Published for SISSA by Springer
Received: December 14, 2011
Accepted: January 30, 2012
Published: February 22, 2012

Duality invariant actions and generalised geometry

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Abstract: We construct the non-linear realisation of the semi-direct product of E_{11} and its first fundamental representation at lowest order and appropriate to spacetime dimensions four to seven. This leads to a non-linear realisation of the duality groups and introduces fields that depend on a generalised space which possess a generalised vielbein. We focus on the part of the generalised space on which the duality groups alone act and construct an invariant action.

Keywords: M-Theory, String Duality, Supergravity Models

ArXiv ePrint: 1111.0459

 doi:10.1007/JHEP02(2012)108

Contents

1 Introduction

Nobody really knows what M-theory is, although quite a lot is known about its various limits. These include the five ten-dimensional string theories, along with eleven-dimensional supergravity which describes the low energy effective action of the IIA string at strong coupling. In fact the low energy effective actions of the different string theories given by their respective supergravities contain both nonperturbative and perturbative information. As such, the U-duality web relating these theories can be tested in detail using the supergravity description. Common to all these theories is a notion of spacetime described either by a vielbein or a metric together with various gauge fields and fermions which propagate in the spacetime. It seems strange that in a theory that is supposed to unify the forces of nature, one treats the gravitational field geometrically whereas others are painted on to the geometrical spacetime. Our aim here is to develop a more democratic approach.

Such an approach was advocated in [1] where it was conjectured that the non-linear realisation of a certain Kac-Moody algebra called E_{11} is an extension of eleven dimensional supergravity. In [1], spacetime is not encoded in an E_{11} covariant way. Spacetime can be introduced by considering the non-linear realisation of the semi-direct product of E_{11} with its a fundamental representation, usually called the first fundamental representation [2]. This semi-direct product is explained in detail later. Semi-direct product constructions are well known, for example, the Poincaré group is just the semi-direct product of the Lorentz group and its vector representation, that is the spacetime translations. The first fundamental representation contains as its first component the spacetime translations, then a two and five form as well as an infinite number of other objects. There is considerable evidence to suggest that all brane charges are contained in this representation [2–5] and for each field in the E_{11} part of the non-linear realisation, there is a corresponding element in this representation [3]. The inclusion of the first fundamental representation in the non-linear realisation leads to a generalised spacetime with a coordinate for every brane charge and for every field. Thus for the metric we find the usual coordinates x^a of spacetime, for the three form new coordinates $x_{a_1 a_2}$ and for the six form new coordinates $x_{a_1 \ldots a_6}$ and so on [2]. The E_{11} part of the formulation is also democratic in the sense that E_{11} contains all the duality symmetries together with all the corresponding fields [6].

To understand better this development, it is useful to recall some of the background. In the early days of particle physics, with the recognition of the importance of symmetries, non-linear realisations played an important role. In particular, Goldstone's theorem states that if a rigid symmetry G is spontaneously broken to a subgroup H, then there are $(\text{dim}G - \text{dim}H)$ massless particles. Furthermore it was realised that the dynamics of these particles is controlled by the non-linear realisation of G with local subgroup H. In the case of the chiral symmetry, the group G is $SU(2) \otimes SU(2)$, the subgroup H is the diagonal subgroup $SU(2)$ and the three massless particles are the three pions in the limit of zero mass. The dynamics of the pions can be accounted for by this non-linear realisation [7–11]. The general formulation of such non-linear realisation for any group is given in references [12–14].

Of course it was only later that the importance of gauge symmetries was understood, and it was realised that pions were made of quarks subject to forces controlled by an SU(3) gauge theory. However, this only serves to illustrate that in the context of spontaneously broken symmetries, non-linear realisations provide a way of finding the underlying symmetry even though the fundamental degrees of freedom are not known.

The non-linear realisations used in the early days of particle physics, and just discussed above, are essentially a coset construction of G with respect to H and spacetime is a dummy variable as far as group theory is concerned. The sigma model usually describes this coset construction. However, one can also construct non-linear realisations in which the group contains generators associated with spacetime and in particular the spacetime translations. For these non-linear realisations spacetime arises naturally as it parametrises the part of the group element that includes the generators associated with spacetime. One early paper using this method was [15] where the non-linear realisation with $G = \text{GL}(4, \mathbb{R})$ and $H = O(3,1)$ was studied in the context of general relativity. However, it was Borisov and Ogievetsky [16] who showed that general relativity in four dimensions could be reformulated as a non-linear realisation of the groups $G = \text{GL}(4, \mathbb{R}) \ltimes I^4$ and $H = O(3,1)$. Here $\text{GL}(4, \mathbb{R}) \ltimes I^4$ is the semi-direct product of the groups $\text{GL}(4, \mathbb{R})$ and the group I^4 of spacetime translation generators. It is the inclusion of the latter that lead to the presence of the spacetime coordinates in the theory. In fact the dynamics of this non-linear realisation was only unique up to a few constants and these were fixed to precisely the

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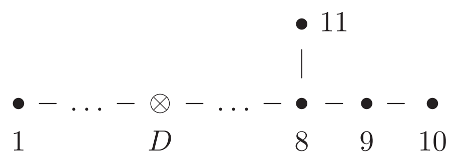

Figure 1. The E_{11} Dynkin diagram with node D deleted.

right values if one demanded that the theory be also invariant under the conformal group, also non-linearly realised. Another use of such non-linear realisations was by Volkov and Akulov [17] who used it to compute the dynamics of the massless fermion that results from the breaking of supersymmetry and postulated that it could be a neutrino.

The E_{11} conjecture arises from the recognition that the eleven dimensional supergravity theory is a non-linear realisation and that this leads to an algebra including the spacetime translations [18]. When the spacetime translations are omitted from the algebra, it can be extended to a Kac-Moody algebra and the smallest such algebra is E_{11} [1]. As the non-linear realisation involves the spacetime generators, it cannot be a sigma model. To include the spacetime translations in a covariant manner the first fundamental representation of E_{11}, denoted by l_1, is considered. This is the smallest E_{11} representation that contains spacetime translations. The original and early papers introduce the spacetime generators by hand and so only included the first component of the l_1 representation.

An earlier work that formulates the gauge fields of the maximal supergravity theories as a non-linear realisation using a graded algebra is [19–21]. The non-linear realisation in [19–21] does not contain any spacetime generators.

A theory in d dimensions[1] can be found [1, 4, 22–26] by taking the non-linear realisation of $E_{11} \ltimes l_1$ with the decomposition of E_{11} into the subalgebra $GL(D) \otimes E_d$, where $D = 11 - d$. This can be done by deleting node D in the E_{11} Dynkin diagram in figure 1.

The E_d factor in the subalgebra $GL(D) \otimes E_d$ is the well known E_d symmetry[2] [27–29] which has been known to be a symmetry of the maximal supergravity theory in D dimensions for many years. Thus these symmetries naturally emerge. The $GL(D)$ factor in the subalgebra, together with the spacetime translations in D dimensions which are contained in the l_1 representation give rise to gravity in D dimensions as they should according to [16]. Indeed this confirms that we have found a theory in D dimensions. In the decomposition of E_{11} into E_d one finds the expected fields of D dimensional supergravity as well as a hierarchy of form fields [23, 30], which play an important role in gauged supergravities, as well as an infinite number of higher level fields. The l_1 representation is also decomposed into representations of $GL(D) \otimes E_d$ and in addition to the spacetime translations in D dimensions one finds an infinite number of coordinates beginning with some coordinates, which are scalars under GL(D) but transform under E_d indeed in $d = 4, 5, 6, 7, 8$ dimensions they belong to the 10, $\overline{16}$, $\overline{27}$, 56 and $248 \oplus 1$ representations of

[1]In this paper d corresponds to the directions in which the duality acts. In [1, 4, 22–26], the complementary view is taken whereby d is $11 - d$ of this paper.

[2]Throughout this paper, we are considering the split forms of the exceptional groups, usually denoted $E_{d(d)}$.

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D	d	G	H
3	8	E_8	SO(16)
4	7	E_7	SU(8)
5	6	E_6	USp(8)
6	5	SO(5,5)	SO(5)×SO(5)
7	4	SL(5)	SO(5)
8	3	SL(3)×SL(2)	SO(3)×SO(2)
9	2	SL(2)	SO(2)
10	1	SO(1,1)	1

Table 1. The duality groups that appear on the reduction of 11-dimensional supergravity to $D-$dimensions.

SL(5), SO(5,5), E_6, E_7 and E_8 respectively [4, 31]. The non-linear realisation of $E_{11} \ltimes l_1$ not only gives rise to generalised spacetime, but it also leads to a generalised vielbein which is determined in terms of the E_{11} fields and depends on the generalised spacetime. In this paper, the theory in d dimensions is considered. We explicitly construct the generalised vielbeins and the corresponding dynamics.

For future reference, in table 1 we recall the U-duality groups in the various dimensions.

In fact one can formulate the dynamics of strings, membranes etc in the presence of the background fields as an $E_{11} \ltimes l_1$ non-linear realisation [31]. The difference compared to the non-linear realisation used to construct the supergravity theories was in the choice of local subalgebra. In [31] the coordinates of the generalised spacetime specifies the dynamics of the brane.

An enlarged spacetime also appeared in the context of the first quantised string [32–34] and membrane [35] where the usual spacetime is extended to include additional coordinates describing string winding modes. The aim in the case of the string is to make the T-duality symmetry manifest by introducing additional coordinates corresponding to string winding modes. This is then extended to the membrane in [35], where new coordinates are introduced corresponding to membrane windings, so that the U-duality group is made manifest. The work in [35] is further developed in [36] for the SL(5) duality group to a give duality-invariant dynamics for fields living on a space whose coordinates belong to the ten dimensional representation of SL(5). The invariant dynamics is constructed using a generalised metric given in terms of the background supergravity fields, and later extended to the duality group SO(5,5) in [37]. The usual way in which duality groups appear is where one dimensionally reduces eleven dimensional supergravity. The duality group then acts on the components of the fields in the Kaluza-Klein directions. In [36, 37] the opposite approach is taken; the duality group acts on the space where the fields have spacetime dependence, i.e. no Killing directions are assumed.

In [38], the non-linear realisation of $E_{11} \ltimes l_1$ decomposed to $GL(4) \otimes E_7$ is constructed. The part of the l_1 representation that is kept leads to the usual coordinates of the four dimensional spacetime and also the coordinates which are scalars under GL(4) but transform as a 56 dimensional representation of E_7. The non-linear realisation is then used in [38] to construct an invariant action.

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In the present paper, we will show how the results of [36, 37] can be derived in a very straightforward way from the $E_{11} \ltimes l_1$ non-linear realisation discussed above. Indeed we construct the non-linear realisation $E_{11} \ltimes l_1$ decomposed to $GL(D) \otimes E_d$ suitable to d dimensions. We restrict the l_1 representation to contain only the coordinates that are scalars under GL(D), which turn out to transform as the $\overline{10}$, 16, 27 and 56 dimensional representations of SL(5), SO(5,5), E_6 and E_7 respectively. We construct invariant actions where the fields are defined on these generalised spacetimes.

In section 2, we revisit four dimensions and the SL(5) duality group. The generalised metric in this case was constructed in [36] using M2-brane considerations. In section 2, the non-linear realisation of the SL(5) motion group is used to construct the generalised metric, which is the same as in [36] up to a conformal factor. The purpose of this section is to illustrate non-linear realisation for a familiar group before considering the non-linear realisation of $E_{11} \ltimes l_1$. We give a review of E_{11} and its first fundamental representation in section 3. In this section, we also review the non-linear realisation of $E_{11} \ltimes l_1$. In section 4, the example of four dimensions is revisited to show how the non-linear realisation of $E_{11} \ltimes l_1$ can be used to find the generalised metric and the dynamics. The non-linear realisation of $E_{11} \ltimes l_1$ produces a generalised metric that differs from the generalised metric in section 2 by a crucial conformal factor. The precise value of the conformal factor in the latter case is such that the generalised metric cannot be used to construct the dynamics. This suggests that the duality groups in lower dimensions must be viewed as subgroups of a larger group. We show that the generalised metric that is derived from the non-linear realisation of $E_{11} \ltimes l_1$ can be used to construct the dynamics and naturally incorporates the correct measure. In sections 5, 6 and 7, we proceed to carry the same procedure in five, six and seven dimensions. In each case we find the generalised metric and formulate the dynamics in terms of this object to give a duality invariant action that reproduces the usual 11-dimensional supergravity action. In appendix B, we outline the difference between constructing the non-linear realisation of the duality groups in $d = 4, 5, 6$ and 7 dimensions and the non-linear realisation of $E_{11} \ltimes l_1$.

2 SL(5) generalised metric

In this section we consider in detail the duality group SL(5) and give a rather pedestrian presentation. This will allow us to study the SL(5) duality group and the ten dimensional spacetime that occurs in this case in isolation. We will just present the algebra rather than derive it from $E_{11} \ltimes l_1$, we will explain in detail the way the non-linear realisation leads to the generalised metric and the corresponding dynamics. This will allow one to gain some understanding of the technical aspects of the non-linear realisations used without all the complications of the $E_{11} \ltimes l_1$ algebra.

The starting point of the non-linear realisation method is the duality group, from which we form the corresponding motion group. The semi-direct product of a group with a representation of the group defines the motion group [39–43]. For example, the Poincaré group is the motion group of the Lorentz group. The SL(5) algebra itself is given by 24 tracefree generators. In the fundamental representation of SL(5), the generators can be chosen to be

$$\left(M^I{}_J\right)^P{}_Q = -\delta^P_J \delta^I_Q + \frac{1}{5}\delta^I_J \delta^P_Q.$$

The indices $I, J = 1, \ldots 5$ and are the generator labels, while P, Q are matrix indices which also run from 1 to 5 because we are in the fundamental representation. It can be explicitly checked that the generators satisfy the expected SL(5) commutation relations

$$[M^I{}_J, M^K{}_L] = \delta^K_J M^I{}_L - \delta^I_L M^K{}_J. \tag{2.1}$$

We will construct the motion group of SL(5) where the translation generators form a ten-dimensional representation. This is similar to the construction of the Poincaré group from the Lorentz group. The translation generators form the 10 of SL(5), which we denote by P_{IJ}, where the indices again run from 1 to 5 and P is antisymmetric in these indices so that we have ten generators.

The translation generators all commute with each other, and their commutation relations with the group generators and the translation generators are

$$[M^I{}_J, P_{KL}] = -2\,\delta^I_{[K} P_{|J|L]} + \frac{2}{5}\delta^I_J P_{KL}. \tag{2.2}$$

The coefficient of the first term on the right-hand side is fixed by the Jacobi identities, while the coefficient of the second is determined by the requirement that the generators M are tracefree.

The SL(5) duality group first appeared when a Kaluza-Klein reduction of eleven-dimensional supergravity was made on a flat 4-torus. In our picture, SL(5) appears as a group which controls the geometry of the 4-manifold itself. Unlike Kaluza-Klein reduction, the 4-manifold is not associated with any Killing vectors. The fields *depend* on the coordinates in the directions of the 4-manifold. We will ignore the dependence of all fields on directions orthogonal to the 4-manifold. This is *opposite* to Kaluza-Klein reduction. Thus, if we were considering eleven-dimensional supergravity there will be seven directions that are ignored. However, as was found in [36], the four directions must be augmented by the six winding directions associated with the M2 branes charges. There a total of ten dimensions of the extended space associated with the four physical spatial directions. The ultimate interpretation of these extra dimensions is presently a little unclear but is discussed in [44], where the local symmetries of M-theory is explored in the context of generalised geometry. This approach leads to the physical section condition for M-theory generalised geometry. The extra dimensions are M-theoretic generalisations of the winding coordinates found in doubled field theory [45–48].

To make the relation to the usual fields and coordinates clear, we will decompose the SL(5) group into its SL(4) × U(1) subgroup. The SL(4) corresponds to the usual four spatial directions. We let

$$M^I{}_J = \begin{cases} M^i{}_j \\ M^5{}_j = \frac{1}{6}\epsilon_{jklm} R^{klm} \\ M^i{}_5 = \frac{1}{6}\epsilon^{iklm} R_{klm} \end{cases}. \tag{2.3}$$

The indices labelled by $i, j, \dots$ are GL(4) indices that run from 1 to 4. Note that

$$M^5{}_5 = -\sum_{i=1}^{4} M^i{}_i,$$

by the tracelessness of $M^I{}_J$. The generator $\sum M^i{}_i$ which we will denote by M, gives the scaling of generators in the GL(4) decomposition and so determines their U(1) charge. The generator $M^i{}_j$ can be shifted by M, and indeed we will shift

$$M^i{}_j \to K^i{}_j = M^i{}_j - \delta^i_j M. \tag{2.4}$$

The dilatation is now given by

$$K \equiv \sum K^i{}_i = -3M,$$

which generates the U(1) subgroup of SL(5). With this choice,

$$[K, R^{klm}] = 3\, R^{klm}, \qquad [K, R_{klm}] = -3\, R_{klm} \qquad \text{and} \qquad [K, K^i{}_j] = 0,$$

so that K counts the index of the GL(4) representations, in other words, its U(1) charge. Other choices can of course be made, but these will result in more complicated commutation relations between the $K^i{}_j$ generator and generalised translation generators.

We can now rewrite the SL(5) algebra in terms of the GL(4) and U(1) generators

$$[K^i_{\ j}, K^l_{\ m}] = \delta^l_j K^i_{\ m} - \delta^i_m K^l_{\ j}, \qquad [R^{i_1\dots i_3}, R_{j_1\dots j_3}] = 18\,\delta^{[i_1 i_2}_{[j_1 j_2} K^{i_3]}_{\ \ j_3]} - 2\,\delta^{i_1\dots i_3}_{j_1\dots j_3} K, \tag{2.5}$$

$$[K^i_{\ j}, R_{k_1\dots k_3}] = -3\,\delta^i_{[k_1} R_{k_2 k_3]j}, \qquad [K^i_{\ j}, R^{k_1\dots k_3}] = 3\,\delta^{[k_1}_j R^{k_2 k_3]i}; \tag{2.6}$$

all other commutators vanish. The fully antisymmetrised Kronecker delta function is defined to be

$$\delta^{i_1\dots i_p}_{j_1\dots j_p} = \delta^{[i_1}_{j_1} \dots \delta^{i_p]}_{j_p} = \frac{1}{p!}\Big(\delta^{i_1}_{j_1} \dots \delta^{i_p}_{j_p} + (\text{all remaining even permutations of } i_1 \dots i_p) \\ - (\text{all odd permutations of } i_1 \dots i_p)\Big),$$

making a total of $p!$ terms in the parentheses.

Now that we have the SL(5) algebra, we similarly write the translation generators

$$P_{IJ} = \begin{cases} P_{i5} = P_i \\ P_{ij} = \dfrac{1}{2}\epsilon_{ijkl} Z^{kl} \end{cases}. \tag{2.7}$$

The 10-dimensional representation in terms of a GL(4) decomposition is made in order to relate the translation generators to the generators of ordinary spatial translations in four-dimensions P_i, together with the generalised translations Z^{ij}, which correspond to windings of the M2-brane.

Now, from equation (2.2), the rest of the commutation relations of the algebra are

$$[K^i{}_j, P_k] = -\delta^i_k P_j - \frac{1}{5}\delta^i_j P_k, \qquad [K^i{}_j, Z^{kl}] = 2\,\delta^{[k}_j Z^{|i|l]} - \frac{1}{5}\delta^i_j Z^{kl}, \tag{2.8}$$

$$[R_{ijk}, P_l] = 0, \qquad [R_{ijk}, Z^{mn}] = 3!\,\delta^{mn}_{[ij} P_{k]},$$
$$[R^{ijk}, P_l] = 3\,\delta^{[i}_l Z^{jk]}, \qquad [R^{ijk}, Z^{mn}] = 0. \tag{2.9}$$

Note that for the translation generators the U(1) generator K does not count the index of the generator as it did for the SL(5) generators;

$$[K, P_i] = \frac{9}{5} P_i \qquad \text{and} \qquad [K, Z^{ij}] = \frac{11}{5} Z^{ij}.$$

In figure 2, the weight diagram of the ten-dimensional representation of SL(5) is presented. The weight diagram is generated by subtracting positive roots from the weights (equivalently adding negative roots to the weights). The generators

$$K^i{}_j, R^{k_1\ldots k_3}, R_{k_1\ldots k_3}$$

are associated to the roots of SL(5)

$$\alpha_{ij}, \alpha_{k_1\ldots k_3}, -\alpha_{k_1\ldots k_3}.$$

The root lattice is generated by adding arbitrary multiples of positive roots to these. For example,

$$\alpha_{12} + \alpha_{23} = \alpha_{13} \qquad \text{and} \qquad \alpha_{12} + \alpha_{234} = \alpha_{134},$$

from which the commutators

$$[K^1{}_2, K^2{}_3] = K^1{}_3 \qquad \text{and} \qquad [K^1{}_2, R^{234}] = R^{134}$$

can be constructed. Similarly, the translation generators P_i and Z^{ij} are associated to the weights labelled by x^i and x_{ij} in figure 2. The x^i and x_{ij} then become coordinates of the extended space. The commutation relations of the motion group of SL(5) encode how the roots act on the weights. The negative roots

$$\alpha_{ij}, \quad \text{for } i < j, \qquad \text{and} \qquad \alpha_{k_1\ldots k_3}$$

act on the 10-dimensional weight diagram by lowering the weights, while the positive roots

$$\alpha_{ij}, \quad \text{for } i > j, \qquad \text{and} \qquad -\alpha_{k_1\ldots k_3}$$

raise the weights. In figure 2, for example, α_{23} acts on the weight x_{34} to give x_{24}. In terms of a commutation relation, this is

$$[K^2{}_3, Z^{34}] = Z^{24},$$

which is consistent with the second equation in (2.8).

