Superstring field theory with open and closed strings

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ABSTRACT: We construct Lorentz invariant and gauge invariant 1PI effective action for closed and open superstrings and demonstrate that it satisfies the classical BV master equation. We also construct the quantum master action for this theory satisfying the quantum BV master equation and generalize the construction to unoriented theories. The extra free field needed for the construction of closed superstring field theory plays a crucial role in coupling the closed strings to D-branes and orientifold planes.

Contents

1 Introduction and summary

We now have a consistent formulation of closed superstring field theory (see [1] for a review). Our goal in this paper is to extend this construction to interacting theory of open and closed strings. Such systems arise naturally in the presence of D-branes.

As has been described in [1], full quantum string field theory can be studied at different levels. The basic formulation involves the quantum master action satisfying quantum BV master equation [2–4]. However, without giving up any information, one can have various equivalent formulations that are useful for specific studies. One of them is the 1PI effective action, suitable for studying the problem of mass renormalization, vacuum shift and computing S-matrix in the shifted vacuum. Another is the Wilsonian effective action, suitable for studying the dynamics of string states below certain mass scale, but without making any low energy approximation. On the other hand the quantum master action is useful for studying problems that require making all loop momenta integration manifest, e.g. study of unitarity [6] and analyticity [7] properties of the amplitudes. Of these the 1PI effective action has the simplest gauge transformation properties, in that it should be invariant under suitable gauge transformation. As a consequence of this, it satisfies the classical BV master equation [2]. In contrast the quantum BV master action and the Wilsonian effective action are not invariant under any gauge transformation, since the gauge non-invariance of the action needs to compensate for the gauge non-invariance of the path integral measure [2–4]. What remains invariant is the combination $d\mu \, e^{2S}$ where $d\mu$ is the integration measure in the space of fields and anti-fields and S is the action [5]. This is ensured by the fact that the action satisfies quantum BV master equation.

For this reason we begin our study by first constructing the 1PI effective action. To experts in quantum field theory, it may appear strange that we construct the 1PI effective action before writing down the actual action whose off-shell 1PI amplitudes would give the interaction terms of the 1PI effective action. However as explained in [1], since in string theory we already know the formal expressions for on-shell amplitudes (ignoring effects of vacuum shift and mass renormalization), we can first generalize this to construct off-shell amplitudes, given as integrals over moduli spaces of punctured Riemann surfaces. By appropriately restricting the region of integration over the moduli spaces, we can construct either 1PI amplitudes or amplitudes corresponding to elementary vertices of the quantum master action. The former requires us to remove certain regions of integration around separating type degenerations in the moduli space, whereas the latter requires us to remove certain regions of integration around all degenerations.

After constructing the 1PI effective action and checking its desired properties, namely gauge invariance and validity of classical BV master equation, we also describe the construction of quantum master action satisfying the quantum BV master equation. This construction only requires a few changes from its 1PI counterpart. We also generalize our results to unoriented string theory, required to describe the interacting theory of closed and open strings in the presence of orientifold planes. Throughout our work, we shall follow the general strategy that has been used in the formulation of bosonic string field theory [8–10], but extending this to superstring field theory requires a few additional ingredients that we shall explain.

We now give a summary of our result for the 1PI effective action of superstring field theory of open and closed strings. We define $\mathcal{H}_{m,n}$ to be the vector space of GSO even closed string states $|s\rangle$, carrying (left, right) picture numbers (m, n) and satisfying the constraints [8]:[1]

$$b_0^- |s\rangle = 0, \quad L_0^- |s\rangle = 0, \quad L_0^\pm \equiv L_0 \pm \bar{L}_0, \quad b_0^\pm \equiv b_0 \pm \bar{b}_0, \quad c_0^\pm \equiv \frac{1}{2}(c_0 \pm \bar{c}_0). \tag{1.1}$$

We also denote by $\mathcal{H}_m$ the vector space of GSO even open string states of picture number m. We now introduce two subspaces in the closed string Hilbert space and two subspaces in open string Hilbert space as follows:

$$\begin{aligned}
\mathcal{H}^c &\equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1/2,-1} \oplus \mathcal{H}_{-1,-1/2} \oplus \mathcal{H}_{-1/2,-1/2}, \\
\widetilde{\mathcal{H}}^c &\equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-3/2,-1} \oplus \mathcal{H}_{-1,-3/2} \oplus \mathcal{H}_{-3/2,-3/2}, \\
\mathcal{H}^o &\equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-1/2}, \\
\widetilde{\mathcal{H}}^o &\equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-3/2}.
\end{aligned} \tag{1.2}$$

For $A_i^c \in \mathcal{H}^c$ and $A_i^o \in \mathcal{H}^o$, we define $\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\}$ to be the off-shell 1PI amplitude, summed over all genera and all number of boundaries, with external closed string

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[1]These restrictions on the off-shell closed string states are needed to get well-defined off-shell amplitudes, which require us to choose local coordinates at the punctures that are used to insert the vertex operators of the off-shell string states into the correlation function. Such choice of local coordinates at the punctures are possible globally only if we ignore the phase of the local coordinate. The requirement that the vertex operators must be invariant under such phase rotations leads to (1.1).

states $A_1^c, \cdots, A_N^c$ and external open string states $A_1^o, \cdots A_M^o$, *excluding the one point function of closed string states on the disc from the definition of* $\{A^c;\}$. In computing this amplitude, we also need to insert in the correlation function appropriate combination of picture changing operators (PCO) and ghosts. For $\widetilde{A}^c \in \widetilde{\mathcal{H}}^c$, we also define $\{\widetilde{A}^c\}_D$ to be the disc one point function of $\widetilde{A}^c$ with appropriate insertion of PCOs and c-ghosts. More detailed definitions of $\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\}$ and $\{\widetilde{A}^c\}_D$ have been given in section 2.

Note the exclusion of the one point function on the disc from the definition of $\{A^c;\}$. If A^c is in the RR sector, then the contribution to $\{A^c;\}$ from the disc does not exist, since the RR vertex operators carry total picture number -1, and therefore it is impossible to satisfy picture number conservation on the disc, requiring total picture number -2, by inserting PCOs which carry positive picture number. For A^c belonging to the NSNS sector the disc one point function is non-vanishing, but we shall still exclude its contribution from the definition of $\{A^c;\}$ for uniformity and define $\{\widetilde{A}^c\}_D$ to contain contributions from one point function on the disc both for NSNS and RR sector $\widetilde{A}^c$. As we shall comment at the end of this section, whether to include the contribution to the disc one point function of NSNS sector closed string states into the definition of $\{A^c;\}$ or $\{\widetilde{A}^c\}_D$ is a matter of convention, but for RR states there is no such option. For one point function of closed string states in $\mathcal{H}^c$ on surfaces with more boundaries/genera, there is no problem with picture number conservation, and we include their contribution in the definition of $\{A^c;\}$.

We shall take the closed string states A_i^c's and $\widetilde{A}^c$ to be even elements of the grassmann algebra and the open string states A_i^o's and $\widetilde{A}_i^o$'s to be odd elements of the grassmann algebra. This means that if ζ is an odd c-number element of the grassmann algebra, then we have

$$\zeta A_i^c = A_i^c \zeta, \quad \zeta \widetilde{A}^c = \widetilde{A}^c \zeta, \quad \zeta A_i^o = -A_i^o \zeta, \quad \zeta \widetilde{A}_i^o = -\widetilde{A}_i^o \zeta, \tag{1.3}$$

where in (1.3), A_i^c, $\widetilde{A}^c$, A_i^o and $\widetilde{A}^o$ refer to the vertex operators corresponding to the states.[2] Unless mentioned otherwise, all our subsequent equations will assume this assignment of grassmann parities. In this case, $\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\}$ and $\{\widetilde{A}^c\}_D$ can be shown to be even elements of the grassmann algebra. Furthermore it follows from the definition given in section 2 that $\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\}$ is invariant under arbitrary exchanges $A_i^c \leftrightarrow A_j^c$ and $A_k^o \leftrightarrow A_\ell^o$. Note that there is no additional sign even for the grassmann odd vertex operators A_k^o. More discussion on this can be found in comment 1 at the end of this section.

We also define

$$[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o]^c \in \widetilde{\mathcal{H}}^c, \qquad [A_1^c \cdots A_N^c; A_1^o \cdots A_M^o]^o \in \widetilde{\mathcal{H}}^o, \qquad [\ ]_D \in \mathcal{H}^c, \tag{1.4}$$

via

$$\langle A_0^c | c_0^- | [A_1^c \cdots A_N^c; A_1^o \cdots A_M^o]^c \rangle = \{A_0^c A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\}, \quad \forall |A_0^c\rangle \in \mathcal{H}^c,$$
$$\langle A_0^o | [A_1^c \cdots A_N^c; A_1^o \cdots A_M^o]^o \rangle = \{A_1^c \cdots A_N^c; A_0^o A_1^o \cdots A_M^o\}, \quad \forall |A_0^o\rangle \in \mathcal{H}^o,$$
$$\langle \widetilde{A}^c | c_0^- | [\ ]_D \rangle = \{\widetilde{A}^c\}_D. \tag{1.5}$$

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[2]Ref. [10] assigned even grassmann parity to the open string field by taking the open string vacuum $|0\rangle$ to have odd grassmann parity and identified the grassmann parity of the string field as that of the corresponding ket state. In contrast we take the grassmann parity of the string field to that of the corresponding vertex operator.

Here $\langle A|$ denotes the BPZ conjugate of $|A\rangle$, generated by $z \to 1/z$ for closed strings and $z \to -1/z$ for open strings. We shall see in (2.3) that $[\ ]_D$ is related to the boundary state of the D-brane system via appropriate picture changing operation and rescaling. It follows from (1.5), and the fact that $\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\}$ is grassmann even, that $[A_1^c \cdots A_M^o]^c$ is grassmann odd and $[A_1^c \cdots A_M^o]^o$ is grassmann even. Similarly one can show that $|[\ ]_D\rangle$ is grassmann odd. Other useful identities involving $\{\cdots\}$, $[\cdots]$, $\{\widetilde{A}^c\}_D$ and $[\ ]_D$ are:

$$\{A_1^c \cdots A_k^c \, \mathcal{G}[B_1^c \cdots B_\ell^c; C_1^o \cdots C_m^o]^c; D_1^o \cdots D_n^o\}$$
$$= \{B_1^c \cdots B_\ell^c \, \mathcal{G}[A_1^c \cdots A_k^c; D_1^o \cdots D_n^o]^c; C_1^o \cdots, C_m^o\}, \tag{1.6}$$

$$\{A_1^c \cdots A_k^c; \mathcal{G}[B_1^c \cdots B_\ell^c; C_1^o \cdots C_m^o]^o D_1^o \cdots D_n^o\}$$
$$= \{B_1^c \cdots B_\ell^c; \mathcal{G}[A_1^c \cdots, A_k^c; D_1^o \cdots D_n^o]^o C_1^o \cdots C_m^o\}. \tag{1.7}$$

and

$$\{A_1^c \cdots A_N^c[\ ]_D; A_1^o \cdots A_M^o\} = \{[A_1^c \cdots A_N^c; A_1^o \cdots A_M^o]^c\}_D. \tag{1.8}$$

In (1.6), (1.7), $\mathcal{G}$ is given by:

$$\mathcal{G}|s^o\rangle = \begin{cases} |s^o\rangle & \text{if } |s^o\rangle \in \mathcal{H}_{-1} \\ \frac{1}{2}(\mathcal{X}_0 + \bar{\mathcal{X}}_0)\,|s^o\rangle & \text{if } |s^o\rangle \in \mathcal{H}_{-3/2} \end{cases},$$

$$\mathcal{G}|s^c\rangle = \begin{cases} |s^c\rangle & \text{if } |s^c\rangle \in \mathcal{H}_{-1,-1} \\ \mathcal{X}_0\,|s^c\rangle & \text{if } |s^c\rangle \in \mathcal{H}_{-1,-3/2} \\ \bar{\mathcal{X}}_0\,|s^c\rangle & \text{if } |s^c\rangle \in \mathcal{H}_{-3/2,-1} \\ \mathcal{X}_0\bar{\mathcal{X}}_0\,|s^c\rangle & \text{if } |s^c\rangle \in \mathcal{H}_{-3/2,-3/2} \end{cases}, \tag{1.9}$$

with

$$\mathcal{X}_0 \equiv \oint \frac{dz}{z}\,\mathcal{X}(z), \quad \bar{\mathcal{X}}_0 \equiv \oint \frac{d\bar{z}}{\bar{z}}\,\bar{\mathcal{X}}(\bar{z}), \tag{1.10}$$

$\mathcal{X}$ and $\bar{\mathcal{X}}$ being holomorphic and anti-holomorphic PCOs. $\oint$ includes multiplicative factors of $\pm(2\pi i)^{-1}$.

In the following we shall also need to deal with states of wrong grassmann parity — closed string states which are grassmann odd and open string states which are grassmann even. To derive the relevant relations, we multiply each grassmann odd closed string state and grassmann even open string state by grassmann odd c-numbers so that they acquire standard grassmann parities and therefore obey the standard symmetry properties and other identities described above. We can now bring the grassmann odd c-numbers to the extreme left in both sides of the equations keeping track of the signs picked up during this process. In doing this we follow the convention that a grassmann odd c-number can be passed through $\{$ and $\langle$ without any extra sign, — the physical origin of these rules will be described above (2.6). Once this is done, we can remove these c-numbers from both sides of the equations and derive the relevant identities. To avoid confusion, we shall use the convention that string states labelled by roman letters, like A_i^c, $\widetilde{A}^c$, A_i^o and $\widetilde{A}^o$, carry the correct grassmann parity — even for A_i^c, $\widetilde{A}^c$ and odd for A_i^o, $\widetilde{A}^o$. When we need to use states of general grassmann parity, we shall use caligraphic letters $\mathcal{A}_i^c$, $\widetilde{\mathcal{A}}^c$, $\mathcal{A}_i^o$ and $\widetilde{\mathcal{A}}^o$. In

this convention, in an expression like $Q_B A_i^c$, it will be understood that A_i^c is grassmann even and therefore $Q_B A_i^c$ is grassmann odd.

