Published for SISSA by Springer

Received: October 22, 2019
Revised: January 11, 2020
Accepted: January 20, 2020
Published: February 4, 2020

Non-linear supersymmetry and $T\bar{T}$-like flows

Christian Ferko,[a] **Hongliang Jiang,**[b] **Savdeep Sethi**[a]
and Gabriele Tartaglino-Mazzucchelli[b,c]

[a] *Enrico Fermi Institute & Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, IL 60637, U.S.A.*

[b] *Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland*

[c] *School of Mathematics and Physics, University of Queensland, St Lucia, Brisbane, Queensland 4072, Australia*

E-mail: cferko@uchicago.edu, jiang@itp.unibe.ch, sethi@uchicago.edu, g.tartaglino-mazzucchelli@uq.edu.au

Abstract: The $T\bar{T}$ deformation of a supersymmetric two-dimensional theory preserves the original supersymmetry. Moreover, in several interesting cases the deformed theory possesses additional non-linearly realized supersymmetries. We show this for certain $\mathcal{N} = (2,2)$ models in two dimensions, where we observe an intriguing similarity with known $\mathcal{N} = 1$ models in four dimensions. This suggests that higher-dimensional models with non-linearly realized supersymmetries might also be obtained from $T\bar{T}$-like flow equations. We show that in four dimensions this is indeed the case for $\mathcal{N} = 1$ Born-Infeld theory, as well as for the Goldstino action for spontaneously broken $\mathcal{N} = 1$ supersymmetry.

Keywords: Extended Supersymmetry, Superspaces, Supersymmetry Breaking, Supersymmetric Gauge Theory

ArXiv ePrint: 1910.01599

https://doi.org/10.1007/JHEP02(2020)016

JHEP02(2020)016

Contents

1 Introduction **1**

2 $D = 2$ $\mathcal{N} = (2, 2)$ flows and non-linear $\mathcal{N} = (2, 2)$ Supersymmetry **3**
- 2.1 $T\bar{T}$ deformations with $\mathcal{N} = (2, 2)$ supersymmetry 3
- 2.2 The $T\bar{T}$-deformed twisted-chiral model and partial-breaking 7
- 2.3 The $T\bar{T}$-deformed chiral model and partial-breaking 12

3 $D = 4$ T^2 deformations and their supersymmetric extensions **15**
- 3.1 Comments on the T^2 operator in $D = 4$ 15
- 3.2 $D = 4$ $\mathcal{N} = 1$ supercurrent-squared operator 17

4 Bosonic Born-Infeld as a T^2 flow **19**

5 Supersymmetric Born-Infeld from supercurrent-squared deformation **21**
- 5.1 $D = 4$ $\mathcal{N} = 1$ supersymmetric BI and non-linear supersymmetry 21
- 5.2 Bosonic truncation 24
- 5.3 Supersymmetric Born-Infeld as a supercurrent-squared flow 25

6 $D = 4$ Goldstino action from supercurrent-squared deformation **27**
- 6.1 $D = 4$ Goldstino actions 28
- 6.2 $D = 4$ KS Goldstino model as a supercurrent-squared flow 29
  - 6.2.1 KS model 29
  - 6.2.2 $D = 4$ Goldstino action as a supercurrent-squared flow 30

7 Conclusions and outlook **31**

A Deriving a useful on-shell identity **34**

1 Introduction

There has been considerable recent excitement about quantum field theories in two dimensions deformed by the irrelevant operator $T\bar{T}$ [1, 2]. Part of the reason for excitement is that the deformed theory appears to be a new structure, which is neither a local quantum field theory nor a full-fledged string theory. There are many basic issues yet to be resolved, like how to define observables in the theory. What is understood, however, is the finite volume spectrum [2, 3] and the structure of the S-matrix [4, 5]. For a recent overview, see the review [6].

Another reason for excitement is apparent at the classical level. The $T\bar{T}$ deformation of a two-dimensional Lagrangian leads to a classical flow equation for the deformed Lagrangian $\mathcal{L}_\lambda(x)$ of the form

$$\frac{d}{d\lambda}\mathcal{L}_\lambda = -\frac{1}{8}T\bar{T} \propto \det\left(T_{\mu\nu}[\mathcal{L}_\lambda]\right), \tag{1.1}$$

where $T_{\mu\nu}[\mathcal{L}_\lambda]$ is the stress-energy tensor for the deformed theory at value λ of the flow parameter. When the undeformed theory is a free scalar theory,

$$S = \frac{1}{2}\int d^2x\, \partial_\mu\phi\partial^\mu\phi\,, \tag{1.2}$$

the deformed theory is the gauge-fixed Nambu-Goto string with string tension determined by the deformation parameter λ [3, 7]:

$$S = \int d^2x \left(-\frac{1}{2\lambda} + \frac{1}{2\lambda}\sqrt{1+2\lambda\partial_\sigma\phi\partial^\sigma\phi}\right) . \tag{1.3}$$

This is a beautiful connection between $T\bar{T}$ deformations and a field theory which classically possesses a non-linearly realized $D = 3$ Lorentz symmetry; for other connections between $T\bar{T}$ and classical string theory, see, for example, [8–15].

The magic of $T\bar{T}$ in two dimensions is that this composite operator is well-defined at the quantum level. This property does not currently extend to higher-dimensional candidates without some additional ingredients. One such potential ingredient is supersymmetry. Deforming a supersymmetric $D = 2$ theory with $T\bar{T}$ preserves the original supersymmetry of the theory. The supercurrent-squared operators that make the original supersymmetry manifest have been explicitly constructed for various theories in [15–18]. The usual $T\bar{T}$ operator is found as a supersymmetric descendant of supercurrent-squared up to equations of motion and total derivatives.

Some of the simplest examples studied so far are $T\bar{T}$ deformations of supersymmetric free theories. A remarkable feature of the deformed models is that the resulting interacting higher-derivative actions possess a set of hidden non-linear supersymmetries, in addition to their linearly realized ones. The deformed actions with $\mathcal{N} = (0, 1)$, $(1, 1)$ and $(0, 2)$ supersymmetry [15–17] coincide with gauge-fixed supersymmetric Nambu-Goto models, which exhibit various partial supersymmetry breaking patterns [19].

This connection between $T\bar{T}$ and structures which are central in string theory leads to a natural question: are more general classes of theories with non-linear symmetries related to flow equations for some analogue of $T\bar{T}$? One recent set of examples are the $\mathcal{N} = (2, 2)$ supersymmetric $T\bar{T}$-deformed actions of [18]. Do they also admit non-linear supersymmetries? The answer is yes! Following the ideas of [20], in this work we explicitly construct two models describing the partial supersymmetry breaking pattern $\mathcal{N} = (4, 4) \to \mathcal{N} = (2, 2)$ in $D = 2$. These models have manifest $\mathcal{N} = (2, 2)$ supersymmetry from the superspace structure used in their construction, but they also admit another hidden non-linear $\mathcal{N} = (2, 2)$ supersymmetry. It turns out the resulting actions are exactly the same as the $\mathcal{N} = (2, 2)$ chiral and twisted chiral $T\bar{T}$-deformed actions of [18]. The intriguing relation between non-linear supersymmetry and $T\bar{T}$ therefore persists for models with

JHEP02(2020)016

manifest $\mathcal{N} = (2,2)$ supersymmetry. Interestingly, even the $D = 2$ Volkov-Akulov action, describing the dynamics of the Goldstinos which arise from the spontaneous breaking of $\mathcal{N} = (2,2)$ supersymmetry, satisfies a $T\bar{T}$ flow equation [21].

This collection of examples motivates us to see whether any higher-dimensional theories with non-linear supersymmetries might also satisfy $T\bar{T}$-like flow equations. It has been known for more than two decades that the Bagger-Galperin action for the $D = 4$ $\mathcal{N} = 1$ Born-Infeld theory describes $\mathcal{N} = 2 \to \mathcal{N} = 1$ partial supersymmetry breaking [22]. Does the Bagger-Galperin action arise from a $T\bar{T}$-like deformation of $\mathcal{N} = 1$ Maxwell theory? That the linear order deformation is given by a supercurrent-squared operator was noted long ago in [23]. Much more recently, bosonic Born-Infeld theory was shown to satisfy a T^2 flow equation, where T^2 is an operator quadratic in the stress-energy tensor [24]. In this work, we explicitly show that the Bagger-Galperin action indeed satisfies a supercurrent-squared flow equation, generalizing the observation of [23] to all orders in the deformation parameter. The supercurrent-squared deformation operator is constructed from supercurrent multiplets, but its top component contains other currents besides the stress-energy tensor. This is different from $D = 2$ where the top component of the supercurrent-squared operator is exactly the standard $T\bar{T}$ operator on-shell.

This paper is organized as follows: in section 2, we show that $D = 2$ $\mathcal{N} = (2,2)$ deformed models of either free chiral or twisted chiral multiplets possess additional non-linearly realized $\mathcal{N} = (2,2)$ supersymmetries. In section 3, we describe a particular four-dimensional analogue of $T\bar{T}$ motivated by [24], and generalize it to a supercurrent-squared operator for theories with $\mathcal{N} = 1$ supersymmetry. Section 4 reviews the argument that relates bosonic Born-Infeld theory to the solution of a T^2 flow equation [24]. In section 5, we show that $\mathcal{N} = 1$ Born-Infeld theory satisfies a supercurrent-squared flow equation to all orders in the deformation parameter. In section 6 we show that a particular form of the $D = 4$ Goldstino action also satisfies a supercurrent-squared flow, generalizing the $D = 2$ result of [21]. We end with concluding thoughts in section 7. Appendix A contains a useful result for the analysis of section 5.

2 $D = 2$ $\mathcal{N} = (2,2)$ flows and non-linear $\mathcal{N} = (2,2)$ Supersymmetry

The $\mathcal{N} = (2,2)$ supersymmetric extension of $T\bar{T}$ was recently studied in [18], where the existence of extra non-linearly realized supersymmetries for some solutions of the $T\bar{T}$ flow equation was briefly discussed. In this section, we are going to explore in detail how these non-linear supersymmetries arise for the simplest $\mathcal{N} = (2,2)$ $T\bar{T}$ flows. The undeformed models are supersymmetrized theories of free scalars, while the deformed models are $\mathcal{N} = (2,2)$ supersymmetric extensions of the $D = 4$ gauge-fixed Nambu-Goto string studied in [18]. Before entering into the details of how the non-linear supersymmetry arises, let us review some of the results of [18] that are relevant for the analysis in this section.

2.1 $T\bar{T}$ deformations with $\mathcal{N} = (2,2)$ supersymmetry

The composite operator

$$T\bar{T}(x) = T_{++++}(x)\, T_{----}(x) - \big[\Theta(x)\big]^2 , \tag{2.1}$$

written here in light-cone coordinates, possesses several remarkable features. Although it is an irrelevant operator, it is quantum mechanically well-defined and preserves many of the symmetries of the undeformed theory [1–3].

In particular, $T\bar{T}$ deformations preserve supersymmetry along the flow [15–18, 25]. More specifically, the $T\bar{T}(x)$ operator of a supersymmetric theory is related to a supersymmetric descendant operator $\mathcal{T}\bar{\mathcal{T}}(x)$,

$$\mathcal{T}\bar{\mathcal{T}}(x) = T\bar{T}(x) + \text{EOM} + \partial_{++}(\cdots) + \partial_{--}(\cdots) \,. \tag{2.2}$$

The previous equation states the equivalence of $T\bar{T}(x)$ and $\mathcal{T}\bar{\mathcal{T}}(x)$ up to total derivatives and terms that vanish on-shell, which we have indicated with "EOM". When $\mathcal{N} = (2,2)$ supersymmetry is linearly realized and preserved along the flow, which is the case of interest for this analysis, $\mathcal{T}\bar{\mathcal{T}}(x)$ is expressed as a D-term, or full superspace integral, of a supercurrent-squared primary operator [18]:

$$\mathcal{T}\bar{\mathcal{T}}(x) = \int d^4\theta\, \mathcal{O}^{\text{FZ}}(x,\theta)\,, \quad \mathcal{O}^{\text{FZ}}(x,\theta) := -\mathcal{J}_{++}(x,\theta)\mathcal{J}_{--}(x,\theta) + 2\mathcal{V}(x,\theta)\bar{\mathcal{V}}(x,\theta)\,. \tag{2.3}$$

Here $\mathcal{J}_{\pm\pm}(x,\theta)$, $\mathcal{V}(x,\theta)$ and its complex conjugate $\bar{\mathcal{V}}(x,\theta)$ are the local operators describing the Ferrara-Zumino (FZ) supercurrent multiplet for $D = 2$ $\mathcal{N} = (2,2)$ supersymmetry [26, 27].[1] These operators satisfy the following conservation equations

$$\bar{D}_{\pm}\mathcal{J}_{\mp\mp} = \pm D_{\mp}\mathcal{V}\,, \qquad \bar{D}_{\pm}\mathcal{V} = 0\,, \tag{2.4}$$

together with their complex conjugates. In superspace, assuming the supersymmetric Lagrangian $\mathcal{L}_\lambda(x)$ along the flow is given by

$$\mathcal{L}_\lambda(x) = \int d^4\theta\, \mathcal{A}_\lambda(x,\theta)\,, \tag{2.5}$$

with $\mathcal{A}_\lambda(x,\theta)$ the full superspace Lagrangian, the flow equation can be rewritten in a manifestly $\mathcal{N} = (2,2)$ supersymmetric form:

$$\frac{d}{d\lambda}\mathcal{A}_\lambda = -\frac{1}{8}\mathcal{O}^{\text{FZ}} = \frac{1}{8}\left(\mathcal{J}_{++}\mathcal{J}_{--} - 2\mathcal{V}\bar{\mathcal{V}}\right)\,. \tag{2.6}$$

In [18] supersymmetric flows for various theories were studied. The simplest cases, on which we will focus in this section, are $T\bar{T}$-deformed theories of free scalars, fermions and auxiliary fields. In the case of $D = 2$ $\mathcal{N} = (2,2)$ supersymmetry, a scalar multiplet can have several different off-shell representations [28–31]. The two cases we will consider here are chiral and twisted-chiral supermultiplets, which are the most commonly studied cases.

In $\mathcal{N} = (2,2)$ superspace, parametrized by coordinates $\zeta^M = (x^{\pm\pm}, \theta^\pm, \bar{\theta}^\pm)$, let the complex superfields $X(x,\theta)$ and $Y(x,\theta)$ satisfy chiral and twisted-chiral constraints, respectively,

$$\bar{D}_{\pm}X = 0\,, \quad \bar{D}_{+}Y = D_{-}Y = 0\,. \tag{2.7}$$

[1] For simplicity, we have assumed that the $D = 2$ $\mathcal{N} = (2,2)$ theory possesses a well-defined FZ multiplet. For a description of the more general case where one needs to use the $\mathcal{N} = (2,2)$ $\mathcal{S}$-multiplet of currents, discussed in [27], to define the the supercurrent-squared operator we refer to the original analysis of [18].

JHEP02(2020)016

Here the supercovariant derivatives and supercharges are[2]

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\partial_{\pm\pm}\,, \quad \bar D_\pm = -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\partial_{\pm\pm}\,, \tag{2.8a}$$

$$Q_\pm = i\frac{\partial}{\partial\theta^\pm} + \bar\theta^\pm\partial_{\pm\pm}\,, \quad \bar Q_\pm = -i\frac{\partial}{\partial\bar\theta^\pm} - \theta^\pm\partial_{\pm\pm}\,, \tag{2.8b}$$

and they satisfy

$$D_\pm^2 = \bar D_\pm^2 = 0\,, \qquad \{D_\pm, \bar D_\pm\} = -2i\partial_{\pm\pm}\,, \qquad [D_\pm, \partial_{\pm\pm}] = [\bar D_\pm, \partial_{\pm\pm}] = 0\,, \tag{2.9a}$$

$$Q_\pm^2 = \bar Q_\pm^2 = 0\,, \qquad \{Q_\pm, \bar Q_\pm\} = -2i\partial_{\pm\pm}\,, \qquad [Q_\pm, \partial_{\pm\pm}] = [\bar Q_\pm, \partial_{\pm\pm}] = 0\,. \tag{2.9b}$$

There is one more caveat worth mentioning: in much of the $\mathcal{N} = (2,2)$ literature, twisted-chiral multiplets, often denoted Σ in this context, naturally arise as field strengths for $\mathcal{N} = (2,2)$ vector superfields V. The lowest component of such a superfield is a complex scalar, but the top component proportional to $\bar\theta^-\theta^+$ encodes the gauge-field strength along with a real auxiliary field. On the other hand, there are twisted chiral superfields denoted Y whose bottom component is a complex scalar and whose top component is just a complex auxiliary field. It is to this latter case that we restrict. The free Lagrangians for these supermultiplets are given by

$$\mathcal{L}_0^{\rm c} = \int d^4\theta\, X\bar X\,, \qquad \mathcal{L}_0^{\rm tc} = -\int d^4\theta\, Y\bar Y\,. \tag{2.10}$$

In [18] it was shown that the following Lagrangian

$$\mathcal{L}_\lambda^{\rm c} = \int d^4\theta \left(X\bar X + \frac{\lambda D_+ X \bar D_+ \bar X D_- X \bar D_- \bar X}{1 - \frac{1}{2}\lambda A + \sqrt{1 - \lambda A + \frac{1}{4}\lambda^2 B^2}} \right)\,, \tag{2.11a}$$

with

$$A = \partial_{++}X\partial_{--}\bar X + \partial_{++}\bar X\partial_{--}X\,, \quad B = \partial_{++}X\partial_{--}\bar X - \partial_{++}\bar X\partial_{--}X\,, \tag{2.11b}$$

is a solution of the flow equation (2.6) on-shell, and hence describes the $T\bar T$-deformation (1.1) of the free chiral supermultiplet Lagrangian (2.10).

A simple way to generate the $T\bar T$-deformation of the free twisted-chiral theory is to remember that a twisted-chiral multiplet can be obtained from a chiral one by acting with a $\mathbb{Z}_2$ automorphism on the Grassmann coordinates of $\mathcal{N} = (2,2)$ superspace:

$$\theta^+ \leftrightarrow \theta^+\,, \quad \theta^- \leftrightarrow -\bar\theta^-\,. \tag{2.12}$$

This leaves the D_+ and $\bar D_+$ derivatives invariant while it exchanges D_- with $\bar D_-$. As a result, the chiral and twisted-chiral differential constraints (2.7) are mapped into each others under the automorphism (2.12).[3]

[2]The reader should be aware that in this section we follow the notation of [32], which is slightly different from the notation used in [18].

[3]In the literature this $\mathbb{Z}_2$ automorphism (2.12) is often called the "mirror-map" or "mirror-image" because it exchanges the vector and axial U(1) R-symmetries.

