

# On the six-dimensional origin of the AGT correspondence

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**ABSTRACT:** We argue that the six-dimensional  $(2, 0)$  superconformal theory defined on  $M \times C$ , with  $M$  being a four-manifold and  $C$  a Riemann surface, can be twisted in a way that makes it topological on  $M$  and holomorphic on  $C$ . Assuming the existence of such a twisted theory, we show that its chiral algebra contains a  $W$ -algebra when  $M = \mathbb{R}^4$ , possibly in the presence of a codimension-two defect operator supported on  $\mathbb{R}^2 \times C \subset M \times C$ . We expect this structure to survive the  $\Omega$ -deformation.

**KEYWORDS:** Supersymmetric gauge theory, Conformal and W Symmetry

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**1 Introduction**

In recent years there have been remarkable advances in our understanding of  $\mathcal{N} = 2$  supersymmetric gauge theories in four dimensions. One highlight is a conjectural relation between these theories and conformal field theories (CFT) in two dimensions, formulated by Alday, Gaiotto and Tachikawa [1] (AGT) and generalized thereafter by others [2–6]. In essence, the AGT conjecture asserts a correspondence between certain quantities in two types of theories. One is  $\mathcal{N} = 2$  theory obtained by compactifying the six-dimensional (2,0) superconformal theory on a Riemann surface  $C$  [7–9]. The other is CFT on  $C$  with W-algebra symmetry [10].

In view of the nature of the  $\mathcal{N} = 2$  theory involved, it is clear that the AGT correspondence should have its origin in six dimensions. A nice explanation would be as follows. Take the (2,0) theory defined on the product  $\mathbb{R}^4 \times C$ . Compactified on  $C$ , the theory reduces to an  $\mathcal{N} = 2$  theory on  $\mathbb{R}^4$ . If instead we somehow “compactify” it on  $\mathbb{R}^4$ , we get a theory on  $C$ . This latter theory is, presumably, a CFT with W-algebra symmetry. Unlike an overall scaling of the metric, scaling the metric of  $\mathbb{R}^4$  or  $C$  separately is not a symmetry. Certain quantities are, however, protected under separate compactification, hence can be computed in either effective theory. The comparison would then lead to the correspondence.

This scenario sounds plausible, and there are pieces of evidence supporting its validity [11–13]. Nevertheless, it seems that we still lack a satisfactory explanation of how exactly the alleged W-algebra arises from six dimensions. In this paper we address this issue.

Our setup is the following. We consider the (2,0) theory of type  $\mathfrak{g}$  on  $M \times C$ , where  $\mathfrak{g}$  is a simply-laced real simple Lie algebra and  $M$  is a four-manifold. If  $M$  and  $C$  are curved, to preserve at least one supersymmetry we need to twist the theory. In section 2, we argue that the theory becomes topological on  $M$  and holomorphic on  $C$  after the twisting is done; in other words, it depends on the geometry of the spacetime only through the smooth structure of  $M$  and the conformal structure of  $C$ . Roughly speaking, this means

that we obtain a chiral CFT on  $C$  “with values in a topological field theory on  $M$ ” [14]. The chiral algebra of this theory is the object of our interest.

In section 3, we show that the chiral algebra contains a W-algebra when  $M = \mathbb{R}^4$ , if the twisted theory indeed has the above property. The type of the W-algebra is precisely the one relevant for the AGT correspondence, namely the one that results from the quantum Drinfeld-Sokolov reduction of the affine Lie algebra  $\widehat{\mathfrak{g}}$  with respect to a principal  $\mathfrak{sl}_2$  embedding [15, 16]; see appendix for a brief review of quantum Drinfeld-Sokolov reduction. Since the chiral algebra is protected under compactification, the same W-algebra symmetry must be present in the effective theory on  $C$ , justifying the crucial assumption in the aforementioned argument. Our reasoning also applies to the case where there are a number of supersymmetric codimension-two defect operators inserted at points on  $C$ . This is actually part of the original conjecture. Furthermore, we can place another such defect operator on  $\mathbb{R}^2 \times C \subset M \times C$ , in which case the relevant W-algebra changes. This covers the generalization of the conjecture proposed in [5, 6], which involves  $\mathcal{N} = 2$  theories with a surface operator extending along  $\mathbb{R}^2 \subset \mathbb{R}^4$ .

Therefore, by studying the chiral algebra of the  $(2, 0)$  theory, we gain a fairly clear picture of the origin of W-algebras that appear in the AGT correspondence and its generalization incorporating surface operators. We leave some questions unanswered, however. The most important one is about the  $\Omega$ -deformation [17]. For a complete treatment, we must tame infrared divergences coming from the noncompactness of the spacetime. We will assume that this is done by some regularization procedure,<sup>1</sup> but the AGT conjecture picks a particular one. That is to turn on the  $\Omega$ -deformation, which confines quantum excitations within an effectively compact region in  $\mathbb{R}^4$ . What special role does the  $\Omega$ -deformation play in our story, other than merely providing an infrared regulator? Although we propose a possible explanation in section 4, a definitive answer will have to wait until the appearance of a six-dimensional realization of the  $\Omega$ -deformation.

