

Penner type matrix model and Seiberg-Witten theory

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ABSTRACT: We discuss the Penner type matrix model recently proposed by Dijkgraaf and Vafa for a possible explanation of the relation between four-dimensional gauge theory and Liouville theory by making use of the connection of the matrix model to two-dimensional CFT. We first consider the relation of gauge couplings defined in UV and IR regimes of $N_f = 4$, $\mathcal{N} = 2$ supersymmetric gauge theory being related as $q_{UV} = \vartheta_2(q_{IR})^4 / \vartheta_3(q_{IR})^4$. We then use this relation to discuss the action of modular transformation on the matrix model and determine its spectral curve.

We also discuss the decoupling of massive flavors from the $N_f = 4$ matrix model and derive matrix models describing asymptotically free $\mathcal{N} = 2$ gauge theories. We find that the Penner type matrix theory reproduces correctly the standard results of $\mathcal{N} = 2$ supersymmetric gauge theories.

KEYWORDS: Matrix Models, Extended Supersymmetry, Supersymmetric gauge theory, Duality in Gauge Field Theories

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1 Introduction

Recently an interesting observation has been made on the relationship between the $\mathcal{N} = 2$ supersymmetric gauge theory and the 2d conformal field theory. In [1], it has been shown that Nekrasov’s partition function [2] of $\mathcal{N} = 2$ SU(2) superconformal gauge theory can be identified with the correlation function of 2d Liouville theory. This relation was further studied and checked from various directions [3–26]. For general discussions on S-duality in $\mathcal{N} = 2$ superconformal gauge theories, see [27–39].

In ref. [40], the following matrix model with a Penner like action

$$W(M) = \sum_{i=1}^3 \mu_i \log(M - q_i), \quad q_1 = 0, q_2 = 1, q_3 = q \quad (1.1)$$

has been proposed to describe $\mathcal{N} = 2$ SU(2) superconformal gauge theory with four hypermultiplets. It is suggested that this matrix theory explains the correspondence of [1], by making use of the CFT description of the matrix model [41, 42]. (This matrix model corresponds to $c = 1$ (or $b = i$) case in the Liouville theory.) The four mass parameters of the gauge theory are identified with μ_i in the matrix model action and μ_0 which corresponds to the charge at infinity. These parameters satisfy the following relation [40]

$$\sum_{i=1}^3 \mu_i + \mu_0 = -2g_s N, \quad (1.2)$$

where N is the size of the matrix.

We study this matrix model in details in this article and show that they in fact correctly reproduce the physics of SU(2) Seiberg-Witten theory [43, 44].

In section 2 we first consider the M-theory curve for the $N_f = 4$, $\mathcal{N} = 2$ supersymmetric SU(2) gauge theory based on the brane construction [45]. We study its period integrals and by comparing with those of the standard Seiberg-Witten curve we obtain the relation between gauge coupling constants at UV regime and IR regime as $q_{UV} = \vartheta_2(q_{IR})^4/\vartheta_3(q_{IR})^4$. The parameter q which appears in the matrix model action (1.1) is identified as the UV coupling $q = q_{UV}$.

We discuss in section 3 the modular property of the matrix model (1.1) by making use of the IR-UV relation of gauge coupling constants. We study the modular transformation of the spectral curve and determine its precise mass-dependence by imposing modular invariance.

In sections 4, we consider the decoupling of massive flavors and find matrix model actions corresponding to asymptotically free gauge theories with $N_f = 2, 3$. We check that the discriminants of the spectral curve and free energy of these matrix models match well with those of Seiberg-Witten theories.

2 $\mathcal{N} = 2$ gauge theory and UV and IR gauge coupling constants

2.1 M-theory curve of $N_f = 4$, SU(2) gauge theory

$\mathcal{N} = 2$ SU(2) gauge theory with four hypermultiplets is known to be scale invariant and has the exactly marginal coupling constant

$$\tau_{UV} = \frac{\theta_{UV}}{\pi} + \frac{8\pi i}{g_{UV}^2}. \tag{2.1}$$

The flavor symmetry of the theory is $SO(8)$ whose maximal subgroup is $SU(2)^4$. While introducing the masses of the hypermultiplets breaks the conformal invariance, the theory is modular invariant involving the triality of $SO(8)$ which rotates the mass parameters [44]. We introduce the mass parameters of four hypermultiplets m_{\pm} and \tilde{m}_{\pm} . By combining these, we define

$$m_0 = \frac{1}{2}(m_+ - m_-), \quad m_2 = \frac{1}{2}(m_+ + m_-), \quad m_1 = \frac{1}{2}(\tilde{m}_+ - \tilde{m}_-), \quad m_3 = \frac{1}{2}(\tilde{m}_+ + \tilde{m}_-), \tag{2.2}$$

each of which is a mass parameter associated with each SU(2) flavor symmetry.

This gauge theory can be obtained by considering the intersecting brane system in type IIA string theory [45]. The SU(2) gauge part is induced by two D4-branes suspended between two NS5-branes. The D4-branes occupy the $x^{0,1,2,3}$ and x^6 directions and the NS5-branes occupy $x^{0,1,2,3}$ and $x^{4,5}$ which are denoted by the complex coordinate v . A massive hypermultiplet can be introduced by a semi-infinite D4-brane attached to an NS5-brane. We here choose the configuration such that two semi-infinite D4-branes are attached to the left NS5-brane and extended to $x^6 = -\infty$ and two more D4-branes are attached to the right NS5-brane extending to $x^6 = +\infty$. By the M-theory uplift, the Seiberg-Witten curve [43] of this theory becomes a hypersurface on $(t, v) \in \mathbb{C}^2$ where $t = e^{-(x^6 + ix^{10})/R}$.

$$(v - m_+)(v - m_-)t^2 + c_1(v^2 + Mv - U)t + c_2(v - \tilde{m}_+)(v - \tilde{m}_-) = 0, \tag{2.3}$$

where M and U are constants which depend on the masses and the Coulomb moduli u . The first and the third terms are determined as follows: in the large t , the first term is dominant and we obtain $v \sim m_+, m_-$ which should be the masses of the hypermultiplets induced by the left semi-infinite D4-branes (large t corresponds to $x^6 \rightarrow -\infty$). On the other hand, in the small t , the third term is dominant and $v \sim \tilde{m}_+, \tilde{m}_-$ which are the mass parameters induced by the right semi-infinite D4-branes (small t corresponds to $x^6 \rightarrow +\infty$). The dimensional analysis and the regularity constraint in the massless limit show that M is linear in the mass parameters and U is linear in the Coulomb moduli parameter u and also contains additional terms in mass squared. Also, c_1 and c_2 are the constants which parametrize the gauge coupling constant. Then, the curve can be written as

$$v^2(t-1)(t-q) = v((m_+ + m_-)t^2 + (1+q)Mt + q(\tilde{m}_+ + \tilde{m}_-)) - m_+m_-t^2 - (1+q)Ut - q\tilde{m}_+\tilde{m}_-, \quad (2.4)$$

where we have chosen that $c_1 = -(1+q)$ and $c_2 = q$.