Under the $\mathbb{Z}_2$ automorphism (2.12), the Lagrangian (2.11a) turns into the following twisted-chiral Lagrangian

$$\mathcal{L}_\lambda^{\mathrm{tc}} = -\int d^4\theta \left(Y\bar{Y} + \frac{\lambda D_+ Y \bar{D}_+ \bar{Y} \bar{D}_- Y D_- \bar{Y}}{1 - \frac{1}{2}\lambda A + \sqrt{1 - \lambda A + \frac{1}{4}\lambda^2 B^2}} \right) , \tag{2.13a}$$

where

$$A = \partial_{++} Y \partial_{--} \bar{Y} + \partial_{++} \bar{Y} \partial_{--} Y , \quad B = \partial_{++} Y \partial_{--} \bar{Y} - \partial_{++} \bar{Y} \partial_{--} Y . \tag{2.13b}$$

Thanks to the map (2.12), by construction the Lagrangian (2.13a) is a $T\bar{T}$-deformation (1.1) and its superspace Lagrangian $\mathcal{A}_\lambda^{\mathrm{tc}}$, $\mathcal{L}_\lambda^{\mathrm{tc}} = \int d^4\theta\, \mathcal{A}_\lambda^{\mathrm{tc}}$, is an on-shell solution of the following flow equation

$$\frac{d}{d\lambda} \mathcal{A}_\lambda^{\mathrm{tc}} = \frac{1}{8} \left(\mathcal{R}_{++} \mathcal{R}_{--} - 2\mathcal{B}\bar{\mathcal{B}} \right) . \tag{2.14}$$

Here $\mathcal{R}_{\pm\pm}(x,\theta)$, $\mathcal{B}(x,\theta)$ and its complex conjugate $\bar{\mathcal{B}}(x,\theta)$ are the local operators describing the $\mathcal{R}$-multiplet of currents for $D = 2$ $\mathcal{N} = (2,2)$ supersymmetry that arise by applying (2.12) to the FZ multiplet of the chiral theory (2.11a) [18]. They satisfy the conservation equations,

$$\bar{D}_+ \mathcal{R}_{--} = i\bar{D}_- \mathcal{B} , \qquad D_- \mathcal{R}_{++} = iD_+ \mathcal{B} , \qquad \bar{D}_+ \mathcal{B} = D_- \mathcal{B} = 0 , \tag{2.15}$$

together with their complex conjugates. Like the case of the FZ-multiplet, the supercurrent-squared operator

$$\mathcal{T}\bar{\mathcal{T}}(x) = \int d^4\theta\, \mathcal{O}^{\mathcal{R}}(x,\theta) , \quad \mathcal{O}^{\mathcal{R}}(x,\theta) := -\mathcal{R}_{++}(x,\theta)\mathcal{R}_{--}(x,\theta) + 2\mathcal{B}(x,\theta)\bar{\mathcal{B}}(x,\theta) , \tag{2.16}$$

satisfies (2.2); namely, $\mathcal{T}\bar{\mathcal{T}}(x)$ is equivalent to $T\bar{T}(x)$ up to total derivatives and EOM [18].

Note that the bosonic truncation of both (2.11a) and (2.13a) give the Lagrangian

$$\mathcal{L}_{\lambda,\mathrm{bos}} = \frac{\sqrt{1 + 2\lambda a + \lambda^2 b^2} - 1}{4\lambda} = \frac{a}{4} - \lambda \frac{\partial_{++}\phi \partial_{--}\phi \partial_{++}\bar{\phi}\partial_{--}\bar{\phi}}{1 + \lambda a + \sqrt{1 + 2\lambda a + \lambda^2 b^2}} , \tag{2.17}$$

where

$$a = \partial_{++}\phi\partial_{--}\bar{\phi} + \partial_{++}\bar{\phi}\partial_{--}\phi , \qquad b = \partial_{++}\phi\partial_{--}\bar{\phi} - \partial_{++}\bar{\phi}\partial_{--}\phi , \tag{2.18}$$

and ϕ is either $\phi = X|_{\theta=0}$ or $\phi = Y|_{\theta=0}$. This is the Lagrangian for the gauge-fixed Nambu-Goto string in four dimensions [3].

The aim of the remainder of this section is to show that the Lagrangians (2.11a) and (2.13a) are structurally identical to the Bagger-Galperin action for the $D = 4$ $\mathcal{N} = 1$ supersymmetric Born-Infeld theory [22], which we will analyse in detail in section 5. Since the Bagger-Galperin action possesses a second non-linearly realized $D = 4$ $\mathcal{N} = 1$ supersymmetry, we will show that the theories described by (2.11a) and (2.13a) also possess an extra set of non-linearly realized $\mathcal{N} = (2,2)$ supersymmetries.

JHEP02(2020)016

2.2 The $T\bar{T}$-deformed twisted-chiral model and partial-breaking

Let us start with the twisted-chiral Lagrangian (2.13a) which, as we will show, is the one more directly related to the $D = 4$ Bagger-Galperin action. In complete analogy to the $D = 4$ case, we are going to show that (2.13a) is a model for a Nambu-Goldstone multiplet of $D = 2$ $\mathcal{N} = (4, 4) \to \mathcal{N} = (2, 2)$ partial supersymmetry breaking. The analysis is similar in spirit to the $D = 4$ construction of the Bagger-Galperin action using $D = 4$ $\mathcal{N} = 2$ superspace proposed by Roček and Tseytlin [20]; see also [33–35] for more recent analysis.

To describe manifest $\mathcal{N} = (4, 4)$ supersymmetry we can use $\mathcal{N} = (4, 4)$ superspace which augments the $\mathcal{N} = (2, 2)$ superspace coordinates $\zeta^M = (x^{\pm\pm}, \theta^\pm, \bar{\theta}^\pm)$ of the previous section with the following additional complex Grassmann coordinates $(\eta^\pm, \bar{\eta}^\pm)$. The extra supercovariant derivatives and supercharges are given by

$$\mathcal{D}_+ = \frac{\partial}{\partial\eta^+} + i\bar{\eta}^+\partial_{++}\,, \quad \bar{\mathcal{D}}_+ = -\frac{\partial}{\partial\bar{\eta}^+} - i\eta^+\partial_{++}\,, \tag{2.19a}$$

$$\mathcal{Q}_+ = i\frac{\partial}{\partial\eta^+} + \bar{\eta}^+\partial_{++}\,, \quad \bar{\mathcal{Q}}_+ = -i\frac{\partial}{\partial\bar{\eta}^+} - \eta^+\partial_{++}\,, \tag{2.19b}$$

with similar expressions for $\mathcal{D}_-$ and $\mathcal{Q}_-$. They satisfy

$$\mathcal{D}_\pm^2 = \bar{\mathcal{D}}_\pm^2 = 0\,, \qquad \{\mathcal{D}_\pm, \bar{\mathcal{D}}_\pm\} = -2i\partial_{\pm\pm}\,, \qquad [\mathcal{D}_\pm, \partial_{\pm\pm}] = [\bar{\mathcal{D}}_\pm, \partial_{\pm\pm}] = 0\,, \tag{2.20a}$$

$$\mathcal{Q}_\pm^2 = \bar{\mathcal{Q}}_\pm^2 = 0\,, \qquad \{\mathcal{Q}_\pm, \bar{\mathcal{Q}}_\pm\} = -2i\partial_{\pm\pm}\,, \qquad [\mathcal{Q}_\pm, \partial_{\pm\pm}] = [\bar{\mathcal{Q}}_\pm, \partial_{\pm\pm}] = 0\,, \tag{2.20b}$$

while they (anti-)commute with all the usual $D_\pm$ and $Q_\pm$ operators.

Two-dimensional $\mathcal{N} = (4, 4)$ supersymmetry can also be usefully described in the language of $\mathcal{N} = (2, 2)$ superspace. In this section, we will largely refer to [32] for such a description. In this approach from the full $(4, 4)$ supersymmetry, one copy of $(2, 2)$ is manifest while a second $(2, 2)$ is hidden. For our goal of describing a model of partial supersymmetry breaking, we view the hidden $(2, 2)$ supersymmetry as broken and non-linearly realized. We will derive such a description starting from $\mathcal{N} = (4, 4)$ superspace and describe the broken/hidden supersymmetry using the $\eta^\pm$ directions.

The hidden supersymmetry transformation of a generic $D = 2$ $\mathcal{N} = (4, 4)$ superfield $U = U(x^{\pm\pm}, \theta^\pm, \bar{\theta}^\pm, \eta^\pm, \bar{\eta}^\pm)$ under the hidden $(2, 2)$ supersymmetry is

$$\delta U = i(\epsilon^+\mathcal{Q}_+ + \epsilon^-\mathcal{Q}_- - \bar{\epsilon}^+\bar{\mathcal{Q}}_+ - \bar{\epsilon}^-\bar{\mathcal{Q}}_-)U\,. \tag{2.21}$$

The $(2, 2)$ supersymmetry, generated by the $Q_\pm$ and $\bar{Q}_\pm$ operators, will always be manifest and preserved, so we will not bother to discuss it in detail. For convenience, we also introduce the chiral coordinate $y^{\pm\pm} = x^{\pm\pm} + i\eta^\pm\bar{\eta}^\pm$. Using this coordinate, the spinor covariant derivatives and supercharges take the form

$$\mathcal{D}_\pm = \frac{\partial}{\partial\eta^\pm} + 2i\bar{\eta}^\pm\frac{\partial}{\partial y^{\pm\pm}}\,, \qquad \bar{\mathcal{D}}_\pm = -\frac{\partial}{\partial\bar{\eta}^\pm}\,, \tag{2.22a}$$

$$\mathcal{Q}_\pm = i\frac{\partial}{\partial\eta^\pm}\,, \qquad \bar{\mathcal{Q}}_\pm = -i\frac{\partial}{\partial\bar{\eta}^\pm} - 2\eta^\pm\frac{\partial}{\partial y^{\pm\pm}}\,. \tag{2.22b}$$

After this technical introduction, let us turn to our main construction. Consider a $(4,4)$ superfield which is chiral under the hidden $(2,2)$ supersymmetry:

$$\bar{\mathcal{D}}_\pm \boldsymbol{\mathcal{X}} = 0 \,. \tag{2.23}$$

We can expand it in terms of hidden fermionic coordinates,

$$\boldsymbol{\mathcal{X}} = X + \eta^+ X_+ + \eta^- X_- + \eta^+\eta^- F \,, \tag{2.24}$$

where $X = X(y^{\pm\pm}, \theta^\pm, \bar\theta^\pm)$, $X_\pm = X_\pm(y^{\pm\pm}, \theta^\pm, \bar\theta^\pm)$ and $F = F(y^{\pm\pm}, \theta^\pm, \bar\theta^\pm)$ are themselves $(2,2)$ superfields. In the following discussion, we will keep the $\theta^\pm$, $\bar\theta^\pm$ dependence implicit. The hidden $(2,2)$ supersymmetry transformation rules can then be straightforwardly computed using (2.21) and (2.24). They take the form

$$\delta X = -\epsilon^+ X_+ - \epsilon^- X_- \,, \tag{2.25a}$$
$$\delta X_\pm = \mp\epsilon^\mp F - 2i\bar\epsilon^\pm \partial_{\pm\pm} X, \tag{2.25b}$$
$$\delta F = -2i\bar\epsilon^- \partial_{--} X_+ + 2i\bar\epsilon^+ \partial_{++} X_- \,. \tag{2.25c}$$

The $\boldsymbol{\mathcal{X}}$ superfield is still reducible under $\mathcal{N} = (4,4)$ supersymmetry so we can put additional constraints on the $(2,2)$ superfields X, $X_\pm$ and F. Here we will consider $(4,4)$ twisted multiplets, and refer the reader to [28, 36–41] for a more detailed analysis. For this discussion, we will follow the $\mathcal{N} = (2,2)$ superspace description of [32]. One type of twisted multiplet with $(4,4)$ supersymmetry can be defined by setting

$$X_+ = \bar D_+ \bar Y, \qquad X_- = -\bar D_- Y \,, \tag{2.26}$$

where X and Y are chiral and twisted-chiral, respectively, under the manifest $(2,2)$ supersymmetry:

$$\bar D_+ X = \bar D_- X = \bar D_+ Y = D_- Y = 0 \,, \qquad D_+ \bar X = D_- \bar X = D_+ \bar Y = \bar D_- \bar Y = 0 \,. \tag{2.27}$$

The superfield (2.24) becomes

$$\boldsymbol{\mathcal{X}} = X + \eta^+ \bar D_+ \bar Y - \eta^- \bar D_- Y + \eta^+\eta^- F \,. \tag{2.28}$$

The supersymmetry transformation rules then become

$$\delta X = -\epsilon^+ \bar D_+ \bar Y + \epsilon^- \bar D_- Y \,, \tag{2.29a}$$
$$\delta F = -2i\bar\epsilon^- \partial_{--} \bar D_+ \bar Y - 2i\bar\epsilon^+ \partial_{++} \bar D_- Y \,, \tag{2.29b}$$

while $\delta X_\pm$ remains the same as (2.25b). By using the conjugation property for two fermions, $\overline{\chi\xi} = \bar\xi\bar\chi = -\bar\chi\bar\xi$, and the conjugation property $\overline{D_+ A} = \bar D_+ \bar A$ for a bosonic superfield A, it follows that

$$\delta\bar X = \bar\epsilon^+ D_+ Y - \bar\epsilon^- D_- \bar Y \,. \tag{2.30}$$

One can check that

$$\delta \bar D^2 \bar X = \bar D^2 \delta \bar X = \bar D^2 \left(\bar\epsilon^+ D_+ Y - \bar\epsilon^- D_- \bar Y\right) = 2i\bar\epsilon^+ \partial_{++} \bar D_- Y + 2i\bar\epsilon^- \partial_{--} \bar D_+ Y \,, \tag{2.31}$$

where $\bar{D}^2 = \bar{D}_+\bar{D}_-$. Note that in the first equality we made use of the fact that the manifest and hidden $(2,2)$ supersymmetries are independent. The supersymmetry transformation rule for $-\bar{D}^2 X$ is then exactly that of the auxiliary field F. Thus we can consistently set

$$F = -\bar{D}^2\bar{X}\,, \tag{2.32}$$

which is the last constraint necessary to describe a version of the $(4,4)$ twisted multiplet in terms of a chiral and twisted-chiral $\mathcal{N} = (2,2)$ superfields. The resulting $(4,4)$ superfield $\boldsymbol{\mathcal{X}}$, expanded in terms of the hidden $(2,2)$ fermionic coordinates, takes the form

$$\boldsymbol{\mathcal{X}} = X + \eta^+\bar{D}_+\bar{Y} - \eta^-\bar{D}_-Y - \eta^+\eta^-\bar{D}^2\bar{X}\,, \tag{2.33}$$

which closely resembles the expansion of a $D = 4$ $\mathcal{N} = 2$ vector multiplet when one identifies the analogue of the $D = 4$ $\mathcal{N} = 1$ chiral vector multiplet field strength W_α with the $(2,2)$ chiral superfields $\bar{D}_+\bar{Y}$ and $\bar{D}_-Y$. Note in particular that $\boldsymbol{\mathcal{X}}$ turns to be $(4,4)$ chiral:

$$\bar{D}_\pm\boldsymbol{\mathcal{X}} = 0\,, \quad \bar{\mathcal{D}}_\pm\boldsymbol{\mathcal{X}} = 0\,. \tag{2.34}$$

To summarize: the entire $(4,4)$ off-shell twisted multiplet is described in terms of one chiral and one twisted-chiral $(2,2)$ superfield, which possess the following hidden $(2,2)$ supersymmetry transformations:

$$\delta X = -\epsilon^+\bar{D}_+\bar{Y} + \epsilon^-\bar{D}_-Y\,, \tag{2.35a}$$
$$\delta Y = \bar{\epsilon}^-D_-X + \epsilon^+\bar{D}_+\bar{X}\,. \tag{2.35b}$$

Let us now introduce the action for a free $\mathcal{N} = (4,4)$ twisted multiplet. Taking the square of $\boldsymbol{\mathcal{X}}$ in (2.33) we obtain

$$\boldsymbol{\mathcal{X}}^2 = \eta^+\eta^-\Big(-2X\bar{D}^2\bar{X} + 2\bar{D}_+\bar{Y}\bar{D}_-Y\Big) + \dots\,, \tag{2.36}$$

where the ellipses denote terms that are not important for our analysis. Since $\boldsymbol{\mathcal{X}}$ and therefore $\boldsymbol{\mathcal{X}}^2$ are chiral superfields, we can consider the chiral integral in the hidden direction

$$\int d\eta^+d\eta^-\boldsymbol{\mathcal{X}}^2 = 2X\bar{D}^2\bar{X} - 2\bar{D}_+\bar{Y}\cdot\bar{D}_-Y\,. \tag{2.37}$$

Note also that, since X and Y are chiral and twisted-chiral under the manifest supersymmetry (2.27), it follows that

$$\begin{aligned}\int d^2x\, d\theta^+d\theta^-d\bar{\theta}^+d\bar{\theta}^-(X\bar{X} - Y\bar{Y}) &= \int d^2x\, d\theta^+d\theta^-\bar{D}_+\bar{D}_-(X\bar{X} - Y\bar{Y})\,,\\ &= \int d^2x\, d\theta^+d\theta^-\Big(X\bar{D}_+\bar{D}_-\bar{X} - \bar{D}_+\bar{Y}\cdot\bar{D}_-Y\Big)\,,\end{aligned} \tag{2.38}$$

which can also be rewritten as

$$\int d^2x\, d\theta^+d\theta^-d\bar{\theta}^+d\bar{\theta}^-(X\bar{X} - Y\bar{Y}) = \int d^2x\, d\bar{\theta}^+d\bar{\theta}^-\Big(\bar{X}D_+D_-X - D_+Y\cdot D_-\bar{Y}\Big)\,. \tag{2.39}$$

The sum of the two equations above yields

$$4\int d^2x\, d\theta^+ d\theta^- d\bar\theta^+ d\bar\theta^- (X\bar X - Y\bar Y) = \int d^2x\, d\theta^+ d\theta^- d\eta^+ d\eta^- \boldsymbol{\mathcal{X}}^2 + c.c.\,. \tag{2.40}$$

The left-hand side has an enhanced $\mathcal{N}=(4,4)$ supersymmetry as discussed in [32]. This becomes manifest from our $(4,4)$ superspace construction on the right-hand side.

