Flat space compressible fluid as holographic dual of black hole with curved horizon

Xin Hao, Bin Wu and Liu Zhao

School of Physics, Nankai University,
Tianjin 300071, China

E-mail: shanehowe@mail.nankai.edu.cn, wubin@mail.nankai.edu.cn, lzhao@nankai.edu.cn

ABSTRACT: We consider the fluid dual of $(d+2)$-dimensional vacuum Einstein equation either with or without a cosmological constant. The background solutions admit black hole event horizons and the spatial sections of the horizons are conformally flat. Therefore, a d-dimensional flat Euclidean space $\mathbb{E}^d$ is contained in the conformal class of the spatial section of the black hole horizon. A compressible, forced, stationary and viscous fluid system can be constructed on the product (Newtonian) spacetime $\mathbb{R} \times \mathbb{E}^d$ as the lowest order fluctuation modes around such black hole background. This construction provides the first example of holographic duality which is beyond the class of bulk/boundary correspondence.

Contents

1 Introduction

Since 't Hooft [1] and Susskind [2] proposed the so-called holographic principle about twenty years ago, the study of holographic dual of gravitational systems has become one of the major subjects of study in the area of high energy theories. Holography is a property which matches one system in the bulk involving gravity to another system on the boundary without gravity. The most well understood realization of holographic duality is the AdS/CFT correspondence [3–5], which establishes an equivalence between the superstring theory on $AdS_5 \times S^5$ and the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory on the boundary of the AdS_5. However, there are accumulating evidences indicating that holography can be realized in situations which requires neither AdS in the bulk nor CFT on the boundary. In this respect, the Gravity/Fluid correspondence plays a very instructive example.

The relationship between gravity and fluid system was first known through the membrane paradigm, see [6] and [7–9] for recent applications of this approach. Later on, such relationship is rediscovered as the long wavelength limit of AdS/CFT correspondence [10–16]. In both approaches, the ratio of the viscosity to the entropy density of the dual fluid takes a universal value $\frac{1}{4\pi}$ [17, 18]. Considerable efforts have been made to clarify that the results from the membrane paradigm and from the long wavelength limit of AdS/CFT are related by RG flow [18–21].

Renewed interests in Gravity/Fluid correspondence arise following the work [22], in which a fluid dual on a finite cutoff in a Rindler background has been constructed. This is the first successful attempt in constructing Gravity/Fluid correspondence beyond the framework of AdS/CFT. Subsequent works revealed that similar construction also works

in black hole backgrounds in Einstein gravity [23–26] as well as in higher curvature gravity [27–29], and the viscosity of the dual fluid is calculated in [30, 31]. Numerous examples of this correspondence was studied extensively [32–36].

Besides the membrane paradigm and the AdS/CFT approach, the Gravity/Fluid correspondence can be realized either using a boost-rescaling technique [23–29, 31–36] or by the introduction of Petrov I boundary conditions [37–46], which is mathematically much simpler. In all known example cases, there are two remarkable features (or drawbacks) in the Gravity/Fluid correspondence. Firstly, just like in any other realizations of holographic duality, the boundary (or holographic screen) must be chosen such that it is an equipotential hypersurface in the bulk spacetime. This requires, in particular, that if one starts from a spherically symmetric solution of the gravitational field equations, the final fluid dual must also live in a spacetime with spherical spatial sections. Thus, if one would like to understand flat space fluid mechanics from a gravitational perspective, then the only choice seems to be starting from an AdS bulk and choose a solution with flat horizon. Secondly, the dual fluid is always incompressible due to the fact that to the lowest nontrivial order, the conservation of the Brown-York tensor on the boundary implies a divergence-free condition of the velocity field of the dual fluid.

In this work, we are aimed to realize a duality relationship between gravity solutions with spatially non-flat horizons and a compressible fluid system living in a flat, Newtonian spacetime with one less dimensions. Evidently, a flat Newtonian spacetime cannot be realized as an equipotential hypersurface in the bulk spacetime unless the gravity solution is plane symmetric. In fact, the flat Newtonian space may not be a subspace of the bulk spacetime at all. Therefore, the duality relationship as described above is highly nontrivial not only because it evades the two drawbacks of general Gravity/Fluid correspondence, but also because it provides a remarkable example of holographic duality which is beyond the class of bulk/boundary dualities.

2   Static black holes in $(d+2)$-dimensions

Let us start by introducing a particular class of static vacuum solutions to the Einstein equation

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{2.1}$$

in $(d+2)$-dimensions, where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the bulk Einstein tensor, Λ denotes a possible cosmological constant which may be positive, zero or negative.

Under the coordinates $x^\mu = (t, r, x^i)$ $(i = 1, \cdots, d)$, the static solution to the vacuum Einstein equation can be described by the line element

$$\mathrm{d}s^2_{d+2} = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 e^{\Phi(x^i)}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad f(r) = \kappa - \frac{\omega}{r^{d-1}} - \frac{2\Lambda r^2}{d(d+1)}, \tag{2.2}$$

where $\kappa = 1$ if $\Lambda \geq 0$ and $\kappa = -1, 0, 1$ if $\Lambda < 0$. To ensure that (2.2) is a solution to the vacuum Einstein equation (2.1), the function $\Phi(x^i)$ must obey a set of complicated

differential equations,

$$\delta^{jk}\partial_j\partial_k\Phi + \frac{d-2}{2}\left(2\partial_i^2\Phi + \delta^{jk}\partial_j\Phi\partial_k\Phi - (\partial_i\Phi)^2\right) + 2\kappa(d-1)e^\Phi = 0,$$

$$\text{(no summations over fixed } i = 1, \cdots, d) \qquad (2.3)$$

$$(d-2)\left(\partial_i\partial_j\Phi - \frac{1}{2}\partial_i\Phi\partial_j\Phi\right) = 0 \qquad (i \neq j). \qquad (2.4)$$