Lastly, let us point out that in principle our chiral algebra can be much larger than just the W-algebra, and may contain other interesting structures. It deserves to be explored more deeply.

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<sup>1</sup>Without a regularization one may still find a W-algebra in the classical sense. For instance, classical W-algebras can be obtained by turning on the  $\Omega$ -deformation and then removing it. In the presence of the  $\Omega$ -deformation, it is known that the level of  $\widehat{\mathfrak{g}}$  is given by  $k = -h^\vee + \epsilon_2/\epsilon_1$ , where  $h^\vee$  is the dual Coxeter number of  $\mathfrak{g}$  and  $\epsilon_1, \epsilon_2$  are deformation parameters. In the limit  $\epsilon_1 \rightarrow 0$ , we have  $k \rightarrow \infty$  and the quantum W-algebra reduces to a classical W-algebra. The opposite limit  $\epsilon_2 \rightarrow 0$  is also a classical limit, via the duality of W-algebras sending  $k + h^\vee \rightarrow (k + h^\vee)^{-1}$  [10]. (The limit in which both  $\epsilon_1$  and  $\epsilon_2$  are taken to zero but  $\epsilon_2/\epsilon_1$  remains finite appears to be more subtle.) In general, the level would depend on a specific regularization scheme that we employ.

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## 2 Chiral algebra from the (2,0) theory

Let us see how a chiral algebra can arise from the (2, 0) theory on  $M \times C$ . First of all, we need to twist the theory.

The theory has the R-symmetry  $\text{Spin}(5)_R$  under which the sixteen supercharges transform in the spinor representation. Due to the product structure of the spacetime the holonomy is  $\text{Spin}(4)_M \times \text{Spin}(2)_C \cong \text{SU}(2)_l \times \text{SU}(2)_r \times \text{U}(1)_C$ . Correspondingly, we split  $\text{Spin}(5)_R$  into  $\text{Spin}(3)_R \cong \text{SU}(2)_R$  and  $\text{Spin}(2)_R \cong \text{U}(1)_R$ . Under  $\text{SU}(2)_l \times \text{SU}(2)_r \times \text{U}(1)_C \times \text{SU}(2)_R \times \text{U}(1)_R$ , the supercharges transform as

$$((2, 1)_{1/2} \oplus (1, 2)_{-1/2}) \otimes (2_{1/2} \oplus 2_{-1/2}). \tag{2.1}$$

The twisting is done in two steps. The first step is to identify the diagonal  $\text{U}(1)'_C \subset \text{U}(1)_C \times \text{U}(1)_R$  with the holonomy group of  $C$ . This gives eight supercharges that are scalars on  $C$  and so preserved by the curvature of  $C$ . These transform under  $\text{SU}(2)_l \times \text{SU}(2)_r \times \text{SU}(2)_R$  as

$$(2, 1, 2) \oplus (1, 2, 2). \tag{2.2}$$

Thus we get  $\mathcal{N} = 2$  supersymmetry on  $M$  (which is generally broken by the curvature of  $M$ ). The second step is to replace  $\text{SU}(2)_r$  by the diagonal  $\text{SU}(2)'_r \subset \text{SU}(2)_r \times \text{SU}(2)_R$ . Then we are left with one supercharge that is a singlet under  $\text{SU}(2)_l \times \text{SU}(2)'_r \times \text{U}(1)'_C$ . We call it  $Q$ ; it has  $\text{U}(1)_R$  charge  $1/2$ .

It is crucial to understand whether  $Q$  obeys  $Q^2 = 0$  or not. One way to determine this is to note that if we compactify the theory on  $C$  right after the first step, then the second step is nothing but the familiar Donaldson-Witten twist [18] applied to the  $\mathcal{N} = 2$  theory on  $M$ . In that case we know that  $Q^2$  is not zero, but equal to the gauge transformation generated by an adjoint scalar  $\sigma$ . For the  $\text{U}(1)_R$  charges to match,  $\sigma$  must come from a scalar  $\Phi$  of charge 1 in the (2, 0) theory. After the twisting,  $\Phi$  is a one-form which can be written as  $\Phi = \Phi_z dz$ , where  $z$  is a local holomorphic coordinate on  $C$ . Hence we expect that in six dimensions,  $Q^2$  is given by some sort of gauge transformation specified by the one-form  $\Phi$ , that reduces in four dimensions to the gauge transformation by  $\sigma$ .

In the abelian case, such a symmetry is indeed known. The abelian (2, 0) theory has a two-form “gauge field”  $B$  (with values in a vector bundle) whose field strength  $H = dB$  is self-dual. (More precisely,  $B$  is something called a connection on an abelian gerbe.) The symmetry in question acts by

$$B \rightarrow B + d\Lambda \tag{2.3}$$

with  $\Lambda$  a one-form. If  $B_{iz}$  is identified with the component  $A_i$  of the gauge field  $A$  of the  $\mathcal{N} = 2$  theory on  $M$ , this transformation reduces to the ordinary gauge transformation  $A \rightarrow A + d_M \Lambda_z$  upon dimensional reduction on  $C$ . Since there are no other conceivable symmetry to which  $Q^2$  can be equated, we should have  $Q^2 = 0$  up to the two-form gauge transformation (2.3) with  $\Lambda = \Phi$ .