It can be shown that with this prescription, the generalizations of (1.5) remain relatively simple even for states of wrong grassmann parity, provided we use the prescription that [for $[\cdots]^c$ behaves as a grassmann odd object and [for $[\cdots]^o$ behaves as a grassmann even object. In that case, when we replace A_i^c, $\widetilde{A}^c$, A_i^o and $\widetilde{A}^o$ by $\mathcal{A}_i^c$, $\widetilde{\mathcal{A}}^c$, $\mathcal{A}_i^o$ and $\widetilde{\mathcal{A}}^o$, the first and the third equations of (1.5) remain the same, and the second equation also remains the same if either $\mathcal{A}_0^o$ or the product $\mathcal{A}_1^c \cdots \mathcal{A}_N^c$ is grassmann even. In case both of these are grassmann odd, we have an extra minus sign on the right hand side of the second equation. To see the necessity of assigning odd grassmann parity for [in $[\cdots]^c$, let us use the generalization of the first equation of (1.5) to get

$$\{\mathcal{A}_0^c \zeta \cdots\} = \langle \mathcal{A}_0^c | c_0^- | [\zeta \cdots]^c \rangle, \tag{1.11}$$

for a grassmann odd c-number ζ. Now on the left hand side we can bring ζ to the extreme left by picking up a multiplicative factor given by the grassmann parity of $\mathcal{A}_0^c$. On the right hand side we shall get an additional minus sign while moving ζ through c_0^-. Therefore to compensate for this we need the rule:

$$[\zeta \cdots]^c = -\zeta [\cdots]^c. \tag{1.12}$$

There is no such factor for $[\cdots]^o$ due to the absence of c_0 factor in the second line of (1.5).

Another rule we need to follow is to never move a grassmann odd variable through $]^{c,o}$ or $\}$. To see this consider $\{; A_1^o A_2^o A_3^o\}$ and move a grassmann odd c-number ζ through this from left to right. We may conclude that this operation will pick up a minus sign due to three grassmann odd open string vertex operators inserted in between $\{\cdots\}$. But this may not be the correct result, e.g. the correlation function of three grassmann odd operators on a disc is a grassmann even number. Furthermore, this cannot be compensated by simply assigning an odd grassmann parity to $\}$ since the result depends on the number of boundaries. The same rule will be followed for matrix elements like $\langle A^o | \mathcal{O} | B^o \rangle$ for any operator $\mathcal{O}$ acting on the open string states. Any grassmann odd c-number from inside the matrix element will be taken outside the matrix element from the left. However in this case we could allow grassmann odd c-numbers to be taken out from the right by picking an extra minus sign, i.e. by treating the ket vacuum as grassmann odd.

We are now ready to describe the form of the action. We introduce two sets of grassmann even closed string fields $\Psi^c \in \mathcal{H}^c$ and $\widetilde{\Psi}^c \in \widetilde{\mathcal{H}}^c$ and two sets of grassmann odd open string fields $\Psi^o \in \mathcal{H}^o$ and $\widetilde{\Psi}^o \in \widetilde{\mathcal{H}}^o$. The 1PI effective action is given by:

$$S_{1PI} = -\frac{1}{2\,g_s^2} \langle \widetilde{\Psi}^c | c_0^- Q_B \mathcal{G} | \widetilde{\Psi}^c \rangle + \frac{1}{g_s^2} \langle \widetilde{\Psi}^c | c_0^- Q_B | \Psi^c \rangle - \frac{1}{2\,g_s} \langle \widetilde{\Psi}^o | Q_B \mathcal{G} | \widetilde{\Psi}^o \rangle + \frac{1}{g_s} \langle \widetilde{\Psi}^o | Q_B | \Psi^o \rangle$$
$$+ \{\widetilde{\Psi}^c\}_D + \sum_{N \geq 0} \sum_{M \geq 0} \frac{1}{N!\,M!} \{(\Psi^c)^N ; (\Psi^o)^M\}, \tag{1.13}$$

where Q_B represents the closed string BRST operator on $\mathcal{H}^c$ and $\widetilde{\mathcal{H}}^c$ and open string BRST operator on $\mathcal{H}^o$ and $\widetilde{\mathcal{H}}^o$. We show in section 3 that the action is invariant under the gauge

transformation:

$$|\delta\Psi^c\rangle = Q_B|\Lambda^c\rangle + g_s^2 \sum_{N,M} \frac{1}{N!\,M!}\mathcal{G}[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^c + g_s^2 \sum_{N,M} \frac{1}{N!\,M!}\mathcal{G}[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^c,$$

$$|\delta\Psi^o\rangle = Q_B|\Lambda^o\rangle - g_s \sum_{N,M} \frac{1}{N!\,M!}\mathcal{G}[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^o - g_s \sum_{N,M} \frac{1}{N!\,M!}\mathcal{G}[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^o,$$

$$|\delta\widetilde{\Psi}^c\rangle = Q_B|\widetilde{\Lambda}^c\rangle + g_s^2 \sum_{N,M} \frac{1}{N!\,M!}[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^c + g_s^2 \sum_{N,M} \frac{1}{N!\,M!}[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^c,$$

$$|\delta\widetilde{\Psi}^o\rangle = Q_B|\widetilde{\Lambda}^o\rangle - g_s \sum_{N,M} \frac{1}{N!\,M!}[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^o - g_s \sum_{N,M} \frac{1}{N!\,M!}[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^o,$$

$$(1.14)$$

where $|\Lambda^c\rangle \in \mathcal{H}^c$, $|\Lambda^o\rangle \in \mathcal{H}^o$, $|\widetilde{\Lambda}^c\rangle \in \widetilde{\mathcal{H}}^c$, $|\widetilde{\Lambda}^o\rangle \in \widetilde{\mathcal{H}}^o$, are gauge transformation parameters. Λ^c and $\widetilde{\Lambda}^c$ are grassmann odd while Λ^o and $\widetilde{\Lambda}^o$ are grassmann even. We also check in section 4 that this form of the gauge transformation is consistent with what we obtain from the BV formalism.

Besides the various identities mentioned earlier, we need another set of identities, known as the 'main identities' [8], to prove gauge invariance of the action. For grassmann even $\widetilde{A}^c$, A_i^c and grassmann odd A_i^o, these identities take the form:

$$\{(Q_B\widetilde{A}^c)\}_D = 0\,, \tag{1.15}$$

and,

$$\sum_{i=1}^{N}\{A_1^c\cdots A_{i-1}^c(Q_BA_i^c)A_{i+1}^c\cdots A_N^c;A_1^o\cdots A_M^o\}$$

$$+\sum_{j=1}^{M}\{A_1^c\cdots A_N^c;A_1^o\cdots A_{j-1}^o(Q_BA_j^o)A_{j+1}^o\cdots A_M^o\}(-1)^{j-1}$$

$$=-\frac{1}{2}\sum_{k=0}^{N}\sum_{\{i_1\cdots,i_k\}\subset\{1,\cdots,N\}}\sum_{\ell=0}^{M}\sum_{\{j_1,\cdots,j_\ell\}\subset\{1,\cdots,M\}}\Big(g_s^2\,\{A_{i_1}^c\cdots A_{i_k}^c\mathcal{B}^c;A_{j_1}^o\cdots A_{j_\ell}^o\}$$

$$+g_s\,\{A_{i_1}^c\cdots A_{i_k}^c;\mathcal{B}^oA_{j_1}^o\cdots A_{j_\ell}^o\}\Big)$$

$$-g_s^2\{[A_c^1\cdots A_N^c;A_1^o\cdots A_M^o]^c\}_D\,,$$

$$\mathcal{B}^c \equiv \mathcal{G}[A_{\bar{i}_1}^c\cdots A_{\bar{i}_{N-k}}^c;A_{\bar{j}_1}^o\cdots A_{\bar{j}_{M-\ell}}^o]^c, \qquad \mathcal{B}^o \equiv \mathcal{G}[A_{\bar{i}_1}^c\cdots A_{\bar{i}_{N-k}}^c;A_{\bar{j}_1}^o\cdots A_{\bar{j}_{M-\ell}}^o]^o,$$

$$\{i_1,\cdots,i_k\}\cup\{\bar{i}_1,\cdots,\bar{i}_{N-k}\}=\{1,\cdots,N\}, \quad \{j_1,\cdots,j_\ell\}\cup\{\bar{j}_1,\cdots,\bar{j}_{M-\ell}\}=\{1,\cdots,M\}\,.$$

$$(1.16)$$

The term proportional to $\{[A_c^1\cdots A_N^c;A_1^o\cdots A_M^o]^c\}_D$ is not present in bosonic open-closed string field theory. There we include the contribution of the disc one point function in the definition of $\{\mathcal{B}^c;\}$. As mentioned below (1.2), this is not an option for RR sector $\mathcal{B}^c$ in superstring theory due to an obstruction associated with the picture number conservation.

Note that on the left hand side of (1.16), $Q_B A_i^c$ and $Q_B A_i^o$ are string states of 'wrong grassmann parity', while on the right hand side $\mathcal{B}^c$ and $\mathcal{B}^o$ are also of wrong grassmann parity. Therefore the corresponding objects $\{\cdots\}$ will have to be defined by multiplying these wrong parity objects by grassmann odd c-number ζ. It is useful to include this grassmann odd c-number ζ explicitly in the identity so that each term in the identity has only states of correct grassmann parity. This takes the form:

$$
\begin{aligned}
&\sum_{i=1}^{N} \{A_1^c \cdots A_{i-1}^c (\zeta Q_B A_i^c) A_{i+1}^c \cdots A_N^c; A_1^o \cdots A_M^o\} \\
&+ \sum_{j=1}^{M} \{A_1^c \cdots A_N^c; A_1^o \cdots A_{j-1}^o (\zeta Q_B A_j^o) A_{j+1}^o \cdots A_M^o\} \\
&= -\frac{1}{2} \sum_{k=0}^{N} \sum_{\{i_1 \cdots, i_k\} \subset \{1, \cdots, N\}} \sum_{\ell=0}^{M} \sum_{\{j_1, \cdots, j_\ell\} \subset \{1, \cdots, M\}} \left(g_s^2 \{A_{i_1}^c \cdots A_{i_k}^c B^c; A_{j_1}^o \cdots A_{j_\ell}^o\} \right. \\
&\left. + g_s \{A_{i_1}^c \cdots A_{i_k}^c; B^o A_{j_1}^o \cdots A_{j_\ell}^o\} \right) \\
&- g_s^2 \{\zeta [A_c^1 \cdots A_N^c; A_1^o \cdots A_M^o]^c\}_D, \\
&B^c \equiv \mathcal{G}\zeta [A_{i_1}^c \cdots A_{i_{N-k}}^c; A_{j_1}^o \cdots A_{j_{M-\ell}}^o]^c, \qquad B^o \equiv \mathcal{G}\zeta [A_{i_1}^c \cdots A_{i_{N-k}}^c; A_{j_1}^o \cdots A_{j_{M-\ell}}^o]^o, \\
&\{i_1, \cdots, i_k\} \cup \{\bar{i}_1, \cdots, \bar{i}_{N-k}\} = \{1, \cdots, N\}, \quad \{j_1, \cdots, j_\ell\} \cup \{\bar{j}_1, \cdots, \bar{j}_{M-\ell}\} = \{1, \cdots, M\}.
\end{aligned}
\tag{1.17}
$$

The $(-1)^{j-1}$ factor on the left hand side of (1.16) will be generated when we pull ζ through the open string vertex operators A_i^o to the extreme left.

The equations of motion following from the 1PI effective action are given by

$$
|\tilde{\Psi}^c\rangle \quad : \quad Q_B\big(|\Psi^c\rangle - \mathcal{G}|\tilde{\Psi}^c\rangle\big) + g_s^2 |[\ ]_D\rangle = 0, \tag{1.18}
$$

$$
|\Psi^c\rangle \quad : \quad Q_B|\tilde{\Psi}^c\rangle + g_s^2 \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \frac{1}{(N-1)!M!} [(\Psi^c)^{N-1}; (\Psi^o)^M]^c = 0, \tag{1.19}
$$

$$
|\tilde{\Psi}^o\rangle \quad : \quad Q_B\big(|\Psi^o\rangle - \mathcal{G}|\tilde{\Psi}^o\rangle\big) = 0, \tag{1.20}
$$

$$
|\Psi^o\rangle \quad : \quad Q_B|\tilde{\Psi}^o\rangle + g_s \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!(M-1)!} [(\Psi^c)^N; (\Psi^o)^{M-1}]^o = 0. \tag{1.21}
$$

Now, multiplying the second equation by $\mathcal{G}$ and adding to the first equation gives

$$
Q_B|\Psi^c\rangle + g_s^2 \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \frac{1}{(N-1)!M!} \mathcal{G}[(\Psi^c)^{N-1}; (\Psi^o)^M]^c + g_s^2 |[\ ]_D\rangle = 0. \tag{1.22}
$$

Similarly, multiplying the 4th equation by $\mathcal{G}$ and adding to the 3rd equation gives

$$
Q_B|\Psi^o\rangle + g_s \sum_{N=0}^{\infty} \sum_{M=1}^{\infty} \frac{1}{N!(M-1)!} \mathcal{G}[(\Psi^c)^N; (\Psi^o)^{M-1}]^o = 0. \tag{1.23}
$$

Eqs. (1.22) and (1.23) give the interacting equations of motion for the physical string fields Ψ^c and Ψ^o. Once we pick a solution to these equations and fix $|\Psi^c\rangle$ and $|\Psi^o\rangle$, we can determine $\widetilde{\Psi}^c$ and $\widetilde{\Psi}^o$ using (1.19) and (1.21) respectively. The only freedom in obtaining solution to (1.19) is to add solutions to $Q_B|\widetilde{\Psi}^c\rangle = 0$ to an existing solution. Similarly the only freedom in obtaining solution to (1.21) is to add solutions to $Q_B|\widetilde{\Psi}^o\rangle = 0$ to an existing solution. These represent free field degrees of freedom. Furthermore the choice of solutions to these free field equations of motion does not affect the interacting field equations (1.22) and (1.23) since they do not involve $\widetilde{\Psi}^c$ and $\widetilde{\Psi}^o$. Therefore the degrees of freedom associated with $\widetilde{\Psi}^c$ and $\widetilde{\Psi}^o$ represent free fields that completely decouple from the interacting part of the theory, and they have no observable signature.

