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Four-dimensional noncommutative deformations of U(1) gauge theory and L_∞ bootstrap.

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Abstract: We construct a family of four-dimensional noncommutative deformations of U(1) gauge theory following a general scheme, recently proposed in JHEP 08 (2020) 041 for a class of coordinate-dependent noncommutative algebras. This class includes the $\mathfrak{su}(2)$, the $\mathfrak{su}(1,1)$ and the angular (or λ-Minkowski) noncommutative structures. We find that the presence of a fourth, commutative coordinate x^0 leads to substantial novelties in the expression for the deformed field strength with respect to the corresponding three-dimensional case. The constructed field theoretical models are Poisson gauge theories, which correspond to the semi-classical limit of fully noncommutative gauge theories. Our expressions for the deformed gauge transformations, the deformed field strength and the deformed classical action exhibit flat commutative limits and they are exact in the sense that all orders in the deformation parameter are present. We review the connection of the formalism with the L_∞ bootstrap and with symplectic embeddings, and derive the L_∞-algebra, which underlies our model.

Keywords: Gauge Symmetry, Non-Commutative Geometry

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1 Introduction

Noncommutative gauge and field theories have been widely studied over more than twenty years. Much has been written about physical motivations for considering space-time to be "quantum" and physical models to be described in terms of noncommuting observables, if one wants to go beyond the dichotomy between classical gravity and quantum physics. There exist excellent reviews for that, see for instance [1–3]. Despite the large efforts made, there is however no general consensus about the appropriate noncommutative generalisation of field theory, mainly because, except for very few models, all attempts proposed present formal and/or interpretative problems, which render the results not fully satisfactory. Nonetheless, the problems addressed with the promise of providing effective models of space-time quantisation and compatible gauge theories maintain their validity.

One main motivation for confronting once again with noncommutative gauge theory is a series of recent publications proposing the framework of L_∞ algebras and a bootstrap approach as appropriate for formulating gauge noncommutativity in a consistent way [4, 5]. Moreover, an interesting connection has been established between the L_∞ bootstrap and symplectic embeddings of non-commutative algebras [6, 7]. It is worth mentioning that the role of L_∞ algebras in gauge and field theory is already investigated in [8–10], see also [11–16] for recent progresses in studies of the L_∞ structures in the field theoretic context.

In the present paper, we shall take advantage of the constructive approach proposed in [17] which, starting from the request that gauge theory be compatible with the desired space-time noncommutativity *and* be equivalent to the standard one in the commutative limit, yields recursive equations for field dependent gauge transformations and deformed field strength. As we shall explicitly discuss, the procedure is closely related to the L_∞ bootstrap and symplectic embedding approaches.

The only exact (all orders in the non-commutativity parameter) nontrivial[1] models, which have been constructed so far along the lines of [17], and exhibiting the flat commutative limit, are the three-dimensional U(1) theory with $\mathfrak{su}(2)$ noncommutativity [17] and the two-dimensional U(1) model with kappa-Minkowski[2] noncommutativity [19]. Therefore, the construction of four-dimensional models of this kind seems to be a valuable and timely problem.

In the present work we fill this gap: we construct exact noncommutative four-dimensional deformations of U(1) gauge theory, implementing several three-dimensional noncommutative structures within the general framework proposed in [17], and adding one more commutative coordinate. As we shall see below, such an addition brings somewhat more than a naive generalisation of the corresponding three-dimensional setup. For the non-trivial sector of the algebra we shall consider explicitly the angular (or λ-Minkowski) noncommutativity [22–28],

$$[\hat{x}^3, \hat{x}^1] = -\mathrm{i}\,\lambda\hat{x}^2, \quad [\hat{x}^3, \hat{x}^2] = \mathrm{i}\,\lambda\hat{x}^1, \quad [\hat{x}^1, \hat{x}^2] = 0, \tag{1.1}$$

and the $\mathfrak{su}(2)$ noncommutativity [29–38],

$$[\hat{x}^k, \hat{x}^l] = -\mathrm{i}\,\lambda\,\varepsilon^{kls}\,\delta_{sp}\,\hat{x}^p. \tag{1.2}$$

The latter may be easily generalised to $\mathfrak{su}(1,1)$, while the time variable $\hat{x}^0$ stays commutative for all cases considered. We shall use the Greek letters μ, ν, ..., and the Latin letters a, b, c, ..., to denote the four-dimensional and the three-dimensional (i.e. the spatial) coordinates respectively. The three-dimensional deformation[3] of U(1) gauge theory, based on the $\mathfrak{su}(2)$ noncommutativity (1.2) has already been studied in detail in [17]. We shall see, however, that an addition of time as a fourth commutative coordinate extends the results of [17] in a nontrivial way. For a given starting space-time $\mathcal{M}$, we shall indicate with $\mathcal{A}_\Theta = (\mathcal{F}(\mathcal{M}), \star)$ the noncommutative algebra of functions representing noncommutative space-time, equipped by some noncommutative star product

$$f \star g \neq g \star f, \qquad f, g \in \mathcal{A}_\Theta \tag{1.3}$$

which, for coordinate functions, reproduces the linear algebras (1.1) and (1.2). Noncommutativity is therefore specified by the $x-$dependent skew-symmetric matrix $\Theta(x)$:

$$[x^\mu, x^\nu]_\star = \mathrm{i}\,\Theta^{\mu\nu}(x), \tag{1.4}$$

which we assume to be a Poisson bivector in order to maintain associativity of the star-product. The symbol $[\ ,\ ]_\star$ denotes the star commutator, defined as follows

$$[f, g]_\star \equiv f \star g - g \star f, \qquad \forall f, g \in \mathcal{A}_\Theta. \tag{1.5}$$

[1] I.e. with coordinate-dependent noncommutativity.

[2] The deformed gauge transformations and the deformed field strength for the kappa-Minkowski noncommutativity have been constructed in [19] in arbitrary dimension d, however, the gauge invariant classical action exhibits the *flat* commutative limit at $d = 2$ only.

[3] Our notations do not coincide with the ones of [17]: $\lambda_{\text{our}} = -2\theta_{\text{Ref.}[17]}$.

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Standard U(1) gauge transformations, $\delta^0_f A = \partial f$, with $f \in \mathcal{F}(\mathcal{M})$, close an Abelian algebra, $[\delta^0_f, \delta^0_g] = 0$. For non-Abelian gauge theories where gauge parameters are valued in a non-Abelian Lie algebra, $\mathbf{f} = f_i \tau^i$, we have instead $\delta^0_{\mathbf{f}} A = \partial \mathbf{f} - i[A, \mathbf{f}]$ so that

$$[\delta^0_{\mathbf{f}}, \delta^0_{\mathbf{g}}] A = \partial[\mathbf{f}, \mathbf{g}] - i[A, [\mathbf{f}, \mathbf{g}]] = \delta^0_{[\mathbf{f},\mathbf{g}]} A.$$

