

# Finite derivation of the one-loop sine-Gordon soliton mass

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**ABSTRACT:** Calculations of quantum corrections to soliton masses generally require both the vacuum sector and the soliton sector to be regularized. The finite part of the quantum correction depends on the assumed relation between these regulators when both are taken to infinity. Recently, in the case of quantum kinks, a manifestly finite prescription for the calculation of the quantum corrections has been proposed, which uses the kink creation operator to relate the two sectors. In this note, we test this new prescription by calculating the one-loop correction to the sine-Gordon soliton mass, reproducing the well-known result which has been derived using integrability.

**KEYWORDS:** Solitons Monopoles and Instantons, Field Theories in Lower Dimensions, Nonperturbative Effects

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**Contents**

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Pöschl-Teller potential</b>	<b>2</b>
2.1	Vacuum state and the soliton	2
2.2	Shifted Hamiltonian	3
<b>3</b>	<b>Solutions to the Pöschl-Teller Hamiltonian</b>	<b>4</b>
<b>4</b>	<b>Mode expansion</b>	<b>6</b>
4.1	PT annihilation and creation operators	6
4.2	Contributions of continuum and bound states	7
4.3	Diagonalized Hamiltonian	8
<b>5</b>	<b>Conclusion</b>	<b>9</b>

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**1 Introduction**

In general, quantum corrections to soliton masses can be computed using the WKB approximation introduced in ref. [1]. In ref. [2] this method was applied to the sine-Gordon soliton and it was found to yield the exact answer of [3], as was confirmed using integrability in ref. [4].

The soliton mass is defined to be the difference between the lowest energy configurations in the one-soliton and vacuum sectors. These two energies are themselves both infinite, and so both must be regularized and then the regulators must be taken to infinity. The result of this calculation depends on the relation between the regulators when this limit is taken [5], and it is in general not known which relation yields the right answer. For example, identifying modes in a compactified theory yields a different mass than an identification of momentum cutoffs. Supersymmetric and integrable models are the exception, as the soliton mass can be computed using supersymmetry and integrability and so one can determine which relation between regulators agrees with this answer. For example a regulator which preserves the supersymmetry is guaranteed to yield the correct answer. Therefore it may appear as though the WKB method can only be used to compute soliton masses which are already known.

A resolution to this problem was proposed in ref. [6]. It was noted that the vacuum and one-soliton sectors are related by the operator which creates the soliton, and so this operator provides the correct identification of the regulators. As scalar theories in 1+1 dimensions can be rendered finite by normal-ordering, the vacuum Hamiltonian was normal ordered and corresponding one-soliton sector Hamiltonian was directly computed using this

identification. The one-soliton sector Hamiltonian was not normal ordered when written in terms of the eigenfunctions of its kinetic term, but simply commuting the corresponding creation operators to the left produced a constant term which was precisely equal to the result of ref. [1] for the one-loop correction to the mass.

In this paper we test the method introduced in ref. [6] to derive the one-loop correction to the mass of the sine-Gordon soliton. This correction has been derived using integrability in ref. [4], with no arbitrary choice of regulator matching, and so it provides a robust test of the method.

First of all, we shift the scalar field by the classical soliton solution to derive the one-soliton sector Hamiltonian. We find that only the quadratic terms contribute to the soliton mass at one-loop and we identify these terms with the Pöschl-Teller Hamiltonian. We use the classical solutions of this Hamiltonian to exactly diagonalize it, providing the desired soliton mass as well as the Hamiltonian describing the excited states in the soliton sector as a sum of quantum harmonic oscillator states.

## 2 Pöschl-Teller potential

### 2.1 Vacuum state and the soliton

The sine-Gordon Hamiltonian is

$$H = \int dx \mathcal{H}(x), \quad \mathcal{H}(x) = \frac{1}{2} : \pi(x) \pi(x) : + \frac{1}{2} : \partial_x \phi(x) \partial_x \phi(x) : - \frac{m^2}{\lambda} : \left( \cos(\sqrt{\lambda} \phi(x)) - 1 \right) : \quad (2.1)$$

where  $m$  and  $\lambda$  are positive numbers. The field  $\phi$  has dimensions of [action]<sup>1/2</sup>,  $m$  has dimensions of [mass] and  $\lambda$  has dimensions of [action]<sup>-1</sup> therefore the only dimensionless constant is  $\lambda \hbar$ . Our loop expansion will therefore be an expansion in  $\lambda \hbar$ . We however set  $\hbar = 1$  everywhere.

