Analytic epsilon expansion of three-loop on-shell master integrals up to four-loop transcendentality weight

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ABSTRACT: We evaluate analytically higher terms of the ϵ-expansion of the three-loop master integrals corresponding to three-loop quark and gluon form factors and to the three-loop master integrals contributing to the electron $g-2$ in QED up to the transcendentality weight typical to four-loop calculations, i.e. eight and seven, respectively. The calculation is based on a combination of a method recently suggested by one of the authors (R.L.) with other techniques: sector decomposition implemented in FIESTA, the method of Mellin-Barnes representation, and the PSLQ algorithm.

Contents

1 Introduction

One year ago a method of multiloop calculations [1] based on the use of dimensional recurrence relations [2] and analytic properties of Feynman integrals as functions of the parameter of dimensional regularization, $d = 4 - 2\epsilon$, was suggested (the DRA method). Then it was successfully applied [3] to the analytic evaluation of the terms of order ϵ^0 of the ϵ-expansion of two most complicated master integrals corresponding to the three-loop quark and gluon form factors. These results provided the possibility to obtain completely analytic expressions for the form factors. The power of the DRA method was further demonstrated in ref. [4] where the three previously unknown terms of the expansion in ϵ, i.e. up to transcendentality weight ten, of the three-loop non-planar massless propagator diagram were evaluated. Some additional details of the method were presented in ref. [5].

The main point of ref. [4] was to emphasize that once analytic expressions, in terms of well convergent series, are obtained within the DRA method it is easy to evaluate extra terms of the expansion in ϵ. In the present paper we further exploit this nice feature of the method by evaluating, up to the transcendentality weight typical to four-loop calculations, higher terms of the ϵ-expansion of the three-loop master integrals corresponding to three-loop quark and gluon form factors and to the three-loop master integrals contributing to the electron $g - 2$ in QED.

In the first of these two problems, all the master integrals apart from three most complicated ones were evaluated in refs. [6, 7]. In fact, the word *evaluated* means here the evaluation up to the order of ϵ which appears in the finite part of the form factors. Mathematically, this means the evaluation up to transcendentality weight six. Then one of the three most complicated master integrals (called $A_{9,1}$ in refs. [6, 7] and in the present paper) and the pole parts of $A_{9,4}$ and $A_{9,2}$ (shown in figure 1 in the next section) were evaluated analytically, while the ϵ^0 parts of $A_{9,4}$ and $A_{9,2}$ were evaluated numerically — see refs. [8, 9]. These two missing ingredients were calculated in ref. [3]. Motivated by future four-loop calculations, the authors of ref. [10] presented one more term of the ϵ-expansion for all the master integrals but $A_{9,1}$, $A_{9,2}$ and $A_{9,4}$. For $A_{9,1}$, this was done in [4]. In the present paper we evaluate all the master integrals up to transcendentality

weight eight which is intrinsic to a four-loop evaluation of the form factors. Explicitly, the constants present in results for highest powers of ϵ are linear combinations of $\pi^8, \zeta_3^2 \pi^2, \zeta_5 \zeta_3$ and $\zeta_{-6,-2}$.

The second family of master integrals we are evaluating is connected with the anomalous magnetic moment of the muon, i.e. $g - 2$ factor. The three-loop evaluation was performed in refs. [11, 12]. The evaluation at the four-loop level, within a pure numerical approach where each of the Feynman integrals involved is evaluated numerically, without a reduction to master integrals, was already performed — see ref. [13] and references therein. Even the numerical evaluation in five loops has been started — see a recent paper [14] and references therein.

In ref. [15] the status of partially analytic evaluation of the four-loop $g - 2$ factor was presented. In this approach, an integration by parts (IBP) reduction [16] to master integrals is used. However, it was pointed out that the analytic evaluation of the corresponding master integrals was not possible for the moment. To evaluate the master integrals numerically with a high precision it was supposed to use the method developed in refs. [17, 18] and based on difference equations. Keeping in mind that all the three-loop $g - 2$ master integrals will inevitably appear in a four-loop calculation because they are present in factorizable four-loop diagrams where three-loop master integrals enter in products with one-loop integrals having poles in ϵ, Laporta calculated numerically [19] several higher order terms of the ϵ-expansion of the three-loop $g - 2$ master integrals, with the use of the method of refs. [17, 18].

We are more optimistic about the possibility to evaluate four-loop $g-2$ master integrals analytically. To make a step in this direction we have performed an analytical calculation corresponding to the numerical calculation of ref. [19]. More precisely, we have evaluated analytically higher terms of the ϵ-expansion of the three-loop $g - 2$ master integrals up to transcendentality weight seven which is intrinsic to a four-loop evaluation of the electron $g - 2$. So, the highest terms we present involve $\zeta_7, \zeta_5 \pi^2, \mathrm{Li}_7(1/2), \ln^7 2, \zeta_{-5,1,1}$, etc. In our calculation we use the DRA method, obtain results for the master integrals with a very high precision ($\sim 350 - 500$ digits) and then apply the so-called PSLQ algorithm [20] to arrive at analytical values. There are fifty four independent transcendental constants in the four loop basis so that this high precision is mandatory. (Typically, one needs the accuracy of at least seven digits per constant. Observe that this high accuracy is not provided by the method of ref. [17, 18].)

The three-loop $g-2$ master integrals were analytically evaluated in refs. [11, 12, 21, 22]. In fact, in refs. [21, 22], one additional master integral was considered, which is important for the analytic calculation of the three-loop renormalization constants previously obtained numerically in refs. [23, 24]. This three-loop evaluation was typically performed up to transcendentality weight five, with some exceptions where some master integrals were expanded in ϵ up to weight four.

In the next two sections we present results for the master integrals for the three-loop form factors and for the $g - 2$ factor, respectively.
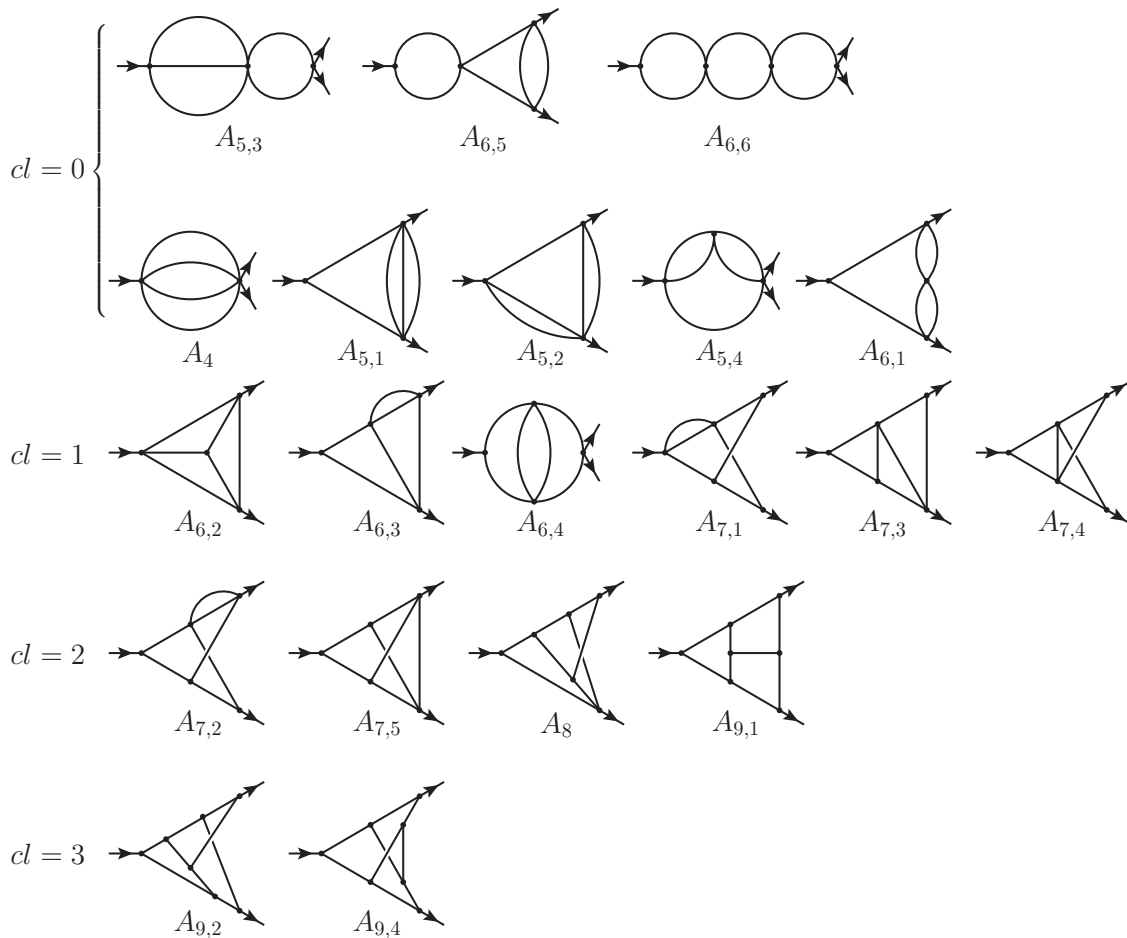
Figure 1. Master integrals for the three-loop form factors. Internal lines denote massless propagators $1/(k^2 + i0)$.

2 Master integrals for quark and gluon form factors

The master integrals for the three-loop form factors are shown in figure 1. Two external momenta are on the light cone, $p_1^2 = p_2^2 = 0$. In ref. [4] the DRA method was applied to the evaluation of these master integrals. The corresponding results for arbitrary d are obtained in the form of multiple well-convergent series. Here we present our analytical results on the higher terms of the ϵ expansion which were obtained by calculating these series with a high precision and then applying the PSLQ algorithm [20].

The high orders of ϵ-expansion allowed us to guess, for all master integrals, except $A_{6,4}$ and the three most complicated ones ($A_{9,1}$, $A_{9,2}$, and $A_{9,4}$), the factors which makes the expansion homogeneous in the transcendentality weight. Taking into account the fact that these four master integrals can be replaced by the integrals with numerators [9] (or denominator squared for $A_{6,4}$), see figure 2, which are also homogeneous in the transcendentality weight, we have a complete basis of master integrals with homogeneous transcendentality weight. Let us present the results for the expansion of the integrals in this basis. The loop
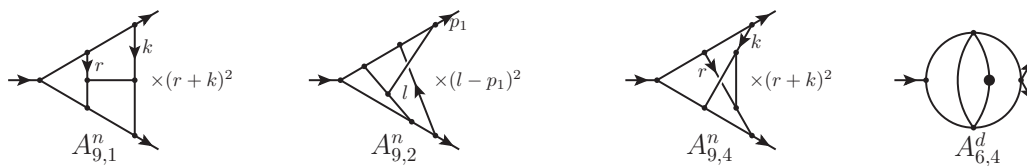
Figure 2. Master integrals with a homogeneous transcendentality weight to replace the corresponding integrals without numerator.

integration measure is taken as

$$\frac{d^d k}{i\pi^{d/2}} .\tag{2.1}$$

We arrange the results in the order of increasing *complexity level* [4]. This notion can be determined recursively. The complexity level of the master integral J is equal to $n + 1$, where n is the maximum complexity level of the master integrals, which can be obtained from J by shrinking some of its internal lines. The integrals which nullify upon shrinking of any internal line have zero complexity level.