Namely, the algebra of gauge transformations closes with respect to a non-Abelian Lie bracket. Noncommutative U(1) gauge theory, with gauge parameters now belonging to $\mathcal{A}_\Theta$ behaves very much like non-Abelian theories. Therefore, according to [17], we shall require that the algebra of gauge transformations closes with respect to the star commutator, namely

$$[\delta_f, \delta_g] A = \delta_{-\mathrm{i}[f,g]_\star} A. \tag{1.6}$$

However, if gauge transformations are defined as a natural generalisation of the non-Abelian case,

$$A' = A + \partial f - \mathrm{i}[A, f]_\star, \tag{1.7}$$

it is known that, by composing two such transformations, we get the result (1.6) only if ∂ is a derivation of the star commutator, which in general is not the case. Hence, the guiding principle in [17] was the definition of the infinitesimal gauge transformations,

$$A_\mu \longrightarrow A_\mu + \delta_f A_\mu, \tag{1.8}$$

in such a way that they close the noncommutative algebra (1.6) and reduce to the standard U(1) transformations in the commutative limit,

$$\lim_{\lambda \to 0} \delta_f A_\mu = \partial_\mu f. \tag{1.9}$$

The star commutator, which enters in (1.6) has the following structure:

$$[f, g]_\star = \mathrm{i}\,\{f, g\} + \dots, \tag{1.10}$$

where $\{f, g\}$ stands for the Poisson bracket of f and g,

$$\{f, g\} \equiv \Theta^{\mu\nu}\, \partial_\mu f\, \partial_\nu g, \tag{1.11}$$

while the remaining terms, denoted through "…", contain higher derivatives. From now on we neglect these terms, namely we consider the semi-classical limit [6, 7]. Therefore our noncommutative gauge algebra becomes the Poisson gauge algebra:

$$[\delta_f, \delta_g] = \delta_{\{f,g\}}. \tag{1.12}$$

We shall consider in what follows a two-parameter family of Poisson structures:

$$\Theta^{0\mu} = 0 = \Theta^{\mu 0}, \quad \Theta^{jk} = -\lambda\, \varepsilon^{jks}\, \check{\alpha}_{sl}\, x^l, \tag{1.13}$$

where the 3×3 matrix $\check{\alpha}$ is defined as follows:

$$\check{\alpha} := \mathrm{diag}\,\{1,\, 1,\, \alpha\}, \quad \alpha \in \mathbb{R}. \tag{1.14}$$

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At $\alpha = 0$ we get the Poisson structure which corresponds to the angular noncommutativity,

$$\{x^3, x^1\} = -\lambda x^2, \quad \{x^3, x^2\} = \lambda x^1, \quad \{x^1, x^2\} = 0, \quad \{x^j, x^0\} = 0, \tag{1.15}$$

while at $\alpha = 1$ the three-dimensional bivector Θ^{jk} is nothing but the Poisson structure of the $\mathfrak{su}(2)$ case,

$$\{x^k, x^l\} = -\lambda\, \varepsilon^{kls}\, \delta_{sp}\, x^p. \tag{1.16}$$

Another interesting case is represented by $\alpha = -1$ which corresponds to the Lie algebra $\mathfrak{su}(1,1)$. We emphasise however that the Jacobi identity,

$$f_\mu^{\xi\lambda} f_\lambda^{\nu\phi} + f_\mu^{\nu\lambda} f_\lambda^{\phi\xi} + f_\mu^{\phi\lambda} f_\lambda^{\xi\nu} = 0, \qquad f_\mu^{\xi\lambda} \equiv \partial_\mu \Theta^{\xi\lambda}, \tag{1.17}$$

is satisfied for *any* α, not just at $\alpha = 0$ and $\alpha = \pm 1$. Introducing the projector $\boldsymbol{\delta}_\mu^\nu$ on the three-dimensional space,

$$\boldsymbol{\delta}_\mu^\nu := \delta_\mu^\nu - \delta_\mu^0 \delta_0^\nu, \tag{1.18}$$

we get an explicit formula for the structure constants:

$$f_\rho^{\mu\nu} = -\lambda\, \boldsymbol{\delta}_j^\mu\, \boldsymbol{\delta}_k^\nu\, \boldsymbol{\delta}_\rho^l\, \varepsilon^{jks}\, \breve{\alpha}_{sl}, \tag{1.19}$$

yielding

$$\{x^j, x^k\} = -\lambda\, \varepsilon^{jks}\, \breve{\alpha}_{sl}\, x^s, \qquad \{x^j, x^0\} = 0. \tag{1.20}$$

The paper is organised as follows. Section 2 is devoted to deformed gauge transformations. Moreover, a connection with the symplectic embedding approach is discussed. In section 3 we present relevant aspects of the L_∞ bootstrap approach to gauge theories and establish the L_∞ algebra, which corresponds to our gauge transformations. In section 4 we introduce a deformed field strength and a suitable classical action.

2 Noncommutative gauge transformations

According to [17] the infinitesimal deformed gauge transformations, which close the algebra (1.12), and reproduce the correct undeformed limit (1.9), can be constructed by allowing for a field-dependent deformation as follows:

$$\delta_f A_\mu = \gamma_\mu^\nu(A) \partial_\nu f + \{A_\mu, f\}. \tag{2.1}$$

This variation satisfies the following derivation property [6]:

$$\delta_{fg} A_\mu = g \delta_f A_\mu + f \delta_g A_\mu. \tag{2.2}$$

For (1.12) to be satisfied the 4×4 matrix γ has to solve the master equation[4]

$$\gamma_\mu^\nu \partial_A^\mu \gamma_\lambda^\xi - \gamma_\mu^\xi \partial_A^\mu \gamma_\lambda^\nu + \Theta^{\nu\mu} \partial_\mu \gamma_\lambda^\xi - \Theta^{\xi\mu} \partial_\mu \gamma_\lambda^\nu - \gamma_\lambda^\mu \partial_\mu \Theta^{\nu\xi} = 0, \tag{2.3}$$

[4]We use the notation $\partial_A^\mu \equiv \frac{\partial}{\partial A_\mu}$.

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moreover, it has to reduce to the identity at the commutative limit,

$$\lim_{\lambda\to 0}\gamma_\nu^\mu = \delta_\nu^\mu. \tag{2.4}$$

The last requirement guarantees that the noncommutative transformations (2.1) reproduce the standard Abelian gauge transformations (1.9) in the undeformed theory. A general result has been established in [6] in the context of symplectic embeddings, that is valid for any Θ which is linear in x. It suggests a solution of eq. (2.3) in the form[5]

$$\gamma_\mu^\nu(A) = -\frac{1}{2}f_\mu^{\nu\lambda}A_\lambda + \delta_\mu^\nu + \chi\left(-\frac{M}{2}\right)_\mu^\nu, \tag{2.5}$$

where the matrix M is defined by

$$M_\mu^\nu = f_\xi^{\nu\lambda}f_\mu^{\xi\phi}A_\lambda A_\phi \tag{2.6}$$

and the function χ reads

$$\chi(u) = \sqrt{\frac{u}{2}}\cot\sqrt{\frac{u}{2}} - 1 = \sum_{n=1}^{\infty}\frac{(-2)^n B_{2n}\,u^n}{(2n)!}. \tag{2.7}$$

In the last equalities the quantities B_{2n} are the Bernoulli numbers, and $\chi\left(\frac{M}{2}\right)_\mu^\nu$ have to be understood as the matrix elements of $\chi(M/2)$.

By replacing the structure constants (1.19) in the definition (2.6) we obtain

$$M_\mu^\nu = -\lambda^2\,\boldsymbol{Z}\,\hat{M}_\mu^\nu, \tag{2.8}$$

where the matrix $\hat{M}$, defined by

$$\hat{M}_\mu^\nu = \delta_\mu^\nu - \frac{\boldsymbol{A}_{\boldsymbol{\alpha}}^\nu \boldsymbol{A}_\mu}{\boldsymbol{Z}}, \tag{2.9}$$

is a *projector*, i.e. $\hat{M}^2 = \hat{M}$, and hence

$$\hat{M}^n = \hat{M}, \qquad \forall n \in \mathbb{N}. \tag{2.10}$$

From now on we shall make use of the notations:[6]

$$\begin{aligned}
\boldsymbol{A}_\mu &:= \boldsymbol{\delta}_\mu^\nu A_\nu,\\
\boldsymbol{A}_{\boldsymbol{\alpha}}^\mu &:= \hat{\alpha}^{\mu\xi}A_\xi, \quad \hat{\alpha} := \mathrm{diag}\{\alpha,\,\alpha,\,1,\,0\},\\
\boldsymbol{Z} &:= A_\mu \boldsymbol{A}_{\boldsymbol{\alpha}}^\mu = \alpha\cdot(A_1)^2 + \alpha\cdot(A_2)^2 + (A_3)^2\,.
\end{aligned} \tag{2.11}$$