For generic d, explicit solution for the function $\Phi(x^i)$ is guaranteed to exist because the well-known Schwarzschild-Tangherlini-(A)dS solution to the vacuum Einstein equation can be written in the form (2.2). In particular, when $d = 2$, the eqs. (2.3)–(2.4) degenerate into the Euclideanized Liouville equation (or Laplacian equation if $\kappa = 0$) [47]

$$(\partial_1^2 + \partial_2^2)\Phi + 2\kappa e^\Phi = 0. \qquad (2.5)$$

It is evident that the line element (2.2) possesses a black hole event horizon provided $\omega \neq 0$. The event horizon is located at one of the zeros $r = r_h$ of the metric function $f(r)$, which is the largest (if Λ is non-positive) or the second largest (if Λ is positive) root of $f(r)$. The spatial section of the horizon surface has a conformally flat metric

$$\mathrm{d}s_d^2 = r_h^2 e^{\Phi(x^i)}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

which contains the d-dimensional flat Euclidean space $\mathbb{E}^d$ with metric δ_{ij} in its conformal class.

In what follows, it is desirable to rewrite the line element (2.2) in the Eddington-Finkelstein coordinates (u, r, x^i) with the lightlike coordinate u defined by

$$u = t + \int_0^r \frac{\mathrm{d}r'}{f(r')}. \qquad (2.6)$$

Doing so, the line element (2.2) can be rearranged into the following form

$$\mathrm{d}s_{d+2}^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -f(r)du^2 + 2dudr + r^2 e^{\Phi(x^i)}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \qquad (2.7)$$

3 Hypersurface and Brown-York tensor

Now consider a $(d+1)$-dimensional timelike hypersurface Σ_c located at $r = r_c$. The geometry of this embedding hypersurface is best characterized by its first and second fundamental forms. The first fundamental form is provided by the restriction of the bulk line element on the hypersurface, i.e.

$$\begin{aligned}
ds_{d+1}^2 = \gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b &= -f(r_c)du^2 + r_c^2 e^\Phi\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \\
&= -(\mathrm{d}x^0)^2 + r_c^2 e^\Phi\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \\
&= -\frac{1}{\lambda^2}\mathrm{d}\tau^2 + r_c^2 e^\Phi\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j,
\end{aligned} \qquad (3.1)$$

where $x^a = (u, x^i)$, $\tau = \lambda x^0 = (\lambda \sqrt{f_c}) u$ and the rescaling parameter λ is introduced to facilitate the forthcoming analysis on the non-relativistic limit.[1] Here and below we use the notations $f_c = f(r_c)$, $f'_c = f'(r)|_{r=r_c}$ etc. The notations f'_h, f''_h will also be used, which are similar to f'_c, f''_c but with r_c replaced by r_h.

One can easily promote the hypersurface tensor γ_{ab} to a bulk tensor $\gamma_{\mu\nu}$ by adding a raw and a column in the r-direction which are both full of zeros. The second fundamental form is the extrinsic curvature of the hypersurface, which is defined as

$$K_{\mu\nu} = \frac{1}{2} \mathscr{L}_n \gamma_{\mu\nu}, \tag{3.2}$$

where

$$n^\mu = \left(\frac{1}{\sqrt{f}}, \sqrt{f}, 0, \cdots, 0 \right)$$

is a unit vector field which is normal to Σ_c at $r = r_c$ and is written in the coordinate (u, r, x^i).

The projection of the bulk Einstein equation gives rise to the so-called momentum and Hamiltonian constraints,

$$(G_{\mu\nu} + \Lambda g_{\mu\nu}) \gamma^\mu_{\ b} n^\nu |_{\Sigma_c} = 0,$$
$$(G_{\mu\nu} + \Lambda g_{\mu\nu}) n^\mu n^\nu |_{\Sigma_c} = 0.$$

In terms of the two fundamental forms introduced above, these can be reformulated in the form

$$D_a(K^a_{\ b} - \gamma^a_{\ b} K) = 0, \tag{3.3}$$
$$\hat{R} + K^{ab} K_{ab} - K^2 = 2\Lambda, \tag{3.4}$$

where $\hat{R}$ is the Ricci scalar of Σ_c, D_a is the covariant derivative that is compatible with γ_{ab}. Using the definition [50]

$$t_{ab} = \gamma_{ab} K - K_{ab} \tag{3.5}$$

for the Brown-York stress energy tensor t_{ab} on hypersurface, the momentum constraints (3.3) becomes that of the covariant divergence free condition

$$D_a t^a_{\ b} = 0 \tag{3.6}$$

for t_{ab}, and the Hamiltonian constraint becomes

$$\hat{R} + t^a_{\ b} t^b_{\ a} - \frac{t^2}{d} = 2\Lambda. \tag{3.7}$$

All these are the standard material for the construction of Gravity/Fluid correspondence.

---

[1]One can think of $1/\lambda$ as the speed of light, hence $\lambda \to 0$ corresponds to infinite light speed, i.e. the non-relativistic limit.

Before moving on to the construction of dual fluid, let us mention that the lightlike coordinate u (and hence τ) in the bulk spacetime becomes naturally timelike on the hypersurface Σ_c. The time evolution of the dual fluid will be defined with respect to this coordinate. However, the spacetime in which the dual fluid lives will not be Σ_c (which is in general curved) but rather the product space $\mathbb{R} \times \mathbb{E}^d$ in which the first factor represents the time direction and $\mathbb{E}^d$ is the d-dimensional Euclidean space which lies in the conformal class of the spatial section of Σ_c. In other words, the dual fluid will be living in a $(d+1)$-dimensional Newtonian spacetime.