How to generalize the above symmetry to the nonabelian case is a bit mysterious. Still, given the fact that the nonabelian theory can be perturbed to an abelian theory or compactified to a nonabelian  $\mathcal{N} = 2$  theory, it is very likely that a generalization does exist. So let us assume that we have  $Q^2 = 0$  up to some “gauge transformation.” With this relation at hand, we can now define the  $Q$ -cohomology of “gauge-invariant” operators or states. The  $Q$ -cohomology classes of operators and states are the physical objects in the twisted theory.

Since the twisted theory can be compactified to a topologically twisted  $\mathcal{N} = 2$  theory on  $M$ , we expect it to be also topological on  $M$ . We can see how the physics depends on the geometry of  $C$  by reversing the steps in the twisting. If we twist first along  $M$ , then we get two supercharges that are scalars on  $M$ . These transform under  $U(1)_C \times U(1)_R$  as

$$(-1/2, \pm 1/2), \tag{2.4}$$

showing that we have  $(0, 2)$  supersymmetry in two dimensions. The twisting along  $C$  then turns one of the supercharges into a scalar on  $C$ , which we call  $Q$ . Twisted  $(0, 2)$  theories have the antiholomorphic degrees decoupled. Hence, we expect that the twisted theory is holomorphic on  $C$ .

The theory being topological on  $M$  and holomorphic on  $C$ ,  $Q$ -cohomology classes of local operators are independent of the position in  $M$  and vary holomorphically on  $C$ . Moreover, two of them can be multiplied by operator product expansion (OPE), with the coefficients being holomorphic functions on  $C$ . Therefore, these  $Q$ -cohomology classes form a chiral algebra, an OPE algebra of holomorphic fields, in the sense of two-dimensional CFT. The locality on  $M$  actually plays no role here, so we may also include in the chiral algebra  $Q$ -cohomology classes of operators that are local on  $C$  but nonlocal on  $M$ . Such nonlocal operators will be important to us.

So far we have made two assumptions (apart from the very existence of the  $(2, 0)$  theory and some of their properties), that we have  $Q^2 = 0$  up to some “gauge transformation” by  $\Phi$ , and that the twisted theory is topological on  $M$  and holomorphic on  $C$ . A strong support for these assumptions comes from the existence of an analogous twist for  $\mathcal{N} = 2$  superconformal gauge theory on the product  $\Sigma \times C$  of two Riemann surfaces, introduced by Kapustin [19] in the course of generalizing geometric Langlands duality. When  $M = \Sigma \times \Sigma'$ , our twist reduces to that of Kapustin via compactification on  $\Sigma'$ . Kapustin’s theory has  $Q^2 = 0$ , which is consistent with our formula for the abelian case since the transformation (2.3) for  $\Lambda = \Phi$  gives a trivial gauge transformation when dimensionally reduced to  $\Sigma \times C$ . Furthermore, Kapustin’s theory is topological on  $\Sigma$  and holomorphic on  $C$ .

### 3 Identifying the W-algebra

Our chiral algebra originated from twisted  $(0, 2)$  supersymmetry in two dimensions. This is encouraging, because the chiral algebra of a twisted  $(0, 2)$  theory often contains a W-algebra. For example, the chiral algebra of the A-model contains the Virasoro algebra, which is the  $W_2$  algebra. (In fact, it contains the  $(2, 0)$  superconformal algebra [20].) A more interesting example is provided by the theory obtained from the A-model by killing

the left-moving fermions. If we take the target space to be the flag manifold of a simple Lie group, the chiral algebra of this model contains the corresponding affine Lie algebra of critical level at the level of perturbation theory [20, 21].<sup>2</sup>

That said, we ask: how does a W-algebra associated to  $\widehat{\mathfrak{g}}$  arise in the chiral algebra of the twisted theory for  $M = \mathbb{R}^4$ ?

To answer this question, let us think of the  $(2,0)$  theory as if it were a gauge theory, with gauge group  $G$  whose Lie algebra is  $\mathfrak{g}$ , and with all the fields valued in the adjoint representation. We choose a framing of field configurations at the infinity of  $\mathbb{R}^4$ . That is to say, we regard two configurations to be physically identical if and only if they are related by a gauge transformation that is identity at infinity. Let  $\mathcal{G}^\infty$  be the group of gauge transformations that are global on  $\mathbb{R}^4$ , which is just the group of maps from  $C$  to  $G$ . This is a physical symmetry of the theory, rather than a gauge symmetry. As we now see, the conserved currents associated with this symmetry give rise to the affine currents of  $\widehat{\mathfrak{g}}$  in the chiral algebra.