Using the identity (2.10) we can easily calculate the nontrivial term of eq. (2.5):

$$\begin{aligned}
\chi\left(-\frac{M}{2}\right) &= \sum_{n=1}^{\infty}\frac{(-2)^n B_{2n}}{(2n)!}\left(-\frac{M}{2}\right)^n = \sum_{n=1}^{\infty}\frac{(-2)^n B_{2n}}{(2n)!}\left(\frac{\lambda^2\boldsymbol{Z}}{2}\right)^n\cdot\underbrace{\hat{M}^n}_{\hat{M}}\\
&= \left(\sum_{n=1}^{\infty}\frac{(-2)^n B_{2n}}{(2n)!}\left(\frac{\lambda^2\boldsymbol{Z}}{2}\right)^n\right)\cdot\hat{M} = \chi\left(\frac{\lambda^2\boldsymbol{Z}}{2}\right)\cdot\hat{M}.
\end{aligned} \tag{2.12}$$

[5] Note that our notations differ from the ones of [6]. In order to obtain our eq. (2.5) one has to set $t = 1$, $p = A$ and replace γ by $\gamma - \mathbb{1}$ in eq. (6.3) of [6].

[6] Do not confuse $\hat{\alpha}$ and $\check{\alpha}$!

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Substituting this result in the general formula (2.5), we arrive at the final expression for γ,

$$\gamma^{\nu}_{\mu}(A) = -\frac{1}{2}\, f^{\nu\lambda}_{\mu}\, A_{\lambda} + \delta^{\nu}_{\mu} + \frac{\lambda^2}{4}\,\hat{\chi}\left(\frac{\boldsymbol{Z}\,\lambda^2}{4}\right)\left(\boldsymbol{Z}\,\boldsymbol{\delta}^{\nu}_{\mu} - \boldsymbol{A}^{\nu}_{\alpha}\boldsymbol{A}_{\mu}\right), \tag{2.13}$$

where we introduced another form factor,

$$\hat{\chi}(v) := \frac{1}{v}(\sqrt{v}\cot\sqrt{v} - 1), \tag{2.14}$$

in order to confront our results with the ones of [17]. Setting $\alpha = 1$, one can easily see that the three-dimensional part of (2.13), viz γ^{i}_{j}, coincides with the known three-dimensional result (2.11) of [17] for the $\mathfrak{su}(2)$-case.

Interestingly, the field-dependent deformation of gauge transformations (2.1) has been derived in [6] as the result of a symplectic embedding of the Poisson manifold $(\mathcal{M}, \Theta)$ into the symplectic manifold $(T^*\mathcal{M}, \omega)$, where $T^*\mathcal{M}$ denotes the cotangent bundle and ω an appropriate symplectic form such that $\pi_*\omega^{-1} = \Theta$, $\pi : T^*\mathcal{M} \to \mathcal{M}$ being the projection map.

Shortly, the idea of symplectic embeddings of Poisson manifolds is a generalization of symplectic realizations [20, 21], which consist in the following. One considers the canonical symplectic form ω_0 on $T^*\mathcal{M}$, which is locally given by $\omega_0 = d\lambda_0 = dp_\mu \wedge dx^\mu$ with $\lambda_0 = p_\mu dx^\mu$ the Liouville one-form. The contraction of λ_0 with the Poisson tensor Θ defined on $\mathcal{M}$ yields a vector field, $X^\Theta = \Theta(\lambda_0, \ ) = \Theta^{\mu\nu}(x)p_\mu\partial_{x^\nu}$ whose flow we shall indicate with $\varphi^\Theta_t, t \in \mathbb{R}$. In terms of the latter, it is possible to endow, at least locally, the cotangent space with a new symplectic form, ω whose inverse naturally projects down to the Poisson tensor Θ on $\mathcal{M}$ through the projection map $\pi : T^*\mathcal{M} \to \mathcal{M}$. According to [20, 21] such a form is given by the integrated pull-back of the canonical symplectic form ω_0 through the flow associated with the vector field X^Θ,

$$\omega := \int_0^1 (\varphi^\Theta_t)^*(\omega_0)\, dt \tag{2.15}$$

which in coordinates reads $\omega = dy^\mu \wedge dp_\mu$ with $y^\mu(x,p) = \int_0^1 x^\mu \circ \varphi^\Theta_t\, dt$. The Jacobian matrix $J = (\partial y^\mu/\partial x^\nu)$ is formally invertible. On denoting its inverse by $\gamma(x,p)$ the symplectic Poisson tensor $\omega^{-1} = \frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial p_\mu}$ is given in terms of the original variables (x,p) and the matrix $\gamma(x,p)$, according to

$$\omega^{-1} = \Theta^{\mu\nu}\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu} - \gamma^\mu_\nu(x,p)\frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial p_\nu}. \tag{2.16}$$

Let's pose $\omega^{-1} = \Lambda$. A generalization of the previous procedure consists in *defining* Λ as a deformation of Θ, according to eq. (2.16) and imposing that it satisfies Jacobi identity, provided Θ does. This amounts to compute the Schouten bracket, $[\Lambda, \Lambda]$, and impose that it be zero. We obtain the following equation for the matrix γ,

$$\gamma^\nu_\mu \frac{\partial}{\partial p_\mu}\gamma^\xi_\lambda - \gamma^\xi_\mu \frac{\partial}{\partial p_\mu}\gamma^\nu_\lambda + \Theta^{\nu\mu}\frac{\partial}{\partial x^\mu}\gamma^\xi_\lambda - \Theta^{\xi\mu}\frac{\partial}{\partial x^\mu}\gamma^\nu_\lambda - \gamma^\mu_\lambda \frac{\partial}{\partial x^\mu}\Theta^{\nu\xi} = 0, \tag{2.17}$$

where $[\Theta, \Theta] = 0$ has been used. The latter is exactly the master equation (2.3) after replacing derivatives with respect to p_μ by derivatives with respect to A_μ, which is, however, a non-trivial difference, since A_μ is itself a function of x, while p_μ is obviously not.

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The relation between the two approaches, which has been established in [6, 7], may be summarised as follows.

Let us first consider the standard setting with $\Theta = 0$. Then, the cotangent bundle $T^*\mathcal{M}$ is endowed with the canonical symplectic form ω_0. The gauge field $A \in \Omega^1(U)$, $U \subset \mathcal{M}$ is associated with a local section $s_A : U \to T^*U$, through a local trivialisation, $\psi_U^{-1}(s_A(x)) = (x, A(x))$. The image of s_A is a submanifold of T^*U. Let

$$\xi_A = \lambda_0 - \pi^* A \tag{2.18}$$

be a local one-form on T^*U with λ_0 the Liouville form. We have

$$s_A^*(\xi_A) = 0 \tag{2.19}$$

it being $s_A^*(\lambda_0) = A = (\pi \circ s_A)^*(A)$. This means that ξ_A vanishes exactly on the submanifold $\mathrm{im}(s_A) \subset T^*U$. Therefore the latter is identified by the constraint (2.19), which in turn amounts to fix the fibre coordinate at x, p to its value $A(x)$ identified by the section s_A. Then, the infinitesimal gauge transformation of the gauge potential A, with gauge parameter f, may be defined in terms of the canonical Poisson bracket ω_0^{-1} as follows

$$\delta_f A_\nu(x) = s_A^*\{\pi^* f, \xi_{A_\nu}\}_{\omega_0^{-1}} = \frac{\partial f}{\partial x^\mu}\frac{\partial \xi_{A_\nu}}{\partial p_\mu} = \partial_\nu f. \tag{2.20}$$

