

Duality in $\mathcal{N} = 2$ minimal model holography

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ABSTRACT: Recently a duality between a family of $\mathcal{N} = 2$ supersymmetric higher spin theories on AdS_3 , and the 't Hooft like limit of a class of Kazama-Suzuki models (that are parametrised by N and k) was proposed. The higher spin theories can be described by a Chern-Simons theory based on the infinite-dimensional Lie algebra $\text{shs}[\mu]$, and under the duality, μ is to be identified with $\lambda = \frac{N}{N+k+1}$. Here we elucidate the structure of the (quantum) asymptotic symmetry algebra $\mathcal{W}_\infty[\mu]$ for arbitrary μ and central charge c . In particular, we show that for each value of the central charge, there are generically four different values of μ that describe the same \mathcal{W}_∞ algebra. Among other things this proves that the quantum symmetries on both sides of the duality agree; this equivalence does not just hold in the 't Hooft limit, but even at finite N and k .

KEYWORDS: Supersymmetry and Duality, Extended Supersymmetry, AdS-CFT Correspondence, Conformal and W Symmetry

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1 Introduction and summary

During the last few years dualities between higher spin theories on AdS_{d+1} and weakly coupled d -dimensional conformal field theories have attracted a lot of attention. The original idea that dualities of this kind should appear in the free field theory limit of the usual AdS/CFT correspondence was already noted some time ago [1–4]. However, a concrete proposal was only made by Klebanov & Polyakov [5] (and generalised shortly afterwards by Sezgin & Sundell [6]) who suggested that the large N limit of the $O(N)$ vector model in 3 dimensions is dual to Vasiliev’s (parity preserving) higher spin gravity on AdS_4 [7, 8]. More recently, highly non-trivial evidence in favour of this proposal has been obtained by comparing correlation functions of the two theories [9, 10]. By now, the structure of these correlation functions has been understood conceptually [11, 12], and various further generalisations (in particular to parity violating theories) have been proposed and studied [13–17].

In a somewhat different development, a lower dimensional version of this duality was proposed in [18] and further refined in [19]. It relates a one-parameter family of higher spin theories on AdS_3 to a 't Hooft like large N limit of 2-dimensional $\mathcal{W}_{N,k}$ minimal model CFTs.¹ This proposal was inspired by the asymptotic symmetry analysis (à la Brown-Henneaux [21]) of the higher spin theories [22, 23] (see [24, 25] for subsequent developments), and has, by now, been tested in a variety of ways [26–32]. There have also been interesting results concerning the construction of black holes for these higher spin theories, as well as their dual CFT interpretation [33–36].

The proposal of [18] was generalised to the case where instead of the $\mathfrak{su}(N)$ based \mathcal{W} -algebras, one considers the $\mathfrak{so}(2N)$ series [37, 38]. More recently, a $\mathcal{N} = 2$ supersymmetric generalisation has been proposed [39], relating a family of Kazama-Suzuki models [40, 41] to the supersymmetric higher spin theory of [42, 43], and various aspects of it have been confirmed [44–47].

As was already alluded to above, for the formulation of the original duality [18] the determination of the asymptotic symmetry algebra of the higher spin theory [22–25] was crucial since it defines, by the usual AdS/CFT correspondence, also the symmetry of the dual CFT. Originally, this analysis was done classically, i.e. the algebra was determined as a commutative Poisson algebra. However, since the resulting algebra is non-linear, i.e. a \mathcal{W} -algebra, the naive quantisation does not lead to a consistent Lie algebra since normal-ordering contributions from commutators of non-linear terms spoil the Jacobi identities. Recently, it was understood [19] how to overcome this limitation for the original bosonic case. To this end the most general \mathcal{W}_∞ algebra with the field content predicted by the asymptotic symmetry analysis was studied. (For the bosonic case, the algebra is generated by one primary field for each spin $s = 2, 3, 4, \dots$) It was found that the Jacobi identities fix the structure of this algebra up to two free parameters (see also [48]): the central charge c , as well as the coupling constant γ of the spin $s = 4$ field in the OPE of two spin $s = 3$ fields. This is what one would have expected for the quantisation of the classical asymptotic symmetry algebra since the latter has also two free parameters: the size of AdS in Planck units — this is directly related to the central charge by the familiar Brown-Henneaux relation [21] $c = \frac{3\ell}{2G}$ — and the parameter characterising the Lie algebra $\mathfrak{hs}[\mu]$ whose Chern-Simons theory defines the higher spin theory. In order to determine the exact relation between γ and μ , the representation theory of the two algebras was compared (see also [49] for earlier work), using in particular that for $\mu = N$ integer, the higher spin algebra truncates to $\mathfrak{sl}(N) \cong \mathfrak{hs}[N]/\chi_N$ for which the quantum representation theory is known; this then led to an explicit dictionary between γ and μ , see eq. (2.15) of [19].

As it turned out, this relation is not one-to-one, meaning that there are different values of μ — generically there are three distinct values — that lead to the *same* γ , and hence to the *same* \mathcal{W}_∞ algebra; thus from the point of view of $\mathcal{W}_\infty[\mu]$ there is a ‘trinality’ of identifications. One of these identifications then implies the equivalence between the quantum symmetry of the bulk higher spin theory, and the chiral algebra of the minimal

¹This is the natural generalisation of the vector models in 3 dimensions since, for vanishing 't Hooft coupling, the theory is indeed equivalent to the singlet sector of a free theory [20].

model CFTs, and thus proves an important aspect of the conjectured duality of [18]. In fact, the equivalence even holds for finite N and k , and hence makes a prediction for the ‘quantum corrections’ of the higher spin theory that appear at finite c .

In this paper we analyse the quantum \mathcal{W} algebra of the $\mathcal{N} = 2$ supersymmetric higher spin theory that was conjectured to be dual to the ’t Hooft limit of the Kazama-Suzuki models in [39]. In this case, the higher spin algebra whose Chern-Simons action defines the higher spin theory is $\text{shs}[\mu]$, which truncates, for $\mu = -N$, to the superalgebra $\mathfrak{sl}(N+1|N)$. The classical asymptotic symmetry algebra was already partially determined in [45, 46], and first steps towards analysing the quantum algebra were taken in [47], see also [50] for earlier work. Here we follow the same strategy as in [19]: we first study the most general $\mathcal{N} = 2$ supersymmetric algebra \mathscr{W}_∞ whose field content agrees with that predicted by the asymptotic symmetry analysis of the higher spin theory (see section 2). Due to the complexity of the algebra, we can only study the first few commutators, but they already suggest that the algebra \mathscr{W}_∞ is also characterised by two parameters: the central charge c , and the coupling constant γ of the Virasoro primary spin $s = 2$ field W^2 in the OPE of W^2 with itself.

In order to find the relation between γ and μ we then use (in section 3) that the Drinfel’d-Sokolov reduction of $\mathfrak{sl}(N+1|N)$ is equivalent to a Kazama-Suzuki model [51]. The representation theory of the Kazama-Suzuki models follows directly from their coset description, and thus again by comparing representations, we can identify the exact relation between γ and μ , see eq. (3.30) below. We also check (see section 3.4) that the wedge subalgebra of $\mathscr{W}_\infty[\mu]$ agrees indeed with $\text{shs}[\mu]$, as has to be the case if the former is the Drinfel’d-Sokolov reduction of the latter [52].

As in the bosonic case of [19], the relation is not one-to-one, and there are now generically four different values of μ that define the same algebra (see section 4). Among other things this leads to the familiar level-rank duality of Kazama-Suzuki models that was already observed in [53]. More importantly in our context, the identifications also imply that the quantum algebra $\mathscr{W}_\infty[\lambda]$ of the higher spin gravity theory is equivalent to the chiral algebra of the Kazama-Suzuki models. As before, this relation does not just hold in the ’t Hooft limit, but even for finite N and k ; this establishes therefore an important aspect of the duality of [39].

The structure of the quantum $\mathscr{W}_\infty[\mu]$ algebra also implies how the various representations behave as a function of c ; for the case of the two minimal representations that are dual to the scalar fields of [39] this is studied in section 4.1. Following the same logic as in [19] this analysis suggests that in the present case both ‘scalar’ fields should be thought of as describing non-perturbative solutions in the semiclassical limit. We also suggest a possible explanation of this somewhat surprising conclusion.

Finally, there are three appendices where we have collected some of the more technical material: appendix A contains the commutation relations of the modes of the low lying fields, while in appendix B we give explicit expressions for the first few composite fields. Finally, appendix C describes the structure constants of the superalgebra $\text{shs}[\mu]$, as well as the relation between its generators and those of the wedge subalgebra of $\mathscr{W}_\infty[\mu]$.

2 The structure of the $\mathcal{N} = 2$ \mathcal{W} -algebras

Let us begin by studying the structure of the superconformal \mathcal{W} -algebras that are relevant for the $\mathcal{N} = 2$ version of minimal model holography [18] proposed in [39]. These algebras, which we shall denote by \mathscr{W}_∞ in the following, are generated, in addition to the $\mathcal{N} = 2$ superconformal algebra, by a single $\mathcal{N} = 2$ primary field for every integer spin $s \geq 2$. The analysis of [39, 44–47] suggests that, for each value of the central charge c , there is a one-parameter family of such algebras that are labelled by the 't Hooft parameter λ of the Kazama-Suzuki models. Here we want to show that, with the above field content, the Jacobi identities fix the algebra precisely up to two free parameters that we can identify with the central charge c , and the self-coupling γ of the spin-2 $\mathcal{N} = 2$ -primary field. In the next section we shall then explain how γ can be expressed in terms of λ and c .

Recall that each $\mathcal{N} = 2$ multiplet contains 4 Virasoro primary fields. Indeed, if we denote by W^s the $\mathcal{N} = 2$ primary field of spin s and U(1)-charge zero, then the four fields are simply

$$W^{s0} = W^s, \quad W^{s\pm} = G_{-\frac{1}{2}}^\pm W^s, \quad W^{s1} = \frac{1}{4}(G_{-\frac{1}{2}}^+ G_{-\frac{1}{2}}^- - G_{-\frac{1}{2}}^- G_{-\frac{1}{2}}^+) W^s. \quad (2.1)$$

Here we have used the usual conventions for the $\mathcal{N} = 2$ superconformal algebra which we review for the convenience of the reader in appendix A. The fields $W^{s\pm}$ have spin $s + \frac{1}{2}$ and U(1)-charge ± 1 , while the field W^{s1} has spin $s + 1$ and U(1)-charge zero. The fact that they lie in an $\mathcal{N} = 2$ multiplet means that the various components satisfy the OPEs

$$\begin{aligned} G^\pm(z)W^{s0}(w) &\sim \mp \frac{W^{s\pm}(w)}{z-w}, & G^\pm(z)W^{s\pm}(w) &\sim 0, & (2.2) \\ G^\pm(z)W^{s\mp}(w) &\sim \pm \left[\frac{2sW^{s0}(w)}{(z-w)^2} + \frac{\partial W^{s0}(w)}{z-w} \right] + \frac{2W^{s1}(w)}{z-w}, \\ G^\pm(z)W^{s1}(w) &\sim \frac{1}{2} \left[\frac{(2s+1)W^{s\pm}(w)}{(z-w)^2} + \frac{\partial W^{s\pm}(w)}{z-w} \right], \\ J(z)W^{s1}(w) &\sim s \frac{W^{s0}(w)}{(z-w)^2}. \end{aligned}$$

Using the usual expansion of fields in terms of modes

$$W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-h}, \quad (2.3)$$

where h is the conformal dimension of W , the OPEs (2.2) can also be converted into commutation relations for the corresponding modes; the resulting formulae are given in (A.3).

We shall collectively denote the fields in the $\mathcal{N} = 2$ multiplet (2.1) by $W^{(s)}$; in order to have a coherent notation, we shall also denote the generating fields J , G^\pm , and T of the $\mathcal{N} = 2$ Virasoro algebra by $W^{(1)}$.