Let  $\{t_a\}$  be an antihermitian basis of  $\mathfrak{g}$ , and consider a gauge transformation  $\exp(\epsilon f t_a) \in \mathcal{G}^\infty$  with  $\epsilon$  an infinitesimal parameter and  $f$  a real function on  $C$ . Under this transformation the gauge field changes by  $\delta A = \epsilon f [t_a, A] + \epsilon d_C f t_a$ . To find the corresponding conserved current, we promote  $\epsilon$  to an arbitrary function supported on an open set in  $\mathbb{R}^4 \times C$ , and look at the coefficient of  $d\epsilon$  in the variation of the action. By gauge invariance, we may as well compute the variation under the transformation  $\delta A = \epsilon d_C f t_a - d(\epsilon f) t_a = d\epsilon f t_a$ , which is the difference between the transformation with  $\epsilon$  promoted to a function afterwards (hence no longer a symmetry) and the gauge transformation  $\exp(\epsilon f t_a)$ . This makes it clear that the conserved current takes the form  $f j_a$ . Define operators

$$J_a = \int_{\mathbb{R}^4} \star_4 j_{a,z}, \quad \bar{J}_a = \int_{\mathbb{R}^4} \star_4 j_{a,\bar{z}}, \tag{3.1}$$

where  $\star_4$  is the four-dimensional Hodge star operator, sending 1 to the volume form of  $\mathbb{R}^4$ . If we choose a local cylindrical coordinate  $w = \sigma + i\tau$  such that  $z = \exp(-iw)$  and regard  $\tau$  as time, then the integral

$$\int d\sigma f(z J_a + \bar{z} \bar{J}_a) \tag{3.2}$$

gives the conserved charge. Since  $Q$  is a gauge singlet, this commutes with  $Q$  for any choice of  $f$ , which implies that  $z J_a + \bar{z} \bar{J}_a$  is  $Q$ -closed. But  $J_a$  and  $\bar{J}_a$  are functionals of the fields with no explicit dependence on the coordinate, so it must be that they are both  $Q$ -closed. Then  $\bar{J}_a$  must be  $Q$ -exact. For it transforms nontrivially under antiholomorphic scaling on  $C$ , which would contradict the holomorphy of the twisted theory if it did not vanish in the  $Q$ -cohomology. Going back to the expression (3.2) and setting  $f = 1$ , we find that the zero mode of  $J_a$  acts by  $t_a$  in the  $Q$ -cohomology.

Therefore, for each  $t_a$ , we obtain in the chiral algebra a holomorphic current  $J_a$  whose zero mode acts by  $t_a$ . The collection  $\{J_a\}$  of such currents generate  $\widehat{\mathfrak{g}}$ , as promised. A

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<sup>2</sup>Nonperturbatively, the chiral algebra vanishes once instanton corrections are taken into account [21–23]. The chiral algebra of the flag manifold model apparently has an intimate connection with geometric Langlands duality [24–26].

similar construction was in fact found by Johansen [27] in the context of holomorphically twisted  $\mathcal{N} = 1$  theories defined on the product of two Riemann surfaces. For those theories, each flavor symmetry gives rise to an affine Lie algebra in the  $Q$ -cohomology.

We have given a heuristic argument that the chiral algebra of the  $(2, 0)$  theory on  $\mathbb{R}^4 \times C$ , framed at infinity, contains  $\widehat{\mathfrak{g}}$  as a subalgebra. Of course, the  $(2, 0)$  theory is not really a gauge theory, and the above reasoning does not apply as it is. But it does apply once we compactify the theory on a circle — for then we have five-dimensional maximally supersymmetric Yang-Mills theory with gauge group  $G$ ! Hence, if  $M = S^1 \times \mathbb{R}^3$  for example, we do get the affine currents  $J_a$ , but this time defined by integration over  $\mathbb{R}^3$  in the five-dimensional theory.

For  $M = \mathbb{R}^4$ , however, a similar construction may not give the whole  $\widehat{\mathfrak{g}}$ . One way to compactify the theory on a circle in  $\mathbb{R}^4$  is to bend  $\mathbb{R}^4$  into the product of a cigar and  $\mathbb{R}^2$ , and make the cigar very narrow. Such a geometry was considered in [28] in relation to the quantum Hitchin system, and also in [29] in relation to Khovanov homology. This procedure reduces  $\mathbb{R}^4$  to  $\mathbb{R}_+ \times \mathbb{R}^2$ , so introduces a boundary at the origin of the half-line  $\mathbb{R}_+$ . The boundary condition here is not our choice; it is specified by the six-dimensional theory since there was no boundary at the beginning. Global symmetries can change this boundary condition. This is not a problem if one wants to deal with the conserved currents placed at a point away from the boundary, in which case one derives Ward identities by considering local transformations supported in the neighborhood of that point. But if one considers the currents placed on the boundary, the Ward identities are no longer the same because of the boundary contribution. This is the situation we face if we define the  $J_a$  by integration over  $\mathbb{R}_+ \times \mathbb{R}^2$ , a submanifold which intersects with the boundary. In order not to spoil our argument, we should project out those  $J_a$  that act nontrivially on the boundary state created by the compactification. It is this projection that we will find implements quantum Drinfeld-Sokolov reduction.