Zero complexity level. Master integrals with the zero complexity level nullify upon shrinking of any internal line. As we have already argued in ref. [4], they are always expressed in terms of the Γ-functions. We present their expansion here only for convenience of the reader.

$$
\begin{aligned}
A_{5,3}(4-2\epsilon) &= -\frac{\Gamma(1-\epsilon)^5\Gamma(\epsilon)\Gamma(2\epsilon-1)}{\Gamma(3-3\epsilon)\Gamma(2-2\epsilon)} \\
&= \frac{e^{-3\gamma\epsilon}}{(1-3\epsilon)(1-2\epsilon)^2(1-3\epsilon/2)}\Bigg\{\frac{1}{4\epsilon^2} - \frac{\pi^2}{16} - \frac{13\epsilon\zeta_3}{4} - \frac{17\pi^4\epsilon^2}{384} \\
&\quad -\epsilon^3\left(\frac{303\zeta_5}{20} - \frac{13\pi^2\zeta_3}{16}\right) + \epsilon^4\left(\frac{169\zeta_3^2}{8} - \frac{9439\pi^6}{483840}\right) + \epsilon^5\left(\frac{221\pi^4\zeta_3}{384}\right. \\
&\quad \left.+\frac{303\pi^2\zeta_5}{80} - \frac{2439\zeta_7}{28}\right) + \epsilon^6\left(-\frac{169}{32}\pi^2\zeta_3^2 + \frac{3939\zeta_5\zeta_3}{20} - \frac{76397\pi^8}{7741440}\right) \\
&\quad +O\left(\epsilon^7\right)\Bigg\},
\end{aligned}\tag{2.2}
$$

$$
\begin{aligned}
A_{6,5}(4-2\epsilon) &= \frac{\Gamma(1-2\epsilon)^2\Gamma(1-\epsilon)^4\Gamma(\epsilon)^2\Gamma(2\epsilon)}{\Gamma(2-3\epsilon)\Gamma(2-2\epsilon)^2} \\
&= \frac{e^{-3\gamma\epsilon}}{(1-3\epsilon)(1-2\epsilon)^2}\Bigg\{\frac{1}{2\epsilon^3} + \frac{\pi^2}{24\epsilon} - \frac{11\zeta_3}{2} - \frac{77\pi^4\epsilon}{960} - \epsilon^2\left(\frac{11\pi^2\zeta_3}{24}\right. \\
&\quad \left.+\frac{273\zeta_5}{10}\right) - \epsilon^3\left(\frac{5233\pi^6}{80640} - \frac{121\zeta_3^2}{4}\right) - \epsilon^4\left(-\frac{847}{960}\pi^4\zeta_3 + \frac{91\pi^2\zeta_5}{40}\right. \\
&\quad \left.+\frac{2313\zeta_7}{14}\right) + \epsilon^5\left(\frac{121}{48}\pi^2\zeta_3^2 + \frac{3003\zeta_5\zeta_3}{10} - \frac{2326579\pi^8}{58060800}\right) \\
&\quad +O\left(\epsilon^6\right)\Bigg\},
\end{aligned}\tag{2.3}
$$

$$A_{6,6}(4-2\epsilon) = \frac{\Gamma(1-\epsilon)^6 \Gamma(\epsilon)^3}{\Gamma(2-2\epsilon)^3}$$

$$= \frac{e^{-3\gamma\epsilon}}{(1-2\epsilon)^3} \left\{ \frac{1}{\epsilon^3} - \frac{\pi^2}{4\epsilon} - 7\zeta_3 - \frac{37\pi^4\epsilon}{480} + \epsilon^2 \left(\frac{7\pi^2\zeta_3}{4} - \frac{93\zeta_5}{5} \right) \right.$$

$$+ \epsilon^3 \left(\frac{49\zeta_3^2}{2} - \frac{943\pi^6}{120960} \right) + \epsilon^4 \left(\frac{259\pi^4\zeta_3}{480} + \frac{93\pi^2\zeta_5}{20} - \frac{381\zeta_7}{7} \right)$$

$$\left. + \epsilon^5 \left(-\frac{49}{8}\pi^2\zeta_3^2 + \frac{651\zeta_5\zeta_3}{5} + \frac{6527\pi^8}{9676800} \right) + O\left(\epsilon^6 \right) \right\}, \tag{2.4}$$

$$A_4(4-2\epsilon) = \frac{\Gamma(1-\epsilon)^4 \Gamma(3\epsilon-2)}{\Gamma(4-4\epsilon)}$$

$$= \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(3-4\epsilon)(1-3\epsilon)(2-3\epsilon)(1-2\epsilon)} \left\{ \frac{1}{6\epsilon} - \frac{\pi^2\epsilon}{24} - \frac{29\epsilon^2\zeta_3}{6} \right.$$

$$- \frac{71\pi^4\epsilon^3}{960} - \epsilon^4 \left(\frac{421\zeta_5}{10} - \frac{29\pi^2\zeta_3}{24} \right) + \epsilon^5 \left(\frac{841\zeta_3^2}{12} - \frac{11539\pi^6}{145152} \right)$$

$$- \epsilon^6 \left(-\frac{2059}{960}\pi^4\zeta_3 - \frac{421\pi^2\zeta_5}{40} + \frac{6189\zeta_7}{14} \right) + \epsilon^7 \left(-\frac{841}{48}\pi^2\zeta_3^2 + \frac{12209\zeta_5\zeta_3}{10} \right.$$

$$\left. \left. - \frac{737687\pi^8}{8294400} \right) + O\left(\epsilon^8 \right) \right\}, \tag{2.5}$$

$$A_{5,1}(4-2\epsilon) = -\frac{\Gamma(2-3\epsilon)^2 \Gamma(1-\epsilon)^3 \Gamma(2\epsilon-1) \Gamma(3\epsilon-1)}{\Gamma(3-4\epsilon) \Gamma(3-3\epsilon)}$$

$$= \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(2-3\epsilon)(1-2\epsilon)^2} \left\{ -\frac{1}{12\epsilon^2} - \frac{\pi^2}{16} + \frac{23\epsilon\zeta_3}{12} - \frac{7\pi^4\epsilon^2}{1152} \right.$$

$$+ \epsilon^3 \left(\frac{23\pi^2\zeta_3}{16} + \frac{351\zeta_5}{20} \right) + \epsilon^4 \left(\frac{65243\pi^6}{1451520} - \frac{529\zeta_3^2}{24} \right) + \epsilon^5 \left(\frac{161\pi^4\zeta_3}{1152} \right.$$

$$\left. + \frac{1053\pi^2\zeta_5}{80} + \frac{5503\zeta_7}{28} \right) + \epsilon^6 \left(-\frac{529}{32}\pi^2\zeta_3^2 - \frac{8073\zeta_5\zeta_3}{20} + \frac{75527\pi^8}{860160} \right)$$

$$\left. + O\left(\epsilon^7 \right) \right\}, \tag{2.6}$$

$$A_{5,2}(4-2\epsilon) = -\frac{\Gamma(2-3\epsilon) \Gamma(1-2\epsilon) \Gamma(1-\epsilon)^4 \Gamma(\epsilon) \Gamma(3\epsilon-1)}{\Gamma(3-4\epsilon) \Gamma(2-2\epsilon)^2}$$

$$= \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(1-2\epsilon)^3} \left\{ \frac{1}{6\epsilon^2} + \frac{\pi^2}{24} - \frac{23\epsilon\zeta_3}{6} - \frac{25\pi^4\epsilon^2}{576} - \epsilon^3 \left(\frac{23\pi^2\zeta_3}{24} \right. \right.$$

$$\left. + \frac{351\zeta_5}{10} \right) - \epsilon^4 \left(\frac{66041\pi^6}{725760} - \frac{529\zeta_3^2}{12} \right) - \epsilon^5 \left(-\frac{575}{576}\pi^4\zeta_3 + \frac{351\pi^2\zeta_5}{40} \right.$$

$$\left. + \frac{5503\zeta_7}{14} \right) - \epsilon^6 \left(-\frac{529}{48}\pi^2\zeta_3^2 - \frac{8073\zeta_5\zeta_3}{10} + \frac{1513373\pi^8}{11612160} \right)$$

$$\left. + O\left(\epsilon^7 \right) \right\}, \tag{2.7}$$

$$A_{5,4}(4-2\epsilon) = -\frac{\Gamma(2-3\epsilon)\Gamma(1-\epsilon)^5\Gamma(\epsilon)^2\Gamma(3\epsilon-1)}{\Gamma(3-4\epsilon)\Gamma(2-2\epsilon)^2\Gamma(2\epsilon)}$$

$$= \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(1-2\epsilon)^3}\left\{ \frac{1}{3\epsilon^2} - \frac{\pi^2}{12} - \frac{23\epsilon\zeta_3}{3} - \frac{11\pi^4\epsilon^2}{96} - \epsilon^3\left(\frac{351\zeta_5}{5} \right.\right.$$

$$\left. -\frac{23\pi^2\zeta_3}{12} \right) - \epsilon^4\left(\frac{49199\pi^6}{362880} - \frac{529\zeta_3^2}{6} \right) - \epsilon^5\left(-\frac{253}{96}\pi^4\zeta_3 - \frac{351\pi^2\zeta_5}{20} \right.$$

$$\left. +\frac{5503\zeta_7}{7} \right) + \epsilon^6\left(-\frac{529}{24}\pi^2\zeta_3^2 + \frac{8073\zeta_5\zeta_3}{5} - \frac{1006669\pi^8}{5806080} \right)$$

$$\left. +O\left(\epsilon^7\right) \right\}, \tag{2.8}$$

$$A_{6,1}(4-2\epsilon) = \frac{\Gamma(1-3\epsilon)^2\Gamma(1-\epsilon)^4\Gamma(\epsilon)^2\Gamma(3\epsilon)}{\Gamma(2-4\epsilon)\Gamma(2-2\epsilon)^2}$$

$$= \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(1-2\epsilon)^2}\left\{ \frac{1}{3\epsilon^3} + \frac{\pi^2}{4\epsilon} - \frac{17\zeta_3}{3} + \frac{83\pi^4\epsilon}{1440} - \epsilon^2\left(\frac{17\pi^2\zeta_3}{4} \right.\right.$$

$$\left. +\frac{281\zeta_5}{5} \right) - \epsilon^3\left(\frac{44651\pi^6}{362880} - \frac{289\zeta_3^2}{6} \right) - \epsilon^4\left(\frac{1411\pi^4\zeta_3}{1440} + \frac{843\pi^2\zeta_5}{20} \right.$$

$$\left. +\frac{4817\zeta_7}{7} \right) - \epsilon^5\left(-\frac{289}{8}\pi^2\zeta_3^2 - \frac{4777\zeta_5\zeta_3}{5} + \frac{960457\pi^8}{3225600} \right)$$

$$\left. +O\left(\epsilon^6\right) \right\}. \tag{2.9}$$

Non-zero complexity level. Integrals with complexity level equal to $n > 0$ are expressed in arbitrary dimension d in terms of an n-fold series. We present these integrals in the same order as in figure 1. We also present results for $A_{6,4}^d$, $A_{9,1}^n$, $A_{9,2}^n$, and $A_{9,3}^n$ which have the uniform transcendentality and can be used as master integrals instead of $A_{6,4}$, $A_{9,1}$, $A_{9,2}$, and $A_{9,3}$, respectively.