To describe $\mathcal{N}=(4,4)\to\mathcal{N}=(2,2)$ supersymmetry breaking we can appropriately deform the $(4,4)$ twisted multiplet. Analogous to the case of a $D=4$ $\mathcal{N}=2$ vector multiplet deformed by a magnetic Fayet-Iliopoulos term [42] (see also [20, 33–35, 43]), we add a deformation parameter to the auxiliary field F of $\boldsymbol{\mathcal{X}}$, which is deformed to

$$\boldsymbol{\mathcal{X}}_{\rm def} = X + \eta^+ D_+\bar Y - \eta^- \bar D_- Y - \eta^+\eta^-\left(\bar D^2\bar X + \kappa\right)\,. \tag{2.41}$$

Assuming that the auxiliary field F gets a VEV, $\langle F\rangle = \kappa$ or equivalently $\langle \bar D^2\bar X\rangle = 0$, then by looking at the supersymmetry transformations of $X_\pm$ for the deformed multiplet

$$\delta X_\pm = \pm\epsilon^\mp\left(\bar D^2\bar X + \kappa\right) - 2i\bar\epsilon^\pm\partial_{\pm\pm}X\,, \tag{2.42}$$

we can see the $\mathcal{N}=(4,4)\to\mathcal{N}=(2,2)$ supersymmetry breaking pattern arises; specifically, the hidden $\mathcal{N}=(2,2)$ is spontaneously broken and non-linearly realized. For later use, it is important to stress that, though the hidden transformations of $\delta X_\pm$ are modified by the non-linear term proportional to κ, the hidden transformation of X remains the same as in the undeformed case given in eq. (2.29a).

In analogy to the $D=4$ case of [20, 33, 34], to describe the Goldstone multiplet associated to partial supersymmetry breaking we impose the following nilpotent constraint on the deformed $(4,4)$ twisted superfield:

$$\boldsymbol{\mathcal{X}}^2_{\rm def} = 0 = -2\eta^+\eta^-\left(X(\kappa+\bar D^2\bar X) - \bar D_+\bar Y\cdot\bar D_- Y\right) + \dots\,. \tag{2.43}$$

This implies the constraint

$$X\left(\kappa+\bar D^2\bar X\right) - \bar D_+\bar Y\cdot\bar D_- Y = 0\,, \tag{2.44}$$

which requires

$$X = \frac{\bar D_+\bar Y\cdot\bar D_- Y}{\kappa+\bar D^2\bar X} = \frac{W^2}{\kappa+\bar D^2\bar X}\,, \tag{2.45}$$

and its conjugate

$$\bar X = -\frac{D_+ Y\cdot D_-\bar Y}{\kappa + D^2 X} = \frac{\bar W^2}{\kappa+D^2X}\,. \tag{2.46}$$

Here $\bar D^2 = \bar D_+\bar D_-$, $D^2 = -D_+D_-$ and we have introduced the superfields:

$$W^2 = -X_+X_- = \bar D_+\bar Y\cdot\bar D_- Y = \bar D_+\bar D_-(Y\bar Y) = \bar D^2(Y\bar Y)\,, \tag{2.47a}$$

$$\bar W^2 = \bar X_+\bar X_- = -D_+Y\cdot D_-\bar Y = -D_+D_-(Y\bar Y) = D^2(Y\bar Y)\,. \tag{2.47b}$$

The constraint (2.44) is the $D = 2$ analogue of the Bagger-Galperin constraint for a Maxwell-Goldstone multiplet for $D = 4$ $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking [22]. Combining (2.45) and (2.46) gives

$$\kappa X = \bar{D}^2(Y\bar{Y} - X\bar{X}) = \bar{D}^2\left[Y\bar{Y} - \frac{\bar{D}_+\bar{Y} \cdot \bar{D}_-Y \cdot D_-\bar{Y} \cdot D_+Y}{(\kappa + D^2X)(\kappa + \bar{D}^2\bar{X})}\right], \tag{2.48}$$

which is consistent thanks to the κ terms in the denominator. Because of the four fermion coupling in the numerator of the last term, no fermionic terms can appear in the denominator. So effectively we have the equation

$$(\kappa + D^2X)_{\text{eff}} = \left(\kappa + D^2\frac{W^2}{\kappa + \bar{D}^2\bar{X}}\right)_{\text{eff}} = \kappa + \frac{D^2W^2}{\kappa + (\bar{D}^2\bar{X})_{\text{eff}}}\,, \tag{2.49}$$

and its conjugate

$$(\kappa + \bar{D}^2\bar{X})_{\text{eff}} = \kappa + \frac{\bar{D}^2\bar{W}^2}{\kappa + (D^2X)_{\text{eff}}}\,. \tag{2.50}$$

Solving them we get

$$(D^2X)_{\text{eff}} = \frac{B - \kappa^2 + \sqrt{B^2 + 2\kappa^2A + \kappa^4}}{2\kappa}\,, \tag{2.51a}$$

$$(\bar{D}^2\bar{X})_{\text{eff}} = \frac{-B - \kappa^2 + \sqrt{B^2 + 2\kappa^2A + \kappa^4}}{2\kappa}\,. \tag{2.51b}$$

Substituting these expressions into (2.48) gives

$$X = \frac{1}{\kappa}\bar{D}^2\Upsilon\,, \qquad \bar{X} = \frac{1}{\kappa}D^2\Upsilon\,, \qquad \Upsilon = \bar{\Upsilon} = Y\bar{Y} - \frac{2W^2\bar{W}^2}{A + \kappa^2 + \sqrt{B^2 + 2\kappa^2A + \kappa^4}}\,, \tag{2.52}$$

where

$$A = D^2W^2 + \bar{D}^2\bar{W}^2 = \{D^2, \bar{D}^2\}(Y\bar{Y}) = \partial_{++}Y\partial_{--}\bar{Y} + \partial_{++}\bar{Y}\partial_{--}Y\,, \tag{2.53a}$$

$$B = D^2W^2 - \bar{D}^2\bar{W}^2 = [D^2, \bar{D}^2](Y\bar{Y}) = \partial_{++}Y\partial_{--}\bar{Y} - \partial_{++}\bar{Y}\partial_{--}Y\,. \tag{2.53b}$$

The result is that the $\mathcal{N} = (2,2)$ chiral part X of the $\mathcal{N} = (4,4)$ twisted multiplet is expressed in terms of the $(2,2)$ twisted-chiral superfield Y. Thanks to the linearly realized construction in terms of $(4,4)$ superfields, it is straightforward to obtain the non-linearly realized $\mathcal{N} = (2,2)$ supersymmetry transformations for Y. In particular, it suffices to look at the transformations of D_+Y and $\bar{D}_-Y$ that can be obtained by substituting back the composite expression for $X = X[Y]$ into the transformations (2.42). By construction, these expressions ensure that δX transforms according to (2.29a).

Since X is chiral under the manifest $(2,2)$ supersymmetry (2.27), we can consider the chiral integral

$$\begin{aligned} S_{\kappa^2} &= -\frac{1}{2}\kappa\int d^2x\, d\theta^+ d\theta^- X + \text{c.c.} = -\frac{1}{2}\int d^2x\, d\theta^+ d\theta^- \bar{D}^2\Upsilon + \text{c.c.} \\ &= -\int d^2x\, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \Upsilon\,. \end{aligned} \tag{2.54}$$

A remarkable property of this action is that it is invariant under the hidden non-linearly realized supersymmetry. Using (2.29a), we see that

$$\delta S_{\kappa^2} = -\frac{1}{2}\kappa \int d^2x D_+ D_- \delta X\Big|_{\theta=\bar\theta=0} + \text{c.c.}\,, \tag{2.55a}$$

$$= -\frac{1}{2}\kappa \int d^2x \Big(-2i\epsilon^- \partial_{--} D_+ Y - 2i\epsilon^+ \partial_{++} D_- \bar Y \Big)\Big|_{\theta=\bar\theta=0} + \text{c.c.} = 0\,, \tag{2.55b}$$

where we used the fact that Y is a twisted-chiral superfield (2.27).

Explicitly, the action reads

$$S_{\kappa^2} = -\int d^2x\, d\theta^+ d\theta^- d\bar\theta^+ d\bar\theta^- \left(Y\bar Y - \frac{2W^2\bar W^2}{\kappa^2 + A + \sqrt{\kappa^4 + 2\kappa^2 A + B^2}} \right), \tag{2.56}$$

which precisely matches the model of eq. (2.13a) if we identify the coupling constants:

$$\lambda = -\frac{2}{\kappa^2}\,. \tag{2.57}$$

This shows explicitly that the $T\bar T$-deformation of the free twisted-chiral action possesses a non-linearly realized $\mathcal{N} = (2,2)$ hidden supersymmetry.

2.3 The $T\bar T$-deformed chiral model and partial-breaking

Let us now turn to the $T\bar T$ deformation of the free chiral model of eq. (2.11a). The construction follows the previous subsection with the difference that we will start with a different formulation of the $(4,4)$ twisted multiplet described in terms of $(2,2)$ superfields. Consider again an $\mathcal{N} = (4,4)$ superfield which is chiral under the hidden (2,2) supersymmetry:

$$\bar{\mathcal{D}}_+ \boldsymbol{\mathcal{Y}} = \bar{\mathcal{D}}_- \boldsymbol{\mathcal{Y}} = 0\,. \tag{2.58}$$

Its expansion in hidden superspace variables is

$$\boldsymbol{\mathcal{Y}} = Y + \eta^+ Y_+ + \eta^- Y_- + \eta^+\eta^- G\,, \tag{2.59}$$

where $Y = Y(y^{\pm\pm}, \theta^\pm, \bar\theta^\pm)$, $Y_\pm = Y_\pm(y^{\pm\pm}, \theta^\pm, \bar\theta^\pm)$ and $G = G(y^{\pm\pm}, \theta^\pm, \bar\theta^\pm)$ are themselves superfields with manifest $(2,2)$ supersymmetry. The hidden $(2,2)$ supersymmetry transformation rules of the components are

$$\delta Y = -\epsilon^+ Y_+ - \epsilon^- Y_-\,, \tag{2.60a}$$

$$\delta Y_\pm = \mp\epsilon^\mp G - 2i\bar\epsilon^\pm \partial_{\pm\pm} Y\,, \tag{2.60b}$$

$$\delta G = -2i\bar\epsilon^- \partial_{--} Y_+ + 2i\bar\epsilon^+ \partial_{++} Y_-\,. \tag{2.60c}$$

This representation of $(4,4)$ off-shell supersymmetry is again reducible so we can impose constraints. As in the construction of the previous section, we impose

$$Y_+ = \bar D_+ \bar X\,, \qquad Y_- = D_- X\,, \tag{2.61}$$

then

$$\boldsymbol{\mathcal{Y}} = Y + \eta^+ \bar D_+ \bar X + \eta^- D_- X + \eta^+\eta^- G\,. \tag{2.62}$$

Here X and Y are consistently chosen to be chiral and twisted-chiral under the manifest $(2,2)$ supersymmetry:

$$\bar{D}_+X = \bar{D}_-X = \bar{D}_+Y = D_-Y = 0\,, \qquad D_+\bar{X} = D_-\bar{X} = D_+\bar{Y} = \bar{D}_-\bar{Y} = 0\ . \tag{2.63}$$

Then we have

$$\delta Y = -\epsilon^+\bar{D}_+\bar{X} - \epsilon^-D_-X\,, \tag{2.64}$$

as well as its conjugate

$$\delta\bar{Y} = \bar{\epsilon}^+D_+X + \bar{\epsilon}^-\bar{D}_-\bar{X}\ . \tag{2.65}$$

Hence it follows that

$$\delta(\bar{D}_+D_-\bar{Y}) = \bar{D}_+D_-\delta\bar{Y} = 2i\bar{\epsilon}^+\partial_{++}D_-X - 2i\bar{\epsilon}^-\partial_{--}\bar{D}_+\bar{X}\ . \tag{2.66}$$

This should be compared with

$$\delta G = 2i\bar{\epsilon}^+\partial_{++}D_-X - 2i\bar{\epsilon}^-\partial_{--}\bar{D}_+\bar{X}\,, \tag{2.67}$$

showing that $\bar{D}_+D_-\bar{Y}$ transforms exactly like the auxiliary field G. This enables us to further constrain the $(4,4)$ multiplet by setting

$$G = \bar{D}_+D_-\bar{Y}\ . \tag{2.68}$$

Imposing these conditions gives a $(4,4)$ twisted superfield

$$\boldsymbol{\mathcal{Y}} = Y + \eta^+\bar{D}_+\bar{X} + \eta^-D_-X + \eta^+\eta^-\bar{D}_+D_-\bar{Y}\,, \tag{2.69}$$

which by construction is twisted-chiral and chiral with respect to the manifest and hidden $(2,2)$ supersymmetries, respectively:

$$\bar{D}_+\boldsymbol{\mathcal{Y}} = D_-\boldsymbol{\mathcal{Y}} = 0\,, \quad \bar{\mathcal{D}}_\pm\boldsymbol{\mathcal{Y}} = 0\ . \tag{2.70}$$

Its free dynamical action can be easily constructed by considering its square

$$\boldsymbol{\mathcal{Y}}^2 = 2\eta^+\eta^-\Big(Y\bar{D}_+D_-\bar{Y} - \bar{D}_+\bar{X}\cdot D_-X\Big) + \dots\ . \tag{2.71}$$

In fact, the following relations hold:

$$\begin{aligned}\int d^2x\,d\theta^+d\theta^-d\bar{\theta}^+d\bar{\theta}^-(X\bar{X}-Y\bar{Y}) &= \int d^2x\,d\theta^+d\bar{\theta}^-\bar{D}_+D_-(X\bar{X}-Y\bar{Y})\,,\\ &= \int d^2x\,d\theta^+d\bar{\theta}^-\Big(D_+X\cdot\bar{D}_-\bar{X} - \bar{Y}D_+\bar{D}_-Y\Big)\ .\end{aligned} \tag{2.72}$$

Alternatively,

$$\begin{aligned}\int d^2x\,d\theta^+d\theta^-d\bar{\theta}^+d\bar{\theta}^-(X\bar{X}-Y\bar{Y}) &= \int d^2x\,d\bar{\theta}^+d\theta^-D_+\bar{D}_-(X\bar{X}-Y\bar{Y})\,,\\ &= \int d^2x\,d\bar{\theta}^+d\theta^-\Big(\bar{D}_+\bar{X}\cdot D_-X - Y\bar{D}_+D_-\bar{Y}\Big)\ .\end{aligned} \tag{2.73}$$

JHEP02(2020)016

These relations imply

$$4\int d^2x\, d\theta^+ d\theta^- d\bar\theta^+ d\bar\theta^- (X\bar X - Y\bar Y) = \int d^2x\, d\theta^+ d\bar\theta^- d\eta^+ d\eta^-\; \boldsymbol{\mathcal{Y}}^2 + \text{c.c.}\ . \tag{2.74}$$

Once again the $(4,4)$ supersymmetry of the left hand side becomes manifest on the right hand side.

As in the $(4,4)$ twisted multiplet considered in the previous subsection, we can deform this representation to induce the partial breaking. The deformed multiplet is described by the following $(4,4)$ superfield:

$$\boldsymbol{\mathcal{Y}}_{\text{def}} = Y + \eta^+ \bar D_+ \bar X + \eta^- D_- X + \eta^+\eta^- \left(\bar D_+ D_- \bar Y + \kappa\right) . \tag{2.75}$$

The hidden supersymmetry transformations of the component $(2,2)$ superfields can be straightforwardly computed using the arguments of the previous subsection. For the goal of this section, it is enough to mention that δY is the same as the undeformed case of eq. (2.64).

To eliminate half of the degrees of freedom of $\boldsymbol{\mathcal{Y}}_{\text{def}}$ and describe a Goldstone multiplet for $\mathcal{N} = (4,4) \to \mathcal{N} = (2,2)$ partial supersymmetry breaking, we again impose the nilpotent constraint

$$\boldsymbol{\mathcal{Y}}^2_{\text{def}} = 0 = 2\eta^+\eta^- \left(Y(\kappa + \bar D_+ D_- \bar Y) - \bar D_+ \bar X \cdot D_- X\right) + \dots\ . \tag{2.76}$$

This yields the following constraint for the $(2,2)$ superfields

$$Y(\kappa + \bar D_+ D_- \bar Y) - \bar D_+ \bar X \cdot D_- X = 0\,, \tag{2.77}$$

which is equivalent to

$$Y = \frac{\bar D_+ \bar X \cdot D_- X}{\kappa + \bar D_+ D_- \bar Y} = \frac{\widetilde W^2}{\kappa + \bar{\widetilde D}^2 \bar Y}\,, \qquad \bar Y = \frac{\bar D_- \bar X \cdot D_+ X}{\kappa + \bar D_+ D_- \bar Y} = \frac{\bar{\widetilde W}^2}{\kappa + \widetilde D^2 \bar Y}\ . \tag{2.78}$$

Here $\bar{\widetilde D}^2 = \bar D_+ D_-$, $\widetilde D^2 = -D_+ \bar D_-$ and we have introduced the following bilinears:

$$\widetilde W^2 \equiv \bar D_+ \bar X \cdot D_- X = \bar{\widetilde D}^2 (X\bar X), \qquad \bar{\widetilde W}^2 \equiv \bar D_- \bar X \cdot D_+ X = \widetilde D^2 (X \bar X)\ . \tag{2.79}$$

Using exactly the same tricks as before and inspired by the $D = 4$ Bagger-Galperin model, we can solve the constraints (2.77) to find

$$Y = \frac{1}{\kappa}\bar{\widetilde D}^2 \widetilde\Upsilon\,, \quad \bar Y = \frac{1}{\kappa}\widetilde D^2 \widetilde\Upsilon\,, \qquad \widetilde\Upsilon = \bar{\widetilde\Upsilon} = X\bar X - \frac{2\widetilde W^2 \bar{\widetilde W}^2}{\widetilde A + \kappa^2 + \sqrt{\widetilde B^2 + 2\kappa^2 \widetilde A + \kappa^4}}\,, \tag{2.80}$$

where

$$\widetilde A = \widetilde D^2 \widetilde W^2 + \bar{\widetilde D}^2 \bar{\widetilde W}^2 = \{\widetilde D^2, \bar{\widetilde D}^2\}(X\bar X) = \partial_{++} X \partial_{--}\bar X + \partial_{++}\bar X \partial_{--} X\ , \tag{2.81a}$$

$$\widetilde B = \widetilde D^2 \widetilde W^2 - \bar{\widetilde D}^2 \bar{\widetilde W}^2 = [\widetilde D^2, \bar{\widetilde D}^2](X\bar X) = \partial_{++} X \partial_{--}\bar X - \partial_{++}\bar X \partial_{--} X\ . \tag{2.81b}$$

JHEP02(2020)016

Since Y is twisted-chiral under the manifest $(2,2)$ supersymmetry (2.63), we can consider the twisted-chiral integral

$$S_{\kappa^2} = \frac{1}{2}\kappa \int d^2x\, d\theta^+ d\bar{\theta}^- Y + c.c = \frac{1}{2}\int d^2x\, d\theta^+ d\bar{\theta}^- \bar{\widetilde{D}}^2 \widetilde{\Upsilon} + \text{c.c.} = \int d^2x\, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \widetilde{\Upsilon} \,. \tag{2.82}$$

By using arguments analogous to those around eqs. (2.55) of the previous subsection, the action (2.82) proves to be $\mathcal{N} = (4,4)$ supersymmetric.