Thus, we are in need of understanding the boundary conditions of the maximally supersymmetric Yang-Mills theory. There are really two kinds of boundaries in the spacetime  $\mathbb{R}_+ \times \mathbb{R}^2 \times C$ . One is the boundary coming from six dimensions. This is  $\{\infty\} \times \mathbb{R}^2 \times C \cup \mathbb{R}_+ \times \{\infty\} \times C$ , located at the infinity of  $\mathbb{R}_+ \times \mathbb{R}^2$ . The other is the boundary created by the compactification. This is  $\{0\} \times \mathbb{R}^2 \times C$ , located at the origin of  $\mathbb{R}_+$ . Relevant to the projection is only the latter, as the conserved currents used to define the  $J_a$  are never placed at infinity which, strictly speaking, is not part of the spacetime. But understanding the boundary condition for the former tells us something important about the latter, so we turn to it first.

At infinity, we impose half-BPS boundary condition (that is, preserving half of the supersymmetry) because we want  $\mathcal{N} = 2$  supersymmetry when we compactify the theory on  $C$ . This is given as follows [8]. The R-symmetry  $\text{Spin}(5)_R$  acts on the five scalars  $\phi_i$  of the theory by  $\text{SO}(5)$  rotations. In order to twist the theory, we have split  $\text{Spin}(5)_R$  into  $\text{Spin}(3)_R$ , rotating  $(\phi_1, \phi_2, \phi_3)$ , and  $\text{Spin}(2)_R$ , rotating  $(\phi_4, \phi_5)$ . After the twisting,  $\phi_4 + i\phi_5$  becomes the component of a  $(1, 0)$ -form  $\varphi$  along  $C$ . The boundary condition at infinity is that  $A_i = \phi_i = 0$  for  $i = 1, 2, 3$ , and for the remaining components the pair  $(A, \varphi)$  solves

the Hitchin equations on  $C$ :

$$F_A + [\varphi, \varphi^\dagger] = 0, \quad \bar{\partial}_A \varphi = 0. \tag{3.3}$$

Notice that this boundary condition is invariant under  $\text{Spin}(3)_R$ .

The boundary condition at the origin of  $\mathbb{R}_+$  is determined by the boundary condition of the original six-dimensional theory, whose dimensional reduction we just described. As such, it must be invariant under any symmetry (except the gauge symmetry, which is not a physical symmetry) preserved by the boundary condition at infinity, especially  $\text{Spin}(3)_R$ . It turns out that we have a slightly unusual condition here: as the three scalars  $\phi_1, \phi_2, \phi_3$  approach the origin  $y = 0$ , they develop a singularity according to the Nahm equation

$$\frac{d\phi_i}{dy} + \epsilon_{ijk}[\phi_j, \phi_k] = 0. \tag{3.4}$$

More precisely, the three scalars must behave near  $y = 0$  as

$$\phi_i \sim \frac{t_i}{y} \tag{3.5}$$

up to an  $\text{Spin}(3)_R$  transformation, where the  $t_i \in \mathfrak{g}$  form a standard basis of a principal  $\mathfrak{su}_2$  subalgebra satisfying the commutation relations  $[t_i, t_j] = \epsilon_{ijk}t_k$  [29, 30].

Now we look for a W-algebra inside the chiral algebra. From our previous discussion, we guess that it is generated by those  $J_a$  that kill the boundary state at the origin of  $\mathbb{R}_+$ , specified by the Nahm pole (3.5). The residue of the pole defines via complexification an embedding  $\rho: \mathfrak{sl}_2 \rightarrow \mathfrak{g}_{\mathbb{C}}$ , which is a basic ingredient of quantum Drinfeld-Sokolov reduction. This strongly suggests that the W-algebra associated to the pair  $(\widehat{\mathfrak{g}}, \rho)$  is hidden somewhere in the chiral algebra.

Obvious candidates for the generators are the  $J_a$  corresponding to the elements of  $\mathcal{G}^\infty$  that leave the Nahm pole invariant, but these only generate the  $\mathfrak{sl}_2$ -invariant subalgebra of  $\widehat{\mathfrak{g}}$ . We can find more generators if we exploit the  $\text{Spin}(3)_R$  freedom in the boundary condition. To see what this freedom makes the total set of generators be, we use a little trick. First, we enlarge the theory by a subgroup  $\mathcal{H}$  of the complexification  $\mathcal{G}_{\mathbb{C}}^\infty$  of  $\mathcal{G}^\infty$ . By this we mean the following. The boundary conditions define in the field configuration space a subspace over which the path integral is performed. We consider the orbit of this subspace in the complexified field space generated by the action of  $\mathcal{H}$ , and perform the path integral over it. After we enlarge the theory, we gauge  $\mathcal{H}$  to account for the overcounting. For this operation not to change the physics, no real elements of  $\mathcal{H}$  should leave invariant the boundary conditions (of the pre-enlarged theory); otherwise, gauging  $\mathcal{H}$  would identify different configurations satisfying the same boundary conditions, which are physically distinct in the original description. And yet, for the resulting gauge-fixed algebra to be nicely described, we want  $\mathcal{H}$  to be large enough so that the whole  $\widehat{\mathfrak{g}}$  lies in the chiral algebra of the enlarged theory.