4 Petrov I boundary condition

As usual in Gravity/Fluid correspondence, we introduce the Petrov I [48, 49] boundary condition on Σ_c, i.e.

$$C_{(l)i(l)j} = l^\mu (m_i)^\nu l^\rho (m_j)^\sigma C_{\mu\nu\rho\sigma} = 0, \qquad (4.1)$$

where

$$l^2 = k^2 = 0, \, , (k,l) = 1, \, , (l, m_i) = (k, m_i) = 0, \, , (m_i, m_j) = \delta_{ij} \qquad (4.2)$$

are a set of Newman-Penrose basis vector fields, and $C_{\mu\nu\rho\sigma}$ is the bulk Weyl tensor. We choose the basis vector fields to be

$$l^\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{f}}(\partial_u)^\mu - n^\mu \right) = \frac{1}{\sqrt{2}}\left((\partial_0)^\mu - n^\mu \right),$$

$$k^\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{f}}(\partial_u)^\mu + n^\mu \right) = \frac{1}{\sqrt{2}}\left((\partial_0)^\mu + n^\mu \right),$$

$$(m_i)^\mu = r^{-1} e^{-\frac{1}{2}\Phi}(\partial_i)^\mu. \qquad (4.3)$$

then the boundary condition becomes

$$C_{0i0j} + C_{0ij(n)} + C_{0ji(n)} + C_{i(n)j(n)} = 0, \qquad (4.4)$$

where C_{abcd}, $C_{abc(n)}$, $C_{a(n)b(n)}$ are projections of the bulk Weyl tensor

$$C_{abcd} = \gamma^\mu_a \gamma^\nu_b \gamma^\sigma_c \gamma^\rho_d C_{\mu\nu\sigma\rho},$$

$$C_{abc(n)} = \gamma^\mu_a \gamma^\nu_b \gamma^\sigma_c n^\rho C_{\mu\nu\sigma\rho},$$

$$C_{a(n)b(n)} = \gamma^\mu_a n^\nu \gamma^\sigma_c n^\rho C_{\mu\nu\sigma\rho},$$

and all these can be expressed in terms of the the fundamental forms of Σ_c:

$$C_{abcd} = \hat{R}_{abcd} + K_{ad}K_{bc} - K_{ac}K_{bd} - \frac{4\Lambda}{d(d+1)}\gamma_{a[c}\gamma_{d]b},$$

$$C_{abc(n)} = D_a K_{bc} - D_b K_{ac},$$

$$C_{a(n)b(n)} = -\hat{R}_{ab} + K K_{ab} - K_{ac}K^c_{\ b} + \frac{2\Lambda}{(d+1)}\gamma_{ab}. \qquad (4.5)$$

Here, of course, $\hat{R}_{abcd}$ and $\hat{R}_{ab}$ are the Riemann and Ricci tensors of Σ_c.

Then we rewrite the Brown-York tensor (3.5) in components

$$K^\tau{}_\tau = \frac{t}{d} - t^\tau{}_\tau, \qquad\qquad K^\tau{}_i = -t^\tau{}_i,$$
$$K^i{}_j = \frac{t}{d}\delta^i{}_j - t^i{}_j, \qquad\qquad K = \frac{t}{d}.$$

Inserting these relations as well as (4.5) into (4.4), the boundary conditions finally become

$$\frac{2}{\lambda^2}t^\tau{}_i t^\tau{}_j + \frac{t^2}{d^2}\gamma_{ij} + \frac{2\Lambda}{d}\gamma_{ij} - (t^\tau{}_\tau - 2\lambda D_\tau)\left(\frac{t}{d}\gamma_{ij} - t_{ij}\right)$$
$$- \frac{2}{\lambda}D_{(i}t^\tau{}_{j)} - t_{ik}t^k{}_j - \hat{R}_{ij} = 0, \tag{4.6}$$

where the explicit appearance of the parameter λ comes from the rescaling of the coordinate $x^0 \to \tau/\lambda$ which is made in (3.1).

5 Near horizon and non-relativistic limit

Now let us place the hypersurface Σ_c very close to the black hole event horizon at $r = r_h$. This means that $r_c - r_h$ is a small parameter, and we take this parameter to be $r_c - r_h = \alpha^2\lambda^2$, where λ is the same rescaling parameter appeared in (3.1) and α is a finite constant which must be present to balance the dimensionality. Note that such an identification implies that the near horizon limit $\lambda \to 0$ is simultaneously the non-relativistic limit. The near horizon nature of Σ_c allows us to expand f_c in power series of λ,

$$f(r_c) = f'(r_h)(r_c - r_h) + \frac{1}{2}f''(r_h)(r_c - r_h)^2 + \cdots$$
$$= f'_h \cdot (\alpha^2\lambda^2) + \frac{1}{2}f''_h \cdot (\alpha^2\lambda^2)^2 + \cdots, \tag{5.1}$$

which is crucial in the following constructions.

To realize the fluid dual of the gravitational theory, it is insufficient to consider only the background metric. Rather, it is necessary to consider small fluctuations around the background solution. So, on the hypersurface Σ_c, the metric can be expanded in power series in λ,

$$\gamma_{ab} = \gamma_{ab}^{(B)} + \sum_{n=1}^{\infty}\gamma_{ab}^{(n)}\lambda^n, \tag{5.2}$$

where $\gamma_{ab}^{(B)}$ represents the background metric and $\gamma_{ab}^{(n)}$ are the fluctuation modes. Consequently, both the Ricci curvature $\hat{R}_{ab}$ and the Brown-York tensor $t^a{}_b$ will also be subject to fluctuations, i.e.

$$\hat{R}_{ab} = \hat{R}_{ab}^{(B)} + \sum_{n=1}^{\infty}\lambda^n\hat{R}_{ab}^{(n)}, \tag{5.3}$$

$$t^a{}_b = t^{a(B)}_b + \sum_{n=1}^{\infty}\lambda^n t^{a(n)}_b, \tag{5.4}$$

where the superscripts (B) indicate contributions from the background geometry.