The theory has a series of degenerate ground states  $|0\rangle_k$  with

$${}_k \langle 0 | \phi | 0 \rangle_k = \frac{2\pi}{\sqrt{\lambda}} k, \quad k \in \mathbb{Z} \quad (2.2)$$

and without loss of generality we will be interested in solitons which connect the adjacent ground states  $|0\rangle_0$  and  $|0\rangle_1$ .

Performing the standard expansion about the ground state  $|0\rangle_0$

$$\phi(x) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\omega_p}} \left( a_p^\dagger + a_{-p} \right) e^{-ipx}, \quad \pi(x) = i \int \frac{dp}{2\pi} \frac{\sqrt{\omega_p}}{\sqrt{2}} \left( a_p^\dagger - a_{-p} \right) e^{-ipx} \quad (2.3)$$

where

$$\omega_p = \sqrt{m^2 + p^2} \quad (2.4)$$

the canonical commutation relations satisfied by  $\phi$  and  $\pi$  imply

$$[a_p, a_q^\dagger] = 2\pi \delta(p - q). \quad (2.5)$$

The normal ordering in eq. (2.1) is defined with respect to this  $a$  and  $a^\dagger$ .

Let  $E_0$  and  $E_K$  be the Hamiltonian eigenvalues of the vacua  $|0\rangle_k$  and the one-soliton sector ground state  $|K\rangle$

$$H|0\rangle_k = E_0|0\rangle_k, \quad H|K\rangle = E_K|K\rangle. \quad (2.6)$$

The soliton mass is defined to be

$$M_K = E_K - E_0. \quad (2.7)$$

$E_0$  can be calculated in perturbation theory as in ref. [7]. The leading contributions appear at two loops and are of order  $O(\lambda^2)$ . We will see that they are therefore not relevant to the one-loop soliton mass which is of order  $O(\lambda^0)$ . Therefore, at the one-loop order considered here,  $E_0 = 0$ .

The classical equation of motion derived from (2.1) is

$$\frac{\partial^2 \phi_{cl}(x, t)}{\partial t^2} - \frac{\partial^2 \phi_{cl}(x, t)}{\partial x^2} = -\frac{m^2}{\sqrt{\lambda}} \sin\left(\sqrt{\lambda} \phi_{cl}(x, t)\right) \quad (2.8)$$

which has a stationary soliton solution

$$\phi_{cl}(x, t) = f(x), \quad f(x) = \frac{4}{\sqrt{\lambda}} \arctan e^{mx}. \quad (2.9)$$

At leading order in the semiclassical expansion one expects that this will be the form factor of the soliton ground state [8]

$$\langle K|\phi(x)|K\rangle = f(x) + O(\hbar). \quad (2.10)$$

## 2.2 Shifted Hamiltonian

Following ref. [9], eq. (2.10) would be solved if  $|K\rangle = \mathcal{D}_f|0\rangle_0 + O(\hbar)$  where  $\mathcal{D}_f$  is the displacement operator

$$\mathcal{D}_f = \exp\left(-i \int dx f(x) \pi(x)\right) \quad (2.11)$$

which satisfies [6]

$$[\mathcal{D}_f, \phi(y)] = -f(y)\mathcal{D}_f, \quad : F[\pi(x), \phi(x)] : \mathcal{D}_f = \mathcal{D}_f : F[\pi(x), \phi(x) + f(x)] : \quad (2.12)$$

where  $F$  is any function of two variables.