By eliminating the terms linear in v and changing the coordinate as $v = xt$, the curve can be written as the following form [27]:

$$x^2 = \left(\frac{m_2t^2 + (1+q)\frac{M}{2}t + m_3q}{t(t-1)(t-q)} \right)^2 + \frac{(m_0^2 - m_2^2)t^2 - (1+q)Ut + (m_1^2 - m_3^2)q}{t^2(t-1)(t-q)}, \quad (2.5)$$

where x^2dt^2 is considered as a quadratic differential on a sphere (t is a coordinate on the sphere) and has double poles at $t = 0, 1, q, \infty$. In this coordinate, the Seiberg-Witten one-form is

$$\lambda_{\text{SW}} = \frac{xdt}{2\sqrt{2}\pi i}, \quad (2.6)$$

where we have divided by the factor $2\sqrt{2}\pi i$ in order to be consistent with the convention in later sections. Here, (t, x) are local coordinates on the cotangent bundle of the sphere. The Seiberg-Witten curve is the double cover of this sphere with four punctures.

The moduli space of the sphere with four punctures is parametrized by q above. As discussed in [27, 45], this moduli space can be identified with the parameter space of the exactly marginal operator of 4d gauge theory, that is the gauge coupling constant (2.1). Therefore, we identify q with $q_{\text{UV}} = e^{\pi i \tau_{\text{UV}}}$. Note that this is the UV gauge coupling constant. The effective gauge coupling at IR is derived using the full information of the Seiberg-Witten theory. We will discuss this point in the next subsection.

The masses of the flavors can be read from the residues of λ_{SW} at the punctures. As seen in [27], each puncture is associated with each $\text{SU}(2)$ flavor symmetry. Indeed, we can easily see that the residues at $t = 0, \infty$ are $\pm m_1$ and $\pm m_0$ where the signs represent their value on the upper and the lower sheets respectively. (More precisely, the residues are $\pm \frac{m_{0,1}}{2\sqrt{2}\pi i}$, but we will ignore the factor $2\sqrt{2}\pi i$ in what follows when discussing the residues.) Also, the requirement that the residues at $t = 1, q_{\text{UV}}$ should be $\pm m_2, \pm m_3$ determines the form of M as follows:

$$M = -\frac{2q_{\text{UV}}}{1+q_{\text{UV}}}(m_2 + m_3). \quad (2.7)$$

The Coulomb branch vev a can be obtained by the period integral

$$a = \oint_A \lambda_{\text{SW}}. \tag{2.8}$$

This is the function of the Coulomb moduli u . Also, the dual B cycle integral computes the derivative of the prepotential

$$a_D = \frac{\partial \mathcal{F}}{\partial a} = \oint_B \lambda_{\text{SW}}. \tag{2.9}$$

The moduli derivative of λ_{SW} is a holomorphic one-form

$$\omega = \frac{\partial \lambda_{\text{SW}}}{\partial u} = -\frac{(1 + q_{\text{UV}})}{4\pi i \sqrt{2P_4(t)}} \frac{\partial U}{\partial u} dt, \tag{2.10}$$

whose A and B periods give $\frac{\partial a}{\partial u}$ and $\frac{\partial a_D}{\partial u}$. $P_4(t)$ is a degree-four polynomial whose precise form does not concern us. As we will see in subsection 2.2, $\frac{\partial U}{\partial u}$ becomes 1 in the weak coupling limit: $q_{\text{UV}} \rightarrow 0$.

We now consider a way to extract the parameter U (essentially $\langle \text{Tr } \phi^2 \rangle$) from the M-theory curve above. In the pure $\mathcal{N} = 2$ $\text{SU}(N)$ gauge theory, it is known that such an operation exists [46–48]

$$\langle \text{Tr } \phi^2 \rangle = \frac{1}{2\pi i} \oint v \lambda_{\text{SW}} = \frac{1}{2\pi i} \oint x^2 t dt, \tag{2.11}$$

where v is the coordinate in (2.3). Also in the $\text{SU}(2)$ gauge theory with four massive flavors, we can use this kind of integral to extract the U parameter. Actually, since the term including U in x^2 is

$$-\frac{(1 + q_{\text{UV}})U}{t(t-1)(t-q_{\text{UV}})}, \tag{2.12}$$

the integral

$$\frac{1}{2\pi i} \oint_{C_\infty} x^2 t^n dt, \quad \text{for } n \geq 2 \tag{2.13}$$

is linear in U and contains terms involving mass parameters. We denote a contour around infinity in the counterclockwise direction as C_∞ . In the simplest case ($n = 2$), we easily obtain

$$\frac{1}{2\pi i} \oint_{C_\infty} x^2 t^2 dt = (1 + q_{\text{UV}})U - (1 + q_{\text{UV}})m_0^2 + (q_{\text{UV}} - 1)m_2^2 + 2q_{\text{UV}}m_2m_3. \tag{2.14}$$

This relation becomes an important point when we compare the gauge theory with the matrix model in section 3.

2.2 UV and IR gauge coupling constants

The M-theory curve (2.5) which we have seen in the previous subsection is different from the standard Seiberg-Witten curve introduced in [44]. In this subsection, we will see that the comparison of the M-theory curve and the standard Seiberg-Witten curve leads to a relation between the UV and IR gauge coupling constants [1, 25, 26, 49, 50]. We first consider the massless case for simplicity.

On the one hand, we have the standard form of the Seiberg-Witten curve [44]

$$y^2 = 4x^3 - g_2 u^2 x - g_3 u^3, \tag{2.15}$$

where u parametrizes the Coulomb moduli and is related to $\langle \text{Tr } \phi^2 \rangle$. The IR gauge coupling constant

$$\tau_{\text{IR}} \equiv \frac{\theta_{\text{IR}}}{\pi} + \frac{8\pi i}{g_{\text{IR}}^2} \tag{2.16}$$

is calculated from the period integrals of the holomorphic one-form ω which is defined as

$$\omega = \frac{\sqrt{2}}{4\pi} \frac{dx}{y} = \frac{\sqrt{2}}{4\pi} \frac{dx}{\sqrt{4x^3 - g_2 u^2 x - g_3 u^3}}, \tag{2.17}$$

where g_2 and g_3 are the functions of $q_{\text{IR}} = e^{\pi i \tau_{\text{IR}}}$:

$$\begin{aligned} g_2(\omega_1, q_{\text{IR}}) &= \left(\frac{\pi}{\omega_1}\right)^4 \frac{1}{24} (\vartheta_3(q_{\text{IR}})^8 + \vartheta_2(q_{\text{IR}})^8 + \vartheta_4(q_{\text{IR}})^8), \\ g_3(\omega_1, q_{\text{IR}}) &= \left(\frac{\pi}{\omega_1}\right)^6 \frac{1}{432} (\vartheta_4(q_{\text{IR}})^4 - \vartheta_2(q_{\text{IR}})^4) (2\vartheta_3(q_{\text{IR}})^8 + \vartheta_4(q_{\text{IR}})^4 \vartheta_2(q_{\text{IR}})^4), \end{aligned} \tag{2.18}$$

and $2\omega_1 = \pi$ in this case. A -cycle integral of ω gives

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{4\pi} \oint_A \frac{dx}{\sqrt{4x^3 - g_2 u^2 x - g_3 u^3}} = \frac{1}{2\sqrt{2u}}, \tag{2.19}$$

which leads to $a = \frac{1}{2}\sqrt{2u}$.