Besides the results reviewed above, this paper also contains the following results:

1. We show in section 4 that the action (1.13) satisfies the classical BV master equation $(S, S) = 0$ where (F, G) denotes the anti-bracket between two functions F and G of the string field, as defined in (4.7), (4.8).

2. We construct in section 5 the quantum BV master action of superstring field theory of open and closed strings. This action has the same form as (1.13), but with the interaction vertices $\{\cdots\}$ replaced by slightly modified vertices $\{\!\{\cdots\}\!\}$. Also the main identity satisfied by the new vertices now has additional terms given by the last two terms in (5.5). Due to these additional terms in the vertices the action now satisfies the quantum BV master equation (5.2), with Δ defined in (5.3).

3. In section 6 we describe how to generalize the construction of the 1PI action and the BV master action to unoriented open closed string field theory. The structure of the action remains the same but the definitions of the interaction terms change. In particular $\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\}$ now gets additional contribution from non-orientable surfaces, $\{\widetilde{A}^c\}_D$ gets additional contribution from the projective sphere and there are additional normalization factors in the definitions of these quantities.

We end this section with a few comments.

1. In most formulation of classical open string field theory an associative $*$-product and its generalization known as the A_∞ algebra plays a significant role [11–22]. This is not manifest in the formulation of the quantum action described above. Instead what plays a central role here is the associated Lie algebra and its infinite dimensional generalization — the L_∞ algebra. For example, if we consider Witten's open bosonic string field theory [11], then our definition of $[; A^o B^o]^o$ at the tree level corresponds to $A^o * B^o - (-1)^{AB} B^o * A^o$ in the language of $*$-product. The price we pay in giving up the A_∞ structure is that we can no longer extract color ordered amplitudes from the theory without digging into the detailed definition of $\{\cdots\}$.[3] If we want to make manifest the information on color ordering, we need to follow a more elaborate approach described in [10, 23, 24].

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[3]We thank Theodore Erler for pointing this out to us.

2. A special feature of the 1PI effective action (1.13) is the linear term $\{\widetilde{\Psi}^c\}_D$ that involves a closed string field from $\widetilde{\mathcal{H}}^c$. Physically this is related to the fact that Ψ^c contains RR field strengths while $\widetilde{\Psi}^c$ contains the RR potential. Since D-branes carry RR charge; we need the RR potential, hidden inside $\widetilde{\Psi}^c$, to describe the coupling of closed string fields to D-branes.

3. As has already been alluded to before, we can set $\{\widetilde{A}^c\}_D$ to zero if $\widetilde{A}^c$ belongs to the NSNS sector, provided we include this contribution into the definition of $\{A^c; \}$. This corresponds to removing part of $[\,]_D$ that belongs to the NSNS sector and absorbing this into the definition of $[\,;\,]^c$. Under this (1.22) remains unchanged. This describes a superstring field theory that is equivalent to the original superstring field theory.

4. Using the open-closed superstring field theory one can construct gauge invariant 1PI effective actions for theories that are apparently anomalous. For example if we consider type IIB string theory with certain number of space-filling D9-branes, its spectrum will contain, besides the usual closed string fields, additional chiral fermions from the gauge supermultiplet on the D9-branes. This theory is known to suffer from gravitational anomaly [25]. However there is no difficulty in writing down a gauge invariant 1PI effective action for this theory. In the latter description, the inconsistency shows up due to the presence of a term in the action that is linear in the RR 10-form field, encoded in the $\{\widetilde{\Psi}^c\}_D$ term in (1.13). Due to the presence of this term, the theory does not have a vacuum solution to the equations of motion [27]. However in some cases we may be able to cancel the effect of the $\{\widetilde{\Psi}^c\}_D$ term, leading to the last term on the left hand side of (1.22), by switching on other background fields contributing to the second term on the left hand side of (1.22). Examples of this kind can be found in compactification of type IIB string theory on Calabi-Yau manifolds where the space-filling D3-brane charge can be cancelled by flux of 3-form fields along the internal 3-cycles of the Calabi-Yau manifold.

5. The action (1.13) contains insertion of c_0^- in several places — in the kinetic term as well as in the definition of $\{\widetilde{\Psi}^c\}_D$ given in (2.2). This can be traced to the presence of conformal Killing vectors on the associated Riemann surfaces — a sphere with two punctures and a line integral of the BRST current, and a disk with one bulk puncture. We can avoid this by including c_0^- in the definition of $\widetilde{\Psi}^c$, declaring $\widetilde{\mathcal{H}}^c$ to be the subspace of states annihilated by c_0^- instead of b_0^-. This will introduce a b_0^- in the $\widetilde{\Psi}^c$-$\widetilde{\Psi}^c$ kinetic term, but this is a more natural operator since the anti-commutator of Q_B with b_0^- generates L_0^-. It has in fact been argued in [26] that in this formalism we can take Ψ^c and $\widetilde{\Psi}^c$ to be unconstrained elements in the Hilbert space of closed string states except for the restriction on the picture numbers. This introduces some additional free field degrees of freedom which decouple from the interacting part of the theory.

6. As has been mentioned already, (1.22) and (1.23) can be regarded as the equations of motion of the physical string fields Ψ^c and Ψ^o. One important question is: given

a solution to these equations, can we always find a solution to the equations of motion (1.19) and (1.21) for the additional fields $\widetilde{\Psi}^c$ and $\widetilde{\Psi}^o$? If the answer is in the negative, this may impose further constraints on the physical fields Ψ^c and Ψ^o. Now, using (1.16) it is straightforward to show that once (1.22) and (1.23) are satisfied, the second terms in (1.19) and (1.21) are BRST invariant. Therefore the question reduces to whether they describe non-trivial elements of the BRST cohomology, since as long as they are BRST trivial one can always find $\widetilde{\Psi}^c$ and $\widetilde{\Psi}^o$ satisfying (1.19) and (1.21). This can be studied separately in the zero momentum sector and the non-zero momentum sector. We shall analyze this question for (1.19), — the analysis of (1.21) will be similar and in fact simpler. In the sector carrying non-zero momentum along the non-compact space-time directions, the contribution from $[\ ]_D$ can be expressed as $Q_B b_0^+ (L_0^+)^{-1} [\ ]_D$, and therefore the second term in (1.22), being equal to $-Q_B |\Psi^c\rangle - Q_B b_0^+ (L_0^+)^{-1} [\ ]_D$, is BRST trivial. It follows from the analysis of [27] that the second term in (1.19) is also BRST trivial, — one can use the inverse picture changing operator introduced in [27] to map elements of the BRST cohomology from $\mathcal{H}^c$ to $\widetilde{\mathcal{H}}^c$ in the non-zero momentum sector. On the other hand, it was shown in [27] that in $\widetilde{\mathcal{H}}^c$, the BRST cohomology in the zero momentum sector is trivial.[4] Therefore the BRST invariance of the second term in (1.19) implies BRST exactness of this term. This, in turn, implies that (1.19) always has a solution when (1.22) and (1.23) are satisfied. A similar conclusion follows for (1.21). Therefore (1.19) and (1.21) do not impose any additional constraint on Ψ^c and Ψ^o besides the ones implied by (1.22) and (1.23).

7. Once a consistent superstring field theory for open and closed strings has been formulated, it can be used to systematically study various aspects of string theory that are not easily amenable to the standard world-sheet approach. This includes for example the study of mass renormalization or vacuum shift [1], or studying superstring theory in RR background [28].

8. The superstring field theory of open and closed strings constructed here does not suffer from any ultra-violet divergence. However this theory suffers from all the usual infra-red divergences that a quantum field theory suffers. These have physical origin and need to be dealt with as in a quantum field theory.

9. Superstring field theory action that we write down can be formulated around any background associated with a superconformally invariant world-sheet theory in the NSR formalism. Even for a given background the theory is not unique — it depends on the choice of local coordinates at the punctures and the choice of PCO locations that we have to make in defining the interaction terms of the action. For superstring field theories of closed strings it is known that apparently different string field theories, that one gets by making different choices, are all related by field redefinition. We expect a similar result to hold for the theory described here, but we have not

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[4]For this one needs to use x_0 cohomology where one allows polynomials in the zero modes of the non-compact space-time coordinate fields to appear in the state.

attempted to give a complete proof. Another feature of the theory that we would like to prove is background independence — if we have different backgrounds related by marginal deformation of the world-sheet theory, then the superstring field theories formulated around these different backgrounds should also be related by field redefinition. This analysis is expected to be more complicated than the corresponding analysis for closed superstring field theory, since under marginal deformation of the bulk world-sheet theory we also need to deform the D-brane system appropriately [10]. We hope to return to these problems in the future.

2 Construction of 1PI vertices and their properties

We shall now describe the construction of the 1PI vertices $\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\}$ and $\{\widetilde{A}^c\}_D$ satisfying the various identities described in section 1. Since the construction proceeds more or less in the same way as in the case of closed superstring field theory reviewed in [1], we shall only emphasize the differences.

We begin by describing our convention for correlation functions of open string vertex operators. If there are N D-branes, not necessarily of the same kind, then a general open string state is described by an $N \times N$ matrix, with the i-j matrix element describing the state of an open string whose left end is on the i-th D-brane and the right end is on the j-th D-brane. A correlation function of open and closed string vertex operators on a general Riemann surface with multiple boundaries include taking traces over the $N \times N$ Chan-Paton matrices on each boundary of the Riemann surface. These traces will not be written explicitly, but will be understood as part of the definition of the correlation function. We shall follow the standard convention of referring to the locations of the vertex operators on the Riemann surface as punctures, with closed string vertex operators inserted at the bulk punctures and the open string vertex operators inserted at the boundary punctures.

We shall first define $\{\widetilde{A}^c\}_D$. We take this to be the one point function of $g_s^{-1} c_0^- \widehat{\mathcal{G}} \widetilde{A}^c$ on the unit disc $|z| \leq 1$, with the vertex operator inserted at the center of the disc using local coordinate $z\, e^{\beta}$. Here β is some positive number and[5]

$$\widehat{\mathcal{G}} \equiv \begin{cases} 1 \text{ on } \mathcal{H}_{-1,-1} \\ \frac{1}{2}(\mathcal{X}_0 + \bar{\mathcal{X}}_0) \text{ on } \mathcal{H}_{-3/2,-3/2} \end{cases}, \tag{2.1}$$

where $\mathcal{X}_0$, $\bar{\mathcal{X}}_0$ are zero modes of the PCOs as defined in (1.10). The c_0^- factor is needed due to the presence of conformal Killing vector on the disc, which makes one point function of any operator vanish if the operator is annihilated by b_0^-. The $\widehat{\mathcal{G}}$ factor is needed for picture number conservation. Since $\widetilde{A}^c$ has picture number -2 in the NSNS sector and -3 in the RR sector, $\widehat{\mathcal{G}} A^c$ has picture number -2 in both sectors. This is the correct picture number for getting a non-vanishing one point function on the disc. The identity (1.15) can now be proved by deforming the integration contour defining Q_B to the boundary of the disc. On

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[5]Since there is no insertion on the boundary, $\mathcal{X}_0$ and $\bar{\mathcal{X}}_0$ could be evaluated by taking their defining integration contours on the boundary. Since on the boundary the holomorphic and anti-holomorphic PCOs are equal, $\mathcal{X}_0$ and $\bar{\mathcal{X}}_0$ are identical. Therefore we could replace $\widehat{\mathcal{G}}$ by $\mathcal{X}_0$ or $\bar{\mathcal{X}}_0$.

the boundary the holomorphic and the anti-holomorphic components of Q_B cancel each other. There is no additional contribution from having to pass Q_B through $\widehat{\mathcal{G}}$ since $[Q_B, \widehat{\mathcal{G}}]$ vanishes. The term containing $\{Q_B, c_0^-\}$ vanishes since $\{Q_B, c_0^-\}$ does not contain a c_0^- and therefore integration over the zero mode c_0^- associated with the conformal Killing vector vanishes.

Given this definition of $\{\widetilde{A}^c\}_D$, we can define $[\ ]_D$ via (1.5). We can also express $[\ ]_D$ in terms of the boundary state [29–32] of the D-brane system as follows. Let $|B\rangle$ be the boundary state of the D-brane system under consideration so that $\langle\phi^c|c_0^-|B\rangle$ describes the one point function of the closed string vertex operator $c_0^-\phi^c$, inserted at the center of the unit disc $|z| \leq 1$ in the z coordinate system. This gives

$$\{\widetilde{A}^c\}_D = g_s^{-1} \langle\widetilde{A}^c|c_0^- \, \widehat{\mathcal{G}} \, e^{-\beta(L_0 + \bar{L}_0)}|B\rangle \,. \tag{2.2}$$

Since one point function on the disc is non-zero only for vertex operators of picture number -2, ϕ^c must have picture number -2 for getting non-vanishing $\langle\phi^c|c_0^-|B\rangle$. On the other hand, $\langle\phi^c|c_0^-|B\rangle$, being related to a sphere correlation function of ϕ^c and B, is non-zero only when the total picture number of ϕ^c and B add up to -4. Therefore $|B\rangle$ has picture number -2. Comparing the last equation of (1.5) and (2.2), we can express $[]_D \in \mathcal{H}^c$ as

$$[]_D = g_s^{-1}\mathbf{P} \, \widehat{\mathcal{G}} \, e^{-\beta(L_0 + \bar{L}_0)}|B\rangle \,, \tag{2.3}$$

where $\mathbf{P}$ denotes projection operator into $\mathcal{H}^c$. Its role is to pick the $(-1, -1)$ and $(-1/2, -1/2)$ components of $\widehat{\mathcal{G}}|B\rangle$, which otherwise has states in $\mathcal{H}_{m,n}$ for all m, n with $m + n = -2$ in the NSNS sector and $m + n = -1$ in the RR sector.