Eq. (2.10) leads us to rewrite the soliton ground state as

$$|K\rangle = \mathcal{D}_f \mathcal{O}|0\rangle_0 \quad (2.13)$$

where  $\mathcal{O}$  is equal to the identity plus corrections of order  $O(\hbar)$ . We now define the soliton sector Hamiltonian  $H_K$  by the similarity transform

$$H\mathcal{D}_f = \mathcal{D}_f H_K. \quad (2.14)$$

Then a quick calculation shows

$$H_K \mathcal{O}|0\rangle_0 = \mathcal{D}_f^{-1} H|K\rangle_0 = E_K \mathcal{O}|0\rangle_0. \quad (2.15)$$

Therefore instead of searching for the eigenstate  $|K\rangle$  of  $H$ , we may equivalently search for the eigenstate  $\mathcal{O}|0\rangle_0$  of  $H_K$ . Although  $H$  and  $H_K$  are related by a similarity transformation, the second problem can be treated in ordinary perturbation theory as  $\mathcal{O}$  is equal to the identity plus loop corrections.

$H_K$  can be evaluated using (2.12)

$$H_K[\pi(x), \phi(x)] = H[\pi(x), \phi(x) + f(x)] \quad (2.16)$$

and so

$$H_K = E_{cl} + \int dx [\mathcal{H}_{PT} + \mathcal{H}_I] \quad (2.17)$$

where the classical energy is

$$E_{cl} = \int dx \left[ \frac{1}{2} (\partial_x f(x))^2 + \frac{m^2}{\lambda} \left( 1 - \cos(\sqrt{\lambda} f(x)) \right) \right] = \frac{8m}{\lambda} \quad (2.18)$$

the interaction terms are

$$\mathcal{H}_I = \frac{m^2}{\sqrt{\lambda}} \sin(\sqrt{\lambda} f(x)) \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{(2n+1)!} : \phi^{2n+1}(x) : - \frac{m^2}{\lambda} \cos(\sqrt{\lambda} f(x)) \sum_{n=2}^{\infty} \frac{(-\lambda)^n}{2n!} : \phi^{2n}(x) : \quad (2.19)$$

and the Poschl-Teller (PT) Hamiltonian density is

$$\mathcal{H}_{PT} = \frac{: \pi^2(x) :}{2} + \frac{: \partial_x \phi(x) \partial_x \phi(x) :}{2} + \left( \frac{m^2}{2} - m^2 \operatorname{sech}^2(mx) \right) : \phi^2(x) : \quad (2.20)$$

Recall that our loop expansion is an expansion in  $\lambda$ . The classical energy is of order  $O(\lambda^{-1})$ . Therefore the one-loop correction will be  $\lambda$ -independent. As the PT terms are  $\lambda$ -independent, any correction derived from them will appear at one loop. The  $\mathcal{H}_I$  terms on the other hand are all of at least order  $O(\lambda^{1/2})$ , and so only contribute at two loops and beyond. Thus, to calculate the one-loop soliton mass, we may drop  $\mathcal{H}_I$  leaving

$$H' = E_{cl} + H_{PT}, \quad H_{PT} = \int dx \mathcal{H}_{PT}. \quad (2.21)$$

In the remainder of this note we will explicitly diagonalize  $H'$  and so obtain the one-loop soliton mass as well as its excitation spectrum at one loop.

### 3 Solutions to the Pöschl-Teller Hamiltonian

In this section we will calculate the inverse Fourier transforms of the eigenfunctions of the Pöschl-Teller wave equation. To find the eigenstates of  $H_{PT}$ , we insert the factorization Ansatz

$$\phi_{cl}(x, t) = \psi_k(x) e^{-i\omega_k t} \quad (3.1)$$

into the corresponding classical equations of motion to obtain

$$0 = \partial_x^2 \psi_k(x) + (k^2 + 2m^2 \operatorname{sech}^2(mx)) \psi_k(x), \quad k^2 = \omega_k^2 - m^2. \quad (3.2)$$

There will be a bound solution  $\psi_B$  corresponding to the Goldstone mode of the soliton and also, at each  $k$  an even an odd continuum solution given by the hypergeometric functions [10]

$$\begin{aligned}\psi_k^e(x) &= \cosh^2(mx)F\left(\frac{2+ik/m}{2}, \frac{2-ik/m}{2}; \frac{1}{2}; -\sinh^2(mx)\right) \\ \psi_k^o(x) &= \cosh^2(mx)\sinh(mx)F\left(\frac{3+ik/m}{2}, \frac{3-ik/m}{2}; \frac{3}{2}; -\sinh^2(mx)\right).\end{aligned}\tag{3.3}$$