As pointed out in [49], however, the gauge coupling constant receives the correction due to instanton effects even in $N_f = 4$ theory and, therefore, $q_{\text{IR}} \neq q_{\text{UV}}$. As we saw in the previous subsection, we obtain the following curve in the massless limit of the M-theory curve:

$$x^2 = -\frac{(1 + q_{\text{UV}})U}{t(t-1)(t - q_{\text{UV}})}, \tag{2.20}$$

where q_{UV} is the UV gauge coupling constant. Note that the parameter U in the above curve would be different from u in the curve (2.15). Dimensional analysis shows that U is proportional to u :

$$U = Au, \tag{2.21}$$

where A depends only on the UV gauge coupling q_{UV} . The holomorphic one-form is given by

$$\omega = \frac{1}{4\sqrt{2}\pi i} \sqrt{\frac{-(1 + q_{\text{UV}})A}{u}} \frac{dt}{\sqrt{t(t-1)(t - q_{\text{UV}})}}. \tag{2.22}$$

By changing the coordinate as $t \rightarrow \tilde{t} + \frac{1+q_{UV}}{3}$ and by rescaling as $\tilde{t} \rightarrow 4z$, $t(t-1)(t-q_{UV})$ is transformed to

$$16 \left(4z^3 - \frac{1}{12}(1 - q_{UV} + q_{UV}^2)z - \frac{1}{432}(2 - 3q_{UV} - 3q_{UV}^2 + 2q_{UV}^3) \right) \equiv 16y_0^2. \quad (2.23)$$

Then, the holomorphic one-form reads as

$$\omega = \frac{1}{4\sqrt{2}\pi} \sqrt{\frac{(1 + q_{UV})A}{u}} \frac{dz}{y_0}. \quad (2.24)$$

Now, we want to compare this with the one-form (2.17). In order to reproduce the A period $\frac{1}{2\sqrt{2}u}$, we should obtain

$$\oint_A \frac{dz}{y_0} = \frac{2\pi}{\sqrt{(1 + q_{UV})A}}. \quad (2.25)$$

Therefore, the following relations must be satisfied:

$$\begin{aligned} \frac{1}{12}(1 - q_{UV} + q_{UV}^2) &= g_2(\omega_1, q_{IR}), \\ \frac{1}{432}(2 - 3q_{UV} - 3q_{UV}^2 + 2q_{UV}^3) &= g_3(\omega_1, q_{IR}), \end{aligned} \quad (2.26)$$

and the period is given by

$$2\omega_1 = \frac{2\pi}{\sqrt{(1 + q_{UV})A}}. \quad (2.27)$$

By using formula $\vartheta_4^4 = \vartheta_3^4 - \vartheta_2^4$, the right hand sides of (2.26) can be calculated as

$$\begin{aligned} g_2 &= \frac{1}{12}(1 + q_{UV})^2 A^2 \vartheta_3^8 \left(1 - \frac{\vartheta_2^4}{\vartheta_3^4} + \left(\frac{\vartheta_2^4}{\vartheta_3^4} \right)^2 \right), \\ g_3 &= \frac{1}{432}(1 + q_{UV})^3 A^3 \vartheta_3^{12} \left(2 - 3 \frac{\vartheta_2^4}{\vartheta_3^4} - 3 \left(\frac{\vartheta_2^4}{\vartheta_3^4} \right)^2 + 2 \left(\frac{\vartheta_2^4}{\vartheta_3^4} \right)^3 \right). \end{aligned} \quad (2.28)$$

This shows that

$$q_{UV} = \frac{\vartheta_2(q_{IR})^4}{\vartheta_3(q_{IR})^4} = 16q_{IR} - 128q_{IR}^2 + 704q_{IR}^3 - 3072q_{IR}^4 + \dots \quad (2.29)$$

and

$$A = \frac{1}{\vartheta_2(q_{IR})^4 + \vartheta_3(q_{IR})^4}, \quad (2.30)$$

which implies that $U = u + \mathcal{O}(q_{UV}) = \langle \text{Tr } \phi^2 \rangle + \mathcal{O}(q_{UV})$ in the weak coupling limit. The relation (2.29) between the UV and IR gauge coupling constants has already been suggested in [50] from the topological string analysis and derived in [25, 26] along the similar line as above.

Let us next consider the case with the massive flavors. In this case, the parameter U appeared in the M-theory curve (2.5) receives additional contributions due to mass terms. However, the form of A (2.30) should not change. Therefore, we obtain

$$\frac{\partial U}{\partial u} = \frac{1}{\vartheta_2^4 + \vartheta_3^4}. \tag{2.31}$$

It follows from this that the holomorphic one-form (2.10) becomes

$$\omega = -\frac{dt}{4\pi i \vartheta_3^4 \sqrt{2P_4(t)}}. \tag{2.32}$$

The A -cycle integral of this one-form can be translated into the following standard form of the elliptic integral of the first kind, by the coordinate transformation:

$$\int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \tag{2.33}$$

up to a constant factor, where k is a function determined from $P_4(t)$. We also obtain the similar form for the B cycle integral. The IR gauge coupling constant is given by the period of this one form: $\tau_{\text{IR}} = \frac{\oint_B \omega}{\oint_A \omega}$ and this in turn leads to

$$k^2 = \frac{\vartheta_2(q_{\text{IR}})^4}{\vartheta_3(q_{\text{IR}})^4}. \tag{2.34}$$

This determines the IR gauge coupling constant in terms of the UV one.

3 Matrix model and modular invariance

In this section, we analyze the matrix model with the action (1.1) which was proposed to describe $\mathcal{N} = 2$ SU(2) gauge theory with four massive hypermultiplets and study its modular properties. We first review the general technique of the one matrix model and then consider the modular invariance of the spectral curve.