In the near horizon limit, the background contributions will also depend on the parameter λ, thanks to the expansion (5.1). So, we need to evaluate the background values of the Brown-York and the Ricci tensors on Σ_c and expand the results near the horizon $r = r_h$. By direct calculations, we can get

$$
K^\tau{}_\tau = \frac{f'_c}{2\sqrt{f_c}}, \qquad\qquad K^\tau{}_i = 0
$$
$$
K^i{}_j = \frac{\sqrt{f_c}}{r_c}\delta^i{}_j, \qquad\qquad K = \frac{f'_c}{2\sqrt{f_c}} + \frac{d\sqrt{f_c}}{r_c}. \tag{5.5}
$$

These in turn lead to the background Brown-York tensor

$$
t^{\tau(B)}{}_\tau = \frac{d\sqrt{f_c}}{r_c},
$$
$$
t^{\tau(B)}{}_i = 0,
$$
$$
t^{i(B)}{}_j = \left(\frac{f'_c}{2\sqrt{f_c}} + \frac{(d-1)\sqrt{f_c}}{r_c} \right)\delta^i{}_j,
$$
$$
t^{(B)} = \frac{d}{2}\frac{f'_c}{\sqrt{f_c}} + d^2\frac{\sqrt{f_c}}{r_c}, \tag{5.6}
$$

where $t^{(B)}$ is the trace of $t^{a(B)}{}_b$. Using (5.1), we have the following expansions for $\frac{\sqrt{f_c}}{r_c}$ and $\frac{f'_c}{\sqrt{f_c}}$:

$$
\frac{\sqrt{f_c}}{r_c} = (\alpha\lambda)\frac{\sqrt{f'_h}}{r_h} + \frac{1}{2}(\alpha\lambda)^3\frac{\sqrt{f''_h}}{r_h} + \cdots,
$$
$$
\frac{f'_c}{\sqrt{f_c}} = \frac{\sqrt{f'_h}}{\alpha\lambda} + \alpha\lambda\frac{f''_h}{\sqrt{f'_h}} + \cdots. \tag{5.7}
$$

Inserting the expansions in (5.7) into (5.6) and then into (5.4), we get

$$
t^\tau{}_\tau = \frac{d\alpha\lambda\sqrt{f'_h}}{r_h} + \lambda t^{\tau(1)}{}_\tau + \cdots,
$$
$$
t^\tau{}_i = 0 + \lambda t^{\tau(1)}{}_i + \cdots,
$$
$$
t^i{}_j = \left(\frac{1}{2}\frac{\sqrt{f'_h}}{\alpha\lambda} + \frac{\alpha\lambda f''_h}{2\sqrt{f'_h}} + \frac{(d-1)\alpha\lambda\sqrt{f'_h}}{r_h} \right)\delta^i{}_j + \lambda t^{i(1)}{}_j + \cdots,
$$
$$
t = d\left(\frac{1}{2}\frac{\sqrt{f'_h}}{\alpha\lambda} + \frac{\alpha\lambda f''_h}{2\sqrt{f'_h}} + \frac{d\alpha\lambda\sqrt{f'_h}}{r_h} \right) + \lambda t^{(1)} + \cdots. \tag{5.8}
$$

We shall also make use of the ij components of the Ricci tensor $\hat{R}_{ab}$ on Σ_c. By explicit calculations, we find

$$
\hat{\Gamma}^{\tau(B)}_{ab} = \hat{\Gamma}^{a(B)}_{\tau b} = 0,
$$
$$
\hat{\Gamma}^{k(B)}_{ij} = \frac{1}{2}\left(\delta^k{}_i \partial_j\Phi + \delta^k{}_j \partial_i\Phi - \delta_{ij}\partial^k\Phi \right),
$$

– 7 –

where $\hat{\Gamma}_{ab}^{c(B)}$ are components of the Christoffel connection under the background geometry of Σ_c. It is remarkable that the components of $\hat{\Gamma}_{ab}^{c(B)}$ are independent of the position of Σ_c. Consequently, the background Ricci tensor $\hat{R}_{ab}^{(B)}$ will also independent of r_c. Explicitly, we have

$$\hat{R}_{\tau a}^{(B)} = 0, \qquad \hat{R}_{ij}^{(B)} = \kappa(d-1)e^{\Phi}\delta_{ij}, \qquad (5.9)$$

where use have been made of the equations (2.3) and (2.4). So, in the near horizon limit, $\hat{R}_{ij}^{(B)}$ will not develop λ dependences. However, since the metric γ_{ab} on Σ_c may fluctuate due to (5.2), the fluctuation parts $\hat{R}_{ab}^{(n)}$ in (5.3) will in general be nonzero. Moreover, due to the fluctuations of the metric, the covariant derivatives such as $D_j t^{\tau}{}_k$ will also receive fluctuating corrections which are at least $\mathcal{O}(\lambda^1)$ because

$$\hat{\Gamma}_{ab}^c = \hat{\Gamma}_{ab}^{c(B)} + \mathcal{O}(\lambda^1). \qquad (5.10)$$