These hypergeometric functions may be calculated as in the appendix of ref. [6] to obtain

$$\begin{aligned}F\left(\frac{2+ik}{2}, \frac{2-ik}{2}; \frac{1}{2}; -\sinh^2(x)\right) &= \frac{\cos(kx) - \frac{m}{k}\sin(kx)\tanh(mx)}{\cosh^2(mx)} \\ F\left(\frac{3+ik/m}{2}, \frac{3-ik/m}{2}; \frac{3}{2}; -\sinh^2(mx)\right) &= \frac{\left(\frac{\cos(kx)}{\cosh(mx)} + \frac{k}{m}\frac{\sin(kx)}{\sinh(mx)}\right)}{\cosh^2(mx)(1+k^2/m^2)}.\end{aligned}\tag{3.4}$$

Substituting these back into eq. (3.3) and changing the normalization by a  $k$ -dependent factor one obtains the solutions

$$\begin{aligned}\psi_k^e(x) &= \frac{k}{m}\cos(kx) - \sin(kx)\tanh(mx) \\ \psi_k^o(x) &= \cos(kx)\tanh(mx) + \frac{k}{m}\sin(kx)\end{aligned}\tag{3.5}$$

which are normalized so that

$$\int dx \psi_{k_1}^i(x) \psi_{k_2}^j(x) = \pi \delta^{ij} C_{k_1}^2 \delta(k_1 - k_2), \quad C_k = \sqrt{1+k^2/m^2}, \quad i, j \in \{e, o\}.\tag{3.6}$$

The inverse Fourier transform of

$$g_k(x) = \psi_k^e(x) + i\psi_k^o(x)\tag{3.7}$$

is

$$\tilde{g}_k(p) = \int dx g_k(x) e^{ipx} = \frac{2k}{m} \pi \delta(p - k) - \frac{\pi}{m} \operatorname{csch}\left(\frac{\pi(p - k)}{2m}\right)\tag{3.8}$$

which is normalized so that

$$\int \frac{dp}{2\pi} \tilde{g}_{k_1}(p) \tilde{g}_{k_2}(p) = \int dx g_{k_1}(x) g_{k_2}(-x) = 2\pi C_{k_1}^2 \delta(k_1 - k_2).\tag{3.9}$$

The delta function results from the fact that asymptotically the eigenfunctions of  $H_{PT}$  and  $H_0$  (defined in (4.2)) are equal. There is no  $\delta(p + k)$  term because with the coefficient in (2.20) the PT potential is reflectionless [10]. Inserting

$$\omega_B = 0, \quad k_B = im\tag{3.10}$$

into (3.5) one finds the bound solution

$$g_B(x) = i\pi \operatorname{sech}\left(\frac{pm}{2m}\right)\tag{3.11}$$

which corresponds to the Goldstone mode of the soliton. It satisfies the normalization condition

$$\int dx |g_B(x)|^2 = C_B^2, \quad C_B = \sqrt{2} \quad (3.12)$$

and has inverse Fourier transform

$$\tilde{g}_B(p) = \int dx g_B(x) e^{ipx} = \frac{\pi p}{m^2} \operatorname{csch}\left(\frac{\pi p}{2m}\right). \quad (3.13)$$

## 4 Mode expansion

### 4.1 PT annihilation and creation operators

To diagonalize  $H_{PT}$ , first we decompose it

$$H_{PT} = H_0 + \tilde{H}_{PT} \quad (4.1)$$

where  $H_0$  is the usual free Hamiltonian

$$H_0 = \int dx \left[ \frac{1}{2} : \pi(x)\pi(x) : + \frac{1}{2} : \partial_x \phi(x) \partial_x \phi(x) : + \frac{m^2}{2} : \phi^2(x) : \right] = \int \frac{dp}{2\pi} \omega_p a_p^\dagger a_p. \quad (4.2)$$

