authors of refs. [75, 76] showed that the Regge limit of the ℓ -loop four-graviton amplitude is given to all orders in ϵ by

$$M^{(\ell)} = \frac{1}{\ell!} \left(-\frac{i\pi}{\epsilon} \right)^\ell G^{(\ell)}(\epsilon) + \mathcal{O}(x) \quad (6.7)$$

where

$$\begin{aligned} G^{(\ell)}(\epsilon) &= \frac{\Gamma^\ell(1-2\epsilon)\Gamma(1+\ell\epsilon)}{\Gamma^{\ell-1}(1-\epsilon)\Gamma^\ell(1+\epsilon)\Gamma(1-(\ell+1)\epsilon)} \\ &= 1 - \frac{1}{3}\ell(2\ell^2 + 3\ell - 5)\zeta_3\epsilon^3 + \mathcal{O}(\epsilon^4). \end{aligned} \quad (6.8)$$

In particular, eq. (6.7) gives (in the Regge limit)

$$\begin{aligned} M^{(1)} &= -\frac{i\pi}{\epsilon} + \mathcal{O}(x), \\ M^{(2)} &= \left(-\frac{\pi^2}{2\epsilon^2} \right) \frac{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)\Gamma(1-3\epsilon)} + \mathcal{O}(x), \\ M^{(3)} &= i\pi \left(\frac{\pi^2}{6\epsilon^3} - \frac{11\pi^2\zeta_3}{3} - \frac{11\pi^6}{180}\epsilon \right) + \mathcal{O}(\epsilon^2) + \mathcal{O}(x). \end{aligned} \quad (6.9)$$

We observe that (up to signs) these are precisely the one-, two-, and three-loop values of the coefficients $B_{\ell\ell}^{(\ell)}$ of the $\mathcal{N} = 4$ SYM four-gluon amplitude given in eq. (5.41):

$$B_{11}^{(1)} = -M^{(1)} + \mathcal{O}(x), \quad B_{22}^{(2)} = M^{(2)} + \mathcal{O}(x), \quad B_{33}^{(3)} = -M^{(3)} + \mathcal{O}(\epsilon) + \mathcal{O}(x) \quad (6.10)$$

where the one- and two-loop relations hold to all orders in ϵ , and the three-loop relation holds to the accuracy of the $\mathcal{N} = 4$ SYM calculation. This motivates the all-loop-orders conjecture:

$$\boxed{B_{\ell\ell}^{(\ell)} = (-1)^\ell M^{(\ell)} + \mathcal{O}(x)} \quad (6.11)$$

relating the Regge limits of the $\mathcal{N} = 8$ supergravity amplitude and the $\mathcal{N} = 4$ SYM amplitude. Comparing the leading infrared-divergent contribution (5.40) with eq. (6.7) confirms

¹³Related by $\tilde{\eta} = \alpha_{GS}/(-t)^\epsilon$ to α_G defined in refs. [75, 76].

that eq. (6.11) is valid for any ℓ to at least the first three orders in the Laurent expansion in ϵ . Based on all this, we conjecture that eq. (6.11) holds to all orders in ϵ for any ℓ .

We can express the SYM/supergravity relation (6.11) directly in terms of color-ordered amplitudes using eqs. (4.26)–(4.28)

$$\begin{aligned}
 M^{(1)} &= \frac{A_1^{(1,0)} + A_3^{(1,0)}}{A_1^{(0)} \tilde{a}} + \mathcal{O}(x), & \ell = 1, \\
 M^{(\ell)} &= \frac{\frac{1}{2} \left(A_1^{(\ell,\ell)} - A_3^{(\ell,\ell)} \right)}{3^{\ell/2} \cdot A_1^{(0)} \tilde{a}^\ell} + \mathcal{O}(x), & \text{even } \ell, \\
 M^{(\ell)} &= \frac{- \left(A_1^{(\ell,\ell-3)} + A_3^{(\ell,\ell-3)} \right) + \frac{1}{4} \left(A_4^{(\ell,\ell-2)} + A_6^{(\ell,\ell-2)} + A_1^{(\ell,\ell-1)} + A_3^{(\ell,\ell-1)} \right)}{3^{(\ell-1)/2} \cdot A_1^{(0)} \tilde{a}^\ell} + \mathcal{O}(x), & \text{odd } \ell > 1.
 \end{aligned}
 \tag{6.12}$$