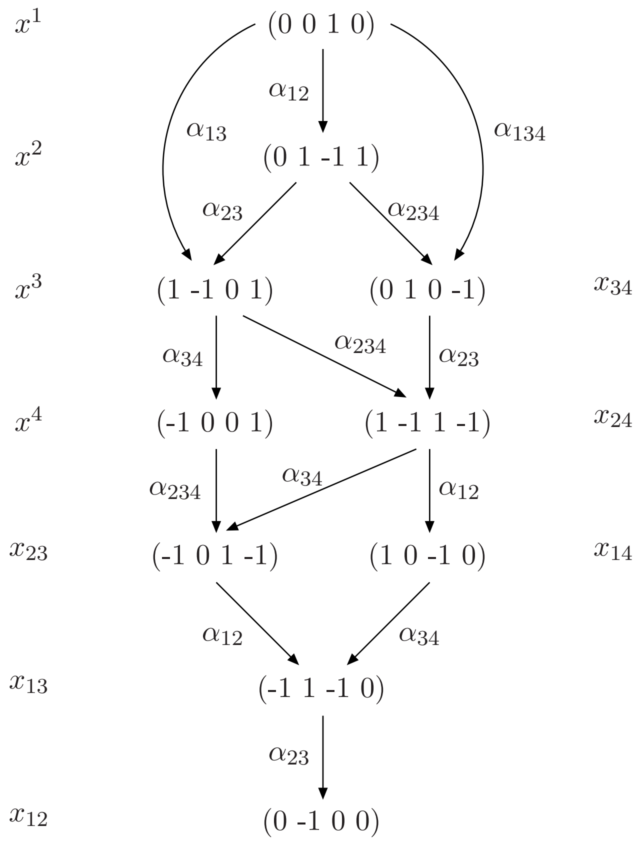

Figure 2. The weight diagram of the 10-dimensional representation of SL(5).

We need to find the normalisation of the translation generators, which set the conventions for the tangent space metric. Let[3]

$$\mathrm{tr}(P_{IJ}P_{KL}) = 2\,\delta_{IJ,KL} = (\delta_{IK}\delta_{JL} - \delta_{IL}\delta_{JK}),$$

and by inserting the translation generators given in equation (2.7) we find that

$$\mathrm{tr}(P_iP_j) = \delta_{ij}, \qquad \mathrm{tr}(Z^{ij}Z^{kl}) = 2\,\delta^{ij,kl}, \qquad \mathrm{tr}(P_iZ^{kl}) = 0, \tag{2.10}$$

where

$$\delta^{ij,kl} = \frac{1}{2}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}).$$

The generalised metric is constructed using the non-linear realisation method. We start by writing the group element

$$g_l = \mathrm{e}^{x^iP_i}\mathrm{e}^{\frac{1}{\sqrt{2}}x_{kl}Z^{kl}},$$

[3]Our treatment of the normalisation of generators in this section is motivated purely by convenience. A more rigorous treatment involves the definition of the Cartan involution of P and is described in appendix A.

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where x^i are the conventional coordinates, and x_{kl} are the "winding coordinates." The coefficient of the each exponent is such that the tangent space metric takes the canonical form, i.e.

$$\mathrm{tr}(g_l^{-1}\mathrm{d}g_l g_l^{-1}\mathrm{d}g_l) = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \delta_{kl,mn}\mathrm{d}x_{kl}\mathrm{d}x_{mn}. \tag{2.11}$$

The group element that defines the fields is

$$g_E = \mathrm{e}^{h_i{}^j K^i{}_j}\mathrm{e}^{\frac{1}{3!}C_{ijk}R^{ijk}}.$$

C_{ijk} is the 3-form potential of M-theory restricted to the 4-space and $h_i{}^j$ determines the vielbein.

The generalised vielbein, E, is given by the Maurer-Cartan form of g_l conjugated by g_E

$$L_A E_\Pi{}^A \mathrm{d}z^\Pi = g_E^{-1} g_l^{-1}\mathrm{d}g_l g_E, \tag{2.12}$$

where $L_A = (P_i, Z^{kl}/\sqrt{2})$ and $\mathrm{d}z^\Pi = (\mathrm{d}x^\mu, \mathrm{d}x_{\mu\nu})$. Latin letters indicate tangent space indices, while Greek letters label spacetime indices. The normalisation of L_A has been arranged so that $\mathrm{tr}(L_A L_B) = \delta_{AB}$. In terms of the generalised vielbein, the generalised line element is given by

$$\mathrm{Tr}(L_A E_\Pi{}^A \mathrm{d}z^\Pi L_B E_\Sigma{}^B \mathrm{d}z^\Sigma) = E_\Pi{}^A E_\Sigma{}^B \delta_{AB}\mathrm{d}z^\Pi\mathrm{d}z^\Sigma.$$

Consequently, the generalised metric is

$$M_{\Pi\Sigma} = E_\Pi{}^A E_\Sigma{}^B \delta_{AB}. \tag{2.13}$$

One can regard the 1-forms $E_\Pi{}^A\mathrm{d}z^\Pi$ as an orthonormal basis in our generalised tangent space.

The Cartan metric of g_l gives the generalised tangent space metric, equation (2.11). It can be thought of as the generalised metric of flat space with vanishing 3-form potential. Conjugating the Maurer-Cartan form by $\mathrm{e}^{h_i{}^j K^i{}_j}$ gives the vielbein for curved space and further conjugation by $\mathrm{e}^{\frac{1}{3!}C_{ijk}R^{ijk}}$ gives the dependence of the generalised vielbein on the 3-form potential.

We now find the result of conjugating $g_l^{-1}\mathrm{d}g_l$ by the group element corresponding to the K generator. The Maurer-Cartan form of g_l is

$$g_l^{-1}\mathrm{d}g_l = \mathrm{d}x^i P_i + \frac{1}{\sqrt{2}}\mathrm{d}x_{kl}Z^{kl}. \tag{2.14}$$

Using the Hadamard formula[4]

$$\mathrm{e}^X Y \mathrm{e}^{-X} = \mathrm{e}^{\mathrm{ad}X}Y,$$

[4]The adjoint map ad is defined by $(\mathrm{ad}^n X)Y = [X[X[X\ldots[X,Y]]]\ldots]$, where there are n commutators [49].

we can evaluate

$$
\begin{aligned}
\mathrm{e}^{-h_i{}^j K^i{}_j} \mathrm{d}x^m P_m \mathrm{e}^{h_k{}^l K^k_l} &= \mathrm{d}x^k \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} h_{i_1}{}^{j_1} \dots h_{i_n}{}^{j_n} [K^{i_n}{}_{j_n}, [\dots [K^{i_1}{}_{j_1}, P_k] \dots]], \\
&= \mathrm{d}x^i \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \frac{1}{5^m} \binom{n}{m} (\mathrm{tr}h)^m (h^{n-m})_i{}^j P_j, \\
&= \mathrm{d}x^i \sum_{m=0}^{\infty} \frac{1}{5^m m!} (\mathrm{tr}h)^m \sum_{n=0}^{\infty} \frac{1}{n!} (h^n)_i{}^j P_j, \\
&= \det(\mathrm{e}^h)^{1/5} (\mathrm{e}^h)_\mu{}^j \, \mathrm{d}x^\mu P_j,
\end{aligned} \tag{2.15}
$$

where in going to the second line we have used the first commutation relation in the line of equations labelled (2.8), and in the last equality we have used $\det(\mathrm{e}^h) = \mathrm{e}^{\mathrm{tr}(h)}$. We can identify e^h with the vielbein corresponding to usual spatial metric. In the last line, we have used a Greek letter as an index on $\mathrm{d}x$ because a distinction should be made between the index on the translation generator which should be thought of as a tangent space index, and the index on the $\mathrm{d}x$, which is a space index. Space is thus endowed with the metric

$$
g_{\mu\nu} = (\mathrm{e}^h)_\mu{}^i (\mathrm{e}^h)_\nu{}^j \delta_{ij}. \tag{2.16}
$$

The remaining term in the Maurer-Cartan form, (2.14), can be conjugated by the group element of the K generator in a similar way. For the $\mathrm{d}x_{kl} Z^{kl}$ term we can again use the Hadamard formula and find

$$
\begin{aligned}
&\mathrm{e}^{-h_i{}^j K^i{}_j} \mathrm{d}x_{mn} Z^{mn} \mathrm{e}^{h_k{}^l K^k_l} \\
&\quad = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}x_{mn} \left(\mathrm{ad}(hK)\right)^n Z^{mn}, \\
&\quad = \mathrm{d}x_{mn} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{m=0}^{n} \sum_{p=0}^{m} \binom{n}{m} \binom{m}{p} (h^p)_i{}^m (h^{m-p})_j{}^n \left(-\frac{1}{5}\mathrm{tr}h\right)^{n-m} Z^{ij}.
\end{aligned} \tag{2.17}
$$

The easiest way to prove the second equality is to use induction on n. We then interchange the summations in equation (2.17), taking care of the limits of the summations, to write the expression on the right-hand side as a product of three exponentials

$$
\mathrm{e}^{-h_i{}^j K^i{}_j} \mathrm{d}x_{mn} Z^{mn} \mathrm{e}^{h_k{}^l K^k_l} = \det(\mathrm{e}^h)^{1/5} (\mathrm{e}^{-h})_i{}^\mu (\mathrm{e}^{-h})_j{}^\nu \, \mathrm{d}x_{\mu\nu} Z^{ij}. \tag{2.18}
$$

As above, the indices on the translation generators are tangent space indices and the indices on the differential 2-form are space indices. $(\mathrm{e}^{-h})_i{}^\mu$ is the inverse vielbein corresponding to the metric g in equation (2.16).

We have constructed the generalised vielbein in a space with metric g. To find the dependence of the generalised vielbein, and metric, on the 3-form potential C, we will conjugate by the group element corresponding to the R^{ijk} generator. The commutation relations of the R^{ijk} generator with the translation generators are given in equations (2.9), from which it can be seen that R^{ijk} sends the translation generators into one another —

more precisely, P is sent to Z. The generator R_{ijk} has the opposite effect. Therefore, unlike before when conjugation by the group element corresponding to K leads to an infinite series, in this case the sum will truncate because the commutation relation of R^{ijk} and Z^{mn} vanishes. So there will only be a finite order dependence on the 3-form potential. We begin by conjugating the term proportional to P_i, (2.15),

$$\begin{aligned}
&\mathrm{e}^{-\frac{1}{3!}C_{j_1\cdots j_3}R^{j_1\cdots j_3}}\mathrm{e}^{-h_k{}^l K^k{}_l}\mathrm{d}x^i P_i \mathrm{e}^{h_k{}^l K^k{}_l}\mathrm{e}^{\frac{1}{3!}C_{j_1\cdots j_3}R^{j_1\cdots j_3}} \\
&\quad= \det(\mathrm{e}^h)^{1/5}(\mathrm{e}^h)_\mu{}^i\,\mathrm{d}x^\mu\left(P_i - \frac{1}{3!}C_{j_1\ldots j_3}[R^{j_1\ldots j_3},P_i]\right. \\
&\qquad\qquad\left. + \frac{1}{2}\frac{1}{(3!)^2}C_{j_1\ldots j_3}C_{k_1\ldots k_3}[R^{j_1\ldots j_3},[R^{k_1\ldots k_3},P_i]] + \ldots\right) \\
&\quad= \det(\mathrm{e}^h)^{1/5}(\mathrm{e}^h)_\mu{}^i\,\mathrm{d}x^\mu\left(P_i - \frac{1}{2}C_{ijk}Z^{jk}\right),
\end{aligned} \tag{2.19}$$

using commutation relations (2.9). As stressed earlier, the series truncates.

The conjugation of the term proportional to Z^{ij} is trivial because $[R^{ijk}, Z^{mn}] = 0$.

$$\begin{aligned}
g_h^{-1}g_l^{-1}\mathrm{d}g_l g_h &= \det(\mathrm{e}^h)^{1/5}(\mathrm{e}^h)_\mu{}^i\,\mathrm{d}x^\mu\left(P_i - \frac{1}{2}C_{ijk}Z^{jk}\right) \\
&\quad + \frac{1}{\sqrt{2}}\det(\mathrm{e}^h)^{1/5}(\mathrm{e}^{-h})_i{}^\mu(\mathrm{e}^{-h})_j{}^\nu\,\mathrm{d}x_{\mu\nu}Z^{ij}.
\end{aligned} \tag{2.20}$$

To find the generalised vielbein we need to compare the above expression with equation (2.12). Hence the generalised vielbein is

$$E_\Pi{}^A = (\det e)^{1/5}\begin{pmatrix} e_\mu{}^i & -\frac{1}{\sqrt{2}}e_\mu{}^j C_{ji_1i_2} \\ 0 & e^{\mu_1}{}_{[i_1}e^{\mu_2}{}_{i_2]}\end{pmatrix}. \tag{2.21}$$

Tangent space indices are written with Latin letters and Greek letters are spatial indices. We have also abbreviated the spatial vielbein e^h to e. The position of the indices on e indicate whether it is the spatial vielbein or inverse vielbein. If the spatial index is lowered, i.e. $e_\mu{}^i$, then this is the vielbein, and if the spatial index is raised, i.e. $e^\mu{}_i$, then this is the inverse vielbein.

Now from the generalised vielbein we can easily calculate the generalised metric, using equation (2.13),

$$M_{KL} = g^{1/5}\begin{pmatrix} g_{\mu\nu} + \frac{1}{2}C_\mu{}^{ij}C_{\nu ij} & -\frac{1}{\sqrt{2}}C_\mu{}^{\nu_1\nu_2} \\ -\frac{1}{\sqrt{2}}C^{\mu_1\mu_2}{}_\nu & g^{\mu_1\mu_2,\nu_1\nu_2}\end{pmatrix}, \tag{2.22}$$

where $g = (\det e)^2$ is the determinant of the metric $g_{\mu\nu}$. This is the same metric as in [36, 50, 51] except for the factor of $g^{1/5}$. This latter factor comes from the term proportional to δ^i_j in the commutation relations of $[K^i{}_j, P_k]$ and $[K^i{}_j, Z^{kl}]$, equation (2.8). The precise value of this coefficient was fixed by requiring that the SL(5) generator $M^I{}_J$ is traceless in equation (2.2). It is important to note that for this particular coefficient, i.e. power of g multiplying the metric, we obtain a generalised metric that does not describe the dynamical theory (see appendix B). In the next sections, we will consider the groups

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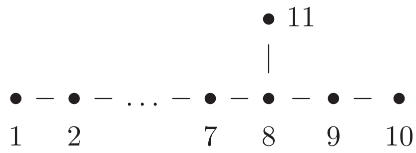

Figure 3. The E_{11} Dynkin diagram

governing generalised geometry as coming from E_{11}, in which case the factor of the term proportional δ^i_j in the commutators of K and P, Z is different. This results in a change in the factor multiplying the metric to $g^{-1/2}$, rather than $g^{1/5}$. The corresponding generalised metric can be used to construct the dynamics and naturally incorporates the measure in precisely the correct way.

We will now review the non-linear realisation of $E_{11} \ltimes l_1$ and find the generalised metrics for the SL(5), SO(5,5), E_6 and E_7 duality groups from the $E_{11} \ltimes l_1$ non-linear realisation.

3 A review of E_{11} and its first fundamental representation and their non-linear realisation

In this section, we will review previous work on the original E_{11} conjecture [1]: its application to ten [1, 22, 52] and lower dimensions [4, 23–26]; the development of E_{11} as an algebra [53, 54]; its first fundamental representation and its relation to brane charges [2–5, 31]; and finally the non-linear realisation of $E_{11} \ltimes l_1$ [2, 24, 31, 55, 56]. We collect together results that are found in different papers in a single place and we will take the opportunity to give a user friendly presentation. Some of this review is taken from the forthcoming book [57].

The E_{11} algebra consists of an infinite number of generators and, like all Kac-Moody algebras, it is completely determined by its Cartan matrix, or equivalently its Dynkin diagram given in figure 3.

Upon deleting the eleventh node of the E_{11} Dynkin diagram we find the Dynkin diagram for SL(11). We can therefore classify the generators of E_{11} in terms of this subalgebra, or to put it another way, we can decompose the adjoint representation of E_{11} into representations of SL(11). The resulting decomposition of E_{11} can be labelled in terms of a grading usually termed the level. Generators with non-negative levels are given, in increasing order, by [1, 54]

$$K^a{}_b(0), R^{a_1a_2a_3}(1), R^{a_1a_2\ldots a_6}(2), R^{a_1a_2\ldots a_8,b}(3), \ldots, \tag{3.1}$$

where $a, a_1, a_2, \ldots, b, \ldots = 1, 2, \ldots, 11$ and the number in brackets is the level of the respective generator. The last generator satisfies the constraint

$$R^{[a_1a_2\ldots a_8,b]} = 0.$$

Of course, the sequence does not terminate, reflecting the fact that E_{11} is infinite dimensional. The level zero generators $K^a{}_b$ obey the GL(11) algebra; the enlargement of SL(11) to GL(11) arises in the same way as the SL(5) case in section 2. The Cartan subalgebra generator associated with node eleven remains as part of the group even though that node in the Dynkin diagram has been deleted. The algebra of the GL(11) generators is given by

$$[K^a{}_b, K^c{}_d] = \delta_b^c K^a{}_d - \delta_d^a K^c{}_b. \tag{3.2}$$

The E_{11} algebra also contains an infinite number of generators of negative level which are partners of those with positive level but have their indices downstairs;

$$R_{a_1a_2a_3}(-1), R_{a_1a_2\ldots a_6}(-2), R_{a_1a_2\ldots a_8,b}(-3), \ldots, \tag{3.3}$$

with an identical constraint on the last generator. The generators of positive level are associated with negative roots in the Chevalley-Serre basis. Similarly, those of negative level are associated to positive roots. Those of zero level contain both positive and negative roots as well as the entire Cartan subalgebra.

By construction the generators of equations (3.1) and (3.3) belong to representations of GL(11) and so their commutators with the generators $K^a{}_b$ are as their index structure suggests. We list the commutators of the first few generators with K below [1]:

$$[K^a{}_b, R^{c_1\ldots c_3}] = 3\delta_b^{[c_1} R^{|a|c_2c_3]}, \tag{3.4}$$

$$[K^a{}_b, R_{c_1\ldots c_3}] = -3\delta^a_{[c_1} R_{|b|c_2c_3]}, \tag{3.5}$$

$$[K^a{}_b, R^{c_1\ldots c_6}] = 6\delta_b^{[c_1} R^{|a|c_2\ldots c_6]}, \tag{3.6}$$

$$[K^a{}_b, R_{c_1\ldots c_6}] = -6\delta^a_{[c_1} R_{|b|c_2\ldots c_6]}, \tag{3.7}$$

$$[K^a{}_b, R^{c_1\ldots c_8,d}] = 8\delta_b^{[c_1} R^{|a|c_2\ldots c_8],d} + \delta_b^d R^{c_1\ldots c_8,a}, \tag{3.8}$$

$$[K^a{}_b, R_{c_1\ldots c_8,d}] = -8\delta^a_{[c_1} R_{|b|c_2\ldots c_8],d} - \delta_d^a R_{c_1\ldots c_8,b}. \tag{3.9}$$

The commutators of E_{11} preserve the level, and it turns out that all the positive level generators can just be found from the multiple commutators of $K^a{}_b$ and $R^{a_1a_2a_3}$ and all the negative generators from the multiple commutators of $K^a{}_b$ and $R_{a_1a_2a_3}$. The commutators of some of the positive level generators are given by

$$[R^{c_1\ldots c_3}, R^{c_4\ldots c_6}] = 2R^{c_1\ldots c_6}, \qquad [R^{a_1\ldots a_6}, R^{b_1\ldots b_3}] = 3R^{a_1\ldots a_6[b_1b_2,b_3]}. \tag{3.10}$$

Similarly some of the commutators of the negative definite level generators are given by

$$[R_{c_1\ldots c_3}, R_{c_4\ldots c_6}] = 2R_{c_1\ldots c_6}, \qquad [R_{a_1\ldots a_6}, R_{b_1\ldots b_3}] = 3R_{a_1\ldots a_6[b_1b_2,b_3]}. \tag{3.11}$$

Finally, the commutation relations between the positive and negative generators of up to level three are given by [2]

$$[R^{a_1\ldots a_3}, R_{b_1\ldots b_3}] = 18\delta^{[a_1a_2}_{[b_1b_2} K^{a_3]}{}_{b_3]} - 2\delta^{a_1a_2a_3}_{b_1b_2b_3} D, \tag{3.12}$$

$$[R_{b_1\ldots b_3}, R^{a_1\ldots a_6}] = \frac{5!}{2}\delta^{[a_1a_2a_3}_{b_1b_2b_3} R^{a_4a_5a_6]}, \tag{3.13}$$

$$[R^{a_1\ldots a_6}, R_{b_1\ldots b_6}] = -5!.3.3\delta^{[a_1\ldots a_5}_{[b_1\ldots b_5} K^{a_6]}{}_{b_6]} + 5!\delta^{a_1\ldots a_6}_{b_1\ldots b_6} D, \tag{3.14}$$

$$[R_{a_1\ldots a_3}, R^{b_1\ldots b_8,c}] = 8.7.2(\delta^{[b_1b_2b_3}_{a_1a_2a_3} R^{b_4\ldots b_8]c} - \delta^{[b_1b_2|c|}_{a_1a_2a_3} R^{b_3\ldots b_8]}), \tag{3.15}$$

$$[R_{a_1\ldots a_6}, R^{b_1\ldots b_8,c}] = \frac{7!.2}{3}(\delta^{[b_1\ldots b_6}_{a_1\ldots a_6} R^{b_7b_8]c} - \delta^{c[b_1\ldots b_5}_{a_1\ldots a_6} R^{b_6b_7b_8]}), \tag{3.16}$$

where $D = \sum_b K^b{}_b$. There are similar formulae when higher or lower level generators are involved.