2.1 The strategy

Next we want to study the OPEs of the fields $W^{s\alpha}$, $\alpha = 0, \pm, 1$ with one another. These OPEs are constrained by the requirement that they must be associative; translated into

modes this is believed to be equivalent to the condition that the corresponding commutators satisfy the Jacobi identity.

We shall proceed in two steps. First we present the most general ansatz for the singular part of the OPEs $W^{s_1 \alpha_1}(z)W^{s_2 \alpha_2}(w)$ with $s_1, s_2 \geq 2$ that is compatible with the full $\mathcal{N} = 2$ superconformal symmetry and with the assumed spectrum of \mathscr{W}_∞ . This step is actually the technical core of our calculation: we have worked in terms of Virasoro primaries, using the *Mathematica* packages `OPEdefs` and `OPEconf` of Thielemans² [54, 55]. The compatibility with the $\mathcal{N} = 2$ superconformal symmetry can then be implemented by requiring the associativity of the OPE with the $\mathcal{N} = 2$ superconformal generators. This fixes the coefficients of the various Virasoro primaries relative to one another.

In a second step we then require that these different OPEs are associative, i.e. that they satisfy

$$\left(\left(W^{s_1 \alpha_1}(x)W^{s_2 \alpha_2}(y) \right) W^{s_3 \alpha_3}(z) \right) = \left(W^{s_1 \alpha_1}(x) \left(W^{s_2 \alpha_2}(y)W^{s_3 \alpha_3}(z) \right) \right). \quad (2.4)$$

It is believed that requiring (2.4) is equivalent to demanding the Jacobi identities

$$\left[W_{m_1}^{s_1 \alpha_1}, \left[W_{m_2}^{s_2 \alpha_2}, W_{m_3}^{s_3 \alpha_3} \right] \right] + \text{cycl.} = 0 \quad \text{for all } m_1, m_2, m_3. \quad (2.5)$$

Note that in order to study (2.4), one has to work with the *full* OPEs, rather than just their singular part. The precise way in which this calculation can be done is explained in detail in the thesis of Thielemans [55]. His *Mathematica* package `OPEdefs` allows to compute these associativity constraints very efficiently by using a built-in function called `OPEJacobi`.

We shall present our answers mostly in a rather compact form, namely by grouping together the fields that appear in the same $\mathcal{N} = 2$ superconformal representation. However, as mentioned before, we have actually carried out the calculations by working in terms of Virasoro primaries and then realising the $\mathcal{N} = 2$ superconformal symmetry by solving the $W^{(1)} \times W^{(s_1)} \times W^{(s_2)}$ associativity constraints.

For the case at hand, there are infinitely many $\mathcal{N} = 2$ superconformal primary fields, and therefore infinitely many associativity constraints to check. The full problem is therefore too hard to be solved completely. However, we have studied the low-lying OPEs in detail, and they already suggest that there is indeed exactly one free parameter beyond the central charge, that characterises these \mathscr{W}_∞ algebras.

2.2 Enumerating $\mathcal{N} = 2$ primary fields

Before we can make the most general ansatz for the various OPEs, we first need to understand how many $\mathcal{N} = 2$ primary fields \mathscr{W}_∞ contains. (In particular, we need to determine how many composite $\mathcal{N} = 2$ primary fields there are.) This information can be easily read off from the vacuum character³ of \mathscr{W}_∞

$$\chi_\infty(q, z) = \text{Tr}_0(q^{L_0} z^{J_0}) = \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \frac{(1 + z q^{n+\frac{1}{2}})(1 + z^{-1} q^{n+\frac{1}{2}})}{(1 - q^n)(1 - q^{n+1})}. \quad (2.6)$$

²The latest versions of `OPEdefs` and `OPEconf` are available directly from the author.

³In the definition of the various characters we drop for convenience the overall $-\frac{c}{24}$ exponent.

We want to decompose $\chi_\infty(q, z)$ in terms of characters of irreducible $\mathcal{N} = 2$ Virasoro representations. The character of the $\mathcal{N} = 2$ vacuum representation equals

$$\chi_0(q, z) = \prod_{n=1}^{\infty} \frac{(1 + z q^{n+\frac{1}{2}})(1 + z^{-1} q^{n+\frac{1}{2}})}{(1 - q^n)(1 - q^{n+1})}, \quad (2.7)$$

while for a generic irreducible $\mathcal{N} = 2$ representation with highest weight (h, Q) with respect to (L_0, J_0) we have instead

$$\begin{aligned} \chi_{(h,Q)}(q, z) &= q^h z^Q \prod_{n=1}^{\infty} \frac{(1 + z q^{n-\frac{1}{2}})(1 + z^{-1} q^{n-\frac{1}{2}})}{(1 - q^n)^2} \\ &= q^h z^Q \frac{(1 + z q^{\frac{1}{2}})(1 + z^{-1} q^{\frac{1}{2}})}{(1 - q)} \chi_0(q, z). \end{aligned} \quad (2.8)$$

The multiplicity $d(h, Q)$ of an $\mathcal{N} = 2$ primary field with quantum numbers (h, Q) in \mathscr{W}_∞ is then simply determined by the decomposition

$$\chi_\infty(q, z) = \chi_0(q, z) + \sum_{h \in \frac{1}{2}\mathbb{N}} \sum_{Q \in \mathbb{Z}} d(h, Q) \chi_{(h,Q)}(q, z). \quad (2.9)$$

Writing $\chi_\infty(q, z) = \chi_0(q, z) \cdot \chi_{\text{HS}}(q, z)$ with

$$\chi_{\text{HS}}(q, z) = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{(1 + z q^{n+\frac{1}{2}})(1 + z^{-1} q^{n+\frac{1}{2}})}{(1 - q^n)(1 - q^{n+1})}, \quad (2.10)$$

and dividing the whole expression by $\chi_0(q, z)$, the generating function for $d(h, Q)$ turns out to equal

$$P(q, z) \equiv \sum_{h \in \frac{1}{2}\mathbb{N}} \sum_{Q \in \mathbb{Z}} d(h, Q) q^h z^Q = \frac{(1 - q)(\chi_{\text{HS}}(q, z) - 1)}{(1 + z q^{\frac{1}{2}})(1 + z^{-1} q^{\frac{1}{2}})}. \quad (2.11)$$

The first few terms are explicitly

$$P(q, z) = q^2 + q^3 + 2q^4 + 3q^5 + (2z + 2z^{-1})q^{\frac{11}{2}} + 7q^6 + \dots \quad (2.12)$$

Since in \mathscr{W}_∞ there is one simple $\mathcal{N} = 2$ primary field for every spin $s \geq 2$, we read off from (2.12) that the first composite $\mathcal{N} = 2$ primary field appears at spin 4 and U(1) charge zero; this field is essentially the normal ordered product of $W^{(2)}$ with itself. The higher terms can be similarly interpreted.

2.3 Constraining the OPE

Let us illustrate our method with the example of the OPEs of the $\mathcal{N} = 2$ supermultiplet $W^{(2)}$ (whose $\mathcal{N} = 2$ primary is the spin 2 field $W^{(2)}$). In terms of $\mathcal{N} = 2$ multiplets, the singular part of the OPE has the general form

$$W^{(2)} \times W^{(2)} \sim n_2 I + c_{22,2} W^{(2)} + c_{22,3} W^{(3)}, \quad (2.13)$$

where on the right-hand-side also the corresponding $\mathcal{N} = 2$ superconformal descendants are included (if they contribute to the singular part of the OPE).

In order to see that this is the most general ansatz recall that in the singular part of the OPE of two Virasoro primary fields of conformal dimension h_1 and h_2 , only Virasoro primary fields with $h \leq h_1 + h_2 - 1$ can appear. However, to apply this general rule to our current context, we need to remember that each $\mathcal{N} = 2$ multiplet actually contains 4 Virasoro primaries, see eq. (2.1). Thus each $\mathcal{N} = 2$ OPE gives actually rise to 16 OPEs of Virasoro primaries; the condition that an $\mathcal{N} = 2$ multiplet appears in the OPE then requires that all its 4 Virasoro primaries of eq. (2.1) appear among the 16 Virasoro primary OPEs.⁴ Obviously, the $\mathcal{N} = 2$ superconformal symmetry relates the structure constants of some of these Virasoro primaries to one another, but the explicit expressions are somewhat complicated, see [58].

Given the structure of (2.12) it follows that the singular part of the OPE can, apart from the identity I of spin zero, at most contain the $\mathcal{N} = 2$ multiplets $W^{(s)}$ of spin $s = 2, 3, 4$. The $\mathcal{N} = 2$ multiplet of spin $s = 4$, however, cannot actually appear, since it contains the Virasoro primary W^{41} of spin $s = 5$. However, W^{41} can only appear in the OPE $W^{21} \times W^{21}$, and then the conformal symmetry requires that the coefficient of W_{m+n}^{41} in the commutator $[W_m^{21}, W_n^{21}]$ is independent of m and n . Since the commutator must be anti-symmetric in $m \leftrightarrow n$, the overall coefficient must therefore vanish. Thus we arrive at (2.13).

We also need to be specific about what we precisely mean by the various structure constants, given that the $\mathcal{N} = 2$ primary does not necessarily appear in the OPE of the two $\mathcal{N} = 2$ primaries. We define $c_{22,2}$ by the OPE

$$\begin{aligned}
 W^{20}(z)W^{20}(w) \sim & \frac{n_2}{(z-w)^4} + \frac{n_2}{c(c-1)} \frac{(4cT - 6J^2)(w)}{(z-w)^2} + c_{22,2} \frac{W^{20}(w)}{(z-w)^2} \\
 & + \frac{n_2}{c(c-1)} \frac{(2c\partial T - 6\partial JJ)(w)}{(z-w)} + \frac{c_{22,2}}{2} \frac{\partial W^{02}(w)}{z-w},
 \end{aligned}
 \tag{2.14}$$

where we have for once given *all* the singular terms. (In the following we shall not be so explicit any more, see however appendix A for the commutators of $W^{2\alpha}$.) The structure constant $c_{22,3}$ can be similarly defined by a coefficient in the OPE

$$W^{20}(z)W^{2-}(w) \sim \frac{c_{22,2}}{2} \frac{W^{2-}(w)}{(z-w)^2} + c_{22,3} \frac{W^{3-}(w)}{z-w} + \dots
 \tag{2.15}$$

So far we have only used the constraints that come from the $\mathcal{N} = 2$ superconformal symmetry, i.e. the conditions that follow from the Jacobi identities (2.5) associated to $s_1 = 1$ and $s_2 = s_3 = 2$. The next step is to study (2.5) (or the associativity of (2.4)) for $s_1 = s_2 = s_3 = 2$. However, since the OPE of $W^{(2)}$ with itself generates $W^{(3)}$, we also need to make an ansatz for $W^{(2)} \times W^{(3)}$. Using the same arguments as above, one finds that the most general ansatz for the singular part of that OPE is (again the $\mathcal{N} = 2$ superconformal

⁴Another way of saying this is that the $\mathcal{N} = 2$ primary of the right-hand-side does not necessarily have to appear in the OPE of the two $\mathcal{N} = 2$ primaries on the left-hand-side. Indeed, this is the origin of the so-called odd fusion rules of [56], see also [57].

descendants are included where they contribute to the singular part)

$$W^{(2)} \times W^{(3)} \sim c_{23,2} W^{(2)} + c_{23,3} W^{(3)} + c_{23,4} W^{(4)} + a_{23,4} A^{(4)} + c_{23,5} W^{(5)} + a_{23,5} A^{(5)}, \quad (2.16)$$

where A^4 and A^5 are the uncharged composite $\mathcal{N} = 2$ primaries of spin 4 and 5, respectively. Note that it would seem from (2.12) that three fields of spin 5 should generically appear on the right-hand-side of (2.16). However, one of them is just the normal ordered product of W^2 with W^3 (appropriately completed to make it $\mathcal{N} = 2$ primary), which does not contribute to the singular part of the OPE. The composite fields A^4 and A^5 have the leading terms

$$A^4 = (W^{20})^2 + \dots, \\ A^5 = W^{20}W^{21} + \frac{(5c-24)}{8c} W^{2+}W^{2-} - \frac{6}{c} J(W^{20})^2 + \dots$$

that have to be completed to make them $\mathcal{N} = 2$ primaries; explicit formulae for them are given in appendix B. The structure constants in the ansatz (2.16) are again defined by OPE coefficients as

$$W^{2+}(z)W^{3-}(w) \sim c_{23,2} \frac{20W^{20}(w)}{(z-w)^4} + c_{23,3} \frac{2W^{30}(w)}{(z-w)^3} - c_{23,4} \frac{8W^{40}(w)}{(z-w)^2} - a_{23,4} \frac{8A^{40}(w)}{(z-w)^2} + c_{23,5} \frac{W^{50}(w)}{z-w} + a_{23,5} \frac{A^{50}(w)}{z-w} + \dots \quad (2.17)$$

It follows from the permutation symmetry of the 3-point functions together with the $\mathcal{N} = 2$ superconformal symmetry, see the appendix of [58], that we have the relation

$$10 c_{23,2} n_2 = -3 c_{22,3} n_3, \quad (2.18)$$

where n_s are the normalisation constants

$$\langle W^{s0}(z)W^{s0}(w) \rangle = \frac{n_s}{(z-w)^{2s}}. \quad (2.19)$$

This fixes the normalisation of the other fields in the multiplet as well.