$$A_{6,2}(4-2\epsilon) = \frac{e^{-3\gamma\epsilon}}{(1-5\epsilon)(1-4\epsilon)}\left\{ \frac{2\zeta_3}{\epsilon} + \frac{7\pi^4}{180} - \epsilon\left(\frac{7\pi^2\zeta_3}{6} - 10\zeta_5 \right) \right.$$

$$\left. -\epsilon^2\left(78\zeta_3^2 + \frac{473\pi^6}{15120} \right) - \epsilon^3\left(\frac{415\pi^4\zeta_3}{144} + \frac{5\pi^2\zeta_5}{2} + \frac{445\zeta_7}{2} \right) \right.$$

$$\left. -\epsilon^4\left(-\frac{992}{3}\zeta_{-6,-2} - \frac{253}{6}\pi^2\zeta_3^2 + \frac{2576\zeta_5\zeta_3}{5} + \frac{425689\pi^8}{1814400} \right) \right.$$

$$\left. +O\left(\epsilon^5\right) \right\}, \tag{2.10}$$

$$A_{6,3}(4-2\epsilon) = \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(1-3\epsilon)(1-2\epsilon)}\left\{\frac{1}{6\epsilon^3} + \frac{\pi^2}{8\epsilon} - \frac{35\zeta_3}{6} - \frac{77\pi^4\epsilon}{2880}\right.$$

$$-\epsilon^2\left(\frac{49\pi^2\zeta_3}{24} + \frac{651\zeta_5}{10}\right) + \epsilon^3\left(\frac{1141\zeta_3^2}{12} - \frac{93451\pi^6}{725760}\right)$$

$$-\epsilon^4\left(\frac{713\pi^2\zeta_5}{40} - \frac{511}{320}\pi^4\zeta_3 + \frac{9017\zeta_7}{14}\right) + \epsilon^5\left(\frac{623}{48}\pi^2\zeta_3^2\right.$$

$$\left.\left.-\frac{544}{9}\zeta_{-6,-2} + \frac{11195\zeta_5\zeta_3}{6} - \frac{2022493\pi^8}{11612160}\right) + O\left(\epsilon^6\right)\right\}, \qquad (2.11)$$

$$A_{6,4}^d(4-2\epsilon) = \frac{6e^{-3\gamma\epsilon}}{1-2\epsilon}\left\{\frac{\zeta_3}{\epsilon} + \frac{\pi^4}{60} + \epsilon\left(-\frac{1}{4}\pi^2\zeta_3 + 17\zeta_5\right) + \epsilon^2\left(\frac{577\pi^6}{15120} - 20\zeta_3^2\right)\right.$$

$$+\epsilon^3\left(-\frac{301}{480}\pi^4\zeta_3 - \frac{17\pi^2\zeta_5}{4} + \frac{471\zeta_7}{2}\right) + \epsilon^4\left(\frac{25579\pi^8}{604800} + 5\pi^2\zeta_3^2\right.$$

$$\left.\left.-\frac{1838\zeta_3\zeta_5}{5} + 48\zeta_{-6,-2}\right) + O\left(\epsilon^5\right)\right\}, \qquad (2.12)$$

$$A_{6,4}(4-2\epsilon) = e^{-3\gamma\epsilon}\left\{\frac{1}{3\epsilon^3} + \frac{7}{3\epsilon^2} + \epsilon^{-1}\left(\frac{31}{3} - \frac{\pi^2}{12}\right) + \left(\frac{103}{3} - \frac{7\pi^2}{12} + \frac{7\zeta_3}{3}\right)\right.$$

$$+\epsilon\left(\frac{235}{3} - \frac{31\pi^2}{12} + \frac{49\zeta_3}{3} + \frac{5\pi^4}{96}\right) + \epsilon^2\left(\frac{19}{3} - \frac{103\pi^2}{12} + \frac{289\zeta_3}{3}\right.$$

$$\left.+\frac{35\pi^4}{96} - \frac{7\pi^2\zeta_3}{12} + \frac{599\zeta_5}{5}\right) + \epsilon^3\left(-\frac{3953}{3} - \frac{235\pi^2}{12} + \frac{1729\zeta_3}{3}\right.$$

$$\left.+\frac{967\pi^4}{480} - \frac{49\pi^2\zeta_3}{12} + \frac{4193\zeta_5}{5} + \frac{108481\pi^6}{362880} - \frac{599\zeta_3^2}{6}\right) + \epsilon^4\left(-\frac{31889}{3}\right.$$

$$-\frac{19\pi^2}{12} + \frac{10213\zeta_3}{3} + \frac{5263\pi^4}{480} - \frac{289\pi^2\zeta_3}{12} + \frac{20609\zeta_5}{5} + \frac{108481\pi^6}{51840}$$

$$\left.-\frac{4193\zeta_3^2}{6} - \frac{1553\pi^4\zeta_3}{480} - \frac{599\pi^2\zeta_5}{20} + \frac{13593\zeta_7}{7}\right) + \epsilon^5\left(-\frac{188141}{3}\right.$$

$$+\frac{3953\pi^2}{12} + \frac{57445\zeta_3}{3} + \frac{28723\pi^4}{480} - \frac{1729\pi^2\zeta_3}{12} + \frac{90257\zeta_5}{5}$$

$$+\frac{3695263\pi^6}{362880} - \frac{21449\zeta_3^2}{6} - \frac{10871\pi^4\zeta_3}{480} - \frac{4193\pi^2\zeta_5}{20} + 13593\zeta_7$$

$$\left.\left.+\frac{1913939\pi^8}{5806080} + \frac{599}{24}\pi^2\zeta_3^2 - \frac{9847\zeta_3\zeta_5}{5} + 576\zeta_{-6,-2}\right) + O\left(\epsilon^6\right)\right\}, \qquad (2.13)$$

$$A_{7,1}(4-2\epsilon) = \frac{e^{-3\gamma\epsilon}}{1-2\epsilon}\left\{-\frac{1}{4\epsilon^5} + \frac{11\pi^2}{48\epsilon^3} + \frac{41\zeta_3}{4\epsilon^2} + \frac{227\pi^4}{1152\epsilon} + \left(\frac{1763\zeta_5}{20} - \frac{355\pi^2\zeta_3}{48}\right)\right.$$

$$-\epsilon\left(\frac{1649\zeta_3^2}{8} - \frac{32759\pi^6}{483840}\right) - \epsilon^2\left(\frac{43847\pi^4\zeta_3}{5760} + \frac{4971\pi^2\zeta_5}{80} - \frac{16397\zeta_7}{28}\right)$$

$$-\epsilon^3\left(-352\zeta_{-6,-2} - \frac{12251}{96}\pi^2\zeta_3^2 + \frac{60603\zeta_5\zeta_3}{20} + \frac{23355197\pi^8}{116121600}\right)$$

$$\left.+O\left(\epsilon^4\right)\right\}, \qquad (2.14)$$

$$A_{7,2}(4-2\epsilon) = \frac{e^{-3\gamma\epsilon}}{1-2\epsilon}\left\{-\frac{\pi^2}{12\epsilon^3} - \frac{2\zeta_3}{\epsilon^2} - \frac{17\pi^4}{180\epsilon} + \left(\frac{9\pi^2\zeta_3}{4} - 15\zeta_5\right)\right.$$

$$+\epsilon\left(75\zeta_3^2 - \frac{1199\pi^6}{362880}\right) + \epsilon^2\left(\frac{2959\pi^4\zeta_3}{720} + \frac{262\pi^2\zeta_5}{15} + \frac{1383\zeta_7}{8}\right)$$

$$+\epsilon^3\left(-\frac{2912}{9}\zeta_{-6,-2} - \frac{883}{24}\pi^2\zeta_3^2 + \frac{12493\zeta_5\zeta_3}{15} + \frac{21377\pi^8}{75600}\right)$$

$$\left. +O\left(\epsilon^4\right)\right\}, \tag{2.15}$$

$$A_{7,3}(4-2\epsilon) = e^{-3\gamma\epsilon}\left\{\epsilon^{-1}\left(\frac{\pi^2\zeta_3}{6} + 10\zeta_5\right) + \left(\frac{31\zeta_3^2}{2} + \frac{119\pi^6}{2160}\right)\right.$$

$$+\epsilon\left(\frac{61\pi^4\zeta_3}{360} + \frac{23\pi^2\zeta_5}{3} + \frac{1279\zeta_7}{4}\right) + \epsilon^2\left(\frac{656}{3}\zeta_{-6,-2}\right.$$

$$\left.\left. -\frac{149}{24}\pi^2\zeta_3^2 + 775\zeta_5\zeta_3 + \frac{48707\pi^8}{604800}\right) + O\left(\epsilon^3\right)\right\}, \tag{2.16}$$

$$A_{7,4}(4-2\epsilon) = \frac{e^{-3\gamma\epsilon}}{1-6\epsilon}\left\{-\frac{6\zeta_3}{\epsilon^2} - \frac{11\pi^4}{90\epsilon} - \left(46\zeta_5 - \frac{7\pi^2\zeta_3}{2}\right)\right.$$

$$+\epsilon\left(288\zeta_3^2 + \frac{109\pi^6}{1080}\right) + \epsilon^2\left(\frac{8381\pi^4\zeta_3}{720} + \frac{11\pi^2\zeta_5}{2} + 1309\zeta_7\right)$$

$$+\epsilon^3\left(-\frac{7712}{3}\zeta_{-6,-2} - 160\pi^2\zeta_3^2 + \frac{3668\zeta_5\zeta_3}{5} + \frac{72323\pi^8}{43200}\right)$$

$$\left. +O\left(\epsilon^4\right)\right\}, \tag{2.17}$$

$$A_{7,5}(4-2\epsilon) = \frac{e^{-3\gamma\epsilon}}{1-6\epsilon}\left\{-\left(2\pi^2\zeta_3 + 10\zeta_5\right) - \epsilon\left(18\zeta_3^2 + \frac{11\pi^6}{162}\right)\right.$$

$$-\epsilon^2\left(\frac{13\pi^4\zeta_3}{30} - \frac{7\pi^2\zeta_5}{6} + \frac{307\zeta_7}{2}\right) + \epsilon^3\left(\frac{992}{9}\zeta_{-6,-2}\right.$$

$$\left.\left. +\frac{185}{2}\pi^2\zeta_3^2 + \frac{1690\zeta_5\zeta_3}{3} - \frac{2431\pi^8}{680400}\right) + O\left(\epsilon^4\right)\right\}, \tag{2.18}$$

$$A_8(4-2\epsilon) = \frac{e^{-3\gamma\epsilon}}{3\epsilon+1}\left\{-\frac{8\zeta_3}{3\epsilon^2} - \frac{5\pi^4}{27\epsilon} - \left(\frac{352\zeta_5}{3} - \frac{58\pi^2\zeta_3}{9}\right)\right.$$

$$+\epsilon\left(\frac{340\zeta_3^2}{3} - \frac{5261\pi^6}{34020}\right) + \epsilon^2\left(\frac{5431\pi^4\zeta_3}{540} + \frac{256\pi^2\zeta_5}{9} - \frac{2443\zeta_7}{3}\right)$$

$$+\epsilon^3\left(-\frac{9728}{9}\zeta_{-6,-2} - \frac{1775}{9}\pi^2\zeta_3^2 + \frac{9968\zeta_5\zeta_3}{5} + \frac{1625251\pi^8}{4082400}\right)$$

$$\left. +O\left(\epsilon^4\right)\right\}, \tag{2.19}$$

$$A_{9,1}^n(4-2\epsilon) = e^{-3\gamma\epsilon}\Bigg\{ \frac{1}{36\epsilon^6} + \frac{7\pi^2}{144\epsilon^4} + \frac{55\zeta_3}{36\epsilon^3} + \frac{5329\pi^4}{51840\epsilon^2} + \epsilon^{-1}\left(\frac{1171\pi^2\zeta_3}{432} + \frac{1199\zeta_5}{60} \right)$$

$$+ \left(\frac{353\zeta_3^2}{8} + \frac{2606843\pi^6}{13063680} \right) + \epsilon\left(\frac{61831\pi^4\zeta_3}{51840} + \frac{46657\pi^2\zeta_5}{2160} + \frac{180625\zeta_7}{252} \right)$$

$$+ \epsilon^2\left(-\frac{23680}{81}\zeta_{-6,-2} + \frac{85381\zeta_3\zeta_5}{108} - \frac{80579\pi^2\zeta_3^2}{864} + \frac{1527522487\pi^8}{3135283200} \right)$$

$$+ O\left(\epsilon^3\right) \Bigg\}, \tag{2.20}$$

$$A_{9,1}(4-2\epsilon) = e^{-3\gamma\epsilon}\Bigg\{ \frac{1}{18\epsilon^5} - \frac{1}{2\epsilon^4} + \epsilon^{-3}\left(\frac{53}{18} + \frac{29\pi^2}{216} \right) - \epsilon^{-2}\left(\frac{29}{2} + \frac{149\pi^2}{216} - \frac{35\zeta_3}{18} \right)$$