The construction of $\{\cdots\}$ proceeds as in the case of closed superstring field theory — therefore we shall be brief, emphasizing only the new aspects of this construction. We denote by $\mathcal{M}_{g,b,m_c,n_c,p_c,q_c,m_o,n_o}$ the moduli space of Riemann surfaces with genus g, b boundaries, m_c NSNS punctures, n_c NSR punctures, p_c RNS punctures, q_c RR punctures, m_o NS-sector punctures on the boundary and n_o R-sector punctures on the boundary, with the understanding that the integration over $\mathcal{M}_{g,b,m_c,n_c,p_c,q_c,m_o,n_o}$ includes sum over spin structures. It will also be understood that the Ramond punctures carry picture number $-1/2$, i.e. when we insert a vertex operator at the puncture, it belongs to $\mathcal{H}^c$ or $\mathcal{H}^o$. We denote by $\widetilde{\mathcal{P}}_{g,b,m_c,n_c,p_c,q_c,m_o,n_o}$ a fiber bundle over this moduli space, with the fiber directions specifying the choice of local coordinates at the punctures and also the locations of the

$$\wp \equiv 4g + 2b - 4 + 2m_c + 3(n_c + p_c)/2 + q_c + m_o + n_o/2 \,, \tag{2.4}$$

PCOs. Note that for Riemann surfaces with boundary, the picture number in the holomorphic and anti-holomorphic sectors are not separately conserved, but only the total picture number is conserved. For this reason, we have specified only the total picture number. Only for $b = 0$, we have to have $2g - 2 + m_c + n_c + (p_c + q_c)/2$ anti-holomorphic PCOs and $2g - 2 + m_c + p_c + (n_c + q_c)/2$ holomorphic PCOs. Therefore for $b \neq 0$ the fiber has $\wp + 1$ different branches, with the r-th branch having r holomorphic PCOs and $(\wp - r)$ anti-holomorphic PCOs.

In order to simplify notation we shall define the formal sum

$$\widetilde{\mathcal{P}}_{g,b,N,M} \equiv \sum_{\substack{m_c,n_c,p_c,q_c,m_o,n_o \\ m_c+n_c+p_c+q_c=N, m_o+n_o=M}} \widetilde{\mathcal{P}}_{g,b,m_c,n_c,p_c,q_c,m_o,n_o}, \tag{2.5}$$

and similarly $\mathcal{M}_{g,b,N,M}$. Following the same procedure reviewed in [1] for closed string theory, given a set of closed string states $A_1^c, \cdots, A_N^c$ and open string states $A_1^o, \cdots, A_M^o$, we can construct a p-form $\Omega_p^{g,b,N,M}(A_1^c, \cdots, A_N^c; A_1^o, \cdots, A_M^o)$ on $\widetilde{\mathcal{P}}_{g,b,N,M}$ for any positive integer p, in terms of appropriate correlation functions of b-ghosts, PCOs and of vertex operators $\{A_i^c\}$, $\{A_i^o\}$ on the corresponding Riemann surfaces. In this construction the local coordinates used in the insertion of off-shell vertex operators and the locations of the PCOs are determined by the point in $\widetilde{\mathcal{P}}_{g,b,N,M}$ where we compute the p-form. The specific form of the ghost insertions is determined by the tangent vectors of $\widetilde{\mathcal{P}}_{g,b,N,M}$ with which we contract $\Omega_p^{g,b,N,M}$, i.e. the particular components of $\Omega_p^{g,b,N,M}$ that we want to compute. The sign rules of section 1, that tells us that we can take a grassmann odd c-number out of $\{\cdots\}$ from the left, implicitly assumes that in computing the correlation functions that define $\Omega_p^{g,b,N,M}$, we insert the vertex operators for the external states first in the order they appear inside $\{\cdots\}$ and then insert all the ghosts and PCOs. Similarly the sign rule that we can move a grassmann odd c-number through $\langle$ without extra sign corresponds to treating the bra vacuum $\langle 0|$ as grassmann even. $\Omega_p^{g,b,N,M}$ satisfies the useful property:

$$\sum_{i=1}^{N} \Omega_p^{g,b,N,M}(A_1^c, \cdots, A_{i-1}^c, Q_B A_i^c, A_{i+1}^c, \cdots A_N^c; A_1^o, \cdots, A_M^o)$$
$$+ \sum_{j=1}^{M} (-1)^{j-1} \Omega_p^{g,b,N,M}(A_1^c, \cdots, A_N^c; A_1^o, \cdots, A_{j-1}^o, Q_B A_j^o, A_{j+1}^o, \cdots, A_M^o)$$
$$= \kappa(g, b, N, M, p) \, d\Omega_{p-1}^{g,b,N,M}(A_1^c, \cdots, A_N^c; A_1^o, \cdots, A_M^o), \tag{2.6}$$

where κ is an appropriate sign factor about which we shall say more later. The identity is derived by deforming the integration contour used in defining Q_B away from the vertex operators and making it act on the ghosts / PCO insertions. This generates insertion of stress tensor in the correlation function which in turn has the interpretation of an exterior derivative acting on Ω_{p-1}. The phase κ could in principle differ from the corresponding result in the closed string case from having to move odd operators through open string vertex operators. It can be determined by careful analysis as in [10], but we shall extract the relevant information using an indirect approach to be described later. A special role will be played by $\Omega_{6g-6+3b+2N+M}^{g,b,N,M}$ since the dimension of the moduli space $\mathcal{M}_{g,b,N,M}$ is given by $6g - 6 + 3b + 2N + M$.

As in the case of closed (super-)string field theory reviewed in [1], we introduce the notion of a generalized section of $\widetilde{\mathcal{P}}_{g,b,N,M}$ by extending the notion of a section. A generalized section can be a formal weighted average of many sections — with the understanding that integral over such a generalized section will be given by the weighted average of the integral over the corresponding sections. Unlike a regular section, a generalized section may also contain vertical segments across which the PCO locations jump discontinuously. The

integral of $\Omega_{6g-6+3b+2N+M}^{g,b,N,M}$ over such vertical segments will have to be defined by adding to the integral over the continuous part of the section some correction terms described in [33, 34].

The ability to include vertical segments in the generalized section plays a crucial role in open-closed string field theory — it allows us to choose generalized sections that can jump between the different branches of $\widetilde{\mathcal{P}}_{g,b,N,M}$ mentioned earlier. This is done by moving one or more PCOs from the bulk to a boundary across a vertical segment [33, 34]. Since on the boundary the holomorphic and anti-holomorphic PCOs are identical, we can replace holomorphic PCOs by anti-holomorphic PCOs (or vice versa) and then move them to the desired positions in the bulk across another vertical segment. Such jumps may be necessary in order to ensure that near various boundaries of the moduli space, the generalized section factorizes correctly into the direct product of the sections on the component Riemann surfaces to which the original Riemann surface degenerates.

For defining $\{\cdots\}$ we also need the notion of a section segment where we remove from the base $\mathcal{M}_{g,b,N,M}$ certain codimension zero regions and then erect a generalized section on this truncated space. Even though we drop the word generalized for brevity, it should be understood that the section segments we shall be working with refer to generalized section segments, allowing us to take weighted averages and vertical segments. Each such section segment represents a family of punctured Riemann surfaces equipped with a choice of local coordinates at the punctures and the choice of PCO locations. We shall now define two operations on section segments that will be important for us:

1. **Sewing.** Let us take a pair of section segments, one in $\widetilde{\mathcal{P}}_{g,b,N,M}$ and the other in $\widetilde{\mathcal{P}}_{g',b',N',M'}$. Each represents a family of Riemann surfaces equipped with choice of local coordinates at the punctures and PCO locations. We can now construct a new section segment by sewing them at a pair of punctures — one from each section segment. If the sewing is done at a bulk puncture, then this means that we take a Riemann surface from one family and sew one of its punctures to a puncture on the Riemann surface from the other family by making the identification

$$w_1 w_2 = e^{-s-i\theta}, \quad 0 \le s \le \infty, \quad 0 \le \theta < 2\pi, \tag{2.7}$$

where w_1 and w_2 are the local coordinates at the respective punctures. For sewing at a pair of boundary punctures the analog of (2.7) takes the form

$$w_1 w_2 = -e^{-s}. \tag{2.8}$$

In both cases, we also insert a factor of $\mathcal{G}$ defined in (1.9) around the origin of the w_1 (or the w_2) coordinate system. Doing this operation for each element of the first section segment and each element of the second section segment, we generate a family of new Riemann surfaces equipped with local coordinates at the punctures and choice of PCO locations, producing a new section segment. When the sewing is done at a pair of bulk punctures via (2.7), the resulting family of Riemann surfaces gives a section segment of $\widetilde{\mathcal{P}}_{g+g',b+b',N+N'-2,M+M'}$. On the other hand for sewing at a pair of boundary punctures via (2.8), we get a section segment of $\widetilde{\mathcal{P}}_{g+g',b+b'-1,N+N',M+M'-2}$.

2. **Hole creation.** Another way of producing a new section segment from a given one is to sew a Riemann surface belonging to a section segment to a disc $|z| \leq 1$ with one bulk puncture, via the relation

$$w_1 w_2 = e^{-s} \,, \tag{2.9}$$

where w_1 is the local coordinate at the bulk puncture on the Riemann surface that is being sewed and w_2 is the local coordinate at the bulk puncture at the center of the disc. We also need to insert a factor of $\widehat{\mathcal{G}}$ defined in (2.3) around the origin of the w_1 or w_2 coordinate system. Note the absence of the phase $e^{-i\theta}$ even though we sew two closed string punctures — this is related to the presence of a conformal Killing vector on the one punctured disc. We shall take w_2 to be related to the standard coordinate z on the unit disc by $w_2 = e^{\beta} z$, where β is the positive constant that appeared in the definition of $\{\widetilde{A}^c\}_D$. In that case, hole creation is equivalent to inserting at a bulk puncture on the Riemann surface the state (see (A.16))

$$-g_s^2 \, e^{-s(L_0+\bar{L}_0)}(b_0 + \bar{b}_0)|[\,]\rangle_D = -g_s \, e^{-(s+\beta)(L_0+\bar{L}_0)} \, \mathbf{P} \, \widehat{\mathcal{G}} \, (b_0 + \bar{b}_0)|B\rangle \,, \quad 0 \leq s \leq \infty \,. \tag{2.10}$$

The parameter s labels the extra modulus that appears when we replace a closed string puncture by a boundary. If the original section segment belonged to $\widetilde{\mathcal{P}}_{g,b,N,M}$, then the new section segment obtained by hole creation belongs to $\widetilde{\mathcal{P}}_{g,b+1,N-1,M}$.

In bosonic open-closed string field theory the hole creation need not be described as a separate operation, — it can be included in the sewing of a Riemann surface with punctures to the disc with one bulk puncture. In open-closed superstring field theory the disc with one bulk puncture requires special treatment since picture number conservation makes the disc one point function of vertex operators in $\mathcal{H}^c$ vanish and we need to pick vertex operators from $\widetilde{\mathcal{H}}^c$.

The definition of $\{\cdots\}$ requires choice of section segments $\mathcal{R}_{g,b,N,M}$ of $\widetilde{\mathcal{P}}_{g,b,N,M}$ satisfying certain properties:

1. The projection of $\mathcal{R}_{g,b,N,M}$ on the base $\mathcal{M}_{g,b,N,M}$ must not contain any separating type degeneration.

2. $\mathcal{R}_{g,b,N,M}$ must be symmetric under the exchange of punctures, separately for closed strings and for open strings.

3. Given a set of section segments $\{\mathcal{R}_{g,b,N,M}\}$, we can generate new section segments from them by repeated application of sewing and hole creation. The demand we make on $\mathcal{R}_{g,b,N,M}$ is that the formal sum of all of these section segments produces a full generalized section whose projection to the base covers the full moduli space $\mathcal{M}_{g,b,N,M}$.

As in the case of closed superstring field theory described in [1], we can systematically construct the section segments $\mathcal{R}_{g,b,N,M}$'s satisfying the above requirements as follows. Since the sewing and hole creation operation always increase the dimension of the section segment due to appearance of new parameters s or θ, we shall set up a recursive procedure

for constructing $\mathcal{R}_{g,b,N,M}$ with the dimension of $\mathcal{R}_{g,b,N,M}$ as the recursion parameter. We begin by making choices of $\mathcal{R}_{0,0,3,0}$, $\mathcal{R}_{0,1,0,3}$ and $\mathcal{R}_{0,1,1,1}$. In each of these cases the moduli space is 0-dimensional, so the only choice we have to make is the choice of local coordinates and PCO locations.[6] We choose the local coordinates w_i so that they are related to some natural local coordinate z on these Riemann surfaces by large scaling: $w_i = e^{\beta_i} z$ with large $|\beta_i|$, so that $|w_i| = 1$ describes a small circle around the puncture in the natural coordinates. This is known as adding long stubs to the string vertices [35]. Similarly we choose the parameter β appearing in the definition of $\{\widetilde{A}^c\}_D$ in (2.3) to be large. In order to make $\mathcal{R}_{g,b,N,M}$ symmetric under the exchange of punctures, we may need to average over different choices of local coordinates and / or PCO locations — this is allowed since we only require $\mathcal{R}_{g,b,N,M}$ to be a generalized section segment. We now focus on $\mathcal{R}_{g,b,N,M}$'s whose expected dimension is 1, e.g. $\mathcal{R}_{0,1,0,4}$, $\mathcal{R}_{0,1,1,2}$, $\mathcal{R}_{0,1,2,0}$ and $\mathcal{R}_{0,2,0,1}$. For any such g, b, N, M, we first determine all section segments obtained by sewing or hole creation operation involving zero dimensional $\mathcal{R}_{g',b',N',M'}$. As long as the zero dimensional $\mathcal{R}_{g',b',N',M'}$ have been constructed by adding long stubs, sewing and hole creation of the corresponding section segments will generate one dimensional section segments of $\widetilde{\mathcal{P}}_{g,b,N,M}$, whose projection on the base covers only small regions in $\mathcal{M}_{g,b,N,M}$ around separating type degenerations, leaving behind large gaps. We now choose one dimensional $\mathcal{R}_{g,b,N,M}$ to 'fill the gap' so that together we have a complete generalized section of $\widetilde{\mathcal{P}}_{g,b,N,M}$. There is clearly a lot of freedom since the section segments generated by sewing or hole creation of zero dimensional $\mathcal{R}_{g',b',M',N'}$ only fix the boundaries of one dimensional $\mathcal{R}_{g,b,N,M}$, by requiring them to match the $s = 0$ boundaries of the sewing operation (2.8) or hole creation operation (2.9). In the interior we can choose local coordinates / PCO locations arbitrarily, subject to the restriction that the local coordinates should carry long stubs, and the PCOs must avoid spurious poles [36] by finite margin [1]. We now repeat the process, generating all the two dimensional section segments by sewing and hole creation of the section segments of lower dimensional $\mathcal{R}_{g',b',N',M'}$'s and then choose the two dimensional $\mathcal{R}_{g,b,N,M}$'s by filing the gap. This procedure can be repeated to generate all the $\mathcal{R}_{g,b,N,M}$'s.