The validity of eq. (6.12) for $\ell = 1$ and $\ell = 2$ is not really surprising since they are the Regge limits of more general exact relations. The one-loop relation is the Regge limit of the exact one-loop relation [1, 23]

$$\frac{\mathcal{M}^{(1)}}{\mathcal{M}^{(0)} \tilde{\eta}} = (1-x) \frac{\left(A_1^{(1,0)} + A_2^{(1,0)} + A_3^{(1,0)} \right)}{A_1^{(0)} \tilde{a}}
 \tag{6.13}$$

and the $\ell = 2$ relation is the Regge limit of the more general exact two-loop relation [23]

$$\frac{\mathcal{M}^{(2)}}{\mathcal{M}^{(0)} \tilde{\eta}^2} = \left(\frac{1-x}{6} \right) \frac{\left[(1-x) A_1^{(2,2)} + x A_2^{(2,2)} - A_3^{(2,2)} \right]}{A_1^{(0)} \tilde{a}^2}
 \tag{6.14}$$

where we have expressed both relations using the notation of the current paper. Both eqs. (6.13) and (6.14) are proved by expressing the amplitudes in terms of planar and non-planar integrals [2, 3, 23]. Although previous attempts at finding exact SYM/supergravity relations beyond two loops were not successful, there may nevertheless exist exact relations of which eq. (6.12) is the Regge limit.

7 Conclusions

We presented in this paper an all-loop-order basis of color factors C_{ik} suitable for writing the Regge limit of the $\mathcal{N} = 4$ SYM four-gluon amplitude. These color factors have well-defined signature under crossing symmetry $u \leftrightarrow s$; specifically, C_{ik} has negative/positive signature for k even/odd.

We found that the coefficients $B_{ik}^{(\ell)}$ of the Regge limit of the $\mathcal{N} = 4$ SYM amplitude in this basis are polynomials of order $\ell - k$ in $L = \log |s/t| - \frac{1}{2} i\pi$. Thus $B_{00}^{(\ell)}$ alone contributes at leading log (LL) order, $B_{i1}^{(\ell)}$ starts at NLL, $B_{i2}^{(\ell)}$ at NNLL, etc. The coefficients of color factors with negative/positive signature are real/imaginary respectively (when expressed in terms of L), as shown on general grounds in ref. [50]. Using results from ref. [25], we computed these coefficients explicitly through three-loop order, verifying consistency with

all the expected IR divergences [14–16, 26–35, 45–50], as well as with certain IR-finite NLL and NNLL predictions [49, 50].

Based on our explicit results, we conjectured an all-loop-orders equivalence (up to sign) between the coefficients $B_{\ell\ell}^{(\ell)}$ and the Regge limit of the ℓ -loop $\mathcal{N} = 8$ supergravity four-point amplitude. This equivalence was proven to be valid to all orders in ϵ at one and two loops, through $\mathcal{O}(\epsilon^0)$ at three loops, and for the first three terms in the Laurent expansion in ϵ at ℓ loops.

Naturally, it would be nice to establish the validity of the conjectured SYM/supergravity relation more generally, perhaps via an eikonal exponentiation approach [75, 76], or via known representations of these amplitudes in terms of planar and nonplanar integrals [2–7, 7, 8, 37].

It would also be interesting to know if the SYM/supergravity relations for $\ell > 2$ are the Regge limits of more general exact SYM/supergravity relations, as is the case for $\ell = 1$ and $\ell = 2$.

Finally, it would be intriguing to discover all-orders-in- ϵ expressions for the other coefficients $B_{ik}^{(\ell)}$ of the Regge basis of color factors.

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