We define the free energy of the matrix model as follows:

$$\begin{aligned} \exp\left(\frac{F_m}{g_s^2}\right) &= \int dM \exp\left(\frac{1}{g_s} W(M)\right) \\ &= \int \left(\prod_{I=1}^N d\lambda_I\right) \exp\left[\frac{1}{g_s} \left(\sum_{I=1}^N W(\lambda_I) + g_s \sum_{I < J} \log(\lambda_I - \lambda_J)^2\right)\right], \end{aligned} \tag{3.1}$$

where M is the hermitian matrix whose size is N and $W(M)$ is (1.1). In the second line, we have switched to integrals over the eigenvalues λ_I . The last term comes from the Vandermonde determinant. The critical points are determined by the equation of motion

$$\sum_{i=1}^3 \frac{\mu_i}{\lambda_I - q_i} + 2g_s \sum_{J(\neq I)} \frac{1}{\lambda_I - \lambda_J} = 0. \tag{3.2}$$

By ignoring the second term, we obtain two critical points e_p ($p = 1, 2$). The positions q_i will be chosen as $q_1 = 0, q_2 = 1$ and $q_3 = q$. Let each N_p ($p = 1, 2$) be the number of the eigenvalues which are classically at e_p . Let us define the resolvent of the matrix model as

$$R_m(z) = g_s \text{tr} \frac{1}{z - M}. \tag{3.3}$$

In what follows, we take the large N limit while μ_p and the filling fractions $\nu_i \equiv g_s N_p$ are fixed. The loop equation in the large N limit is written as

$$\langle R_m(z) \rangle^2 = -\langle R_m(z) \rangle W'(z) + \frac{f(z)}{4}, \tag{3.4}$$

where

$$f(z) \equiv 4g_s \text{tr} \left\langle \frac{W'(z) - W'(M)}{z - M} \right\rangle = \sum_{i=1}^3 \frac{c_i}{z - q_i} \tag{3.5}$$

has simple poles at $z = q_i$. Note that this function is not a polynomial in contrast to the ordinary case of polynomial action. c_i are functions of μ_i and the filling fractions but they satisfy $\sum_i c_i = 0$ which follows from the equations of motion (3.2).

Let us define the meromorphic one-form $\lambda_m = \frac{x(z)dz}{2\sqrt{2\pi i}}$ such that

$$x(z)^2 \equiv (2\langle R_m(z) \rangle + W'(z))^2 = W'(z)^2 + f(z). \tag{3.6}$$

This one-form has simple poles at $z = q_i, \infty$ with the residues μ_i, μ_0 . (As in the field theory analysis, the residues are precisely $\frac{\mu_i}{2\sqrt{2\pi i}}$ by the definition of λ_m . But, for convenience, we will ignore the factor in the denominator when discussing the residues.) Note that the residue of the pole at $z = \infty$ can be evaluated by observing $\langle R_m \rangle \sim \frac{g_s N}{z}$ and $W'(z) \sim \frac{\sum_{i=1}^3 \mu_i}{z}$ at large z and by using the relation (1.2).

The spectral curve of the matrix model (3.6) looks like the M-theory curve. The former includes the parameters μ_i which will be identified with the masses m_i of the latter. These are the residues of the meromorphic one-form λ_m and λ_{SW} at the simple poles. Hence, the identification $q = q_{\text{UV}}$ is needed. To compare more precisely, we should add a dimensionful parameter by $g_s \rightarrow g_s \epsilon$ since the parameters in the matrix model are dimensionless. In the analysis of the gauge theory, the parameter U is obtained from $\oint x^2 t^n dt$ for $n \geq 2$ (2.14). In the matrix model language, the similar integral gives

$$\frac{1}{2\pi i} \oint_{C_\infty} x^2 z^2 dz = \frac{1}{2\pi i} \oint_{C_\infty} (2R_m + W')^2 z^2 dz = 4\mu_0 g_s \langle \text{tr} M \rangle + 2\mu_0 (\mu_2 + q\mu_3), \tag{3.7}$$

where C_∞ is the contour around infinity. Therefore, the precise correspondence between the matrix model and the gauge theory is given by the following identification:

$$4\mu_0 g_s \langle \text{tr} M \rangle + 2\mu_0 (\mu_2 + q\mu_3) = (1 + q_{\text{UV}})U - (1 + q_{\text{UV}})m_0^2 + (q_{\text{UV}} - 1)m_2^2 + 2q_{\text{UV}}m_2m_3, \tag{3.8}$$

and also $\mu_i = m_i, q = q_{\text{UV}}$.

We can rewrite $\langle \text{tr} M \rangle$ in terms of the coefficients c_i in (3.5). First of all, the residue at $z = \infty$ imposes the following constraint on c_i :

$$c_2 + qc_3 = \mu_0^2 - \left(\sum_{i=1}^3 \mu_i \right)^2. \quad (3.9)$$

Therefore, by recalling $\sum_i c_i = 0$, only one of c_i 's is independent. By using (3.9) and $\sum c_i = 0$, the spectral curve can be written as

$$x^2 = \left(\frac{\mu_1}{z} + \frac{\mu_2}{z-1} + \frac{\mu_3}{z-q} \right)^2 + \frac{\left(\mu_0^2 - \left(\sum_{i=1}^3 \mu_i \right)^2 \right) z + qc_1}{z(z-1)(z-q)}. \quad (3.10)$$

Also, the filling fractions can be obtained by the A-cycle integrals of the one-form:

$$\frac{\nu_p}{\sqrt{2\pi i}} = \frac{1}{2\pi i} \oint_{A_p} \lambda_m, \quad p = 1, 2, \quad (3.11)$$

where A_p are the cycles around the branch cuts (corresponding to two critical points). From this, we can in principle determine ν_p as the function of μ_i and c_1 . In other words, c_1 can be treated as an independent parameter.

It follows from the dimensional analysis that c_1 has mass dimension two. Therefore, it is natural to consider that c_1 is a linear function in U of the M-theory curve. Indeed, by substituting the explicit expression (3.10) into the left hand side of (3.7), we obtain

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_\infty} x^2 z^2 dz &= -qc_1 - (1+q)\mu_0^2 + (1+q)\mu_1^2 + (q-1)\mu_2^2 \\ &\quad + (1-q)\mu_3^2 + 2q\mu_1\mu_2 + 2\mu_3\mu_1. \end{aligned} \quad (3.12)$$

This and (3.8) lead to

$$\begin{aligned} q_{UV}c_1 &= (1+q_{UV})m_1^2 + (1-q_{UV})m_3^2 + 2q_{UV}m_1m_2 \\ &\quad - 2q_{UV}m_2m_3 + 2m_3m_1 - (1+q_{UV})U. \end{aligned} \quad (3.13)$$

We can show that under this relation, $\mu_i = m_i$ and $q = q_{UV}$, the meromorphic one-form λ_m is equal to λ_{SW} . Therefore, the A and B -periods of λ_m and λ_{SW} coincide, e.g., $\sqrt{2}\nu_1(u) = a(u)$.