Note that allowing $\mathcal{R}_{g,b,N,M}$ to be generalized section segments is crucial for this construction. We have already mentioned that making it symmetric under the exchange of punctures may require averaging over different choices of local coordinates and/or PCO locations. Furthermore the boundaries of $\mathcal{R}_{g,b,N,M}$, fixed by sewing or hole creation in lower dimensional $\mathcal{R}_{g',b,M',N'}$'s, are often generalized sections since they often have insertion of the operator $\mathcal{G}$ or $\widehat{\mathcal{G}}$ involving average of PCO insertions on a circle.

For the purpose of our analysis we shall not need the explicit form of $\mathcal{R}_{g,b,N,M}$'s. Explicit construction of such $\mathcal{R}_{g,b,N,M}$'s can be done using minimal area metric [8] or hyperbolic metric [37–39], but any other choice satisfying the above requirements will also be acceptable for our construction. We now define:

$$\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\} \tag{2.11}$$
$$\equiv \sum_{\substack{g,b \geq 0 \\ (g,b) \neq (0,1) \text{ for } (N,M)=(1,0)}} (g_s)^{2g-2+b} \int_{\mathcal{R}_{g,b,N,M}} \Omega_{6g-6+3b+2N+M}^{g,b,N,M}(A_1^c, \cdots, A_N^c; A_1^o, \cdots, A_M^o).$$

---

[6]Note that $\mathcal{R}_{0,0,n,0}$ and $\mathcal{R}_{0,1,0,n}$ are empty for $n \leq 2$.

The proof of the main identity (1.16) can now be given as follows. It is clear from (2.11) that the left hand side of the main identity is given by

$$\sum_{\substack{g,b\geq 0 \\ (g,b)\neq(0,1)\,\text{for}\,(N,M)=(1,0)}} (g_s)^{2g-2+b} \int_{\mathcal{R}_{g,b,N,M}} \text{l.h.s. of (2.6) with } p = 6g-6+3b+2N+M\,. \tag{2.12}$$

We can now use (2.6) to express this as

$$\sum_{\substack{g,b\geq 0 \\ (g,b)\neq(0,1)\,\text{for}\,(N,M)=(1,0)}} (g_s)^{2g-2+b} \int_{\partial\mathcal{R}_{g,b,N,M}} \kappa_{g,b,N,M}\,\Omega^{g,b,N,M}_{6g-7+3b+2N+M}(A^c_1,\cdots,A^c_N;A^o_1,\cdots,A^o_M)\,, \tag{2.13}$$

where $\kappa_{g,b,N,M} \equiv \kappa(g,b,N,M,6g-6+3b+2N+M)$. From the definition of $\mathcal{R}_{g,b,N,M}$ given above, it follows that $\partial\mathcal{R}_{g,b,N,M}$ must match onto the $s=0$ boundary of one of the three sewing operations (2.7), (2.8), (2.9) acting on (a pair of) $\mathcal{R}_{g',b',N',M'}$. The standard arguments in conformal field theory now show that up to signs, the $s=0$ boundary of the operation (2.7) produces the term involving $\mathcal{B}^c$ on the right hand side of (1.16), the $s=0$ boundary of the operation (2.8) produces the term involving $\mathcal{B}^o$ on the right hand side of (1.16), and the $s=0$ boundary of the operation (2.9) produces the term involving $\{[\cdots]\}_D$ on the right hand side of (1.16). The signs can be determined by careful analysis as in [10] since there is no essential difference between bosonic and superstring theories here, the open string fields being grassmann odd in both cases. However we shall determine the signs by an indirect argument. We outline below the general strategy, leaving the detailed analysis to appendix A.

First of all, the fact that using (1.16) we can prove gauge invariance of the action, as shown in section 3, provides an indirect evidence for the correctness of the signs of the terms on the right hand side of (1.16). We can provide a more direct argument for these signs using the factorization property of string amplitudes as follows. Factorization property tells us that near the separating type degeneration, the integrand of a string amplitude breaks into a sum of products of the integrands associated with the component Riemann surfaces into which the original Riemann surface degenerates. This property is needed to ensure that the contribution to the amplitude from the region near a separating type degeneration has the interpretation of a pair of string amplitudes connected by a propagator. This can be used to fix any phase ambiguity in the integrand of a string amplitude by relating it to the product of the phases of amplitudes at lower genus / with lower number of punctures. We shall show in appendix A that if the phases of the amplitudes are fixed this way, then (2.13) gives precisely the right hand side of the main identity (1.16) without any extra sign.

Other properties of $\{\cdots\}$ and $[\cdots]^{c,o}$ described in section 1 can be proved easily. For example, symmetry of $\{A^c_1\cdots A^c_N;A^o_1\cdots A^o_M\}$ under arbitrary exchanges $A^c_i \leftrightarrow A^c_j$ and $A^o_k \leftrightarrow A^o_\ell$ follows immediately from the symmetry of $\mathcal{R}_{g,b,N,M}$ under permutation of bulk punctures and of boundary punctures. Put another way, the definition of $\{A^c_1\cdots A^c_N;A^o_1\cdots A^o_M\}$ involves explicit symmetrization under $A^c_i \leftrightarrow A^c_j$ and $A^o_k \leftrightarrow A^o_\ell$. The fact that $\{A^c_1\cdots A^c_N;A^o_1\cdots A^o_M\}$ and $\{\tilde{A}^c\}_D$ are grassmann even can be proved iteratively by starting with an amplitude with no external states (or, for low genus, amplitudes

with minimal number of external states needed for removing the conformal Killing vectors) and then noting that the addition of a grassmann even closed string state is accompanied by two insertions of b-ghosts in the correlator, while the addition of a grassmann odd open string is accompanied by one insertion of b-ghost. Therefore the grassmann parity of the correlator remains unchanged under these operations. This is also consistent with the fact that in order to get a non-zero result for an amplitude, the total number of grassmann odd component fields (coming from space-time ghosts, fermions etc.) must be even. The identities (1.6)–(1.8) can be proved using (1.5). For example we can express the left hand side of (1.6) as:

$$
\begin{aligned}
&\{A_1^c \cdots A_k^c \, \mathcal{G}[B_1^c \cdots B_\ell^c; C_1^o \cdots C_m^o]^c; D_1^o \cdots D_n^o\} \\
&= \langle \mathcal{G}[B_1^c \cdots B_\ell^c; C_1^o \cdots C_m^o]^c | c_0^- | [A_1^c \cdots A_k^c; D_1^o \cdots D_n^o]^c \rangle \\
&= \langle [B_1^c \cdots B_\ell^c; C_1^o \cdots C_m^o]^c | c_0^- \, \mathcal{G} | [A_1^c \cdots A_k^c; D_1^o \cdots D_n^o]^c \rangle \\
&= \langle \mathcal{G}[A_1^c \cdots A_k^c; D_1^o \cdots D_n^o]^c | c_0^- \, | [B_1^c \cdots B_\ell^c; C_1^o \cdots C_m^o]^c \rangle \\
&= \{B_1^c \cdots B_\ell^c \, \mathcal{G}[A_1^c \cdots A_k^c; D_1^o \cdots D_n^o]^c; C_1^o \cdots C_m^o\} \,,
\end{aligned}
\tag{2.14}
$$

where in the second step we have used the fact that the both $[\cdots]^c$ in (2.14) are grassmann odd. This establishes the symmetry property (1.6). The other identities (1.7), (1.8) can be proven in a similar manner.

3 Gauge invariance of the 1PI action

In this section we shall prove the invariance of the 1PI action given in (1.13) under the gauge transformation (1.14). The corresponding result for bosonic string field theory of open and closed strings follows from this by setting $\widetilde{\Psi} = \Psi$, $\widetilde{\Lambda} = \Lambda$ and $\mathcal{G} = 1$.

Under the gauge transformation (1.14), the variation in the action (1.13) is given by

$$
\begin{aligned}
\delta S_{1PI} = {}& -\frac{1}{g_s^2}\langle \widetilde{\Psi}^c | c_0^- Q_B \mathcal{G} | \delta \widetilde{\Psi}^c \rangle + \frac{1}{g_s^2}\langle \delta \widetilde{\Psi}^c | c_0^- Q_B | \Psi^c \rangle + \frac{1}{g_s^2}\langle \widetilde{\Psi}^c | c_0^- Q_B | \delta \Psi^c \rangle \\
& -\frac{1}{g_s}\langle \widetilde{\Psi}^o | Q_B \mathcal{G} | \delta \widetilde{\Psi}^o \rangle + \frac{1}{g_s}\langle \delta \widetilde{\Psi}^o | Q_B | \Psi^o \rangle + \frac{1}{g_s}\langle \widetilde{\Psi}^o | Q_B | \delta \Psi^o \rangle \\
& + \{\delta \widetilde{\Psi}^c\}_D + \sum_{N=1}^\infty \sum_{M=0}^\infty \frac{N}{N! M!} \{(\Psi^c)^{N-1} \delta \Psi^c; (\Psi^o)^M\} \\
& + \sum_{N=0}^\infty \sum_{M=1}^\infty \frac{M}{N! M!} \{(\Psi^c)^N; \delta \Psi^o (\Psi^o)^{M-1}\} \,,
\end{aligned}
\tag{3.1}
$$

where we used the identity

$$
\begin{aligned}
\langle \delta \widetilde{\Psi}^c | c_0^- Q_B \mathcal{G} | \Psi^c \rangle &= \langle \widetilde{\Psi}^c | c_0^- Q_B \mathcal{G} | \delta \widetilde{\Psi}^c \rangle \,, \\
\langle \delta \widetilde{\Psi}^o | Q_B \mathcal{G} | \widetilde{\Psi}^o \rangle &= \langle \widetilde{\Psi}^o | Q_B \mathcal{G} | \delta \widetilde{\Psi}^o \rangle \,.
\end{aligned}
\tag{3.2}
$$

We shall analyze separately the effect of the transformations generated by $\widetilde{\Lambda}^c$, $\widetilde{\Lambda}^o$, Λ^c and Λ^o. First we consider the gauge transformation parametrized by $\widetilde{\Lambda}^c$ in (1.14). Using $Q_B^2 = 0$

we see that the variation of the action is given by

$$\widetilde{\delta}_c S_{1PI} = \{Q_B \widetilde{\Lambda}^c\}_D. \tag{3.3}$$

But this vanishes by the identity (1.15). Similarly, the variation of the action parametrized by $\widetilde{\Lambda}^o$ in (1.14) vanishes identically:

$$\widetilde{\delta}_o S_{1PI} = 0. \tag{3.4}$$

For the gauge transformation parametrized by Λ^c, we have, from (1.14),

$$
\begin{aligned}
\delta_c S_{1PI} = &-\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\widetilde{\Psi}^c|c_0^- Q_B\mathcal{G}|[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^c\rangle \\
&+\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^c|c_0^- Q_B|\Psi^c\rangle \\
&+\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\widetilde{\Psi}^c|c_0^- Q_B\mathcal{G}|[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^c\rangle \\
&+\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\widetilde{\Psi}^o|Q_B\mathcal{G}|[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^o\rangle \\
&-\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^o|Q_B|\Psi^o\rangle \\
&-\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\widetilde{\Psi}^o|Q_B\mathcal{G}|[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^o\rangle \\
&+g_s^2\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\{[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^c\}_D \\
&+\sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\frac{1}{(N-1)!M!}\{(\Psi^c)^{N-1}Q_B\Lambda^c;(\Psi^o)^M\} \\
&+\sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\sum_{P\geq 0}\sum_{Q\geq 0}\frac{g_s^2}{(N-1)!M!P!Q!}\{(\Psi^c)^{N-1}\mathcal{G}[\Lambda^c(\Psi^c)^P;(\Psi^o)^Q]^c;(\Psi^o)^M\} \\
&-\sum_{N=0}^{\infty}\sum_{M=1}^{\infty}\sum_{P\geq 0}\sum_{Q\geq 0}\frac{g_s}{N!(M-1)!P!Q!}\{(\Psi^c)^N;\mathcal{G}[\Lambda^c(\Psi^c)^P;(\Psi^o)^Q]^o(\Psi^o)^{M-1}\}.
\end{aligned} \tag{3.5}
$$

The first and 3rd terms cancel each other while 4th and 6th terms cancel each other. After using (1.5) we are left with

$$
\begin{aligned}
\delta_c S_{1PI} = &\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\{(Q_B\Psi^c)\Lambda^c(\Psi^c)^N;(\Psi^o)^M\} \\
&-\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\{\Lambda^c(\Psi^c)^N;(Q_B\Psi^o)(\Psi^o)^M\} \\
&+g_s^2\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\{[\Lambda^c(\Psi^c)^N;(\Psi^o)^M]^c\}_D
\end{aligned} \tag{3.6}
$$

$$+ \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \frac{1}{(N-1)!M!} \{ (\Psi^c)^{N-1} Q_B \Lambda^c ; (\Psi^o)^M \}$$

$$+ \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \sum_{P \geq 0} \sum_{Q \geq 0} \frac{g_s^2}{(N-1)!M!P!Q!} \{ (\Psi^c)^{N-1} \mathcal{G}[\Lambda^c (\Psi^c)^P ; (\Psi^o)^Q]^c ; (\Psi^o)^M \}$$

$$- \sum_{N=0}^{\infty} \sum_{M=1}^{\infty} \sum_{P \geq 0} \sum_{Q \geq 0} \frac{g_s}{N!(M-1)!P!Q!} \{ (\Psi^c)^N ; \mathcal{G}[\Lambda^c (\Psi^c)^P ; (\Psi^o)^Q]^o (\Psi^o)^{M-1} \} \,.$$