Let us define  $t_\pm = t_1 \pm it_2$  and  $t_0 = t_3$  so that they obey the standard  $\mathfrak{sl}_2$  commutation relations, and split  $\mathfrak{g}_{\mathbb{C}}$  as  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$  according to the eigenvalue of  $t_0$ , just as one does in quantum Drinfeld-Sokolov reduction. Then, a good choice of  $\mathcal{H}$  is  $\mathcal{G}_+^\infty$ , the



subgroup of  $\mathcal{G}_C^\infty$  corresponding to  $\mathfrak{g}_+$ . Clearly its elements change the boundary condition at  $y = 0$ , so enlarging by  $\mathcal{G}_+^\infty$  and gauging it does not change the physics. Moreover, the enlarged theory has the whole  $\widehat{\mathfrak{g}}$  in the chiral algebra. To see this, note that  $J_a \in \widehat{\mathfrak{g}}_+$  lie in the chiral algebra since the boundary condition of the enlarged theory is manifestly invariant under  $\mathcal{G}_+^\infty$ , and so does  $J_- = 2J_1 - J_+$  since  $J_1$  acts on the Nahm pole by an  $\text{Spin}(3)_R$  transformation rotating around the 1-axis. Thus, by applying  $J_-$  successively on  $J_a \in \widehat{\mathfrak{g}}_+$ , we can generate all the  $J_a$  that belong to an  $\mathfrak{sl}_2$  multiplet of nonzero highest weight. Adding the  $\mathfrak{sl}_2$  singlets, we obtain  $\widehat{\mathfrak{g}}$ .

Now gauging  $\mathcal{G}_+^\infty$ , we find that the chiral algebra contains a subalgebra given by the BRST cohomology computed in  $\widehat{\mathfrak{g}}$ . We can fix the gauge, for example, by setting the  $\mathfrak{g}_+$ -valued part of the gauge field  $A$  to zero at infinity. The gauge that is directly related to quantum Drinfeld-Sokolov reduction is defined by requiring that the operator  $J = J^a t_a$  take the form

$$J = t_+ + \sum_{t_a \in \mathfrak{g}_0 \oplus \mathfrak{g}_-} J^a t_a. \tag{3.6}$$

Assuming that the level  $k$  of  $\widehat{\mathfrak{g}}$  is nonzero and finite (which we believe is generically true from the AGT conjecture),  $J/k$  transforms under  $\mathcal{G}_+^\infty$  in the same way as  $A_z$  does, so we can first set the  $\mathfrak{g}_+$ -valued part of  $J$  to zero. Then we can set  $J^+ = 1$  to reach the desired form, locally on  $C$ . To set  $J^+ = 1$  globally, in general we need to twist the theory further by identifying the diagonal of  $U(1)_C \times U(1)_\rho$  with the holonomy of  $C$ , where  $U(1)_\rho$  is the  $U(1)$  subgroup of  $\mathcal{G}^\infty$  generated by  $t_0$ . After that,  $J^+$  can be thought of as a section of a trivial bundle and equated with a global section. The residual gauge freedom is the antiholomorphic elements of  $\mathcal{G}_+^\infty$  which we could apply right after we set the  $\mathfrak{g}_+$ -part of  $J$  to zero. We can fix it if we want, but this is not necessary.

The above gauge-fixing procedure precisely reproduces<sup>3</sup> the constraints imposed by the quantum Drinfeld-Sokolov reduction of  $\widehat{\mathfrak{g}}$  with respect to the principal  $\mathfrak{sl}_2$  embedding  $\rho$ . Therefore, the gauge-fixed algebra is the W-algebra associated to  $(\widehat{\mathfrak{g}}, \rho)$ . This was what we wanted to see.

Our argument readily generalizes to the case where the  $(2, 0)$  theory has a number of half-BPS codimension-two defect operators [8, 9] inserted at points on  $C$ . This is simply because we can define the  $J_a$  away from those points. (If we place  $J_a$  in the neighborhood of one of the insertion points, then we can use the Ward identity to deduce the action of  $J_a$  on the defect operator there.) In the five-dimensional description, the defects create singularities in the gauge field so that the fields transform by nontrivial monodromies as they go around the insertion points. Defect operators of this type change the  $\mathcal{N} = 2$  theory on the four-dimensional side of the AGT correspondence, while introducing vertex operators on the two-dimensional side.