Finally, substituting (5.8) and (5.3) (with (5.9) inserted) into (4.6), we get in the first nontrivial order $\lambda^{(0)}$ the following identity,

$$\frac{\sqrt{f_h'}}{\alpha}t^i{}_j^{(1)} = 2\gamma^{ik(0)}t^{\tau(1)}{}_k t^{\tau(1)}{}_j - 2\gamma^{ik(0)}\zeta_{kj} + \frac{\sqrt{f_h'}}{d\alpha}t^{(1)}\delta^i{}_j, \qquad (5.11)$$

where we have introduced the shorthand notations

$$\gamma^{ik(0)} = r_h^{-2}e^{-\Phi}\delta^{ik},$$
$$\zeta_{kj} = \partial_{(k}t^{\tau(1)}{}_{j)} - \partial_{(k}\Phi t^{\tau(1)}{}_{j)} + \frac{1}{2}\delta_{kj}\delta^{lm}\partial_l\Phi t^{\tau(1)}{}_m, \qquad (5.12)$$

which are, respectively, the inverse of the near horizon background metric on Σ_c and the leading term in $D_{(i}t^{\tau}{}_{j)}$:

$$D_{(i}t^{\tau}{}_{j)} = \lambda\zeta_{ij} + \mathcal{O}(\lambda^2).$$

Some terms which ought to appear in (5.11) cancels out because

$$\frac{2\Lambda}{d} + \frac{f_h'}{r_h} - \kappa\frac{d-1}{r_h^2} = 0.$$

Besides eq. (5.11), which is the lowest nontrivial order of the Petrov I boundary condition (4.6), we also need to consider the fluctuation modes in the covariant conservation condition (3.6) and the Hamiltonian constraint (3.7). Using (5.8) and (5.10), we can evaluate the τ component of (3.6), which reads

$$D_a t^a{}_{\tau} = \partial_\tau t^{\tau}{}_{\tau} + \partial_i t^i{}_{\tau} + \hat{\Gamma}_{ij}^i t^j{}_{\tau}$$
$$= -\frac{1}{\lambda}\left[\gamma^{ij(0)}\left(\partial_i + \frac{d-2}{2}\partial_i\Phi\right)t^{\tau(1)}{}_j\right] + \mathcal{O}(\lambda^{-1}). \qquad (5.13)$$

Therefore, at order λ^{-1}, we get

$$\delta^{ij}\left(\partial_i + \frac{d-2}{2}\partial_i\Phi\right)t^{\tau(1)}{}_j = 0. \qquad (5.14)$$

Similarly, we can also evaluate the spatial components of (3.6), which yields, at the first nontrivial order $\mathcal{O}(\lambda^1)$, the following equation,

$$\partial_\tau t^{\tau(1)}_i - \frac{1}{2}(t^{(1)} - t^{\tau(1)}_\tau)\partial_i\Phi + \left(\partial_j + \frac{d}{2}\partial_j\Phi\right)t^{j(1)}_i = 0. \tag{5.15}$$

The first non-vanishing order of the Hamiltonian constraint is at $\mathcal{O}(\lambda^0)$:

$$t^{\tau(1)}_\tau = -2\gamma^{ij(0)}t^{\tau(1)}_i t^{\tau(1)}_j. \tag{5.16}$$

In the next section we will show that the equations (5.11), (5.14), (5.15) and (5.16) give rise to the Navier-Stokes equation of a forced, stationary and compressible fluid system.

6 Dual fluid in flat space

In this section we study the dual fluid equations that arise from the fluctuation modes described in the last section. For this purpose, we need to insert (5.11) into (5.15) and simplify the result. The term $\partial_j t^{j(1)}_i$ can be evaluated as follows,

$$\partial_j t^{j(1)}_i = \frac{\alpha}{\sqrt{f'_h}}\partial_j\left(2\gamma^{jk(0)}t^{\tau(1)}_k t^{\tau(1)}_i\right) - \frac{\alpha}{\sqrt{f'_h}}\partial_j\left(2\gamma^{jk(0)}\zeta_{ki}\right) + \frac{1}{d}\partial_i t^{(1)}, \tag{6.1}$$

where

$$\partial_j\left(2\gamma^{jk(0)}t^{\tau(1)}_k t^{\tau(1)}_i\right) = -d\gamma^{jk(0)}\partial_j\Phi t^{\tau(1)}_k t^{\tau(1)}_i + 2\gamma^{jk(0)}t^{\tau(1)}_j\partial_k t^{\tau(1)}_i, \tag{6.2}$$

and

$$\partial_j\left(2\gamma^{jk(0)}\zeta_{ki}\right) = \partial_i(\gamma^{jk(0)}\partial_j t^{\tau(1)}_k) + \gamma^{jk(0)}(\partial_j\partial_k - \partial_j\partial_k\Phi - 2\partial_j\Phi\partial_k + \partial_j\Phi\partial_k\Phi)t^{\tau(1)}_i. \tag{6.3}$$

Inserting (6.2) and (6.3) into (6.1), we get

$$\partial_j t^{j(1)}_i = \frac{\alpha}{\sqrt{f'_h}}\Big(-d\gamma^{jk(0)}\partial_j\Phi t^{\tau(1)}_k t^{\tau(1)}_i + 2\gamma^{jk(0)}t^\tau_j\partial_k t^{\tau(1)}_i - \partial_i(\gamma^{jk(0)}\partial_j t^{\tau(1)}_k)$$
$$- \gamma^{jk(0)}(\partial_j\partial_k - \partial_j\partial_k\Phi - 2\partial_j\Phi\partial_k + \partial_j\Phi\partial_k\Phi)t^{\tau(1)}_i\Big) + \frac{1}{d}\partial_i t^{(1)}. \tag{6.4}$$