$$+ \epsilon^{-1}\left(\frac{129}{2} + \frac{139\pi^2}{72} - \frac{307\zeta_3}{18} + \frac{5473\pi^4}{25920} \right) - \left(\frac{537}{2} + \frac{19\pi^2}{8} - \frac{1103\zeta_3}{18} \right.$$

$$+ \frac{3125\pi^4}{5184} - \frac{871\pi^2\zeta_3}{216} - \frac{793\zeta_5}{10} \Bigg) + \epsilon\left(\frac{2133}{2} - \frac{97\pi^2}{8} - \frac{287\zeta_3}{2} + \frac{4717\pi^4}{2880} \right.$$

$$+ \frac{2969\pi^2\zeta_3}{216} - \frac{8251\zeta_5}{30} + \frac{76801\pi^6}{186624} + \frac{5521\zeta_3^2}{36} \Bigg) - \epsilon^2\left(\frac{8181}{2} - \frac{969\pi^2}{8} \right.$$

$$- \frac{195\zeta_3}{2} + \frac{1333\pi^4}{320} + \frac{5887\pi^2\zeta_3}{72} - \frac{22487\zeta_5}{30} + \frac{4286603\pi^6}{6531840} - \frac{799\zeta_3^2}{4}$$

$$- \frac{138403\pi^4\zeta_3}{25920} + \frac{11987\pi^2\zeta_5}{1080} - \frac{228799\zeta_7}{126} \Bigg) + \epsilon^3\left(\frac{30537}{2} - \frac{5589\pi^2}{8} \right.$$

$$+ \frac{2685\zeta_3}{2} + \frac{9163\pi^4}{960} + \frac{2047\pi^2\zeta_3}{8} - \frac{14613\zeta_5}{10} + \frac{9379\pi^6}{16128} - \frac{73819\zeta_3^2}{36}$$

$$+ \frac{125549\pi^4\zeta_3}{25920} + \frac{295667\pi^2\zeta_5}{1080} - \frac{542683\zeta_7}{126} + \frac{325463531\pi^8}{313528320}$$

$$- \frac{70259}{432}\pi^2\zeta_3^2 + \frac{38845\zeta_3\zeta_5}{54} - \frac{78272}{81}\zeta_{-6,-2} \Bigg) + O\left(\epsilon^4\right) \Bigg\}, \tag{2.21}$$

$$A_{9,2}^n(4-2\epsilon) = e^{-3\gamma\epsilon}\Bigg\{ \frac{2}{9\epsilon^6} - \frac{17\pi^2}{54\epsilon^4} - \frac{31\zeta_3}{3\epsilon^3} - \frac{119\pi^4}{432\epsilon^2} + \left(\frac{341\pi^2\zeta_3}{36} - \frac{2507\zeta_5}{15} \right)\epsilon^{-1}$$

$$+ \left(29\zeta_3^2 - \frac{195551\pi^6}{544320} \right) + \left(-\frac{5963\pi^4\zeta_3}{4320} + \frac{8183\pi^2\zeta_5}{60} - \frac{43329\zeta_7}{14} \right)\epsilon$$

$$+ \epsilon^2\left(\frac{20752}{9}\zeta_{-6,-2} + \frac{101288\zeta_3\zeta_5}{15} - \frac{6419\pi^2\zeta_3^2}{36} - \frac{24178127\pi^8}{14515200} \right)$$

$$+ O\left(\epsilon^3\right) \Bigg\}, \tag{2.22}$$

$$A_{9,2}(4-2\epsilon) = e^{-3\gamma\epsilon}\Bigg\{-\frac{2}{9\epsilon^6} - \frac{5}{6\epsilon^5} + \epsilon^{-4}\left(\frac{20}{9} + \frac{17\pi^2}{54}\right) + \epsilon^{-3}\left(-\frac{50}{9} + \frac{181\pi^2}{216}\right.$$

$$\left.+\frac{31\zeta_3}{3}\right) + \epsilon^{-2}\left(\frac{110}{9} - \frac{17\pi^2}{9} + \frac{347\zeta_3}{18} + \frac{119\pi^4}{432}\right) + \epsilon^{-1}\left(-\frac{170}{9}\right.$$

$$\left.+\frac{19\pi^2}{6} - \frac{514\zeta_3}{9} + \frac{163\pi^4}{960} - \frac{341\pi^2\zeta_3}{36} + \frac{2507\zeta_5}{15}\right) + \left(-\frac{130}{9} + \frac{\pi^2}{2}\right.$$

$$\left.+\frac{1516\zeta_3}{9} - \frac{943\pi^4}{1080} - \frac{737\pi^2\zeta_3}{24} + \frac{2783\zeta_5}{6} + \frac{195551\pi^6}{544320} - 29\zeta_3^2\right)$$

$$+\epsilon\left(\frac{2950}{9} - \frac{83\pi^2}{2} - \frac{4444\zeta_3}{9} + \frac{8801\pi^4}{2160} + \frac{1357\pi^2\zeta_3}{18} - \frac{2830\zeta_5}{3}\right.$$

$$\left.+\frac{2416889\pi^6}{2177280} + \frac{19169\zeta_3^2}{36} + \frac{5963\pi^4\zeta_3}{4320} - \frac{8183\pi^2\zeta_5}{60} + \frac{43329\zeta_7}{14}\right)$$

$$+\epsilon^2\left(-\frac{19090}{9} + \frac{569\pi^2}{2} + \frac{12916\zeta_3}{9} - \frac{7795\pi^4}{432} - \frac{1433\pi^2\zeta_3}{9} + \frac{3112\zeta_5}{3}\right.$$

$$-\frac{64733\pi^6}{54432} - \frac{1214\zeta_3^2}{9} + \frac{58517\pi^4\zeta_3}{1728} - \frac{146521\pi^2\zeta_5}{360} + \frac{580805\zeta_7}{42}$$

$$\left.+\frac{24178127\pi^8}{14515200} + \frac{6419}{36}\pi^2\zeta_3^2 - \frac{101288\zeta_3\zeta_5}{15} - \frac{20752}{9}\zeta_{-6,-2}\right)$$

$$+O\left(\epsilon^3\right)\Bigg\}, \tag{2.23}$$

$$A_{9,4}^n(4-2\epsilon) = e^{-3\gamma\epsilon}\Bigg\{-\frac{1}{9\epsilon^6} + \frac{43\pi^2}{108\epsilon^4} + \frac{109\zeta_3}{9\epsilon^3} - \frac{481\pi^4}{12960\epsilon^2} + \epsilon^{-1}\left(\frac{3463\zeta_5}{45} - \frac{2975\pi^2\zeta_3}{108}\right)$$

$$+\left(-\frac{3115\zeta_3^2}{6} - \frac{247613\pi^6}{466560}\right) + \epsilon\left(-\frac{38903\pi^4\zeta_3}{2592} - \frac{113629\pi^2\zeta_5}{540} + \frac{8564\zeta_7}{63}\right)$$

$$+\epsilon^2\left(\frac{76288}{81}\zeta_{-6,-2} - \frac{730841\zeta_3\zeta_5}{135} + \frac{152299\pi^2\zeta_3^2}{216} - \frac{30535087\pi^8}{31352832}\right)$$

$$+O\left(\epsilon^3\right)\Bigg\}, \tag{2.24}$$

$$
\begin{aligned}
A_{9,4}(4-2\epsilon) = e^{-3\gamma\epsilon}\Bigg\{ & -\frac{1}{9\epsilon^6} - \frac{8}{9\epsilon^5} + \epsilon^{-4}\left(1 + \frac{43\pi^2}{108}\right) + \epsilon^{-3}\left(\frac{14}{9} + \frac{53\pi^2}{27} + \frac{109\zeta_3}{9}\right) \\
& + \epsilon^{-2}\left(-17 - \frac{311\pi^2}{108} + \frac{608\zeta_3}{9} - \frac{481\pi^4}{12960}\right) - \epsilon^{-1}\bigg(-84 - \frac{11\pi^2}{18} \\
& + \frac{949\zeta_3}{9} - \frac{85\pi^4}{108} + \frac{2975\pi^2\zeta_3}{108} - \frac{3463\zeta_5}{45}\bigg) - \bigg(339 - \frac{77\pi^2}{4} - \frac{434\zeta_3}{9} \\
& + \frac{2539\pi^4}{2592} + \frac{299\pi^2\zeta_3}{3} - \frac{7868\zeta_5}{15} + \frac{247613\pi^6}{466560} + \frac{3115\zeta_3^2}{6}\bigg) \\
& - \epsilon\bigg(-1242 + 112\pi^2 - 589\zeta_3 + \frac{487\pi^4}{432} - \frac{19499\pi^2\zeta_3}{108} + \frac{30067\zeta_5}{45} \\
& + \frac{25567\pi^6}{30240} + \frac{18512\zeta_3^2}{9} + \frac{38903\pi^4\zeta_3}{2592} + \frac{113629\pi^2\zeta_5}{540} - \frac{8564\zeta_7}{63}\bigg) \\
& - \epsilon^2\bigg(4293 - \frac{1887\pi^2}{4} + 3756\zeta_3 - \frac{491\pi^4}{32} + \frac{4019\pi^2\zeta_3}{18} + \frac{7874\zeta_5}{15} \\
& - \frac{9901847\pi^6}{3265920} - \frac{26291\zeta_3^2}{6} + \frac{9037\pi^4\zeta_3}{135} + \frac{35728\pi^2\zeta_5}{45} - \frac{72537\zeta_7}{14} \\
& + \frac{30535087\pi^8}{31352832} - \frac{152299}{216}\pi^2\zeta_3^2 + \frac{730841\zeta_3\zeta_5}{135} - \frac{76288}{81}\zeta_{-6,-2}\bigg) \\
& + O\left(\epsilon^3\right)\Bigg\}.
\end{aligned}
\tag{2.25}
$$

Here $\gamma = 0.577\ldots$ is the Euler constant, $\zeta_m = \zeta(m)$, and $\zeta_{m_1,m_2} = \zeta(m_1, m_2)$ are multiple zeta values (see, e.g., ref. [25])

$$
\zeta(m_1,\ldots,m_k) = \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_{k-1}-1}\prod_{j=1}^{k}\frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}.
$$

Our result for $A_{6,2}$ agrees with ref. [26] where the epsilon expansion up to transcendentality weight eight was obtained.

3 Master integrals for the $g - 2$ factor

The master integrals for the three-loop $g-2$ factor are shown in figure 3. The solid internal lines correspond to massive propagators $1/(-k^2+m^2)$, while the dotted lines correspond to massless ones, $1/(-k^2)$. The external solid lines denote the incoming and outgoing on-shell momentum p, so that $p^2 = m^2$. We set $m = 1$ for simplicity. Again, the ordering in figure 3 corresponds to increasing complexity level.