2.4 Structure constants

Now we have everything in place to study the associativity of the OPE (2.4) for $s_1 = s_2 = s_3 = 2$. The calculation is somewhat tedious, but the end result is simple: the structure constants appearing in (2.13) and (2.16) must satisfy

$$c_{22,3} c_{23,2} = -\frac{6(c+3)(5c-12)}{5c(c+6)(2c-3)} n_2 - \frac{3(c-15)(c-1)}{10(c+3)(5c-12)} (c_{22,2})^2 \quad (2.20)$$

$$c_{23,3} = \frac{3(c+6)(2c-3)}{(c+3)(5c-12)} c_{22,2} \quad (2.21)$$

$$c_{23,5} = a_{23,5} = 0. \quad (2.22)$$

Before proceeding we should note that there is one non-trivial consistency check we can easily perform: if we require all $W^{(s)}$ multiplets with $s \geq 3$ to vanish, our algebra should reduce to that constructed explicitly by Romans in [59]. In order to be able to decouple all of these fields, we need in particular that $c_{22,3} = 0$ in eq. (2.13). Eq. (2.20) then fixes $c_{22,2}$ in terms of n_2 , and the solution coincides exactly with the one obtained in [59] if we normalise the W^{20} field as he does, namely with $n_2 = \frac{c}{2}$.

Next we observe that eqs. (2.18) and (2.20) – (2.22) fix all the structure constants in the OPEs (2.13) and (2.16), except for $c_{22,2}$, $c_{23,4}$ and $a_{23,4}$. The fact that the latter two structure constants are unconstrained at this stage is not surprising: it simply reflects that we can rescale W^4 and A^4 arbitrarily. (Furthermore, there is the freedom of redefining W^4 by adding to it a multiple of A^4 — as a consequence, these two structure constants must always appear together in Jacobi identities [60].) Thus there is only one free parameter at this stage, namely the structure constant $c_{22,2}$. Note that our normalisation convention (2.19) only fixes W^2 up to a sign, leading to a sign ambiguity in the definition of $c_{22,2}$.

Thus the structure seems to be rather similar to that of the non-supersymmetric \mathcal{W} -algebra $\mathcal{W}_\infty[\mu]$, that is determined (for each central charge c) by a single structure constant [19], see also [48]. Reasoning by analogy with [19] we therefore conjecture that for every value of the central charge c , there exists a one-parameter family of non-isomorphic \mathcal{W}_∞ algebras which are parametrised by

$$\gamma = (c_{22,2})^2. \tag{2.23}$$

In the next section we want to relate these algebras to those that appear in the supersymmetric minimal model holography of [39].

3 Minimal representation

In the application to minimal model holography [18, 39], the above \mathcal{W}_∞ algebras should arise as the Drinfel’d-Sokolov (DS) reduction of the infinite dimensional Lie algebra $\text{shs}[\mu]$, which can be constructed as

$$\text{shs}[\mu] \oplus \mathbb{C} = \frac{U(\mathfrak{osp}(1|2))}{\langle C^{\text{osp}} - \frac{1}{4}\mu(\mu - 1)\mathbf{1} \rangle}. \tag{3.1}$$

Here we have normalised the Casimir operator C^{osp} so that for the $\mathfrak{osp}(1|2)$ representation of dimension $4j + 1$ it takes the value $C^{\text{osp}} = j(j + \frac{1}{2})$. We can think of $\text{shs}[\mu]$ as

$$\text{shs}[\mu] \cong \mathfrak{sl}(1 - \mu | -\mu) \tag{3.2}$$

since, for $\mu = -N$ with $N \in \mathbb{N}$, we have⁵

$$\text{shs}[-N]/\chi_N \cong \mathfrak{sl}(N + 1 | N). \tag{3.3}$$

Here χ_N is the maximal (infinite-dimensional) ideal that appears for these values of μ . Let us denote the DS-reduction of $\text{shs}[\mu]$ by $\mathcal{W}_\infty[\mu]$,

$$\mathcal{W}_\infty[\mu] \equiv \text{Drinfel’d-Sokolov reduction of shs}[\mu]. \tag{3.4}$$

⁵Note that by definition $\text{shs}[-\mu] \cong \text{shs}[1 + \mu]$.

We want to understand how to relate γ in (2.23) to μ .

It was shown by Ito in [51] that the Drinfel'd-Sokolov reduction of $\mathfrak{sl}(N+1|N)$ is equivalent to the Kazama-Suzuki coset

$$\frac{\mathrm{SU}(N+1)_k \times \mathrm{SO}(2N)_1}{\mathrm{SU}(N)_{k+1} \times \mathrm{U}(1)_\kappa}, \quad (3.5)$$

where $\kappa = N(N+1)(N+k+1)$, see [44] for our notation. Thus it follows that (3.5) is equivalent to (a quotient of) $\mathscr{W}_\infty[-N]$. For future use we also recall that the central charge of (3.5) equals

$$c_{N,k} = \frac{3kN}{k+N+1}. \quad (3.6)$$

In order to relate γ to μ we can now use the same idea as in [19, 49]. From their coset description, it is clear that the Kazama-Suzuki models possess *minimal* representations whose Virasoro character agrees to low orders in q with

$$\chi_{\min}(q) = \frac{q^h(1+q^{\frac{1}{2}})}{(1-q)} \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \frac{(1+q^{n+\frac{1}{2}})^2}{(1-q^n)(1-q^{n+1})}. \quad (3.7)$$

Indeed, in the notation of [44] where the coset representations are labelled by $(\Lambda; \Xi, l)$, this is the case for the 4 representations

$$(\mathfrak{f}; 0, N), \quad (\bar{\mathfrak{f}}; 0, -N), \quad (0; \mathfrak{f}, -(N+1)), \quad (0; \bar{\mathfrak{f}}, (N+1)), \quad (3.8)$$

where \mathfrak{f} and $\bar{\mathfrak{f}}$ is the fundamental and anti-fundamental representation of $\mathrm{SU}(N)$ or $\mathrm{SU}(N+1)$, respectively. Their conformal dimensions equal

$$\begin{aligned} h((\mathfrak{f}; 0, N)) &= h((\bar{\mathfrak{f}}; 0, -N)) = \frac{N}{2(N+k+1)} \\ h((0; \mathfrak{f}, -(N+1))) &= h((0; \bar{\mathfrak{f}}, N+1)) = \frac{k}{2(N+k+1)}, \end{aligned} \quad (3.9)$$

and all of them are ‘chiral primaries’, i.e. have $Q = \pm 2h$. Indeed, this is immediate from their character formula (3.7), which implies that each of these representations has a null-vector of the form $G_{-1/2}^\pm |h, Q\rangle$. Furthermore, since the character has only a single state at conformal weight $h + \frac{1}{2}$, all other $(-1/2)$ -descendants have to be proportional to $G_{-1/2}^\mp P^0$, i.e. the representation generated from P^0 has to have very many null-vectors.

These null-vectors are only compatible with the commutation relations of \mathscr{W}_∞ (that depend on γ) if h solves an equation in terms of γ and c . We can then compare this to the solutions (3.9) that arise for $\mu = -N$ and $c = c_{N,k}$, and this will allow us to determine the N (and c) dependence of γ ; analytically continuing in N will then finally lead to the desired relation between μ and γ .

3.1 Ansatz for OPEs

Actually, it will be more convenient to work out the equation for h in terms of γ not directly using the commutation relations (as was done in the bosonic case in [19]), but rather by

exploiting the associativity of the OPE $W^{(2)} \times W^{(2)} \times P$. (Incidentally, this is also the approach that was taken in the original analysis of [49].) In order to do so, we first need to translate the above statements about the structure of the P -representation, into the OPE language. For definiteness, let us consider the case $Q = +2h$, so that $G_{-1/2}^+ P^0 = 0$, where $P^0 = |h, Q\rangle$ is the $\mathcal{N} = 2$ primary. (The case $Q = -2h$ works analogously.) Let us also denote the non-vanishing $G_{-1/2}^-$ descendant of conformal weight $h + \frac{1}{2}$ by P^- . The above statements about the null-vectors then imply that we have the OPEs

$$G^-(z)P^0(w) \sim \frac{P^-}{z-w}, \quad G^+(z)P^0(w) \sim 0, \quad (3.10)$$

$$G^+(z)P^-(w) \sim \frac{4hP^0}{(z-w)^2} + \frac{2\partial P^0}{z-w}, \quad G^-(z)P^-(w) \sim 0. \quad (3.11)$$

In order to work out the most general ansatz for the OPEs of the higher spin fields $W^{(s)}$ with P , we need to understand again the decomposition of the minimal representation in terms of irreducible $\mathcal{N} = 2$ representations. As before in section 2.2, this can be read off from the character (3.7). Including the $U(1)$ -chemical potential z , the character of the minimal representation (with $Q = 2h$) has the form

$$\begin{aligned} \chi_{\min}(q, z) &= q^h z^{2h} \frac{(1 + z^{-1}q^{\frac{1}{2}})}{(1-q)} \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} \frac{(1 + zq^{n+\frac{1}{2}})(1 + z^{-1}q^{n+\frac{1}{2}})}{(1-q^n)(1-q^{n+1})} \\ &= q^h z^{2h} \chi_0 \left[\frac{(1 + z^{-1}q^{\frac{1}{2}})}{(1-q)} + \sum_{s \in \frac{1}{2}\mathbb{N}} \sum_{Q \in \mathbb{Z}} d_{\min}(s, Q) q^s z^Q \frac{(1 + zq^{\frac{1}{2}})(1 + z^{-1}q^{\frac{1}{2}})}{(1-q)} \right], \end{aligned} \quad (3.12)$$

where χ_0 was defined in (2.7), and $d_{\min}(s, Q)$ is the multiplicity of the $\mathcal{N} = 2$ representation with conformal dimension $h' = h + s$ and $U(1)$ -eigenvalue $Q' = 2h + Q$. Their generating function is now

$$P_{\min}(q, z) = \sum_{s \in \frac{1}{2}\mathbb{N}} \sum_{Q \in \mathbb{Z}} d_{\min}(s, Q) q^s z^Q = \frac{\chi_{\text{HS}}(q, z) - 1}{1 + zq^{\frac{1}{2}}}, \quad (3.13)$$

where χ_{HS} was defined in (2.10). The first few terms are explicitly

$$P_{\min}(q, z) = q^2 + z^{-1}q^{\frac{5}{2}} + 2q^3 + 2z^{-1}q^{\frac{7}{2}} + \dots \quad (3.14)$$