Explicitly, the action reads

$$S_{\kappa^2} = \int d^2x\, d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \left(X\bar{X} - \frac{2\widetilde{W}^2 \bar{\widetilde{W}}^2}{\kappa^2 + \widetilde{A} + \sqrt{\kappa^4 + 2\kappa^2 \widetilde{A} + \widetilde{B}^2}} \right), \tag{2.83}$$

which precisely matches the model of eq. (2.11a) if we identify the coupling constants:

$$\lambda = -\frac{2}{\kappa^2} \,. \tag{2.84}$$

This shows explicitly that the $T\bar{T}$ deformation of the free chiral action possesses a non-linearly realized $\mathcal{N} = (2,2)$ supersymmetry.

3 $D = 4$ T^2 deformations and their supersymmetric extensions

In section (2) we exhibited the non-linear supersymmetry possessed by two $D = 2$ $\mathcal{N} = (2,2)$ models constructed in [18] from the $T\bar{T}$ deformation of free actions. The striking relationship with the $D = 4$ supersymmetric Born-Infeld (BI) theory naturally makes one wonder whether some kind of $T\bar{T}$ flow equation is satisfied by supersymmetric $D = 4$ BI, and related actions. We will spend the rest of the paper exploring this possibility. In this section, we start with a few general observations on T^2 or supercurrent-squared operators in $D > 2$.

3.1 Comments on the T^2 operator in $D = 4$

In two dimensions, by $T\bar{T}$ we mean the operator $T_{\mu\nu}T^{\mu\nu} - (T^\mu_\mu)^2$, which is proportional to $\det[T_{\mu\nu}]$ [1–3]. One can attempt to generalize this structure to $D > 2$. In general, one could consider the following stress-tensor squared operator

$$O^{[r]}_{T^2} = T^{\mu\nu}T_{\mu\nu} - r\,\Theta^2 \,, \qquad \Theta \equiv T^\mu_\mu \,, \tag{3.1}$$

with r a real constant parameter. In two dimensions, the unique choice $r = 1$ yields a well defined operator which is free of short distance singularities [1, 2]. However, to the best of our knowledge, there is no analogous argument in higher dimensions that guarantees a well-defined irrelevant operator $O^{[r]}_{T^2}$ at the quantum level. Nevertheless, in a D-dimensional space-time, one possible extension is given by $O^{[r]}_{T^2}$ with $r = 1/(D-1)$, which reduces to the $T\bar{T}$ operator in two dimensions.

This operator has received some attention recently since it is motivated by a particular holographic picture in $D > 2$ [44, 45]. We will not enter into a detailed discussion of the physical properties enjoyed by $O_{T^2}^{[1/(D-1)]}$, but simply make two brief comments. First, this combination is invariant under a set of improvement transformations of the stress-energy tensor. Indeed it is easy to show that such a T^2 operator transforms by,

$$O_{T^2}^{[1/(D-1)]} \to O_{T^2}^{[1/(D-1)]} + \text{total derivatives}\,, \tag{3.2}$$

if the (symmetric) stress-energy tensor shifts by the following improvement transformation,

$$T_{\mu\nu} \to T_{\mu\nu} + \left(\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2\right) u\,, \tag{3.3}$$

for an arbitrary scalar field u.

Second, for any operator $O_{T^2}^{[r]}$ in $D > 2$ dimensions (or on a curved space), the original argument of [1] for the factorization of the $T\bar{T}$ operator will no longer hold. For this reason, such an operator will not enjoy the same properties at the quantum level as the original $D = 2$ operator introduced by Zamolodchikov. In the approach to defining a higher-dimensional $T\bar{T}$ deformation through cut-off holography, one can sidestep this issue by taking a large N limit, in which the factorization property is expected to hold [45, 46]; the precise relative coefficient $r = 1/(D-1)$ can be derived from considerations of bulk gravitational physics.

In four dimensions, there is another choice of interest, specifically $r = 1/2$. In fact, it was shown in [24] that the bosonic Born-Infeld action can be obtained by deforming the free Maxwell theory with the operator $O_{T^2}^{[1/2]}$.[4] In this work, we are going to use $O_{T^2}^{[1/2]}$ as our deforming operator. Once generalized to the supersymmetric case, we will see that this operator plays a central role for various models possessing non-linearly realized symmetries.

One interesting property enjoyed by $O_{T^2}^{[1/2]}$ is its invariance under a shift of the Lagrangian density of the theory, or equivalently a shift of the zero point energy. This can serve as motivation for this particular combination. Under a constant shift of the Lagrangian density $\mathcal{L}$, and correspondingly its stress-energy tensor,

$$\mathcal{L} \to \mathcal{L} + c\,, \qquad T^{\mu\nu} \to T^{\mu\nu} - c\,\eta^{\mu\nu}\,, \tag{3.4}$$

the composite operator $O_{T^2}^{[r]}$ transforms in the following way:

$$O_{T^2}^{[r]} \to O_{T^2}^{[r]} + 2c(2r-1)\Theta + 4c^2(1-r)\ . \tag{3.5}$$

When the theory is not conformal, which is the general situation at an arbitrary point in the flow since the deformation introduces a scale, and $r \neq 1/2$, the operator $O_{T^2}^{[r]}$ always transforms in a non-trivial way because of the extra trace term. This implies that under a constant shift in the Lagrangian, the dynamics is modified which is certainly peculiar since the shift is trivial in the undeformed theory.[5]

However if $r = \frac{1}{2}$, $O_{T^2}^{[r]}$ is unaffected up to an honest field-independent cosmological constant term. The shift of the vacuum energy does not affect the dynamics of the theory,

[4] It is worth mentioning that another type of higher-dimensional generalization of $T\bar{T}$-deformations, specifically the operator $|\det T|^{1/(D-1)}$, was studied in [7, 47].

[5] It is worth noting that $T\bar{T}$ in $D = 2$ shares this peculiarity.

as long as the theory is not coupled to gravity. This property is especially interesting, since the $D = 4$ $\mathcal{N} = 1$ Goldstino action, which we will study in section 6 in the context of T^2 flows, is the low-energy description of supersymmetry breaking which can generate a cosmological constant. For these reasons, we will study the particular operator quadratic in stress-energy tensors given by

$$O_{T^2} \equiv T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}\Theta^2 \,, \tag{3.6}$$

in the remainder of the paper.

3.2 $D = 4$ $\mathcal{N} = 1$ supercurrent-squared operator

We would like to find the $\mathcal{N} = 1$ supersymmetric extension of the O_{T^2} operator in four dimensions. As reviewed in section 2, in two dimensions the manifestly supersymmetric $T\bar{T}$ deformation is roughly given by the square of the supercurrent superfields. One might suspect that a similar construction holds in four dimensions.

For the remainder of this work, we will assume that the $D = 4$ $\mathcal{N} = 1$ supersymmetric theories under our consideration admit a Ferrara-Zumino (FZ) multiplet of currents [26]. Generalizations of this case involving the supercurrent multiplets described in [27, 48–54] might be possible, but merit separate investigation. The operator content of the FZ multiplet, which has 12+12 component fields, includes the conserved supersymmetry current $S_{\mu\alpha}$, its conjugate $\bar{S}_\mu{}^{\dot\alpha}$ and the conserved symmetric energy-momentum tensor $T_{\mu\nu}$:

$$T_{\mu\nu} = T_{\nu\mu} \,, \qquad \partial^\mu T_{\mu\nu} = 0 \,, \qquad \partial^\mu S_\mu = \partial^\mu \bar{S}_\mu = 0 \,. \tag{3.7}$$

The FZ multiplet also includes a complex scalar field x, as well as the R-current vector field j_μ, which is not necessarily conserved [26].

In $D = 4$ $\mathcal{N} = 1$ superspace, the FZ multiplet is described by a vector superfield $\mathcal{J}_\mu$ and a complex scalar superfield $\mathcal{X}$ satisfying the following constraints:[6]

$$\bar{D}^{\dot\alpha}\mathcal{J}_{\alpha\dot\alpha} = D_\alpha \mathcal{X} \,, \qquad \bar{D}_{\dot\alpha}\mathcal{X} = 0 \,. \tag{3.9}$$

The constraints can be solved, and the FZ supercurrents expressed in terms of its $12 + 12$ independent components read[7]

$$\begin{aligned}
\mathcal{J}_\mu(x) = {} & j_\mu + \theta\left(S_\mu - \frac{1}{\sqrt{2}}\sigma_\mu\bar\chi\right) + \bar\theta\left(\bar{S}_\mu + \frac{1}{\sqrt{2}}\bar\sigma_\mu\chi\right) + \frac{i}{2}\theta^2\partial_\mu\bar{\mathsf{x}} - \frac{i}{2}\bar\theta^2\partial_\mu\mathsf{x} \\
& +\theta\sigma^\nu\bar\theta\left(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}\Theta - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\right) \\
& -\frac{i}{2}\theta^2\bar\theta\left(\bar{\partial\!\!\!/}S_\mu + \frac{1}{\sqrt{2}}\bar\sigma_\mu\partial\!\!\!/\bar\chi\right) - \frac{i}{2}\bar\theta^2\theta\left(\partial\!\!\!/\bar{S}_\mu - \frac{1}{\sqrt{2}}\sigma_\mu\bar{\partial\!\!\!/}\chi\right) \\
& +\frac{1}{2}\theta^2\bar\theta^2\left(\partial_\mu\partial^\nu j_\nu - \frac{1}{2}\partial^2 j_\mu\right) ,
\end{aligned} \tag{3.10}$$

[6]We follow the conventions of [55] except for the conversion between vector and bi-spinor indices. Following [27], we will use the convention $v_{\alpha\dot\alpha} = -2\sigma^\mu_{\alpha\dot\alpha}v_\mu$, $v_\mu = \frac{1}{4}\bar\sigma^{\dot\alpha\alpha}v_{\alpha\dot\alpha}$. Then it follows that

$$\mathcal{J}_{\alpha\dot\alpha} = -2\sigma^\mu_{\alpha\dot\alpha}\mathcal{J}_\mu, \qquad \mathcal{J}^\mu = \frac{1}{4}\mathcal{J}_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}, \qquad \mathcal{J}^2 \equiv \eta^{\mu\nu}\mathcal{J}_\mu\mathcal{J}_\nu = -\frac{1}{8}\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\mathcal{J}_{\alpha\dot\alpha}\mathcal{J}_{\beta\dot\beta} \,. \tag{3.8}$$

[7]For convenience, we have rescaled the supersymmetry current compared to [27]: $S^{\text{here}}_\mu = -iS^{\text{there}}_\mu$.

and

$$\mathcal{X}(y) = \mathsf{x}(y) + \sqrt{2}\theta\chi(y) + \theta^2 \mathsf{F}(y)\,, \tag{3.11a}$$

$$\chi_\alpha = \frac{\sqrt{2}}{3}(\sigma^\mu)_{\alpha\dot\alpha}\bar S_\mu^{\dot\alpha}\,, \qquad \mathsf{F} = \frac{2}{3}\Theta + i\partial_\mu j^\mu\,, \tag{3.11b}$$

where the chiral coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$, and we used $\partial\!\!\!/ = \sigma^\mu\partial_\mu$, $\bar{\partial\!\!\!/} = \bar\sigma^\mu\partial_\mu$.

If we seek a manifestly supersymmetric completion of the operator (3.6) by using combinations of the supercurrent superfields with dimension 4, it is clear that the only possibility is the full superspace integral of a linear combination of $\mathcal{J}^2$ and $\mathcal{X}\bar{\mathcal{X}}$. Up to total derivatives and terms that vanish by using the supercurrent conservation equations, or equivalently that vanish on-shell, the D-terms of $\mathcal{J}^2$ and $\mathcal{X}\bar{\mathcal{X}}$ are given by[8]

$$\begin{aligned}\mathcal{J}^2|_{\theta^2\bar\theta^2} \equiv \eta^{\mu\nu}\mathcal{J}_\mu\mathcal{J}_\nu|_{\theta^2\bar\theta^2} &= -\frac{1}{2}\Big(2T_{\mu\nu} - \frac{2}{3}\eta_{\mu\nu}\Theta - \frac{1}{2}\epsilon_{\nu\mu\rho\sigma}\partial^\rho j^\sigma\Big)^2 + j^\mu\Big(\partial_\mu\partial^\nu j_\nu - \frac{1}{2}\partial^2 j_\mu\Big)\\ &\quad + \frac{1}{2}\partial_\mu \mathsf{x}\partial^\mu\bar{\mathsf{x}} + \frac{i}{2}\Big(A - \bar A\Big) &(3.13a)\\ &= -2(T_{\mu\nu})^2 + \frac{4}{9}\Theta^2 - \frac{5}{4}\Big(\partial_\mu j^\mu\Big)^2 - \frac{3}{4}j_\mu\partial^2 j^\mu + \frac{1}{2}\partial_\mu\bar{\mathsf{x}}\partial^\mu \mathsf{x}\\ &\quad + i\Big(S_\mu\partial\!\!\!/\bar S^\mu - \bar\chi\bar{\partial\!\!\!/}\chi\Big) + \text{total derivatives} + \text{EOM}\,, &(3.13b)\end{aligned}$$

and

$$\begin{aligned}\mathcal{X}\bar{\mathcal{X}}|_{\theta^2\bar\theta^2} &= \mathsf{F}\bar{\mathsf{F}} - \partial_\mu\mathsf{x}\partial^\mu\bar{\mathsf{x}} - i\bar\chi\bar{\partial\!\!\!/}\chi + \text{total derivatives} &(3.14a)\\ &= \frac{4}{9}\Theta^2 + (\partial_\mu j^\mu)^2 - \partial_\mu\mathsf{x}\partial^\mu\bar{\mathsf{x}} - i\bar\chi\bar{\partial\!\!\!/}\chi + \text{total derivatives}\,. &(3.14b)\end{aligned}$$

To get a manifestly supersymmetric extension of $O_{T^2} = T^2 - \frac{1}{2}\Theta^2$, we have to consider the following linear combination[9]

$$\mathcal{O}_{T^2} = -\frac{1}{2}\Big(\eta^{\mu\nu}\mathcal{J}_\mu\mathcal{J}_\nu + \frac{5}{4}\mathcal{X}\bar{\mathcal{X}}\Big) = \frac{1}{16}\mathcal{J}^{\alpha\dot\alpha}\mathcal{J}_{\alpha\dot\alpha} - \frac{5}{8}\mathcal{X}\bar{\mathcal{X}}\,. \tag{3.16}$$

In fact, the supersymmetric descendant of the supercurrent-squared operator $\mathcal{O}_{T^2}$ is

$$\begin{aligned}\mathscr{O}_{T^2} &= \int d^4\theta\, \mathcal{O}_{T^2} &(3.17a)\\ &= T^2 - \frac{1}{2}\Theta^2 + \frac{3}{8}j_\mu\partial^2 j^\mu + \frac{3}{8}\partial_\mu\mathsf{x}\partial^\mu\bar{\mathsf{x}} - \frac{i}{2}\Big(S_\mu\partial\!\!\!/\bar S^\mu - \frac{9}{4}\bar\chi\bar{\partial\!\!\!/}\chi\Big)\\ &\quad + \text{total derivatives} + \text{EOM}\,. &(3.17b)\end{aligned}$$

[8]The composite A (and analogously its conjugate $\bar A$) is given by

$$A = \Big(S_\mu - \frac{1}{\sqrt{2}}\sigma_\mu\bar\chi\Big)\Big(\partial\!\!\!/\bar S^\mu - \frac{1}{\sqrt{2}}\sigma^\mu\bar{\partial\!\!\!/}\chi\Big) = S_\mu\partial\!\!\!/\bar S^\mu - \bar\chi\bar{\partial\!\!\!/}\chi + \sqrt{2}\bar S^\mu\partial_\mu\bar\chi + \text{total derivatives}\,. \tag{3.12}$$

The equality can be obtained with some algebra. Note that the last term drops after integration by parts because of the conservation equation for S_μ.

[9]More generally for the operator in (3.1), the supersymmetric generalization is given by

$$\mathcal{O}^{[r]}_{T^2} = -\frac{1}{2}\Big(\eta^{\mu\nu}\mathcal{J}_\mu\mathcal{J}_\nu + \frac{9r-2}{2}\mathcal{X}\bar{\mathcal{X}}\Big)\,. \tag{3.15}$$

This result shows that $\mathscr{O}_{T^2}$ is the natural supersymmetric extension of O_{T^2}. However, it is worth emphasizing that in the $D = 4$ case, the supersymmetric descendent $\mathscr{O}_{T^2}$ of $\mathcal{O}_{T^2}$ has extra non-trivial contributions from other currents. This should be contrasted with the $D = 2$ case where $\mathscr{O}_{T^2} = O_{T^2}$ up to EOM and total derivatives, see eq. (2.2).

It actually does not seem possible to find a linear combination of $\mathcal{J}^2$ and $\mathcal{X}\bar{\mathcal{X}}$ such that an analogue of eq. (2.2) holds in $D = 4$. This suggests that, in contrast with the $D = 2$ case, deformations of a Lagrangian triggered by the operators O_{T^2} and $\mathscr{O}_{T^2}$ will in general lead to different flows: one manifestly supersymmetric, while the other not.

4 Bosonic Born-Infeld as a T^2 flow

It was shown in [24] that the $D = 4$ Born-Infeld action arises from a $D > 2$ generalization of the $T\bar{T}$ deformation. Specifically, the operator driving the flow equation was shown to be the O_{T^2} defined in eq. (3.6) of the preceding section. In this section we review this result in detail as it is a primary inspiration for our supersymmetric extensions.