Now, we specialize the identity (1.16) to the following case

$$\begin{aligned}
A_i^c &= \Psi^c &&\text{for } i = 1, \cdots, N-1 \,, \\
A_N^c &= \zeta \Lambda^c \,, \\
A_j^o &= \Psi^o &&\text{for } j = 1, \cdots, M \,,
\end{aligned} \tag{3.7}$$

where ζ is a grassmann odd c-number. This gives

$$\begin{aligned}
&(N-1)\{Q_B \Psi^c (\Psi^c)^{N-2} \zeta \Lambda^c ; (\Psi^o)^M \} + \{ (\Psi^c)^{N-1} Q_B \zeta \Lambda^c ; (\Psi^o)^M \} \\
&+ M \{ (\Psi^c)^{N-1} \zeta \Lambda^c ; (Q_B \Psi^o)(\Psi^o)^{M-1} \} \\
&= - \sum_{k=0}^{N-1} \sum_{\ell=0}^{M} \binom{N-1}{k} \binom{M}{\ell} \Big(g_s^2 \{ (\Psi^c)^k \mathcal{G}[(\Psi^c)^{N-1-k} \zeta \Lambda^c ; (\Psi^o)^{M-\ell}]^c ; (\Psi^o)^\ell \} \\
&\quad + g_s \{ (\Psi^c)^k ; \mathcal{G}[(\Psi^c)^{N-1-k} \zeta \Lambda^c ; (\Psi^o)^{M-\ell}]^o (\Psi^o)^\ell \} \Big) \\
&\quad - g_s^2 \{ [(\Psi^c)^{N-1} \zeta \Lambda^c ; (\Psi^o)^M]^c \}_D \,.
\end{aligned} \tag{3.8}$$

In writing the above equation, we have used the identities (1.6), (1.7). Now, bringing the grassmann odd parameter ζ to the extreme left, multiplying the expression by $\{(N-1)!M!\}^{-1}$, and summing over M and N, we obtain

$$\begin{aligned}
0 &= \sum_{M \geq 0} \sum_{N \geq 2} \frac{1}{(N-2)!M!} \{ Q_B \Psi^c (\Psi^c)^{N-2} \Lambda^c ; (\Psi^o)^M \} \\
&\quad + \sum_{M \geq 0} \sum_{N \geq 1} \frac{1}{(N-1)!M!} \{ (\Psi^c)^{N-1} Q_B \Lambda^c ; (\Psi^o)^M \} \\
&\quad - \sum_{M \geq 0} \sum_{N \geq 1} \frac{1}{(N-1)!(M-1)!} \{ (\Psi^c)^{N-1} \Lambda^c ; (Q_B \Psi^o)(\Psi^o)^{M-1} \} \\
&\quad + \sum_{M \geq 0} \sum_{N \geq 1} \sum_{k=0}^{N-1} \sum_{\ell=0}^{M} \frac{1}{(N-1-k)!(M-\ell)!\ell!k!} \Big(g_s^2 \{ (\Psi^c)^k \mathcal{G}[(\Psi^c)^{N-1-k} \Lambda^c ; (\Psi^o)^{M-\ell}]^c ; (\Psi^o)^\ell \} \\
&\quad - g_s \{ (\Psi^c)^k ; \mathcal{G}[(\Psi^c)^{N-1-k} \Lambda^c ; (\Psi^o)^{M-\ell}]^o (\Psi^o)^\ell \} \Big) \\
&\quad + g_s^2 \sum_{M \geq 0} \sum_{N \geq 1} \frac{1}{(N-1)!M!} \{ [(\Psi^c)^{N-1} \Lambda^c ; (\Psi^o)^M]^c \}_D \,.
\end{aligned} \tag{3.9}$$

After redefining the sums appropriately, we see that the right hand side of the above equation is precisely the gauge transformation $\delta_c S_{1PI}$ given in (3.6). This proves gauge

invariance under the transformation generated by Λ^c:

$$\delta_c S_{1PI} = 0\,. \tag{3.10}$$

Finally, we consider the variation of the action under the gauge transformation parametrized by Λ^o. The variation of the action is given by

$$\begin{aligned}
\delta_o S_{1PI} = {}& -\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\widetilde{\Psi}^c|c_0^- Q_B\mathcal{G}|[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^c\rangle \tag{3.11}\\
& +\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^c|c_0^- Q_B|\Psi^c\rangle\\
& +\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\widetilde{\Psi}^c|c_0^- Q_B\mathcal{G}|[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^c\rangle\\
& +\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\widetilde{\Psi}^o|Q_B\mathcal{G}|[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^o\rangle\\
& -\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^o|Q_B|\Psi^o\rangle\\
& -\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\widetilde{\Psi}^o|Q_B\mathcal{G}|[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^o\rangle\\
& +g_s^2\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\{[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^c\}_D\\
& +\sum_{N=0}^{\infty}\sum_{M=1}^{\infty}\frac{1}{(M-1)!N!}\{(\Psi^c)^N;Q_B\Lambda^o(\Psi^o)^{M-1}\}\\
& +\sum_{N=1}^{\infty}\sum_{M=0}^{\infty}\sum_{P\geq 0}\sum_{Q\geq 0}\frac{g_s^2}{(N-1)!M!P!Q!}\{(\Psi^c)^{N-1}\mathcal{G}[(\Psi^c)^P;\Lambda^o(\Psi^o)^Q]^c;(\Psi^o)^M\}\\
& -\sum_{N=0}^{\infty}\sum_{M=1}^{\infty}\sum_{P\geq 0}\sum_{Q\geq 0}\frac{g_s}{N!(M-1)!P!Q!}\{(\Psi^c)^N;\mathcal{G}[(\Psi^c)^P;\Lambda^o(\Psi^o)^Q]^o(\Psi^o)^{M-1}\}\,.
\end{aligned}$$

Again, the first and third terms cancel each other and the fourth and sixth terms cancel each other. After using (1.5), we are left with

$$\begin{aligned}
\delta_o S_{1PI} = {}& \sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\langle\{(Q_B\Psi^c)(\Psi^c)^N;\Lambda^o(\Psi^o)^M\} \tag{3.12}\\
& -\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\{(\Psi^c)^N;(Q_B\Psi^o)\Lambda^o(\Psi^o)^M\}\\
& +g_s^2\sum_{N\geq 0}\sum_{M\geq 0}\frac{1}{N!M!}\{[(\Psi^c)^N;\Lambda^o(\Psi^o)^M]^c\}_D\\
& +\sum_{N=0}^{\infty}\sum_{M=1}^{\infty}\frac{1}{(M-1)!N!}\{(\Psi^c)^N;Q_B\Lambda^o(\Psi^o)^{M-1}\}
\end{aligned}$$

$$+g_s^2 \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \sum_{P \geq 0} \sum_{Q \geq 0} \frac{1}{(N-1)!M!P!Q!} \{(\Psi^c)^{N-1}\mathcal{G}[(\Psi^c)^P; \Lambda^o(\Psi^o)^Q]^c; (\Psi^o)^M\}$$

$$-g_s \sum_{N=0}^{\infty} \sum_{M=1}^{\infty} \sum_{P \geq 0} \sum_{Q \geq 0} \frac{1}{N!(M-1)!P!Q!} \{(\Psi^c)^N; \mathcal{G}[(\Psi^c)^P; \Lambda^o(\Psi^o)^Q]^o(\Psi^o)^{M-1}\} \,.$$

Now, we specialize the identity (1.16) to the following case

$$\begin{aligned} A_i^o &= \Psi^o && \text{for } i = 1, \cdots, M-1\,, \\ A_N^o &= \zeta \Lambda^o\,, \\ A_j^c &= \Psi^c && \text{for } j = 1, \cdots, N\,, \end{aligned} \tag{3.13}$$

for some grassmann odd c-number ζ. This gives

$$N\{Q_B\Psi^c(\Psi^c)^{N-1}; \zeta\Lambda^o(\Psi^o)^{M-1}\} + \{(\Psi^c)^N; (Q_B\zeta\Lambda^o)(\Psi^o)^{M-1}\}$$
$$+ (M-1)\{(\Psi^c)^N; (Q_B\Psi^o)\zeta\Lambda^o(\Psi^o)^{M-1}\}$$
$$= -\sum_{k=0}^{N} \sum_{\ell=0}^{M-1} \binom{N}{k}\binom{M-1}{\ell} \Big(g_s^2\{(\Psi^c)^k\mathcal{G}[(\Psi^c)^{N-k}; \zeta\Lambda^o(\Psi^o)^{M-\ell-1}]^c; (\Psi^o)^\ell\}$$
$$+ g_s\{(\Psi^c)^k; \mathcal{G}[(\Psi^c)^{N-1-k}; \zeta\Lambda^o(\Psi^o)^{M-\ell-1}]^o(\Psi^o)^\ell\}\Big)$$
$$- g_s^2\{[(\Psi^c)^N; \zeta\Lambda^o(\Psi^o)^{M-1}]^c\}_D\,. \tag{3.14}$$

Bringing the grassmann odd parameter ζ to extreme left and summing over M and N after multiplying with $\{N!(M-1)!\}^{-1}$, we get

$$0 = \sum_{N \geq 1} \sum_{M \geq 1} \frac{1}{(N-1)!(M-1)!} \{(Q_B\Psi^c)(\Psi^c)^{N-1}; \Lambda^o(\Psi^o)^{M-1}\}$$

$$+ \sum_{N \geq 0} \sum_{M \geq 1} \frac{1}{N!(M-1)!} \{(\Psi^c)^N; (Q_B\Lambda^o)(\Psi^o)^{M-1}\}$$

$$- \sum_{N \geq 0} \sum_{M \geq 2} \frac{1}{N!(M-2)!} \{(\Psi^c)^N; (Q_B\Psi^o)\Lambda^o(\Psi^o)^{M-2}\}$$

$$+ \sum_{N \geq 0} \sum_{M \geq 1} \sum_{k=0}^{N} \sum_{\ell=0}^{M-1} \frac{1}{(N-k)!(M-\ell-1)!k!\ell!} \Big(g_s^2\{(\Psi^c)^k\mathcal{G}[(\Psi^c)^{N-k}; \Lambda^o(\Psi^o)^{M-\ell-1}]^c; (\Psi^o)^\ell\}$$

$$- g_s\{(\Psi^c)^k; \mathcal{G}[(\Psi^c)^{N-1-k}; \Lambda^o(\Psi^o)^{M-\ell-1}]^o(\Psi^o)^\ell\}\Big)$$

$$+ g_s^2 \sum_{N \geq 0} \sum_{M \geq 1} \frac{1}{N!(M-1)!} \{[(\Psi^c)^N; \Lambda^o(\Psi^o)^{M-1}]^c\}_D\,. \tag{3.15}$$

After redefining the sums and comparing with (3.12), we find

$$\delta_o S_{1PI} = 0\,. \tag{3.16}$$

This completes the proof of gauge invariance of the 1PI effective action.

4 Classical BV master equation for the 1PI action

We shall now show that the 1PI effective action satisfies the classical BV master equation [8, 10, 40, 41]. For this, we first identify the fields and anti-fields of the theory. This is done by dividing the Hilbert spaces as follows

$$
\begin{aligned}
\mathcal{H}^c &= \mathcal{H}^c_+ \oplus \mathcal{H}^c_- \,, & \widetilde{\mathcal{H}}^c &= \widetilde{\mathcal{H}}^c_+ \oplus \widetilde{\mathcal{H}}^c_- \,, \\
\mathcal{H}^o &= \mathcal{H}^o_+ \oplus \mathcal{H}^o_- \,, & \widetilde{\mathcal{H}}^o &= \widetilde{\mathcal{H}}^o_+ \oplus \widetilde{\mathcal{H}}^o_- \,,
\end{aligned}
\tag{4.1}
$$

such that the states in $\mathcal{H}^c_+$ and $\widetilde{\mathcal{H}}^c_+$ have world-sheet ghost number ≥ 3, the states in $\mathcal{H}^c_-$ and $\widetilde{\mathcal{H}}^c_-$ have world-sheet ghost number ≤ 2, the states in $\mathcal{H}^o_+$ and $\widetilde{\mathcal{H}}^o_+$ have world-sheet ghost number ≥ 2 and the states in $\mathcal{H}^o_-$ and $\widetilde{\mathcal{H}}^o_-$ have the world-sheet ghost number ≤ 1. We denote the basis states of $\mathcal{H}^c_+$, $\mathcal{H}^c_-$, $\widetilde{\mathcal{H}}^c_+$, and $\widetilde{\mathcal{H}}^c_-$ by $|\varphi^r_+\rangle, |\varphi^-_r\rangle, |\tilde\varphi^r_+\rangle$ and $|\tilde\varphi^-_r\rangle$ respectively. They are chosen to satisfy orthonormality and completeness conditions:

$$
\langle \tilde\varphi^r_+ | c^-_0 | \varphi^-_s \rangle = \delta^r_s = \langle \varphi^-_s | c^-_0 | \tilde\varphi^r_+ \rangle, \quad \langle \varphi^r_+ | c^-_0 | \tilde\varphi^-_s \rangle = \delta^r_s = \langle \tilde\varphi^-_s | c^-_0 | \varphi^r_+ \rangle,
$$
$$
|\varphi^-_r\rangle\langle\tilde\varphi^r_+| + |\varphi^r_+\rangle\langle\tilde\varphi^-_r| = b^-_0 = |\tilde\varphi^r_+\rangle\langle\varphi^-_r| + |\tilde\varphi^-_r\rangle\langle\varphi^r_+| \,.
\tag{4.2}
$$

Similarly, we denote the basis states of $\mathcal{H}^o_+$, $\mathcal{H}^o_-$, $\widetilde{\mathcal{H}}^o_+$, and $\widetilde{\mathcal{H}}^o_-$ by $|\phi^r_+\rangle, |\phi^-_r\rangle, |\tilde\phi^r_+\rangle$ and $|\tilde\phi^-_r\rangle$ respectively. They satisfy orthonormality and completeness conditions:

$$
\langle \tilde\phi^r_+ | \phi^-_s \rangle = \delta^r_s = \langle \phi^-_s | \tilde\phi^r_+ \rangle, \quad \langle \phi^r_+ | \tilde\phi^-_s \rangle = \delta^r_s = \langle \tilde\phi^-_s | \phi^r_+ \rangle,
$$
$$
|\phi^-_r\rangle\langle\tilde\phi^r_+| + |\phi^r_+\rangle\langle\tilde\phi^-_r| = 1 = |\tilde\phi^r_+\rangle\langle\phi^-_r| + |\tilde\phi^-_r\rangle\langle\phi^r_+| \,.
\tag{4.3}
$$