With these preparations, we can now make the most general ansatz for the singular part of the OPE of P with the higher spin fields

$$W^{(2)} \times P \sim w_2 P, \quad (3.15)$$

$$W^{(3)} \times P \sim w_3 P + a P^{(2)} + b P^{(\frac{5}{2})}, \quad (3.16)$$

where $P^{(2)}$ and $P^{(\frac{5}{2})}$ are the composite $\mathcal{N} = 2$ (non-chiral) primary fields corresponding to the first two terms in eq. (3.14); their corresponding $\mathcal{N} = 2$ primary states are of the form

$$P^{20} = W_{-2}^{20} P^0 + \dots, \quad P^{\frac{5}{2}0} = W_{-2}^{20} P^- - h W_{-\frac{5}{2}}^{20} P^0 + \dots, \quad (3.17)$$

completed to make them $\mathcal{N} = 2$ primary, see appendix B for the full expressions. The parameters appearing in (3.15) and (3.16) are defined in terms of OPE coefficients of the component fields as

$$W^{20}(z) P^0(w) \sim w_2 \left[\frac{P^0(w)}{(z-w)^2} + \frac{1}{h(c-6h)} \frac{c\partial P^0(w) - 6h(JP^0)(w)}{z-w} \right] \quad (3.18)$$

$$W^{30}(z) P^0(w) \sim w_3 \frac{P^0(w)}{(z-w)^3} + a \frac{P^{20}(w)}{z-w} + \dots \quad (3.19)$$

$$W^{30}(z) P^-(w) \sim b \frac{P^{\frac{5}{2}0}(w)}{(z-w)} + \dots \quad (3.20)$$

In particular, w_2 and w_3 are therefore the eigenvalue of P^0 with respect to W_0^{20} and W_0^{30} , respectively.

3.2 Structure constants

Now we are in the position to study the associativity of the OPEs $W^{(2)} \times W^{(2)} \times P$. After a tedious but straightforward calculation it leads to the constraints on the parameters appearing in eqs. (3.15) and (3.16)

$$w_2^2 = \frac{h^2(1+2h)(c-6h)(c-3+12h)n_2}{c(c-1)[c(1-h)+3h]} \quad (3.21)$$

$$w_3 = -\frac{12h(1+h)(c-1)(c+3)(c-12h)(c-6+18h)n_2}{c(c+6)(2c-3)(5c-12)[c(1-h)+3h]c_{22,3}} \quad (3.22)$$

$$a = \frac{72(1+h)(c-1)(c-6+18h)w_2}{(1+2h)(5c-12)(c-6h)(c-3+12h)c_{22,3}} \quad (3.23)$$

$$b = -\frac{54(2h-1)(c-1)(c-12h)w_2}{h(1+2h)(5c-12)(c-6h)(c-3+12h)c_{22,3}} \quad (3.24)$$

Furthermore, the conformal dimension h of the minimal representation must be related to the parameter $c_{22,2}$ of the \mathscr{W}_∞ algebra by

$$c_{22,2} = -\frac{2(c+3)[c(1-4h)-12h^2]w_2}{h(1+2h)(c-6h)(c-3+12h)} \quad (3.25)$$

Note that it follows that the structure constants of eqs. (3.15) and (3.16) are uniquely determined (up to the sign of w_2) by the associativity of the OPEs $W^{(2)} \times W^{(2)} \times P$. In terms of the parameter (2.23) and using eq. (3.21) for w_2^2 , the relation (3.25) then finally becomes

$$\gamma = \frac{4(c+3)^2[c(1-4h)-12h^2]^2 n_2}{(1+2h)c(c-1)(c-6h)(c-3+12h)[c(1-h)+3h]} \quad (3.26)$$

3.3 The desired relation

Next we plug into eq. (3.26) the expression for the central charge $c = c_{N,k}$ (3.6), and one of the conformal dimensions h in eq. (3.9); this leads to a relation for γ in terms of N and k

$$\gamma = \frac{8(1+k)^2(k-N)^2(1+N)^2(1+k+N)n_2}{(k-1)k(N-1)N(1+2k+N)(1+k+2N)(3kN-N-k-1)} \quad (3.27)$$

Note that the same formula is obtained, independent of which of the two solutions in (3.9) one considers; this is a non-trivial consistency check on our analysis. Next we want to replace k in favour of c ; unlike the bosonic case considered in [19], here the central charge (3.6) uniquely determines the level k , and we get

$$k = \frac{c(N+1)}{3N-c}. \tag{3.28}$$

Plugging this relation into eq. (3.27) we then get an expression for γ as a function of N and c

$$\gamma = -\frac{8(c+3)^2(c+2cN-3N^2)^2n_2}{c(c-1)(c-3-6N)(N-1)(c+3N)(2c-3N+cN)}. \tag{3.29}$$

Finally, we can replace N by $-\mu$ in the above equation, and analytically continue μ ; this leads to our central relation

$$\gamma(\mu, c) = \frac{8(c+3)^2(c-2c\mu-3\mu^2)^2n_2}{c(c-1)(c-3+6\mu)(\mu+1)(c-3\mu)(2c+3\mu-c\mu)} \tag{3.30}$$

establishing the connection between the γ -parameter of the \mathscr{W}_∞ algebra, and the μ -parameter in $\mathscr{W}_\infty[\mu]$, i.e. in the DS reduction of $\text{shs}[\mu]$. Note that the n_2 factor on the right-hand-side simply reflects the fact that $\gamma = (c_{22,2})^2$ depends on the normalisation of $W^{(2)}$.

We should mention that the expressions (3.27), (3.29) for γ are compatible with those obtained in [47, 50, 59] for the finitely generated algebras $\mathscr{W}_\infty(1, 2, \dots, N)$ with $N = 2, 3, 4$, and conjectured in [50] for arbitrary N .

3.4 Wedge subalgebra

As another consistency check we can analyse whether the ‘wedge subalgebra’ [52] of $\mathscr{W}_\infty[\mu]$ agrees indeed with $\text{shs}[\mu]$, as must be the case if $\mathscr{W}_\infty[\mu]$ is the Drinfel’d-Sokolov reduction of $\text{shs}[\mu]$. Recall that the wedge subalgebra of $\mathscr{W}_\infty[\mu]$ is defined by restricting the modes $W_m^{s,\alpha}$ to the wedge $|m| < s$, and taking the limit $c \rightarrow \infty$. As can be seen from eq. (3.30), the $c \rightarrow \infty$ limit of γ is zero unless we take n_2 to be proportional to the central charge; we can choose

$$n_2 = -\frac{c}{6}(\mu+1)(\mu-2), \tag{3.31}$$

so that the structure constant $c_{22,2}$ equals

$$c_{22,2} = \frac{2}{\sqrt{3}}(1-2\mu) + \mathcal{O}(c^{-1}). \tag{3.32}$$

Note that this then reproduces the result of [46]. With this normalisation convention we have checked that the other structure constants, that are determined by the OPEs (2.13) and (2.16) (with the coefficients given by (2.20) – (2.22)), agree indeed with those of $\text{shs}[\mu]$; the details are described in appendix C.

4 Dualities

As in the bosonic case [19], the actual \mathscr{W}_∞ algebra only depends on γ and c . However, since the map $\mu \mapsto \gamma(\mu, c)$ is not injective, there are in general different values of μ that lead to the same γ , and hence to the same algebra. Indeed, if we fix γ and c , then (3.30) leads to a quartic equation for μ . This means that we have a 4-fold equivalence of algebras

$$\mathscr{W}_\infty[\mu_1] \cong \mathscr{W}_\infty[\mu_2] \cong \mathscr{W}_\infty[\mu_3] \cong \mathscr{W}_\infty[\mu_4], \quad (4.1)$$

where the relation between the four parameters takes the remarkably simple form

$$\mu_1 = \mu, \quad \mu_2 = \frac{c - c\mu}{c + 3\mu}, \quad \mu_3 = \frac{c + 3\mu}{3(\mu - 1)}, \quad \mu_4 = -\frac{c}{3\mu}. \quad (4.2)$$

Note that these relations break the classical $\mu \mapsto 1 - \mu$ symmetry (that is obvious from the definition of $\text{shs}[\mu]$, see (3.1)) at finite c ; this is analogous to what happened in the bosonic analysis of [19], where the $\mu \mapsto -\mu$ symmetry was similarly broken.

It is also useful to think about these identifications in the (N, k) -parametrisation, i.e. in terms of the Kazama-Suzuki cosets

$$\frac{\text{SU}(N+1)_k \times \text{SO}(2N)_1}{\text{SU}(N)_{k+1} \times \text{U}(1)_\kappa}. \quad (4.3)$$

Then we have

$$N_1 = N, \quad k_1 = k, \quad (4.4)$$

$$N_2 = k, \quad k_2 = N, \quad (4.5)$$

$$N_3 = -\frac{N}{N+k+1}, \quad k_3 = -\frac{k}{N+k+1}, \quad (4.6)$$

$$N_4 = -\frac{k}{N+k+1}, \quad k_4 = -\frac{N}{N+k+1}. \quad (4.7)$$

Note that the equivalence between (4.4) and (4.5) (and similarly between (4.6) and (4.7)) is the familiar level-rank duality of the Kazama-Suzuki models [53]. On the other hand, the relation between (4.4) and (4.6) explains the agreement of symmetries between the 't Hooft limit of the Kazama-Suzuki models, and the higher spin theory based on $\text{shs}[\lambda]$ with

$$\lambda = \frac{N}{N+k+1}. \quad (4.8)$$

Indeed, (4.4) is the standard Kazama-Suzuki coset which is equivalent to (4.6), and hence to

$$\frac{\text{SU}(N+1)_k \times \text{SO}(2N)_1}{\text{SU}(N)_{k+1} \times \text{U}(1)_\kappa} \cong \mathscr{W}_\infty[\lambda] \quad \text{for } c = c_{N,k}, \quad (4.9)$$

where we have used the dictionary $\mu_3 = -N_3$. In the 't Hooft limit, this proves that the two dual theories have equivalent symmetries, but (4.9) is actually a stronger statement since it applies also to finite N and k .

4.1 Analytic continuation

Given the detailed understanding of the structure of the algebra, we can also ask how the conformal dimensions of the various representations behave in the ‘semiclassical’ regime, i.e. for large c . This analysis is of significance in order to determine which of the states of the CFT should correspond to perturbative or non-perturbative higher spin excitations, respectively, see [19].

First we note that a generic $\mathcal{W}_\infty[\mu]$ algebra has four minimal representations, since (3.25) has always four solutions for h . Plugging in the expression for γ in terms of μ and c , the four solutions are

$$h_1 = \frac{\mu}{2}, \quad h_2 = \frac{c(1-\mu)}{2(c+3\mu)}, \quad h_3 = -\frac{c}{6\mu}, \quad h_4 = -\frac{(c+3\mu)}{6(1-\mu)}. \quad (4.10)$$

As an aside, we can also write these formulae in terms of (N, k) , where they take the form

$$h_1 = -\frac{N}{2}, \quad h_2 = -\frac{k}{2}, \quad h_3 = \frac{k}{2(N+k+1)}, \quad h_4 = \frac{N}{2(N+k+1)}. \quad (4.11)$$

If we fix μ and consider the semi-classical limit ($c \rightarrow \infty$), two of the solutions in (4.10), namely h_1 and h_2 , are ‘perturbative’ (since they remain finite in this limit), while two solutions, namely h_3 and h_4 , are ‘non-perturbative’ — they are proportional to c and go to $-\infty$ in this limit (for $0 < \mu < 1$).