The $D = 4$ bosonic BI action on a flat background is given by

$$\begin{aligned} S_{\rm BI} &= \frac{1}{\alpha^2}\int d^4x \left[1 - \sqrt{-\det(\eta_{\mu\nu} + \alpha F_{\mu\nu})}\right] \\ &= \frac{1}{\alpha^2}\int d^4x \left[1 - \sqrt{1 + \frac{\alpha^2}{2}F^2 - \frac{\alpha^4}{16}(F\tilde{F})^2}\,\right] \\ &= -\frac{1}{4}\int d^4x\, F^2 + \text{higher derivative terms}\,, \end{aligned} \tag{4.1}$$

where $F_{\mu\nu} = (\partial_\mu v_\nu - \partial_\nu v_\mu)$ is the field strength for an Abelian gauge field v_μ, and

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu}, \qquad F\tilde{F} \equiv F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\ . \tag{4.2}$$

The stress-energy tensor for the BI action can be computed straightforwardly and it reads [56]

$$T^{\mu\nu} = -\frac{F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{1}{\alpha^2}\left(\sqrt{1 + \frac{\alpha^2}{2}F^2 - \frac{\alpha^4}{16}(F\tilde{F})^2} - 1 - \frac{\alpha^2}{2}F^2\right)\eta^{\mu\nu}}{\sqrt{1 + \frac{\alpha^2}{2}F^2 - \frac{\alpha^4}{16}(F\tilde{F})^2}}\ . \tag{4.3}$$

This can be written in the following useful form

$$T^{\mu\nu} = \frac{T^{\mu\nu}_{\rm Maxwell}}{\sqrt{1 + 2A + B^2}} + \frac{\eta^{\mu\nu}}{\alpha^2\sqrt{1 + 2A + B^2}}\frac{A^2 - B^2}{1 + A + \sqrt{1 + 2A + B^2}}\,, \tag{4.4}$$

where we used the stress-energy tensor for the Maxwell theory

$$T^{\mu\nu}_{\rm Maxwell} = -F^{\mu\lambda}F^{\nu}{}_{\lambda} + \frac{1}{4}F^2\eta^{\mu\nu}\,, \tag{4.5}$$

while A and B are defined by

$$A = \frac{1}{4}\alpha^2F^2\,, \qquad B = \frac{i}{4}\alpha^2F\tilde{F}\ . \tag{4.6}$$

It is easy to compute the trace of the stress-energy tensor

$$\Theta = T^{\mu\nu}\eta_{\mu\nu} = \frac{4}{\alpha^2\sqrt{1+2A+B^2}}\frac{A^2-B^2}{1+A+\sqrt{1+2A+B^2}}\,, \tag{4.7}$$

where, interestingly, the combination (A^2-B^2) proves to be related to the square of $T^{\mu\nu}_{\mathrm{Maxwell}}$. Using the identity

$$(F\tilde{F})^2 = \frac{1}{4}(\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma})^2 = 4F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - 2(F^2)^2\,, \tag{4.8}$$

we see that

$$T^2_{\mathrm{Maxwell}} = F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} - \frac{1}{4}(F^2)^2 = \frac{1}{4}\Big((F^2)^2 + (F\tilde{F})^2\Big) = \frac{4}{\alpha^4}(A^2-B^2)\,. \tag{4.9}$$

Using tracelessness of the free Maxwell stress-energy tensor, the O_{T^2} operator can be easily computed:

$$O_{T^2} = T^2 - \frac{1}{2}\Theta^2 = \frac{4(A^2-B^2)}{\alpha^4\sqrt{1+2A+B^2}^2}\left(1 - \frac{A^2-B^2}{(1+A+\sqrt{1+2A+B^2})^2}\right), \tag{4.10a}$$

$$= \frac{4(A^2-B^2)}{\alpha^4\sqrt{1+2A+B^2}^2}\left(1 - \frac{1+A-\sqrt{1+2A+B^2}}{1+A+\sqrt{1+2A+B^2}}\right), \tag{4.10b}$$

$$= \frac{8(A^2-B^2)}{\alpha^4\sqrt{1+2A+B^2}}\frac{1}{1+A+\sqrt{1+2A+B^2}}\,, \tag{4.10c}$$

$$= \frac{8(1+A-\sqrt{1+2A+B^2})}{\alpha^2\sqrt{1+2A+B^2}}\,. \tag{4.10d}$$

The variation of the BI Lagrangian with respect to the parameter α^2 can be readily computed, and it is given by

$$\frac{\partial\mathcal{L}_\alpha}{\partial\alpha^2} = \frac{1+\frac{1}{4}\alpha^2F^2 - \sqrt{1+\frac{1}{2}\alpha^2F^2 - \frac{1}{16}\alpha^4(F\tilde{F})^2}}{\alpha^2\sqrt{1+\frac{1}{2}\alpha^4F^2 - \frac{1}{16}\alpha^4(F\tilde{F})^2}}\,. \tag{4.11}$$

Once we use (4.6) it is clear that (4.10a) and (4.11) have exactly the same structure and satisfy the following equivalence equation

$$\frac{\partial\mathcal{L}_\alpha}{\partial\alpha^2} = \frac{1}{8}O_{T^2}\,, \tag{4.12}$$

showing that the BI Lagrangian satisfies a T^2-flow driven by the operator O_{T^2}.

Before turning to $D=4$ supersymmetric analysis, it is worth mentioning that the structure of the computation relating the O_{T^2} operator to the bosonic BI theory, which we just reviewed, is quite similar to what we saw in section 2 for the $D=2$ $\mathcal{N}=(2,2)$ supersymmetric $T\bar{T}$ flows. For example, in the deformation of the free twisted-chiral multiplet action, the analogue of the A and B combinations of (4.6) is given by (2.53), but the square root structure of the actions is completely analogous. This fact, together with the non-linearly realized supersymmetry we investigated in section 2, naturally lead to the guess that the $D=4$ $\mathcal{N}=1$ supersymmetric Born-Infeld (BI) theory may also satisfy a T^2 flow. The next section is devoted to explaining how this is the case.

5 Supersymmetric Born-Infeld from supercurrent-squared deformation

In section 2 we proved, by analogy and extension of the $D = 4$ results of [22], that two $D = 2$ supercurrent-squared flows possess additional non-linearly realized supersymmetry. In this section we reverse the logic. We will look at a well-studied model, namely the Bagger-Galperin construction [22] of $D = 4$ $\mathcal{N} = 1$ Born-Infeld theory [23, 57], and show that it satisfies a supercurrent-squared flow equation.

5.1 $D = 4$ $\mathcal{N} = 1$ supersymmetric BI and non-linear supersymmetry

Let us review some well known results about the $D = 4$ $\mathcal{N} = 1$ Born-Infeld theory [23], the Bagger-Galperin action [22], the non-linearly realized second supersymmetry, and its precise $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking pattern. For more detail, we refer to the following references on the subject [20, 22, 23, 33–35].

We start with the following $\mathcal{N} = 2$ superfield,

$$\mathcal{W}(y,\theta,\tilde\theta) = X(y,\theta) + \sqrt{2}i\tilde\theta W(y,\theta) - \tilde\theta^2 G(y,\theta)\,, \qquad y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta + i\tilde\theta\sigma^\mu\bar{\tilde\theta}\,, \tag{5.1}$$

which is chiral with respect to both supersymmetries:[10]

$$\bar D_{\dot\alpha}\mathcal{W} = \bar{\tilde D}_{\dot\alpha}\mathcal{W} = 0\ . \tag{5.3}$$

Since we are ultimately interested in partial $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking, we will mostly use $\mathcal{N} = 1$ superfields associated to the θ Grassmann variables to describe manifest supersymmetry, while we use the $\tilde\theta$ variable for the hidden non-linearly realized supersymmetry. The $\mathcal{N} = 1$ superfields X, W_α, and G of eq. (5.1) are chiral under the manifest $\mathcal{N} = 1$ supersymmetry. Under the additional hidden $\mathcal{N} = 1$ supersymmetry, they transform as follows:

$$\tilde\delta X = \sqrt{2}i\epsilon W\,, \tag{5.4a}$$

$$\tilde\delta W = \sqrt{2}\sigma^\mu\bar\epsilon\partial_\mu X + \sqrt{2}i\epsilon G\,, \tag{5.4b}$$

$$\tilde\delta G = -\sqrt{2}\partial_\mu W\sigma^\mu\bar\epsilon\ . \tag{5.4c}$$

The superfield (5.1) has 16+16 independent off-shell components and is reducible. It contains the degrees of freedom of an $\mathcal{N} = 2$ vector and tensor multiplet. To reduce the degrees of freedom and describe an irreducible $\mathcal{N} = 2$ off-shell vector multiplet, we impose the following conditions on the $\mathcal{N} = 1$ components of $\mathcal{W}$:

(i) First that W_α is the field-strength superfield of an $\mathcal{N} = 1$ vector multiplet satisfying,

$$D^\alpha W_\alpha - \bar D_{\dot\alpha}\bar W^{\dot\alpha} = 0\,, \tag{5.5}$$

[10] We follow the conventions of [55]. The $D = 4$, $\mathcal{N} = 2$ superspace is parametrised by bosonic coordinates x^μ and the Grasmannian coordinates $(\theta^\alpha, \bar\theta^{\dot\alpha})$ and $(\tilde\theta^\alpha, \bar{\tilde\theta}^{\dot\alpha})$. In terms of the chiral coordinate y^μ introduced in (5.1), the supercovariant derivatives are given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial y^\mu}\,, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\,, \tag{5.2}$$

and similarly for $\tilde D_\alpha, \bar{\tilde D}_{\dot\alpha}$.

(ii) and that

$$G = \frac{1}{4}\bar{D}^2\bar{X} \ . \tag{5.6}$$

The latter condition can easily be seen to be consistent since it is straightforward to verify that $\frac{1}{4}\bar{D}^2\bar{X}$ transforms in the same way as G given in (5.4c). Therefore we can impose (5.6) without violating $\mathcal{N}=2$ supersymmetry.

Since $\mathcal{W}$ is chiral with respect to both sets of supersymmetries, we can consider the following Lagrangian,

$$\mathcal{L}^{\mathcal{N}=2}_{\mathcal{W}^2} = \frac{1}{4}\int d^2\theta d^2\tilde{\theta}\,\mathcal{W}^2 + \text{c.c.} = \frac{1}{4}\int d^2\theta\left(W^2 - \frac{1}{2}X\bar{D}^2\bar{X}\right) + \text{c.c.} \ . \tag{5.7}$$

On the other hand, the $\mathcal{N}=2$ Maxwell theory written in terms of the $\mathcal{N}=1$ chiral superfields X and W_α is given by

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=2}_{\text{Maxwell}} &= \int d^2\theta d^2\bar{\theta}\,\bar{X}X + \frac{1}{4}\int d^2\theta\,W^2 + \frac{1}{4}\int d^2\theta\,\bar{W}^2\,,\\ &= \frac{1}{4}\int d^2\theta\left(W^2 - \frac{1}{2}X\bar{D}^2\bar{X}\right) + \text{c.c.} + \text{total derivative} \ .\end{aligned} \tag{5.8}$$

We see that these two Lagrangians are the same, confirming that the extra constraint imposed on $\mathcal{W}$ is correct. The off-shell $\mathcal{N}=2$ vector multiplet can therefore be described in term of the following $\mathcal{N}=2$ superfield

$$\mathcal{W}(y,\theta,\tilde{\theta}) = X(y,\theta) + \sqrt{2}i\tilde{\theta}W(y,\theta) - \frac{1}{4}\tilde{\theta}^2\bar{D}^2\bar{X}(y,\theta)\,, \tag{5.9}$$

where X and W_α are $\mathcal{N}=1$ chiral and vector multiplets, respectively. Their component expansion reads:

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha \mathsf{D} - i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha\,, \tag{5.10a}$$

$$X = x + \sqrt{2}\theta\chi - \theta^2\,\mathsf{F} \ . \tag{5.10b}$$

Following [20] (see also [33–35]), we break $\mathcal{N}=2$ supersymmetry by considering a Lorentz and $\mathcal{N}=1$ invariant condensate with a non-trivial dependence on the hidden Grassmann variables $\langle\mathcal{W}\rangle = \mathcal{W}_{\text{def}} \propto \tilde{\theta}^2 \neq 0$, such that

$$\mathcal{W} \to \mathcal{W}_{\text{new}} = \langle\mathcal{W}\rangle + \mathcal{W} = \mathcal{W} + \mathcal{W}_{\text{def}}\,, \tag{5.11a}$$

$$\mathcal{W}_{\text{new}} = X + \sqrt{2}i\tilde{\theta}W - \frac{1}{4}\tilde{\theta}^2\left(\bar{D}^2\bar{X} + \frac{2}{\kappa}\right) . \tag{5.11b}$$

The hidden supersymmetry transformations of the $\mathcal{N}=1$ components of the deformed $\mathcal{N}=2$ vector multiplet turn out to be

$$\tilde{\delta}X = \sqrt{2}i\epsilon W\,, \tag{5.12a}$$

$$\tilde{\delta}W = \frac{i}{\sqrt{2}\kappa}\epsilon + \frac{i}{2\sqrt{2}}\epsilon\bar{D}^2\bar{X} + \sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu X \ . \tag{5.12b}$$

Assuming the model under consideration preserves the manifest $\mathcal{N}=1$ supersymmetry, which implies $\langle \bar{D}^2 X\rangle = 0$, the explicit non-linear κ-dependent term in the transformation of the fermionic W_α signals the spontaneous partial breaking $\mathcal{N}=2 \to \mathcal{N}=1$ of the hidden supersymmetry.

To describe the Maxwell-Goldstone multiplet for the partial breaking $\mathcal{N}=2 \to \mathcal{N}=1$, we can impose the following nilpotent constraint on the deformed $\mathcal{N}=2$ superfield strength $\mathcal{W}_{\text{new}}$ [20]

$$(\mathcal{W}_{\text{new}})^2 = 0\ . \tag{5.13}$$

Once reduced to $\mathcal{N}=1$ superfields, following the expansion (5.11b), this constraint implies the Bagger-Galperin constraint [22]

$$\frac{1}{\kappa} X = W^2 - \frac{1}{2} X \bar{D}^2 \bar{X}\,, \tag{5.14}$$

which can be solved to eliminate X in terms of $W^2 = W^\alpha W_\alpha$ and its complex conjugate $\bar{W}^2 = \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}$:

$$X = \kappa W^2 - \kappa^3 \bar{D}^2\left[\frac{W^2 \bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}-\mathcal{B}^2}}\right], \tag{5.15}$$

where we have introduced:

$$\mathcal{A} = \frac{\kappa^2}{2}(D^2 W^2 + \bar{D}^2 \bar{W}^2) = \overline{\mathcal{A}}\,, \qquad \mathcal{B} = \frac{\kappa^2}{2}(D^2 W^2 - \bar{D}^2 \bar{W}^2) = -\overline{\mathcal{B}}\ . \tag{5.16}$$

For later use we denote the lowest components of the composite superfields $\mathcal{A}$ and $\mathcal{B}$

$$A = \mathcal{A}|_{\theta=0}\,, \qquad B = \mathcal{B}|_{\theta=0}\ . \tag{5.17}$$

We will not repeat the derivation of (5.15) which can be found in the original paper [22], and was reviewed and slightly modified in section 2 for our analysis in two dimensions.

The $\mathcal{N}=1$ supersymmetric BI action can be constructed using the following $\mathcal{N}=1$ (anti-)chiral Lagrangian linear in X:

$$\mathcal{L}_\kappa = \frac{1}{4\kappa}\left(\int d^2\theta X + \int d^2\bar\theta \bar{X}\right)\ . \tag{5.18}$$

The second hidden supersymmetry eq. (5.12a) written in terms of the unconstrained real vector multiplet V, where $W_\alpha = -1/4\bar{D}^2 D_\alpha V$, takes the form:

$$\tilde{\delta} X = -\frac{1}{4}\sqrt{2} i \epsilon^\alpha \bar{D}^2 D_\alpha V\ . \tag{5.19}$$

Using the fact that $D^2 \bar{D}^2 D_\alpha \propto \partial_{\alpha\dot\alpha} D^2 \bar{D}^{\dot\alpha}$, one can immediately see that the supersymmetry variation of $\mathcal{L}_\kappa$ in (5.18) is a total derivative. Therefore this supersymmetric BI action is invariant under the second hidden non-linear supersymmetry.

Using the solution (5.15), the supersymmetric BI Lagrangian takes the explicit form

$$\begin{aligned}\mathcal{L}_\kappa &= \frac{1}{4\kappa}\int d^2\theta \left(\kappa W^2 - \kappa^3 \bar{D}^2\left[\frac{W^2\bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\right]\right) + \text{c.c.} \\ &= \frac{1}{4}\int d^2\theta\, W^2 + \frac{1}{4}\int d^2\bar\theta\, \bar{W}^2 + 2\kappa^2 \int d^2\theta d^2\bar\theta\, \frac{W^2 \bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\,,\end{aligned} \tag{5.20}$$

which makes it clear that the supersymmetric BI is a non-linear deformation of the free $\mathcal{N}=1$ Maxwell theory. This supersymmetric extension of BI was first constructed by Bagger and Galperin in [22]. In this work when we refer to the supersymmetric BI theory, we will always mean the Bagger-Galperin action.

We can easily calculate the flow under the κ^2 coupling constant,

$$\frac{\partial \mathcal{L}_\kappa}{\partial \kappa^2} = 2\int d^2\theta d^2\bar{\theta}\, \frac{W^2\bar{W}^2}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\frac{1}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\,. \tag{5.21}$$

Our goal is now to show that the right hand side of this flow equation on-shell is the specific supercurrent bilinear (3.16) that we introduced earlier. This will establish a supercurrent-squared flow for the supersymmetric BI action.

Before turning to the core of this analysis let us recall that at the leading order in κ^2, the fact that $D=4$ $\mathcal{N}=1$ BI satisfies a supercurrent-squared flow was already noticed in [23]. This result was also highlighted recently in the introduction of [16]. In fact, note that in the free limit $\alpha=\kappa=0$, the Lagrangian (5.20) becomes the $\mathcal{N}=1$ supersymmetric Maxwell theory. Its supercurrent multiplet is

$$\mathcal{J}_{\alpha\dot{\alpha}} = -4W_\alpha\bar{W}_{\dot{\alpha}}\,, \qquad \mathcal{X}=0\,, \tag{5.22}$$

where $\mathcal{X}=0$ because super-Maxwell theory is scale invariant. The supersymmetric T^2 deformation operator (3.16) is then simply given by

$$\mathcal{O}_{T^2} = \frac{1}{16}\mathcal{J}_{\alpha\dot{\alpha}}\mathcal{J}^{\alpha\dot{\alpha}} - \frac{5}{8}\mathcal{X}\bar{\mathcal{X}} = W^2\bar{W}^2\,, \tag{5.23}$$

and to leading order (5.21) turns into [23]

$$\frac{\partial \mathcal{L}_\kappa}{\partial \kappa^2} = \int d^2\theta d^2\bar{\theta}\, W^2\bar{W}^2 + \mathcal{O}(\kappa^2) = \int d^2\theta d^2\bar{\theta}\, \mathcal{O}_{T^2} + \mathcal{O}(\kappa^2)\,. \tag{5.24}$$

This shows that the supercurrent-squared flow equation is satisfied at this order. The rest of this section is devoted to demonstrating the full non-linear extension of this result. First, we are going to look at the bosonic truncation of (5.20) and (5.21).

5.2 Bosonic truncation

In the pure bosonic case the gauginos are set to zero in (5.10a), $\lambda=\bar{\lambda}=0$, and $W^2,\bar{W}^2$ only have $\theta^2,\bar{\theta}^2$ components, so $\mathcal{A},\mathcal{B}$ can only contribute the lowest components:

$$A = \mathcal{A}|_{\theta=0} = 2\kappa^2\Big(F^2-2\mathsf{D}^2\Big)\,, \qquad B=\mathcal{B}|_{\theta=0} = 2\kappa^2 iF\tilde{F}\,. \tag{5.25}$$

Therefore the supersymmetric BI Lagrangian reduces to

$$\mathcal{L} = \frac{1}{8\kappa^2}\left[1-\sqrt{1+4\kappa^2\Big(F^2-2\mathsf{D}^2\Big)-4\kappa^4\Big(F\tilde{F}\Big)^2}\,\right]. \tag{5.26}$$

The auxiliary field $\mathsf{D}=0$ after using its EOM, and the Lagrangian is equivalent to the bosonic BI Lagrangian (4.1) with the identification $\alpha^2=8\kappa^2$. This immediately implies that on-shell the bosonic truncation of the supersymmetric BI satisfies a T^2 flow equation driven by the O_{T^2} operator (3.6), as we discussed in (4.10a). A similar story is going to hold for the complete supersymmetric model of (5.20) and (5.21).