By examining the above commutators one can see that the level is nothing more that the number of times the generator $R^{a_1a_2a_3}$ minus the number of times the generator $R_{a_1a_2a_3}$ occurs. For the purposes of this paper this definition will suffice, but a precise description of the level is as follows. Each generator is associated with a root of E_{11}, which can be expressed as a sum of simple roots. Each node of the Dynkin diagram is associated with a simple root. The level refers to the GL(11) decomposition which picks out the eleventh node in figure 3. Associated to the eleventh node is the simple root α_{11}. The level is the coefficient of α_{11} when the root associated to that generator is written as the sum of simple roots.

For the purposes of this paper all that is required to know about the E_{11} algebra is the above commutation relations. The reader who is interested in a more detailed account of E_{11} from the definition of a Kac-Moody algebra may consult [1] and the later papers on E_{11} referenced in this paper. As we will see shortly, the non-linear realisation of the E_{11} algebra leads to the fields found in the massless bosonic sector of M-theory.

In early papers on E_{11}, in addition to the group element belonging to E_{11}, spacetime was introduced into the group element by including a factor of $e^{x^a P_a}$, where P_a are the generators of spacetime translations. The generators P_a were taken to have non-trivial commutators with the GL(11) generators $K^a{}_b$ of E_{11}, but trivial commutators with all the non-zero level generators. It was realised from the beginning that this was an ad hoc and incomplete step.

Later, it was proposed to incorporate spacetime by using a representation of E_{11} [2], which was denoted by the l_1 representation. This representation generalises the notion of spacetime translation generators. The l_1 representation, when decomposed into representations of SL(11), has the content [2–4]

$$\begin{aligned} L_A = \{&P_a, (0); Z^{a_1a_2}(1); Z^{a_1\ldots a_5}(2); Z^{a_1\ldots a_7,b}(3), Z^{a_1\ldots a_8}(3);\\ &Z^{a_1\ldots a_8,b_1b_2b_3}(4), Z^{a_1\ldots a_9,(bc)}(4), Z^{a_1\ldots a_9,b_1b_2}(4), Z^{a_1\ldots a_{10},b}(4), Z^{a_1\ldots a_{11}}(4);\\ &Z^{a_1\ldots a_9,b_1\ldots b_4}(5), Z^{a_1\ldots a_8,b_1\ldots b_6}(5), Z^{a_1\ldots a_9,b_1\ldots b_5}(5), \ldots\}. \end{aligned} \tag{3.17}$$

The numbers in brackets are the levels of the generators which just counts the number of times the generator R^{abc} acts on the highest value component in P_a. One sees that at the very lowest level it contains the spacetime translations P_a and then some generators that have the index structure to be the central charges in the eleven dimensional supersymmetry algebra as well as an infinite number of higher level objects. From the mathematical viewpoint, the l_1 representation has the highest weight Λ_1 which obeys the relations $(\Lambda_1, \alpha_a) = \delta_{a1}$ where α_a are the simple roots of E_{11}. This is just the fundamental representation associated with node one. The deduction of the above content, (3.17), from this definition is explained in [2–4].

At the lowest levels the l_1 representation contains objects that have the correct index structure to be the brane charges; that is P_a,Z^{ab},$Z^{a_1\ldots a_5}\ldots$ associated with the point particle, M2 brane and M5 brane, respectively. At level three $Z^{a_1\ldots a_7,b}$ probably represents the KK monopole (or D6-brane) charge. It has been conjectured that the l_1 representation contains all brane charges and there is now a substantial amount of evidence for this conjecture [2–5].

The generators of equations (3.1) and (3.3) correspond to the SL(11) decomposition of E_{11}, which is the one appropriate to the eleven dimensional theory. To find the theory in d dimensions we should carry out the decomposition of the adjoint representation of E_{11} into representations of the direct product of the duality group in d dimensions and GL(D), where $D = 11 - d$ [4, 23–26]. This can be found from equations (3.1) and (3.3) by simply carrying out the dimensional reduction by hand as will be done in this paper. Deleting the $D-$th node, for $D = 1, \ldots 8$, we obtain direct products of the duality groups $E_{10}, E_9, E_8, E_7, E_6$, SO(5,5), SL(5) and SL(2)×SL(3) with GL(D), respectively. The same decomposition is required for the l_1 representation and the results [4, 5, 31] are given in table 2. Some of the entries in the table agree with those previously found by taking an explicit charge and using U-duality to find the other members of the multiplet [58–60].

It was proposed [2] that the dynamics should be a non-linear realisation of semi-direct product of E_{11} and generators that belonged to the l_1 representation, the motion group of E_{11}; denoted by $E_{11} \ltimes l_1$. This algebra contains the generators of equations (3.1), (3.3) and those of equation (3.17) which we now take to be generators and call the generalised translation generators. The commutators for the low level generators of the l_1 representation with $R^{a_1a_2a_3}$ are determined up to constants by demanding that the levels match and so we can take [2]

$$[R^{a_1a_2a_3}, P_b] = 3\delta_b^{[a_1} Z^{a_2a_3]}, \tag{3.18}$$

$$[R^{a_1a_2a_3}, Z^{b_1b_2}] = Z^{a_1a_2a_3b_1b_2}, \tag{3.19}$$

$$[R^{a_1a_2a_3}, Z^{b_1\ldots b_5}] = Z^{b_1\ldots b_5[a_1a_2,a_3]} + Z^{b_1\ldots b_5a_1a_2a_3} \tag{3.20}$$

The normalisation of the generators is fixed by these relations, see appendix A for a detailed explanation. The commutators of the generalised translation generators with those of GL(11) are given by

$$[K^a{}_b, P_c] = -\delta_c^a P_b + \frac{1}{2}\delta_b^a P_c, \tag{3.21}$$

$$[K^a{}_b, Z^{c_1c_2}] = 2\delta_b^{[c_1} Z^{|a|c_2]} + \frac{1}{2}\delta_b^a Z^{c_1c_2}, \tag{3.22}$$

$$[K^a{}_b, Z^{c_1\ldots c_5}] = 5\delta_b^{[c_1} Z^{|a|c_2\ldots c_5]} + \frac{1}{2}\delta_b^a Z^{c_1\ldots c_5}. \tag{3.23}$$

Some of the remaining commutators are given by [2]

$$[R_{a_1a_2a_3}, P_b] = 0, \tag{3.24}$$

$$[R_{a_1a_2a_3}, Z^{b_1b_2}] = 6\delta_{[a_1a_2}^{b_1b_2} P_{a_3]}, \tag{3.25}$$

$$[R_{a_1a_2a_3}, Z^{b_1\ldots b_5}] = \frac{5!}{2}\delta_{a_1a_2a_3}^{[b_1b_2b_3} Z^{b_4b_5]}, \tag{3.26}$$

$$[R_{a_1a_2a_3}, Z^{b_1\ldots b_7,d}] = 378\delta_{a_1a_2a_3}^{d[b_1b_2} Z^{b_3\ldots b_7]}, \tag{3.27}$$

D	G	Z	Z^a	$Z^{a_1a_2}$	$Z^{a_1\ldots a_3}$	$Z^{a_1\ldots a_4}$	$Z^{a_1\ldots a_5}$	$Z^{a_1\ldots a_6}$	$Z^{a_1\ldots a_7}$
8	$SL(3)\otimes SL(2)$	$(\mathbf{3},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{1},\mathbf{3})$ $(\mathbf{8},\mathbf{1})$ $(\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$ $(\mathbf{6},\mathbf{2})$	$(\mathbf{6},\mathbf{1})$ $(\mathbf{18},\mathbf{1})$ $(\mathbf{3},\mathbf{1})$ $(\mathbf{6},\mathbf{1})$ $(\mathbf{3},\mathbf{3})$
7	$SL(5)$	**10**	$\overline{\mathbf{5}}$	**5**	$\overline{\mathbf{10}}$	**24** **1**	**40** **15** **10**	**70** **50** **45** **5**	— — — —
6	$SO(5,5)$	$\overline{\mathbf{16}}$	**10**	**16**	**45** **1**	$\overline{\mathbf{144}}$ **16**	**320** **126** **120**	— — —	— — —
5	E_6	$\overline{\mathbf{27}}$	**27**	**78** **1**	$\overline{\mathbf{351}}$ $\overline{\mathbf{27}}$	**1728** **351** **27**	— — —	— — —	— — —
4	E_7	**56**	**133** **1**	**912** **56**	**8645** **1539** **133** **1**	— — — —	— — — —	— — — —	— — — —
3	E_8	**248** **1**	**3875** **248** **1**	**147250** **30380** **3875** **248** **1**	— — — — —	— — — — —	— — — — —	— — — — —	— — — — —

Table 2. Table giving the representations of the symmetry group G of the form charges in the l multiplet up to and including rank $D-1$ in $D=8$ dimensions and below [4, 5, 31].

$$[R^{a_1\ldots a_6}, P_b] = -3\delta_b^{[a_1} Z^{\ldots a_6]}, \tag{3.28}$$

$$[R^{a_1\ldots a_6}, Z^{b_1b_2}] = -Z^{b_1b_2[a_1\ldots a_5,a_6]} - Z^{b_1b_2a_1\ldots a_6}. \tag{3.29}$$

These commutators can be largely determined by demanding that the level is preserved and that the Jacobi identities hold. The factor of $\frac{1}{2}$ in the terms proportional to δ_b^a in equations (3.21)–(3.23) are fixed by the Jacobi identities once it is found to be present in the first equation, (3.21). These terms follow from the fact that the l_1 representation is a highest weight representation of E_{11}. If one considers the analogous representation for subalgebras such as E_{10}, or even the finite dimensional E_n series one finds factors other than $\frac{1}{2}$. Indeed the corresponding factor for E_n is $\frac{1}{n-9}$ $(n \neq 9)$. E_9 is an exception because it's an affine algebra, so its Cartan matrix has vanishing determinant.

To carry out explicit computations of the $E_{11} \ltimes l_1$ non-linear realisation at low levels, one only needs the above commutators and one does not have to absorb the general theory of Kac-Moody algebras.

As explained in the introduction the non-linear realisation we are using here is not a sigma model as the l_1 representation are generators associated with spacetime and they introduce the coordinates of the generalised spacetime into the theory. How to construct such non-linear realisations is illustrated by example in [1, 18] and many of the later papers on E_{11} even though only the generators of spacetime translations P_a are used. The non-linear realisation of $E_{11} \ltimes l_1$ was used in [24] to construct all five dimensional gauged supergravities and in [55] and [56] to construct the IIA ten dimensional supergravity in the NS-NS and R-R sectors respectively. The next section uses it to construct the four dimensional theory at level zero but keeping only the scalar coordinates in the ten of SL(5). That section may be read at the same time as the abstract material below. The material in this section may also be compared with section 2 which covers the SL(5) case without the complications of the $E_{11} \ltimes l_1$ algebra.

The non-linear realisation is built from the group element

$$g = g_l g_E. \tag{3.30}$$

In eleven dimensions the group element g_E takes the form

$$g_E = \dots e^{\frac{1}{6!}C^{a_1\dots a_6}R_{a_1\dots a_6}} e^{\frac{1}{3!}C^{a_1a_2a_3}R_{a_1a_2a_3}} e^{h_a{}^bK^a{}_b} e^{\frac{1}{3!}C_{a_1a_2a_3}R^{a_1a_2a_3}} e^{\frac{1}{6!}C_{a_1\dots a_6}R^{a_1\dots a_6}} \dots . \tag{3.31}$$

Using equation (3.17), the group element g_l, in eleven dimensions, takes the form

$$g_l = e^{x^aP_a} e^{\frac{1}{\sqrt{2}}x_{ab}Z^{ab}} e^{\frac{1}{\sqrt{5}}x_{a_1\dots a_5}Z^{a_1\dots a_5}} \dots . \tag{3.32}$$

The precise choice of the normalisation is explained in appendix A.

Thus the non-linear realisation of $E_{11} \ltimes l_1$ introduces a generalised spacetime with coordinates [2, 24, 31]

$$z^\Pi = \{x^a, ; x_{a_1a_2}; x_{a_1\dots a_5}; x_{a_1\dots a_7,b}, x_{a_1\dots a_8}; x_{a_1\dots a_8,b_1b_2b_3}, x_{a_1\dots a_9,(bc)}, x_{a_1\dots a_9,b_1b_2}, \\ x_{a_1\dots a_{10},b}, x_{a_1\dots a_{11}}; x_{a_1\dots a_9,b_1\dots b_4,c}, x_{a_1\dots a_8,b_1\dots b_6}, x_{a_1\dots a_9,b_1\dots b_5}, \dots\}, \tag{3.33}$$

where the first coordinate x^a is the coordinate of the spacetime we are so used to. However the multiplet contains an infinite number of additional coordinates. As a result of the way they have arisen, there is a one to one correspondence between the generators of equation (3.17) and the coordinates of equation (3.33) so that each coordinate is automatically associated with a brane charge. In particular, the usual coordinates x^a are associated with the generators P_a of spacetime translations, the coordinates x_{ab} with the charge Z^{ab} of the M2 brane and so on. One can show [3] that for every field there is a corresponding

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brane charge, for example ${h_a}^b, C_{a_1a_2a_3}, \ldots$ correspond to $P_a, Z^{ab}, \ldots$, respectively. As a result every field now has a corresponding coordinate associated with it; we can think of the usual spacetime coordinates x^a as being associated with the metric, the coordinates x_{ab} as associated with the three form field $C_{a_1a_2a_3}$, etc. Thus this construction generalises spacetime to take account of the objects within it. Einstein's theory corresponds to the lowest level. We take the fields ${h_a}^b, C_{a_1a_2a_3}, \ldots$ to depend on all of the coordinates x^a, x_{ab} etc. Introducing the generator P_a on its own, as mentioned above, is just the lowest order approximation.

The group element in lower dimensions is easily written down using the generators of E_{11} as decomposed into representations of $GL(D) \otimes E_d$ where $D = 11 - d$. As mentioned above, elements of l_1 are given in table 2. We find, in table 2, the scalar, vector, and higher rank generators in $D-$dimensions contained in the l_1 representation. In particular, we find that the scalar charges in the l_1 representation in $d = 4, 5, 6, 7$ dimensions belong to the 10, $\overline{16}$, $\overline{27}$ and 56 representations of SL(5), SO(5,5), E_6 and E_7 respectively [4, 31]. In this paper we will be interested in the non-linear realisation at level zero with respect to the deleted node. With this restriction only a finite number of fields and coordinates will remain.

A non-linear realisation is specified by a choice of algebra and subalgebra, called the local subalgebra. In our case the algebra is E_{11} and we will denote the local subalgebra by $I(E_{11})$. By definition, the non-linear realisation is just a dynamics which is invariant under the transformations

$$g \to g_0 g, \;\; g_0 \in E_{11} \ltimes l_1, \quad \text{and} \quad g \to gh, \;\; h \in I(E_{11}) \tag{3.34}$$

In this equation g_0 is a rigid transformation, and so does not depend on the generalised spacetime, while h is a local transformation which does depend on the generalised spacetime. The local subalgebra $I(E_{11})$ is taken to be a maximal subalgebra that is invariant under Cartan involution. This subalgebra of E_{11} is generated by

$$\begin{aligned} {K^a}_b - \eta_{bc}\eta^{ad}{K^c}_d, \qquad\qquad & R^{a_1a_2a_3} - \eta^{a_1b_1}\eta^{a_2b_2}\eta^{a_3b_3}R_{b_1b_2b_3}, \\ & R^{a_1a_2\ldots a_6} + \eta^{a_1b_1}\ldots\eta^{a_6b_6}R_{b_1b_2\ldots b_6}, \ldots, \end{aligned} \tag{3.35}$$

where η is the Minkowski metric. The Cartan involution invariant subgroups of the groups SL(n), SO(n,n), E_6 and E_7 are their maximally compact subgroups, which are SO(n), $SO(n) \otimes SO(n)$, USp(8) and SU(8) respectively, provided the d dimensions are all spacelike. Hence at the lowest level the local subalgebra is just the Lorentz group. We may therefore use the local transformation of equation (3.34) to bring the group element g_E in eleven dimensions into the form

$$g_E = e^{{h_a}^b {K^a}_b} e^{\frac{1}{3!}C_{a_1a_2a_3}R^{a_1a_2a_3}} \ldots . \tag{3.36}$$

This mostly contains the generators of the Borel subalgebra of E_{11} which are the generators given in equation (3.1). The exception is the field at level zero, i.e. ${h_a}^b$ where we have chosen not to fix all of the local Lorentz group. The Cartan involution I takes, up to a sign, a generator with a positive level to a generator with a negative level and with the same set of indices but downstairs, that is it takes a contravariant to a contragredient SL(11) representation. More technically it takes a generator with a positive root α to a generator with the negative root $-\alpha$, for example $I(R^{a_1\ldots a_3}) = -R_{a_1\ldots a_3}$ and $I(R^{a_1\ldots a_6}) = R_{a_1\ldots a_6}$. Furthermore, it maps the generators of the Cartan subalgebra into themselves. For a more formal definition see, for example, [1] and many later papers on E_{11} .

As we explained to find the non-linear realisation in eleven dimensions we delete node eleven and decompose $E_{11} \ltimes l_1$. At level zero this algebra becomes $GL(11) \ltimes P_\mu$ where P_μ are just the usual spacetime translations in eleven dimensions. At level zero $I(E_{11})$ is just the Lorentz group. Thus in this case the generalised spacetime has the coordinates x^a and so is just our familiar spacetime. The only fields are ${h_a}^b$. In fact the non-linear realisation, after the adjustment of a few constants that are not determined, leads to eleven dimensional gravity. It turns out that e^h viewed as a matrix is just the vielbein [16] just as was shown in section 2. In what follows it will be useful to recall that the non-linear realisation of the semi-direct product of $GL(d)$ and spacetime translations leads to d-dimensional gravity, as was shown long ago for the case of four dimensions [16].

Under a rigid $g_0 \in E_{11}$ and a local $H \in I(E_{11})$, the different parts of the group element transform as

$$g_l \to g_0 g_l (g_0)^{-1}, \qquad \text{and} \qquad g_E \to g_0 g_E \tag{3.37}$$

$$g_l \to g_l, \qquad \text{and} \qquad g_E \to g_E h, \tag{3.38}$$

respectively, as the l_1 generators form a realisation of E_{11}. As a result the coordinates transform under G as

$$z^\Pi L_\Pi \to g_0 z^\Pi L_\Pi (g_0)^{-1} \tag{3.39}$$

To give a more concrete meaning to the above rigid transformations we will carry them out for $g_0 = e^{\frac{1}{3!}a_{a_1a_2a_3}R^{a_1a_2a_3}}$ where $a_{a_1a_2a_3}$ is a constant parameter. Using equation (3.37) and equations (3.32) and (3.36), we find that

$$\begin{aligned} &\delta x^a = 0, \qquad \delta x_{ab} = \frac{1}{\sqrt{2}} a_{abc} x^c, \qquad \delta {h_a}^b = 0, \\ &\delta C_{a_1a_2a_3} = a_{a_1a_2a_3} - 3 a_{b[a_1a_2} {h_{a_3]}}^b, \qquad \delta C_{a_1\ldots a_6} = 0. \end{aligned} \tag{3.40}$$

To construct the dynamics from the non-linear realisation, it is usual to first construct the Cartan form. The Cartan form belongs to the Lie algebra and so in our case the algebra $E_{11} \ltimes l_1$. As such, it can be written as

$$\mathcal{V} \equiv g^{-1} dg = dz^\Pi {E_\Pi}^A L_A + dz^\Pi G_{\Pi,*} R^* \tag{3.41}$$

where L_A are the generators of the l_1 representation and R^* are the generators of E_{11} in equation (3.1) and (3.36) with $*$ denoting the appropriate set of indices. When we write the sums involving the L_A generators we are including the square root of the combinatorial factors that occur in the group element in equation (3.32). Since the generators L_A form a representation of E_{11}, the Cartan form is given by

$$\mathcal{V} = g_E^{-1} dz^A L_A g_E + g_E^{-1} dg_E \tag{3.42}$$

where we have assumed that the generators L_A mutually commute. We may write

$$dz^\Pi E_\Pi{}^A L_A = g_E^{-1} dz^A L_A g_E = dz^T \cdot E \cdot L \tag{3.43}$$

where in the last line we have used an obvious matrix notation in that the matrix E has the elements $E_\Pi{}^A$. The remaining part of the Cartan form is given by

$$dz^\Pi G_{\Pi,*} R^* = g_E^{-1} dg_E \tag{3.44}$$

and it is just the Cartan form of E_{11}.