Recall that the operators  $a$  and  $a^\dagger$  were defined in (2.3) by decomposing  $\phi$  and  $\pi$  into plane waves, which are solutions of the wave equation corresponding to  $H_0$ . To diagonalize  $H_{PT}$ , we instead decompose  $\phi$  and  $\pi$  into the basis of constant frequency solutions of the PT equation. In particular they will contain continuum and bound state contributions

$$\phi(x) = \phi_C(x) + \phi_B(x), \quad \pi(x) = \pi_C(x) + \pi_B(x) \quad (4.3)$$

which, following [6], may be decomposed into the PT oscillator basis

$$\begin{aligned} \phi_C(x) &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} (b_k^\dagger + b_{-k}) \frac{g_k(x)}{C_k}, & \phi_B(x) &= \phi_0 \frac{g_B(x)}{C_B}, \\ \pi_C(x) &= i \int \frac{dk}{2\pi} \sqrt{\frac{\omega_k}{2}} (b_k^\dagger - b_{-k}) \frac{g_k(x)}{C_k}, & \pi_B(x) &= \pi_0 \frac{g_B(x)}{C_B} \end{aligned} \quad (4.4)$$

where we have introduced the operators  $\phi_0$  for  $\pi_0$  which are just the position and momentum operators of the soliton.

These definitions are easily inverted

$$b_k^\dagger = \int dx \left[ \sqrt{\frac{\omega_k}{2}} \phi(x) - \frac{i}{\sqrt{2\omega_k}} \pi(x) \right] \frac{g_k^*(x)}{C_k}, \quad b_{-k} = \int dx \left[ \sqrt{\frac{\omega_k}{2}} \phi(x) + \frac{i}{\sqrt{2\omega_k}} \pi(x) \right] \frac{g_k^*(x)}{C_k} \quad (4.5)$$

from which one sees that the continuum  $b$  operators satisfy the Heisenberg algebra

$$[b_{k_1}, b_{k_2}^\dagger] = 2\pi \delta(k_1 - k_2) \quad (4.6)$$

while the bound state

$$\phi_0 = \int dx \phi(x) \frac{g_B^*(x)}{C_B}, \quad \pi_0 = \int dx \pi(x) \frac{g_B^*(x)}{C_B}. \quad (4.7)$$

satisfies the canonical algebra

$$[\phi_0, \pi_0] = i. \quad (4.8)$$

We cannot directly write  $H_{PT}$  in terms of  $b$  and  $b^\dagger$  because it is the  $a$  and  $a^\dagger$  operators which are normal ordered. Thus we must first write it in terms of  $a$  and then convert these to  $b$ . To do this one first inverts (2.3)

$$a_p^\dagger = \int dx \left[ \sqrt{\frac{\omega_p}{2}} \phi(x) - \frac{i}{\sqrt{2\omega_p}} \pi(x) \right] e^{ipx}, \quad a_{-p} = \int dx \left[ \sqrt{\frac{\omega_p}{2}} \phi(x) + \frac{i}{\sqrt{2\omega_p}} \pi(x) \right] e^{ipx} \quad (4.9)$$

and decomposes the  $a$  operators as

$$a_p^\dagger = a_{C,p}^\dagger + a_{BE,p}^\dagger, \quad a_p = a_{C,p} + a_{BE,p}. \quad (4.10)$$

As we know  $a$  as a function of  $\phi$ , which is a known function of  $b$ , we can write the Bogoliubov transformation which relates the  $a$  and  $b$  oscillator modes

$$\begin{aligned} a_{C,p}^\dagger &= \int \frac{dk}{2\pi} \frac{\tilde{g}_k(p)}{2C_k} \left( \frac{\omega_p + \omega_k}{\sqrt{\omega_p \omega_k}} b_k^\dagger + \frac{\omega_p - \omega_k}{\sqrt{\omega_p \omega_k}} b_{-k} \right) \\ a_{C,-p} &= \int \frac{dk}{2\pi} \frac{\tilde{g}_k(p)}{2C_k} \left( \frac{\omega_p - \omega_k}{\sqrt{\omega_p \omega_k}} b_k^\dagger + \frac{\omega_p + \omega_k}{\sqrt{\omega_p \omega_k}} b_{-k} \right) \\ a_{BE,p}^\dagger &= \frac{\tilde{g}_B(p)}{C_B} \left[ \sqrt{\frac{\omega_p}{2}} \phi_0 - \frac{i}{\sqrt{2\omega_p}} \pi_0 \right], \quad a_{BE,-p} = \frac{\tilde{g}_B(p)}{C_B} \left[ \sqrt{\frac{\omega_p}{2}} \phi_0 + \frac{i}{\sqrt{2\omega_p}} \pi_0 \right]. \end{aligned} \quad (4.11)$$

Note that the delta function terms in (3.8) can be directly integrated, using the delta function, and one sees that they do not mix  $a$  with  $b^\dagger$ . This will imply that they do not affect the one-loop mass corrections of the soliton.