Now let us consider the case $\Theta \neq 0$, namely, $(\mathcal{M}, \Theta)$ is a Poisson manifold. A symplectic embedding is performed as described above, with symplectic form now given by the inverse of (2.16), while the image of $U \subset \mathcal{M}$ through the local section s_A is still defined by the constraint (2.18). Then, the infinitesimal gauge transformation of the gauge potential is formally the same as in the previous case, (2.20), except for the fact that ω_0^{-1} is to be replaced by the Poisson tensor ω^{-1} defined by (2.16). Therefore we have

$$\delta_f A_\mu := s_A^*\{\pi^* f, \xi_{A_\nu}\}_{\omega^{-1}} = \Theta^{\rho\sigma}\frac{\partial f}{\partial x^\rho}\frac{\partial \xi_{A_\mu}}{\partial x^\sigma} - \gamma^\rho_\sigma(x, A)\frac{\partial f}{\partial x^\rho}\frac{\partial \xi_{A_\mu}}{\partial p_\sigma} = \{A_\mu, f\}_\Theta + \gamma^\rho_\mu(x, A)\frac{\partial f}{\partial x^\rho}, \tag{2.21}$$

which is precisely the starting assumption (2.1). The master equation for γ, eq. (2.17), yields therefore eq. (2.3) once the constraint (2.18) has been imposed. Notice that, in order to compare with the results of [6] we have to pose $t = 1$ and substitute γ by $\gamma - \mathbb{1}$.

Let us remark that the solution of the master equation eq. (2.3) is not unique, i.e. one may construct other deformed gauge transformations, which close the algebra (1.12). In the approach just described, this is due to the freedom in choosing different symplectic embeddings for the Poisson manifold $(\mathcal{M}, \Theta)$ [6, 7], and gives rise to a field redefinition, which maps gauge orbits of the original fields onto gauge orbits of the new fields [39].

In the next section we will see that the present construction is closely related to the L_∞-bootstrap. In that setting, the ambiguity mentioned above corresponds to a quasi-isomorphism of the underlying L_∞ algebra, which is *unique* (up to quasi-isomorphisms) [39].

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3 Relation to L_∞ algebras and bootstrap

L_∞ algebras are homotopy generalisations of Lie algebras defined on a graded vector space $V = \oplus_k V_k, k \in \mathbb{Z}$, with multi-linear n-brackets[7]

$$\ell_n : (v_1, \cdots, v_n) \in V^{\otimes n} \to v \in V. \tag{3.1}$$

$k \in \mathbb{Z}$ denotes the grading of the subspace V_k, so that $\deg(v) = k \iff v \in V_k$. By definition

$$\deg(\ell_n(v_1, \cdots, v_n)) = n - 2 + \sum_{i=1}^{n} \deg(v_i). \tag{3.2}$$

The brackets are graded anti-symmetric

$$\ell_n(v_1, \cdots, v_j, v_{j+1}, \cdots) = (-1)^{1+\deg(v_j)\deg(v_{j+1})} \ell_n(v_1, \cdots, v_{j+1}, v_j, \cdots), \tag{3.3}$$

and satisfy the generalised Jacobi identities,

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma, v)\, l_j(l_i(v_{\sigma(1)}, \ldots, v_{\sigma(i)}), v_{\sigma(i+1)}, \ldots, v_{\sigma(n)}) = 0, \quad n \in \mathbb{N} \tag{3.4}$$

where $\sum_\sigma$ denotes the sum over permutations, σ, of the variables $v_1, \ldots, v_n$ such that,

$$\sigma(1) < \cdots < \sigma(i), \qquad \sigma(i+1) < \cdots < \sigma(n), \tag{3.5}$$

$(-1)^\sigma$ takes care of the signature of the permutation and $\epsilon(\sigma, v)$ stands for the Koszul sign, which takes into account the degree of the permuted entries (see [9] for details).

L_∞ algebras and gauge transformations are related in the following way [8, 9]. Consider a graded space V such that the only nonempty subspaces are V_0 and V_{-1}. By construction the former is identified with a space of the gauge parameters, $f \in V_0$ whilst the latter contains the gauge fields, $A = A_\mu \, \mathrm{d}x^\mu \in V_{-1}$. We shall look for the deformed gauge transformation in the form of a series expansion, as follows:

$$\delta_f A = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-2)}{2}} l_{n+1}(f, A, \cdots, A). \tag{3.6}$$

By setting

$$l_1(f) = \mathrm{d}f = (\partial_\mu f)\, \mathrm{d}x^\mu, \tag{3.7}$$

$$l_2(f, g) = -\{f, g\}, \qquad \forall f, g \in V_0, \tag{3.8}$$

and determining the remaining brackets, l_k, from the requirement of closure of the L_∞-algebra, one can build the gauge transformation (3.6). Such a "completion" is referred to as the L_∞ bootstrap [4, 5]. General properties of the L_∞-construction automatically insure that the condition (1.12) is satisfied, see e.g. [5, 18].

In the previous section we have constructed the deformed gauge transformations without any reference to the L_∞ algebras, however, Proposition 5.9 of [6] guarantees, that for

[7] We are considering the so called l-picture [9].

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any symplectic embedding, related[8] to the deformed gauge transformation (2.1), the L_∞ algebra is indeed there, and can be constructed as follows.

- Expanding the right hand side of the transformation (2.1), presented as

$$\delta_f A = \left(\gamma^\nu_\mu(A)\partial_\nu f + \{A_\mu, f\}\right) \mathrm{d}x^\mu, \tag{3.9}$$

  in powers of A, and comparing with the right-hand side of (3.6) one finds all the brackets of the form $l_n(f, A, \cdots, A)$. All other brackets, which depend on a single argument f and $n-1$ arguments A can be, obviously, recovered from the mentioned ones by the graded antisymmetry (3.3).

- The only nonzero bracket, which involves *two* arguments $f, g \in V_0$, is given by eq. (3.8).

- All other brackets are identically equal to zero.

The proposition, mentioned above, also asserts that L_∞-algebras which correspond to different choices of γ (i.e. different symplectic embeddings), associated with the same Poisson bivector Θ via eq. (2.3), are necessarily connected by L_∞-quasi-isomorphisms. From this point of view the L_∞ structure, which underlies a given deformed gauge transformation of the form (2.1) is "unique".

Applying the prescription presented above to the matrix γ, given by eq. (2.13), we get

$$\begin{aligned}\delta_f A = {} & (\partial_\mu f)\mathrm{d}x^\mu + \{A_\mu, f\}\,\mathrm{d}x^\mu - \frac{1}{2} f^{\nu\lambda}_\mu (\partial_\nu f) A_\lambda \,\mathrm{d}x^\mu \\ & + \sum_{n=1}^{\infty} \frac{(-2)^n B_{2n}}{(2n)!} \frac{\lambda^{2n} \boldsymbol{Z}^{n-1}}{2^n} \left(\boldsymbol{Z}\,\boldsymbol{\delta}^\nu_\mu - \boldsymbol{A}^\nu_{\boldsymbol{\alpha}} \boldsymbol{A}_\mu\right) (\partial_\nu f)\,\mathrm{d}x^\mu,\end{aligned} \tag{3.10}$$

therefore the only non-zero brackets of the underlying L_∞ algebra are given by

$$\begin{aligned} l_1(f) &= (\partial_\mu f)\,\mathrm{d}x^\mu, \\ l_2(f, A) &= \{A_\mu, f\}\mathrm{d}x^\mu - \frac{1}{2} f^{\nu\lambda}_\mu (\partial_\nu f)\,\mathrm{d}x^\mu A_\lambda, \\ l_{2n+1}(f, A, \cdots, A) &= \frac{B_{2n}}{(2n)!} \cdot \lambda^{2n} \boldsymbol{Z}^{n-1} \cdot \left(\boldsymbol{Z}\,\boldsymbol{\delta}^\nu_\mu - \boldsymbol{A}^\nu_{\boldsymbol{\alpha}} \boldsymbol{A}_\mu\right) (\partial_\nu f)\,\mathrm{d}x^\mu, \quad n \in \mathbb{N}, \\ l_2(f, g) &= -\{f, g\}, \qquad \forall f, g \in V_0. \end{aligned} \tag{3.11}$$

We remind that the structure constants are given by eq. (1.19), and the quantity $\boldsymbol{Z}$ is defined by (2.11). This result is a direct generalisation of the L_∞ algebra, presented in the Example 6.4 of [6] for the three-dimensional $\mathfrak{su}(2)$-case.