The closed string fields are expanded as[7]

$$
|\widetilde{\Psi}^c\rangle = g_s \sum_r (-1)^{\tilde\varphi^-_r} (\widetilde{\psi}^c)^r |\widetilde{\varphi}^-_r\rangle - g_s \sum_r (\psi^c)^*_r |\widetilde{\varphi}^r_+\rangle \,,
$$
$$
|\Psi^c\rangle - \frac{1}{2}\mathcal{G}|\widetilde{\Psi}^c\rangle = g_s \sum_r (-1)^{\varphi^-_r} (\psi^c)^r |\varphi^-_r\rangle - g_s \sum_r (\widetilde{\psi}^c)^*_r |\varphi^r_+\rangle \,,
\tag{4.4}
$$

where φ in the exponent for any state $|\varphi\rangle$ denotes the grassmann parity of the vertex operator φ, taking value 0 for even operators and 1 for odd operators. We define the target space ghost number of the coefficient fields by $g = 2 - G$ where G denotes the world-sheet ghost number of the corresponding basis states. This means that the coefficients $(\psi^c)^r, (\widetilde{\psi}^c)^r$ have target space ghost numbers ≥ 0 whereas the coefficients $(\psi^c)^*_r$ and $(\widetilde{\psi}^c)^*_r$ have target space ghost numbers ≤ -1. In the BV quantization, the $(\psi^c)^r$ and $(\widetilde{\psi}^c)^r$ will be interpreted as fields whereas $(\psi^c)^*_r$ and $(\widetilde{\psi}^c)^*_r$ will be interpreted as anti-fields. The factors of g_s in (4.4) ensure that in the action $(\psi^c)^r$, $(\widetilde{\psi}^c)^r$, $(\psi^c)^*_r$ and $(\widetilde{\psi}^c)^*_r$ have conventionally normalized kinetic terms.

In a similar way, we expand the open string fields as

$$
|\widetilde{\Psi}^o\rangle = g_s^{1/2} \sum_r (\widetilde{\psi}^o)^r |\tilde\phi^-_r\rangle + g_s^{1/2} \sum_r (-1)^{\tilde\phi^r_+ + 1} (\psi^o)^*_r |\tilde\phi^r_+\rangle \,,
$$
$$
|\Psi^o\rangle - \frac{1}{2}\mathcal{G}|\widetilde{\Psi}^o\rangle = g_s^{1/2} \sum_r (\psi^o)^r |\phi^-_r\rangle + g_s^{1/2} \sum_r (-1)^{\phi^r_+ + 1} (\widetilde{\psi}^o)^*_r |\phi^r_+\rangle \,.
\tag{4.5}
$$

---

[7] in these equations * denotes anti-fields and not complex conjugation.

We define the target space ghost number of the coefficient fields by $g = 1 - G$ where G denotes the world-sheet ghost number of the corresponding basis states. This means that the coefficients $(\psi^o)^r, (\widetilde{\psi}^o)^r$ have target space ghost numbers ≥ 0 whereas the coefficients $(\psi^o)^*_r$ and $(\widetilde{\psi}^o)^*_r$ have target space ghost numbers ≤ -1. In the BV quantization, $(\psi^o)^r$ and $(\widetilde{\psi}^o)^r$ will be interpreted as fields whereas $(\psi^o)^*_r$ and $(\widetilde{\psi}^o)^*_r$ will be interpreted as anti-fields.

The BV anti-bracket between any two functions F and G of the fields is given by

$$(F, G) = \frac{\partial_R F}{\partial \psi^r} \frac{\partial_L G}{\partial \psi^*_r} - \frac{\partial_R F}{\partial \psi^*_r} \frac{\partial_L G}{\partial \psi^r} , \tag{4.6}$$

where ψ^r stand for all the fields $(\psi^c)^r$, $(\widetilde{\psi}^c)^r$, $(\psi^o)^r$ and $(\widetilde{\psi}^o)^r$ and ψ^*_r stand for all the anti-fields $(\psi^c)^*_r$, $(\widetilde{\psi}^c)^*_r$, $(\psi^o)^*_r$ and $(\widetilde{\psi}^o)^*_r$. ∂_L and ∂_R denotes left and right derivatives respectively. If for an arbitrary function $F(\Psi^c, \widetilde{\Psi}^c, \Psi^o, \widetilde{\Psi}^o)$, one expresses the first order variation as

$$\begin{aligned}
\delta F &= \langle F^c_R | c^-_0 | \delta \widetilde{\Psi}^c \rangle + \langle \widetilde{F}^c_R | c^-_0 | \delta \Psi^c \rangle + \langle F^o_R | \delta \widetilde{\Psi}^o \rangle + \langle \widetilde{F}^o_R | \delta \Psi^o \rangle \\
&= \langle \delta \widetilde{\Psi}^c | c^-_0 | F^c_L \rangle + \langle \delta \Psi^c | c^-_0 | \widetilde{F}^c_L \rangle + \langle \delta \widetilde{\Psi}^o | F^o_L \rangle + \langle \delta \Psi^o | \widetilde{F}^o_L \rangle ,
\end{aligned} \tag{4.7}$$

then using (4.4), (4.5), (4.6), one can express the anti-bracket as

$$\begin{aligned}
(F, G) =\ &-g_s^2 \Big(\langle F^c_R | c^-_0 | \widetilde{G}^c_L \rangle + \langle \widetilde{F}^c_R | c^-_0 | G^c_L \rangle + \langle \widetilde{F}^c_R | c^-_0 \mathcal{G} | \widetilde{G}^c_L \rangle \Big) \\
&-g_s \Big(\langle F^o_R | \widetilde{G}^o_L \rangle + \langle \widetilde{F}^o_R | G^o_L \rangle + \langle \widetilde{F}^o_R | \mathcal{G} | \widetilde{G}^o_L \rangle \Big) .
\end{aligned} \tag{4.8}$$

Our goal will be to verify that the action (1.13) satisfies the classical BV master equation:

$$(S_{1PI}, S_{1PI}) = 0 . \tag{4.9}$$

For this we need to compute the quantities $S^{c,o}_R, S^{c,o}_L, \widetilde{S}^{c,o}_R$ and $\widetilde{S}^{c,o}_L$ using the definition (4.7). The variation of the action (1.13) under an arbitrary deformation of the fields is given by

$$\begin{aligned}
\delta S_{1PI} =\ &-\frac{1}{g_s^2} \langle \delta \widetilde{\Psi}^c | c^-_0 Q_B \mathcal{G} | \widetilde{\Psi}^c \rangle + \frac{1}{g_s^2} \langle \delta \widetilde{\Psi}^c | c^-_0 Q_B | \Psi^c \rangle + \frac{1}{g_s^2} \langle \delta \Psi^c | c^-_0 Q_B | \widetilde{\Psi}^c \rangle \\
&-\frac{1}{g_s} \langle \delta \widetilde{\Psi}^o | Q_B \mathcal{G} | \widetilde{\Psi}^o \rangle + \frac{1}{g_s} \langle \delta \widetilde{\Psi}^o | Q_B | \Psi^o \rangle + \frac{1}{g_s} \langle \delta \Psi^o | Q_B | \widetilde{\Psi}^o \rangle + \langle \delta \widetilde{\Psi}^c | c^-_0 | [\ ]_D \rangle \\
&+\sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \frac{1}{(N-1)! M!} \langle \delta \Psi^c | c^-_0 | [(\Psi^c)^{N-1}; (\Psi^o)^M]^c \rangle \\
&+\sum_{N=0}^{\infty} \sum_{M=1}^{\infty} \frac{1}{N! (M-1)!} \langle \delta \Psi^o | [(\Psi^c)^N; (\Psi^o)^{M-1}]^o \rangle .
\end{aligned} \tag{4.10}$$

This gives,

$$
\begin{aligned}
|S_L^c\rangle &= -\frac{1}{g_s^2} Q_B \mathcal{G} |\widetilde{\Psi}^c\rangle + \frac{1}{g_s^2} Q_B |\Psi^c\rangle + |[\ ]_D\rangle \,, \\
|\widetilde{S}_L^c\rangle &= \frac{1}{g_s^2} Q_B |\widetilde{\Psi}^c\rangle + \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \big| [(\Psi^c)^N ; (\Psi^o)^M]^c \big\rangle \,, \\
|S_L^o\rangle &= -\frac{1}{g_s} Q_B \mathcal{G} |\widetilde{\Psi}^o\rangle + \frac{1}{g_s} Q_B |\Psi^o\rangle \,, \\
|\widetilde{S}_L^o\rangle &= \frac{1}{g_s} Q_B |\widetilde{\Psi}^o\rangle + \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \big| [(\Psi^c)^N ; (\Psi^o)^M]^o \big\rangle \,.
\end{aligned}
\tag{4.11}
$$

The variation (4.10) can also be written as

$$
\begin{aligned}
\delta S_{1PI} &= -\frac{1}{g_s^2} \langle \widetilde{\Psi}^c | c_0^- Q_B \mathcal{G} |\delta \widetilde{\Psi}^c\rangle + \frac{1}{g_s^2} \langle \Psi^c | c_0^- Q_B |\delta \widetilde{\Psi}^c\rangle + \frac{1}{g_s^2} \langle \widetilde{\Psi}^c | c_0^- Q_B |\delta \Psi^c\rangle \\
&\quad -\frac{1}{g_s} \langle \widetilde{\Psi}^o | Q_B \mathcal{G} |\delta \widetilde{\Psi}^o\rangle + \frac{1}{g_s} \langle \Psi^o | Q_B |\delta \widetilde{\Psi}^o\rangle + \frac{1}{g_s} \langle \widetilde{\Psi}^o | Q_B |\delta \Psi^o\rangle + \langle [\ ]_D | c_0^- |\delta \widetilde{\Psi}^c\rangle \\
&\quad + \sum_{N=1}^{\infty} \sum_{M=0}^{\infty} \frac{1}{(N-1)!M!} \langle [(\Psi^c)^{N-1} ; (\Psi^o)^M]^c | c_0^- |\delta \Psi^c\rangle \\
&\quad + \sum_{N=0}^{\infty} \sum_{M=1}^{\infty} \frac{1}{N!(M-1)!} \langle [(\Psi^c)^N ; (\Psi^o)^{M-1}]^o |\delta \Psi^o\rangle \,.
\end{aligned}
\tag{4.12}
$$

This gives

$$
\begin{aligned}
\langle S_R^c | &= \frac{1}{g_s^2} \langle \widetilde{\Psi}^c | Q_B \mathcal{G} - \frac{1}{g_s^2} \langle \Psi^c | Q_B + \langle [\ ]_D | \,, \\
\langle \widetilde{S}_R^c | &= -\frac{1}{g_s^2} \langle \widetilde{\Psi}^c | Q_B + \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \langle [(\Psi^c)^N ; (\Psi^o)^M]^c | \,, \\
\langle S_R^o | &= -\frac{1}{g_s} \langle \widetilde{\Psi}^o | Q_B \mathcal{G} + \frac{1}{g_s} \langle \Psi^o | Q_B \,, \\
\langle \widetilde{S}_R^o | &= \frac{1}{g_s} \langle \widetilde{\Psi}^o | Q_B + \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \langle [(\Psi^c)^N ; (\Psi^o)^M]^o | \,,
\end{aligned}
\tag{4.13}
$$

in the convention that the operators Q_B and $\mathcal{G}$ act on the right even though they are to the right of a bra state. Using (4.8), (1.5), (1.15) and $Q_B^2 = 0$, we now get

$$
\begin{aligned}
-(S_{1PI}, S_{1PI}) &= 2 \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \{ Q_B \Psi^c (\Psi^c)^N ; (\Psi^o)^M \} \\
&\quad + 2 \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \{ (\Psi^c)^N ; Q_B \Psi^o (\Psi^o)^M \} \\
&\quad + g_s^2 \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \sum_{P=0}^{\infty} \sum_{Q=0}^{\infty} \frac{1}{N!M!P!Q!} \{ \mathcal{G} [(\Psi^c)^N ; (\Psi^o)^M]^c (\Psi^c)^P ; (\Psi^o)^Q \}
\end{aligned}
$$

$$+g_s \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \sum_{P=0}^{\infty} \sum_{Q=0}^{\infty} \frac{1}{N!M!P!Q!} \{(\Psi^c)^P; \mathcal{G}[(\Psi^c)^N; (\Psi^o)^M]^o (\Psi^o)^Q\}$$

$$+2\, g_s^2 \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \{[(\Psi^c)^N; (\Psi^o)^M]^c\}_D \,. \tag{4.14}$$

If we specialize the main identity (1.16) to the case

$$\begin{aligned} A_i^c &= \Psi^c, & i = 1, \cdots, N, \\ A_i^o &= \Psi^o, & i = 1, \cdots, M, \end{aligned} \tag{4.15}$$

we obtain

$$N\{Q_B \Psi^c (\Psi^c)^{N-1}; (\Psi^o)^M\} + M\{(\Psi^c)^N; Q_B \Psi^o (\Psi^o)^{M-1}\}$$

$$= -\frac{1}{2} \sum_{k=0}^{N} \sum_{\ell=0}^{M} \binom{N}{k} \binom{M}{\ell} \Big(g_s^2 \{(\Psi^c)^k \mathcal{G}[(\Psi^c)^{N-k}; (\Psi^o)^{M-\ell}]^c; (\Psi^o)^\ell\}$$

$$+ g_s \{(\Psi^c)^k; \mathcal{G}[(\Psi^c)^{N-k}; (\Psi^o)^{M-\ell}]^o (\Psi^o)^\ell\} \Big)$$

$$- g_s^2 \{[(\Psi^c)^N; (\Psi^o)^M]^c\}_D \,. \tag{4.16}$$

Multiplying by $\frac{1}{N!M!}$ and summing over N and M from 0 to ∞, we get

$$0 = 2 \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \{Q_B \Psi^c (\Psi^c)^N; (\Psi^o)^M\} + 2 \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \{(\Psi^c)^N; Q_B \Psi^o (\Psi^o)^M\}$$

$$+ \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \sum_{k=0}^{N} \sum_{\ell=0}^{M} \frac{1}{(N-k)!(M-\ell)!k!\ell!} \Big(g_s^2 \{(\Psi^c)^k \mathcal{G}[(\Psi^c)^{N-k}; (\Psi^o)^{M-\ell}]^c; (\Psi^o)^\ell\}$$

$$+ g_s \{(\Psi^c)^k; \mathcal{G}[(\Psi^c)^{N-k}; (\Psi^o)^{M-\ell}]^o (\Psi^o)^\ell\} \Big)$$

$$+ 2 g_s^2 \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \frac{1}{N!M!} \{[(\Psi^c)^N; (\Psi^o)^M]^c\}_D \,. \tag{4.17}$$

By redefining the sums in the second line and comparing with the expression of anti-bracket (S_{1PI}, S_{1PI}), we see that

$$(S_{1PI}, S_{1PI}) = 0 \,. \tag{4.18}$$

Therefore the 1PI effective action satisfies the classical BV master equation.