According to [19], the semiclassical limit is obtained by working with the higher spin theory based on $\mathfrak{sl}(N+1|N)$, and taking $c \rightarrow \infty$ while keeping N fixed. In order to understand what happens in this limit, we should write the conformal dimensions of the two minimal representations of the coset CFT (3.9) in terms of N and c ; this leads to

$$\begin{aligned} h(\mathfrak{f}; 0, N) &= h(\bar{\mathfrak{f}}; 0, -N) = \frac{N}{2(N+k+1)} = \frac{3N-c}{6(N+1)} \\ h(0; \mathfrak{f}, -(N+1)) &= h(0; \bar{\mathfrak{f}}, (N+1)) = \frac{k}{2(N+k+1)} = \frac{c}{6N}, \end{aligned} \quad (4.12)$$

where we have used (3.6) to express k in terms of c (and N). These two solutions agree with h_3 and h_4 from (4.10), (4.11) for $\mu = -N$, and hence are *both* non-perturbative. While this may sound somewhat surprising at first, it actually ties in nicely with the results of [19].⁶ To see this, recall that the $\mathcal{N} = 2$ multiplet based on either $(\mathfrak{f}; 0)$ or $(0; \mathfrak{f})$ actually contains a scalar field that is quantised using the alternate $(-)$ quantisation, see figure 3 of [44].⁷ From a bosonic point of view, we expect the fields with this alternate boundary condition to become non-perturbative in the semiclassical limit [19], and thus both $\mathcal{N} = 2$ multiplets must show this behaviour, in agreement with the above. It is tempting to speculate that the non-perturbative characteristic of these fields is related to the fact that, in 4 dimensions, the scalar field with the alternate boundary condition actually breaks the higher spin symmetry at finite N [62], see also [63].

Our result also suggests that the generalisation of the analysis of [30] to the supersymmetric case should lead to more classical solutions; it would be very interesting to confirm this.

⁶We thank Rajesh Gopakumar for the following suggestion.

⁷Incidentally, the same phenomenon also occurs in one dimension higher, see [61].

5 Conclusion

In this paper we have analysed the structure of the \mathscr{W}_∞ algebra that underlies the higher spin — CFT duality of [39]. This algebra is generated, in addition to the $\mathcal{N} = 2$ superconformal algebra, by exactly one $\mathcal{N} = 2$ primary field for every integer spin $s \geq 2$. In particular, we have solved the Jacobi identities arising from the first few OPEs, and we have found that the \mathscr{W}_∞ algebra is characterised by the central charge, as well as one free parameter that can be taken to describe the self-coupling γ of the spin $s = 2$ primary field.

From the point of view of the bulk AdS₃ theory, \mathscr{W}_∞ should be the quantum Drinfel'd-Sokolov reduction of the infinite superalgebra $\text{shs}[\mu]$ that appears in the Chern-Simons description of the higher spin theory. For $\mu = -N$, this DS algebra is equivalent [51] to a specific Kazama-Suzuki model [40, 41], whose representation theory is known from the coset description. By comparing representations we have determined, as in [19], the exact relation between μ and the coupling constant γ characterising \mathscr{W}_∞ , see (3.30). This identification agrees with various results that had been previously determined in the literature [50, 59], and it is compatible with the requirement that the wedge subalgebra of $\mathscr{W}_\infty[\mu]$ reduces to $\text{shs}[\mu]$.

It follows from (3.30) that there are generically 4 different values of μ , see (4.2), that correspond to the same γ , and hence define the *same* \mathscr{W}_∞ algebra. The 4-fold equivalence of the $\mathscr{W}_\infty[\mu]$ algebras (4.1) explains, among other things, the level-rank duality [53] among the Kazama-Suzuki cosets, see (4.4) vs. (4.5). More importantly, it also establishes the equivalence of the quantum asymptotic symmetry algebra $\mathscr{W}_\infty[\mu]$ of the higher spin theory on AdS₃, to the Kazama-Suzuki models in the duality of [39]. As in [19] this equivalence is actually true at finite N and k , and hence makes definitive predictions about the quantum corrections of the higher spin theory. It would be very interesting to reproduce these quantum corrections directly from the higher spin theory.

Among other things, our improved understanding of the quantum symmetry algebra $\mathscr{W}_\infty[\mu]$ also allows us to study the semi-classical (large c at fixed $\mu = N$) behaviour of the two complex scalar fields that appear in the duality of [39]. Quite surprisingly, the conformal dimension of both dual fields is proportional to the central charge, thus suggesting that neither should be thought of as a perturbative scalar. Instead one should expect that they have an interpretation in terms of ‘non-perturbative’ classical solutions of the type found in [30] for the bosonic case; it would be very interesting to check this in detail. A possible explanation for this non-perturbative behaviour of the ‘scalar fields’ is that all $\mathcal{N} = 2$ matter multiplets involve a scalar that is quantised in the alternate (–) manner; such scalars turned out to be non-perturbative in the bosonic analysis of [19]. It would be very interesting to understand this issue better, in particular, if there is a relation to the fact that in the AdS₄/CFT₃ duality the scalar field with the (–) boundary condition breaks the higher spin symmetry at finite N .

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A Commutation relations

The $\mathcal{N} = 2$ Virasoro algebra is generated by the energy momentum tensor $T(z)$, a U(1) current $J(z)$ and two fermionic currents $G^\pm(z)$. Their OPEs take the familiar form

$$\begin{aligned}
 T(z)T(w) &\sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \\
 T(z)J(w) &\sim \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \quad T(z)G^\pm(w) \sim \frac{3G^\pm(w)}{2(z-w)^2} + \frac{\partial G^\pm(w)}{z-w}, \\
 J(z)J(w) &\sim \frac{c}{3(z-w)^2}, \quad J(z)G^\pm(w) \sim \pm \frac{G^\pm(w)}{z-w}, \\
 G^+(z)G^-(w) &\sim \frac{2c}{3(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w}, \quad G^\pm(z)G^\pm(w) \sim 0.
 \end{aligned} \tag{A.1}$$

The commutation relations of the corresponding modes are then

$$\begin{aligned}
 [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
 [L_m, J_n] &= -nJ_{m+n}, \quad [L_m, G_r^\pm] = \left(\frac{m}{2} - r\right)G_{m+r}^\pm, \\
 [J_m, J_n] &= \frac{c}{3}\delta_{m+n,0}, \quad [J_m, G_r^\pm] = \pm G_{m+r}^\pm, \\
 [G_r^+, G_s^-] &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}, \quad [G_r^\pm, G_s^\pm] = 0.
 \end{aligned} \tag{A.2}$$

Similarly, the modes of the fields in (2.2) satisfy

$$\begin{aligned}
 [L_m, W_n^{s0}] &= [(s-1)m - n]W_{m+n}^{s0}, \quad [L_m, W_r^{s\pm}] = \left[\left(s - \frac{1}{2}\right)m - r\right]W_{m+r}^{s\pm}, \\
 [L_m, W_n^{s1}] &= [sm - n]W_{m+n}^{s1}, \quad [J_m, W_n^{s0}] = 0, \quad [J_m, W_r^{s\pm}] = \pm W_{m+r}^{s\pm}, \\
 [J_m, W_n^{s1}] &= smW_{m+n}^{s0}, \quad [G_r^\pm, W_n^{s0}] = \mp W_{r+n}^{s\pm}, \\
 [G_r^\pm, W_t^{s\mp}] &= \pm[(2s-1)r - t]W_{r+t}^{s0} + 2W_{r+t}^{s1}, \quad [G_r^\pm, W_t^{s\pm}] = 0, \\
 [G_r^\pm, W_n^{s1}] &= \frac{1}{2}[(2s+1)r - n]W_{r+n}^{s\pm}.
 \end{aligned} \tag{A.3}$$

Finally, the complete commutators $[W_m^{2\alpha}, W_n^{2\beta}]$ corresponding to the OPEs (2.13) take the form

$$\begin{aligned}
 [W_m^{20}, W_n^{20}] &= (m-n)\mathcal{A}_{m+n}^{[2]} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\
 [W_m^{21}, W_n^{21}] &= \frac{c}{48}m(m^2-1)(m^2-4)\delta_{m+n,0} + (m-n)(2m^2 - mn + 2n^2 - 8)\mathcal{B}_{m+n}^{[2]} \\
 &\quad + (m-n)\left(\mathcal{B}_{m+n}^{[4]} - 2c_{22}^3W_{m+n}^{31}\right) \\
 [W_m^{20}, W_n^{21}] &= \mathcal{C}_{m+n}^{[4]} + (2m-n)\left(\mathcal{C}_{m+n}^{[3]} - c_{22}^3W_{m+n}^{30}\right) \\
 &\quad + (6m^2 - 3mn + n^2 - 4)\mathcal{C}_{m+n}^{[2]} + m(m^2-1)\mathcal{C}_{m+n}^{[1]}
 \end{aligned}$$

$$\begin{aligned}
 [W_r^{2+}, W_s^{2-}] &= \left(\mathcal{D}_{r+s}^{[4]} - 2c_{22}^3 W_{r+s}^{31} \right) + (r-s) \left(\mathcal{D}_{r+s}^{[3]} - 3c_{22}^3 W_{r+s}^{30} \right) \\
 &\quad + \left(3r^2 - 4rs + 3s^2 - \frac{9}{2} \right) \mathcal{D}_{r+s}^{[2]} + (r-s) \left(r^2 + s^2 - \frac{5}{2} \right) \mathcal{D}_{r+s}^{[1]} \\
 &\quad + \frac{c}{12} \left(r^2 - \frac{1}{4} \right) \left(r^2 - \frac{9}{4} \right) \delta_{r+s,0} \\
 [W_r^{2+}, W_s^{2+}] &= \mathcal{E}_{r+s}^{[4]} \\
 [W_m^{20}, W_r^{2+}] &= \left(\Phi_{m+r}^{[7/2]} - c_{22}^3 W_{m+r}^{3+} \right) + \left(\frac{3}{2}m - r \right) \Phi_{m+r}^{[5/2]} \\
 &\quad + \left(3m^2 - 2mr + r^2 - \frac{9}{4} \right) \Phi_{m+r}^{[3/2]} \\
 [W_r^{2+}, W_m^{21}] &= \Psi_{r+m}^{[9/2]} + \left(2r - \frac{3}{2}m \right) \left(\Psi_{r+m}^{[7/2]} - c_{22}^3 W_{r+m}^{3+} \right) \\
 &\quad + \left(2r^2 - 2rm + m^2 - \frac{5}{2} \right) \Psi_{r+m}^{[5/2]} \\
 &\quad + \left(4r^3 - 3r^2m + 2rm^2 - m^3 - 9r + \frac{19}{4}m \right) \Psi_{r+m}^{[3/2]}.
 \end{aligned}$$

Here $\mathcal{A}^{[s]}$, $\mathcal{B}^{[s]}$, $\mathcal{C}^{[s]}$, $\mathcal{D}^{[s]}$, $\mathcal{E}^{[s]}$, $\Phi^{[s]}$, $\Psi^{[s]}$ are defined exactly as in the paper of Romans [59], except that his coupling constant κ should be regarded as a free parameter and identified with $\frac{1}{2}c_{22,2}$.

B Composite fields

In this appendix we give the explicit form of the composite $\mathcal{N} = 2$ primaries that appear in our ansatz for the OPEs (2.16) and (3.16). The main reason for doing so, besides fixing the normalisation of their structure constants, is to provide some non-trivial expressions that can be checked at intermediate steps of the calculation.