[8] Here "related" means that both the deformed gauge transformation and the symplectic embedding are defined via the same matrix γ, which is a solution of the master equation (2.3).

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4 Deformed field strength

According to [17, 18] the deformed field strength, which transforms in a covariant way under the noncommutative transformations (2.1),

$$\delta_f \mathcal{F}_{\mu\nu} = \{\mathcal{F}_{\mu\nu}, f\}, \tag{4.1}$$

may be searched by adapting the usual definition of the non-Abelian field strength to our Poisson gauge algebra. This yields [17, 18]:

$$\mathcal{F}_{\mu\nu} = R_{\mu\nu}{}^{\rho\lambda}\left(2\gamma^{\xi}_{\rho}\,\partial_\xi A_\lambda + \{A_\rho, A_\lambda\}\right), \tag{4.2}$$

with the unknown $R_{\mu\nu}{}^{\xi\lambda}$ satisfying appropriate conditions. It would certainly be interesting to derive this result from symplectic embeddings, as we did for the gauge potential, however, such a connection is still missing. Therefore the field strength is here obtained, as in [17, 18], in a more direct way.

By imposing that (4.1) be satisfied, one gets an equation for the coefficient function $R_{\mu\nu}{}^{\xi\lambda}$, which we name the second master equation,

$$\gamma^{\xi}_{\lambda}\,\partial^{\lambda}_{A} R_{\mu\nu}{}^{\rho\omega} + \Theta^{\xi\lambda}\,\partial_\lambda R_{\mu\nu}{}^{\rho\omega} + R_{\mu\nu}{}^{\rho\lambda}\,\partial^{\omega}_{A}\gamma^{\xi}_{\lambda} + R_{\mu\nu}{}^{\lambda\omega}\,\partial^{\rho}_{A}\gamma^{\xi}_{\lambda} = 0. \tag{4.3}$$

The latter exhibits the following undeformed limit:

$$\lim_{\lambda\to 0} R_{\mu\nu}{}^{\rho\omega} = \frac{1}{2}\left(\delta^{\rho}_{\mu}\delta^{\omega}_{\nu} - \delta^{\omega}_{\mu}\delta^{\rho}_{\nu}\right). \tag{4.4}$$

This requirement together with the relation (2.4) ensures that the noncommutative field strength reduces to the commutative one in the undeformed theory:

$$\lim_{\lambda\to 0} \mathcal{F}_{\mu\nu} = F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.5}$$

A solution of eq. (4.3), which satisfies the condition (4.4), is known for arbitrary Poisson bivector Θ up to $\mathcal{O}(\Theta^2)$ terms [17], see appendix **A**. Substituting our data (1.13) in these formulae, and using the straightforward identities,

$$\begin{aligned} f^{\rho\lambda}_{\nu} f^{\xi\omega}_{\lambda} &= \lambda^2(\hat{\alpha}^{\rho\omega}\boldsymbol{\delta}^{\xi}_{\nu} - \hat{\alpha}^{\rho\xi}\boldsymbol{\delta}^{\omega}_{\nu}), \\ (f^{\xi\rho}_{\mu} f^{\sigma\omega}_{\nu} - f^{\xi\rho}_{\nu} f^{\sigma\omega}_{\mu}) A_\xi A_\sigma &= \lambda^2(\boldsymbol{\delta}^{\rho}_{\mu}\boldsymbol{\delta}^{\omega}_{\nu} - \boldsymbol{\delta}^{\rho}_{\nu}\boldsymbol{\delta}^{\omega}_{\mu})\,\boldsymbol{Z} \\ &\quad + \lambda^2(\boldsymbol{A}^{\omega}_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu\boldsymbol{\delta}^{\rho}_{\nu} - \boldsymbol{A}^{\omega}_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu\boldsymbol{\delta}^{\rho}_{\mu} - \boldsymbol{A}^{\rho}_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu\boldsymbol{\delta}^{\omega}_{\nu} + \boldsymbol{A}^{\rho}_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu\boldsymbol{\delta}^{\omega}_{\mu}), \end{aligned} \tag{4.6}$$

we get:

$$\begin{aligned} R_{\mu\nu}{}^{\rho\omega}(A) = {} & \frac{1}{2}\left(\boldsymbol{\delta}^{\rho}_{\mu}\boldsymbol{\delta}^{\omega}_{\nu} - \boldsymbol{\delta}^{\omega}_{\mu}\boldsymbol{\delta}^{\rho}_{\nu}\right)\cdot\left(1 - \frac{1}{12}\lambda^2\boldsymbol{Z}\right) \\ & + \frac{1}{2}\left(\delta^0_\mu\delta^{\rho}_0\boldsymbol{\delta}^{\omega}_{\nu} - \delta^0_\nu\delta^{\rho}_0\boldsymbol{\delta}^{\omega}_{\mu} + \delta^0_\nu\delta^{\omega}_0\boldsymbol{\delta}^{\rho}_{\mu} - \delta^0_\mu\delta^{\omega}_0\boldsymbol{\delta}^{\rho}_{\nu}\right)\cdot\left(1 - \frac{1}{6}\lambda^2\boldsymbol{Z}\right) \\ & + \frac{1}{4}\left(f^{\xi\omega}_{\nu}\delta^{\rho}_{\mu} - f^{\xi\omega}_{\mu}\delta^{\rho}_{\nu} + f^{\xi\rho}_{\mu}\delta^{\omega}_{\nu} - f^{\xi\rho}_{\nu}\delta^{\omega}_{\mu}\right)A_\xi \\ & - \frac{\lambda^2}{24}\left(\boldsymbol{\delta}^{\rho}_{\mu}\boldsymbol{A}^{\omega}_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu - \boldsymbol{\delta}^{\rho}_{\nu}\boldsymbol{A}^{\omega}_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu - \boldsymbol{\delta}^{\omega}_{\mu}\boldsymbol{A}^{\rho}_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu + \boldsymbol{\delta}^{\omega}_{\nu}\boldsymbol{A}^{\rho}_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu\right) \\ & + \frac{\lambda^2}{12}\left(\boldsymbol{A}^{\omega}_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu\delta^0_\mu\delta^{\rho}_0 - \boldsymbol{A}^{\omega}_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu\delta^0_\nu\delta^{\rho}_0 - \boldsymbol{A}^{\rho}_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu\delta^0_\mu\delta^{\omega}_0 + \boldsymbol{A}^{\rho}_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu\delta^0_\nu\delta^{\omega}_0\right) + \mathcal{O}(\lambda^3). \end{aligned} \tag{4.7}$$