It is known [11, 12] that at the three-loop level the $g - 2$ factor is determined by seventeen master integrals. In fact, in an earlier paper [11] there was one more master integral (I_{11} in the notation of [11]) having the same set of denominators as $G_{4,6}$ — see figure 4. In a later paper [12] it was recognized that this additional integral can be expressed via $G_{4,4}$ and G_3, at least, up to a high power of ϵ. The DRA method is based on the consideration of relations exact in d, so that we needed an exact relation between I_{11},
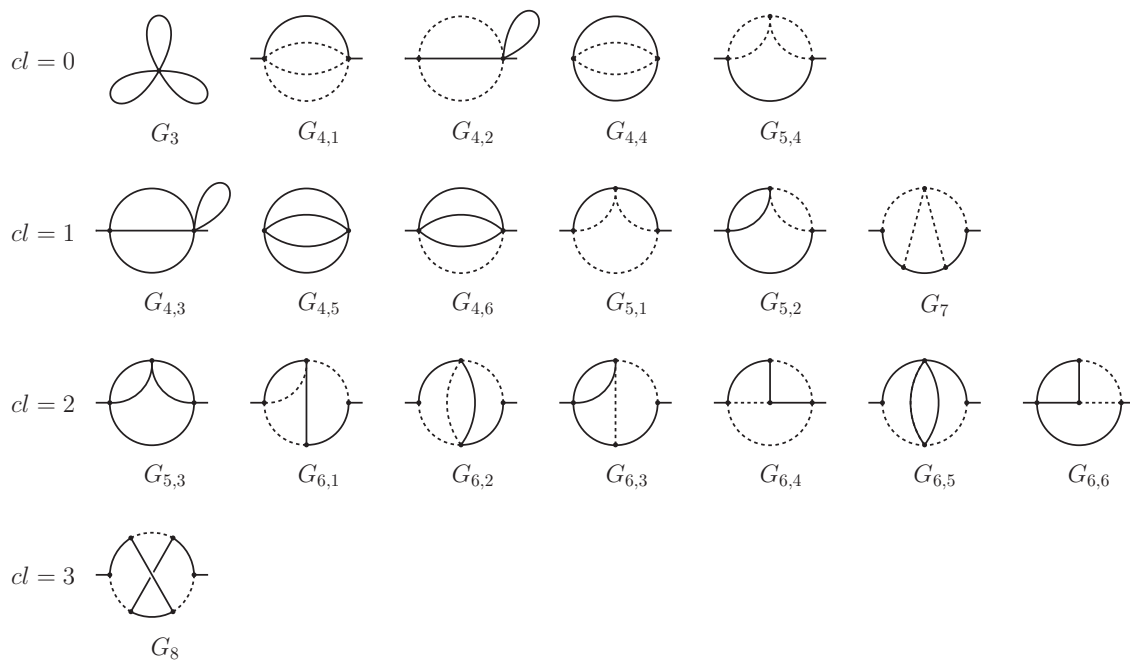
Figure 3. Master integrals for the three-loop $g - 2$ factor. Each line corresponds to a definite complexity level indicated to the left.
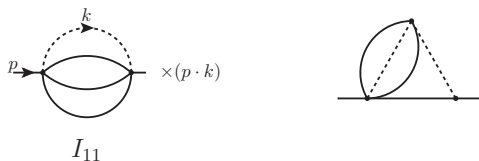

Figure 4. The integral I_{11} (left) and the auxiliary diagram used for its reduction (right).

$G_{4,4}$, and G_3. For this purpose, we have performed an IBP reduction of the integrals with denominators determined by the diagram depicted in figure 4. This reduction gave us the expected relation:

$$I_{11}(d) = \frac{2d - 5}{2d - 4} G_{4,4}(d) - \frac{1}{4} G_3(d).$$ (3.1)

Since we are oriented at a future four-loop calculation, we consider two more integrals, $G_{5,4}$ and G_7, which also appear in the factorizable four-loop master integrals. The application of the DRA method allowed us to express all master integrals in d dimensions in terms of one-, two-, and threefold series with maximal nested depth equal to the complexity level of the integral. These representations are available upon request from the authors. Here we present expansions of the master integrals near $d = 4$ obtained by numerical summation of the series followed by the application of the PSLQ algorithm. Again, for convenience of the reader, we start from listing the results for zero complexity level master integrals expressible via Γ-function.

$$G_3(4 - 2\epsilon) = \Gamma(\epsilon - 1)^3,$$ (3.2)

$$G_{4,1}(4-2\epsilon) = \frac{\Gamma(5-6\epsilon)\Gamma(1-\epsilon)^3\Gamma(2\epsilon-1)\Gamma(3\epsilon-2)}{\Gamma(4-4\epsilon)\Gamma(3-3\epsilon)}$$

$$= \frac{(1-6\epsilon)\Gamma(1+\epsilon)^3}{(1-4\epsilon)(3-4\epsilon)(1-3\epsilon)(2-3\epsilon)(1-2\epsilon)}\left\{ -\frac{1}{\epsilon^2} - 2\pi^2 - 32\epsilon\zeta_3 - \frac{74\pi^4\epsilon^2}{15} \right.$$

$$\left. -\epsilon^3\left(64\pi^2\zeta_3 + 1248\zeta_5 \right) - \epsilon^4\left(\frac{4636\pi^6}{315} + 512\zeta_3^2 \right) - \epsilon^5\left(\frac{2368\pi^4\zeta_3}{15} + 2496\pi^2\zeta_5 \right.\right.$$

$$\left.\left. +37008\zeta_7 \right) + O\left(\epsilon^6 \right) \right\}, \tag{3.3}$$

$$G_{4,2}(4-2\epsilon) = \frac{\Gamma(3-4\epsilon)\Gamma(1-\epsilon)^2\Gamma(\epsilon-1)\Gamma(\epsilon)\Gamma(2\epsilon-1)}{\Gamma(3-3\epsilon)\Gamma(2-2\epsilon)}$$

$$= \frac{(1-4\epsilon)\Gamma(1+\epsilon)^3}{(1-3\epsilon)(2-3\epsilon)(1-2\epsilon)(1-\epsilon)}\left\{ \frac{1}{\epsilon^3} + \frac{2\pi^2}{3\epsilon} + 8\zeta_3 + \frac{32\pi^4\epsilon}{45} + \epsilon^2\left(\frac{16\pi^2\zeta_3}{3} \right.\right.$$

$$\left.\left. +144\zeta_5 \right) + \epsilon^3\left(\frac{916\pi^6}{945} + 32\zeta_3^2 \right) + \epsilon^4\left(\frac{256\pi^4\zeta_3}{45} + 96\pi^2\zeta_5 + 1992\zeta_7 \right) \right.$$

$$\left. +O\left(\epsilon^5 \right) \right\}, \tag{3.4}$$

$$G_{4,4}(4-2\epsilon) = \frac{2^{1-2\epsilon}\Gamma(2-\epsilon)\Gamma(\epsilon-1)^2\Gamma\left(\epsilon-\frac{1}{2}\right)\Gamma(3\epsilon-2)}{\Gamma\left(2\epsilon-\frac{1}{2}\right)}$$

$$= \frac{(1-4\epsilon)\Gamma(1+\epsilon)^3}{(1-3\epsilon)(2-3\epsilon)(1-2\epsilon)(1-\epsilon)}\left\{ \frac{2}{3\epsilon^3} + \frac{16\zeta_3}{3} - \frac{4\pi^4\epsilon}{15} + 96\epsilon^2\zeta_5 \right.$$

$$\left. -\epsilon^3\left(\frac{8\pi^6}{21} - \frac{64\zeta_3^2}{3} \right) + \epsilon^4\left(-\frac{32}{15}\pi^4\zeta_3 + 1328\zeta_7 \right) + O\left(\epsilon^5 \right) \right\} \tag{3.5}$$

$$G_{5,4}(4-2\epsilon) = \frac{\Gamma(3-6\epsilon)\Gamma(1-\epsilon)^4\Gamma(\epsilon)^2\Gamma(3\epsilon-1)}{\Gamma(3-4\epsilon)\Gamma(2-2\epsilon)^2}$$

$$= \frac{(1-6\epsilon)\Gamma(1+\epsilon)^3}{(1-4\epsilon)(1-2\epsilon)^3}\left\{ -\frac{1}{3\epsilon^3} - \frac{2\pi^2}{3\epsilon} - \frac{38\zeta_3}{3} - \frac{151\pi^4\epsilon}{90} - \epsilon^2\left(\frac{76\pi^2\zeta_3}{3} + 430\zeta_5 \right) \right.$$

$$\left. -\epsilon^3\left(\frac{4729\pi^6}{945} + \frac{722\zeta_3^2}{3} \right) - \epsilon^4\left(\frac{2869\pi^4\zeta_3}{45} + 860\pi^2\zeta_5 + 12434\zeta_7 \right) + O\left(\epsilon^5 \right) \right\} \tag{3.6}$$

$$G_{4,3}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \left\{ \frac{3}{2\epsilon^3} + \frac{23}{4\epsilon^2} + \frac{105}{8\epsilon} + \left(\frac{275}{16} + \frac{4\pi^2}{3} \right) + \epsilon \left(-\frac{567}{32} + 10\pi^2 \right.\right.$$

$$\left. -8\pi^2\ln2 + 28\zeta_3 \right) + \epsilon^2 \left(-\frac{14917}{64} + \frac{145\pi^2}{3} - 60\pi^2\ln2 + 210\zeta_3 - \frac{62\pi^4}{45} + 16\pi^2\ln^22 \right.$$

$$\left. +8\ln^42 + 192a_4 \right) + \epsilon^3 \left(-\frac{144015}{128} + \frac{385\pi^2}{2} - 290\pi^2\ln2 + 1015\zeta_3 - \frac{31\pi^4}{3} \right.$$

$$\left. +120\pi^2\ln^22 + 60\ln^42 + 1440a_4 + \frac{124}{15}\pi^4\ln2 - 32\pi^2\ln^32 - \frac{48\ln^52}{5} + 1152a_5 \right.$$

$$\left. -\frac{40\pi^2\zeta_3}{3} - 930\zeta_5 \right) + \epsilon^4 \left(-\frac{1108525}{256} + \frac{8281\pi^2}{12} - 1155\pi^2\ln2 + \frac{8085\zeta_3}{2} - \frac{899\pi^4}{18} \right.$$

$$\left. +580\pi^2\ln^22 + 290\ln^42 + 6960a_4 + 62\pi^4\ln2 - 240\pi^2\ln^32 - 72\ln^52 + 8640a_5 \right.$$

$$\left. -100\pi^2\zeta_3 - 6975\zeta_5 - \frac{562\pi^6}{135} - \frac{124}{5}\pi^4\ln^22 + 48\pi^2\ln^42 + \frac{48\ln^62}{5} + 6912a_6 \right.$$

$$\left. +2880s_6 + 80\pi^2\ln2\,\zeta_3 - 1220\zeta_3^2 \right) + \epsilon^5 \left(-\frac{7710087}{512} + \frac{18585\pi^2}{8} - \frac{8281}{2}\pi^2\ln2 \right.$$

$$\left. +\frac{57967\zeta_3}{4} - \frac{2387\pi^4}{12} + 2310\pi^2\ln^22 + 1155\ln^42 + 27720a_4 + \frac{899}{3}\pi^4\ln2 \right.$$

$$\left. -1160\pi^2\ln^32 - 348\ln^52 + 41760a_5 - \frac{1450\pi^2\zeta_3}{3} - \frac{67425\zeta_5}{2} - \frac{281\pi^6}{9} - 186\pi^4\ln^22 \right.$$

$$\left. +360\pi^2\ln^42 + 72\ln^62 + 51840a_6 + 21600s_6 + 600\pi^2\ln2\,\zeta_3 - 9150\zeta_3^2 \right.$$

$$\left. +\frac{784}{45}\pi^6\ln2 + \frac{248}{5}\pi^4\ln^32 - \frac{288}{5}\pi^2\ln^52 - \frac{288\ln^72}{35} + 41472a_7 - \frac{55680}{7}\ln2\,s_6 \right.$$

$$\left. +\frac{4300\pi^4\zeta_3}{63} - 240\pi^2\ln^22\,\zeta_3 + \frac{69600}{7}\ln2\,\zeta_3^2 + \frac{32086\pi^2\zeta_5}{7} + 16740\ln^22\,\zeta_5 \right.$$

$$\left.\left. -\frac{579651\zeta_7}{7} + \frac{55680s_{7a}}{7} - \frac{65280s_{7b}}{7} \right) + O\left(\epsilon^6\right) \right\}, \tag{3.7}$$