In our conventions, the normal ordered product $O_1 O_2$ of two operators O_1 and O_2 is the coefficient of the $(z-w)^0$ term in the OPE $O_1(z)O_2(w)$, see [55] for some properties of this normal ordered product. (In particular, it is not associative.) In the expressions below, a normal ordered product of multiple operators is always nested from the right, i.e. $O_1 O_2 \cdots O_n O_{n-1} := (O_1(O_2(\cdots(O_{n-1}O_n)\cdots)))$. With these conventions, the explicit expressions for A^4 , A^5 , P^2 and $P^{\frac{5}{2}}$ are

$$\begin{aligned}
 A^4 &= (W^{20})^2 - \frac{9(22c^2 - 3c + 9)n_2}{c(c-1)(c+1)(c+6)(2c-3)(5c-9)} J^4 + \frac{18(c-33)n_2}{(c+1)(c+6)(2c-3)(5c-9)} JG^+G^- \\
 &\quad - \frac{2(c+3)(44c^2 - 129c + 99)n_2}{(c-1)(c+1)(c+6)(2c-3)(5c-9)} T^2 + \frac{12(22c^2 - 3c + 9)n_2}{(c-1)(c+1)(c+6)(2c-3)(5c-9)} TJ^2 \\
 &\quad + \frac{3(c-33)cn_2}{(c+1)(c+6)(2c-3)(5c-9)} G^+\partial G^- + \frac{9(4c^c + 3^2 - 51c + 54)n_2}{2c(c+1)(c+6)(2c-3)(5c-9)} (\partial J)^2 \\
 &\quad - \frac{18(c-33)n_2}{(c+1)(c+6)(2c-3)(5c-9)} \partial TJ - \frac{3(c-33)cn_2}{(c+1)(c+6)(2c-3)(5c-9)} \partial G^+G^- \\
 &\quad + \frac{6(108 - 108c - 6c^2 - 25c^3 + 3c^4)n_2}{c(c-1)(c+1)(c+6)(2c-3)(5c-9)} \partial^2 JJ + \frac{3(45 + 11c)c_{22,2}}{(5c-12)(c^2 + 18c - 51)} J^2 W^{20} \\
 &\quad + \frac{18(c-33)(c-1)c_{22,2}}{(c+3)(5c-12)(c^2 + 18c - 51)} JW^{21} - \frac{2(594 - 477c + 96c^2 + 11c^3)c_{22,2}}{(c+3)(5c-12)(c^2 + 18c - 51)} TW^{20}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{3(c-33)(c-1)cc_{22,2}}{2(c+3)(5c-12)(c^2+18c-51)} G^- W^{2+} - \frac{3(c-33)(c-1)cc_{22,2}}{2(c+3)(5c-12)(c^2+18c-51)} G^+ W^{2-} \\
 & - \frac{18c_{22,3}}{7c-18} JW^{30} + \frac{2cc_{22,3}}{7c-18} W^{31} - \frac{3(27-42c+c^2+2c^3)n_2}{(c+1)(c+6)(2c-3)(5c-9)} \partial^2 T \\
 & - \frac{3(c-1)(108-87c+10c^2+c^3)c_{22,2}}{4(c+3)(5c-12)(c^2+18c-51)} \partial^2 W^{20} - \frac{(c-33)(c-6)n_2}{2(c+1)(c+6)(2c-3)(5c-9)} \partial^3 J, \\
 A^5 = & W^{20} W^{21} + \frac{(5c-24)}{8c} W^{2+} W^{2-} - \frac{6}{c} J (W^{20})^2 + \frac{3(c-4)(-42+12c+c^2)c_{22,3}}{4c(c^2+26c-75)} \partial^2 W^{30} \\
 & - \frac{27(11c-64)c_{22,3}}{4c(c^2+26c-75)} J^2 W^{30} + \frac{3(-72-47c+5c^2)c_{22,3}}{4c(c^2+26c-75)} JW^{31} + \frac{9(72-81c+17c^2)c_{22,3}}{4c(c^2+26c-75)} TW^{30} \\
 & - \frac{9(200-103c+17c^2)c_{22,3}}{16c(c^2+26c-75)} G^- W^{3+} + \frac{9(200-103c+17c^2)c_{22,3}}{16c(c^2+26c-75)} G^+ W^{3-} \\
 & + \frac{(5c-24)c_{22,3}}{8c} \partial W^{31} + \frac{9(-576-1956c+257c^2)c_{22,2}}{2c(5c-12)(7c+6)(2c^2+9c-36)} J^3 W^{20} \\
 & + \frac{3(-15552+4968c+15192c^2-2397c^3+64c^4)c_{22,2}}{2c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} J^2 W^{21} + \frac{3(c-15)(5c-24)c_{22,2}}{8c(c+3)(5c-12)} J \partial W^{21} \\
 & - \frac{3(108864-13608c-11178c^2+3195c^3-1065c^4+242c^5)c_{22,2}}{16c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} J \partial^2 W^{20} \\
 & + \frac{(139968-69336c-42174c^2+37809c^3-6951c^4-174c^5+8c^6)c_{22,2}}{8c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} \partial^2 W^{21} \\
 & - \frac{3(-46656+193320c-46782c^2-16179c^3+2333c^4+414c^5)c_{22,2}}{16c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} \partial J \partial W^{20} \\
 & - \frac{3(-31104-25056c+22878c^2-10599c^3+656c^4)c_{22,2}}{8c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} J G^- W^{2+} \\
 & - \frac{3(62208-19872c-19080c^2+13872c^3-3061c^4+58c^5)c_{22,2}}{8c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} G^- \partial W^{2+} \\
 & + \frac{3(-31104-25056c+22878c^2-10599c^3+656c^4)c_{22,2}}{8c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} J G^+ W^{2-} \\
 & + \frac{3(c-15)(5c-24)c_{22,2}}{8c(c+3)(5c-12)} \partial J W^{21} - \frac{27(c-1)(5c-24)c_{22,2}}{8c(c+3)(5c-12)} T \partial W^{20} \\
 & - \frac{3(-31104+4752c+11052c^2-8349c^3+899c^4)c_{22,2}}{2c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} T J W^{20} \\
 & - \frac{(15552-26136c+15570c^2-2757c^3+46c^4)c_{22,2}}{4c(5c-12)(7c+6)(2c^2+9c-36)} T W^{21} - \frac{(c-15)(2c-3)(5c-24)c_{22,2}}{48c(c+3)(5c-12)} \partial^3 W^{20} \\
 & - \frac{3(-77760+76248c+41274c^2-37695c^3+7033c^4)c_{22,2}}{8c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} G^+ G^- W^{20} \\
 & + \frac{3(-46656+51624c+25398c^2-24915c^3+3427c^4+222c^5)c_{22,2}}{8c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} G^+ \partial W^{2-} \\
 & - \frac{3(15552-31320c+8442c^2+6117c^3-3656c^4+315c^5)c_{22,2}}{4c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} \partial T W^{20} \\
 & + \frac{(93312-42768c-22572c^2+15462c^3-219c^4+10c^5)c_{22,2}}{8c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} \partial G^- W^{2+} \\
 & + \frac{(139968+52488c-3618c^2-17667c^3+879c^4+850c^5)c_{22,2}}{8c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} \partial G^+ W^{2-} \\
 & - \frac{(264384-55080c-79812c^2+41211c^3-4827c^4+199c^5)c_{22,2}}{4c(c+3)(5c-12)(7c+6)(2c^2+9c-36)} \partial^2 J W^{20} \\
 & - \frac{(5184-11880c+7182c^2-975c^3-67c^4+10c^5)n_2}{12(c-2)(c-1)c(c+6)(c+12)(2c-3)} \partial^3 T - \frac{3(c-12)(144-30c+19c^2)n_2}{8(c-2)c(c+6)(c+12)(2c-3)} G^+ \partial^2 G^-
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{(31104 - 34992c + 8856c^2 + 2268c^3 - 618c^4 + 25c^5) n_2}{2(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} T \partial^2 J \\
 & + \frac{27(87c^3 - 386c^2 + 1512c - 576) n_2}{4(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} J^2 G^+ G^- - \frac{3(53c^2 - 630c - 1152) n_2}{4(c-2)(c-1)(c+6)(c+12)(2c-3)} J G^+ \partial G^- \\
 & - \frac{3(10368 + 7344c - 7020c^2 + 1344c^3 + 67c^4) n_2}{4(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} J \partial G^+ G^- + \frac{45(11c - 102) n_2}{(c-2)(c-1)(c+6)(c+12)(2c-3)} T J^3 \\
 & - \frac{9(-1728 + 1656c + 102c^2 - 439c^3 + 45c^4) n_2}{2(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} T^2 J - \frac{3(-864 + 1692c - 894c^2 + 157c^3) n_2}{(c-2)(c-1)c(c+6)(c+12)(2c-3)} T G^+ G^- \\
 & + \frac{9(1728 + 4536c - 2982c^2 + 111c^3 + 65c^4) n_2}{8(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} (J')^2 J - \frac{3(-3456 + 5040c - 1686c^2 - 125c^3 + 45c^4) n_2}{2(c-2)(c-1)c(c+6)(c+12)(2c-3)} \partial T T \\
 & + \frac{9(1728 - 1656c - 102c^2 - 71c^3 + 10c^4) n_2}{2(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} T' J^2 - \frac{3(c-3)(3456 - 1008c - 828c^2 + 84c^3 + 25c^4) n_2}{4(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} \partial T \partial J \\
 & - \frac{(2592 - 756c - 162c^2 + 39c^3 + 16c^4) n_2}{(c-2)(c-1)c(c+6)(c+12)(2c-3)} \partial G^+ \partial G^- + \frac{3(6912 + 3780c - 828c^2 + 55c^3) n_2}{4(c-2)(c-1)c(c+6)(c+12)(2c-3)} \partial^2 J J^2 \\
 & + \frac{3(c-15)(5c-24) n_2}{4(c-1)c(c+6)(2c-3)} \partial^2 J \partial J - \frac{9(-1728 + 2232c - 258c^2 - 23c^3 + 5c^4) n_2}{4(c-2)c^2(c+6)(c+12)(2c-3)} \partial^2 T J \\
 & - \frac{3(c-3)(-1728 + 5112c - 1542c^2 - 135c^3 + 22c^4) n_2}{4(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} \partial^2 G^+ G^- + \frac{9(4c+3)(5c-24) n_2}{2(c-1)c^2(c+6)(2c-3)} T \partial J J \\
 & + \frac{(-67392 + 50760c - 2538c^2 - 465c^3 - 26c^4 + 5c^5) n_2}{8(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} \partial^3 J J - \frac{9(4c+3)(5c-24) n_2}{4(c-1)c^2(c+6)(2c-3)} \partial J G^+ G^- \\
 & - \frac{(-41472 + 62208c - 32112c^2 + 4050c^3 + 525c^4 - 117c^5 + 2c^6) n_2}{16(c-2)(c-1)c^2(c+6)(c+12)(2c-3)} \partial^4 J \\
 & - \frac{27(11c - 102) n_2}{(c-2)(c-1)c(c+6)(c+12)(2c-3)} J^5, \\
 P^2 = & W^{20} P - \frac{3(3+5c-12h)w_2}{(c-6h)(c+3-6h)(c-1-4h)} J^2 P - \frac{2(9c-5c^2-18h-6ch+8c^2h-48ch^2)w_2}{(c-6h)(c-1-4h)(1+2h)(c-3+12h)} T P \\
 & + \frac{18(c-1)(-c+3h+5ch+c^2h-6h^2)w_2}{(c-6h)(c+3-6h)(c-1-4h)h(1+2h)(c-3+12h)} J \partial P \\
 & - \frac{3(c-1)(c-12h)(-c+ch-6h^2)w_2}{(c-6h)(c+3-6h)(c-1-4h)h(1+2h)(c-3+12h)} G^+ P^- \\
 & + \frac{3(27-33c+3c^2+3c^3+78ch-14c^2h-468h^2+84ch^2+32c^2h^2-384ch^3+1152h^4)w_2}{(c-6h)(c+3-6h)(c-1-4h)(1+2h)(c-3+12h)} \partial J P \\
 & - \frac{3(c-1)(-9c+4c^2+c^3+12ch-72h^2)w_2}{2(c-6h)(c+3-6h)(c-1-4h)h(1+2h)(c-3+12h)} \partial^2 P,
 \end{aligned}$$

and

$$\begin{aligned}
 P^{\frac{5}{2}} = & W^{20} P^- - h W^{2-} P + \frac{3(-3+5c+12h-26ch-36h^2+36ch^2+72h^3)w_2}{(c-6h)h(1+2h)(c-3+6h)(c-3+12h)} J^2 P^- \\
 & - \frac{6(-3+5c+12h-26ch-36h^2+36ch^2+72h^3)w_2}{(c-6h)(1+2h)(c-3+6h)(c-3+12h)} J G^- P \\
 & - \frac{9(c-1)(-c+3ch+6h^2)w_2}{(c-6h)h(1+2h)(c-3+6h)(c-3+12h)} J \partial P^- \\
 & + \frac{(3+c)(3h-1)w_2}{h(c-3+6h)(c-3+12h)} T P^- + \frac{3(c-1)(2h-1)(-6+c+6h+5ch+18h^2)w_2}{(c-6h)h(1+2h)(c-3+6h)(c-3+12h)} G^- \partial P \\
 & - \frac{3(6+5c-3c^2-78h+7ch+7c^2h+90h^2-36ch^2+2c^2h^2+252h^3+36ch^3-288h^4)w_2}{2(c-6h)h(1+2h)(c-3+6h)(c-3+12h)} \partial J P^- \\
 & + \frac{2(-18+24c-8c^2-3ch+5c^2h+234h^2-156ch^2+6c^2h^2-108h^3+36ch^3-432h^4)w_2}{(c-6h)(1+2h)(c-3+6h)(c-3+12h)} \partial G^- P
 \end{aligned}$$

$$+ \frac{3(c-1)(3+c)(-2+c+2h)w_2}{2(c-6h)h(1+2h)(c-3+6h)(c-3+12h)} \partial^2 P^- .$$