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This formula suggests to look for the complete solution of (4.3) in the form of the following Ansatz

$$
\begin{aligned}
R_{\mu\nu}{}^{\rho\omega}(A) = {} & \frac{1}{2}\left(\boldsymbol{\delta}^\rho_\mu\boldsymbol{\delta}^\omega_\nu - \boldsymbol{\delta}^\omega_\mu\boldsymbol{\delta}^\rho_\nu\right)\cdot\zeta(\lambda\sqrt{\boldsymbol{Z}}) + \frac{1}{2}\left(\delta^0_\mu\delta^\rho_0\boldsymbol{\delta}^\omega_\nu - \delta^0_\nu\delta^\rho_0\boldsymbol{\delta}^\omega_\mu + \delta^0_\nu\delta^\omega_0\boldsymbol{\delta}^\rho_\mu - \delta^0_\mu\delta^\omega_0\boldsymbol{\delta}^\rho_\nu\right)\cdot\Lambda(\lambda\sqrt{\boldsymbol{Z}}) \\
& + \frac{1}{4}\left(f^{\xi\omega}_\nu\delta^\rho_\mu - f^{\xi\omega}_\mu\delta^\rho_\nu + f^{\xi\rho}_\mu\delta^\omega_\nu - f^{\xi\rho}_\nu\delta^\omega_\mu\right)A_\xi\cdot\tilde{\zeta}(\lambda\sqrt{\boldsymbol{Z}}) \\
& + \frac{\lambda^2}{8}\left(\boldsymbol{\delta}^\rho_\mu\boldsymbol{A}^\omega_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu - \boldsymbol{\delta}^\rho_\nu\boldsymbol{A}^\omega_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu - \boldsymbol{\delta}^\omega_\mu\boldsymbol{A}^\rho_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu + \boldsymbol{\delta}^\omega_\nu\boldsymbol{A}^\rho_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu\right)\cdot\phi(\lambda\sqrt{\boldsymbol{Z}}) \\
& + \frac{\lambda^2}{8}\left(\boldsymbol{A}^\omega_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu\delta^0_\mu\delta^\rho_0 - \boldsymbol{A}^\omega_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu\delta^0_\nu\delta^\rho_0 - \boldsymbol{A}^\rho_{\boldsymbol{\alpha}}\boldsymbol{A}_\nu\delta^0_\mu\delta^\omega_0 + \boldsymbol{A}^\rho_{\boldsymbol{\alpha}}\boldsymbol{A}_\mu\delta^0_\nu\delta^\omega_0\right)\cdot\Phi(\lambda\sqrt{\boldsymbol{Z}}),
\end{aligned}
\tag{4.8}
$$

where the form factors ζ, $\tilde{\zeta}$, ϕ, Λ and Φ are unknown functions, which exhibit the following asymptotic behaviour at small λ:

$$
\begin{aligned}
\zeta(\lambda\sqrt{\boldsymbol{Z}}) &= 1 - \frac{1}{12}\lambda^2\boldsymbol{Z} + \mathcal{O}(\lambda^3), \\
\Lambda(\lambda\sqrt{\boldsymbol{Z}}) &= 1 - \frac{1}{6}\lambda^2\boldsymbol{Z} + \mathcal{O}(\lambda^3), \\
\tilde{\zeta}(\lambda\sqrt{\boldsymbol{Z}}) &= 1 + \mathcal{O}(\lambda^2), \\
\phi(\lambda\sqrt{\boldsymbol{Z}}) &= -\frac{1}{3} + \mathcal{O}(\lambda), \\
\Phi(\lambda\sqrt{\boldsymbol{Z}}) &= \frac{2}{3} + \mathcal{O}(\lambda).
\end{aligned}
\tag{4.9}
$$

Substituting the ansatz (4.8) in the master equation (4.3) at $\mu = 1$, $\nu = 2$, $\rho = 3$, $\omega = 2$, $\xi = 3$ we get,

$$
\frac{A_1\lambda^2}{8}\left(\frac{(A_3)^2}{\boldsymbol{Z}}\,u\left[\phi'(u) - \frac{u}{2}\,\phi(u)\,\hat{\chi}\left(\frac{u^2}{4}\right) - \frac{u}{2}\,\zeta(u)\,\hat{\chi}'\left(\frac{u^2}{4}\right)\right] - \left[\zeta(u)\,\chi\left(\frac{u^2}{4}\right) - \phi(u)\right]\right) = 0,
\tag{4.10}
$$

where $u \equiv \lambda\sqrt{\boldsymbol{Z}}$, and, we remind, the function $\hat{\chi}(v)$ is defined by eq. (2.14). This relation is satisfied for all A_j iff

$$
\begin{aligned}
\phi'(u) - \frac{u}{2}\,\phi(u)\,\hat{\chi}\left(\frac{u^2}{4}\right) - \frac{u}{2}\,\zeta(u)\,\hat{\chi}'\left(\frac{u^2}{4}\right) &= 0, \\
\zeta(u)\,\chi\left(\frac{u^2}{4}\right) - \phi(u) &= 0.
\end{aligned}
\tag{4.11}
$$

The solution of this system of equations, which is compatible[9] with the asymptotics (4.9), is given by

$$
\begin{aligned}
\zeta(u) &= 4\left(\frac{\sin\frac{u}{2}}{u}\right)^2, \\
\phi(u) &= \frac{4\left(-2 + 2\cos u + u\,\sin u\right)}{u^4} = \frac{2}{u}\frac{\mathrm{d}\zeta}{\mathrm{d}u}.
\end{aligned}
\tag{4.12}
$$

[9] Actually, one has to use just the initial condition $\phi(0) = -\frac{1}{3}$, while the asymptotic behaviour of ζ can be checked a posteriori.

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In order to determine the remaining three form factors we substitute the ansatz (4.8) in eq. (4.3) at $\mu = 1$, $\nu = 0$, $\rho = 1$, $\omega = 0$, $\xi = 2$:

$$\begin{aligned}0 = \frac{u\,\alpha}{32\,\boldsymbol{Z}^{\frac{5}{2}}}\Bigg(&-\sqrt{\boldsymbol{Z}}\,u^2\,\alpha\left[2\,u\,\hat{\chi}'\left(\frac{u^2}{4}\right)\Lambda(u) + u\,\Phi(u)\,\hat{\chi}\left(\frac{u^2}{4}\right) - 4\,\Phi'(u)\right](A_1)^2 A_2\\ &+u^2\,\boldsymbol{Z}\left[u^2\,\hat{\chi}'\left(\frac{u^2}{4}\right)\tilde{\zeta}(u) + 4\,\hat{\chi}\left(\frac{u^2}{4}\right)\tilde{\zeta}(u) + 2\,\Phi(u)\right]A_1 A_3\\ &-4\,\boldsymbol{Z}^{\frac{3}{2}}\left[u\,\hat{\chi}\left(\frac{u^2}{4}\right)\Lambda(u) - u\,\tilde{\zeta}(u) - 4\Lambda'(u)\right]A_2\Bigg),\end{aligned} \tag{4.13}$$

what leads us to a system of three coupled equations for three undetermined functions $\tilde{\zeta}(u)$, $\Phi(u)$ and $\Lambda(u)$:

$$\begin{aligned}2\,u\,\hat{\chi}'\left(\frac{u^2}{4}\right)\Lambda(u) + u\,\Phi(u)\,\hat{\chi}\left(\frac{u^2}{4}\right) - 4\,\Phi'(u) &= 0\\ u^2\,\hat{\chi}'\left(\frac{u^2}{4}\right)\tilde{\zeta}(u) + 4\,\hat{\chi}\left(\frac{u^2}{4}\right)\tilde{\zeta}(u) + 2\,\Phi(u) &= 0\\ u\,\hat{\chi}\left(\frac{u^2}{4}\right)\Lambda(u) - u\,\tilde{\zeta}(u) - 4\Lambda'(u) &= 0, \quad u \equiv \lambda\sqrt{\boldsymbol{Z}}.\end{aligned} \tag{4.14}$$

Resolving these equations, and imposing the conditions (4.9), we obtain:

$$\begin{aligned}\tilde{\zeta}(u) &= 4\left(\frac{\sin\frac{u}{2}}{u}\right)^2 = \zeta(u),\\ \Lambda(u) &= \frac{\sin u}{u},\\ \Phi(u) &= \frac{4\,(u - \sin u)}{u^3}.\end{aligned} \tag{4.15}$$