Substituting (6.4) as well as (5.11), (5.12) and (5.16) into (5.15), we get

$$\partial_\tau t^{\tau(1)}_i + \frac{1}{d}\partial_i t^{(1)} + r_h^{-2}e^{-\Phi}\delta^{jk}\Big[2t^{\tau(1)}_k\partial_j t^{\tau(1)}_i - \partial_j\partial_k t^{\tau(1)}_i - 2t^{\tau(1)}_j t^{\tau(1)}_k\partial_i\Phi$$
$$+ \left(\partial_j\partial_k\Phi - \frac{d-4}{2}\partial_j\Phi\partial_k + \frac{d-2}{2}\partial_j\Phi\partial_k\Phi\right)t^{\tau(1)}_i$$
$$- \partial_j\Phi\partial_i t^{\tau(1)}_k + \frac{d-2}{2}(t^{\tau(1)}_k\partial_j - \partial_j\Phi t^{\tau(1)}_k)\partial_i\Phi\Big] = 0, \tag{6.5}$$

where we have chosen $\alpha = \sqrt{f'_h}$ to eliminate the constant factors such as $\frac{\alpha}{\sqrt{f'_h}}$.

Unlike the usual construction of fluid dual, we would like to interpret eqs. (5.14) and (6.5) as the continuity and the Navier-Stokes equations respectively in a flat Euclidean

space with spatial coordinates x^i. To achieve this, let us first rewrite (5.14) in the following form:

$$\partial^j\left(e^{\frac{d-2}{2}\Phi}t^{\tau(1)}_{\ j}\right) = 0. \tag{6.6}$$

Adopting the following "holographic dictionary"

$$\rho = r_h^2 e^{\frac{d}{2}\Phi}, \quad \mu = e^{\frac{d-2}{2}\Phi}, \quad \nu = \frac{\mu}{\rho} = r_h^{-2}e^{-\Phi}, \tag{6.7}$$

and

$$t^{\tau(1)}_{\ i} = \frac{v_i}{2\nu}, \qquad \frac{t^{(1)}}{d} = \frac{p}{2\mu}, \tag{6.8}$$

where ρ, μ, v_i, p are respectively the density, viscosity, velocity field and the pressure of the dual fluid (ν is the kinematic viscosity), then eq. (6.6) becomes the continuity equation

$$\partial^j(\rho v_j) = 0, \tag{6.9}$$

and eq. (6.5) becomes the standard Navier-Stokes equation

$$\rho(\partial_\tau v_i + v^j\partial_j v_i) = -\partial_i p + \partial^j d_{ij} + f_i \tag{6.10}$$

for the velocity field of the fluid, where the symmetric traceless tensor

$$d_{ij} = \mu\left(\partial_j v_i + \partial_i v_j - \frac{2}{d}\delta_{ij}\partial^k v_k\right) \tag{6.11}$$

represents the deviatoric stress, which depends only on the derivatives of the velocity field and hence vanishes in the hydrostatic equilibrium limit, and

$$f_i = \partial^j\Phi\left(d_{ij} + \frac{d-2}{2}p\delta_{ij}\right) + \frac{2}{d}v^j v_j\partial_i\rho - \frac{2}{d}(v_j\partial^j\rho)v_i \tag{6.12}$$

represents a body force. It is easy to identify the last term in (6.12) as a linear resistance force, which is proportional to the velocity field v_i and to the directional gradient of the density of the fluid. The first two terms in (6.12) look unusual, because the factor $\partial^j\Phi$ actually is proportional to the gradient of the logarithm of the density of the fluid. Despite the unusual form of the body force, the equations (6.9) and (6.10) constitute the complete system of equations governing the motion of a compressible, forced, stationary and viscous fluid moving in the $(d+1)$-dimensional Newtonian spacetime $\mathbb{R} \times \mathbb{E}^d$.

7 Concluding remarks

Unlike the ordinary Gravity/Fluid correspondence in which the dual fluid always lives on an equipotential hypersurface (usually taken to be a near horizon hypersurface) and is always incompressible, we have constructed fluid dual in a Newtonian spacetime with one less dimension as compared to the gravity system. It looks striking to realize such kind of a holographic dual, because the dual system does not even live on the boundary of the gravitational system. Some of the distinguished features of our construction are summarized below:

- The holographic screen, if one prefers to speak so, is not (necessarily) a boundary of the bulk spacetime. The dual system lines in a flat Newtonian spacetime even if the black hole horizon in the bulk is curved;

- The dual fluid is compressible but stationary, i.e. the density distribution does not change with time;

- The dual system possesses a non-constant viscosity;

- The dual fluid is subject to both the surface stress and a body force, even though the gravity side is free of source.

Going through the construction process, it is clear that our result depends heavily on the fact that the "angular part" of the black hole solution possesses a conformally flat geometry. This is the case for all maximally symmetric solutions of Einstein equation in any dimension as well as all solutions of Einstein equation in the case of $d = 2$ (i.e. four dimensional spacetime) — in the latter case, the angular part is two dimensional and it is known that any two dimensional manifold is conformally flat. To be more concrete, we would like to present the explicit value of the conformal factor e^Φ in the case of arbitrary $d \geq 2$ with $\kappa = 1$. In this case, we have

$$e^{\Phi(x)} = \frac{1}{\left(1 + \frac{1}{4} \sum_i x_i^2\right)^2}.$$

For the particular choice $d = 2$, e^Φ is given by the solutions of the Liouville (or Laplacian) equation (2.5), and there are infinitely many different solutions to such equations.

Clearly, much has been left to do following this work. The first question to be answered is whether similar construction works in the case of other background geometries or starting from other (extended) theories of gravity (ether with or without source fields). Meanwhile, we have chosen to make a near horizon expansion in the intermediate steps of the construction. Whether the near horizon condition is absolutely required is in question, because there are already a number of examples in the ordinary Gravity/Fluid correspondence in which the holographic screen is not taken as a near the horizon hypersurface but rather as a finite cutoff surface [21, 24, 42]. If all these proves to be working, then a further step will be asking whether holographic duality beyond the class of bulk/boundary correspondence can be worked out in more general settings such as Gravity/Condensed Matter Theory or Gravity/QCD correspondences etc. We hope we could have more to say following these lines shortly.