C Structure constants of $\text{shs}[\mu]$

The structure constants of $\text{shs}[\mu]$ were computed in a very explicit form in [64]; to explain the basis that was used there, recall that the bosonic subalgebra of $\text{shs}[\mu]$ equals

$$\text{shs}[\mu]_0 \simeq \text{hs}[1-\mu] \oplus \text{hs}[\mu] \oplus \mathbb{C}, \quad (\text{C.1})$$

where $\text{hs}[\mu]$ is defined as

$$\text{hs}[\mu] \oplus \mathbb{C} = \frac{\text{U}(\mathfrak{sl}(2))}{\langle C^{\mathfrak{sl}} - \frac{\mu^2-1}{4} \rangle}. \quad (\text{C.2})$$

Using the same conventions as in [24], we denote the standard basis for $\text{hs}[1-\mu]$ by T_m^j , and the standard basis for $\text{hs}[\mu]$ by U_m^j . Furthermore, the generator for the $\mathfrak{u}(1)$ factor in (C.1) is denoted by v . The fermionic generators of $\text{shs}[\mu]$ have eigenvalue ± 1 under the adjoint action of v ; we denote the generators with eigenvalue $+1$ by Ψ_r^j , $|r| \leq j$, $j = \frac{1}{2}, \frac{3}{2}, \dots$, and those with eigenvalue -1 by $\bar{\Psi}_r^j$. The commutation relations of $\text{shs}[\mu]$ can then be written as

$$\begin{aligned} [T_m^j, T_{m'}^{j'}] &= \sum_{j'', m''} f_{TTT}^{jj'j''} C_{mm'm''}^{jj'j''} T_{m''}^{j''}, & [U_m^j, U_{m'}^{j'}] &= \sum_{j'', m''} f_{UUU}^{jj'j''} C_{mm'm''}^{jj'j''} U_{m''}^{j''}, \\ [T_m^j, \Psi_{r'}^{j'}] &= \sum_{j'', r''} f_{T\Psi\Psi}^{jj'j''} C_{mr'r''}^{jj'j''} \Psi_{r''}^{j''}, & [T_m^j, \bar{\Psi}_{r'}^{j'}] &= \sum_{j'', r''} f_{T\bar{\Psi}\bar{\Psi}}^{jj'j''} C_{mr'r''}^{jj'j''} \bar{\Psi}_{r''}^{j''}, \\ [U_m^j, \Psi_{r'}^{j'}] &= \sum_{j'', r''} f_{U\Psi\Psi}^{jj'j''} C_{mr'r''}^{jj'j''} \Psi_{r''}^{j''}, & [U_m^j, \bar{\Psi}_{r'}^{j'}] &= \sum_{j'', r''} f_{U\bar{\Psi}\bar{\Psi}}^{jj'j''} C_{mr'r''}^{jj'j''} \bar{\Psi}_{r''}^{j''}, \end{aligned} \quad (\text{C.3})$$

together with

$$\{\Psi_r^j, \bar{\Psi}_{r'}^{j'}\} = \sum_{j'', m''} C_{rr'm''}^{jj'j''} \left(f_{\Psi\bar{\Psi}T}^{jj'j''} T_{m''}^{j''} + f_{\Psi\bar{\Psi}U}^{jj'j''} U_{m''}^{j''} \right), \quad (\text{C.4})$$

where $C_{mm'm''}^{jj'j''}$ are the $\mathfrak{sl}(2)$ Clebsch-Gordan coefficients and the structure constants $f_{AA'A''}^{jj'j''}$ are those given in [64]; explicitly they are⁸

$$f_{TTT}^{jj'j''} = (1 - \epsilon^{jj'j''}) F_{000}^{jj'j''} (1 - \mu), \quad f_{UUU}^{jj'j''} = (1 - \epsilon^{jj'j''}) F_{000}^{jj'j''} (-\mu), \quad (\text{C.5})$$

$$f_{T\Psi\Psi}^{jj'j''} = -\epsilon^{jj'j''} f_{T\Psi\bar{\Psi}}^{jj'j''}, \quad f_{T\bar{\Psi}\bar{\Psi}}^{jj'j''} = +F_{01/21/2}^{jj'j''} (-\mu), \quad (\text{C.6})$$

$$f_{U\Psi\Psi}^{jj'j''} = -\epsilon^{jj'j''} f_{U\Psi\bar{\Psi}}^{jj'j''}, \quad f_{U\bar{\Psi}\bar{\Psi}}^{jj'j''} = -\epsilon^{jj'j''} F_{0-1/2-1/2}^{jj'j''} (1 - \mu), \quad (\text{C.7})$$

$$f_{\Psi\bar{\Psi}T}^{jj'j''} = -\epsilon^{jj'j''} F_{1/2-1/20}^{jj'j''} (1 - \mu), \quad f_{\Psi\bar{\Psi}U}^{jj'j''} = -F_{-1/21/20}^{jj'j''} (-\mu), \quad (\text{C.8})$$

where $\epsilon^{jj'j''} = (-1)^{j+j'-j''}$. The symbols $F_{kk'k''}^{jj'j''}(\nu)$ were defined in [64] as a deformation of the $6j$ -symbols; their arguments must satisfy $|j-j'| \leq j'' \leq j+j'$, $k'' = k+k'$, $|k| \leq j$,

⁸Note that there is a typo (wrong sign) in eq. (27d) of [64].

$|k'| \leq j'$, $|k''| \leq j''$. For this range of parameters one has

$$F_{kk'k''}^{jj'j''}(\nu) = \sqrt{2j''+1} \Delta^{jj'j''} \sum_t (-1)^t \prod_{p=1}^{j+j'-j''-t} (\nu - j'' + k'' - p) \prod_{q=1}^t (\nu + j'' + k'' + q) \\ \times \frac{\sqrt{(j+k)!(j-k)!(j'+k')!(j'-k')!(j''+k'')!(j''-k'')}}{t!(j+j'-j''-t)!(t+j''-j-k')!(t+j''-j'+k)!(j-k-t)!(j'+k'-t)!},$$

where

$$\Delta^{jj'j''} = \sqrt{\frac{(j+j'-j'')!(j+j''-j')!(j'+j''-j)!}{(j+j'+j''+1)!}}.$$

In order to compare this basis to the wedge algebra of $\mathscr{W}_\infty[\mu]$ it is convenient to rescale all generators by the factor

$$\alpha_m^j = \sqrt{\frac{(j-m)!(j+m)!}{(2j)!}}; \tag{C.9}$$

the rescaled generators will be denoted by small letters, e.g. $T_m^j = \alpha_m^j t_m^j$, etc. Then we have the identifications

$$\begin{aligned} J_0 &= v, & G_r^+ &= \sqrt{2}\psi_r^{1/2}, \\ G_r^- &= \sqrt{2}\bar{\psi}_r^{1/2}, & T_m &= -\frac{t_m^1 + u_m^1}{\sqrt{2}}, \\ W_m^{20} &= \frac{(\mu+1)t_m^1 + (\mu-2)u_m^1}{\sqrt{6}}, & W_r^{2+} &= -\sqrt{2}\psi_r^{3/2}, \\ W_r^{2-} &= -\sqrt{2}\bar{\psi}_r^{3/2}, & W_m^{21} &= \frac{t_m^2 + u_m^2}{\sqrt{2}}, \\ W_m^{30} &= \frac{3[(\mu+2)t_m^2 + (\mu-3)u_m^2]}{5\sqrt{6}c_{22,3}}, & W_r^{3+} &= -\frac{3\psi_r^{5/2}}{\sqrt{5}c_{22,3}}, \\ W_r^{3-} &= -\frac{3\bar{\psi}_r^{5/2}}{\sqrt{5}c_{22,3}}, & W_m^{31} &= \frac{3(t_m^3 + u_m^3)}{2\sqrt{5}c_{22,3}}, \\ W_m^{40} &= \frac{3\sqrt{3}[(\mu+3)t_m^3 + (\mu-4)u_m^3]}{14\sqrt{5}c_{22,3}c_{23,4}}, & W_r^{4+} &= -\frac{3\sqrt{6}\psi_r^{7/2}}{\sqrt{35}c_{22,3}c_{23,4}}, \\ W_r^{4-} &= -\frac{3\sqrt{6}\bar{\psi}_r^{7/2}}{\sqrt{35}c_{22,3}c_{23,4}}, & W_m^{41} &= \frac{3\sqrt{6}(t_m^4 + u_m^4)}{2\sqrt{35}c_{22,3}c_{23,4}}. \end{aligned}$$

We have checked that, with these identifications and to the extent to which we have determined the commutation relations of $\mathscr{W}_\infty[\mu]$, the wedge subalgebra of $\mathscr{W}_\infty[\mu]$ agrees indeed with $\text{shs}[\mu]$. Note that since $c_{22,2} = \sqrt{\gamma}$ has (for $c \rightarrow \infty$) a branch point at $\mu = \frac{1}{2}$, we have to be careful about the branch of the square root we choose; we have worked with (3.32) and restricted μ to $\mu < \frac{1}{2}$. Furthermore, we have absorbed $A^{(4)}$ into the definition of $W^{(4)}$, see eq. (2.16), i.e. we have set $a_{23,4} = 0$.