5.3 Supersymmetric Born-Infeld as a supercurrent-squared flow

The supercurrent for the supersymmetric BI action (5.20) was computed in [58] for $\kappa^2 = \frac{1}{2}$. To simplify notation, we will also consider the special case $\kappa^2 = \frac{1}{2}$ in our intermediate computations. The κ-dependence can be restored easily and will appear in the final formulae.

We can straightforwardly use the results of [58] for our supercurrent-squared flow analysis. The FZ multiplet was computed for a class of models described by the following Lagrangian,

$$\mathcal{L} = \frac{1}{4}\int d^2\theta\, W^2 + \frac{1}{4}\int d^2\bar{\theta}\, \bar{W}^2 + \frac{1}{4}\int d^2\theta d^2\bar{\theta}\, W^2\bar{W}^2\Lambda(u,\bar{u})\,, \tag{5.27}$$

where

$$u = \frac{1}{8}D^2W^2\,, \qquad \bar{u} = \frac{1}{8}\bar{D}^2\bar{W}^2\ . \tag{5.28}$$

The action (5.20) turns out to be given by the following choice of $\Lambda(u,\bar{u})$

$$\Lambda(u,\bar{u}) = \frac{4}{1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\,, \tag{5.29}$$

where

$$\mathcal{A} = 2(u+\bar{u})\ , \qquad \mathcal{B} = 2(u-\bar{u})\ . \tag{5.30}$$

Following [58], we also introduce the composite superfields

$$\Gamma(u,\bar{u}) = \frac{\partial(u\Lambda)}{\partial u}\,, \qquad \bar{\Gamma}(u,\bar{u}) = \frac{\partial(\bar{u}\Lambda)}{\partial \bar{u}}\,, \tag{5.31}$$

which, in the case of interest to us where (5.29) holds, satisfy

$$\Gamma+\bar{\Gamma}-\Lambda = \frac{4}{\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\,, \tag{5.32a}$$

$$\bar{u}\Gamma+u\bar{\Gamma} = 1-\frac{1}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}}\ . \tag{5.32b}$$

The supercurrents will also be functionals of the following composite

$$iM_\alpha = W_\alpha\left[1-\frac{1}{4}\bar{D}^2\left(\bar{W}^2\left(\Lambda+\frac{1}{8}D^2\left(W^2\frac{\partial\Lambda}{\partial u}\right)\right)\right)\right]\,, \tag{5.33a}$$

$$= W_\alpha\Big(1-2\bar{u}\Gamma\Big)+W\bar{W}(\cdots)+W^2(\cdots)\,, \tag{5.33b}$$

where $W\bar{W}(\cdots)$ denotes terms which are proportional to $W_\alpha\bar{W}_{\dot{\alpha}}$, while $W^2(\cdots)$ denotes terms proportional to W^2. We will use similar notation with ellipses denoting quantities with bare fermionic terms that will not contribute to the calculation because of nilpotency conditions.

With the ingredients introduced above, the FZ multiplet for the supersymmetric BI action is given by [58]

$$\mathcal{X} = \frac{1}{6}W^2\bar{D}^2\Big(\bar{W}^2(\Gamma+\bar{\Gamma}-\Lambda)\Big)\,, \tag{5.34a}$$

$$\begin{aligned}\mathcal{J}_{\alpha\dot{\alpha}} = &-2iM_\alpha\bar{W}_{\dot{\alpha}}+2iW_\alpha\bar{M}_{\dot{\alpha}}+\frac{1}{12}[D_\alpha,\bar{D}_{\dot{\alpha}}]\Big(W^2\bar{W}^2\Big)\cdot\Big(\Gamma+\bar{\Gamma}-\Lambda\Big)\\ &+W^2\bar{W}(\cdots)+\bar{W}^2W(\cdots)\ .\end{aligned} \tag{5.34b}$$

For our purposes, the superfields X and $\mathcal{J}_{\alpha\dot{\alpha}}$ can be further simplified as follows:

$$\mathcal{X} = \frac{1}{6}W^2\bar{D}^2\bar{W}^2 \cdot \Big(\Gamma + \bar{\Gamma} - \Lambda\Big) + W^2\bar{W}(\cdots)\,, \tag{5.35a}$$

$$= \frac{2W^2\bar{D}^2\bar{W}^2}{3\Big(1 + \mathcal{A} + \sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}\Big)} + W^2\bar{W}(\cdots)\,, \tag{5.35b}$$

and

$$\mathcal{J}_{\alpha\dot{\alpha}} = -4W_\alpha\bar{W}_{\dot{\alpha}}(1 - \bar{u}\Gamma - u\bar{\Gamma}) + \frac{1}{12}[D_\alpha, \bar{D}_{\dot{\alpha}}]\Big(W^2\bar{W}^2\Big) \cdot \Big(\Gamma + \bar{\Gamma} - \Lambda\Big) + W^2\bar{W}(\cdots) + \bar{W}^2W(\cdots)\,, \tag{5.36a}$$

$$= -\frac{4W_\alpha\bar{W}_{\dot{\alpha}}}{\sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}} + \frac{2D_\alpha W^2 \cdot \bar{D}_{\dot{\alpha}}\bar{W}^2}{3\big(1 + \mathcal{A} + \sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}\big)\sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}} + W^2\bar{W}(\cdots) + \bar{W}^2W(\cdots)\,, \tag{5.36b}$$

where we used (5.32a).

The computation of $\mathcal{X}\bar{\mathcal{X}}$ is trivial and receives contributions only from the square of the first term in (5.35b). The computation of $\mathcal{J}^2$ is less trivial. It is obvious that the last two complicated terms in the second line of (5.36b) make no contribution since all the terms are proportional to $W\bar{W}$, and we have the nilpotency property $W_\alpha W_\beta W_\gamma = 0$. The square of the first term is easy to compute, and it is proportional to $W^2\bar{W}^2$. Next we consider the cross product between the first and second term in (5.36b) which leads to the relation:

$$W_\alpha\bar{W}_{\dot{\alpha}} \cdot D^\alpha W^2 \cdot \bar{D}^{\dot{\alpha}}\bar{W}^2 = W^2(DW) \cdot \bar{W}^2(\bar{D}\bar{W}) = 0\ . \tag{5.37}$$

Remarkably, this cross term vanishes since, as shown in appendix A, on-shell it is true that

$$W^2\bar{W}^2DW = 0\ . \tag{5.38}$$

A simple physical interpretation of this condition is that the manifest supersymmetry is preserved on-shell, implying that the auxiliary field $\mathsf{D} \propto D^\alpha W_\alpha|_{\theta=0}$ has no vev, and is at least linear in gaugino fields $\lambda_\alpha \propto W_\alpha|_{\theta=0}$. The vanishing of this cross term can be compared with the pure bosonic case where the cross terms in T^2 vanish because of the tracelessness property of the free Maxwell stress tensor; see section 4. Finally, we compute the square of the second term in (5.36b) which includes the following structure:

$$D^\alpha W^2 \cdot \bar{D}^{\dot{\alpha}}\bar{W}^2 \cdot D_\alpha W^2 \cdot \bar{D}_{\dot{\alpha}}\bar{W}^2 = W^2\bar{W}^2D^2W^2\bar{D}^2\bar{W}^2\ . \tag{5.39}$$

Here we have used $(D_\alpha W_\beta)(D^\alpha W^\beta) = -\frac{1}{2}D^2W^2 + W^\beta D^2W_\beta$ to simplify the result.

In summary, on-shell the contributions to the supercurrent-squared operator $\mathcal{O}_{T^2}$ defined in eq. (3.16) are given by

$$\mathcal{J}^2 = -\frac{1}{8}\left\{\frac{16W^2\bar{W}^2}{\sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}^2} + \frac{4W^2\bar{W}^2D^2W^2\bar{D}^2\bar{W}^2}{9\sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}^2\big(1 + \mathcal{A} + \sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}\big)^2}\right\}, \tag{5.40a}$$

$$\mathcal{X}\bar{\mathcal{X}} = \frac{4}{9}\frac{W^2\bar{W}^2D^2W^2\bar{D}^2\bar{W}^2}{\sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}^2\big(1 + \mathcal{A} + \sqrt{1 + 2\mathcal{A} + \mathcal{B}^2}\big)^2}\ . \tag{5.40b}$$

Adding these results gives the supersymmetric T^2 primary operator $\mathcal{O}_{T^2}$:

$$\mathcal{O}_{T^2} = -\frac{1}{2}\left(\mathcal{J}^2 + \frac{5}{4}\mathcal{X}\bar{\mathcal{X}}\right) = \frac{W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}^2}\left(1 - \frac{D^2W^2\bar{D}^2\bar{W}^2}{4\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)^2}\right) \tag{5.41a}$$

$$= \frac{W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}^2}\left(1 - \frac{\mathcal{A}^2-\mathcal{B}^2}{(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2})^2}\right) \tag{5.41b}$$

$$= \frac{2W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)} \ . \tag{5.41c}$$

It is worth noting that the simplifications occurring in constructing $\mathcal{O}_{T^2}$ from the supercurrents are very similar to the bosonic case of (4.10a).

Comparing with (5.21), we see that eq. (5.41c) proves that the supersymmetric BI action (5.20) is an on-shell solution of the flow equation

$$\frac{\partial \mathcal{L}_\kappa}{\partial \kappa^2} = \int d^2\theta d^2\bar{\theta}^2 \, \frac{2W^2\bar{W}^2}{\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\left(1+\mathcal{A}+\sqrt{1+2\mathcal{A}+\mathcal{B}^2}\right)} \tag{5.42a}$$

$$= \int d^2\theta d^2\bar{\theta}^2 \, \mathcal{O}_{T^2} + \text{total derivatives} + \text{EOM} \ . \tag{5.42b}$$

It therefore describes a supercurrent-squared deformation of the $\mathcal{N}=1$ free Maxwell Lagrangian. This result establishes a relationship between non-linearly realized supersymmetry and supercurrent-squared flow equations in $D=4$.

Before closing this section, we should make a few comments regarding the on-shell condition (5.38) used in establishing the supercurrent-squared flow equation for the $D=4$ $\mathcal{N}=1$ BI action. First it is important to stress that the flow equation is not satisfied by the supersymmetric BI action off-shell. Second, we note that the specific combination of $\mathcal{J}^2$ and $\mathcal{X}\bar{\mathcal{X}}$ studied is the unique choice for which (5.20) satisfies a supercurrent-squared flow equation, even if only on-shell.

Such a non-trivial condition satisfied by the on-shell supersymmetric BI action is intriguing and hints at the existence of appropriate (super)field redefinitions which might lead to a different supersymmetric extension of BI that satisfies the flow equation off-shell. For example, it is know that the dependence of the off-shell extension on the auxiliary field D can be modified by appropriate (super)field redefinitions, as well as redefinitions of the full superspace Lagrangian. We refer to [59–62] for a list of relevant papers on this subject. Under field redefinitions, the hidden supersymmetry will be modified but will remain a non-linearly realized symmetry of the theory. The existence of an off-shell solution of the supercurrent-squared flow is an interesting question for future research.

6 $D=4$ Goldstino action from supercurrent-squared deformation

In section 5 we showed that the Bagger-Galperin action for the $D=4$ $\mathcal{N}=1$ supersymmetric BI theory satisfies a supercurrent-squared flow. It is known that the truncation of this model to fermions describes a Goldstino action for $D=4$ $\mathcal{N}=1$ supersymmetry breaking; see, for example, [61, 63, 64]. The $\mathcal{N}=1$ non-linearly realized supersymmetry arises as the non-linearly realized part of the $\mathcal{N}=2 \to \mathcal{N}=1$ breaking of the supersymmetric BI. We have shown in sections 4 and 5.2 that the bosonic truncation of the supersymmetric BI satisfies a T^2 flow equation. The same should be true for the fermionic truncation. More generally, one might argue that $D=4$ $\mathcal{N}=1$ Goldstino models could satisfy a sort of flow equation that organizes their expansion in the supersymmetry breaking scale parameter.

Note that in the $D=2$ case, the intuition is similar. If we consider the actions analyzed in section 2 that describe Goldstone models for partial $D=2$ $\mathcal{N}=(4,4) \to \mathcal{N}=(2,2)$ supersymmetry breaking, one can immediately argue that their fermionic truncation describes Goldstino actions possessing non-linearly realized $D=2$ $\mathcal{N}=(2,2)$ supersymmetry. These, by construction, are expected to satisfy a $T\bar{T}$-flow equation. In fact, such an argument is in agreement with the very nice recent analysis of [21] where a $D=2$ Goldstino model possessing $\mathcal{N}=(2,2)$ non-linearly realized supersymmetry was shown to satisfy the supercurrent-squared flow equation (2.6).[11] The model analyzed in [21] is the analogue of the $D=4$ model of [66, 67] and related on-shell to the Goldstino model of [68].[12] This section is devoted to showing that these $D=4$ $\mathcal{N}=1$ Goldstino models satisfy a supercurrent-squared flow driven by the operator $\mathcal{O}_{T^2}$ of the supersymmetric BI, in agreement with the arguments given above.

6.1 $D=4$ Goldstino actions

The Volkov-Akulov (VA) action is the low energy description of supersymmetry breaking. There are several representations of the Goldstino action that are equivalent to the Volkov-Akulov form; see [61, 69] for comprehensive discussions. Here we will focus on two models, but we start by reviewing a few general features of Goldstino actions.

The original VA action was obtained by requiring its invariance under the the non-linear supersymmetry transformation [70]

$$\delta_\xi \lambda^\alpha = \frac{1}{\kappa}\xi^\alpha - i\kappa(\lambda\sigma^m\bar{\xi} - \xi\sigma^m\bar{\lambda})\partial_m\lambda^\alpha \ . \tag{6.1}$$

Explicitly, the original Lagrangian was proven to be

$$\mathcal{L}_{\text{VA}} = -\frac{1}{2\kappa^2}\det A = -\frac{1}{2\kappa^2} - \frac{i}{2}(\lambda\sigma^m\partial_m\bar{\lambda} - \partial_m\lambda\sigma^m\bar{\lambda}) + \text{interactions}\,, \tag{6.2}$$

where

$$A_m{}^a = \delta_m{}^a - i\kappa^2\partial_m\lambda\sigma^a\bar{\lambda} + i\kappa^2\lambda\sigma^a\partial_m\bar{\lambda} \ . \tag{6.3}$$

The alternative representation of the Goldstino action that interests us was originally introduced by Casalbuoni et al. in [66], and later rediscovered and made fashionable by Komargodski and Seiberg [67]. This model, which following recent literature we will call the KS model, was constructed by imposing nilpotent superfield constraints as a generalization of Roček's seminal ideas for the Goldstino model described in [68]. After integrating out

[11] We refer to [65] for a discussion of various models possessing non-linearly realized $(2,2)$ supersymmetry.

[12] Note that the Goldstino models of [66–68] were shown in [61, 64] to be identical to the fermionic truncation of the supersymmetric BI action up to a field redefinition of the Goldstino.

an auxiliary field in the KS model, described in more detail in the next section, the explicit form of the Lagrangian is given by the following very simple combination of terms:

$$\mathcal{L}_{\rm KS} = -f^2 - \frac{i}{2}(\psi\sigma^m\partial_m\bar{\psi} - \partial_m\psi\sigma^m\bar{\psi}) - \frac{1}{4f^2}\partial^\mu\bar{\psi}^2\partial_\mu\psi^2 - \frac{1}{16f^6}\psi^2\bar{\psi}^2\partial^2\psi^2\partial^2\bar{\psi}^2 \ . \qquad (6.4)$$

The action is invariant under a quite involved non-linearly realized supersymmetry transformation whose explicit form can be found in [64, 71]. The Goldstino actions described by (6.2) and (6.4) prove to be equivalent off-shell up to a field redefinition [64, 71].

6.2 $D = 4$ KS Goldstino model as a supercurrent-squared flow

The goal in the rest of this section is to straightforwardly generalize the analysis of [21] to $D = 4$ and to show how the KS action satisfies a flow equation arising from a T^2 deformation of the free fermion action.

6.2.1 KS model

Let us start by reviewing the Goldstino model of [66, 67]. Consider the following Lagrangian

$$\mathcal{L}_{\rm KS} = \int d^4\theta\, \bar{\Phi}\Phi + \int d^2\theta\Big(f\Phi + \frac{1}{2}\Lambda\Phi^2\Big) + \int d^2\bar{\theta}\Big(f\bar{\Phi} + \frac{1}{2}\bar{\Lambda}\bar{\Phi}^2\Big)\,, \qquad (6.5)$$

where Φ, $\bar{\Phi}$ are $D = 4$ $\mathcal{N} = 1$ chiral and anti-chiral superfields, satisfying the constraints $\bar{D}_{\dot{\alpha}}\Phi = D_\alpha\bar{\Phi} = 0$. The constant parameter f, which describes the supersymmetry breaking scale, is real. The superfields Λ, $\bar{\Lambda}$ are chiral and anti-chiral Lagrange multipliers whose EOM yield the nilpotent constraints

$$\Phi^2 = \bar{\Phi}^2 = 0 \ . \qquad (6.6)$$

The equation of motion for Φ is

$$\frac{1}{4}\bar{D}^2\bar{\Phi} = \Lambda\Phi + f\,, \qquad \frac{1}{4}D^2\Phi = \bar{\Lambda}\bar{\Phi} + f \ . \qquad (6.7)$$

As a consequence, we also have

$$\Phi\bar{D}^2\bar{\Phi} = 4f\Phi\,, \qquad \bar{\Phi}D^2\Phi = 4f\bar{\Phi}\,, \qquad (6.8)$$

where the nilpotent properties of (6.6) are used. Note that the constraints (6.6) and (6.8) are the ones originally used by Roček to define his Goldstino model [68]. These observations make manifest the on-shell equivalence of the KS model with Roček's Goldstino model in a simple superspace setting. The off-shell equivalence of all these Golstino models up to field redefinitions, including the VA action, was proven in [61].

The Lagrange multiplier in (6.5) imposes the nilpotent constraint $\Phi^2 = 0$ on the chiral superfield Φ. This condition can be solved in terms of the spinor field ψ and the auxiliary field F of the chiral multiplet, [66, 67]:

$$\Phi = \frac{\psi^2}{2F} + \sqrt{2}\theta\psi + \theta^2 F\,, \qquad (6.9)$$

which is sensible assuming that $F \neq 0$. Substituting back into (6.5) gives a Lagrangian expressed in terms of ψ and the auxiliary field F,

$$\mathcal{L}_{\text{KS}} = -\frac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{1}{2}\bar{F}F + \frac{1}{8}\frac{\bar{\psi}^2}{\bar{F}}\partial^2\left(\frac{\psi^2}{F}\right) + fF + \text{c.c.}\ . \tag{6.10}$$

The auxiliary field can then be eliminated using its equation of motion, which can be solved in closed form

$$F = -f\left(1 + \frac{\bar{\psi}^2}{4f^4}\partial^2\psi^2 - \frac{3}{16f^8}\psi^2\bar{\psi}^2\partial^2\psi^2\partial^2\bar{\psi}^2\right), \tag{6.11}$$

together with the complex conjugate expression for $\bar{F}$. Plugging (6.11) into (6.10) gives the Goldstino action (6.4) [66, 67].