The Cartan form (3.41) is obviously inert under the rigid g_0 transformations of equation (3.34). Note that the generators of the l_1 representation can carry either a Π or an A index depending on the context; this is not a change carried out with the vielbein and $L_A = L_\Pi \delta_A^\Pi$. As the l_1 generators form a representation of E_{11} it follows that $dz^\Pi E_\Pi{}^A$ and $dz^\Pi G_{\Pi,*}$ are separately invariant under these rigid transformations. However, the coordinates, and so dz^Π, do transform under g_0 and as a result $E_\Pi{}^A$ and $G_{\Pi,*}$ are not invariant under g_0 transformations.

Under a local transformation $g \to gh$ of equation (3.41) the Cartan forms transform as $\mathcal{V} \to h^{-1}\mathcal{V}h + h^{-1}dh$. To find quantities that only transform under the local subalgebra we can rewrite $\mathcal{V}$ as

$$\mathcal{V} = g^{-1}dg = dz^\Pi E_\Pi{}^A (L_A + G_{A,*} R^*) \tag{3.45}$$

where we recognise that $G_{A,*} = (E^{-1})_A{}^\Pi G_{\Pi,*}$. Since $dz^\Pi E_\Pi{}^A$ and $R^\star$ are inert under rigid g_0 transformations, it follows that $G_{A,*}$ are also inert under g_0 transformations and just transform under local transformations. As such they are useful quantities with which to construct the dynamics as one need only solve the problem of finding objects which are invariant under the local symmetry. For objects in the coset directions of $\mathcal{V}$, the $h^{-1}dh$ terms in the transformation of equation (3.41) are absent and we may think of $G_{A,*}$ as transforming covariantly — in effect they are the covariant derivatives of the fields. Thus working with the Cartan forms one only has to solve the problem of find invariants under the local transformations h. In fact the situation is a little more subtle as we have used the local subgroup to choose our group element to belong to the Borel subgroup and then a g_0 transformation requires a local compensating transformation. However, as the final dynamics is invariant under local transformations these are automatically taken care of.

Since the generators of the l_1 representation transform as a representation of E_{11} we can write equation (3.39) as

$$z^\Pi L_\Pi \to g_0 z^\Pi L_\Pi (g_0)^{-1} = z^\Pi D(g_0^{-1})_\Pi{}^\Lambda L_\Lambda \tag{3.46}$$

where $D(g_0^{-1})_\Pi{}^\Lambda$ is the corresponding matrix representation. More formally we can define the action of the l_1 representation of E_{11}, to which the generators L_Π belong, by

$$U(k)(L_\Pi) \equiv k^{-1} L_\Pi k = D(k)_\Pi{}^\Lambda L_\Lambda,$$

where $k \in E_{11}$. As a result we find that in matrix notation

$$dz^T \to dz^{T\prime} = dz^T D(g_0^{-1}),$$

or putting in the indices

$$dz^{\Pi} \to dz^{\Pi\prime} = dz^{\Lambda} D(g_0^{-1})_{\Lambda}{}^{\Pi}. \tag{3.47}$$

Consequently, the derivative $\partial_\Pi = \frac{\partial}{\partial z^\Pi}$ in the generalised spacetime transforms as

$$\partial'_\Pi = D(g_0)_\Pi{}^\Lambda \partial_\Lambda .$$

Examining equation (3.43), we note that the generalised vielbein E in matrix form is given by

$$E_\Pi{}^\Lambda = D(g_E)_\Pi{}^\Lambda . \tag{3.48}$$

As the Cartan form is inert under rigid transformations, its action on the coordinates must be compensated by a corresponding change on the lower index of E, using equation (3.47), we find this to be given by $E_\Pi{}^{A\prime} = D(g_0)_\Pi{}^\Lambda E_\Lambda{}^A$. Thus the lower index is a world index, while $E_\Pi{}^{A\prime}$ transforms on its upper index by a local h transformation and so we can think of the upper index as a tangent index. Consequently, we can think of $E_\Pi{}^A$ as a generalised vielbein which controls the geometry of the generalised spacetime.

In almost all the E_{11} papers the dynamics has been constructed using the Cartan forms. However, one can also proceed in another way and this was done in [36, 55] which we now follow. Let us define

$$M \equiv g_E I_c(g_E^{-1}), \tag{3.49}$$

where I_c is the Cartan involution. It is easy to see, using equation (3.38) that M is inert under local transformations as by definition $I_c(h) = h$. However, under a rigid transformation $M \to M' = g_0 M I_c(g_0^{-1})$ under rigid transformations. Using $E = D(g_E)$, we find that M in the l_1 representation is given by

$$D(M) = D(g_E) D(I_c(g_E^{-1})) = E E^{\#} \tag{3.50}$$

where $E^{\#} = I_c(D(g_E^{-1}))$, which for many groups it is just the transpose. Writing out the indices explicitly we find that

$$D(M)_{\Pi\Lambda} = (EE^{\#})_{\Pi\Lambda} \tag{3.51}$$

and we can write its rigid transformation as

$$D(M)_{\Pi\Lambda} \to D(g_0^{-1})_\Lambda{}^\Gamma D(M)_{\Gamma\Theta} D(I_c(g_0^{-1}))_\Pi{}^\Theta . \tag{3.52}$$

Using this method, the problem of finding invariants reduces to constructing g_0 invariants from M. In the subsequent sections we will carry this procedure out in detail for the various dimensions. If we restrict ourselves to two spacetime derivatives then the most general invariant Lagrangian up to boundary terms is given by [36, 55]

$$\begin{aligned} L = {} & c_1 M^{ST} \partial_S M^{PQ} \partial_T M_{PQ} + c_2 M^{ST} \partial_S M^{PQ} \partial_P M_{TQ} \\ & + c_3 M^{MN} M^{ST} (M^{PQ} \partial_S M_{PQ})(\partial_M M_{NT}) \\ & + c_4 M^{ST} (M^{MN} \partial_S M_{MN})(M^{PQ} \partial_T M_{PQ}) + c_5 M_{RQ} \partial_S M^{SR} \partial_P M^{PQ} \end{aligned} \tag{3.53}$$

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where $c_1, \ldots, c_5$ are constants, and M^{ST} denotes $(M^{-1})^{ST}$. The term with coefficient c_5 never gives rise to a U(1) gauge-invariant result. One can therefore set c_5 equal to zero with impunity. Boundary terms may be included in terms of the generalised metric [65].

The non-linear realisation introduces the generalised spacetime, but since it also specifies the dynamics, at least up to a few constants, it also determines the geometry of the generalised spacetime. However, it is important to understand that the non-linear realisation as described above, and used in the papers on E_{11}, is not what is usually described as a sigma model. The latter corresponds to a non-linear realisation in which the group contains no generators associated with any spacetime. As a result the coordinates are introduced by hand and act as dummy variables upon which the fields depend. In contrast the non-linear realisation described here has generators which lead to the introduction of spacetime into the group element and so the generalised spacetime plays a central role in the way the dynamics is formulated.

We note that the conjectured theory based on E_{10} [61–64] is quite different. It uses a non-linear realisation that is equivalent to that which is usually known as a sigma model. In this formulation the fields only depend on time and it is hoped that spacetime will emerge at higher levels in the algebra.

The Lagrangian of equation (3.53) contains five undetermined constants and, since it is to be integrated over a generalised spacetime, it is of a rather unfamiliar form. One may like to find a theory that contains only the spacetime that is familiar to us. Although up to this stage the procedure has been very systematic, how to proceed further is not completely clear. One approach used in the non-linear realisation of $GL(D) \ltimes I^4$ to find gravity [16] is to demand some extra symmetries such as conformal symmetry. This step, taken together with the original non-linear realisation is equivalent to demanding general coordinate invariance. This procedure was also followed in the E_{11} approach [1, 18] and many subsequent papers. This procedure has been generalised in the work of [38] which considered the non-linear realisation of $E_{11} \ltimes l_1$ applied to seven dimensions. In [38] the field dependence on the resulting generalised spacetime was restricted to be only over the usual coordinates of spacetime and then the action was required to be invariant under general coordinate invariance and gauge symmetries. This is the strategy we will adopt here. One finds in all known cases that one can adjust the constants so that this is possible.

4 Four dimensions: SL(5) revisited

In this section, we carry out the non-linear realisation of $E_{11} \ltimes l_1$ appropriate to four dimensions at the lowest level. That is we will systematically carry out the method given in the previous section applied to this case. To find the four-dimensional theory we delete node seven of the E_{11} Dynkin diagram to leave the algebra $GL(7) \otimes SL(5)$, see figure 4, and decompose $E_{11} \ltimes l_1$ into this subalgebra. The subalgebra SL(5) is the well known duality group in the reduction to seven dimensions.

In this paper we are interested in the lowest level result. The simplest way to find the low level algebra is to carry out by hand the dimensional reduction on the generators of $E_{11} \ltimes l_1$ given in equations (3.1), (3.3) and (3.17). Letting $i, j, \cdots = 1, 2, 3, 4$ be the

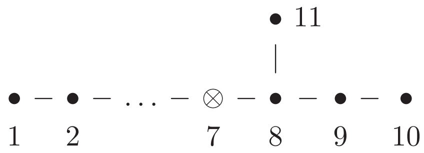

Figure 4. The E_{11} Dynkin diagram appropriate to four dimensions

indices corresponding to the four dimensions we find that the only generators of E_{11}, (3.1) and (3.3), that remain are

$$K^i{}_j, R^{i_1 i_2 i_3}, R_{i_1 i_2 i_3} \quad \text{and} \quad K^a{}_b,\ a, b = 1, 2 \ldots 7 \tag{4.1}$$

of GL(7). We are using the convention that $i, j, k, \ldots$ are tangent indices in the four dimensional space and $a, b, c, \ldots$ are tangent indices in the seven dimensional space. The generators listed in (4.1) have level zero. We observe that the level zero generators have no mixed indices. For the decomposition corresponding to deleting node seven, the generators $K^i{}_a$ ($K^a{}_i$) have level 1 (-1) and multiple commutators of these generators together with the above generators at level zero will lead to all of the E_{11} Kac-Moody algebra. More technically a generator has level n if its corresponding root, when expressed in terms of simple roots, contains the simple root α_7 with factor n.

Keeping only level zero generators we find, using equations (3.2), (3.4), (3.5) and (3.12), that the generators of equation (4.1) obey the algebra

$$[K^i{}_j, K^k{}_l] = \delta^k_j K^i{}_l - \delta^i_l K^k{}_j,$$
$$[K^i{}_j, R^{k_1 k_2 k_3}] = 3\delta^{[k_1}_j R^{|i| k_2 k_3]},$$
$$[K^i{}_j, R_{k_1 k_2 k_3}] = -3\delta^i_{[k_1} R_{|j| k_2 k_3]},$$
$$[R^{i_1 i_2 i_3}, R_{j_1 j_2 j_3}] = 18\delta^{[i_1 i_2}_{[j_1 j_2} K^{i_3]}{}_{j_3]} - 2\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3}\left(\sum_j K^j{}_j + \sum_a K^a{}_a\right);$$
$$[K^a{}_b, K^d{}_c] = \delta^d_b K^a{}_c - \delta^a_c K^d{}_b$$

with all remaining commutators being zero. To see that this really is the algebra $GL(7) \otimes SL(5)$ we should redefine the generators of SL(4) to be $\tilde{K}^i{}_j = K^i{}_j - \frac{1}{5}\delta^i_j \sum_a K^a{}_a$ and then the generators $\tilde{K}^i{}_j, R^{i_1 i_2 i_3}$ and $R_{i_1 i_2 i_3}$ generate SL(5). The generators $K^a{}_b,\ a, b = 1, 2 \ldots 7$ obey the algebra of GL(7) and commute with those of SL(5).

The generators of SL(5) are contained in the generators $M^I{}_J, I, J = 1 \ldots, 5$, the identification with those above being

$$M^I{}_J = \begin{cases} M^i{}_j = \tilde{K}^i{}_j - \dfrac{1}{3}\displaystyle\sum_k \tilde{K}^k{}_k, & i, j = 1, \ldots, 4 \\ M^i{}_5 = \dfrac{1}{3!}\epsilon^{i j_1 j_2 j_3} R_{j_1 j_2 j_3} & j_1, j_2, j_3 = 1, \ldots, 4 \\ M^5{}_i = \dfrac{1}{3!}\epsilon_{i j_1 j_2 j_3} R^{j_1 j_2 j_3} & j_1, j_2, j_3 = 1, \ldots, 4 \end{cases} \tag{4.2}$$

whereupon we find the standard algebra of SL(5), namely

$$[M^I{}_J, M^K{}_L] = \delta^K_J M^I{}_L - \delta^I_L M^K{}_J. \tag{4.3}$$

Since the SL(5) generators $M^I{}_J$ are traceless we have defined $M^5{}_5 = -\sum_{i=1} M^i{}_i$.

We now consider the l_1 representation at lowest level. Carrying out the dimensional reduction on equation (3.17) we find that it contains

$$P_i, Z^{ij}, \quad i,j = 1,2,3,4 \qquad \text{and} \qquad P_a, \quad a = 1,2\ldots 7. \tag{4.4}$$

The commutators of the generators of equation (4.4) are found using equations (3.18), (3.19), (3.21), (3.22), (3.24), (3.25) to be

$$[K^i{}_j, P_l] = -\delta^i_l P_j + \frac{1}{2}\delta^i_j P_l, \tag{4.5}$$

$$[K^i{}_j, Z^{kl}] = 2\delta^{[k}_j Z^{|i|l]} + \frac{1}{2}\delta^i_j Z^{kl}, \tag{4.6}$$

$$[R^{i_1 i_2 i_3}, P_j] = 3\delta^{[i_1}_j Z^{i_2 i_3]}, \tag{4.7}$$

$$[R^{i_1 i_2 i_3}, Z^{kl}] = 0, \tag{4.8}$$

$$[R_{i_1 i_2 i_3}, P_j] = 0, \tag{4.9}$$

$$[R_{i_1 i_2 i_3}, Z^{jk}] = 6\delta^{jk}_{[i_1 i_2} P_{i_3]}; \tag{4.10}$$

$$[K^a{}_b, P_c] = -\delta^a_c P_b + \frac{1}{2}\delta^a_b P_c \tag{4.11}$$

as well as

$$[K^a{}_b, P_l] = \frac{1}{2}\delta^a_b P_l, \qquad [K^a{}_b, Z^{ij}] = \frac{1}{2}\delta^a_b Z^{ij}, \qquad [K^i{}_j, P_a] = \frac{1}{2}\delta^i_j P_a. \tag{4.12}$$

All the remaining commutators are zero. We also take all the generators in the l_1 representation to commute with themselves.

We can package the generators of equation (4.4) with $i,j,\ldots$ indices into $P_{IJ} = -P_{JI}$, $I,J = 1,\ldots,5$, where

$$P_{IJ} = \begin{cases} P_{i5} = P_i & i = 1,\ldots,4 \\ P_{ij} = \frac{1}{2}\epsilon_{ijkl} Z^{kl} & i,j,k,l = 1,\ldots,4 \end{cases}. \tag{4.13}$$

Using equations (4.5)–(4.10), the commutator of P_{IJ} with the generators of SL(5) can be written as

$$[M^I{}_J, P_{LM}] = -\delta^I_L P_{JM} - \delta^I_M P_{LJ} + \frac{2}{5}\delta^I_J P_{LM}. \tag{4.14}$$

We recognise that the generators P_{LM} belong to the 10-dimensional representation of SL(5). Furthermore, one finds that $[M^I{}_J, P_a] = 0$, hence the P_a are SL(5) singlets, but transform as the 7-dimensional representation of GL(7). This is very similar to what is done in section 2. The difference being that here the algebra is derived from E_{11}. We find that it

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includes the extra seven dimensions of spacetime, and some numerical factors in the algebra are different. In particular, comparing the equations in (2.8) to equations (4.5) and (4.6), the coefficient of the terms proportional to δ^i_j are different, $-1/5$ and $1/2$ respectively.

At level zero the non-linear realisation of $E_{11} \ltimes l_1$ reduces to the non-linear realisation of $(GL(7) \otimes SL(5)) \ltimes (P_a \oplus P_{IJ})$. The local subalgebra is generated by $K^a{}_b - \eta^{ad}\eta_{bc}K^c{}_d$ and $K^i{}_j - K^j{}_i$ and $R^{ijk} - R_{ijk}$ respectively. The use of the Minkowski metric η_{ab} to define the local subalgebra leads to the subgroup SO(1,6) rather than SO(7). Thus the local subalgebra is $SO(1,6) \otimes SO(5)$. In fact SO(7) and SO(5) are the standard Cartan involution invariant subalgebras of GL(7) and SL(5) and using the Minkowski metric for the first group results from using a slightly different Cartan involution. The non-linear realisation is built from the group element $g_l g_E$ of equation (3.30) now restricted to level zero. Taking into account the local symmetry, the $GL(7) \otimes SL(5)$ part of the group element can be written as

$$g_E^{(0)} = e^{h_i{}^j K^i{}_j} e^{\frac{1}{3!}C_{i_1 i_2 i_3} R^{i_1 i_2 i_3}} e^{\hat{h}_a{}^b K^a{}_b}. \tag{4.15}$$

The superscript 0 just indicates we are at level zero. Hence we find that the non-linear realisation introduces the fields

$$h_i{}^j, C_{i_1 i_2 i_3}, \quad \text{and} \quad \hat{h}_a{}^b. \tag{4.16}$$

We note that the field $C_{i_1 i_2 i_3}$ was always denoted as $A_{i_1 i_2 i_3}$ in the previous literature on E_{11}.

The part of the group element arising from the l_1 representation is given by

$$g_l^{(0)} = e^{x^i P_i + \frac{1}{\sqrt{2}} x_{ij} Z^{ij}} e^{x^a P_a}. \tag{4.17}$$

As such we see that the $E_{11} \ltimes l_1$ non-linear realisation at level zero introduces a generalised spacetime with the coordinates

$$x^i, x_{ij} \quad \text{and} \quad x^a, \quad a = 1, \ldots, 7. \tag{4.18}$$

The last coordinates are just the usual seven dimensional spacetime and belong to the $\overline{7}$-dimensional representation of GL(7). The first set of coordinates of equation (4.18) are associated with the spacetime translation and the membrane charges, respectively, and transform as a $\overline{10}$ of SL(5); we could write them as $X^{IJ}, I, J = 1, 2, \ldots, 5$. The fields of equation (4.16) are taken to depend on the coordinates of equation (4.18). Thus at lowest level the non-linear realisation involves the group element

$$g^{(0)} = g_l^{(0)} g_E^{(0)} = e^{x^i P_i + \frac{1}{\sqrt{2}} x_{ij} Z^{ij}} e^{x^a P_a} e^{h_i{}^j K^i{}_j} e^{\frac{1}{3!}C_{i_1 i_2 i_3} R^{i_1 i_2 i_3}} e^{\hat{h}_a{}^b K^a{}_b}. \tag{4.19}$$

If would be interesting to construct this non-linear realisation; one would find gravity in seven dimensions coupled to a part that is the non-linear realisation of $SL(5) \ltimes P_{IJ}$. However, in this paper we will consider a simplified non-linear realisation. In a future paper, we will discuss how the other components of the metric and C appear in the non-linear realisation and the action. We note that the generators of SL(5) commute with those of GL(7) and the seven dimensional spacetime translations, i.e with $IGL(7) = GL(7) \ltimes \{P_a\}$. Indeed the only non-trivial commutator between $SL(5) \ltimes \{P_i, Z^{ij}\}$ and IGL(7) is that of the generators of GL(7) which scale the $\{P_i, Z^{ij}\}$ generators by a $\frac{1}{2}$ factor. As such the $SL(5) \ltimes \{P_i, Z^{ij}\}$ transformations of the non-linear realisation do not affect the parts of the group element belonging to IGL(7), that is they do not affect the spacetime coordinate x^a of the gravity field $\hat{h}_a{}^b$. As such it is consistent to set the IGL(7) part of the non-linear realisation to zero, that is set $x^a = 0 = \hat{h}_a{}^b$. This means that we can just consider the non-linear realisation of $SL(5)) \ltimes \{P_i, Z^{ij}\}$ whose corresponding group element is given by

$$g^{(0)\prime} = e^{x^i P_i + \frac{1}{\sqrt{2}} x_{ij} Z^{ij}} e^{h_i{}^j K^i{}_j} e^{\frac{1}{3!} C_{i_1 i_2 i_3} R^{i_1 i_2 i_3}} = g_l^{(0)\prime} g_E^{(0)\prime} \tag{4.20}$$

The prime corresponds to the fact that we have dropped the generators $P_a, K^a{}_b$ and the coordinate x^a and field $\hat{h}_a{}^b$. The remaining fields, namely $h_i{}^j$ and $C_{i_1 i_2 i_3}$ now only depend on the coordinates x^i and x_{ij}. We note that this would not be possible if one were to consider $E_{11} \ltimes l_1$ at higher levels, nonetheless the results provide an interesting laboratory in which to study the generalised spacetime introduced in the non-linear realisation.

Usually when carrying out a Kaluza-Klein reduction to seven dimensions one neglects the dependence of the fields on the spacetime coordinates associated with the upper four dimensions leaving the fields to depend on the seven dimensional spacetime. However, as discussed previously in this paper, a different approach was adopted in the papers [36, 37] where one neglected the dependence on the seven dimensions and kept a dependence on the coordinates associated with the upper space. The simplification of the non-linear realisation we have just carried out corresponds to this latter approach.