$$G_{4,5}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \left\{ \frac{2}{\epsilon^3} + \frac{23}{3\epsilon^2} + \frac{35}{2\epsilon} + \frac{275}{12} + \epsilon\left(-\frac{189}{8} + \frac{112\zeta_3}{3}\right) \right.$$

$$-\epsilon^2\left(\frac{14917}{48} - 280\zeta_3 + \frac{136\pi^4}{45} + \frac{32}{3}\pi^2\ln^2 2 - \frac{32\ln^4 2}{3} - 256a_4\right)$$

$$-\epsilon^3\left(\frac{48005}{32} - \frac{4060\zeta_3}{3} + \frac{68\pi^4}{3} + 80\pi^2\ln^2 2 - 80\ln^4 2 - 1920a_4 - \frac{272}{15}\pi^4\ln 2\right.$$

$$\left.-\frac{64}{3}\pi^2\ln^3 2 + \frac{64\ln^5 2}{5} - 1536a_5 + 1240\zeta_5\right) - \epsilon^4\left(\frac{1108525}{192} - 5390\zeta_3 + \frac{986\pi^4}{9}\right.$$

$$+\frac{1160}{3}\pi^2\ln^2 2 - \frac{1160\ln^4 2}{3} - 9280a_4 - 136\pi^4\ln 2 - 160\pi^2\ln^3 2 + 96\ln^5 2$$

$$-11520a_5 + 9300\zeta_5 + \frac{32\pi^6}{5} + \frac{272}{5}\pi^4\ln^2 2 + 32\pi^2\ln^4 2 - \frac{64\ln^6 2}{5} - 9216a_6$$

$$\left.-3840s_6 + \frac{4880\zeta_3^2}{3}\right) - \epsilon^5\left(\frac{2570029}{128} - \frac{57967\zeta_3}{3} + \frac{1309\pi^4}{3} + 1540\pi^2\ln^2 2\right.$$

$$-1540\ln^4 2 - 36960a_4 - \frac{1972}{3}\pi^4\ln 2 - \frac{2320}{3}\pi^2\ln^3 2 + 464\ln^5 2 - 55680a_5$$

$$+44950\zeta_5 + 48\pi^6 + 408\pi^4\ln^2 2 + 240\pi^2\ln^4 2 - 96\ln^6 2 - 69120a_6 - 28800s_6$$

$$+12200\zeta_3^2 - \frac{3824}{135}\pi^6\ln 2 - \frac{544}{5}\pi^4\ln^3 2 - \frac{192}{5}\pi^2\ln^5 2 + \frac{384\ln^7 2}{35} - 55296a_7$$

$$+\frac{74240}{7}\ln 2\, s_6 - \frac{720\pi^4\zeta_3}{7} - \frac{92800}{7}\ln 2\,\zeta_3^2 - \frac{130360\pi^2\zeta_5}{21} - 22320\ln^2 2\,\zeta_5$$

$$\left.\left.+\frac{772868\zeta_7}{7} - \frac{74240s_{7a}}{7} + \frac{87040s_{7b}}{7}\right) + O\left(\epsilon^6\right)\right\}, \tag{3.8}$$

$$G_{4,6}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \left\{ \frac{1}{\epsilon^3} + \frac{7}{2\epsilon^2} + \frac{253}{36\epsilon} + \frac{2501}{216} - \left(-\frac{59437}{1296} + \frac{64\pi^2}{9} \right)\epsilon \right.$$

$$-\epsilon^2 \left(-\frac{2831381}{7776} + \frac{2272\pi^2}{27} - \frac{256}{3}\pi^2\ln 2 + \frac{1792\zeta_3}{9} \right) - \epsilon^3 \left(-\frac{117529021}{46656} \right.$$

$$+ \frac{49840\pi^2}{81} - \frac{9088}{9}\pi^2\ln 2 + \frac{63616\zeta_3}{27} - \frac{2752\pi^4}{135} + \frac{3584}{9}\pi^2\ln^2 2 + \frac{1024\ln^4 2}{9}$$

$$\left. + \frac{8192 a_4}{3} \right) - \epsilon^4 \left(-\frac{4081770917}{279936} + \frac{875224\pi^2}{243} - \frac{199360}{27}\pi^2\ln 2 + \frac{1395520\zeta_3}{81} \right.$$

$$- \frac{97696\pi^4}{405} + \frac{127232}{27}\pi^2\ln^2 2 + \frac{36352\ln^4 2}{27} + \frac{290816 a_4}{9} + \frac{11008}{45}\pi^4\ln 2$$

$$\left. - \frac{14336}{9}\pi^2\ln^3 2 - \frac{4096\ln^5 2}{15} + 32768 a_5 - \frac{1792\pi^2\zeta_3}{9} - \frac{87296\zeta_5}{3} \right)$$

$$- \epsilon^5 \left(-\frac{125873914573}{1679616} + \frac{13545868\pi^2}{729} - \frac{3500896}{81}\pi^2\ln 2 + \frac{24506272\zeta_3}{243} - \frac{428624\pi^4}{243} \right.$$

$$+ \frac{2791040}{81}\pi^2\ln^2 2 + \frac{797440\ln^4 2}{81} + \frac{6379520 a_4}{27} + \frac{390784}{135}\pi^4\ln 2 - \frac{508928}{27}\pi^2\ln^3 2$$

$$- \frac{145408\ln^5 2}{45} + \frac{1163264 a_5}{3} - \frac{63616\pi^2\zeta_3}{27} - \frac{3099008\zeta_5}{9} - \frac{93184\pi^6}{405} - \frac{22016}{15}\pi^4\ln^2 2$$

$$\left. + \frac{14336}{3}\pi^2\ln^4 2 + \frac{8192\ln^6 2}{15} + 393216 a_6 + 180224 s_6 + \frac{7168}{3}\pi^2\ln 2\,\zeta_3 - \frac{633344\zeta_3^2}{9} \right)$$

$$- \epsilon^6 \left(-\frac{3593750577461}{10077696} + \frac{193770934\pi^2}{2187} - \frac{54183472}{243}\pi^2\ln 2 + \frac{379284304\zeta_3}{729} \right.$$

$$- \frac{37634632\pi^4}{3645} + \frac{49012544}{243}\pi^2\ln^2 2 + \frac{14003584\ln^4 2}{243} + \frac{112028672 a_4}{81} + \frac{1714496}{81}\pi^4\ln 2$$

$$- \frac{11164160}{81}\pi^2\ln^3 2 - \frac{637952\ln^5 2}{27} + \frac{25518080 a_5}{9} - \frac{1395520\pi^2\zeta_3}{81} - \frac{67981760\zeta_5}{27}$$

$$- \frac{3308032\pi^6}{1215} - \frac{781568}{45}\pi^4\ln^2 2 + \frac{508928}{9}\pi^2\ln^4 2 + \frac{290816\ln^6 2}{45} + 4653056 a_6$$

$$+ \frac{6397952 s_6}{3} + \frac{254464}{9}\pi^2\ln 2\,\zeta_3 - \frac{22483712\zeta_3^2}{27} + \frac{719872}{405}\pi^6\ln 2 + \frac{88064}{15}\pi^4\ln^3 2$$

$$- \frac{57344}{5}\pi^2\ln^5 2 - \frac{32768\ln^7 2}{35} + 4718592 a_7 - \frac{19922944}{21}\ln 2\,s_6 + \frac{1789696\pi^4\zeta_3}{189}$$

$$- 14336\pi^2\ln^2 2\,\zeta_3 + \frac{24903680}{21}\ln 2\,\zeta_3^2 + \frac{38040320\pi^2\zeta_5}{63} + 2095104\ln^2 2\,\zeta_5$$

$$\left. \left. - \frac{72259840\zeta_7}{7} + \frac{19922944 s_{7a}}{21} - \frac{25493504 s_{7b}}{21} \right) + O\left(\epsilon^7\right) \right\} , \tag{3.9}$$

$$G_{5,1}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \Bigg\{ -\frac{1}{3\epsilon^3} - \frac{5}{3\epsilon^2} - \epsilon^{-1}\left(4 + \frac{2\pi^2}{3}\right) - \left(-\frac{10}{3} + \frac{7\pi^2}{3} + \frac{26\zeta_3}{3}\right)$$

$$-\epsilon\left(-\frac{302}{3} + \pi^2 + \frac{94\zeta_3}{3} + \frac{35\pi^4}{18}\right) - \epsilon^2\left(-734 - \frac{101\pi^2}{3} + 20\zeta_3\right.$$

$$\left.+\frac{551\pi^4}{90} + \frac{76\pi^2\zeta_3}{3} + 462\zeta_5\right) - \epsilon^3\left(-\frac{12254}{3} - \frac{775\pi^2}{3} - \frac{1232\zeta_3}{3}\right.$$

$$\left.-\frac{28\pi^4}{15} + \frac{236\pi^2\zeta_3}{3} + 1482\zeta_5 + \frac{2353\pi^6}{378} + \frac{482\zeta_3^2}{3}\right) - \epsilon^4\left(-\frac{60346}{3}\right.$$

$$-1383\pi^2 - \frac{9904\zeta_3}{3} - \frac{5249\pi^4}{45} - 32\pi^2\zeta_3 - 252\zeta_5 + \frac{36031\pi^6}{1890}$$

$$\left.+\frac{1510\zeta_3^2}{3} + \frac{3571\pi^4\zeta_3}{45} + 894\pi^2\zeta_5 + 15307\zeta_7\right) + O\left(\epsilon^5\right)\Bigg\}, \tag{3.10}$$

$$G_{5,2}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \Bigg\{ -\frac{2}{3\epsilon^3} - \frac{10}{3\epsilon^2} - \epsilon^{-1}\left(\frac{26}{3} + \frac{\pi^2}{3}\right) - \left(2 + \frac{11\pi^2}{3} + \frac{16\zeta_3}{3}\right)$$

$$-\epsilon\left(-\frac{398}{3} + \frac{73\pi^2}{3} - 16\pi^2\ln 2 + \frac{248\zeta_3}{3} + \frac{13\pi^4}{45}\right) - \epsilon^2\left(-1038 + 129\pi^2\right.$$

$$-160\pi^2\ln 2 + \frac{1888\zeta_3}{3} - \frac{3\pi^4}{5} + \frac{128}{3}\pi^2\ln^2 2 + \frac{64\ln^4 2}{3} + 512a_4 + \frac{8\pi^2\zeta_3}{3} + 96\zeta_5\Bigg)$$

$$-\epsilon^3\left(-\frac{17470}{3} + \frac{1817\pi^2}{3} - 1024\pi^2\ln 2 + 3600\zeta_3 - \frac{751\pi^4}{45} + \frac{1280}{3}\pi^2\ln^2 2 + \frac{640\ln^4 2}{3}\right.$$

$$+5120a_4 + \frac{736}{45}\pi^4\ln 2 - \frac{1024}{9}\pi^2\ln^3 2 - \frac{512\ln^5 2}{15} + 4096a_5 - \frac{8\pi^2\zeta_3}{3} - 2496\zeta_5 + \frac{368\pi^6}{945}$$

$$\left.+\frac{64\zeta_3^2}{3}\right) - \epsilon^4\left(-\frac{85562}{3} + 2649\pi^2 - 5376\pi^2\ln 2 + \frac{53264\zeta_3}{3} - \frac{5849\pi^4}{45} + \frac{8192}{3}\pi^2\ln^2 2\right.$$

$$+\frac{4096\ln^4 2}{3} + 32768a_4 + \frac{1472}{9}\pi^4\ln 2 - \frac{10240}{9}\pi^2\ln^3 2 - \frac{1024\ln^5 2}{3} + 40960a_5$$

$$-\frac{376\pi^2\zeta_3}{3} - 28512\zeta_5 - \frac{274\pi^6}{15} - \frac{3424}{45}\pi^4\ln^2 2 + \frac{2144}{9}\pi^2\ln^4 2 + \frac{2048\ln^6 2}{45}$$