6.2.2 $D = 4$ Goldstino action as a supercurrent-squared flow

One advantage of using the KS model compared to other Goldstino actions is the relatively simple form of the action, thanks to the Lagrange multiplier, which makes the computation of its supercurrent easier. The FZ multiplet resulting from the action (6.5) is

$$\mathcal{J}_{\alpha\dot{\alpha}} = 2D_\alpha\Phi\cdot\bar{D}_{\dot{\alpha}}\bar{\Phi} - \frac{2}{3}[D_\alpha,\bar{D}_{\dot{\alpha}}](\Phi\bar{\Phi}) = \frac{2}{3}D_\alpha\Phi\cdot\bar{D}_{\dot{\alpha}}\bar{\Phi} - \frac{2i}{3}\left(\Phi\partial_{\alpha\dot{\alpha}}\bar{\Phi} - \bar{\Phi}\partial_{\alpha\dot{\alpha}}\Phi\right), \tag{6.12a}$$

$$\mathcal{X} = 4\left(f\Phi + \frac{1}{2}\Lambda\Phi^2\right) - \frac{1}{3}\bar{D}^2(\Phi\bar{\Phi}) = \frac{8}{3}f\Phi + 2\Lambda\Phi^2\ . \tag{6.12b}$$

The composite operators $\mathcal{J}^{\alpha\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}}$ and $\mathcal{X}\bar{\mathcal{X}}$ are then

$$\mathcal{J}^{\alpha\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = \frac{64}{9}f^2\Phi\bar{\Phi} + \text{total derivatives} + \text{EOM}\,, \tag{6.13}$$

and

$$\mathcal{X}\bar{\mathcal{X}} = \frac{64}{9}f^2\Phi\bar{\Phi} + \text{EOM}\,, \tag{6.14}$$

where we used (6.6) and (6.7). The supercurrent-squared operator (3.16) then takes the form

$$\mathcal{O}_{T^2} = \frac{1}{16}\mathcal{J}^{\alpha\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} - \frac{5}{8}\mathcal{X}\bar{\mathcal{X}} = -4f^2\Phi\bar{\Phi} + \text{EOM} + \text{total derivatives}\ . \tag{6.15}$$

The descendant operator $\mathscr{O}_{T^2}$ of eq. (3.17) becomes

$$\mathscr{O}_{T^2} = \int d^2\theta d^2\bar{\theta}\ \mathcal{O}_{T^2} = -4f^2\int d^2\theta d^2\bar{\theta}\ \Phi\bar{\Phi} = 2f^3\int d^2\theta\,\Phi + 2f^3\int d^2\bar{\theta}\,\bar{\Phi}\,, \tag{6.16}$$

where we used (6.8) in the last equality.

From (6.5), it is easy to see that the following relation holds:

$$\frac{\partial\mathcal{L}_{\text{KS}}}{\partial f} = \int d^2\theta\,\Phi + \int d^2\bar{\theta}\,\bar{\Phi}\ . \tag{6.17}$$

By identifying the coupling constants,

$$\gamma = -\frac{1}{4f^2}\,, \tag{6.18}$$

it follows immediately that the KS action,

$$S_\gamma = \int d^4x\, \mathcal{L}_{\mathrm{KS}}\,, \tag{6.19}$$

satisfies the flow equation

$$\frac{\partial S_\gamma}{\partial \gamma} = \int d^4x d^2\theta\, \Phi + \int d^4x d^2\bar{\theta}\, \bar{\Phi} = \int d^4x d^2\theta d^2\bar{\theta}\, \mathcal{O}_{T^2} = \int d^4x\, \mathscr{O}_{T^2}\ . \tag{6.20}$$

This proves that (6.5) satisfies a supercurrent-squared flow (or T^2 flow) equation. Because on-shell the actions (6.4) and (6.5) are equivalent, and the equation

$$\int d^2\theta d^2\bar{\theta}\, \mathcal{O}_{T^2} = \mathscr{O}_{T^2}$$

holds, eq. (6.20) proves that the $D = 4$ $\mathcal{N} = 1$ Goldstino action arises from a supercurrent-squared deformation.[13]

7 Conclusions and outlook

In this work we have explored the relationship between $T\bar{T}$ deformations and non-linear supersymmetry, extending the earlier analysis of [17, 18, 21]. We first showed how two different $D = 2$ $\mathcal{N} = (2,2)$ $T\bar{T}$ deformations of free supersymmetric scalar models, studied in [18], classically possess a hidden non-linearly realized $\mathcal{N} = (2,2)$ supersymmetry. The deformed theories are off-shell supersymmetric extensions of the gauge-fixed Nambu-Goto string in four dimensions. One way to potentially understand the appearance of non-linearly realized symmetries is by relating them to symmetries of the undeformed theories using the field-dependent change of variables discussed in [25, 72].

These $D = 2$ models turn out to be structurally very similar to the Bagger-Galperin action describing a $D = 4$ $\mathcal{N} = 1$ Born-Infeld theory, which possesses extra non-linearly realized $\mathcal{N} = 1$ supersymmetry [22]. Inspired by this similarity and earlier work on the bosonic BI theory [24], we proved that the $\mathcal{N} = 1$ BI action satisfies a supercurrent-squared flow equation to all orders in the deformation parameter, extending the beautiful initial observation of [23].

Moreover, we concluded the paper by showing how the $D = 4$ $\mathcal{N} = 1$ Goldstino action also satisfies the same supercurrent-squared flow. This result extends the recent $D = 2$ analysis of [21] to four dimensions. Our findings hint at an intriguing relation between current-squared deformations and non-linear supersymmetry in various space-time dimensions that calls for a deeper explanation.

For the $D = 2$ case where the $T\bar{T}$ operator is well-defined quantum mechanically, it would be interesting to investigate other examples with various (super-)symmetry breaking

[13] The careful reader may find that the flow can also be satisfied by other supercurrent-squared operators, $\mathcal{J}^2 - r\bar{\mathcal{X}}\mathcal{X}$, with arbitrary r because of the linearity between $\mathcal{J}^2$ and $\bar{\mathcal{X}}\mathcal{X}$. It is worth pointing out the same thing happens in $D = 2$ [21]. We stress that this is not the case for the supercurrent-squared flow satisfied by the $D = 4$ supersymmetric Born-Infeld action.

JHEP02(2020)016

patterns, and analyze the consistency conditions required by the existence of non-linear symmetries at the quantum level.[14]

Another interesting issue related to the quantum properties of $D = 2$ $T\bar{T}$-deformed models concerns their perturbative renormalization behavior. It is well known [4, 5] that the effect of the $T\bar{T}$ deformation in infinite volume is to modify the S-matrix of the undeformed theory by a CDD factor. It is interesting to ask whether one could renormalize the classical deformed action by adding some counter-terms and reproduce the S-matrix at the quantum level. In [73], the authors managed to write down the one-loop effective action of the deformed Lagrangian for a free massive scalar field. In the supersymmetric case it would be interesting to see if with enough supersymmetry one could derive the effective action exactly in λ. For instance, if one deforms an $\mathcal{N} = (2, 2)$ supersymmetric model with a Kähler potential K and superpotential W, it was shown in [18] that the superpotential W is left untouched along the superspace $T\bar{T}$ flow. Moreover, it is known that the superpotential W is protected from perturbative quantum corrections. However, the Kähler potential suffers from quantum corrections and a similar renormalization procedure of the one in [73] should be performed to address the perturbative behavior of such Kähler potential. We leave the detailed analysis of this issue to the future.

For $D > 2$, to the best of our knowledge, there is no complete argument showing that any of the proposed operators $O^{[r]}_{T^2}$ of eq. (3.1), including the holographic operator of [44, 45], possesses any particularly nice quantum properties. By looking at our $D = 4$ $\mathcal{N} = 1$ example, where the flow is controlled by the descendant operator $\mathscr{O}_{T^2}$ of (3.17), it seems clear that any supersymmetric completion of $O^{[r]}_{T^2}$ will involve several other current-squared operators. An important question is to understand whether such extensions have a hope of providing well-defined operators at the quantum level. This seems most promising in models with at least extended $\mathcal{N} > 1$, and more likely maximal, supersymmetry.

In [45, 46], the authors studied the T^2-deformation from an holographic perspective and a particular choice $r = \frac{1}{D-1}$ in the T^2 operator was motivated. In $D = 4$, the supersymmetric generalization of such a T^2-operator is given in (3.15). It is interesting to study the supersymmetric generalization of the holographic setup in [45, 46], and especially understand the role of other currents in supersymmetric T^2-operator arising from the difference between O_{T^2} and $\mathscr{O}_{T^2}$. A purely field theoretical analysis of such a T^2-operator with $r = \frac{1}{D-1}$ and its supersymmetric analog is also of great importance, in particular to match the holographic result in the large-N limit. It is worth mentioning that in the case of $\mathcal{N} = 4$ supersymmetric Yang-Mills in four dimensions, some preliminary interesting results of $T\bar{T}$ like irrelevant deformations preserving integrability were recently presented [74].

Putting aside the quantum properties of these deformations and flows, the connection between non-linear symmetries and $T\bar{T}$ flows might give a novel way to organize interesting low-energy effective actions. The Born-Infeld and Goldstino actions that we have analyzed in this paper are universal low-energy structures in string theory, and in the latter case quantum field theory, precisely because of their non-linear symmetries, which can be geometrically realized via brane physics.

[14]We are grateful to Guzmán Hernández-Chifflet for stimulating comments on this subject.

The study of Volkov-Akulov-Dirac-Born-Infeld actions with extended supersymmetry in various space-time dimensions and their relationship to string theory has received a lot of attention in the past. We refer to the following (incomplete) list of references [20, 22, 57, 59, 60, 75–98]. It would be remarkable if the α' expansion of these models can be reorganized in a simple current-squared flow equation. An efficient way to address the cases we have considered so far in $D = 2$ and $D = 4$ has been via superspace techniques. As a next step, one could try to analyze possible flow equations satisfied by the $D = 4$ $\mathcal{N} = 2$ extensions of the DBI theory, which has been analyzed in superspace; see, for example, [89–95].

Another potentially tractable direction to be explored concerns the possible universality of the operator $\mathscr{O}_{T^2}$ of (3.17) in the context of models with partial supersymmetry breaking. In the literature there are other known models for $D = 4$ $\mathcal{N} = 2 \to \mathcal{N} = 1$ supersymmetry breaking that share structural similarities with the Maxwell-Goldstone model of [22]. Well known are the Goldstone models based on the $D = 4$ $\mathcal{N} = 1$ tensor multiplet [62], see also [60, 61], which have a dual description based on a chiral $\mathcal{N} = 1$ multiplet. It is simple to show that at first order these actions satisfy a supercurrent-squared flow analogous to the Bagger-Galperin action. Whether that result extends beyond leading order is a natural question.

A final avenue for future investigation concerns the relationship between $T\bar{T}$ deformations and amplitudes. In two dimensions, $T\bar{T}$ simply modifies the S-matrix of the undeformed theory by a CDD factor [4], but one might wonder about the S-matrices of higher-dimensional theories deformed by generalizations of $T\bar{T}$. One hint is that theories with non-linearly realized symmetries exhibit enhanced soft behavior — indeed, in the case of non-linearly realized *super*symmetry, there is a proof that such symmetries generically lead to constraints on the soft behavior of the S-matrix [99], a fact which has been applied to the Volkov-Akulov action [100], which satisfies a $T\bar{T}$-like flow as we showed in section 6.

There are also examples involving purely bosonic theories. For instance, in four dimensions, the Dirac action is the unique Lorentz-invariant Lagrangian for a single scalar which is consistent with factorization, has one derivative per field, and exhibits soft degree $\sigma = 2$ for its scattering amplitudes [101]. Similarly, it has been shown that the Born-Infeld action for a vector can be fixed by demanding enhanced soft behavior in a particular multi-soft limit [102], which can be understood in the context of T-duality and dimensional reduction [103]. Given the hints of a deeper relationship between supercurrent-squared deformations, non-linearly realized symmetries, and actions of Dirac or Born-Infeld type, it is natural to ask whether such deformations enhance the soft behavior of scattering amplitudes in a more general context.

Acknowledgments

We are grateful to Chih-Kai Chang and Alessandro Sfondrini for collaboration at early stages of this project. H. J. especially would like to thank Ignatios Antoniadis for stimulating discussions and collaborations on related problems, and inspiration for many of the ideas in the current project. C. F., S. S. and G. T.-M. would also like to thank the organizers and participants of the workshop on *"$T\bar{T}$ and Other Solvable Deformations of*

Quantum Field Theories" for providing a stimulating atmosphere, and the Simons Center for Geometry and Physics for hospitality and partial support during the initial stages of this work. C. F., H. J. and G. T.-M. are also grateful for the support and vively atmosphere during the workshop *"New frontiers of integrable deformations"* in Villa Garbald, Castasegna during the final stage of preparation of this work. C. F. acknowledges support from the University of Chicago's divisional MS-PSD program. C. F. and S. S. are supported in part by NSF Grant No. PHY 1720480. H. J. is supported by Swiss National Science Foundation. The work of G. T.-M. was supported by the Albert Einstein Center for Fundamental Physics, University of Bern, and by the Australian Research Council (ARC) Future Fellowship FT180100353.

A Deriving a useful on-shell identity

This appendix is devoted to deriving the on-shell relation (5.38). We are going to prove this holds for an action of the form (5.27). Let us start by considering the following Lagrangian

$$\mathcal{L} = \frac{1}{4}\int d^2\theta\, W^2 + \frac{1}{4}\int d^2\bar{\theta}\, \bar{W}^2 + \int d^2\theta d^2\bar{\theta}\, W^2\bar{W}^2\, \Omega\big[D^2W^2, \bar{D}^2\bar{W}^2\big]\,. \tag{A.1}$$

Remember that W_α and $\bar{W}_{\dot{\alpha}}$ satisfy the Bianchi identity $D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}$ whose solution is given in terms of a real but otherwise unconstrained scalar prepotential superfield V: $W_\alpha = -1/4\,\bar{D}^2 D_\alpha V$ and $\bar{W}_{\dot{\alpha}} = -1/4\,D^2\bar{D}_{\dot{\alpha}}V$. It is a straightforward calculation to derive the EOM by varying the action (A.1) with respect to the prepotential V. The EOM reads

$$\begin{aligned} 0 = &-D^\alpha W_\alpha + \frac{1}{2}D^\alpha\bar{D}^2\Big(W_\alpha\bar{W}^2\Omega\Big) + \frac{1}{2}\bar{D}_{\dot{\alpha}}D^2\Big(W^2\bar{W}^{\dot{\alpha}}\Omega\Big) \\ &+\frac{1}{2}D^\alpha\left[W_\alpha\bar{D}^2D^2\left(W^2\bar{W}^2\frac{\partial\Omega}{\partial(D^2W^2)}\right)\right] + \frac{1}{2}\bar{D}_{\dot{\alpha}}\left[\bar{W}^{\dot{\alpha}}\left(D^2\bar{D}^2W^2\bar{W}^2\frac{\partial\Omega}{\partial(\bar{D}^2\bar{W}^2)}\right)\right]. \end{aligned} \tag{A.2}$$

Because of the constraint that $W_\alpha W_\beta W_\gamma = 0$ and its complex conjugate, multiplying eq. (A.2) by $W^2\bar{W}^2$ and using the EOM gives the following condition

$$W^2\bar{W}^2(D^\alpha W_\alpha)\Big(1 + f(\Omega)\Big) = 0\,, \tag{A.3}$$

where the functional $f(\Omega)$ is given by

$$\begin{aligned} f(\Omega) := &-\frac{1}{2}(\bar{D}^2\bar{W}^2 + D^2W^2)\Omega \\ &-\frac{1}{2}\left[(D^2W^2)(\bar{D}^2\bar{W}^2)\frac{\partial\Omega}{\partial(D^2W^2)} + (D^2W^2)(\bar{D}^2\bar{W}^2)\frac{\partial\Omega}{\partial(\bar{D}^2\bar{W}^2)}\right]. \end{aligned} \tag{A.4}$$

This implies

$$W^2\bar{W}^2(D^\alpha W_\alpha) = 0\,, \tag{A.5}$$

which is precisely condition (5.38).

JHEP02(2020)016

References

[1] A.B. Zamolodchikov, *Expectation value of composite field $T\bar{T}$ in two-dimensional quantum field theory*, hep-th/0401146 [INSPIRE].

[2] F.A. Smirnov and A.B. Zamolodchikov, *On space of integrable quantum field theories*, Nucl. Phys. **B 915** (2017) 363 [arXiv:1608.05499] [INSPIRE].

[3] A. Cavaglià, S. Negro, I.M. Szécsényi and R. Tateo, *$T\bar{T}$-deformed 2D quantum field theories*, JHEP **10** (2016) 112 [arXiv:1608.05534] [INSPIRE].

[4] S. Dubovsky, V. Gorbenko and M. Mirbabayi, *Natural tuning: towards a proof of concept*, JHEP **09** (2013) 045 [arXiv:1305.6939] [INSPIRE].

[5] S. Dubovsky, V. Gorbenko and M. Mirbabayi, *Asymptotic fragility, near AdS_2 holography and $T\bar{T}$*, JHEP **09** (2017) 136 [arXiv:1706.06604] [INSPIRE].

[6] Y. Jiang, *Lectures on solvable irrelevant deformations of 2d quantum field theory*, arXiv:1904.13376 [INSPIRE].

[7] G. Bonelli, N. Doroud and M. Zhu, *$T\bar{T}$-deformations in closed form*, JHEP **06** (2018) 149 [arXiv:1804.10967] [INSPIRE].

[8] S. Dubovsky, R. Flauger and V. Gorbenko, *Solving the simplest theory of quantum gravity*, JHEP **09** (2012) 133 [arXiv:1205.6805] [INSPIRE].

[9] M. Caselle, D. Fioravanti, F. Gliozzi and R. Tateo, *Quantisation of the effective string with TBA*, JHEP **07** (2013) 071 [arXiv:1305.1278] [INSPIRE].

[10] C. Chen, P. Conkey, S. Dubovsky and G. Hernández-Chifflet, *Undressing confining flux tubes with $T\bar{T}$*, Phys. Rev. **D 98** (2018) 114024 [arXiv:1808.01339] [INSPIRE].

[11] A. Dei and A. Sfondrini, *Integrable spin chain for stringy Wess-Zumino-Witten models*, JHEP **07** (2018) 109 [arXiv:1806.00422] [INSPIRE].

[12] M. Baggio and A. Sfondrini, *Strings on NS-NS backgrounds as integrable deformations*, Phys. Rev. **D 98** (2018) 021902 [arXiv:1804.01998] [INSPIRE].