It is now straightforward to construct the non-linear realisation. The vielbein on the generalised spacetime is given by equation (3.43) which in this case becomes

$$dz \cdot E \cdot L = (g_E^{(0)\prime})^{-1} (dx^i P_i + \frac{1}{\sqrt{2}} dx_{ij} Z^{ij}) g_E^{(0)\prime}. \tag{4.21}$$

From now on we will drop the 0 superscript and the primes on the group elements with the understanding that the group elements are at level zero and do not include the IGL(7) generators. Equation (4.21) is easily evaluated using equations (4.5)–(4.8), and we find that

$$E = (\det e)^{-\frac{1}{2}} \begin{pmatrix} e_\mu{}^i & -\frac{1}{\sqrt{2}} e_\mu{}^j C_{j i_i i_2} \\ 0 & e^{-1}{}_{\mu_1 \mu_2}{}^{i_1 i_2} \end{pmatrix}, \tag{4.22}$$

where $e_\mu{}^i = (e^h)_\mu^i$, and $e^{-1}{}_{\mu_1 \mu_2}{}^{i_1 i_2} = e^{-1}{}_{\mu_1}^{[i_1} e^{-1}{}_{\mu_2}{}^{i_2]}$. We are using $\mu, \nu, \ldots$ as world indices in the four dimensional spacetime. The prefactor follows from the terms with $\frac{1}{2}$ prefactors in equations (4.5) and (4.6), which in turn were inherited from such terms in equations (3.21) and (3.22). As we mentioned there, this precise prefactor arises from the fact that the l_1 is a representation of E_{11}. We note that if one were to just consider it as a ten dimensional representation of SL(5) then the factor would be $-\frac{1}{5}$ rather than $\frac{1}{2}$, as we found in section 2.

We will choose to construct the dynamics from the object defined in equation (3.50) which for simplicity we now denote by M. Using equation (4.22) and $M = EE^\#$, where here $E^\# = E^T$,

$$M = (\det e)^{-1} \begin{pmatrix} g_{\mu\nu} + \frac{1}{2} C_{\mu ij} C_\nu{}^{ij} & -\frac{1}{\sqrt{2}} C_\mu{}^{\nu_1 \nu_2} \\ -\frac{1}{\sqrt{2}} C_\nu{}^{\mu_1 \mu_2} & g^{-1[\mu_1|\nu_1|} g^{-1 \mu_2]\nu_2} \end{pmatrix}, \tag{4.23}$$

where $C_{\mu ij} = e_\mu{}^k C_{kij}$.

The most general action which is quadratic in generalised spacetime derivatives and invariant under the transformations of the non-linear realisation was given in equation (3.53). It involves five constraints and is unfamiliar in that it is defined over the extended space. We now adopt the procedure explained at the end of section three. Dropping the dependence of the fields on x_{ij}, we now evaluate the terms in the action of equation (3.53) to find that

$$
\begin{aligned}
&g^{-\frac{1}{2}} M^{MN}(\partial_M M^{KL})(\partial_N M_{KL}) \\
&\quad = 3g^{\mu\nu}(\partial_\mu g^{\sigma_1\sigma_2})(\partial_\nu g_{\sigma_1\sigma_2}) - \frac{11}{2} g^{\mu\nu}(g^{\sigma_1\sigma_2}\partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2}\partial_\nu g_{\tau_1\tau_2}) \\
&\qquad - g^{\mu\nu} g^{\sigma_1\ldots\sigma_3,\tau_1\ldots\tau_3}(\partial_\mu C_{\sigma_1\ldots\sigma_3})(\partial_\nu C_{\tau_1\ldots\tau_3}), \qquad (4.24)\\
&g^{-1/2} M^{MN}(\partial_N M^{KL})(\partial_L M_{MK}) \\
&\quad = g^{\mu\sigma}(\partial_\mu g^{\nu\tau})(\partial_\nu g_{\sigma\tau}) - (\partial_\mu g^{\mu\nu})(g^{\sigma\tau}\partial_\nu g_{\sigma\tau}) - \frac{1}{4} g^{\mu\nu}(g^{\sigma_1\sigma_2}\partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2}\partial_\nu g_{\tau_1\tau_2}) \\
&\qquad - \frac{1}{2} g^{\mu\tau_1} g^{\sigma_1\sigma_2\sigma_3,\nu\tau_2\tau_3}(\partial_\mu C_{\sigma_1\ldots\sigma_3})(\partial_\nu C_{\tau_1\ldots\tau_3}), \qquad (4.25)\\
&g^{-1/2} M_{RQ}\partial_S M^{SR}\partial_P M^{PQ} \qquad (4.26)\\
&\quad = g_{\sigma\tau}(\partial_\mu g^{\mu\sigma})(\partial_\nu g^{\nu\tau}) + (\partial_\mu g^{\mu\nu})(g^{\sigma\tau}\partial_\nu g_{\sigma\tau}) + \frac{1}{4} g^{\mu\nu}(g^{\sigma_1\sigma_2}\partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2}\partial_\nu g_{\tau_1\tau_2}) \\
&\qquad + \frac{1}{2} g^{\mu\sigma_1} g^{\nu\tau_1} g^{\sigma_2\sigma_3,\tau_2\tau_3}(\partial_\mu C_{\sigma_1\ldots\sigma_3})(\partial_\nu C_{\tau_1\ldots\tau_3}), \qquad (4.27)\\
&g^{-1/2} M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}) \\
&\quad = 49 g^{\mu\nu}(g^{\sigma_1\sigma_2}\partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2}\partial_\nu g_{\tau_1\tau_2}), \qquad (4.28)
\end{aligned}
$$

and

$$
g^{-1/2}\partial_S M^{ST} M^{PQ}\partial_T M^{PQ} = -7(\partial_\mu g^{\mu\nu})(g^{\sigma\tau}\partial_\nu g_{\sigma\tau}) - \frac{7}{2} g^{\mu\nu}(g^{\sigma_1\sigma_2}\partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2}\partial_\nu g_{\tau_1\tau_2}). \tag{4.29}
$$

Carrying out a gauge transformation on the three form field we find the resulting action is gauge invariant if

$$
c_1 = \frac{1}{12}, \quad c_2 = -\frac{1}{2}, \quad c_3 = 0, \quad c_4 = \frac{1}{84}, \quad c_5 = 0.
$$

Up to integration by parts, the action is equal to

$$
\int d^4x \sqrt{g}\left(R - \frac{1}{48} F^{(4)^2}\right), \tag{4.30}
$$

where R is the Ricci scalar of the metric g and $F^{(4)}$ is the field strength of $C, F^{(4)}_{ijkl} = 4\partial_{[i}C_{jkl]}$. We note that it is diffeomorphism invariant as well as U(1) gauge invariant. After integration by parts the neglected boundary piece may be combined with the Gibbons-Hawking term to produce a boundary term for the generalised spacetime [65].

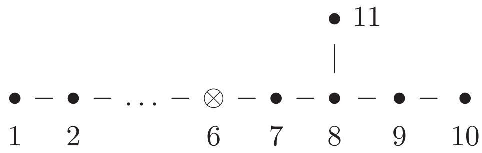

Figure 5. The E_{11} Dynkin diagram appropriate to the SO(5,5) duality

5 Five dimensions: SO(5,5)

The non-linear realisation of the $E_{11} \ltimes l_1$ algebra will now be used to find the generalised metric and construct an SO(5,5) duality manifest dynamics, recovering the result of [37] up to a conformal factor. The conformal factor in [37] was chosen to be a specific value. However, in this section, we see that the conformal factor is determined by the non-linear realisation of $E_{11} \ltimes l_1$. The construction appropriate for the case of five dimensions is found by deleting the sixth node of the E_{11} Dynkin diagram, figure 5, to find the subalgebra GL(6) $\otimes$ SO(5,5). As such we decompose $E_{11} \ltimes l_1$ into representation of GL(6) $\otimes$ SO(5,5).

Consider the lowest level generators of E_{11} given in equations (3.1) and (3.3). The generators that remain when we truncate to five-dimensions are

$$K^i{}_j, R^{ijk}, R_{ijk} \quad \text{and} \quad K^a{}_b,$$

where $i, j, \cdots = 1, \ldots, 5$ and $a, b, \cdots = 1, \ldots, 6$. These generators are all at zero level, and the mixed index generators are all at higher levels as in the case of SL(5). The algebra that these generators satisfy is given by the truncation of the E_{11} algebra, equations (3.2), (3.4), (3.5) and (3.12), to level zero

$$\begin{aligned}
[K^i{}_j, K^k{}_l] &= \delta^k_j K^i{}_l - \delta^i_l K^k{}_j, \\
[K^i{}_j, R^{k_1k_2k_3}] &= 3\delta^{[k_1}_j R^{|i|k_2k_3]}, \\
[K^i{}_j, R_{k_1k_2k_3}] &= -3\delta^i_{[k_1} R_{|j|k_2k_3]}, \\
[R^{i_1i_2i_3}, R_{j_1j_2j_3}] &= 18\delta^{[i_1i_2}_{[j_1j_2} K^{i_3]}_{j_3]} - 2\delta^{i_1i_2i_3}_{j_1j_2j_3}\left(\sum_j K^j{}_j + \sum_a K^a{}_a\right); \\
[K^a{}_b, K^c{}_d] &= \delta^c_b K^a{}_d - \delta^a_d K^c{}_b.
\end{aligned}$$

The K^a_b clearly generate the GL(6) algebra, while $\tilde{K}^i{}_j, R^{ijk}, R_{ijk}$ generate the SO(5,5) algebra, where

$$\tilde{K}^i{}_j = K^i{}_j - \frac{1}{4}\delta^i_j \sum_a K^a{}_a.$$

We make the following identification

$$M^{IJ} = \begin{cases} \frac{1}{3!}\epsilon^{IJklm}R_{klm} & \text{for } I,J = 1,\dots,5, \\ \tilde{K}^I{}_{J-5} - \frac{1}{3}\delta^I_{J-5}\sum_k \tilde{K}^k{}_k & \text{for } I = 1,\dots,5 \text{ and } J = 6,\dots,10, \\ \frac{1}{3!}\epsilon_{(I-5)(J-5)klm}R^{klm} & \text{for } I,J,\dots = 6,\dots,10, \end{cases}$$

where $k,l,m = 1,\dots,5$ in the above. Now, one can see that the generators M^{IJ} satisfy the SO(5,5) algebra

$$[M^{IJ}, M^{KL}] = \eta^{IK}M^{JL} - \eta^{IL}M^{JK} - \eta^{JK}M^{IL} + \eta^{JL}M^{IK},$$

where

$$\eta = \begin{pmatrix} 0 & 1_5 \\ 1_5 & 0 \end{pmatrix}.$$

Similarly taking the l_1 representation generators, given in equation (3.17), and restricting the indices to the case we are considering, at the lowest level we find the generators

$$P_i, Z^{ij}, Z^{i_1\dots i_5} \qquad \text{and} \qquad P_a,$$

where again $i,j,\dots = 1,\dots,5$ and $a,b,\dots = 1,\dots,6$. The first three generators generate the $\overline{16}$ representation of the SO(5,5) group, which we will call $\phi_{\overline{16}}$, while P_a generates translations in the 6-dimensional spacetime. The truncation of the $E_{11} \ltimes l_1$ algebra, equations (3.18)–(3.26), gives the commutation relations of these translation generators with the E_{11} generators

$$\begin{aligned}
[K^i{}_j, P_k] &= -\delta^i_k P_j + \frac{1}{2}\delta^i_j P_k, \\
[K^i{}_j, Z^{kl}] &= 2\delta^{[k}_j Z^{|i|l]} + \frac{1}{2}\delta^i_j Z^{kl}, \\
[K^i{}_j, Z^{k_1\dots k_5}] &= 5\delta^{[k_1}_j Z^{|i|k_2\dots k_5]} + \frac{1}{2}\delta^i_j Z^{k_1\dots k_5} \\
[R^{i_1i_2i_3}, P_j] &= 3\delta^{[i_1}_j Z^{i_2i_3]}, \\
[R^{i_1i_2i_3}, Z^{jl}] &= Z^{i_1i_2i_3jl}, \\
[R^{i_1i_2i_3}, Z^{j_1\dots j_5}] &= 0, \\
[K^a{}_b, P_i] &= \frac{1}{2}\delta^a_b P_i, \\
[K^a{}_b, Z^{ij}] &= \frac{1}{2}\delta^a_b Z^{kl}, \\
[K^a{}_b, Z^{i_1\dots i_5}] &= \frac{1}{2}\delta^a_b Z^{i_1\dots i_5}, \\
[K^i{}_j, P_a] &= \frac{1}{2}\delta^i_j P_a.
\end{aligned}$$

In what follows, we will use the Hodge dual of the $Z^{i_1\ldots i_5}$ generator

$$W = \frac{1}{5!}\epsilon_{i_1\ldots i_5}Z^{i_1\ldots i_5}$$

for which the commutation relations can be easily found from the commutations relations above.

As in the previous section, we will construct the non-linear realisation using the SO(5,5) group element

$$g_E = \mathrm{e}^{h_i{}^j K^i{}_j}\mathrm{e}^{\frac{1}{3!}C_{ijk}R^{ijk}},$$

which introduces the fields $h_i^{\ j}$ and C_{ijk}. Furthermore, the non-linear realisation also requires the group element

$$g_l = \mathrm{e}^{x^i P_i}\mathrm{e}^{\frac{1}{\sqrt{2}}x_{kl}Z^{kl}}\mathrm{e}^{wW},$$

which now has an extra generalised translation generator compared to the SL(5) case. This introduces the coordinates

$$x^i, x_{kl} \text{ and } w,$$

which form the 16 of SO(5,5). In the group elements g_E and g_l, we have, as before, left out the generators $K^a{}_b$ and P_a, respectively. As in the previous section this is a consistent truncation of generators. Using the group elements g_E and g_l we construct the group element of (3.30)

$$g = \mathrm{e}^{x^i P_i + \frac{1}{\sqrt{2}}x_{kl}Z^{kl} + wW}\mathrm{e}^{h_i{}^j K^i{}_j}\mathrm{e}^{\frac{1}{3!}C_{ijk}R^{ijk}}$$

from which the SO(5,5)$\ltimes\phi_{\overline{16}}$ non-linear realisation can be constructed.

The non-linear realisation is carried out in a similar manner to that outlined before, and ultimately one finds

$$\begin{aligned} g_h^{-1}g_l^{-1}\mathrm{d}g_l g_h = &\det(\mathrm{e}^h)^{-1/2}(\mathrm{e}^h)_j{}^i\,\mathrm{d}x^j\left(P_i - \frac{1}{2}C_{ikl}Z^{kl} + \frac{1}{24}C_{ik_1k_2}C_{k_3k_4k_5}\epsilon^{k_1\ldots k_5}W\right)\\ &+\frac{1}{\sqrt{2}}\det(\mathrm{e}^h)^{-1/2}(\mathrm{e}^{-h})_i{}^k(\mathrm{e}^{-h})_j{}^l\,\mathrm{d}x_{kl}\left(Z^{ij} - \frac{1}{6}C_{k_1k_2k_3}\epsilon^{ijk_1k_2k_3}W\right)\\ &+\det(\mathrm{e}^h)^{-3/2}\mathrm{d}wW. \end{aligned} \tag{5.1}$$

The generalised vielbein can be read off from this expression,

$$E_\Pi{}^A = (\det e)^{-1/2}\begin{pmatrix} e_\mu{}^i & -\frac{1}{\sqrt{2}}e_\mu{}^j C_{ji_1i_2} & \frac{1}{4}e_\mu{}^j X_j \\ 0 & e^{\mu_1}{}_{[i_1}e^{\mu_2}{}_{i_2]} & -\frac{1}{\sqrt{2}}e^{\mu_1}{}_{j_1}e^{\mu_2}{}_{j_2}V^{j_1j_2} \\ 0 & 0 & (\det e)^{-1}\end{pmatrix}, \tag{5.2}$$

where

$$V^{ij} = \frac{1}{3!}\epsilon^{ijklm}C_{klm} \qquad \text{and} \qquad X_i = C_{ijk}V^{jk}.$$

The tangent space indices are written with Latin letters and Greek letters indicate space indices. We have also abbreviated the space vielbein e^h to e with the notation that $e_\mu{}^i$ is the vielbein and $e^\mu{}_i$ is the inverse vielbein.

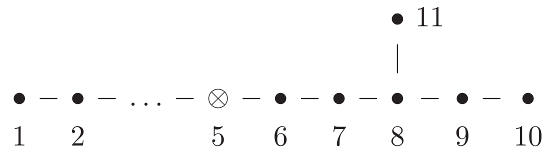

Figure 6. The E_{11} Dynkin diagram appropriate to the E_6 duality.

Hence the generalised metric, M, for the SO(5,5) duality group is

$$M = g^{-1/2} \begin{pmatrix} g_{\mu\nu} + \frac{1}{2} C_\mu{}^{ij} C_{\nu ij} + \frac{1}{16} X_\mu X_\nu & \frac{1}{\sqrt{2}} C_\mu{}^{\nu_1\nu_2} + \frac{1}{4\sqrt{2}} X_\mu V^{\nu_1\nu_2} & \frac{1}{4} g^{-1/2} X_\mu \\ \frac{1}{\sqrt{2}} C^{\mu_1\mu_2}{}_\nu + \frac{1}{4\sqrt{2}} V^{\mu_1\mu_2} X_\nu & g^{\mu_1\mu_2,\nu_1\nu_2} + \frac{1}{2} V^{\mu_1\mu_2} V^{\nu_1\nu_2} & \frac{1}{\sqrt{2}} g^{-1/2} V^{\mu_1\mu_2} \\ \frac{1}{4} g^{-1/2} X_\nu & \frac{1}{\sqrt{2}} g^{-1/2} V^{\nu_1\nu_2} & g^{-1} \end{pmatrix} \tag{5.3}$$

where $g = (\det e)^2$ is the determinant of the metric $g_{\mu\nu}$. This is the same generalised metric as in [37] except for the factor of g. As we mentioned in section 2, and shown in appendix B, multiplying a metric by an overall factor of g does not change the fact that the generalised metric will describe the dynamical theory. However, the factor of $g^{-1/2}$ in the generalised metric, which one obtains by using the truncated $E_{11} \ltimes l_1$ algebra, will naturally lead to the incorporation of the measure in the dynamics.

The generalised metric can now be used to describe the dynamics. The following expression

$$\frac{1}{16} M^{MN} (\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2} M^{MN} (\partial_N M^{KL})(\partial_L M_{MK}) \\ + \frac{11}{1728} M^{MN} (M^{KL} \partial_M M_{KL})(M^{RS} \partial_N M_{RS}),$$

up to integration by parts, leads to the gauge-invariant and diffeomorphism invariant combination

$$\sqrt{g}(R - \frac{1}{48} F^2),$$

where R is the Ricci scalar of the metric g and $F = \mathrm{d}C$ is the field strength of the 3-form potential C.

6 Six dimensions: E_6

The non-linear realisation of $E_{11} \ltimes l_1$ for the case of six dimensions to lowest level follows in the same way as before. We begin by deleting the fifth node of E_{11} Dynkin diagram, see figure 6, to find the subalgebra appropriate to six dimensions, GL(5) $\otimes E_6$.

Truncating the E_{11} generators, equations (3.1) and (3.3), to the six dimensions at the lowest level, we find the group generators

$$K^i{}_j, R^{ijk}, R_{ijk}, R^{i_1 \ldots i_6}, R_{i_1 \ldots i_6} \qquad \text{and} \qquad K^a{}_b,$$

where Latin letters from the middle of the alphabet $i, j, \dots = 1, \dots, 6$, and the start of the alphabet $a, b, \dots = 1, \dots, 5$. These generators are those at level zero as before. The algebra

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satisfied by these generators is found by truncating the E_{11} algebra, equations (3.2), (3.4)–(3.7) and (3.10)–(3.14), appropriately, in which case we find the algebra

$$
\begin{aligned}
[K^i{}_j, K^k{}_l] &= \delta^k_j K^i{}_l - \delta^i_l K^k{}_j, \\
[K^i{}_j, R^{k_1k_2k_3}] &= 3\delta^{[k_1}_j R^{|i|k_2k_3]}, \\
[K^i{}_j, R_{k_1k_2k_3}] &= -3\delta^i_{[k_1} R_{|j|k_2k_3]}, \\
[K^i{}_j, R^{k_1\ldots k_6}] &= 6\delta^{[k_1}_j R^{|i|k_2\ldots k_6]}, \\
[K^i{}_j, R_{k_1\ldots k_6}] &= -6\delta^i_{[k_1} R_{|j|k_2\ldots k_6]}, \\
[R^{i_1i_2i_3}, R^{j_1j_2j_3}] &= 2R^{i_1i_2i_3j_1j_2j_3}, \\
[R_{i_1i_2i_3}, R_{j_1j_2j_3}] &= 2R_{i_1i_2i_3j_1j_2j_3}, \\
[R^{i_1i_2i_3}, R_{j_1j_2j_3}] &= 18\delta^{[i_1i_2}_{[j_1j_2} K^{i_3]}{}_{j_3]} - 2\delta^{i_1i_2i_3}_{j_1j_2j_3}\left(\sum_j K^j{}_j + \sum_\mu K^\mu{}_\mu\right), \\
[R^{i_1\ldots i_6}, R_{j_1\ldots j_6}] &= -5!.3.3\,\delta^{[i_1\ldots i_5}_{[j_1\ldots j_5} K^{i_6]}{}_{j_6]} + 5!\delta^{i_1\ldots i_6}_{j_1\ldots j_6}\left(\sum_j K^j{}_j + \sum_a K^a{}_a\right), \\
[R_{i_1i_2i_3}, R_{j_1\ldots j_6}] &= \frac{5!}{2}\delta^{[j_1j_2j_3}_{i_1i_2i_3} R_{j_4j_5j_6]}; \\
[K^a{}_b, K^c{}_d] &= \delta^c_b K^a{}_d - \delta^a_d K^c{}_b.
\end{aligned}
$$

The $K^a{}_b$ generate the GL(5) algebra, while the generators

$$\tilde{K}^i{}_j = K^i{}_j - \frac{1}{3}\delta^i_j \sum_a K^a{}_a,$$

$R^{ijk}, R_{ijk}, R^{i_1\ldots i_6}$ and $R_{i_1\ldots i_6}$ generate the E_6 algebra.