## 4.2 Contributions of continuum and bound states

Now we are ready to diagonalize  $H_{PT}$  one term at a time. The calculation is very similar to that in ref. [6], except that here there is no odd bound state. Let us first decompose  $H_0$  and  $\tilde{H}_{PT}$  into continuum and bound state contributions

$$H_0 = H_{C,0} + H_{B,0}, \quad \tilde{H}_{PT} = \tilde{H}_C + \tilde{H}_B. \quad (4.12)$$

The continuum contribution is

$$\begin{aligned} H_{C,0} &= \int \frac{dp}{2\pi} \omega_p a_{C,p}^\dagger a_{C,p} \\ &= \frac{1}{4} \int \frac{dk}{2\pi} \frac{I_5(k)}{C_k^2 \omega_k} + \frac{m^2}{2} \int dx \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \operatorname{sech}^2(mx) \frac{g_{k_1}(x) g_{k_2}(x)}{C_{k_1} C_{k_2} \sqrt{\omega_{k_1} \omega_{k_2}}} (b_{k_1}^\dagger b_{k_2}^\dagger + b_{-k_1} b_{-k_2}) \\ &\quad + \int \frac{dk}{2\pi} \omega_k b_k^\dagger b_k + m^2 \int dx \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \operatorname{sech}^2(mx) \frac{g_{k_1}(x) g_{k_2}(x)}{C_{k_1} C_{k_2} \sqrt{\omega_{k_1} \omega_{k_2}}} b_{k_1}^\dagger b_{-k_2} \end{aligned} \quad (4.13)$$

where

$$I_5(k) = \int \frac{dp}{2\pi} (\omega_p - \omega_k)^2 \tilde{g}_k(p) \tilde{g}_k(p). \quad (4.14)$$



Similarly the continuum contribution to the PT potential term is

$$\begin{aligned}
 \tilde{H}_C &= -m^2 \int dx \operatorname{sech}^2(mx) : \phi_C^2(x) : \\
 &= -\frac{m^2}{8} \int dx \int \frac{dp}{2\pi} \int \frac{dq}{2\pi} \frac{\operatorname{sech}^2(\beta x)}{\omega_p \omega_q} e^{-i(p+q)x} \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \frac{\tilde{g}_{k_1}(p) \tilde{g}_{k_2}(q)}{C_{k_1} C_{k_2} \sqrt{\omega_{k_1} \omega_{k_2}}} \\
 &\times \left[ 4\omega_p \omega_q (b_{k_1}^\dagger b_{k_2}^\dagger + b_{-k_1} b_{-k_2}) + 2\omega_q (2\omega_p + \omega_{k_1} + \omega_{k_2}) b_{k_1}^\dagger b_{-k_2} + 2\omega_q (2\omega_p - \omega_{k_1} - \omega_{k_2}) b_{-k_2} b_{k_1}^\dagger \right].
 \end{aligned} \tag{4.15}$$

Combining the two continuum contributions and moving all  $b^\dagger$  to the left using (4.6) we obtain

$$H_C = H_{C,0} + \tilde{H}_C = \int \frac{dk}{2\pi} \omega_k b_k^\dagger b_k + Q_C \tag{4.16}$$

where

$$\begin{aligned}
 Q_C &= \frac{1}{4} \int \frac{dk}{2\pi} \frac{I_5(k)}{C_k^2 \omega_k} + \frac{m^2}{2} \int dx \int \frac{dp}{2\pi} \int \frac{dq}{2\pi} \frac{\operatorname{sech}^2(mx)}{\omega_p} e^{-i(p+q)x} \int \frac{dk}{2\pi} \frac{\tilde{g}_k(p) \tilde{g}_{-k}(q)}{C_k^2} \\
 &\quad - \frac{m^2}{2} \int dx \operatorname{sech}^2(mx) \int \frac{dk}{2\pi} \frac{g_k(x) g_k^*(x)}{C_k^2 \omega_k}.
 \end{aligned} \tag{4.17}$$