We can insert yet another defect operator at the tip of the cigar. Upon compactification on  $C$ , this one becomes a half-BPS surface operator in the  $\mathcal{N} = 2$  theory. It creates a singularity of the form  $A \sim \alpha d\theta$  at the tip, where  $\alpha$  is in the Lie algebra of a maximal

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<sup>3</sup>To recast the standard BRST formalism (see [31] for example) in the form used in [16], one sets the gauge-fixing condition  $F^\alpha = -2i(J^\alpha - \delta^{\alpha+})$  for  $t_\alpha \in \mathfrak{g}_+$ , adds the BRST transformation of  $f^{\alpha\beta}{}_\gamma b_\alpha b_\beta c^\gamma + b_\alpha B^\alpha$  to the action, and integrates  $B_\alpha$  out.

torus  $\mathbb{T} \subset G$  and  $\theta$  is the azimuthal coordinate of the cigar. The gauge group is broken on the surface to the maximal subgroup  $\mathbb{L} \subset G$  commuting with  $\alpha$ , called the Levi subgroup. If the theory is instead compactified on the circle of the cigar, the monodromy is lost but something else happens: the Nahm pole changes. For example, when there is a “full” surface operator [32] for which  $\mathbb{L} = \mathbb{T}$ , the Nahm pole is zero and we get  $\widehat{\mathfrak{g}}$  in the chiral algebra, as is consistent with the results found in [32, 33]. In general, it is believed that the Nahm pole in the presence of a defect operator is one whose  $t_+$  is a principal nilpotent element in the complexification of the Lie algebra of  $\mathbb{L}$ .<sup>4</sup> (In the case of  $\mathfrak{g} = A_N$ , this conclusion was essentially obtained in [34].) This explains the appearance of the W-algebra associated to this  $\mathfrak{sl}_2$  embedding in the generalization of the AGT correspondence proposed in [5, 6].

#### 4 Role of the $\Omega$ -deformation

Even though we have identified the W-algebra in the chiral algebra of the  $(2, 0)$  theory, one mystery remains: what is the role of the  $\Omega$ -deformation in our story? We conclude this paper by giving a possible answer to this question.

Suppose that we can introduce some operation in the  $(2, 0)$  theory on  $\mathbb{R}^4 \times C$  that reduces to the  $\Omega$ -deformation when the theory is compactified on  $C$ . However that is realized, this “six-dimensional  $\Omega$ -deformation” must exploit in some way or another the rotations in two orthogonal two-planes in  $\mathbb{R}^4$ . Then, it would modify the relation  $Q^2 = 0$  (modulo a “gauge transformation”) by adding to the right-hand side the conserved charges generating these rotations. This is what happens in the case of the usual  $\Omega$ -deformation in four dimensions. To define the  $Q$ -cohomology in such a situation, we have to project the algebra of operators to the subalgebra of  $Q^2$ -closed operators. This projection would be harmless if the  $(2, 0)$  theory were a gauge theory, so that we could define the  $J_a$  directly in six dimensions in a manner that is manifestly rotation invariant. In reality, we need first go down to five dimensions, whereby we lose one of the directions in which the  $\Omega$ -deformation is performed. Because of this reduction it is far from obvious whether  $J_a$  would survive the  $\Omega$ -deformation or not.

There is, however, a sufficient condition for a given  $J_a$  to survive. In going down to five dimensions, we took  $M$  to be the product of a cigar and  $\mathbb{R}^2$ , and sent the radius of the cigar to zero. For definiteness, suppose that the cigar was made of a half-cylinder  $\mathbb{R}_+ \times S^1$  capped with a hemisphere. We could turn on the  $\Omega$ -deformation on the cigar using the rotations around its axis. A peculiar property of the  $\Omega$ -deformation is that we can cancel such a deformation on the flat cylinder part by a change of variables [28]. In this “undeformed” description, the effect of the  $\Omega$ -deformation localizes on the hemisphere, so the  $\Omega$ -deformed  $Q$  obeys  $Q^2 = V$  for some conserved charge  $V$  whose current is supported there. After we compactify the theory, the cigar becomes  $\mathbb{R}_+$  and some boundary state  $|\Psi\rangle$  appears at  $y = 0$ . Since  $V$  is now supported at  $y = 0$ , the statement that  $J_a$  is  $V$ -closed is equivalent to saying that  $\langle \Psi' | [V, J_a] | \Psi \rangle = 0$  for any states  $|\Psi'\rangle$  placed at a  $y$ -slice infinitesimally close to  $y = 0$ . Noting that  $|\Psi\rangle$  is  $Q$ - and hence  $V$ -closed as the boundary conditions of the

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<sup>4</sup>I thank Yuji Tachikawa for explaining this point to me.

original theory were chosen that way, we see that this happens if  $J_a|\Psi\rangle = 0$ . Therefore,  $J_a$  is  $V$ -closed if it kills the boundary state created by the compactification.

So it seems to be a good idea to project out those  $J_a$  that act nontrivially on  $|\Psi\rangle$ . But this projection is exactly what we did to avoid the boundary contribution! Then, this could be a more fundamental reason as to why we should carry out the projection: to define the chiral algebra in the presence of the  $\Omega$ -deformation.