The generalised translation generators can be found by considering the generators of the l_1 representation of E_{11}, equation (3.17), at lowest level truncated to six dimensions. The generators that we find in this case are

$$P_i, Z^{ij}, Z^{ijklm} \qquad \text{and} \qquad P_a.$$

The generators with indices labelled by Latin letters from the middle of the alphabet generate the $\overline{27}$ representation of E_6, which we denote $\phi_{\overline{27}}$, while P_a generates translations along the extra 5 directions. From equations (3.18)–(3.29), we can write down the commutation relations for the translation generators, which are

$$[K^i{}_j, P_k] = -\delta^i_k P_j + \frac{1}{2}\delta^i_j P_k, \tag{6.1}$$

$$[K^i{}_j, Z^{kl}] = 2\delta^{[k}_j Z^{|i|l]} + \frac{1}{2}\delta^i_j Z^{kl}, \tag{6.2}$$

$$[K^i{}_j, Z^{k_1\ldots k_5}] = 5\delta^{[k_1}_j Z^{|i|k_2\ldots k_5]} + \frac{1}{2}\delta^i_j Z^{k_1\ldots k_5}, \tag{6.3}$$

$$[R^{i_1i_2i_3}, P_j] = 3\delta_j^{[i_1} Z^{i_2i_3]}, \tag{6.4}$$

$$[R^{i_1i_2i_3}, Z^{jl}] = Z^{i_1i_2i_3jl}, \tag{6.5}$$

$$[R^{i_1i_2i_3}, Z^{j_1\ldots j_5}] = 0, \tag{6.6}$$

$$[R^{i_1\ldots i_6}, P_j] = -3\delta_j^{[i_1} Z^{i_2\ldots i_6]}, \tag{6.7}$$

$$[R^{i_1\ldots i_6}, Z^{jl}] = 0, \tag{6.8}$$

$$[R^{i_1\ldots i_6}, Z^{j_1\ldots j_5}] = 0, \tag{6.9}$$

$$[K^a{}_b, P_i] = \frac{1}{2}\delta_b^a P_i, \tag{6.10}$$

$$[K^a{}_b, Z^{ij}] = \frac{1}{2}\delta_b^a Z^{kl}, \tag{6.11}$$

$$[K^a{}_b, Z^{i_1\ldots i_5}] = \frac{1}{2}\delta_b^a Z^{i_1\ldots i_5}, \tag{6.12}$$

$$[K^i{}_j, P_a] = \frac{1}{2}\delta_j^i P_a. \tag{6.13}$$

For convenience, we will again use the Hodge dual of the Z^{ijklm} generator

$$W_p = \frac{1}{5!}\epsilon_{pijklm} Z^{ijklm}.$$

Now, we are ready to construct the non-linear realisation of $E_6 \ltimes \phi_{\overline{27}}$. The group element of (3.36) is

$$g_E = \mathrm{e}^{h_i{}^j K^i{}_j} \mathrm{e}^{\frac{1}{3!}C_{ijk}R^{ijk}} \mathrm{e}^{\frac{1}{6!}C_{i_1\ldots i_6}R^{i_1\ldots i_6}},$$

which introduces the fields

$$h_i{}^j, C_{ijk} \text{ and } C_{i_1\ldots i_6}.$$

Note that in six dimensions a new field $C_{i_1\ldots i_6}$, which is a 6-form potential, is introduced. This was not present in previous examples because in those cases the dimensions we were considering were less than six. Further to the group element, g_E, there is the group element

$$g_l = \mathrm{e}^{x^i P_i} \mathrm{e}^{\frac{1}{\sqrt{2}}x_{kl}Z^{kl}} \mathrm{e}^{w^i W_i},$$

which introduces the coordinates

$$x^i, x_{kl} \text{ and } w^i.$$

These form the 27 of E_6. It is again consistent to leave out the generators $K^a{}_b$ and P_a from the non-linear realisation.

We now calculate the Maurer-Cartan form for the non-linear realisation and hence the generalised vielbein, equation (3.43). By Hodge dualising equation (6.3), we can find that

$$[K^i{}_j, W_k] = -\delta_k^i W_j + \frac{3}{2}\delta_j^i W_k.$$

Now, using the above commutation relation and equations (6.1) and (6.2), we conjugate the Maurer-Cartan form of g_l by $\mathrm{e}^{h_i{}^j K^i{}_j}$ to obtain

$$\mathrm{e}^{-h_i{}^j K^i{}_j} g_l^{-1} \mathrm{d}g_l \mathrm{e}^{h_k{}^l K^k{}_l} = \det(\mathrm{e}^h)^{-\frac{1}{2}} \Big((\mathrm{e}^h)_\mu{}^i \, \mathrm{d}x^\mu P_i + \frac{1}{\sqrt{2}} (\mathrm{e}^{-h})_i{}^\mu (\mathrm{e}^{-h})_j{}^\nu \, \mathrm{d}x_{\mu\nu} Z^{ij} + \det(\mathrm{e}^h)^{-1} (\mathrm{e}^h)_\mu{}^i \mathrm{d}w^\mu W_i \Big), \tag{6.14}$$

where Greek and Latin letters denote spacetime and tangent space indices, respectively. This gives the dependence of the generalised vielbein on the spacetime metric, and conjugating the above expression by $\mathrm{e}^{\frac{1}{3!}C_{ijk}R^{ijk}}$ we obtain the dependence on the 3-form potential:

$$\begin{aligned} &\mathrm{e}^{-\frac{1}{3!}C_{ijk}R^{ijk}} \mathrm{e}^{-h_i{}^j K^i{}_j} g_l^{-1} \mathrm{d}g_l \mathrm{e}^{h_k{}^l K^k{}_l} \mathrm{e}^{\frac{1}{3!}C_{ijk}R^{ijk}} \\ &\quad = \det(\mathrm{e}^h)^{-1/2} (\mathrm{e}^h)_\mu{}^i \, \mathrm{d}x^\mu P_i \\ &\qquad + \frac{1}{\sqrt{2}} \det(\mathrm{e}^h)^{-1/2} (\mathrm{e}^{-h})_i{}^\mu (\mathrm{e}^{-h})_j{}^\nu \left(\mathrm{d}x_{\mu\nu} - \frac{1}{\sqrt{2}} C_{\mu\nu\rho} \mathrm{d}x^\rho \right) Z^{ij} \\ &\qquad + \det(\mathrm{e}^h)^{-3/2} (\mathrm{e}^h)_\mu{}^i \left(\mathrm{d}w^\mu - \frac{1}{\sqrt{2}} V^{\mu\nu\rho} \mathrm{d}x_{\nu\rho} + \frac{1}{4} C_{\nu kl} V^{\mu kl} \mathrm{d}x^\nu \right) W_i, \end{aligned} \tag{6.15}$$

where we have defined $V^{ijk} = \frac{1}{3!}\epsilon^{ijklmn} C_{lmn}$. In the deriving the above expression we have made use of equation (6.4) and a rewriting of equation (6.5),

$$[R^{ijk}, Z^{mn}] = \epsilon^{pijkmn} W_p.$$

Note that in the truncation to six-dimensions the commutator of R^{ijk} with $Z^{i_1 \dots i_5}$ is zero because $Z^{i_1 \dots l_7, a}$ vanishes.

Finally, we conjugate by the group element given by exponentiation of the $R^{i_1 \dots i_6}$ generator. Note that the only non-vanishing commutation relation of $R^{i_1 \dots i_6}$ with a generalised translation generator is the commutation relation with P_j, equation (6.7), or equivalently

$$[R^{i_1 \dots i_6}, P_d] = 3 \, \delta_d^{[i_1} \epsilon^{i_1 \dots i_6]p} W_p.$$

This gives us the dependence of the generalised vielbein on the 6-form potential. All in all, we obtain

$$\begin{aligned} g_E^{-1} g_l^{-1} \mathrm{d}g_l g_E = {} & \det(\mathrm{e}^h)^{-1/2} (\mathrm{e}^h)_\mu{}^i \, \mathrm{d}x^\mu P_i \\ & + \frac{1}{\sqrt{2}} \det(\mathrm{e}^h)^{-1/2} (\mathrm{e}^{-h})_i{}^\mu (\mathrm{e}^{-h})_j{}^\nu \left(\mathrm{d}x_{\mu\nu} - \frac{1}{\sqrt{2}} C_{\mu\nu\rho} \mathrm{d}x^\rho \right) Z^{ij} \\ & + \det(\mathrm{e}^h)^{-3/2} (\mathrm{e}^h)_\mu{}^i \Big(\mathrm{d}w^\mu - \frac{1}{\sqrt{2}} \det(\mathrm{e}^h) V^{\mu\nu\rho} \mathrm{d}x_{\nu\rho} \\ & \quad + \frac{1}{4} \det(\mathrm{e}^h) C_{kl\nu} V^{\mu kl} \mathrm{d}x^\nu + \frac{1}{2} \det(\mathrm{e}^h) U \mathrm{d}x^\mu \Big) W_i, \end{aligned} \tag{6.16}$$

where U is the Hodge dual of the 6-form potential,

$$U = \frac{1}{6!} \epsilon^{i_1 \dots i_6} C_{i_1 \dots i_6}.$$

Now, we can read off the generalised vielbein from equation (6.16). Using the same notation as before for the ordinary space vielbein, the generalised vielbein is

$$E_\Pi{}^A = (\det e)^{-1/2} \begin{pmatrix} e_\mu{}^i & -\frac{1}{\sqrt{2}} e_\mu{}^j C_{ji_1i_2} & \frac{1}{2} e_\mu{}^{i_3} U + \frac{1}{4} e_\nu{}^{i_3} C_{\mu jk} V^{\nu jk} \\ 0 & e^{\mu_1}{}_{[i_1} e^{\mu_2}{}_{i_2]} & -\frac{1}{\sqrt{2}} e^{\mu_1}{}_{j_1} e^{\mu_2}{}_{j_2} V^{j_1j_2i_3} \\ 0 & 0 & (\det e)^{-1} e^{i_3}_{\mu_3} \end{pmatrix}. \tag{6.17}$$

This generalised vielbein is very similar to the generalised vielbein in the case of the SO(5,5) duality group. In fact the metric and 3-form potential dependence of the two generalised vielbein are identical, except for the obvious difference that third generalised coordinate direction in this case has an index but this is only because we are using the Hodge dual of $Z^{i_1\ldots i_5}$. The dependence of the generalised vielbein on the 3-form potential only changes when there is a new generalised coordinate direction in which case higher order terms in the 3-form potential enter the generalised vielbein. In contrast, though, the generalised vielbein for the E_6 duality group gives the dependence of the generalised vielbein on the 6-form potential.

The generalised metric corresponding to the generalised vielbein, expression (6.17), is constructed using equation (3.50). The dynamics can then be written in terms of this generalised metric. The combination

$$\frac{1}{24} M^{MN}(\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2} M^{MN}(\partial_N M^{KL})(\partial_L M_{MK}) \\ + \frac{19}{9720} M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}), \tag{6.18}$$

again, up to integration by parts, reproduces

$$\sqrt{g}\left(R - \frac{1}{48} F^{(4)^2}\right)$$

when derivatives with respect to the extra generalised coordinates are taken to vanish.

The 6-form potential is not dynamical in 6-dimensions as its gauge-invariant field strength vanishes. When one evaluates the expression in (6.18) one discovers that $C^{(6)}$ cancels completely, verifying that the 6-form potential does not contribute to the action.

7 Seven dimensions: E_7

In this section, we apply the non-linear realisation of $E_{11} \ltimes l_1$ to seven dimensions. This is found by deleting the fourth node of the E_{11} Dynkin diagram, see figure 7, in which case we find the subalgebra GL(4)$\otimes E_7$.

The E_{11} algebra of generators, equations (3.1) and (3.3), at level zero with respect to the deletion of node four are

$$K^i{}_j, R^{ijk}, R_{ijk}, R^{i_1\ldots i_6}, R_{i_1\ldots i_6} \qquad \text{and} \qquad K^a{}_b,$$

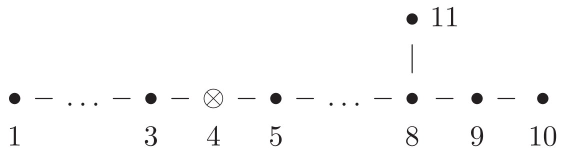

Figure 7. The E_{11} Dynkin diagram appropriate to the E_7 duality

where the indices labelled $i, j, \ldots$ run from 1 to 7, while those labelled by $a, b, \ldots$ run from 1 to 4. The commutation relations between these generators can be read off from the E_{11} algebra, equations (3.2), (3.4)–(3.7) and (3.10)–(3.14),

$$
\begin{aligned}
[K^i{}_j, K^k{}_l] &= \delta^k_j K^i{}_l - \delta^i_l K^k{}_j, \\
[K^i{}_j, R^{k_1k_2k_3}] &= 3\delta^{[k_1}_j R^{|i|k_2k_3]}, \\
[K^i{}_j, R_{k_1k_2k_3}] &= -3\delta^i_{[k_1} R_{|j|k_2k_3]}, \\
[K^i{}_j, R^{k_1\ldots k_6}] &= 6\delta^{[k_1}_j R^{|i|k_2\ldots k_6]}, \\
[K^i{}_j, R_{k_1\ldots k_6}] &= -6\delta^i_{[k_1} R_{|j|k_2\ldots k_6]}, \\
[R^{i_1i_2i_3}, R^{j_1j_2j_3}] &= 2R^{i_1i_2i_3j_1j_2j_3}, \\
[R_{i_1i_2i_3}, R_{j_1j_2j_3}] &= 2R_{i_1i_2i_3j_1j_2j_3}, \\
[R^{i_1i_2i_3}, R_{j_1j_2j_3}] &= 18\delta^{[i_1i_2}_{[j_1j_2} K^{i_3]}{}_{j_3]} - 2\delta^{i_1i_2i_3}_{j_1j_2j_3}\left(\sum_j K^j{}_j + \sum_a K^a{}_a\right), \\
[R^{i_1\ldots i_6}, R_{j_1\ldots j_6}] &= -5!.3.3\,\delta^{[i_1\ldots i_5}_{[j_1\ldots j_5} K^{i_6]}{}_{j_6]} + 5!\delta^{i_1\ldots i_6}_{j_1\ldots j_6}\left(\sum_j K^j{}_j + \sum_a K^a{}_a\right); \\
[R_{i_1i_2i_3}, R^{j_1\ldots j_6}] &= \frac{5!}{2}\delta^{[j_1j_2j_3}_{i_1i_2i_3} R^{j_4j_5j_6]}, \\
[K^a{}_b, K^c{}_d] &= \delta^c_b K^a{}_d - \delta^a_d K^c{}_b.
\end{aligned}
$$

The E_7 algebra derived from Cartan's 56-dimensional representation of E_7 [66–68], see appendix C, can be recovered from these relations by shifting the GL(7) generator, $K^i{}_j$, by the trace of the GL(4) generators $K^a{}_b$

$$\tilde{K}^i{}_j = K^i{}_j - \frac{1}{2}\delta^i{}_j \sum_a K^a{}_a.$$

The list of l_1 generators, equation (3.17), can similarly be truncated to seven dimensions where we find the generators

$$P_i, Z^{ij}, Z^{i_1\ldots i_5}, Z^{i_1\ldots i_7,j} \quad \text{and} \quad P_a.$$

The first four generate the 56 representation of E_7, denoted ϕ_{56}, and P_a generate translations along the four extra directions. The $E_{11} \ltimes l_1$ algebra, equations (3.18)–(3.29) gives the commutation relations of the generalised translation generators with the GL(4)⊗E_7 generators. For convenience, we will use the generators

$$W_{ij} = \frac{1}{5!}\epsilon_{ijk_1\dots k_5}Z^{k_1\dots k_5}, \qquad W^i = \frac{1}{7!}\epsilon_{j_1\dots j_7}Z^{j_1\dots j_7,i}, \tag{7.1}$$

and write the commutation relations in terms of these generators.

$$[K^i{}_j, P_k] = -\delta^i_k P_j + \frac{1}{2}\delta^i_j P_k, \qquad [K^i{}_j, Z^{kl}] = 2\,\delta^{[k}_j Z^{|i|l]} + \frac{1}{2}\delta^i_j Z^{kl},$$

$$[K^i{}_j, W_{kl}] = -2\,\delta^i_{[k} W_{|j|l]} + \frac{3}{2}\delta^i_j W_{kl}, \qquad [K^i{}_j, W^k] = \delta^k_j W^i + \frac{3}{2}\delta^i_j W^k,$$

$$[R_{ijk}, P_l] = 0, \qquad [R_{ijk}, Z^{mn}] = 3!\,\delta^{mn}_{[ij}P_{k]},$$

$$[R_{ijk}, W_{mn}] = \frac{1}{2}\epsilon_{ijkmnpq}Z^{pq}, \qquad [R_{ijk}, W^l] = \frac{1}{560}\delta^l_{[i}W_{jk]},$$

$$[R^{ijk}, P_l] = 3\,\delta^{[i}_l Z^{jk]}, \qquad [R^{ijk}, Z^{mn}] = \frac{1}{2}\epsilon^{ijkmnpq}W_{pq},$$

$$[R^{ijk}, W_{mn}] = 2\,\delta^{[ij}_{mn}W^{k]}, \qquad [R^{ijk}, W^l] = 0,$$

$$[R_{i_1\dots i_6}, P_j] = 0, \qquad [R_{i_1\dots i_6}, Z^{kl}] = 0,$$

$$[R_{i_1\dots i_6}, W_{kl}] = -3\epsilon_{kl[i_1\dots i_5}P_{i_6]}, \qquad [R_{i_1\dots i_6}, Z^j] = -\frac{3}{2}\epsilon_{ki_1\dots i_6}Z^{kj},$$

$$[R^{i_1\dots i_6}, P_j] = \frac{1}{2}\epsilon^{i_1\dots i_6 k}W_{jk}, \qquad [R^{i_1\dots i_6}, Z^{kl}] = \frac{1}{3}\epsilon^{i_1\dots i_6[k}W^{l]},$$

$$[R^{i_1\dots i_6}, W_{kl}] = 0, \qquad [R^{i_1\dots i_6}, W^j] = 0,$$

$$[K^a{}_b, P_i] = \frac{1}{2}\delta^a_b P_i, \qquad [K^a{}_b, Z^{ij}] = \frac{1}{2}\delta^a_b Z^{kl},$$

$$[K^a{}_b, W_{ij}] = \frac{1}{2}\delta^a_b W_{ij}, \qquad [K^a{}_b, W^i] = \frac{1}{2}\delta^a_b W^i$$

$$[K^i{}_j, P_a] = \frac{1}{2}\delta^i_j P_a.$$

In appendix C, we show that the generators

$$\tilde{K}^i{}_j, R^{ijk}, R_{ijk}, R^{i_1\dots i_6}, R_{i_1\dots i_6} \quad \text{and} \quad P_i, Z^{ij}, Z^{i_1\dots i_5}, Z^{i_1\dots i_7,j}$$

do indeed generate the $E_7 \ltimes \phi_{56}$ algebra.

We can now construct the non-linear realisation, equation (3.30), for $E_7 \ltimes \phi_{56}$ and find the generalised metric. The objects from which the non-linear realisation is constructed are the group element, equation (3.31),

$$g_E = \mathrm{e}^{h_i{}^j K^i{}_j}\mathrm{e}^{\frac{1}{3!}C_{ijk}R^{ijk}}\mathrm{e}^{\frac{1}{6!}C_{i_1\dots i_6}R^{i_1\dots i_6}},$$

which introduces the fields

$$h_i{}^j, C_{ijk} \text{ and } C_{i_1\dots i_6},$$

and the group element, equation (3.32),

$$g_l = \mathrm{e}^{x^i P_i}\mathrm{e}^{\frac{1}{\sqrt{2}}x_{ij}Z^{ij}}\mathrm{e}^{\frac{1}{\sqrt{2}}w^{ij}W_{ij}}\mathrm{e}^{\frac{1}{3}w_i W^i},$$

which introduces the generalised coordinates

$$x^i, x_{kl}, w^{kl} \text{ and } w_i.$$

The generalised coordinates are in the $\overline{56}$ of E_7.