Summarising eq. (4.8), eq. (4.12) and eq. (4.15) we arrive at

$$\begin{aligned}R_{\mu\nu}{}^{\rho\omega}(A) = &\frac{1}{2}\left(\boldsymbol{\delta}_\mu^\rho\boldsymbol{\delta}_\nu^\omega - \boldsymbol{\delta}_\mu^\omega\boldsymbol{\delta}_\nu^\rho\right)\cdot\zeta(\lambda\sqrt{\boldsymbol{Z}}) + \frac{1}{2}\left(\delta_\mu^0\delta_0^\rho\boldsymbol{\delta}_\nu^\omega - \delta_\nu^0\delta_0^\rho\boldsymbol{\delta}_\mu^\omega + \delta_\nu^0\delta_0^\omega\boldsymbol{\delta}_\mu^\rho - \delta_\mu^0\delta_0^\omega\boldsymbol{\delta}_\nu^\rho\right)\cdot\Lambda(\lambda\sqrt{\boldsymbol{Z}})\\ &+\frac{1}{4}\left(f_\nu^{\xi\omega}\delta_\mu^\rho - f_\mu^{\xi\omega}\delta_\nu^\rho + f_\mu^{\xi\rho}\delta_\nu^\omega - f_\nu^{\xi\rho}\delta_\mu^\omega\right)A_\xi\cdot\zeta(\lambda\sqrt{\boldsymbol{Z}})\\ &+\frac{\lambda^2}{8}\left(\boldsymbol{\delta}_\mu^\rho\boldsymbol{A}_{\boldsymbol{\alpha}}^\omega\boldsymbol{A}_\nu - \boldsymbol{\delta}_\nu^\rho\boldsymbol{A}_{\boldsymbol{\alpha}}^\omega\boldsymbol{A}_\mu - \boldsymbol{\delta}_\mu^\omega\boldsymbol{A}_{\boldsymbol{\alpha}}^\rho\boldsymbol{A}_\nu + \boldsymbol{\delta}_\nu^\omega\boldsymbol{A}_{\boldsymbol{\alpha}}^\rho\boldsymbol{A}_\mu\right)\cdot\phi(\lambda\sqrt{\boldsymbol{Z}})\\ &+\frac{\lambda^2}{8}\left(\boldsymbol{A}_{\boldsymbol{\alpha}}^\omega\boldsymbol{A}_\nu\delta_\mu^0\delta_0^\rho - \boldsymbol{A}_{\boldsymbol{\alpha}}^\omega\boldsymbol{A}_\mu\delta_\nu^0\delta_0^\rho - \boldsymbol{A}_{\boldsymbol{\alpha}}^\rho\boldsymbol{A}_\nu\delta_\mu^0\delta_0^\omega + \boldsymbol{A}_{\boldsymbol{\alpha}}^\rho\boldsymbol{A}_\mu\delta_\nu^0\delta_0^\omega\right)\cdot\Phi(\lambda\sqrt{\boldsymbol{Z}}),\end{aligned} \tag{4.16}$$

with

$$\begin{aligned}\zeta(u) &= 4\left(\frac{\sin\frac{u}{2}}{u}\right)^2,\\ \Lambda(u) &= \frac{\sin u}{u},\\ \phi(u) &= \frac{2}{u}\frac{\mathrm{d}\lambda}{\mathrm{d}u},\\ \Phi(u) &= \frac{4\,(u - \sin u)}{u^3}.\end{aligned} \tag{4.17}$$

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One can check by direct substitution that our solution is valid for all other combinations of the indexes μ, ν, ρ, ω and ξ.

At $\alpha = 1$ the three-dimensional restriction, $R_{ab}{}^{cd}$, of (4.16) coincides[10] with the known three-dimensional solution for the $\mathfrak{su}(2)$ case [17]. It is remarkable that the presence of the fourth (commutative) coordinate x^0 generalises the mentioned three-dimensional result in a quite nontrivial way, introducing new contributions of the form factors Λ and Φ.

The deformed field strength $\mathcal{F}$, defined by eq. (4.2), allows for a natural definiton of the classical action functional, which remains invariant upon the deformed noncommutative gauge transformations (2.1), and which reproduces correctly the classical limit. Indeed, by defining

$$S[A] := \int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathcal{L}, \qquad \mathcal{L} := -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\xi}\,\eta^{\mu\rho}\eta^{\nu\xi}, \tag{4.18}$$

with

$$\eta = \mathrm{diag}\,(+1,-1,-1,-1), \tag{4.19}$$

we can check that the classical limit is

$$\lim_{\lambda\to 0} S[A] = \int_{\mathbb{R}^4} \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F_{\rho\xi}\,\eta^{\mu\rho}\eta^{\nu\xi}\right), \tag{4.20}$$

thanks to the property (4.5) of the deformed field strength. Moreover, since the deformed field strength $\mathcal{F}$ transforms in a covariant way (eq. (4.1)), the deformed Lagrangian density, being quadratic in $\mathcal{F}$, transforms in a covariant way as well. We have indeed

$$\begin{aligned}
\delta_f\mathcal{L} &\overset{\text{def}}{=} -\frac{1}{4}\lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}\left(\mathcal{F}_{\mu\nu}+\varepsilon\,(\delta_f\mathcal{F}_{\mu\nu})\right)\left(\mathcal{F}_{\rho\xi}+\varepsilon\,(\delta_f\mathcal{F}_{\rho\xi})\right)\eta^{\mu\rho}\eta^{\nu\xi}\\
&= -\frac{1}{4}\Big(\mathcal{F}_{\mu\nu}\,(\delta_f\mathcal{F}_{\rho\xi}) + (\delta_f\mathcal{F}_{\mu\nu})\mathcal{F}_{\rho\xi}\Big)\,\eta^{\mu\rho}\eta^{\nu\xi}\\
&= -\frac{1}{4}\Big(\mathcal{F}_{\mu\nu}\,\{\mathcal{F}_{\rho\xi},f\} + \{\mathcal{F}_{\mu\nu},f\}\mathcal{F}_{\rho\xi}\Big)\,\eta^{\mu\rho}\eta^{\nu\xi} = -\frac{1}{4}\,\{\mathcal{F}_{\mu\nu}\,\mathcal{F}_{\rho\xi},f\}\,\eta^{\mu\rho}\eta^{\nu\xi}\\
&= \{\mathcal{L},f\},
\end{aligned} \tag{4.21}$$

where we have first used the standard definition of first variation of $\mathcal{L}$ upon the variation of $\mathcal{F}$; then we have substituted the explicit expression (4.1) for $\delta_f\mathcal{F}$, and took into account the derivation property of the Poisson bracket. By using eq. (1.11) we thus get

$$\delta_f\mathcal{L} = \partial_\mu(\mathcal{L}\;\partial_\nu f\;\Theta^{\mu\nu}) - \mathcal{L}\;\partial_\nu f\;\partial_\mu\Theta^{\mu\nu}. \tag{4.22}$$

Remark. In order to avoid confusions, we comment on the usage of partial derivatives in this paper. On the one hand, in the master equations (2.3) and (4.3), in the terms $\Theta^{\nu\mu}\partial_\mu\gamma^\xi_\lambda - \Theta^{\xi\mu}\partial_\mu\gamma^\nu_\lambda$ and $\Theta^{\xi\lambda}\,\partial_\lambda R_{\mu\nu}{}^{\rho\omega}$, the partial derivatives act on the *explicit* dependence on x only, whilst $A(x)$ is considered as an independent variable. On the other hand, in all other places of this article, (e.g. in the definition of the Poisson bracket (1.11)), the partial derivatives act on *all* x-dependent objects. In particular, in the last line of eq. (4.22) the partial derivative acts on x, which is present in $\mathcal{L}$ not just explicitly,[11] but also via $A(x)$ and its first derivatives as well. This justifies the Leibnitz rule in the last step of (4.22).

[10] We use slightly different parametrisation of the form factors.