Also, u -derivative of λ_m gives a holomorphic one-form:

$$\omega_m = \frac{\partial \lambda_m}{\partial u} = -\frac{dz}{4\pi i \vartheta_3^4 \sqrt{2P_{m4}(z)}}, \quad (3.14)$$

where $P_{m4}(z)$ is the same degree 4 polynomial as P_4 in (2.10). The effective gauge coupling constant τ_{IR} can be obtained from the period of ω_m .

3.1 Modular invariance

It is known that the standard Seiberg-Witten curve of $SU(2)$ gauge theory with $N_f = 4$ is invariant under modular transformation in [44]. We will see in this subsection that the spectral curve (3.10) which was identified with the M-theory curve (2.5) can be made modular invariant.

To begin with, we consider the massless limit of the curve (3.10)

$$x^2 = -\frac{(1 + q_{UV})U}{z(z-1)(z-q)} = -\frac{\frac{u}{\vartheta_3^4}}{z(z-1)(z-q)}. \quad (3.15)$$

where we have used the relation (2.30). This is invariant under the following transformations

$$I : (z, x) \rightarrow (1 - z, x), \quad q_{UV} \rightarrow 1 - q_{UV}, \quad u \rightarrow -u, \quad S \quad (3.16)$$

$$II : (z, x) \rightarrow \left(\frac{1}{z}, -z^2 x\right), \quad q_{UV} \rightarrow \frac{1}{q_{UV}}, \quad u \rightarrow u, \quad STS \quad (3.17)$$

Since $q_{UV} = \vartheta_2^4 / \vartheta_3^4$ as seen in subsection 2.2, the former transformation is the exchange of ϑ_2^4 and ϑ_4^4 , which is the S-transformation. Also, the latter one can be seen as the STS-transformation: $\vartheta_2^4 \leftrightarrow \vartheta_3^4$. We note that the sign of u changes under S-transformation. This behavior is the same as in the case of standard SW curve. In fact in (2.15) g_2 and g_3 are even and odd and u changes sign under S-transformation.

Next, let us consider the massive case. As analyzed in [44], the modular transformation in this case involves the triality of $SO(8)$ which permutes the $SU(2)$ flavor symmetries. In our notation, this permutes the mass parameters m_i ($i = 0, \dots, 3$) associated with these $SU(2)$'s. Under the S- and STS-transformations, positions and residues of the poles of λ_m are transformed as

$$I : (0, 1, q_{UV}, \infty) \rightarrow (1, 0, 1 - q_{UV}, \infty), \quad m_1 \leftrightarrow m_2, \quad (3.18)$$

$$II : (0, 1, q_{UV}, \infty) \rightarrow \left(\infty, 1, \frac{1}{q_{UV}}, 0\right), \quad m_0 \leftrightarrow m_1. \quad (3.19)$$

Under these transformations, the spectral curve should be invariant. Let us substitute (3.13) into (3.10) and we obtain

$$x^2(z; m_i; q_{UV}) = \left(\frac{m_1}{z} + \frac{m_2}{z-1} + \frac{m_3}{z-q_{UV}}\right)^2 + \frac{\left(m_0^2 - (\sum_{i=1}^3 m_i)^2\right)z + (1 + q_{UV})m_1^2}{z(z-1)(z-q_{UV})} + \frac{(1 - q_{UV})m_3^2 + 2q_{UV}m_1m_2 - 2q_{UV}m_2m_3 + 2m_1m_3 - (1 + q_{UV})U}{z(z-1)(z-q_{UV})}. \quad (3.20)$$

We then impose the conditions

$$x^2(z; m_0, m_1, m_2, m_3; q_{UV}) = x^2(1 - z; m_0, m_2, m_1, m_3; 1 - q_{UV}), \quad (3.21)$$

$$x^2(z; m_0, m_1, m_2, m_3; q_{UV}) = \frac{1}{z^4} x^2\left(\frac{1}{z}; m_1, m_0, m_2, m_3; \frac{1}{q_{UV}}\right). \quad (3.22)$$

Requirement of modular invariance determines completely the mass dependence of the parameter U . The solution to the above conditions is given by

$$(1 + q_{UV})U = \frac{u}{\vartheta_3^4} - q_{UV}(m_2 + m_3)^2 + \frac{1 + q_{UV}}{3} \left(\sum_{i=0}^3 m_i^2 \right). \quad (3.23)$$

3.2 Relation for $\langle \text{tr} M^m \rangle$

In (3.8), we have written down the parameter U in terms of $\langle \text{tr} M \rangle$ by using the integral $\oint x^2 z^2 dz$. As seen in subsection 2.1, we can in principle use the other integrals as

$$\frac{1}{2\pi i} \oint_{C_\infty} x^2 z^n dz, \quad \text{for } n \geq 2, \quad (3.24)$$

in order to extract the parameter U . This in turn gives the relations for $\langle \text{tr} M^m \rangle$'s. For example, we consider $n = 3$ case. From the M-theory curve, we can compute

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_\infty} x^2 t^3 dt &= 4m_0(1 + q_{UV})g_s \langle \text{tr} M \rangle + q_{UV}m_0^2 - q_{UV}m_1^2 + (q_{UV} - 1)m_2^2 \\ &\quad + q_{UV}(1 - q_{UV})m_3^2 + 2(1 + q_{UV})m_0(m_2 + q_{UV}m_3), \end{aligned} \quad (3.25)$$

where we have used (3.8). On the other hand, the same integral for the matrix model is calculated as

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_\infty} x^2 z^3 dz &= 4\mu_0 g_s \langle \text{tr} M^2 \rangle - 4g_s \langle \text{tr} M \rangle (g_s \langle \text{tr} M \rangle + m_2 + q_{UV}m_3) \\ &\quad - (m_2 + q_{UV}m_3)^2 + 2m_0(m_2 + q_{UV}^2 m_3). \end{aligned} \quad (3.26)$$

From these equations, we obtain

$$\begin{aligned} 4\mu_0 g_s \langle \text{tr} M^2 \rangle &= 4g_s \langle \text{tr} M \rangle (g_s \langle \text{tr} M \rangle + (1 + q_{UV})m_0 + m_2 + q_{UV}m_3) \\ &\quad + q_{UV}(m_0^2 - m_1^2 + m_2^2 + m_3^2) + 2q_{UV}(m_2 m_3 + m_0 m_2 + m_0 m_3). \end{aligned} \quad (3.27)$$

Similarly, $\langle \text{tr} M^m \rangle$ for $m \geq 3$ can be written in terms of the lower order ones.