[13] S. Frolov, *TTbar deformation and the light-cone gauge*, arXiv:1905.07946 [INSPIRE].

[14] A. Sfondrini and S.J. van Tongeren, *$T\bar{T}$ deformations as TsT transformations*, arXiv:1908.09299 [INSPIRE].

[15] M. Baggio, A. Sfondrini, G. Tartaglino-Mazzucchelli and H. Walsh, *On $T\overline{T}$ deformations and supersymmetry*, JHEP **06** (2019) 063 [arXiv:1811.00533] [INSPIRE].

[16] C.-K. Chang, C. Ferko and S. Sethi, *Supersymmetry and $T\overline{T}$ deformations*, JHEP **04** (2019) 131 [arXiv:1811.01895] [INSPIRE].

[17] H. Jiang, A. Sfondrini and G. Tartaglino-Mazzucchelli, *$T\bar{T}$ deformations with $\mathcal{N}=(0,2)$ supersymmetry*, Phys. Rev. **D 100** (2019) 046017 [arXiv:1904.04760] [INSPIRE].

[18] C.-K. Chang et al., *$T\bar{T}$ flows and $(2,2)$ supersymmetry*, Phys. Rev. **D 101** (2020) 026008 [arXiv:1906.00467] [INSPIRE].

[19] E. Ivanov, S. Krivonos, O. Lechtenfeld and B. Zupnik, *Partial spontaneous breaking of two-dimensional supersymmetry*, Nucl. Phys. **B 600** (2001) 235 [hep-th/0012199] [INSPIRE].

[20] M. Roček and A.A. Tseytlin, *Partial breaking of global $D = 4$ supersymmetry, constrained superfields and three-brane actions*, Phys. Rev. **D 59** (1999) 106001 [hep-th/9811232] [INSPIRE].

[21] N. Cribiori, F. Farakos and R. von Unge, *2D Volkov-Akulov Model as a $T\overline{T}$ deformation*, Phys. Rev. Lett. **123** (2019) 201601 [arXiv:1907.08150] [INSPIRE].

[22] J. Bagger and A. Galperin, *A new Goldstone multiplet for partially broken supersymmetry*, Phys. Rev. **D 55** (1997) 1091 [hep-th/9608177] [INSPIRE].

[23] S. Cecotti and S. Ferrara, *Supersymmetric Born-Infeld Lagrangians*, Phys. Lett. **B 187** (1987) 335.

[24] R. Conti, L. Iannella, S. Negro and R. Tateo, *Generalised Born-Infeld models, Lax operators and the $T\overline{T}$ perturbation*, JHEP **11** (2018) 007 [arXiv:1806.11515] [INSPIRE].

[25] E.A. Coleman, J. Aguilera-Damia, D.Z. Freedman and R.M. Soni, *$T\overline{T}$-deformed actions and $(1,1)$ supersymmetry*, JHEP **10** (2019) 080 [arXiv:1906.05439] [INSPIRE].

[26] S. Ferrara and B. Zumino, *Transformation properties of the supercurrent*, Nucl. Phys. **B 87** (1975) 207 [INSPIRE].

[27] T.T. Dumitrescu and N. Seiberg, *Supercurrents and brane currents in diverse dimensions*, JHEP **07** (2011) 095 [arXiv:1106.0031] [INSPIRE].

[28] S.J. Gates Jr., C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear σ-models*, Nucl. Phys. **B 248** (1984) 157 [INSPIRE].

[29] T. Buscher, U. Lindström and M. Roček, *New supersymmetric σ models with Wess-Zumino terms*, Phys. Lett. **B 202** (1988) 94 [INSPIRE].

[30] M.T. Grisaru, M. Massar, A. Sevrin and J. Troost, *The quantum geometry of $N = (2,2)$ nonlinear σ-models*, Phys. Lett. **B 412** (1997) 53 [hep-th/9706218] [INSPIRE].

[31] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler manifolds and off-shell supersymmetry*, Commun. Math. Phys. **269** (2007) 833 [hep-th/0512164] [INSPIRE].

[32] E. Ivanov and A. Sutulin, *Diverse $N = (4,4)$ twisted multiplets in $N = (2,2)$ superspace*, Theor. Math. Phys. **145** (2005) 1425 [hep-th/0409236] [INSPIRE].

[33] S.M. Kuzenko and G. Tartaglino-Mazzucchelli, *Nilpotent chiral superfield in $N = 2$ supergravity and partial rigid supersymmetry breaking*, JHEP **03** (2016) 092 [arXiv:1512.01964] [INSPIRE].

[34] I. Antoniadis, J.-P. Derendinger and C. Markou, *Nonlinear $\mathcal{N} = 2$ global supersymmetry*, JHEP **06** (2017) 052 [arXiv:1703.08806] [INSPIRE].

[35] I. Antoniadis, H. Jiang and O. Lacombe, *$\mathcal{N} = 2$ supersymmetry deformations, electromagnetic duality and Dirac-Born-Infeld actions*, JHEP **07** (2019) 147 [arXiv:1904.06339] [INSPIRE].

[36] S.J. Gates Jr., *Superspace formulation of new nonlinear σ-models*, Nucl. Phys. **B 238** (1984) 349 [INSPIRE].

[37] S.J. Gates Jr. and S.V. Ketov, $2D$ $(4,4)$ *hypermultiplets*, Phys. Lett. **B 418** (1998) 111 [hep-th/9504077] [INSPIRE].

[38] S.J. Gates Jr. and S.V. Ketov, $2D$ $(4,4)$ *hypermultiplets. II: field theory origins of dualities*, Phys. Lett. **B 418** (1998) 119 [INSPIRE].

[39] E.A. Ivanov and S.O. Krivonos, $N = 4$ *Super-Liouville equation* (in Russian), J. Phys. **A 17** (1984) L671 [INSPIRE].

[40] E.A. Ivanov and S.O. Krivonos, $N = 4$ *superextension of the Liouville equation with quaternionic structure*, Theor. Math. Phys. **63** (1985) 477 [INSPIRE].

[41] E.A. Ivanov, S.O. Krivonos and V.M. Leviant, *A new class of superconformal σ models with the Wess-Zumino action*, Nucl. Phys. **B 304** (1988) 601 [INSPIRE].

[42] I. Antoniadis, H. Partouche and T.R. Taylor, *Spontaneous breaking of $N = 2$ global supersymmetry*, Phys. Lett. **B 372** (1996) 83 [hep-th/9512006] [INSPIRE].

[43] E. Ivanov and B. Zupnik, *Modifying $N = 2$ supersymmetry via partial breaking*, in the proceedings of the *Theory of elementary particles. 31*st *International Symposium*, September 2–6, Ahrenshoop, Germany (1997), hep-th/9801016 [INSPIRE].

[44] M. Taylor, *TT deformations in general dimensions*, arXiv:1805.10287 [INSPIRE].

[45] T. Hartman, J. Kruthoff, E. Shaghoulian and A. Tajdini, *Holography at finite cutoff with a T^2 deformation*, JHEP **03** (2019) 004 [arXiv:1807.11401] [INSPIRE].

[46] P. Caputa, S. Datta and V. Shyam, *Sphere partition functions & cut-off AdS*, JHEP **05** (2019) 112 [arXiv:1902.10893] [INSPIRE].

[47] J. Cardy, *The $T\overline{T}$ deformation of quantum field theory as random geometry*, JHEP **10** (2018) 186 [arXiv:1801.06895] [INSPIRE].

[48] Z. Komargodski and N. Seiberg, *Comments on supercurrent multiplets, supersymmetric field theories and supergravity*, JHEP **07** (2010) 017 [arXiv:1002.2228] [INSPIRE].

[49] S.J. Gates Jr., M.T. Grisaru and W. Siegel, *Auxiliary field anomalies*, Nucl. Phys. **B 203** (1982) 189 [INSPIRE].

[50] N. Ambrosetti, D. Arnold, J.-P. Derendinger and J. Hartong, *Gauge coupling field, currents, anomalies and $N = 1$ super-Yang–Mills effective actions*, Nucl. Phys. **B 915** (2017) 285 [arXiv:1607.08646] [INSPIRE].

[51] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace or one thousand and one lessons in supersymmetry*, Front. Phys. **58** (1983) 1 [hep-th/0108200] [INSPIRE].

[52] S.M. Kuzenko, *The Fayet-Iliopoulos term and nonlinear self-duality*, Phys. Rev. **D 81** (2010) 085036 [arXiv:0911.5190] [INSPIRE].

[53] K.R. Dienes and B. Thomas, *On the inconsistency of Fayet-Iliopoulos terms in supergravity theories*, Phys. Rev. **D 81** (2010) 065023 [arXiv:0911.0677] [INSPIRE].

[54] S.M. Kuzenko, *Variant supercurrent multiplets*, JHEP **04** (2010) 022 [arXiv:1002.4932] [INSPIRE].

[55] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton, U.S.A. (1992).

[56] D.A. Rasheed, *Nonlinear electrodynamics: zeroth and first laws of black hole mechanics*, hep-th/9702087 [INSPIRE].

[57] S. Deser and R. Puzalowski, *Supersymmetric nonpolynomial vector multiplets and causal propagation*, *J. Phys.* **A 13** (1980) 2501 [INSPIRE].

[58] S.M. Kuzenko and S.A. McCarthy, *Nonlinear selfduality and supergravity*, *JHEP* **02** (2003) 038 [hep-th/0212039] [INSPIRE].

[59] S. Cecotti and S. Ferrara, *Supersymmetric Born-Infeld lagrangians*, *Phys. Lett.* **B 187** (1987) 335 [INSPIRE].

[60] F. Gonzalez-Rey, I.Y. Park and M. Roček, *On dual* 3*-brane actions with partially broken* $N = 2$ *supersymmetry*, *Nucl. Phys.* **B 544** (1999) 243 [hep-th/9811130] [INSPIRE].

[61] S.M. Kuzenko and S.J. Tyler, *On the Goldstino actions and their symmetries*, *JHEP* **05** (2011) 055 [arXiv:1102.3043] [INSPIRE].

[62] J. Bagger and A. Galperin, *The tensor Goldstone multiplet for partially broken supersymmetry*, *Phys. Lett.* **B 412** (1997) 296 [hep-th/9707061] [INSPIRE].

[63] T. Hatanaka and S.V. Ketov, *On the universality of Goldstino action*, *Phys. Lett.* **B 580** (2004) 265 [hep-th/0310152] [INSPIRE].

[64] S.M. Kuzenko and S.A. McCarthy, *On the component structure of* $N = 1$ *supersymmetric nonlinear electrodynamics*, *JHEP* **05** (2005) 012 [hep-th/0501172] [INSPIRE].

[65] F. Farakos, P. Koči and R. von Unge, *Superspace higher derivative terms in two dimensions*, *JHEP* **04** (2017) 002 [arXiv:1612.04361] [INSPIRE].

[66] R. Casalbuoni et al., *Nonlinear realization of supersymmetry algebra from supersymmetric constraint*, *Phys. Lett.* **B 220** (1989) 569 [INSPIRE].

[67] Z. Komargodski and N. Seiberg, *From linear SUSY to constrained superfields*, *JHEP* **09** (2009) 066 [arXiv:0907.2441] [INSPIRE].

[68] M. Roček, *Linearizing the Volkov-Akulov model*, *Phys. Rev. Lett.* **41** (1978) 451 [INSPIRE].

[69] N. Cribiori, G. Dall'Agata and F. Farakos, *Interactions of* N *Goldstini in superspace*, *Phys. Rev.* **D 94** (2016) 065019 [arXiv:1607.01277] [INSPIRE].

[70] D.V. Volkov and V.P. Akulov, *Is the neutrino a Goldstone particle?*, *Phys. Lett.* **B 46** (1973) 109.

[71] S.M. Kuzenko and S.J. Tyler, *Relating the Komargodski-Seiberg and Akulov-Volkov actions: exact nonlinear field redefinition*, *Phys. Lett.* **B 698** (2011) 319 [arXiv:1009.3298] [INSPIRE].

[72] R. Conti, S. Negro and R. Tateo, *The* $T\bar{T}$ *perturbation and its geometric interpretation*, *JHEP* **02** (2019) 085 [arXiv:1809.09593] [INSPIRE].

[73] V. Rosenhaus and M. Smolkin, *Integrability and Renormalization under* $T\bar{T}$, arXiv:1909.02640 [INSPIRE].

[74] L. Rastelli, J. Caetano, and W. Peelaers, *Exploring a famous irrelevant deformation of* $N = 4$ *SYM*, talk given at *Integrability in Gauge and String Theories 2019 (IGST 2019)*, July 15–19, Stockholm, Sweden (2019).

[75] R.R. Metsaev and M. Rakhmanov, *Fermionic terms in the open superstring effective action*, *Phys. Lett.* **B 193** (1987) 202 [INSPIRE].

[76] R.R. Metsaev, M. Rakhmanov and A.A. Tseytlin, *The Born-Infeld action as the effective action in the open superstring theory*, *Phys. Lett.* **B 193** (1987) 207 [INSPIRE].

[77] S. Paban, S. Sethi and M. Stern, *Constraints from extended supersymmetry in quantum mechanics*, *Nucl. Phys.* **B 534** (1998) 137 [hep-th/9805018] [INSPIRE].

[78] S. Paban, S. Sethi and M. Stern, *Supersymmetry and higher derivative terms in the effective action of Yang-Mills theories*, *JHEP* **06** (1998) 012 [hep-th/9806028] [INSPIRE].

[79] Y.-H. Lin, S.-H. Shao, Y. Wang and X. Yin, *Higher derivative couplings in theories with sixteen supersymmetries*, *Phys. Rev.* **D 92** (2015) 125017 [arXiv:1503.02077] [INSPIRE].

[80] W.-M. Chen, Y.-t. Huang and C. Wen, *Exact coefficients for higher dimensional operators with sixteen supersymmetries*, *JHEP* **09** (2015) 098 [arXiv:1505.07093] [INSPIRE].

[81] M.R. Garousi, *Duality constraints on effective actions*, *Phys. Rept.* **702** (2017) 1 [arXiv:1702.00191] [INSPIRE].

[82] M. Heydeman, J.H. Schwarz and C. Wen, *M5-brane and D-brane scattering amplitudes*, *JHEP* **12** (2017) 003 [arXiv:1710.02170] [INSPIRE].

[83] E. Bergshoeff, M. Rakowski and E. Sezgin, *Higher derivative super-Yang-Mills theories*, *Phys. Lett.* **B 185** (1987) 371 [INSPIRE].

[84] A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, hep-th/9908105 [INSPIRE].

[85] E. Bergshoeff et al., *Dirac-Born-Infeld-Volkov-Akulov and deformation of supersymmetry*, *JHEP* **08** (2013) 100 [arXiv:1303.5662] [INSPIRE].

[86] S.V. Ketov, *Many faces of Born-Infeld theory*, in the proceedings of the 7$^{\text{th}}$ *International Wigner Symposium (WigSYM 7)*, August 24–29, College Park, U.S.A. (2001), hep-th/0108189 [INSPIRE].

[87] S.M. Kuzenko and S. Theisen, *Supersymmetric duality rotations*, *JHEP* **03** (2000) 034 [hep-th/0001068] [INSPIRE].

[88] S.F. Kerstan, *Supersymmetric Born-Infeld from the D9-brane*, *Class. Quant. Grav.* **19** (2002) 4525 [hep-th/0204225] [INSPIRE].

[89] S.V. Ketov, *A Manifestly $N = 2$ supersymmetric Born-Infeld action*, *Mod. Phys. Lett.* **A 14** (1999) 501 [hep-th/9809121] [INSPIRE].

[90] S.V. Ketov, *Born-Infeld-Goldstone superfield actions for gauge fixed D5 branes and D3 branes in $6-d$*, *Nucl. Phys.* **B 553** (1999) 250 [hep-th/9812051] [INSPIRE].

[91] S.V. Ketov, *$N = 2$ superBorn-Infeld theory revisited*, *Class. Quant. Grav.* **17** (2000) L91 [hep-th/0005126] [INSPIRE].

[92] S.M. Kuzenko and S. Theisen, *Nonlinear selfduality and supersymmetry*, *Fortsch. Phys.* **49** (2001) 273 [hep-th/0007231] [INSPIRE].

[93] S. Bellucci, E. Ivanov and S. Krivonos, *$N = 2$ and $N = 4$ supersymmetric Born-Infeld theories from nonlinear realizations*, *Phys. Lett.* **B 502** (2001) 279 [hep-th/0012236] [INSPIRE].

[94] S. Bellucci, E. Ivanov and S. Krivonos, *Towards the complete $N = 2$ superfield Born-Infeld action with partially broken $N = 4$ supersymmetry*, *Phys. Rev.* **D 64** (2001) 025014 [hep-th/0101195] [INSPIRE].

[95] S. Bellucci, E. Ivanov and S. Krivonos, *Superbranes and super-Born-Infeld theories from nonlinear realizations*, *Nucl. Phys. Proc. Suppl.* **102** (2001) 26 [hep-th/0103136] [INSPIRE].

JHEP02(2020)016

[96] E.A. Ivanov and B.M. Zupnik, *New representation for Lagrangians of selfdual nonlinear electrodynamics*, in the proceedings of the *Supersymmetries and Quantum Symmetries.* 16th *Max Born Symposium (SQS'01)*, September 21–25, Karpacz, Poland (2001), hep-th/0202203 [INSPIRE].

[97] N. Berkovits and V. Pershin, *Supersymmetric Born-Infeld from the pure spinor formalism of the open superstring*, *JHEP* **01** (2003) 023 [hep-th/0205154] [INSPIRE].

[98] E.A. Ivanov and B.M. Zupnik, *New approach to nonlinear electrodynamics: dualities as symmetries of interaction*, *Phys. Atom. Nucl.* **67** (2004) 2188 [hep-th/0303192] [INSPIRE].

[99] R. Kallosh, *Nonlinear (super)symmetries and amplitudes*, *JHEP* **03** (2017) 038 [arXiv:1609.09123] [INSPIRE].

[100] R. Kallosh, A. Karlsson and D. Murli, *Origin of soft limits from nonlinear supersymmetry in Volkov-Akulov theory*, *JHEP* **03** (2017) 081 [arXiv:1609.09127] [INSPIRE].

[101] C. Cheung, K. Kampf, J. Novotny and J. Trnka, *Effective field theories from soft limits of scattering amplitudes*, *Phys. Rev. Lett.* **114** (2015) 221602 [arXiv:1412.4095] [INSPIRE].

[102] C. Cheung et al., *Vector effective field theories from soft limits*, *Phys. Rev. Lett.* **120** (2018) 261602 [arXiv:1801.01496] [INSPIRE].

[103] H. Elvang, M. Hadjiantonis, C.R.T. Jones and S. Paranjape, *All-multiplicity one-loop amplitudes in Born-Infeld electrodynamics from generalized unitarity*, arXiv:1906.05321 [INSPIRE].

JHEP02(2020)016