$Q_C$  may be simplified using the equation of motion satisfied (3.2) by  $\phi_k$  to obtain

$$Q_C = -\frac{1}{4} \int \frac{dk}{2\pi} \int \frac{dp}{2\pi} \frac{(\omega_p - \omega_k)^2}{\omega_p} \frac{\tilde{g}_k^2(p)}{C_k^2}. \tag{4.18}$$

A similar calculation for the bound state contribution yields

$$H_B = H_{B,0} + \tilde{H}_0 = \frac{\pi_0^2}{2} + Q_B \tag{4.19}$$

where

$$Q_B = -\frac{1}{4} \int \frac{dp}{2\pi} \frac{\tilde{g}_B(p) \tilde{g}_B(p)}{C_B^2} \omega_p. \tag{4.20}$$

Using the fact that the frequency  $\omega_B = 0$  for the Goldstone mode, one sees that this is of the same form as  $Q_C$  in (4.18).

### 4.3 Diagonalized Hamiltonian

Putting everything together, we have diagonalized our one-loop Hamiltonian

$$H_{PT} = \int \frac{dk}{2\pi} \omega_k b_k^\dagger b_k + \frac{\pi_0^2}{2} + Q \tag{4.21}$$

where

$$\begin{aligned}
 Q &= Q_C + Q_B \\
 &= -\frac{1}{4} \int \frac{dk}{2\pi} \int \frac{dp}{2\pi} \frac{(\omega_p - \omega_k)^2}{\omega_p} \frac{\tilde{g}_k^2(p)}{C_k^2} - \frac{1}{4} \int \frac{dp}{2\pi} \frac{\tilde{g}_B(p) \tilde{g}_B(p)}{C_B^2} \omega_p
 \end{aligned} \tag{4.22}$$

is a scalar.

The Hamiltonian is seen to be just a sum of quantum harmonic oscillators described by  $b$  and  $b^\dagger$  plus a center of mass motion described by  $\phi_0$  and  $\pi_0$ . The lowest energy state  $\mathcal{O}|0\rangle$  therefore is the unique state which satisfies

$$b_k \mathcal{O}|0\rangle_0 = \pi_0 \mathcal{O}|0\rangle_0 = 0 \tag{4.23}$$

and it has energy  $E_K = E_{cl} + Q$  by (2.15) and (2.21) because

$$H' \mathcal{O}|0\rangle_1 = (E_{cl} + H_{PT}) \mathcal{O}|0\rangle_0 = (E_{cl} + Q) \mathcal{O}|0\rangle_0. \tag{4.24}$$

The excited states are just the oscillator excitations, made from products of  $b_k^\dagger$ , and arbitrary momenta may be considered within the validity of the one-loop approximation.

Numerically evaluating  $Q$ , we find

$$Q_C = -0.03404m, \quad Q_B = -0.284249m, \quad Q = -0.31825m, \tag{4.25}$$

which agrees with the result  $Q = -m/\pi$  obtained in ref. [4] using, essentially, the integrability [11, 12] of the sine-Gordon model.

## 5 Conclusion

We used the sine-Gordon model to test the method introduced in ref. [6] for the calculation of the one-loop correction to soliton masses. While the WKB method has been applied to both models [1, 2] it suffers from an ambiguity due to a choice of matching of regularization conditions [5]. However in the case of the sine-Gordon model, the soliton mass has been calculated unambiguously using integrability in ref. [4]. Therefore, the case treated in this paper provides a robust test of our method.

The quantum soliton in the sine-Gordon model is also of intrinsic interest. As the sine-Gordon model is understood at strong coupling, where it becomes the massive Thirring model [3], it may be possible to follow the soliton operator to strong coupling. At one loop the operator may be found by solving (4.23) for  $\mathcal{O}$ . Of course it is well-known that in the Thirring model it becomes the fundamental fermion [13], but it would be interesting to see what it becomes in terms of the strongly coupled sine-Gordon model itself. Perhaps this would give a hint as to what becomes of  $\mathcal{N} = 2$  SQCD monopoles [14] when the Higgs mass tends to zero and so the scalar condensate turns off and the infrared coupling becomes strong?

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