3.3 Free energy

So far, we have learned that M-theory curve and spectral curve of matrix model can be identified. However, it is not so obvious that the matrix model free energy coincides with the prepotential of gauge theory, in other words, the B-period of λ_m (differential of the matrix theory) is written as $\frac{\partial F_m}{\partial v_1}$. Let us see this below. We consider the free energy and its derivative with respect to the filling fractions. First of all, we rewrite the free energy as

$$e^{F_m/g_s^2} = \exp \left[\frac{1}{g_s^2} \left(\int d\lambda \rho(\lambda) W(\lambda) + \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log |\lambda - \lambda'| \right) \right], \quad (3.28)$$

where $\rho(\lambda)$ is the eigenvalue distribution function normalized as

$$\int d\lambda \rho(\lambda) = g_s N, \quad (3.29)$$

or $\rho(\lambda) = g_s \sum_I \delta(\lambda - \lambda_I)$. The resolvent can be written as

$$R_m(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda}. \quad (3.30)$$

Now, let us consider the derivative of the free energy with respect to the filling fractions ν_p . Here, we will follow the discussion in [51, 52]. The variations with respect to ν_p can be considered as the shift $\rho(\lambda) \rightarrow \rho(\lambda) + (\delta\nu_p)\delta(\lambda - e_p^+)$ where e_p^+ are any points on the branch cuts (except for the end points of the cuts). Therefore, the free energy is shifted as

$$\begin{aligned} \delta F_m &= (\delta\nu_p) \left[W(e_p^+) + 2 \int d\lambda \rho(\lambda) \log |\lambda - e_p^+| \right] \\ &= (\delta\nu_p) \left[W(e_p^+) + \frac{2}{2\pi i} \oint_{A_1+A_2} \langle R_m(z) \rangle \log(z - e_p^+) dz \right]. \end{aligned} \quad (3.31)$$

The second term is evaluated by deforming the contour to the cycle around the log cut as

$$\begin{aligned} \frac{1}{2\pi i} \oint_{A_1+A_2} \langle R_m(z) \rangle \log(z - e_p^+) &= - \int_{e_p^+}^{\infty} \langle R_m(z) \rangle dz + \frac{g_s N}{2\pi i} \oint_{\infty} \frac{\log(z - e_p^+)}{z} dz \\ &= - \int_{e_p^+}^{\Lambda_0} \langle R_m(z) \rangle dz + g_s N (\log \Lambda_0 + \pi i) + \mathcal{O}\left(\frac{1}{\Lambda_0}\right). \end{aligned} \quad (3.32)$$

Note that $\langle R_m \rangle$ has a pole only at $z = \infty$. We regularized the integral by introducing the cut off Λ_0 . We will take Λ_0 to infinity after the calculation. By changing $\langle R_m \rangle$ into λ_m we obtain

$$\frac{\partial F_m}{\partial \nu_p} = -2\sqrt{2}\pi i \int_{e_i^+}^{\Lambda_0} \lambda_m - \mu_0 \log \Lambda_0 + 2\pi i g_s N + \mathcal{O}\left(\frac{1}{\Lambda_0}\right). \quad (3.33)$$

In order to compare with the prepotential of the gauge theory, we let $g_s \rightarrow g_s \epsilon$ as above, which gives μ_i , μ_0 , ν_p and F_m dimension. Since F_m has mass dimension 2, it is written as

$$2F_m = \sum_{i=1}^3 \mu_i \frac{\partial F_m}{\partial \mu_i} + \sum_{p=1,2} \nu_p \frac{\partial F_m}{\partial \nu_p}. \quad (3.34)$$

The first term is evaluated as

$$\begin{aligned} \frac{\partial F_m}{\partial \mu_i} &= g_s \langle \text{Tr} \log(M - q_i) \rangle = \frac{1}{2\pi i} \oint_{A_1+A_2} \langle R_m(z) \rangle \log(z - q_i) dz \\ &= \frac{\pi i}{\sqrt{2}} \left[\int_{\Lambda_0}^{q_i} + \int_{\tilde{q}_i}^{\tilde{\Lambda}_0} \right] \lambda_m - \frac{1}{2} W(q_i) - \frac{\mu_0}{2} \log \Lambda_0 + g_s N \pi i + \mathcal{O}\left(\frac{1}{\Lambda_0}\right), \end{aligned} \quad (3.35)$$

where we have deformed the contour to the cycle around the log cut as in (3.32). Also, by using (3.33), the second term in (3.34) is

$$\begin{aligned} \sum_{p=1,2} \nu_p \frac{\partial F_m}{\partial \nu_p} &= -\sqrt{2}\pi i \left[\nu_1 \int_{B_1} + \nu_2 \int_{B_2} \right] \lambda_m - g_s N \mu_0 \log \Lambda_0 + 2\pi i (g_s N)^2 + \mathcal{O}\left(\frac{1}{\Lambda_0}\right) \\ &= \frac{\pi i}{\sqrt{2}} \left(\sum_{i=1}^3 \mu_i + \mu_0 \right) \int_{B_2} \lambda_m \\ &\quad + \sqrt{2}\pi i \nu_1 \oint_B \lambda_m - g_s N \mu_0 \log \Lambda_0 + 2\pi i (g_s N)^2 + \mathcal{O}\left(\frac{1}{\Lambda_0}\right), \end{aligned} \quad (3.36)$$

where B_1 and B_2 are the paths from $\tilde{\Lambda}_0$ to Λ_0 through the branch cuts respectively and we define the cycle $B = B_2 - B_1$. Combining these, we finally obtain

$$2F_m = \frac{\pi i}{\sqrt{2}} \left[\sum_{i=1}^3 \mu_i \int_{\tilde{q}_i}^{q_i} + \mu_0 \int_{B_2} \right] \lambda_m + \pi i \oint_{A_1} \lambda_m \oint_B \lambda_m - \frac{1}{2} \sum_{i=1}^3 \mu_i W(q_i) + \frac{1}{2} \mu_0^2 \log \Lambda_0 - \pi i g_s N \mu_0 + \mathcal{O}\left(\frac{1}{\Lambda_0}\right). \quad (3.37)$$

The first line of the above equation is the same as the prepotential of the gauge theory [53–55] up to a factor πi which can be absorbed by the redefinition of the free energy. Note that the divergence of integrals in the first line is canceled by the divergence of terms in the second line and the free energy is finite. There are also finite terms in the second line which depend only on the mass parameter μ_i and μ_0 . However, we should note that the terms in the first line have ambiguities due to the choice of integration paths. By deforming the paths, an additional contribution which is bilinear in the masses can appear. Therefore, we conclude that the free energy of the matrix model is the same as the prepotential up to these moduli independent terms.

4 Matrix model for asymptotically free theories

4.1 Asymptotically free theory with $N_f = 3$

In this and next subsections, we consider the decoupling of heavy flavors from the matrix model (1.1) and introduce the matrix theory with flavors $N_f < 4$ which would describe the $\mathcal{N}=2$ gauge theories in the asymptotically free region.