References

- [1] B. Sundborg, *Stringy gravity, interacting tensionless strings and massless higher spins*, *Nucl. Phys. Proc. Suppl.* **102** (2001) 113 [[hep-th/0103247](#)] [[INSPIRE](#)].
- [2] E. Witten, *Spacetime reconstruction*, talk at the *John Schwarz 60th birthday symposium*, <http://theory.caltech.edu/jhs60/witten/1.html>.
- [3] A. Mikhailov, *Notes on higher spin symmetries*, [hep-th/0201019](#) [[INSPIRE](#)].
- [4] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl. Phys.* **B 644** (2002) 303 [*Erratum ibid.* **B 660** (2003) 403] [[hep-th/0205131](#)] [[INSPIRE](#)].
- [5] I. Klebanov and A. Polyakov, *AdS dual of the critical $O(N)$ vector model*, *Phys. Lett.* **B 550** (2002) 213 [[hep-th/0210114](#)] [[INSPIRE](#)].
- [6] E. Sezgin and P. Sundell, *Holography in 4D (super) higher spin theories and a test via cubic scalar couplings*, *JHEP* **07** (2005) 044 [[hep-th/0305040](#)] [[INSPIRE](#)].
- [7] M.A. Vasiliev, *Higher spin gauge theories in four-dimensions, three-dimensions and two-dimensions*, *Int. J. Mod. Phys.* **D 5** (1996) 763 [[hep-th/9611024](#)] [[INSPIRE](#)].
- [8] M.A. Vasiliev, *Higher spin gauge theories: star product and AdS space*, in *The many faces of the superworld*, M.A. Shifman ed., World Scientific, Singapore (1999), pg. 533 [[hep-th/9910096](#)] [[INSPIRE](#)].
- [9] S. Giombi and X. Yin, *Higher spin gauge theory and holography: the three-point functions*, *JHEP* **09** (2010) 115 [[arXiv:0912.3462](#)] [[INSPIRE](#)].
- [10] S. Giombi and X. Yin, *Higher spins in AdS and twistorial holography*, *JHEP* **04** (2011) 086 [[arXiv:1004.3736](#)] [[INSPIRE](#)].
- [11] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a higher spin symmetry*, [arXiv:1112.1016](#) [[INSPIRE](#)].
- [12] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, [arXiv:1204.3882](#) [[INSPIRE](#)].
- [13] O. Aharony, G. Gur-Ari and R. Yacoby, *$D = 3$ bosonic vector models coupled to Chern-Simons gauge theories*, *JHEP* **03** (2012) 037 [[arXiv:1110.4382](#)] [[INSPIRE](#)].
- [14] S. Giombi et al., *Chern-Simons theory with vector fermion matter*, *Eur. Phys. J.* **C 72** (2012) 2112 [[arXiv:1110.4386](#)] [[INSPIRE](#)].
- [15] C.-M. Chang, S. Minwalla, T. Sharma and X. Yin, *ABJ triality: from higher spin fields to strings*, [arXiv:1207.4485](#) [[INSPIRE](#)].
- [16] O. Aharony, G. Gur-Ari and R. Yacoby, *Correlation functions of large- N Chern-Simons-matter theories and bosonization in three dimensions*, *JHEP* **12** (2012) 028 [[arXiv:1207.4593](#)] [[INSPIRE](#)].
- [17] S. Jain, S.P. Trivedi, S.R. Wadia and S. Yokoyama, *Supersymmetric Chern-Simons theories with vector matter*, *JHEP* **10** (2012) 194 [[arXiv:1207.4750](#)] [[INSPIRE](#)].
- [18] M.R. Gaberdiel and R. Gopakumar, *An AdS_3 dual for minimal model CFTs*, *Phys. Rev.* **D 83** (2011) 066007 [[arXiv:1011.2986](#)] [[INSPIRE](#)].
- [19] M.R. Gaberdiel and R. Gopakumar, *Triality in minimal model holography*, *JHEP* **07** (2012) 127 [[arXiv:1205.2472](#)] [[INSPIRE](#)].

- [20] M.R. Gaberdiel and P. Suchanek, *Limits of minimal models and continuous orbifolds*, *JHEP* **03** (2012) 104 [[arXiv:1112.1708](#)] [[INSPIRE](#)].
- [21] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, *Commun. Math. Phys.* **104** (1986) 207 [[INSPIRE](#)].
- [22] M. Henneaux and S.-J. Rey, *Nonlinear W_∞ as asymptotic symmetry of three-dimensional higher spin Anti-de Sitter gravity*, *JHEP* **12** (2010) 007 [[arXiv:1008.4579](#)] [[INSPIRE](#)].
- [23] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [[arXiv:1008.4744](#)] [[INSPIRE](#)].
- [24] M.R. Gaberdiel and T. Hartman, *Symmetries of holographic minimal models*, *JHEP* **05** (2011) 031 [[arXiv:1101.2910](#)] [[INSPIRE](#)].
- [25] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic W -symmetries in three-dimensional higher-spin gauge theories*, *JHEP* **09** (2011) 113 [[arXiv:1107.0290](#)] [[INSPIRE](#)].
- [26] M.R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju, *Partition functions of holographic minimal models*, *JHEP* **08** (2011) 077 [[arXiv:1106.1897](#)] [[INSPIRE](#)].
- [27] C.-M. Chang and X. Yin, *Higher spin gravity with matter in AdS_3 and its CFT dual*, *JHEP* **10** (2012) 024 [[arXiv:1106.2580](#)] [[INSPIRE](#)].
- [28] K. Papadodimas and S. Raju, *Correlation functions in holographic minimal models*, *Nucl. Phys. B* **856** (2012) 607 [[arXiv:1108.3077](#)] [[INSPIRE](#)].
- [29] C. Ahn, *The coset spin-4 Casimir operator and its three-point functions with scalars*, *JHEP* **02** (2012) 027 [[arXiv:1111.0091](#)] [[INSPIRE](#)].
- [30] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, *Conical defects in higher spin theories*, *JHEP* **02** (2012) 096 [[arXiv:1111.3381](#)] [[INSPIRE](#)].
- [31] M. Ammon, P. Kraus and E. Perlmutter, *Scalar fields and three-point functions in $D = 3$ higher spin gravity*, *JHEP* **07** (2012) 113 [[arXiv:1111.3926](#)] [[INSPIRE](#)].
- [32] C.-M. Chang and X. Yin, *Correlators in W_N minimal model revisited*, *JHEP* **10** (2012) 050 [[arXiv:1112.5459](#)] [[INSPIRE](#)].
- [33] M. Gutperle and P. Kraus, *Higher spin black holes*, *JHEP* **05** (2011) 022 [[arXiv:1103.4304](#)] [[INSPIRE](#)].
- [34] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, *Spacetime geometry in higher spin gravity*, *JHEP* **10** (2011) 053 [[arXiv:1106.4788](#)] [[INSPIRE](#)].
- [35] P. Kraus and E. Perlmutter, *Partition functions of higher spin black holes and their CFT duals*, *JHEP* **11** (2011) 061 [[arXiv:1108.2567](#)] [[INSPIRE](#)].
- [36] M.R. Gaberdiel, T. Hartman and K. Jin, *Higher spin black holes from CFT*, *JHEP* **04** (2012) 103 [[arXiv:1203.0015](#)] [[INSPIRE](#)].
- [37] C. Ahn, *The large- N 't Hooft limit of coset minimal models*, *JHEP* **10** (2011) 125 [[arXiv:1106.0351](#)] [[INSPIRE](#)].
- [38] M.R. Gaberdiel and C. Vollenweider, *Minimal model holography for $SO(2N)$* , *JHEP* **08** (2011) 104 [[arXiv:1106.2634](#)] [[INSPIRE](#)].

- [39] T. Creutzig, Y. Hikida and P.B. Ronne, *Higher spin AdS₃ supergravity and its dual CFT*, *JHEP* **02** (2012) 109 [[arXiv:1111.2139](#)] [[INSPIRE](#)].
- [40] Y. Kazama and H. Suzuki, *New N = 2 superconformal field theories and superstring compactification*, *Nucl. Phys. B* **321** (1989) 232 [[INSPIRE](#)].
- [41] Y. Kazama and H. Suzuki, *Characterization of N = 2 superconformal models generated by coset space method*, *Phys. Lett. B* **216** (1989) 112 [[INSPIRE](#)].
- [42] S. Prokushkin and M.A. Vasiliev, *Higher spin gauge interactions for massive matter fields in 3D AdS space-time*, *Nucl. Phys. B* **545** (1999) 385 [[hep-th/9806236](#)] [[INSPIRE](#)].
- [43] S. Prokushkin and M.A. Vasiliev, *3D higher spin gauge theories with matter*, [hep-th/9812242](#) [[INSPIRE](#)].
- [44] C. Candu and M.R. Gaberdiel, *Supersymmetric holography on AdS₃*, [arXiv:1203.1939](#) [[INSPIRE](#)].
- [45] M. Henneaux, G. Lucena Gomez, J. Park and S.-J. Rey, *Super-W(∞) asymptotic symmetry of higher-spin AdS₃ supergravity*, *JHEP* **06** (2012) 037 [[arXiv:1203.5152](#)] [[INSPIRE](#)].
- [46] K. Hanaki and C. Peng, *Symmetries of holographic super-minimal models*, [arXiv:1203.5768](#) [[INSPIRE](#)].
- [47] C. Ahn, *The large-N 't Hooft limit of Kazama-Suzuki model*, *JHEP* **08** (2012) 047 [[arXiv:1206.0054](#)] [[INSPIRE](#)].
- [48] C. Candu, M.R. Gaberdiel, M. Kelm and C. Vollenweider, *Even spin minimal model holography*, [arXiv:1106.2634](#).
- [49] K. Hornfeck, *Classification of structure constants for W algebras from highest weights*, *Nucl. Phys. B* **411** (1994) 307 [[hep-th/9307170](#)] [[INSPIRE](#)].
- [50] R. Blumenhagen and A. Wisskirchen, *Extension of the N = 2 Virasoro algebra by two primary fields of dimension 2 and 3*, *Phys. Lett. B* **343** (1995) 168 [[hep-th/9408082](#)] [[INSPIRE](#)].
- [51] K. Ito, *Quantum Hamiltonian reduction and N = 2 coset models*, *Phys. Lett. B* **259** (1991) 73 [[INSPIRE](#)].
- [52] P. Bowcock and G. Watts, *On the classification of quantum W algebras*, *Nucl. Phys. B* **379** (1992) 63 [[hep-th/9111062](#)] [[INSPIRE](#)].
- [53] D. Gepner, *Scalar field theory and string compactification*, *Nucl. Phys. B* **322** (1989) 65 [[INSPIRE](#)].
- [54] K. Thielemans, *A Mathematica package for computing operator product expansions*, *Int. J. Mod. Phys. C* **2** (1991) 787 [[INSPIRE](#)].
- [55] K. Thielemans, *An algorithmic approach to operator product expansions, W algebras and W strings*, Ph.D. thesis, KU Leuven, Leuven Belgium (1994) [[hep-th/9506159](#)] [[INSPIRE](#)].
- [56] G. Mussardo, G. Sotkov and M. Stanishkov, *N = 2 superconformal minimal models*, *Int. J. Mod. Phys. A* **4** (1989) 1135 [[INSPIRE](#)].
- [57] M. Gaberdiel, *Fusion rules of chiral algebras*, *Nucl. Phys. B* **417** (1994) 130 [[hep-th/9309105](#)] [[INSPIRE](#)].
- [58] R. Blumenhagen, *N = 2 supersymmetric W algebras*, *Nucl. Phys. B* **405** (1993) 744 [[hep-th/9208069](#)] [[INSPIRE](#)].

- [59] L. Romans, *The $N = 2$ super- W_3 algebra*, *Nucl. Phys. B* **369** (1992) 403 [[INSPIRE](#)].
- [60] M. Kelm and C. Vollenweider, private communication.
- [61] R.G. Leigh and A.C. Petkou, *Holography of the $N = 1$ higher spin theory on AdS_4* , *JHEP* **06** (2003) 011 [[hep-th/0304217](#)] [[INSPIRE](#)].
- [62] S. Giombi and X. Yin, *On higher spin gauge theory and the critical $O(N)$ model*, *Phys. Rev. D* **85** (2012) 086005 [[arXiv:1105.4011](#)] [[INSPIRE](#)].
- [63] T. Hartman and L. Rastelli, *Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT* , *JHEP* **01** (2008) 019 [[hep-th/0602106](#)] [[INSPIRE](#)].
- [64] E. Fradkin and V.Y. Linetsky, *Supersymmetric Racah basis, family of infinite dimensional superalgebras, $SU(\infty + 1|\infty)$ and related 2D models*, *Mod. Phys. Lett. A* **6** (1991) 617 [[INSPIRE](#)].