Acknowledgments

References

[1] G. 't Hooft, *Dimensional reduction in quantum gravity*, in proceedings of *Salamfest*, Trieste Italy (1993), pg. 284 [`gr-qc/9310026`] [ɪɴSPIRE].

[2] L. Susskind, *The World as a hologram*, [*J. Math. Phys.* **36** (1995) 6377](#) [`hep-th/9409089`] [ɪɴSPIRE].

[3] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, [*Int. J. Theor. Phys.* **38** (1999) 1113](#) [`hep-th/9711200`] [ɪɴSPIRE].

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, [*Phys. Lett.* **B 428** (1998) 105](#) [`hep-th/9802109`] [ɪɴSPIRE].

[5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [`hep-th/9802150`] [ɪɴSPIRE].

[6] T. Damour, *Black Hole Eddy Currents*, [*Phys. Rev.* **D 18** (1978) 3598](#) [ɪɴSPIRE].

[7] R.H. Price and K.S. Thorne, *Membrane viewpoint on black holes: properties and evolution of the stretched horizon*, [*Phys. Rev.* **D 33** (1986) 915](#) [ɪɴSPIRE].

[8] T. Damour and M. Lilley, *String theory, gravity and experiment*, `arXiv:0802.4169` [ɪɴSPIRE].

[9] C. Eling and Y. Oz, *Relativistic CFT hydrodynamics from the membrane paradigm*, [*JHEP* **02** (2010) 069](#) [`arXiv:0906.4999`] [ɪɴSPIRE].

[10] G. Policastro, D.T. Son and A.O. Starinets, *From AdS/CFT correspondence to hydrodynamics*, [*JHEP* **09** (2002) 043](#) [`hep-th/0205052`] [ɪɴSPIRE].

[11] S. Bhattacharyya, V.E. Hubeny, S. Minwalla and M. Rangamani, *Nonlinear fluid dynamics from gravity*, [*JHEP* **02** (2008) 045](#) [`arXiv:0712.2456`] [ɪɴSPIRE].

[12] R. Baier, P. Romatschke, D.T. Son, A.O. Starinets and M.A. Stephanov, *Relativistic viscous hydrodynamics, conformal invariance and holography*, [*JHEP* **04** (2008) 100](#) [`arXiv:0712.2451`] [ɪɴSPIRE].

[13] M. Haack and A. Yarom, *Nonlinear viscous hydrodynamics in various dimensions using AdS/CFT*, [*JHEP* **10** (2008) 063](#) [`arXiv:0806.4602`] [ɪɴSPIRE].

[14] S. Bhattacharyya et al., *Forced Fluid Dynamics from Gravity*, [*JHEP* **02** (2009) 018](#) [`arXiv:0806.0006`] [ɪɴSPIRE].

[15] S. Bhattacharyya, S. Minwalla and S.R. Wadia, *The incompressible non-relativistic Navier-Stokes equation from gravity*, [*JHEP* **08** (2009) 059](#) [`arXiv:0810.1545`] [ɪɴSPIRE].

[16] T. Ashok, *Forced fluid dynamics from gravity in arbitrary dimensions*, [*JHEP* **03** (2014) 138](#) [`arXiv:1309.6325`] [ɪɴSPIRE].

[17] G. Policastro, D.T. Son and A.O. Starinets, *The shear viscosity of strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma*, [*Phys. Rev. Lett.* **87** (2001) 081601](#) [`hep-th/0104066`] [ɪɴSPIRE].

[18] P. Kovtun, D.T. Son and A.O. Starinets, *Holography and hydrodynamics: diffusion on stretched horizons*, *JHEP* **10** (2003) 064 [hep-th/0309213] [INSPIRE].

[19] D.T. Son and A.O. Starinets, *Viscosity, Black Holes and Quantum Field Theory*, *Ann. Rev. Nucl. Part. Sci.* **57** (2007) 95 [arXiv:0704.0240] [INSPIRE].

[20] N. Iqbal and H. Liu, *Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm*, *Phys. Rev.* **D 79** (2009) 025023 [arXiv:0809.3808] [INSPIRE].

[21] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, *Wilsonian Approach to Fluid/Gravity Duality*, *JHEP* **03** (2011) 141 [arXiv:1006.1902] [INSPIRE].

[22] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, *From Navier-Stokes To Einstein*, *JHEP* **07** (2012) 146 [arXiv:1101.2451] [INSPIRE].

[23] G. Compere, P. McFadden, K. Skenderis and M. Taylor, *The holographic fluid dual to vacuum Einstein gravity*, *JHEP* **07** (2011) 050 [arXiv:1103.3022] [INSPIRE].

[24] R.-G. Cai, L. Li and Y.-L. Zhang, *Non-relativistic fluid dual to asymptotically AdS gravity at finite cutoff surface*, *JHEP* **07** (2011) 027 [arXiv:1104.3281] [INSPIRE].

[25] C. Eling, A. Meyer and Y. Oz, *The relativistic rindler hydrodynamics*, *JHEP* **05** (2012) 116 [arXiv:1201.2705] [INSPIRE].

[26] G. Compere, P. McFadden, K. Skenderis and M. Taylor, *The relativistic fluid dual to vacuum Einstein gravity*, *JHEP* **03** (2012) 076 [arXiv:1201.2678] [INSPIRE].