In fact, in the absence of the  $\Omega$ -deformation or defect operator, we could do away with the projection altogether by choosing a different way to compactify the theory. For example, the  $U(1)$  action on  $\mathbb{R}^4 \cong \mathbb{C}^2$  defined by  $(z_1, z_2) \rightarrow (e^{i\theta}z_1, e^{i\theta}z_2)$  gives a smooth quotient without boundary,  $\mathbb{C}^2/U(1) \cong \mathbb{R}^3$ . For such a choice the issue of emergent boundary does not occur, and we expect to get the full  $\widehat{\mathfrak{g}}$ .<sup>5</sup> One of the motivations behind our choice was that it would make our construction compatible with this hypothetical  $\Omega$ -deformation of the  $(2, 0)$  theory.

## A Quantum Drinfeld-Sokolov reduction

Given an affine Lie algebra  $\widehat{\mathfrak{g}}$  and an embedding  $\rho: \mathfrak{sl}_2 \rightarrow \mathfrak{g}_{\mathbb{C}}$ , quantum Drinfeld-Sokolov reduction [16] produces a vertex algebra isomorphic, up to a shift in level, to a subalgebra of the universal enveloping algebra  $U(\widehat{\mathfrak{g}})$  of  $\widehat{\mathfrak{g}}$ . This is achieved by imposing constraints on the affine currents by means of BRST gauge fixing. The resulting algebra is the  $W$ -algebra associated to the pair  $(\widehat{\mathfrak{g}}, \rho)$ , denoted by  $W(\widehat{\mathfrak{g}}, \rho)$ .

The choice of the  $\mathfrak{sl}_2$  embedding gives a decomposition of  $\mathfrak{g}_{\mathbb{C}}$  into  $\mathfrak{sl}_2$  multiplets. Take a basis  $\{t_+, t_0, t_-\}$  of the  $\mathfrak{sl}_2$  subalgebra satisfying

$$[t_+, t_-] = 2t_0, \quad [t_0, t_{\pm}] = \pm t_{\pm}, \tag{A.1}$$

and extend it to a complete basis  $\{t_a\}$  of  $\mathfrak{g}_{\mathbb{C}}$ . We assume that  $t_a$  have an integer spin (that is,  $[t_0, t_a] = s_a t_a$  for some  $s_a \in \mathbb{Z}$ ), and write  $\mathfrak{g}_+$ ,  $\mathfrak{g}_0$ , and  $\mathfrak{g}_-$  for the subalgebras of spin positive, zero, and negative, respectively. Then the constraints imposed by the quantum Drinfeld-Sokolov reduction are

$$J^a = 0 \tag{A.2}$$

for all  $t_a \in \mathfrak{g}_+$  except  $t_+$ , and

$$J^+ = 1. \tag{A.3}$$

Here we have raised the index of  $J_a$  using the Killing form. These constraints may be thought of as coming from gauge fixing a certain variant of gauged WZW model [43],

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<sup>5</sup>As a variation of this construction, take  $M$  to be the ALE space obtained by a hyperkähler resolution of the orbifold  $\mathbb{C}^2/\mathbb{Z}_k$ . This space may be thought of as an  $S^1 \times \mathbb{R}$ -fibration over  $\mathbb{C}$ , where the fiber at  $z \in \mathbb{C}$  is given by the equation  $u^2 + v^2 = -\prod_{i=1}^k (z - a_i)$  with  $u, v, a_i \in \mathbb{C}$  and the  $a_i$  distinct. The quotient by the  $S^1$ -action is  $\mathbb{R}^3$ . So again, we expect to get  $\widehat{\mathfrak{g}}$ . This expectation fits nicely with I. Frenkel's conjecture that the cohomology of the moduli space of framed  $G$ -instantons on  $\mathbb{C}^2/\mathbb{Z}_k$  should carry a level  $k$  representation of  $\widehat{\mathfrak{g}}$ , which has been partially proved by Licata [35] and led Braverman and Finkelberg [36] to propose geometric Langlands duality for complex surfaces. See [37–40] for physical explanations on this point. In the case  $k = 1$ , we have argued that turning on the  $\Omega$ -deformation reduces  $\widehat{\mathfrak{g}}$  (which now has a different level) to a  $W$ -algebra. For a general value of  $k$ , it should reduce  $\widehat{\mathfrak{g}}$ , possibly combined with other subalgebras of the chiral algebra, to a parafermionic  $W$ -algebra [41, 42].

hence can be imposed using the BRST formalism. The BRST cohomology computed in  $U(\widehat{\mathfrak{g}})$  is  $W(\widehat{\mathfrak{g}}, \rho)$ .

One subtlety in the above procedure is that the constraint  $J^+ = 1$  breaks conformal invariance if we use the standard Sugawara energy-momentum tensor  $T$ , under which  $J_a$  have conformal weight one. We remedy this problem by adding  $-\partial J_0$  to  $T$ . This shifts the conformal weight of  $J^a$  by  $-s_a$ , thereby making  $J^+$  weight zero.

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