$$+256\pi^2 a_4 + 32768a_6 + 12288s_6 + 352\pi^2\ln 2\,\zeta_3 - \frac{14680\zeta_3^2}{3} + \frac{104\pi^4\zeta_3}{45} + 48\pi^2\zeta_5$$

$$\left.+1328\zeta_7\right) + O\left(\epsilon^5\right)\Bigg\}, \tag{3.11}$$

$$G_7(4-2\epsilon) = \Gamma(1+\epsilon)^3 \Bigg\{ \left(2\pi^2\zeta_3 - 5\zeta_5\right) + \epsilon\left(4\pi^2\zeta_3 - 10\zeta_5 + \frac{16\pi^6}{189} + 7\zeta_3^2\right)$$

$$+\epsilon^2\left(8\pi^2\zeta_3 - 20\zeta_5 + \frac{32\pi^6}{189} + 14\zeta_3^2 - \frac{181\pi^4\zeta_3}{30} + 166\pi^2\zeta_5 - 212\zeta_7\right)$$

$$+O\left(\epsilon^3\right)\Bigg\}, \tag{3.12}$$

$$G_{5,3}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \Bigg\{ -\frac{1}{\epsilon^3} - \frac{16}{3\epsilon^2} - \frac{16}{\epsilon} - \left(20 + \frac{8\pi^2}{3} - 2\zeta_3\right) - \epsilon\left(-\frac{364}{3} + 28\pi^2\right.$$

$$\left. + \frac{200\zeta_3}{3} + \frac{3\pi^4}{10} - 16\pi^2\ln 2\right) - \epsilon^2\left(-1244 + 188\pi^2 + 776\zeta_3 - \frac{46\pi^4}{15} - 21\pi^2\zeta_3 + 126\zeta_5\right.$$

$$\left. - 168\pi^2\ln 2 + \frac{80}{3}\pi^2\ln^2 2 + \frac{64\ln^4 2}{3} + 512a_4\right) - \epsilon^3\left(-7572 + \frac{3100\pi^2}{3} + 5360\zeta_3 - \frac{218\pi^4}{5}\right.$$

$$- \frac{332\pi^2\zeta_3}{3} - 1976\zeta_5 - \frac{22\pi^6}{35} - 332\zeta_3^2 - 1128\pi^2\ln 2 + \frac{128}{5}\pi^4\ln 2 + 126\pi^2\ln 2\,\zeta_3$$

$$+ 280\pi^2\ln^2 2 - 6\pi^4\ln^2 2 - \frac{160}{3}\pi^2\ln^3 2 + 224\ln^4 2 + 6\pi^2\ln^4 2 - \frac{128\ln^5 2}{5} + 5376a_4$$

$$\left. + 144\pi^2 a_4 + 3072a_5\right) + O\left(\epsilon^4\right)\Bigg\}, \tag{3.13}$$

$$G_{6,1}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \Bigg\{ \frac{1}{6\epsilon^3} + \frac{3}{2\epsilon^2} + \epsilon^{-1}\left(\frac{55}{6} - \frac{\pi^2}{3}\right) + \left(\frac{95}{2} - 2\pi^2 - \frac{8\zeta_3}{3} - \frac{\pi^4}{15}\right)$$

$$+ \epsilon\left(\frac{1351}{6} - \frac{17\pi^2}{3} - 14\zeta_3 - \frac{47\pi^4}{45} + 6\pi^2\zeta_3 - 64\zeta_5\right) + \epsilon^2\left(\frac{2023}{2} + \frac{16\pi^2}{3}\right.$$

$$- 16\pi^2\ln 2 + \frac{16\zeta_3}{3} - \frac{457\pi^4}{90} + \frac{26\pi^2\zeta_3}{3} - 342\zeta_5 + \frac{1471\pi^6}{2835} + \frac{8}{3}\pi^4\ln^2 2$$

$$\left. - \frac{8}{3}\pi^2\ln^4 2 - 64\pi^2 a_4 - 56\pi^2\ln 2\,\zeta_3 + 62\zeta_3^2\right) + \epsilon^3\left(\frac{26335}{6} + 187\pi^2 - 224\pi^2\ln 2\right.$$

$$+ 598\zeta_3 - \frac{277\pi^4}{15} + \frac{224}{3}\pi^2\ln^2 2 + \frac{64\ln^4 2}{3} + 512a_4 - \frac{58\pi^2\zeta_3}{3} - 1082\zeta_5$$

$$- \frac{299\pi^6}{378} + 8\pi^4\ln^2 2 - 8\pi^2\ln^4 2 - 192\pi^2 a_4 - 168\pi^2\ln 2\,\zeta_3 + \frac{400\zeta_3^2}{3} - \frac{112}{135}\pi^6\ln 2$$

$$- \frac{64}{3}\pi^4\ln^3 2 + \frac{128}{5}\pi^2\ln^5 2 + 768\pi^2\ln 2\,a_4 + 768\pi^2 a_5 + 1024\ln 2\,s_6$$

$$+ \frac{1651\pi^4\zeta_3}{45} + \frac{1232}{3}\pi^2\ln^2 2\,\zeta_3 - \frac{224}{3}\ln^4 2\,\zeta_3 - 1792a_4\zeta_3 - 1280\ln 2\,\zeta_3^2 - \frac{499\pi^2\zeta_5}{3}$$

$$\left. - 8365\zeta_7 - 1024s_{7a} - 1024s_{7b}\right) + O\left(\epsilon^4\right)\Bigg\}, \tag{3.14}$$

$$G_{6,2}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \left\{ \frac{1}{3\epsilon^3} + \frac{7}{3\epsilon^2} + \frac{31}{3\epsilon} + \left(\frac{103}{3} + \frac{\pi^2}{3} + \frac{2\zeta_3}{3} - \frac{4\pi^4}{45} \right) + \epsilon \left(\frac{235}{3} \right. \right.$$

$$+ 4\pi^2 + \frac{20\zeta_3}{3} - \frac{3\pi^4}{10} + \frac{2\pi^2\zeta_3}{3} - 2\zeta_5 \right) + \epsilon^2 \left(\frac{19}{3} + \frac{91\pi^2}{3} - 16\pi^2\ln 2 + \frac{206\zeta_3}{3} \right.$$

$$+ \frac{14\pi^4}{45} + 2\pi^2\zeta_3 + 6\zeta_5 + \frac{1009\pi^6}{1890} + \frac{8}{3}\pi^4\ln^2 2 - \frac{8}{3}\pi^2\ln^4 2 - 64\pi^2 a_4 - 56\pi^2\ln 2\,\zeta_3$$

$$+ 32\zeta_3^2 \right) + \epsilon^3 \left(-\frac{3953}{3} + 186\pi^2 - 224\pi^2\ln 2 + \frac{1760\zeta_3}{3} + \frac{307\pi^4}{90} + \frac{224}{3}\pi^2\ln^2 2 \right.$$

$$+ \frac{64\ln^4 2}{3} + 512 a_4 + 14\pi^2\zeta_3 + 222\zeta_5 + \frac{979\pi^6}{630} + 8\pi^4\ln^2 2 - 8\pi^2\ln^4 2 - 192\pi^2 a_4$$

$$- 168\pi^2\ln 2\,\zeta_3 + \frac{296\zeta_3^2}{3} - \frac{112}{135}\pi^6\ln 2 - \frac{64}{3}\pi^4\ln^3 2 + \frac{128}{5}\pi^2\ln^5 2 + 768\pi^2\ln 2\,a_4$$

$$+ 768\pi^2 a_5 + 1024\ln 2\,s_6 + \frac{976\pi^4\zeta_3}{45} + \frac{1232}{3}\pi^2\ln^2 2\,\zeta_3 - \frac{224}{3}\ln^4 2\,\zeta_3 - 1792 a_4\zeta_3$$

$$\left. \left. - 1280\ln 2\,\zeta_3^2 - 215\pi^2\zeta_5 - 6114\zeta_7 - 1024 s_{7a} - 1024 s_{7b} \right) + O\left(\epsilon^4\right) \right\}, \qquad (3.15)$$

$$G_{6,3}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \left\{ \frac{1}{6\epsilon^3} + \frac{3}{2\epsilon^2} + \epsilon^{-1}\left(\frac{55}{6} - \frac{\pi^2}{3} \right) + \left(\frac{95}{2} - \frac{7\pi^2}{3} - \frac{14\zeta_3}{3} - \frac{4\pi^4}{45} \right) \right.$$

$$+ \epsilon \left(\frac{1351}{6} - \frac{31\pi^2}{3} + 4\pi^2\ln 2 - 42\zeta_3 - \frac{23\pi^4}{30} + \frac{25\pi^2\zeta_3}{6} - \frac{49\zeta_5}{2} \right) + \epsilon^2 \left(\frac{2023}{2} \right.$$

$$- \frac{103\pi^2}{3} + 32\pi^2\ln 2 - \frac{698\zeta_3}{3} - \frac{148\pi^4}{45} - \frac{40}{3}\pi^2\ln^2 2 - \frac{20\ln^4 2}{3} - 160 a_4 + \frac{109\pi^2\zeta_3}{6}$$

$$- \frac{413\zeta_5}{2} + \frac{23\pi^6}{54} + \frac{7}{3}\pi^4\ln^2 2 - \frac{7}{3}\pi^2\ln^4 2 - 56\pi^2 a_4 - 49\pi^2\ln 2\,\zeta_3 + \frac{191\zeta_3^2}{4} \right)$$

$$+ \epsilon^3 \left(\frac{26335}{6} - \frac{235\pi^2}{3} + 140\pi^2\ln 2 - 994\zeta_3 - \frac{371\pi^4}{30} - 80\pi^2\ln^2 2 - 56\ln^4 2 \right.$$

$$- 1344 a_4 - \frac{182}{45}\pi^4\ln 2 + \frac{368}{9}\pi^2\ln^3 2 + \frac{184\ln^5 2}{15} - 1472 a_5 + \frac{129\pi^2\zeta_3}{2} + \frac{107\zeta_5}{2}$$

$$+ \frac{457\pi^6}{270} + \frac{35}{3}\pi^4\ln^2 2 - \frac{35}{3}\pi^2\ln^4 2 - 280\pi^2 a_4 - 245\pi^2\ln 2\,\zeta_3 + \frac{2641\zeta_3^2}{12}$$

$$- \frac{35}{54}\pi^6\ln 2 - \frac{164}{9}\pi^4\ln^3 2 + \frac{328}{15}\pi^2\ln^5 2 + 656\pi^2\ln 2\,a_4 + 656\pi^2 a_5 + 800\ln 2\,s_6$$

$$+ \frac{3803\pi^4\zeta_3}{180} + \frac{1036}{3}\pi^2\ln^2 2\,\zeta_3 - \frac{175}{3}\ln^4 2\,\zeta_3 - 1400 a_4\zeta_3 - 1000\ln 2\,\zeta_3^2 - \frac{2057\pi^2\zeta_5}{12}$$

$$\left. \left. - \frac{44413\zeta_7}{8} - 800 s_{7a} - 800 s_{7b} \right) + O\left(\epsilon^4\right) \right\}, \qquad (3.16)$$

$$G_{6,4}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \Bigg\{ \frac{2\zeta_3}{\epsilon} - \left(-\frac{\pi^2}{3} - 2\zeta_3 + \frac{7\pi^4}{90} \right) + \epsilon\left(\frac{14\pi^2}{3} - 12\zeta_3 \right.$$

$$- \frac{41\pi^4}{90} - \frac{2\pi^2\zeta_3}{3} + 44\zeta_5 \bigg) - \epsilon^2\left(-\frac{119\pi^2}{3} + 16\pi^2\ln2 + 38\zeta_3 + \frac{14\pi^4}{45} + \frac{26\pi^2\zeta_3}{3} \right.$$