[27] G. Chirco, C. Eling and S. Liberati, *Higher curvature gravity and the holographic fluid dual to flat spacetime*, *JHEP* **08** (2011) 009 [arXiv:1105.4482] [INSPIRE].

[28] X. Bai, Y.-P. Hu, B.-H. Lee and Y.-L. Zhang, *Holographic charged fluid with anomalous current at finite cutoff surface in Einstein-Maxwell gravity*, *JHEP* **11** (2012) 054 [arXiv:1207.5309] [INSPIRE].

[29] D.-C. Zou, S.-J. Zhang and B. Wang, *Holographic charged fluid dual to third order Lovelock gravity*, *Phys. Rev.* **D 87** (2013) 084032 [arXiv:1302.0904] [INSPIRE].

[30] N. Banerjee et al., *Constraints on fluid dynamics from equilibrium partition functions*, *JHEP* **09** (2012) 046 [arXiv:1203.3544] [INSPIRE].

[31] Y.-P. Hu and J.-H. Zhang, *Gravity/Fluid Correspondence and Its Application on Bulk Gravity with* U(1) *Gauge Field*, *Adv. High Energy Phys.* **2014** (2014) 483814 [arXiv:1311.3974] [INSPIRE].

[32] C. Niu, Y. Tian, X.-N. Wu and Y. Ling, *Incompressible Navier-Stokes Equation from Einstein-Maxwell and Gauss-Bonnet-Maxwell Theories*, *Phys. Lett.* **B 711** (2012) 411 [arXiv:1107.1430] [INSPIRE].

[33] R. Nakayama, *The Holographic Fluid on the Sphere Dual to the Schwarzschild Black Hole*, arXiv:1109.1185 [INSPIRE].

[34] R.-G. Cai, L. Li, Z.-Y. Nie and Y.-L. Zhang, *Holographic Forced Fluid Dynamics in Non-relativistic Limit*, *Nucl. Phys.* **B 864** (2012) 260 [arXiv:1202.4091] [INSPIRE].

[35] R.-G. Cai, T.-J. Li, Y.-H. Qi and Y.-L. Zhang, *Incompressible Navier-Stokes Equations from Einstein Gravity with Chern-Simons Term*, *Phys. Rev.* **D 86** (2012) 086008 [arXiv:1208.0658] [INSPIRE].

[36] Y. Tian, X.-N. Wu and H.-b. Zhang, *Poor man's holography: how far can it go?*, *Class. Quant. Grav.* **30** (2013) 125010 [arXiv:1204.2029] [INSPIRE].

[37] V. Lysov and A. Strominger, *From Petrov-Einstein to Navier-Stokes*, arXiv:1104.5502 [INSPIRE].

[38] T.-Z. Huang, Y. Ling, W.-J. Pan, Y. Tian and X.-N. Wu, *From Petrov-Einstein to Navier-Stokes in Spatially Curved Spacetime*, *JHEP* **10** (2011) 079 [arXiv:1107.1464] [INSPIRE].

[39] T.-Z. Huang, Y. Ling, W.-J. Pan, Y. Tian and X.-N. Wu, *Fluid/gravity duality with Petrov-like boundary condition in a spacetime with a cosmological constant*, *Phys. Rev.* **D 85** (2012) 123531 [arXiv:1111.1576] [INSPIRE].

[40] C.-Y. Zhang, Y. Ling, C. Niu, Y. Tian and X.-N. Wu, *Magnetohydrodynamics from gravity*, *Phys. Rev.* **D 86** (2012) 084043 [arXiv:1204.0959] [INSPIRE].

[41] X. Wu, Y. Ling, Y. Tian and C. Zhang, *Fluid/Gravity Correspondence for General Non-rotating Black Holes*, *Class. Quant. Grav.* **30** (2013) 145012 [arXiv:1303.3736] [INSPIRE].

[42] Y. Ling, C. Niu, Y. Tian, X.-N. Wu and W. Zhang, *Note on the Petrov-like boundary condition at finite cutoff surface in gravity/fluid duality*, *Phys. Rev.* **D 90** (2014) 043525 [arXiv:1306.5633] [INSPIRE].

[43] B. Wu and L. Zhao, *Gravity-mediated holography in fluid dynamics*, *Nucl. Phys.* **B 874** (2013) 177 [arXiv:1303.4475] [INSPIRE].

[44] B. Wu and L. Zhao, *Holographic fluid from the nonminimally coupled scalartensor theory of gravity*, *Class. Quant. Grav.* **31** (2014) 105018 [arXiv:1401.6487] [INSPIRE].

[45] R.-G. Cai, Q. Yang and Y.-L. Zhang, *Petrov type-I Spacetime and Dual Relativistic Fluids*, *Phys. Rev.* **D 90** (2014) 041901 [arXiv:1401.7792] [INSPIRE].

[46] R.-G. Cai, Q. Yang and Y.-L. Zhang, *Petrov type-I condition and Rindler fluid in vacuum Einstein-Gauss-Bonnet gravity*, *JHEP* **12** (2014) 147 [arXiv:1408.6488] [INSPIRE].

[47] T. Moskalets and A. Nurmagambetov, *Liouville mode in gauge/gravity duality*, arXiv:1409.4186 [INSPIRE].

[48] A. Coley, R. Milson, V. Pravda and A. Pravdova, *Classification of the Weyl tensor in higher dimensions*, *Class. Quant. Grav.* **21** (2004) L35 [gr-qc/0401008] [INSPIRE].

[49] A. Coley, *Classification of the Weyl Tensor in Higher Dimensions and Applications*, *Class. Quant. Grav.* **25** (2008) 033001 [arXiv:0710.1598] [INSPIRE].

[50] J.D. Brown and J.W. York, *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev.* **D 47** (1993) 1407 [gr-qc/9209012] [INSPIRE].