Before taking the decoupling limit we should recall the precise relationship between u and $\langle \text{Tr } \phi^2 \rangle$ as pointed out in ref [44]

$$u = \langle \text{Tr } \phi^2 \rangle - \frac{1}{6} (\vartheta_4^4 + \vartheta_3^4) \sum_{i=0}^3 m_i^2. \quad (4.1)$$

By substituting (4.1) into (3.23), we obtain

$$-(1 + q_{UV})U = -\frac{\langle \text{Tr } \phi^2 \rangle}{\vartheta_3^4} + 2q_{UV} m_2 m_3 + \frac{1}{2} q_{UV} (m_3^2 - m_1^2) + \frac{1}{2} q_{UV} (m_2^2 - m_0^2). \quad (4.2)$$

In the following subsections we again use the symbol u to denote $\langle \text{Tr } \phi^2 \rangle$ for simplicity. We hope that no confusion should arise from this change of notation.

In order to discuss the decoupling limit we write the mass parameters as follows

$$m_{\pm} = m_2 \pm m_0, \quad \tilde{m}_{\pm} = m_3 \pm m_1, \quad (4.3)$$

and consider the limit $\tilde{m}_- \rightarrow \infty, q_{UV} \rightarrow 0$ and the other masses and $\tilde{m}_- q_{UV} = \Lambda_3$ fixed, where Λ_3 corresponds to the dynamical scale of the gauge theory. In this case, the relation on the mass parameters (1.2) becomes

$$m_0 + m_2 + \tilde{m}_+ = -2g_s N, \quad (4.4)$$

and this remains finite. The matrix model action leads to

$$W(M) = \tilde{m}_+ \log M - \frac{\Lambda_3}{2M} + m_2 \log(M - 1). \quad (4.5)$$

and we obtain the spectral curve for $N_f = 3$ theory

$$x^2 = \frac{\Lambda_3^2}{4z^4} - \frac{\tilde{m}_+ \Lambda_3}{z^3(z-1)} - \frac{u - (m_2 + \frac{1}{2}\tilde{m}_+) \Lambda_3}{z^2(z-1)} + \frac{m_0^2}{z(z-1)} + \frac{m_2^2}{z(z-1)^2} - \frac{m_2 \Lambda_3}{z^2(z-1)}. \quad (4.6)$$

Note that the structure of the singularities is similar to that obtained in [28]. Let us consider u -derivative of the Seiberg-Witten one-form $\lambda_m = \frac{x dz}{2\sqrt{2\pi i}}$:

$$\omega = \frac{\partial \lambda_m}{\partial u} = \frac{dz}{2\sqrt{2\pi i}} \frac{-1}{2xz^2(z-1)} \equiv -\frac{1}{4\pi i \sqrt{2Q_4(z)}}, \quad (4.7)$$

where

$$Q_4(z) = m_0^2 z^4 + \left(-u - m_0^2 + m_2^2 + \frac{1}{2}\tilde{m}_+ \Lambda_3\right) z^3 + \left(u + \frac{\Lambda_3^2}{4} - \frac{3}{2}\tilde{m}_+ \Lambda_3\right) z^2 + \left(-\frac{\Lambda_3^2}{2} + \tilde{m}_+ \Lambda_3\right) z + \frac{\Lambda_3^2}{4}. \quad (4.8)$$

The discriminant of the polynomial above completely matches with that of the Seiberg-Witten curve for $N_f = 3$ in [44]:

$$y^2 = x^2(x-u) - \frac{1}{4}\Lambda_3^2(x-u)^2 - \frac{1}{4}(m_+^2 + m_-^2 + \tilde{m}_+^2)\Lambda_3^2(x-u) + m_+ m_- \tilde{m}_+ \Lambda_3 x - \frac{1}{4}(m_+^2 m_-^2 + m_-^2 \tilde{m}_+^2 + \tilde{m}_+^2 m_+^2)\Lambda_3^2. \quad (4.9)$$

Note that the dynamical scale Λ_3 differs by a factor from Λ_{SW} that appeared in [44]. This is due to the difference of the decoupling limit of ours and that of [44] where they fixed

$$64\tilde{m}_- q_{\text{IR}} \equiv \Lambda_{\text{SW}}. \quad (4.10)$$

As discussed above, $q_{\text{UV}} = 16q_{\text{IR}}$ for the weak coupling region. Thus, we obtain $\Lambda_{\text{SW}} = 4\tilde{m}_- q_{\text{UV}} = 4\Lambda_3$.

For the equal hypermultiplet mass case where $m_0 = 0$, $m_2 = m$ and $\tilde{m}_+ = m$, the discriminant becomes

$$\Delta = \frac{\Lambda_3^2}{32} \left(u - m^2 - \frac{1}{2}m\Lambda_3\right)^3 \left(-32u^2 + 2\Lambda_3^2 u - 48um\Lambda_3 + 3m\Lambda_3^3 + 6m^2\Lambda_3^2 + 128m^3\Lambda_3\right). \quad (4.11)$$

This implies that three (SU(3) triplet) massless particles appear at the singularity $u = m^2 + \frac{1}{2}m\Lambda_3$. This is the correct property of the curve (4.9).

Free energy. Now we consider the free energy of this matrix model. The formula (3.33) for the derivative of the free energy with respect to the filling fractions can be used without change. By restoring the dimension, we obtain the following equation

$$2F_m = \left(\tilde{m}_+ \frac{\partial}{\partial \tilde{m}_+} + m_2 \frac{\partial}{\partial m_2} + \Lambda_3 \frac{\partial}{\partial \Lambda_3} + \sum_{p=1,2} \nu_p \frac{\partial}{\partial \nu_p} \right) F_m. \quad (4.12)$$

By using the result in the previous section, the computation of the first two terms are straightforward: e.g.,

$$\frac{\partial F_m}{\partial \tilde{m}_+} = \frac{\pi i}{\sqrt{2}} \left(\int_{\Lambda_0}^0 + \int_{\tilde{0}}^{\tilde{\Lambda}_0} \right) \lambda_m - \frac{1}{2} W(0) - \frac{m_0}{2} \log \Lambda_0 + \pi i g_s N + \mathcal{O} \left(\frac{1}{\Lambda_0} \right). \quad (4.13)$$

Also, the last term is exactly the same as (3.36) except that $\sum_{i=1}^3 m_i$ becomes $\tilde{m}_+ + m_2$. Only the new ingredient which need a little bit care is the third term in (4.12). In order to compute this, we note that in the spectral curve: $x^2 = W'(z)^2 + f(z)$, $f(z)$ is written as

$$f(z) = \frac{f_1}{z} + \frac{f_2}{z-1} + \frac{f_3}{z^2}, \quad f_3 = -2g_s \Lambda_3 \left\langle \sum_I \frac{1}{\lambda_I} \right\rangle. \quad (4.14)$$

This f_3 can be directly evaluated from the curve (4.6), which leads to

$$\frac{\partial F_m}{\partial \Lambda_3} = -\frac{g_s}{2} \left\langle \sum_I \frac{1}{\lambda_I} \right\rangle = \frac{f_3}{4\Lambda_3} = \frac{1}{4\Lambda_3} \left(u + \left(\frac{1}{2} \tilde{m}_+ + m_2 \right) \Lambda_3 - \tilde{m}_+^2 \right). \quad (4.15)$$

By collecting these, we finally obtain the expression for the free energy

$$2F_m = \frac{\pi i}{\sqrt{2}} \left(\tilde{m}_+ \int_{\tilde{0}}^0 + m_2 \int_{\tilde{1}}^{\tilde{1}} + m_0 \int_{B_2} \right) \lambda_m + \pi i \oint_{A_1} \lambda_m \oint_B \lambda_m + \frac{u}{4} + \dots \quad (4.16)$$

This nicely matches with the prepotential of the theory with $N_f = 3$ [53–55] up to the irrelevant factor πi . The ellipsis is the terms which depend only on the masses and the scale factor. It is reassuring to note that the beta function coefficient $b = 1$ in front of u is correctly recovered in the matrix model computation.