$$+ 42\zeta_5 - \frac{1447\pi^6}{2835} - \frac{8}{3}\pi^4\ln^2 2 + \frac{8}{3}\pi^2\ln^4 2 + 64\pi^2 a_4 + 56\pi^2\ln2\,\zeta_3 - 54\zeta_3^2 \bigg)$$

$$+ \epsilon^3\left(\frac{796\pi^2}{3} - 224\pi^2\ln2 + 256\zeta_3 + \frac{51\pi^4}{10} + \frac{224}{3}\pi^2\ln^2 2 + \frac{64\ln^4 2}{3} + 512 a_4 \right.$$

$$- 26\pi^2\zeta_3 - 574\zeta_5 + \frac{31\pi^6}{630} + 8\pi^4\ln^2 2 - 8\pi^2\ln^4 2 - 192\pi^2 a_4 - 168\pi^2\ln2\,\zeta_3$$

$$+ 104\zeta_3^2 - \frac{112}{135}\pi^6\ln2 - \frac{64}{3}\pi^4\ln^3 2 + \frac{128}{5}\pi^2\ln^5 2 + 768\pi^2\ln2\,a_4 + 768\pi^2 a_5$$

$$+ 1024\ln2\,s_6 + \frac{931\pi^4\zeta_3}{45} + \frac{1232}{3}\pi^2\ln^2 2\,\zeta_3 - \frac{224}{3}\ln^4 2\,\zeta_3 - 1792 a_4\zeta_3$$

$$- 1280\ln2\,\zeta_3^2 - 217\pi^2\zeta_5 - 5655\zeta_7 - 1024 s_{7a} - 1024 s_{7b} \bigg) + O\left(\epsilon^4\right) \Bigg\}, \qquad (3.17)$$

$$G_{6,5}(4-2\epsilon) = \Gamma(1+\epsilon)^3 \Bigg\{ \frac{1}{3\epsilon^3} + \frac{7}{3\epsilon^2} + \frac{31}{3\epsilon} + \left(\frac{103}{3} - \frac{4\zeta_3}{3} - \frac{2\pi^4}{15} \right) + \epsilon\left(\frac{235}{3} \right.$$

$$+ \frac{8\pi^2}{3} + \frac{32\zeta_3}{3} - \frac{3\pi^4}{5} + \frac{28\pi^2\zeta_3}{3} - 78\zeta_5 \bigg) + \epsilon^2\left(\frac{19}{3} + \frac{112\pi^2}{3} - 32\pi^2\ln2 + \frac{692\zeta_3}{3} \right.$$

$$- \frac{164\pi^4}{45} - \frac{16}{3}\pi^2\ln^2 2 + \frac{16\ln^4 2}{3} + 128 a_4 + \frac{140\pi^2\zeta_3}{3} - 414\zeta_5 + \frac{928\pi^6}{945}$$

$$+ \frac{16}{3}\pi^4\ln^2 2 - \frac{16}{3}\pi^2\ln^4 2 - 128\pi^2 a_4 - 112\pi^2\ln2\,\zeta_3 + 169\zeta_3^2 \bigg) + \epsilon^3\left(-\frac{3953}{3} + \frac{952\pi^2}{3} \right.$$

$$- 448\pi^2\ln2 + \frac{6392\zeta_3}{3} - \frac{269\pi^4}{9} + 96\pi^2\ln^2 2 + 96\ln^4 2 + 2304 a_4 + \frac{136}{15}\pi^4\ln2$$

$$+ \frac{32}{3}\pi^2\ln^3 2 - \frac{32\ln^5 2}{5} + 768 a_5 + \frac{532\pi^2\zeta_3}{3} - 2246\zeta_5 + \frac{946\pi^6}{189} + \frac{80}{3}\pi^4\ln^2 2$$

$$- \frac{80}{3}\pi^2\ln^4 2 - 640\pi^2 a_4 - 560\pi^2\ln2\,\zeta_3 + \frac{2519\zeta_3^2}{3} - \frac{266}{135}\pi^6\ln2 - \frac{128}{3}\pi^4\ln^3 2$$

$$+ \frac{256}{5}\pi^2\ln^5 2 + 1536\pi^2\ln2\,a_4 + 1536\pi^2 a_5 + 2432\ln2\,s_6 + \frac{367\pi^4\zeta_3}{5} + \frac{2548}{3}\pi^2\ln^2 2\,\zeta_3$$

$$- \frac{532}{3}\ln^4 2\,\zeta_3 - 4256 a_4\zeta_3 - 3040\ln2\,\zeta_3^2 - 132\pi^2\zeta_5 - \frac{35591\zeta_7}{2} - 2432 s_{7a} - 2432 s_{7b} \bigg)$$

$$+ O\left(\epsilon^4\right) \Bigg\}, \qquad (3.18)$$

$$G_{6,6}(4-2\epsilon) = \Gamma(1+\epsilon)^3\left\{\frac{2\zeta_3}{\epsilon} - \left(\frac{\pi^2}{3} - 10\zeta_3 + \frac{13\pi^4}{90}\right) + \epsilon\left(-\frac{4\pi^2}{3} + 4\pi^2\ln2\right.\right.$$

$$+24\zeta_3 - \frac{13\pi^4}{18} + \frac{49\pi^2\zeta_3}{6} - \frac{85\zeta_5}{2}\bigg) - \epsilon^2\bigg(-7\pi^2 - 16\pi^2\ln2 - 42\zeta_3 + \frac{217\pi^4}{90}$$

$$+\frac{40}{3}\pi^2\ln^2 2 + \frac{20\ln^4 2}{3} + 160a_4 - \frac{245\pi^2\zeta_3}{6} + \frac{425\zeta_5}{2} - \frac{1751\pi^6}{1890} - 5\pi^4\ln^2 2$$

$$+5\pi^2\ln^4 2 + 120\pi^2 a_4 + 105\pi^2\ln2\,\zeta_3 - \frac{495\zeta_3^2}{4}\bigg) + \epsilon^3\bigg(\frac{394\pi^2}{3} - 84\pi^2\ln2$$

$$+268\zeta_3 - \frac{223\pi^4}{18} - \frac{16}{3}\pi^2\ln^2 2 - \frac{104\ln^4 2}{3} - 832a_4 - \frac{182}{45}\pi^4\ln2 + \frac{368}{9}\pi^2\ln^3 2$$

$$+\frac{184\ln^5 2}{15} - 1472a_5 + \frac{313\pi^2\zeta_3}{2} + \frac{431\zeta_5}{2} + \frac{1751\pi^6}{378} + 25\pi^4\ln^2 2 - 25\pi^2\ln^4 2$$

$$-600\pi^2 a_4 - 525\pi^2\ln2\,\zeta_3 + \frac{2475\zeta_3^2}{4} - \frac{133}{90}\pi^6\ln2 - \frac{356}{9}\pi^4\ln^3 2 + \frac{712}{15}\pi^2\ln^5 2$$

$$+1424\pi^2\ln2\,a_4 + 1424\pi^2 a_5 + 1824\ln2\,s_6 + \frac{9259\pi^4\zeta_3}{180} + 756\pi^2\ln^2 2\,\zeta_3 - 133\ln^4 2\,\zeta_3$$

$$-3192a_4\zeta_3 - 2280\ln2\,\zeta_3^2 - \frac{3841\pi^2\zeta_5}{12} - \frac{105133\zeta_7}{8} - 1824s_{7a} - 1824s_{7b}\bigg)$$

$$+O\left(\epsilon^4\right)\bigg\}, \tag{3.19}$$

$$G_8(4-2\epsilon) = \Gamma(1+\epsilon)^3\left\{\left(-\frac{\pi^4}{6} + 4\pi^2\ln^2 2\right) + \epsilon\left(\frac{2\pi^4}{3} - 16\pi^2\ln^2 2 + \frac{166}{45}\pi^4\ln2\right.\right.$$

$$-\frac{208}{9}\pi^2\ln^3 2 - \frac{32\ln^5 2}{15} + 256a_5 + \frac{17\pi^2\zeta_3}{6} - 291\zeta_5\bigg) + \epsilon^2\bigg(-\frac{14\pi^4}{3} + 112\pi^2\ln^2 2$$

$$-\frac{664}{45}\pi^4\ln2 + \frac{832}{9}\pi^2\ln^3 2 + \frac{128\ln^5 2}{15} - 1024a_5 - \frac{34\pi^2\zeta_3}{3} + 1164\zeta_5$$

$$-\frac{21743\pi^6}{11340} - \frac{197}{9}\pi^4\ln^2 2 + \frac{713}{9}\pi^2\ln^4 2 + \frac{64\ln^6 2}{9} - 104\pi^2 a_4 + 5120a_6 + 2688s_6$$

$$-51\pi^2\ln2\,\zeta_3 - 953\zeta_3^2\bigg) + \epsilon^3\bigg(\frac{80\pi^4}{3} - 640\pi^2\ln^2 2 + \frac{4648}{45}\pi^4\ln2 - \frac{5824}{9}\pi^2\ln^3 2$$

$$-\frac{896\ln^5 2}{15} + 7168a_5 + \frac{238\pi^2\zeta_3}{3} - 8148\zeta_5 + \frac{21743\pi^6}{2835} + \frac{788}{9}\pi^4\ln^2 2 - \frac{2852}{9}\pi^2\ln^4 2$$

$$-\frac{256\ln^6 2}{9} + 416\pi^2 a_4 - 20480a_6 - 10752s_6 + 204\pi^2\ln2\,\zeta_3 + 3812\zeta_3^2 + \frac{4868}{189}\pi^6\ln2$$

$$+\frac{8492}{135}\pi^4\ln^3 2 - \frac{7288}{45}\pi^2\ln^5 2 - \frac{4864\ln^7 2}{315} + 1776\pi^2\ln2\,a_4 + 1520\pi^2 a_5 + 77824a_7$$

$$-\frac{106880}{7}\ln2\,s_6 + \frac{4003\pi^4\zeta_3}{21} + \frac{2167}{3}\pi^2\ln^2 2\,\zeta_3 - \frac{316}{3}\ln^4 2\,\zeta_3 - 2528a_4\zeta_3$$

$$+\frac{133600}{7}\ln2\,\zeta_3^2 + \frac{875561\pi^2\zeta_5}{84} + 37200\ln^2 2\,\zeta_5 - \frac{1325727\zeta_7}{7} + \frac{106880s_{7a}}{7}$$

$$-\frac{161920s_{7b}}{7}\bigg) + O\left(\epsilon^4\right)\bigg\}. \tag{3.20}$$

Here

$$a_n = \mathrm{Li}_n\left(1/2\right),$$
$$s_6 = \zeta_{-5,-1} + \zeta_6\,,$$
$$s_{7a} = \zeta_{-5,1,1} + \zeta_{-6,1} + \zeta_{-5,2} + \zeta_{-7}\,,$$
$$s_{7b} = \zeta_7 + \zeta_{5,2} + \zeta_{-6,-1} + \zeta_{5,-1,-1}\,.$$

4   Conclusion

Using the DRA method [1] we have performed the analytic evaluation of two families of the three-loop master integrals in the ϵ-expansion up to the transcendentality weight intrinsic to the corresponding four-loop master integrals. This method is naturally combined with other methods. First, it hardly relies on IBP reduction which is necessary in order to obtain difference equations with respect to dimension in a closed form. To do this we used the code based on ref. [27] and the code called FIRE [28]. To reveal the position and the order of the poles in a basic stripe we used a sector decomposition [29–35] implemented in the code FIESTA [35, 36]. To fix remaining constants in the homogenous solution of dimensional recurrence relations we applied the method of Mellin-Barnes representation [37–42]. When dealing with multiple zeta values we used the code HPL [43]. Finally, to arrive at analytical results for coefficients in the ϵ-expansion we applied the PSLQ algorithm [20]. All results presented in this paper are available from the authors in computer-readable form.

Acknowledgments

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