[11] I.e. via the Poisson bivector $\Theta(x)$, which enters in the definition (4.2) of $\mathcal{F}$ through the Poisson bracket.

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Finally, noticing that the Poisson bivector (1.13) satisfies the identity

$$\partial_\mu \Theta^{\mu\nu} = 0, \tag{4.23}$$

we see that the variation $\delta_f \mathcal{L}$ is a total derivative, therefore the action (4.18) is gauge invariant:

$$\delta_f S[A] = 0. \tag{4.24}$$

5 Summary and outlook

In this article we constructed a family of four-dimensional noncommutative deformations of the U(1) gauge theory, implementing a class of noncommutative spaces (1.13) in the general framework of [17]. This class includes the angular (or λ-Minkowski), the $\mathfrak{su}(2)$ and the $\mathfrak{su}(1,1)$ cases at $\alpha = 0$, $\alpha = +1$ and $\alpha = -1$ respectively. We worked within the semi-classical approximation, so our noncommutative gauge theories are actually Poisson gauge theories.

The first result is the definition (2.1) of deformed gauge transformations, where the matrix γ is given by eq. (2.13). These transformations close the noncommutative algebra (1.12). We also discussed the interpretation of the master equation eq. (2.3), which we used to construct the deformed gauge transformations, as a Jacobi identity for symplectic embeddings [6, 7].

The second result is an explicit L_∞ structure (3.11), which corresponds to our deformed noncommutative gauge transformations in sense of the L_∞ bootstrap.

The third result is an expression for the deformed field strength, eq. (4.2), where the quantity R is given by eq. (4.16). This deformed field strength transforms in a covariant way upon the deformed noncommutative gauge transformations, thereby allowing for a definition of the gauge-invariant classical action (4.18). We stress that the presence of the fourth (commutative) coordinate x^0 brings nontrivial contributions to the deformed strength (via R), which do not look like a simple and intuitive addition to the corresponding three-dimensional result. In particular, the components $\mathcal{F}_{0j}$ exhibit a highly nonlinear dependence on the three-dimensional components of A. This behaviour is different from the one of the matrix γ, where the four-dimensionality does not change the corresponding three-dimensional result that much, since $\gamma^{0\rho} = \delta^{0\rho}$. Let us illustrate the nontriviality on a simple example where the gauge potential does not depend on spatial coordinates x^j. In this situation one may expect that the noncommutativity, being essentially three-dimensional, does not affect the field strength, so $\mathcal{F}_{0j} = \partial_0 A_j$. Our analysis, instead, yields

$$\begin{aligned} \mathcal{F}_{0j} &= 2\, R_{0j}{}^{0k}\, \partial_0 A_k \\ &= \Lambda(\lambda\sqrt{\boldsymbol{Z}})\, \partial_0 A_j + \frac{1}{2}\, \zeta(\lambda\sqrt{\boldsymbol{Z}})\, f^{rk}_j A_r\, \partial_0 A_k + \frac{\lambda^2}{8}\, \Phi(\lambda\sqrt{\boldsymbol{Z}})\, \boldsymbol{A}^k_{\boldsymbol{\alpha}} A_j\, \partial_0 A_k, \end{aligned} \tag{5.1}$$

which disproves the naive expectation. *The nonlinearity which derives from spatial noncommutativity manifests itself in the spatially-homogeneous situation as well, as far as*

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the time-dependence is concerned. One can also check that, substituting an x^j-independent gauge potential A in the equations of motion derived from the classical action (4.18), one gets a nonlinear dynamics, which governs the time-dependence. At the best of our knowledge nothing similar takes place in noncommutative gauge theories which are based on more conventional approaches.

The present research can be continued in various directions. On one hand, one may study various physical consequences of noncommutativity such as the existence of Gribov copies, which has already been established for the noncommutative QED with Moyal type noncommutativity [40–42]. On the other hand, one may focus on purely mathematical structures, which stand behind. In particular, one may wonder whether the field strength $\mathcal{F}$, obtained in this paper, is compatible with the L_∞ bootstrap procedure, related to the *extended* L_∞ algebra [4], that contains one more nonempty subspace V_{-2} of the objects which transform in a covariant way upon the deformed gauge transformations. In particular, one has to check, whether

$$\mathcal{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} l_n(A, \cdots, A), \tag{5.2}$$

where the first bracket corresponds to the undeformed field strength (4.5),

$$l_1(A) = F_{\mu\nu}\, dx^\mu \wedge dx^\nu, \tag{5.3}$$

and all the brackets together fulfil the L_∞ relations.

Finally, an interesting problem which we would like to investigate in the coming future is to understand if it is possible to derive the deformed field strength proposed in this paper within the symplectic embedding approach, and to clarify its geometric nature.

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A General solution for $R_{\mu\nu}{}^{\rho\omega}(x, A)$ up to $\mathcal{O}\left(\Theta^3\right)$ corrections

$$R_{\mu\nu}{}^{\rho\omega}(x, A) = R^{(0)}_{\mu\nu}{}^{\rho\omega}(x, A) + R^{(1)}_{\mu\nu}{}^{\rho\omega}(x, A) + R^{(2)}_{\mu\nu}{}^{\rho\omega}(x, A) + \mathcal{O}(\Theta^3), \tag{A.1}$$

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where

$$
\begin{aligned}
R^{(0)\,\rho\omega}_{\mu\nu}(x,A) &= \frac{1}{2}\left(\delta^\rho_\mu\delta^\omega_\nu - \delta^\omega_\mu\delta^\rho_\nu\right),\\
R^{(1)\,\rho\omega}_{\mu\nu}(x,A) &= \frac{1}{4}\left(\delta^\rho_\mu\,\partial_\nu\Theta^{\xi\omega} - \delta^\omega_\mu\,\partial_\nu\Theta^{\xi\rho} - \delta^\rho_\nu\,\partial_\mu\Theta^{\xi\omega} + \delta^\omega_\nu\,\partial_\mu\Theta^{\xi\rho}\right)A_\xi,\\
R^{(2)\,\rho\omega}_{\mu\nu}(x,A) &= \Big(\frac{1}{12}\delta^\rho_\mu\,\Theta^{\sigma\phi}\,\partial_\nu\partial_\phi\Theta^{\xi\omega} - \frac{1}{12}\delta^\omega_\mu\,\Theta^{\sigma\phi}\,\partial_\nu\partial_\phi\Theta^{\xi\rho}\\
&\quad - \frac{1}{12}\delta^\rho_\nu\,\Theta^{\sigma\phi}\,\partial_\mu\partial_\phi\Theta^{\xi\omega} + \frac{1}{12}\delta^\omega_\nu\,\Theta^{\sigma\phi}\,\partial_\mu\partial_\phi\Theta^{\xi\rho}\\
&\quad + \frac{1}{12}\delta^\rho_\mu\,\partial_\nu\Theta^{\sigma\phi}\,\partial_\phi\Theta^{\xi\omega} - \frac{1}{12}\delta^\omega_\mu\,\partial_\nu\Theta^{\sigma\phi}\,\partial_\phi\Theta^{\xi\rho}\\
&\quad - \frac{1}{12}\delta^\rho_\nu\,\partial_\mu\Theta^{\sigma\phi}\,\partial_\phi\Theta^{\xi\omega} + \frac{1}{12}\delta^\omega_\nu\,\partial_\mu\Theta^{\sigma\phi}\,\partial_\phi\Theta^{\xi\rho}\\
&\quad + \frac{1}{8}\partial_\mu\Theta^{\xi\rho}\,\partial_\nu\Theta^{\sigma\omega} - \frac{1}{8}\partial_\mu\Theta^{\sigma\omega}\,\partial_\nu\Theta^{\xi\rho}\Big)\,A_\xi A_\sigma. \qquad \text{(A.2)}
\end{aligned}
$$

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