Now, the generalised vielbein is constructed from

$$g_E^{-1} g_l^{-1} \mathrm{d} g_l g_E.$$

Similar calculation to the calculations in the previous sections show that the generalised vielbein, ${E_\Pi}^A$, is

$$\mathrm{e}^{-\frac{1}{2}} \begin{pmatrix} {e_\mu}^i & -\frac{1}{\sqrt{2}} {e_\mu}^j C_{j i_1 i_2} & \frac{1}{\sqrt{2}} {e_\mu}^{[i_3} U^{i_4]} + \frac{1}{4\sqrt{2}} {e_\mu}^j X_{j;}{}^{i_3 i_4} & \frac{1}{2} {e_\mu}^j C_{j i_5 k} U^k - \frac{1}{24} {e_\mu}^j X_{j;}{}^{kl} C_{kl i_5} \\ 0 & e^{\mu_1}{}_{[i_1} e^{\mu_2}{}_{i_2]} & -\frac{1}{\sqrt{2}} e^{\mu_1}{}_{j_1} e^{\mu_2}{}_{j_2} V^{j_1 j_2 i_3 i_4} & \frac{1}{\sqrt{2}} e^{\mu_1}{}_{[j} e^{\mu_2}{}_{i_5]} U^j + \frac{1}{4\sqrt{2}} e^{\mu_1}{}_{j_1} e^{\mu_2}{}_{j_2} X_{i_5;}{}^{j_1 j_2} \\ 0 & 0 & \mathrm{e}^{-1} {e_{\mu_3}}^{[i_3} {e_{\mu_4}}^{i_4]} & -\frac{1}{\sqrt{2}} {e_{\mu_3}}^{j_1} {e_{\mu_4}}^{j_2} C_{j_1 j_2 i_5} \\ 0 & 0 & 0 & \mathrm{e}^{-1} e^{\mu_5}{}_{i_5} \end{pmatrix}, \tag{7.2}$$

where e is the determinant of the vielbein e and

$$g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij};$$

$V^{i_1 \dots i_4}, U^i$ are Hodge duals of the 3-form and 6-form potentials, respectively,

$$V^{i_1 \dots i_4} = \frac{1}{3!} \epsilon^{i_1 \dots i_4 j_1 \dots j_3} C_{j_1 \dots j_3}, \qquad U^i = \frac{1}{6!} \epsilon^{i j_1 \dots j_6} C_{j_1 \dots j_6};$$

and

$$X_{i;}{}^{jk} = C_{ilm} V^{jklm}.$$

The indices labelled by Greek indices in the expression for the generalised vielbein are tangent space indices and Latin letters label space indices.

We can find that when we restrict the fields to only depend on ordinary space coordinates then

$$\begin{aligned} g^{-1/2} & M^{MN} (\partial_M M^{KL})(\partial_N M_{KL}) \\ &= 12 g^{\mu\nu} (\partial_\mu g^{\sigma\tau})(\partial_\nu g_{\sigma\tau}) - 62 g^{\mu\nu} (g^{\sigma_1\sigma_2} \partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2} \partial_\nu g_{\tau_1\tau_2}) \\ &\qquad - 4 g^{\mu\nu} g^{\sigma_1 \dots \sigma_3, \tau_1 \dots \tau_3} (\partial_\mu C_{\sigma_1 \dots \sigma_3})(\partial_\nu C_{\tau_1 \dots \tau_3}) \\ &\qquad - \frac{1}{5!} g^{\mu\nu} g^{\sigma_1 \dots \sigma_6, \tau_1 \dots \tau_6} (\partial_\mu C_{\sigma_1 \dots \sigma_6} - 20 C_{[\sigma_1 \dots \sigma_3|} \partial_\mu C_{|\sigma_4 \dots \sigma_6]}) \\ &\qquad\qquad \times (\partial_\nu C_{\tau_1 \dots \tau_6} - 20 C_{[\tau_1 \dots \tau_3|} \partial_\nu C_{|\tau_4 \dots \tau_6]}), \end{aligned} \tag{7.3}$$

$$\begin{aligned} g^{-1/2} & M^{MN} (\partial_N M^{KL})(\partial_L M_{MK}) \\ &= g^{\mu\sigma} (\partial_\mu g^{\nu\tau})(\partial_\nu g_{\sigma\tau}) - (\partial_\mu g^{\mu\nu})(g^{\sigma\tau} \partial_\nu g_{\sigma\tau}) \\ &\qquad - \frac{1}{4} g^{\mu\nu} (g^{\sigma_1\sigma_2} \partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2} \partial_\nu g_{\tau_1\tau_2}) \\ &\qquad - \frac{1}{2} g^{\mu\tau_1} g^{\sigma_1\sigma_2\sigma_3, \nu\tau_2\tau_3} (\partial_\mu C_{\sigma_1 \dots \sigma_3})(\partial_\nu C_{\tau_1 \dots \tau_3}) \\ &\qquad - \frac{1}{4(5!)} g^{\mu\tau_1} g^{\sigma_1 \dots \sigma_6, \nu\sigma_2 \dots \sigma_6} (\partial_\mu C_{\sigma_1 \dots \sigma_6} - 20 C_{[\sigma_1 \dots \sigma_3|} \partial_\mu C_{|\sigma_4 \dots \sigma_6]}) \\ &\qquad\quad \times (\partial_\nu C_{\tau_1 \dots \tau_6} - 20 C_{[\tau_1 \dots \tau_3|} \partial_\nu C_{|\tau_4 \dots \tau_6]}), \end{aligned} \tag{7.4}$$

$$g^{-1/2} M^{MN} (M^{KL} \partial_M M_{KL})(M^{RS} \partial_N M_{RS})$$
$$= 56^2 g^{\mu\nu} (g^{\sigma_1\sigma_2} \partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2} \partial_\nu g_{\tau_1\tau_2}). \qquad (7.5)$$

In the above calculations we made use of the following identities

$$C_{\nu\sigma_1\sigma_2} \partial_\mu V^{\sigma_1\sigma_2\tau_1\tau_2} = V^{\sigma_1\sigma_2\tau_1\tau_2} \partial_\mu C_{\nu\sigma_1\sigma_2} + \frac{2}{3} \delta_\nu^{[\tau_1} V^{\tau_2]\sigma_1\ldots\sigma_3} \partial_\mu C_{\sigma_1\ldots\sigma_3}$$
$$- \frac{1}{2} X_{\nu;}{}^{\tau_1\tau_2} g^{\sigma_1\sigma_2} \partial_\mu g_{\sigma_1\sigma_2},$$
$$C_{\sigma\mu_1\mu_2} V^{\sigma\nu_1\ldots\nu_3} = \frac{3}{2} \delta^{[\nu_1}_{[\mu_1} X_{\mu_2];}{}^{\nu_2\nu_3]}$$

which can be proved by Hodge dualising C and V and then contracting the epsilon tensors. It is also useful to note that

$$C_{\nu_1\ldots\nu_3} V^{\mu\nu_1\ldots\nu_3} = \frac{1}{3!} \epsilon^{\mu\nu_1\ldots\nu_3\sigma_1\ldots\sigma_3} C_{\nu_1\ldots\nu_3} C_{\sigma_1\ldots\sigma_3}$$

vanishes because the epsilon tensor makes exchanging the set of indices $\nu_1 \ldots \nu_3$ and $\sigma_1 \ldots \sigma_3$ an antisymmetric operation.

Now, in equations (7.3)–(7.5), comparing the terms that lead to the Ricci scalar, which is

$$R = \frac{1}{4} g^{\mu\nu} (\partial_\mu g^{\sigma\tau})(\partial_\nu g_{\sigma\tau}) - \frac{1}{2} g^{\mu\sigma} (\partial_\mu g^{\nu\tau})(\partial_\nu g_{\sigma\tau})$$
$$+ \frac{1}{2} (\partial_\mu g^{\mu\nu})(g^{\sigma\tau} \partial_\nu g_{\sigma\tau}) + \frac{1}{4} g^{\mu\nu} (g^{\sigma_1\sigma_2} \partial_\mu g_{\sigma_1\sigma_2})(g^{\tau_1\tau_2} \partial_\nu g_{\tau_1\tau_2}) \qquad (7.6)$$

up to terms that are total derivatives, we conclude that the combination

$$\frac{1}{48} M^{MN} (\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2} M^{MN} (\partial_N M^{KL})(\partial_L M_{MK})$$
$$+ \frac{17}{37632} M^{MN} (M^{KL} \partial_M M_{KL})(M^{RS} \partial_N M_{RS}) \qquad (7.7)$$

leads to the Ricci scalar. In fact, when the fields are allowed to only depend on ordinary space directions, this reduces, up to integration by parts, to

$$\sqrt{g} \left(R - \frac{1}{48} F^{(4)2} - \frac{1}{8!} F^{(7)2} \right),$$

where $\sqrt{g}$ is the measure, $F^{(4)}$ is the field strength of the 3-form potential,

$$F^{(4)}_{\mu_1\ldots\mu_4} = 4 \partial_{[\mu_1} C_{\mu_2\ldots\mu_4]},$$

and $F^{(7)}$ is the field strength of 6-form potential,

$$F^{(7)}_{\mu_1\ldots\mu_7} = 7 \partial_{[\mu_1} C_{\mu_2\ldots\mu_7]} + 140 C_{[\mu_1\ldots\mu_3} \partial_{\mu_4} C_{\mu_5\ldots\mu_7]}.$$

In the full theory in eleven dimensions one knows that the four and seven form field strengths are dual. However, here we are considering the theory in seven dimensions, so we cannot find an eleven-dimensional duality relation. The duality relation between these fields should be recovered if one carries out the non-linear realisation of $E_{11} \ltimes l_1$ in eleven dimensions. If one included *all* the components of $h, C^{(3)}, C^{(6)}$ rather than just those where one has E_7 indices, then one expects to be able to reproduce the duality relation between $F^{(4)}$ and $F^{(7)}$. Indeed, $E_{11} \ltimes l_1$ contains all the fields required to have equations of motion that are only first order in spacetime derivatives.

The generalised vielbein, expression (7.2), is the same as that found in [38], up to factors of det e. In [38], the dynamics is constructed in a different way and the other GL(4) directions are needed in order to construct the action. However, here we formulate the dynamics using the generalised metric and find that imposing gauge invariance automatically results in the action that is invariant under diffeomorphisms, and vice-versa.

Acknowledgments

We would like to thank Gary Gibbons, Mahdi Godazgar, Hugh Osborn and Antony Wassermann for discussions. DSB is supported in part by the Queen Mary STFC rolling grant ST/G000565/1. HG is supported by an STFC grant and thanks St. John's College Cambridge for their support. MJP is in part supported by the STFC rolling grant STJ000434/1. MJP would like to thank the Mitchell foundation and Trinity College Cambridge for their generous support. PW thanks the STFC for support from the rolling grant awarded to King's College. DSB, MJP and PW would like to thank George Mitchell and Sheridan Lorenz for their generous hospitality at Cook's Branch.

A Normalisation of generators

In this appendix, we will derive an invariant scalar product which has implicitly been used to construct the actions given in this paper. Acting with the Cartan involution I_c on the first fundamental representation l_1 we can define a new representation $I_c(l_1)$ by

$$I_c(P_a) = -\bar{P}^a, \;\; I_c(Z^{ab}) = -\bar{Z}_{ab}, \;\; I_c(Z^{a_1 \ldots a_5}) = -\bar{Z}_{a_1 \ldots a_5}, \ldots \tag{A.1}$$

where $\bar{P}^a$, $\bar{Z}_{ab}$, $\bar{Z}_{a_1 \ldots a_5}, \ldots$ are elements of the representation $I_c(l_1)$.

The Cartan involution I_c takes negative root generators to positive root generators up to a sign in such a way as to preserve the algebra. A more fundamental definition can be found in [1], for example. The action of I_c on some of the E_{11} generators is given by

$$I_c(K^a{}_b) = -K^b{}_a, \;\; I_c(R^{a_1 a_2 a_3}) = -R_{a_1 a_2 a_3}, \;\; I_c(R^{a_1 \ldots a_6}) = R_{a_1 \ldots a_6}. \tag{A.2}$$

The Cartan involution interchanges upper and lower indices, and possibly involves a change of sign. Consistency of the commutation rules under I_c determines uniquely the sign. Given equations (A.1) and (A.2) we can derive the commutation relations between E_{11} and those of the $I_c(l_1)$ representation. For example, acting with the Cartan involution on the commutator $[R^{a_1 a_2 a_3}, P_b] = 3\delta_b^{[a_1} Z^{a_2 a_3]}$ we find that

$$[R_{a_1 a_2 a_3}, \bar{P}^b] = -3\delta^b_{[a_1} \bar{Z}_{a_2 a_3]}. \tag{A.3}$$

Using equations (3.19), (3.21), (3.24) and (3.25), we find using similar arguments that

$$[R_{a_1a_2a_3}, \bar{Z}_{b_1b_2}] = -\bar{Z}_{a_1a_2a_3b_1b_2}$$
$$[K^a{}_b, \bar{P}^c] = \delta^c_b \bar{P}^a - \frac{1}{2}\delta^b_a \bar{P}^c,$$
$$[R^{a_1a_2a_3}, \bar{P}^b] = 0,$$
$$[R^{a_1a_2a_3}, \bar{Z}_{b_1b_2}] = -6\delta^{[a_1a_2}_{b_1b_2}\bar{P}^{a_3]}.$$

Given any element A of the l_1 representation and any element B of the $I_c(l_1)$ we can form an invariant scalar product denoted (A,B); the invariance means that

$$([X,A],\bar{B}) = -(A,[X,\bar{B}]), \quad X \in E_{11},\ A \in l_1,\ \bar{B} \in \bar{l}_1. \tag{A.4}$$

Taking $X = R^{a_1a_2a_3}$, $A = P_a$ and $B = \bar{Z}_{b_1b_2}$ we find using equation (A.4) and equation (A.3) that

$$2\delta^{[a_1a_2}_{b_1b_2}(P_c, \bar{P}^{a_3]}) = \delta^{[a_1}_c(Z^{a_1a_2]}, \bar{Z}_{b_1b_2}). \tag{A.5}$$

In fact, choosing our normalisation and using invariance under SL(11) we must set

$$(P_c, \bar{P}^a) = \delta^a_c, \tag{A.6}$$

hence $(Z^{a_1a_2}, \bar{Z}_{b_1b_2}) = 2\delta^{a_1a_2}_{b_1b_2}$. Using similar arguments, and repeating the above result, we find that

$$(P_b, \bar{P}^a) = \delta^a_b, \qquad (Z^{a_1a_2}, \bar{Z}_{b_1b_2}) = 2\delta^{a_1a_2}_{b_1b_2}, \qquad (Z^{a_1\dots a_5}, \bar{Z}_{b_1\dots b_5}) = 5!\delta^{a_1\dots a_5}_{b_1\dots b_5},$$
$$(Z^{a_1\dots a_7,c}, \bar{Z}_{b_1\dots b_7,d}) = 9(7!)\delta^{a_1\dots a_7}_{b_1\dots b_7}\delta^c_d. \tag{A.7}$$

Let us write the scalar product for all generators in the form

$$(L, \bar{L}) = N, \quad L \in l_1, \quad \bar{L} \in I_c(l_1) \tag{A.8}$$

where N is a diagonal matrix.

We will now derive an equation for the object M, that we have used to construct the Lagrangians, in terms of the generalised vielbein E. This will involve the matrix N just introduced. Let us first recall the technical steps given in section 3 leading to the appearance of the generalised vielbein in the non-linear realisation. We can write the group element g_l in the form $g_l = e^{z^T \cdot L'}$ where $L' = CL$ and C is a diagonal matrix which takes account of the possible normalisation factors. The Cartan form contains the terms $g_l^{-1}dg_l = dz^T \cdot L'$. Acting with the Cartan involution we find that $I_c(g_l) = e^{\bar{L}' \cdot \bar{z}^T}$ and so $I_c(g_l^{-1}dg_l) = \bar{L}' \cdot d\bar{z}^T$. It is easy to see that

$$(g_l^{-1}dg_l, I_c(g_l^{-1}dg_l)) = dz^T \cdot CNC \cdot d\bar{z}. \tag{A.9}$$

We take group element $k \in E_{11}$ to act on the generators of the l_1 representation as $k^{-1}L'k = D(k)L'$ and as a result the part of the Cartan form that contains the generalised vielbein E is given by

$$g_E^{-1}(g_l^{-1}dg_l)g_E = dz^T \cdot D(g_E) \cdot L' \equiv dz^T \cdot E \cdot L'. \tag{A.10}$$

Using equation (A.10) and (3.50) we find that

$$(I_c(g_E^{-1}))^{-1}g_E^{-1}(g_l^{-1}dg_l)g_E I_c(g_E^{-1}) = dz^T \cdot D(g_E)D(I_c(g_E^{-1})) \cdot L' \equiv dz^T \cdot M \cdot L'. \quad \text{(A.11)}$$

Let us now consider the object

$$((I_c(g_E^{-1}))^{-1}g_E^{-1}(g_l^{-1}dg_l)g_E I_c(g_E^{-1}), I_c(g_l^{-1}dg_l)) = dz^T \cdot M \cdot CNC \cdot d\bar{z}. \quad \text{(A.12)}$$

Using the invariance of the scalar product (A.4), which is equivalent to

$$(g_0 A g_0^{-1}, \bar{B}) = (A, g_0^{-1}\bar{B}g_0), \quad A \in l_1, \bar{B} \in \bar{l}_1,$$

where g_0 is an E_{11} group element, we find that the object on the left-hand side of equation (A.12) is invariant under both the rigid and local transformations given in equation (3.37) and (3.38). Using again the invariance of the scalar product and equation (A.8) we can evaluate this object to find that

$$\begin{aligned} dz^T \cdot M \cdot CNC \cdot d\bar{z} &= (g_E^{-1}(g_l^{-1}dg_l)g_E, I_c(g_E^{-1})I_c(g_l^{-1}dg_l)(I_c(g_E^{-1}))^{-1}) \\ &= (g_E^{-1}(g_l^{-1}dg_l)g_E, I_c(g_E^{-1}g_l^{-1}dg_l g_E)) \\ &= (dz^T \cdot E \cdot L', (\bar{L}')^T \cdot E^T \cdot d\bar{z}) \\ &= dz^T \cdot ECNC \cdot E^T \cdot d\bar{z}. \end{aligned} \quad \text{(A.13)}$$

Hence we find that $MCNC = ECNCE^T$. We will choose C so that $CNC = I$ and then

$$M = EE^T \quad \text{(A.14)}$$

This choice also implies that

$$(L', \bar{L}') = I \quad \text{(A.15)}$$

and equation (A.9) becomes

$$(g_l^{-1}dg_l, I_c(g_l^{-1}dg_l)) = dx^a d\bar{x}_a + dx^{ab}d\bar{x}_{ab} = \dots. \quad \text{(A.16)}$$

In the case of the SL(5) duality group found in dimension 4, C is the diagonal matrix with diagonal entries

$$\left(1, \frac{1}{\sqrt{2}}\right),$$

so the group element g_l takes the form

$$e^{x^i P_i + \frac{1}{\sqrt{2}} x_{ij} Z^{ij}}$$

in equation (4.17). In dimension 5, the dual of the generator $Z^{a_1 \dots a_5}$ has been used. The normalisation of $W = \frac{1}{5!}\epsilon_{a_1 \dots a_5} Z^{a_1 \dots a_5}$ can easily be found from equation (A.7),

$$(W, \bar{W}) = 1,$$

so in this case C has diagonal entries

$$\left(1, \frac{1}{\sqrt{2}}, 1\right).$$

Similarly, in dimension 6, the dual of the $Z^{a_1 \ldots a_5}$ is $W_a = \frac{1}{5!}\epsilon_{ab_1 \ldots b_5} Z^{b_1 \ldots b_5}$, which from equation (A.7) has the normalisation

$$(W_a, \bar{W}^b) = \delta_a^b,$$

so in six dimensions C also has diagonal entries

$$\left(1, \frac{1}{\sqrt{2}}, 1\right).$$

In seven dimensions, we have used the Hodge dual of two of the translation generators,

$$W_{ab} = \frac{1}{5!}\epsilon_{abc_1 \ldots c_5} Z^{c_1 \ldots c_5}, \ W^a = \frac{1}{7!}\epsilon_{b_1 \ldots b_7} Z^{b_1 \ldots b_7, a}.$$

The normalisation of these generators is found to be

$$(W_{ab}, \bar{W}^{cd}) = 2\delta_{ab}^{cd}, \ (W^a, \bar{W}_b) = 9\delta_b^a.$$

Therefore, in the case of seven dimensions C has diagonal entries

$$\left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{3}\right).$$

B Rescaling of the generalised metric

In this appendix, we show that rescaling a generalised metric by its determinant gives a generalised metric that also reproduces the dynamical theory. There are some important caveats that will be explained. Assume that a generalised metric, M, reproduces the dynamics, when the fields only depend on the ordinary space coordinates and not on the extra generalised coordinates,

$$\begin{aligned} L = {} & c_1\, M^{MN}(\partial_M M^{KL})(\partial_N M_{KL}) + c_2\, M^{MN}(\partial_N M^{KL})(\partial_L M_{MK}) \\ & + c_3\, M^{MN} M^{PQ}(M^{RS}\partial_P M_{RS})(\partial_M M_{NQ}) \\ & + c_4\, M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}), \end{aligned} \tag{B.1}$$

where $c_1, \ldots, c_4$ are known real numbers. We also require that the determinant of M is related to the determinant of the space metric g,

$$\det M = g^a, \tag{B.2}$$

for some real constant a. This is required by gauge-invariance of the theory under gauge transformations of the potential 3-form and 6-form.