4.2 Asymptotically free theory with $N_f = 2$

In this subsection, we propose a matrix model action which corresponds to SU(2) gauge theory with $N_f = 2$. Method of its derivation is similar as the one in the previous section: we consider the limit where z is rescaled as $\frac{\Lambda_3}{\Lambda_2} z$ and $m_- \rightarrow \infty$ while keeping $m_- \Lambda_3 \equiv \Lambda_2^2$. The spectral curve is given by

$$x^2 = \frac{\Lambda_2^2}{4z^4} + \frac{\tilde{m}_+ \Lambda_2}{z^3} + \frac{u}{z^2} + \frac{m_+ \Lambda_2}{z} + \frac{\Lambda_2^2}{4}. \quad (4.17)$$

Here, we have rescaled x as $x \rightarrow \frac{\Lambda_2}{\Lambda_3} x$. This curve is the same form as the first realization of SU(2) gauge theory with $N_f = 2$ in [28]. In this limit, the action of the matrix model (4.5) becomes

$$W(M) = \tilde{m}_+ \log M - \frac{\Lambda_2}{2M} - \frac{\Lambda_2 M}{2}, \quad (4.18)$$

where we have ignored irrelevant constant terms. The relation on the mass parameters (4.4) is, in this limit,

$$\tilde{m}_+ + m_+ = -2g_s N. \quad (4.19)$$

The holomorphic one-form is

$$\omega = \frac{\partial}{\partial u} \frac{xdz}{2\sqrt{2}\pi i} = \frac{dz}{4\sqrt{2}\pi i \sqrt{R_4(z)}}, \quad (4.20)$$

where

$$R_4(z) = \frac{\Lambda_2^2}{4} z^4 + m_+ \Lambda_2 z^3 + uz^2 + \tilde{m}_+ \Lambda_2 z + \frac{\Lambda_2^2}{4}. \quad (4.21)$$

The discriminant of the above polynomial agrees completely with that of the Seiberg-Witten curve for $N_f = 2$ in [44]:

$$y^2 = \left(x^2 - \frac{1}{4}\Lambda_2^2\right)(x-u) + m_+ \tilde{m}_+ \Lambda_2^2 x - \frac{1}{4}(m_+^2 + \tilde{m}_+^2) \Lambda_2^4. \quad (4.22)$$

For the equal mass case where $m_+ = \tilde{m}_+ = m$, the discriminant becomes

$$\Delta = \Lambda_2^4 \left(u + 2m\Lambda_{N_f=2} + \frac{1}{2}\Lambda_2^2\right) \left(u - 2m\Lambda_2 + \frac{1}{2}\Lambda_2^2\right) \left(u - m^2 - \frac{1}{2}\Lambda_2^2\right)^2. \quad (4.23)$$

Free energy. Let us compute the free energy of this model. The procedure is the same as that in section 3 and 4.1. Only non-trivial point is the calculation of Λ_2 derivative:

$$\frac{\partial F_m}{\partial \Lambda_2} = -\frac{g_s}{2} \left\langle \sum_I \left(\frac{1}{\lambda_I} + \lambda_I \right) \right\rangle = \frac{g_2}{4\Lambda_2} - \frac{g_s}{2} \langle \sum_I \lambda_I \rangle, \quad (4.24)$$

where g_2 is the coefficient of $g(z) = \frac{g_1}{z} + \frac{g_2}{z^2}$ in the spectral curve: $x^2 = W'(z)^2 + g(z)$. By comparing this form of the spectral curve with (4.17), we obtain $g_2 = u + \frac{1}{2}\Lambda_2^2 - \tilde{m}_+^2$. The second term is $\langle \sum_I \lambda_I \rangle = \langle \text{tr} M \rangle$. This can be determined, by using the same argument as that in subsection 3.2, as

$$2\Lambda_2 \langle \text{tr} M \rangle = -u - \frac{1}{2}\Lambda_2^2 + m_+^2. \quad (4.25)$$

Therefore, we obtain

$$\Lambda_2 \frac{\partial F_m}{\partial \Lambda_2} = \frac{1}{4} \left(g_2 + u + \frac{1}{2}\Lambda_2^2 - m_+^2 \right) = \frac{u}{2} - \frac{m_+^2 + \tilde{m}_+^2 - \Lambda_2^2}{4}. \quad (4.26)$$

Then, the similar computation as above leads to

$$2F_m = \frac{\pi i}{\sqrt{2}} \left(\tilde{m}_+ \int_{\tilde{0}}^0 + m_+ \int_{B_2} \right) \lambda_m + \pi i \oint_{A_1} \lambda_m \oint_B \lambda_m + \frac{2u}{4} + \dots \quad (4.27)$$

This is the same form as the prepotential of the theory with $N_f = 2$. It is nice to find the appearance of the beta function coefficient $b = 2$ for $N_f = 2$ theory in front of u from the matrix model computation.

5 Conclusion and discussion

In this article we have studied the Penner type matrix model and gave strong evidence that it reproduces correctly the physics of $\mathcal{N}=2$ supersymmetric $SU(2)$ gauge theories. It will be interesting to see if the model can in fact be used to provide a simple derivation of the AGT relation.

Generalization to $SU(2)$ linear quiver gauge theories [27] may be straightforward. In this case, the corresponding action of the matrix model is

$$W(M) = \sum_{i=1}^{k-1} \mu_i \log(M - q_i), \quad (5.1)$$

where k corresponds to the number of the simple poles of the Seiberg-Witten one-form. It will be interesting to study the S-duality group of this matrix theory.

A different type of matrix model has been proposed in ref [56, 57] to derive Seiberg-Witten theory. It is important to see how this model is related to the matrix theory discussed in this paper.

Acknowledgments

We would like to thank K. Hosomichi, Katsushi Ito, H. Itoyama, A. Marshakov, T. Oota, N. Sasakura and M. Taki for discussions. Research of T.E. is supported in part by Grant in Aid from the Japan Ministry of Education, Culture, Sports, Science and Technology. K.M. is supported in part by JSPS Research Fellowships for Young Scientists.

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