Consider rescaling of the generalised metric M by its determinant, or equivalently g,

$$\tilde{M} = g^\alpha M, \tag{B.3}$$

where α is a real number. Therefore, $\tilde{M}^{-1} = g^{-\alpha} M^{-1}$ and so

$$\begin{aligned} \tilde{M}^{MN}(\partial_M \tilde{M}^{KL})(\partial_N \tilde{M}_{KL}) = {} & g^{-\alpha} M^{MN}(\partial_M M^{KL})(\partial_N M_{KL}) \\ & - \frac{\alpha}{a}\left(2 + \frac{\alpha}{a} D\right) g^{-\alpha} M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}), \end{aligned}$$

where D is the dimension of generalised space, and we have used

$$M^{KL}\partial_M M_{KL} = \frac{\partial_M(\det M)}{\det M} = a\frac{\partial_M g}{g}. \tag{B.4}$$

Similarly,

$$\begin{aligned}
&\tilde{M}^{MN}(\partial_N\tilde{M}^{KL})(\partial_L\tilde{M}_{MK})\\
&\quad= g^{-\alpha}M^{MN}(\partial_N M^{KL})(\partial_L M_{MK})\\
&\qquad\qquad -\frac{2\alpha}{a}g^{-\alpha}M^{MN}M^{PQ}(M^{RS}\partial_P M_{RS})(\partial_M M_{NQ})\\
&\qquad\qquad -\frac{\alpha^2}{a^2}g^{-\alpha}M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}),\\
&\tilde{M}^{MN}(\tilde{M}^{KL}\partial_M\tilde{M}_{KL})(\tilde{M}^{RS}\partial_N\tilde{M}_{RS})\\
&\quad= \left(\frac{\alpha}{a}D+1\right)^2 g^{-\alpha}M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}),\\
&\tilde{M}^{MN}\tilde{M}^{PQ}(\tilde{M}^{RS}\partial_P\tilde{M}_{RS})(\partial_M\tilde{M}_{NQ})\\
&\quad= \left(\frac{\alpha}{a}D+1\right)g^{-\alpha}M^{MN}M^{PQ}(M^{RS}\partial_P M_{RS})(\partial_M M_{NQ})\\
&\qquad\qquad +\frac{\alpha}{a}\left(\frac{\alpha}{a}D+1\right)g^{-\alpha}M^{MN}(M^{KL}\partial_M M_{KL})(M^{RS}\partial_N M_{RS}).
\end{aligned}$$

Hence, as long as

$$\left(\frac{\alpha}{a}D+1\right)\neq 0,$$

the rescaled generalised metric also reproduces the action, but with different coefficients for last two terms, i.e. c_1 and c_2 will have the same value, but the value of the constants c_3 and c_4 will change.

The case where

$$\left(\frac{\alpha}{a}D+1\right)$$

vanishes actually corresponds to the case where the generalised metric is derived from the duality group algebra. For example for the SL(5) duality group, let M denote the generalised metric

$$M_{KL} = \begin{pmatrix} g_{\mu\nu}+\frac{1}{2}C_\mu{}^{ij}C_{\nu ij} & -\frac{1}{\sqrt{2}}C_\mu{}^{\nu_1\nu_2} \\ -\frac{1}{\sqrt{2}}C^{\mu_1\mu_2}{}_\nu & g^{\mu_1\mu_2,\nu_1\nu_2}\end{pmatrix}, \tag{B.5}$$

then the generalised metric derived from the SL(5) motion group is $\tilde{M} = g^{1/5}M$, equation (2.22) in section 2.[5] From equation (B.3), $\alpha = 1/5$, and from equation (B.2), or (B.4), $a = -2$. The dimension of the generalised space, D, is 10. Hence

$$\left(\frac{\alpha}{a}D+1\right) = 0.$$

[5] The M in section 2 is $\tilde{M}$ here.

	SL(5)	SO(5,5)	E_6	E_7
α	1/5	1/4	1/3	1/2
a	-2	-4	-9	-28
D	10	16	27	56

Table 3. The values of α, a and D for the duality groups considered in this paper.

In contrast, for the generalised metric from the non-linear realisation of $E_{11} \ltimes l_1$, equation (4.23), the corresponding values of α, a and D are $-1/2, -2$ and 10, so

$$\left(\frac{\alpha}{a}D + 1\right) \neq 0.$$

It can easily be checked that the above statement is also true for the SO(5,5), E_6 and E_7 duality groups. In all these cases, let M denote the generalised metric with no factor of $\det g$ in its top-left entry, and $\tilde{M}$ be the generalised metric from the non-realisation of the duality motion group. Then as can be seen from table 3,

$$\left(\frac{\alpha}{a}D + 1\right) = 0$$

in all these cases. Therefore, the generalised metric constructed from the duality group cannot be used to reproduce the dynamics. However, if the generalised metrics come from the non-linear realisation of larger groups such as E_9, E_{10} or E_{11}, then the value of α is different and the generalised metric can be used to construct the dynamics. The particular advantage of E_{11} is that it not only solves the above problem, but that it also results in the correct overall measure.

C E_7 motion group from Cartan's representation

Here, we will briefly review Cartan's 56-dimensional representation of E_7 [66–68] and use it to find the algebra of the E_7 motion group.[6] We show that the truncation of the $E_{11} \ltimes l_1$ at lowest level to seven dimensions gives the algebra of the E_7 motion group.

We will consider the representation of the exceptional Lie group E_7 on a 56-dimensional space parametrised by bivectors, x^{IJ}, and 2-form y_{IJ}, where I, J run from 1 to 8. The infinitesimal transformations of these under E_7 are

$$x^{IJ} \to x^{IJ} + \Lambda^I{}_K x^{KJ} + \Lambda^J{}_K x^{IK} + \Sigma^{IJKL} y_{KL} \tag{C.1}$$

$$y_{IJ} \to y_{IJ} - \Lambda^K{}_I y_{KJ} - \Lambda^K{}_J y_{IK} + \Sigma_{IJKL} x^{KL}, \tag{C.2}$$

where $\Lambda^I{}_I = 0$, and

$$\Sigma^{IJKL} = \frac{1}{4!}\epsilon^{IJKLMNPQ}\Sigma_{MNPQ}.$$

The Λ and Σ parametrise the infinitesimal E_7 transformations.

[6]See also appendix B of [27] for a complementary account of E_7

To find the commutation relations of the motion group, we denote an E_7 motion group transformation by

$$U(\Lambda, \Sigma; a, b) = \mathrm{e}^{\Lambda^J{}_I M^I{}_J + \Sigma^{IJKL} V_{IJKL} + a^{IJ} X_{IJ} + b_{IJ} Y^{IJ}}, \tag{C.3}$$

where $X^{IJ} y_{KL} = 0$, and $Y_{IJ} x^{KL} = 0$. The generators $M^I{}_J$ and V_{IJKL} generate E_7 transformations parametrised by $\Lambda^I{}_J$ and Σ^{IJKL}, respectively, and X^{IJ} generates translations in the x^{IJ} directions, while Y_{IJ} generates translations in the y_{IJ} directions. The transformation of x^{IJ} and y_{IJ} under the E_7 part of U is given, to first order, in equations (C.1) and (C.2), respectively.

The commutator of two transformations can be used to calculate the commutation relations of the generators. To this end, we calculate the commutator of two transformations on x^{IJ} and y_{IJ} to second order in the infinitesimal parameters

$$\begin{aligned}
&[\tilde{U}(\tilde{\Lambda}, \tilde{\Sigma}; \tilde{a}, \tilde{b}), U(\Lambda, \Sigma; a, b)] x^{IJ} \\
&\quad = \left([\tilde{\Lambda}, \Lambda]^I{}_K - \frac{1}{3}\Theta^I{}_K \right) x^{KJ} + \left([\tilde{\Lambda}, \Lambda]^J{}_K - \frac{1}{3}\Theta^J{}_K \right) x^{IK} \\
&\qquad - 4\left(\tilde{\Lambda}^{[I}{}_K \Sigma^{JMN]K} - \Lambda^{[I}{}_K \tilde{\Sigma}^{JMN]K} \right) y_{MN} + \tilde{\Lambda}^I{}_K a^{KJ} - \Lambda^I{}_K \tilde{a}^{KJ} \\
&\qquad + \tilde{\Lambda}^J{}_K a^{IK} - \Lambda^J{}_K \tilde{a}^{IK} + \tilde{\Sigma}^{IJKL} b_{KL} - \Sigma^{IJKL} \tilde{b}_{KL},
\end{aligned} \tag{C.4}$$

where

$$\Theta^I{}_J = \tilde{\Sigma}^{IKLM} \Sigma_{KLMJ} - \Sigma^{IKLM} \tilde{\Sigma}_{KLMJ}.$$

There is a similar expression for the commutator of two transformations acting on y_{IJ}

$$\begin{aligned}
&[\tilde{U}(\tilde{\Lambda}, \tilde{\Sigma}; \tilde{a}, \tilde{b}), U(\Lambda, \Sigma; a, b)] y_{IJ} \\
&\quad = -\left([\tilde{\Lambda}, \Lambda]^K{}_I - \frac{1}{3}\Theta^K{}_I \right) y_{KJ} - \left([\tilde{\Lambda}, \Lambda]^K{}_J - \frac{1}{3}\Theta^K{}_J \right) y_{IK} \\
&\qquad + 4\left(\tilde{\Lambda}^K{}_{[I} \Sigma_{JMN]K} - \Lambda^K{}_{[I} \tilde{\Sigma}_{JMN]K} \right) x^{MN} + \Lambda^K{}_I \tilde{b}_{KJ} - \tilde{\Lambda}^K{}_I b_{KJ} \\
&\qquad + \Lambda^K{}_J \tilde{b}_{IK} - \tilde{\Lambda}^K{}_J b_{IK} + \tilde{\Sigma}_{IJKL} a^{KL} - \Sigma_{IJKL} \tilde{a}^{KL}.
\end{aligned} \tag{C.5}$$

In the above equations we have used the identity

$$\tilde{\Sigma}^{IJKL} \Sigma_{KLMN} - \Sigma^{IJKL} \tilde{\Sigma}_{KLMN} = -\frac{2}{3} \delta^{[I}_{[M} \Theta^{J]}{}_{N]},$$

which can be proved by Hodge dualising $\tilde{\Sigma}$ and Σ and then contracting the epsilon tensors, and expanding out the antisymmetrisations in the resulting Kronecker delta symbols.

Hence, from the above equations, (C.4) and (C.5), we deduce that the commutator of two transformations $\tilde{U}$ and U is an infinitesimal transformation, as it must be from Lie theory, and the transformation can be written

$$\begin{aligned}
[\tilde{U}, U] = &\left([\tilde{\Lambda}, \Lambda] - \frac{1}{3}\Theta \right)^J{}_I M^I{}_J + 4\left(\tilde{\Lambda}^I{}_M \Sigma^{MJKL} - \Lambda^I{}_M \tilde{\Sigma}^{MJKL} \right) V_{IJKL} \\
&+ \left(2\tilde{\Lambda}^I{}_K a^{KJ} - 2\Lambda^I{}_K \tilde{a}^{KJ} + \tilde{\Sigma}^{IJKL} b_{KL} - \Sigma^{IJKL} \tilde{b}_{KL} \right) X_{IJ} \\
&+ \left(2\Lambda^K{}_I \tilde{b}_{KJ} - 2\tilde{\Lambda}^K{}_I b_{KJ} + \tilde{\Sigma}_{IJKL} a^{KL} - \Sigma_{IJKL} \tilde{a}^{KL} \right) Y^{IJ}.
\end{aligned} \tag{C.6}$$

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Now using the above equation we can find the commutation relations. For example, from the above equation

$$[\tilde{U}(\tilde{\Lambda},0;0,0),U(\Lambda,0;0,0)] = \mathrm{e}^{[\tilde{\Lambda},\Lambda]^J{}_I M^I{}_J}. \tag{C.7}$$

But the $\tilde{U}$ and U can also be written using exponentials, equation (C.3), so the commutator of the two transformations can also be written as

$$\begin{aligned}[\tilde{U}(\tilde{\Lambda},0;0,0),U(\Lambda,0;0,0)] &= \mathrm{e}^{\tilde{\Lambda}^J{}_I M^I{}_J}\mathrm{e}^{\Lambda^L{}_K M^K{}_L} - \mathrm{e}^{\Lambda^L{}_K M^K{}_L}\mathrm{e}^{\tilde{\Lambda}^J{}_I M^I{}_J},\\ &= \tilde{\Lambda}^J{}_I \Lambda^L{}_K [M^I{}_J, M^K{}_L],\end{aligned} \tag{C.8}$$

using the Baker-Campbell-Hausdorff formula

$$\mathrm{e}^X\mathrm{e}^Y = \mathrm{e}^{X+Y+\frac{1}{2}[X,Y]\dots}.$$

Comparing equations (C.7)and (C.8), we deduce that

$$[M^I{}_J, M^K{}_L] = \delta^I_L M^K{}_J - \delta^K_J M^I{}_L. \tag{C.9}$$

The other commutation relations can be found using the same method and are listed below:

$$[M^I{}_J, V_{ABCD}] = 4\,\delta^I_{[A} V_{|J|BCD]} - \frac{1}{2}\,\delta^I_J V_{ABCD}, \tag{C.10}$$

$$[V_{ABCD}, V_{EFGH}] = -\frac{1}{72}\,\left(\delta^J_{[A}\epsilon_{BCD]EFGHI} - \delta^J_{[E}\epsilon_{FGH]ABCDI}\right) M^I{}_J, \tag{C.11}$$

$$[M^I{}_J, X_{KL}] = 2\,\delta^I_{[K} X_{|J|L]} - \frac{1}{4}\,\delta^I_J X_{KL}, \tag{C.12}$$

$$[M^I{}_J, Y^{KL}] = -2\,\delta^{[K}_J Y^{|I|L]} + \frac{1}{4}\,\delta^I_J Y^{KL}, \tag{C.13}$$

$$[V_{ABCD}, X_{IJ}] = \frac{1}{4!}\,\epsilon_{ABCDIJKL} Y^{KL}, \qquad [V_{ABCD}, Y^{IJ}] = \delta^{KL}_{[AB} X_{CD]}. \tag{C.14}$$

These are the commutation relations of SL(8) decomposition of the algebra of the E_7 motion group. The uppercase Latin indices are in fact SL(8) indices, which is why they run from 1 to 8. We are, however, interested in the SL(7) decomposition of the algebra of the E_7 motion group. This is because the E_7 duality appears upon reduction on a 7-torus, so we will make the duality act along these seven spatial directions.

It is not difficult to decompose SL(8) representations in terms of SL(7) representations. We let $I = (i, 8)$, where lowercase Latin letters are SL(7) indices that run from 1 to 7, and we define

$$M^i{}_j = -\tilde{K}^i{}_j + \frac{1}{6}\delta^i_j D, \tag{C.15}$$

$$M^8{}_i = \frac{2}{6!}\epsilon_{ik_1\dots k_6} R^{a_1\dots a_6}, \qquad M^i{}_8 = -\frac{2}{6!}\epsilon^{ik_1\dots k_6} R_{k_1\dots k_6}, \tag{C.16}$$

$$V_{ijk8} = \frac{1}{12} R_{ijk}, \qquad V_{ijkl} = \frac{1}{72}\epsilon_{ijklmnp} R^{mnp}, \tag{C.17}$$

$$X_{i8} = \frac{1}{\sqrt{2}} P_i, \qquad X_{ij} = \frac{1}{\sqrt{2}} W_{ij},$$
$$Y^{i8} = \frac{1}{3\sqrt{2}} W^i, \qquad Y^{ij} = \frac{1}{\sqrt{2}} Z^{ij}, \tag{C.18}$$

where $D = \sum_i \tilde{K}^i{}_j$. The normalisation has been chosen to match the normalisation of the $E_{11} \ltimes l_1$ generators in section 3. In particular, the coefficient of D in the relation between $M^i{}_j$ and $\tilde{K}^i{}_j$, the first equation in the set of equations (C.16), has been chosen so that the commutator of $\tilde{K}^i{}_j$ and $R^{ijk}, R_{ijk}, R^{i_1\dots i_6}$ and $R_{i_1\dots i_6}$ has no trace term.

The commutation relations for the SL(7) decomposition of the E_7 motion group are found by inserting the decomposed generators into the commutation relations (C.10)–(C.14). Whereupon, the E_7 commutation relations are

$$[\tilde{K}^i{}_j, \tilde{K}^k{}_l] = \delta^k_j \tilde{K}^i{}_l - \delta^i_l \tilde{K}^k{}_j, \tag{C.19}$$
$$[\tilde{K}^i{}_j, R_{klm}] = -3\, \delta^i_{[k} R_{|j|lm]},$$
$$[\tilde{K}^i{}_j, R^{klm}] = 3\, \delta^{[k}_j R^{|i|lm]}, \tag{C.20}$$
$$[\tilde{K}^i{}_j, R_{k_1\dots k_6}] = -6\, \delta^i_{[k_1} R_{|j|k_2\dots k_6]},$$
$$[\tilde{K}^i{}_j, R^{k_1\dots k_6}] = 6\, \delta^{[k_1}_j R^{|i|k_2\dots k_6]}, \tag{C.21}$$
$$[R_{i_1\dots i_3}, R_{j_1\dots j_3}] = 2R_{i_1\dots i_3 j_1\dots j_3},$$
$$[R^{i_1\dots i_3}, R^{j_1\dots j_3}] = 2R^{i_1\dots i_3 j_1\dots j_3}, \tag{C.22}$$
$$[R_{i_1\dots i_3}, R^{j_1\dots j_6}] = 60\, \delta^{[j_1\dots j_3}_{i_1\dots i_3} R^{j_4\dots j_6]},$$
$$[R^{i_1\dots i_3}, R_{j_1\dots j_6}] = -60\, \delta^{i_1\dots i_3}_{[j_1\dots j_3} R_{j_4\dots j_6]}, \tag{C.23}$$
$$[R^{i_1\dots i_3}, R_{j_1\dots j_3}] = 18\, \delta^{[i_1 i_2}_{[j_1 j_2} \tilde{K}^{i_3]}{}_{j_3]} - 2\, \delta^{i_1\dots i_3}_{j_1\dots j_3} D, \tag{C.24}$$
$$[R^{i_1\dots i_6}, R_{j_1\dots j_6}] = -5!3.3\, \delta^{[i_1\dots i_5}_{[j_1\dots j_5} \tilde{K}^{i_6]}{}_{j_6]} + 5!\, \delta^{i_1\dots i_6}_{j_1\dots j_6} D. \tag{C.25}$$

Furthermore, the commutation relations of the E_7 generators with the translation generators are

$$[\tilde{K}^i{}_j, P_k] = -\delta^i_k P_j - \frac{1}{2}\delta^i_j P_k, \qquad [\tilde{K}^i{}_j, Z^{kl}] = 2\, \delta^{[k}_j Z^{|i|l]} - \frac{1}{2}\delta^i_j Z^{kl}, \tag{C.26}$$
$$[\tilde{K}^i{}_j, W_{kl}] = -2\, \delta^i_{[k} W_{|j|l]} + \frac{1}{2}\delta^i_j W_{ij}, \qquad [\tilde{K}^i{}_j, W^k] = \delta^k_j W^i + \frac{1}{2}\delta^i_j W^k, \tag{C.27}$$
$$[R_{ijk}, P_k] = 0, \qquad [R_{ijk}, Z^{mn}] = 3!\, \delta^{mn}_{[ij} P_{k]}, \tag{C.28}$$
$$[R_{ijk}, W_{mn}] = \frac{1}{2}\epsilon_{ijkmnpq} Z^{pq}, \qquad [R_{ijk}, W^l] = 9\delta^l_{[i} W_{jk]}, \tag{C.29}$$
$$[R^{ijk}, P_l] = 3\, \delta^{[i}_l Z^{jk]}, \qquad [R^{ijk}, Z^{mn}] = \frac{1}{2}\epsilon^{ijkmnpq} W_{pq}, \tag{C.30}$$
$$[R^{ijk}, W_{mn}] = 2\, \delta^{[ij}_{mn} W^{k]}, \qquad [R^{ijk}, W^l] = 0, \tag{C.31}$$
$$[R_{i_1\dots i_6}, P_j] = 0, \qquad [R_{i_1\dots i_6}, Z^{kl}] = 0, \tag{C.32}$$
$$[R_{i_1\dots i_6}, W_{kl}] = \epsilon_{j i_1\dots i_6} \delta^j_{[k} P_{l]}, \qquad [R_{i_1\dots i_6}, W^k] = -\frac{3}{2}\epsilon_{j i_1\dots i_6} Z^{jk}, \tag{C.33}$$
$$[R^{i_1\dots i_6}, P_k] = -\frac{1}{2}\epsilon^{j i_1\dots i_6} W_{jk}, \qquad [R^{i_1\dots i_6}, Z^{kl}] = \frac{1}{3}\epsilon^{i_1\dots i_6[k} W^{l]}. \tag{C.34}$$

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The E_7 generators $\tilde{K}^i{}_j, R^{ijk}, R_{ijk}, R^{i_1 \dots i_6}, R_{i_1 \dots i_6}$ and the generalised translation generators $P_i, Z^{ij}, Z^{i_1 \dots i_5}, Z^{i_1 \dots i_7, j}$ can be exactly matched to the corresponding generators in section 7, which were derived from the $E_{11} \ltimes l_1$ algebra.

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