The BV formalism also gives a way to derive the gauge transformation laws (1.14). The classical BV master action — and therefore also the 1PI effective action — is known to be invariant under gauge transformations that transform any function F of the string fields $|\Psi^c\rangle$, $|\widetilde{\Psi}^c\rangle$, $|\Psi^o\rangle$ and $|\widetilde{\Psi}^o\rangle$ as [5],

$$\delta F = (F, (S, \Lambda)) \,, \tag{4.19}$$

where Λ is any even function of the fields. Choosing

$$\Lambda = g_s^{-2} \Big(\langle \widetilde{\Psi}^c | c_0^- | \Lambda^c \rangle + \langle \Psi^c | c_0^- | \widetilde{\Lambda}^c \rangle - \langle \widetilde{\Psi}^c | c_0^- \mathcal{G} | \widetilde{\Lambda}^c \rangle \Big) + g_s^{-1} \Big(\langle \widetilde{\Psi}^o | \Lambda^o \rangle + \langle \Psi^o | \widetilde{\Lambda}^o \rangle - \langle \widetilde{\Psi}^o | \mathcal{G} | \widetilde{\Lambda}^o \rangle \Big) \,, \tag{4.20}$$

we reproduce the gauge transformation laws given in (1.14).

5 Quantum BV master action

We can also write down the quantum BV master action for the combined open closed string field theory following the procedure reviewed in [1]. It is given by

$$
S_{BV} = -\frac{1}{2g_s^2}\langle\tilde{\Psi}^c|c_0^- Q_B\mathcal{G}|\tilde{\Psi}^c\rangle \ + \frac{1}{g_s^2}\langle\tilde{\Psi}^c|c_0^- Q_B|\Psi^c\rangle - \frac{1}{2g_s}\langle\tilde{\Psi}^o|Q_B\mathcal{G}|\tilde{\Psi}^o\rangle + \frac{1}{g_s}\langle\tilde{\Psi}^o|Q_B|\Psi^o\rangle
$$

$$
+ \{\!\!\{\tilde{\Psi}^c\}\!\!\}_D + \sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{N!M!}\{\!\!\{(\Psi^c)^N;(\Psi^o)^M\}\!\!\}\,, \tag{5.1}
$$

where $\{\!\!\{\cdots\}\!\!\}$, to be defined shortly, denotes the contribution to the off shell amplitude due to the elementary interaction vertices of superstring field theory. $\{\!\!\{\widetilde{A}^c\}\!\!\}_D$ is defined exactly in the same way as $\{\widetilde{A}^c\}_D$. $\{\!\!\{\cdots\}\!\!\}$ is defined in a way similar to $\{\cdots\}$ given in (2.11), except that the region of integration $\mathcal{R}_{g,b,N,M}$ is replaced by a smaller region $\overline{\mathcal{R}}_{g,b,N,M}$ defined as follows. Recall that we determine $\mathcal{R}_{g,b,N,M}$ by demanding that $\mathcal{R}_{g,b,N,M}$'s, together with all section segments generated from $\mathcal{R}_{g',b',N',M'}$'s by repeated application of hole creation and sewing punctures on different Riemann surfaces, generate complete generalized sections of $\widetilde{\mathcal{P}}_{g,b,N,M}$'s whose bases cover the full moduli spaces $\mathcal{M}_{g,b,N,M}$. For $\overline{\mathcal{R}}_{g,b,N,M}$ we make a similar demand, except that we now also allow sewing two punctures on the same Riemann surface via the sewing relations (2.7) or (2.8). A systematic procedure for constructing the $\overline{\mathcal{R}}_{g,b,N,M}$'s can be developed along the same lines as for the $\mathcal{R}_{g,b,N,M}$'s, as described in section 2. We begin with the dimension zero $\overline{\mathcal{R}}_{g,b,N,M}$'s, which can be taken to be identical to the dimension zero $\mathcal{R}_{g,b,N,M}$'s, and then begin building higher dimensional section segments from the lower dimensional ones by sewing and hole creation operations described in (2.7), (2.8) and (2.9). The only difference from the corresponding procedure for the construction of $\mathcal{R}_{g,b,N,M}$'s is that we also allow sewing of punctures on the same Riemann surface. After constructing all the section segments for a given g, b, N, M this way, we 'fill the gap' by $\overline{\mathcal{R}}_{g,b,N,M}$ so as to generate a full generalized section of $\widetilde{\mathcal{P}}_{g,b,N,M}$.

The quantum BV master action (5.1), constructed this way, satisfies the quantum BV master equation:

$$
\frac{1}{2}\left(S_{BV}, S_{BV}\right) + \Delta S_{BV} = 0\,, \tag{5.2}
$$

where the anti-bracket $(,)$ has been defined in (4.6), and

$$
\Delta S_{BV} = \frac{\partial_R}{\partial\psi^r}\frac{\partial_L S_{BV}}{\partial\psi_r^*}\,. \tag{5.3}
$$

Here ψ^r stand for all the fields $(\psi^c)^r$, $(\widetilde{\psi}^c)^r$, $(\psi^o)^r$ and $(\widetilde{\psi}^o)^r$ and ψ_r^* stand for all the anti-fields $(\psi^c)_r^*$, $(\widetilde{\psi}^c)_r^*$, $(\psi^o)_r^*$ and $(\widetilde{\psi}^o)_r^*$. $-(S_{BV}, S_{BV})$ is given by the right hand side of (4.14) with $\{\ \}$ replaced by $\{\!\!\{\ \}\!\!\}$. Using (5.3) one finds that ΔS is given by:

$$
\Delta S = -\frac{1}{2}g_s^2\sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{N!M!}\{\!\!\{(\Psi^c)^N\varphi_s\varphi_r;(\Psi^o)^M\}\!\!\}\langle\tilde{\varphi}^s|c_0^-\mathcal{G}|\tilde{\varphi}^r\rangle
$$

$$
-\frac{1}{2}g_s(-1)^{\phi_s}\sum_{N=0}^{\infty}\sum_{M=0}^{\infty}\frac{1}{N!M!}\{\!\!\{(\Psi^c)^N;\phi_s\phi_r(\Psi^o)^M\}\!\!\}\langle\tilde{\phi}^s|\mathcal{G}|\tilde{\phi}^r\rangle\,, \tag{5.4}
$$

where $|\varphi_r\rangle$, $|\tilde{\varphi}_r\rangle$, $|\phi_r\rangle$ and $|\tilde{\phi}_r\rangle$ are the basis states in $\mathcal{H}^c$, $\widetilde{\mathcal{H}}^c$, $\mathcal{H}^o$ and $\widetilde{\mathcal{H}}^o$, normalized according to (A.2), (A.9), and $(-1)^{\phi_s}$ denotes grassmann parity of the state $|\phi_s\rangle$. The BV master equation (5.2) can be proved using a modified version of the main identity for $\{\!\{\ \}\!\}$:

$$
\sum_{i=1}^{N} \{\!\{A_1^c \cdots, A_{i-1}^c (Q_B A_i^c) A_{i+1}^c \cdots A_N^c | A_1^o \cdots, A_M^o\}\!\}
$$

$$
+ \sum_{j=1}^{M} (-1)^{j-1} \{\!\{A_1^c \cdots A_N^c | A_1^o \cdots A_{j-1}^o (Q_B A_j^o) A_{j+1}^o \cdots A_M^o\}\!\}
$$

$$
= -\frac{1}{2} \sum_{k=0}^{N} \sum_{\{i_1,\cdots,i_k\}\subset\{1,\cdots,N\}} \sum_{\ell=0}^{M} \sum_{\{j_1,\cdots,j_\ell\}\subset\{1,\cdots,M\}} \Big(g_s^2 \{\!\{A_{i_1}^c \cdots, A_{i_k}^c \mathcal{B}^c | A_{j_1}^o \cdots, A_{j_\ell}^o\}\!\}
$$

$$
+ g_s \{\!\{A_{i_1}^c \cdots, A_{i_k}^c | \mathcal{B}^o A_{j_1}^o \cdots, A_{j_\ell}^o\}\!\} \Big)
$$

$$
- g_s^2 \{\!\{[A_1^c \cdots A_N^c ; A_1^o \cdots A_M^o]_D\}\!\} - \frac{1}{2} g_s^2 \{\!\{A_1^c \cdots A_N^c \varphi_s \varphi_r ; A_1^o \cdots, A_M^o\}\!\} \langle \tilde{\varphi}^s | c_0^- \mathcal{G} | \tilde{\varphi}^r \rangle
$$

$$
- \frac{1}{2} g_s (-1)^{\phi_s} \{\!\{A_1^c \cdots A_N^c ; \phi_s \phi_r A_1^o \cdots, A_M^o\}\!\} \langle \tilde{\phi}^s | \mathcal{G} | \tilde{\phi}^r \rangle. \tag{5.5}
$$

The proof of this follows the same analysis as used in section 2 and appendix A for the proof of (1.16). The last two terms arise due to the fact that $\overline{\mathcal{R}}_{g,b,N,M}$ has two extra sets of boundaries compared to $\mathcal{R}_{g,b,N,M}$, where two bulk punctures or two boundary punctures on a lower dimensional $\overline{\mathcal{R}}_{g',b',N',M'}$ are sewed via (2.7) or (2.8) with $s = 0$.

Given a set of $\overline{\mathcal{R}}_{g,b,N,M}$'s satisfying the necessary conditions, we can construct a set of $\mathcal{R}_{g,b,N,M}$'s by sewing the $\overline{\mathcal{R}}_{g',b',N',M'}$'s with each other / itself via the sewing operations (2.7) and / or (2.8), subject to the constraint that if we omit one such operation, the Riemann surface should not become disconnected. This is precisely the way we build the 1PI amplitudes from elementary vertices using Feynman diagram, with the sewing playing the role of joining vertices by propagators. $\mathcal{R}_{g,b,N,M}$'s constructed this way automatically satisfy the required conditions described in section 2.

6 Unoriented open-closed string field theory

Our construction of the 1PI effective action or BV master action holds for any superconformal field theory that we use to compute the correlation functions of vertex operators that enter the definition of $\{A_1^c \cdots A_N^c ; A_1^o \cdots A_M^o\}$ and $\{\tilde{A}^c\}_D$. Therefore the same construction is valid for any compactification of type IIA or type IIB superstring field theory involving NSNS background. The construction should also generalize to orientifolds where we have unoriented strings, but there will be a few differences. The main difference will be that the definitions of $\mathcal{H}^c$, $\widetilde{\mathcal{H}}^c$, $\mathcal{H}^o$ and $\widetilde{\mathcal{H}}^o$ will automatically include projection by the appropriate orientifold operation. Therefore the sewing and hole creation operation will also have this projection operator. For consistency, now we must also include non-orientable Riemann surfaces in the construction of superstring field theory interaction vertex, — if we start with an oriented Riemann surface and sew two of its punctures with the orientifold projection inserted, we shall generate a non-orientable surface. Together, the oriented and

non-orientable surfaces that will be relevant for us are known as Klein surfaces. Review of the essential results that we shall need can be found in [42–44] and has been summarized in appendix B.

Due to the inclusion of non-orientable surfaces, we encounter a few differences from the corresponding results in oriented open-closed string field theory.

1. Off-shell amplitudes now include sum over the moduli spaces of oriented and non-orientable surfaces. The oriented surfaces are as usual characterized by their genus g and the number of boundaries b, while the non-orientable surfaces are characterized by the number of crosscaps c and the number of boundaries b [42], with the Euler character given by $b+c-2$. Therefore in the definition (2.11) of $\{\cdots\}$, the contribution from the non-orientable surfaces must be weighted by a factor of $(g_s)^{c+b-2}$, whereas the contribution from the oriented surfaces continue to be weighted by $(g_s)^{2g+b-2}$. We shall often express such factors as $(g_s)^{2g+b+c-2}$, with the understanding that oriented surfaces will have $c=0$ and non-orientable surfaces will have $g=0$.

2. We shall now show that in the definition of $\{\cdots\}$ given in (2.11) we must also include an extra factor of

$$2^{-g-(c+b)/2+M/4}\,, \tag{6.1}$$

if we normalize $\Omega_{g,b,c,N,M}$ in the same way as in the case of oriented string theories, e.g. for $c=0$, Ω is defined with the same normalization as for oriented string theory. As we shall explain below, this extra factor (6.1) is needed to ensure that amplitudes factorize correctly near degeneration.[8]

First let us consider the effect of sewing two bulk punctures on an oriented surface. Now the sewing has a projection operator $P = (1+W)/2$ where W is the operation of world-sheet orientation reversal, possibly accompanied by some action on the space-time. Insertion of 1 corresponds to the usual sewing via (2.7) and produces a handle. The resulting Riemann surface is an oriented Riemann surface with two less closed string punctures and one additional genus compared to the original Riemann surface. b, M and $c(= 0)$ remain fixed under this operation. Under this change (6.1) picks up a factor of $1/2$. This correctly accounts for the factor of $1/2$ that appears in the projection operator $(1 + W)/2$.

On the other hand insertion of W changes one of the local coordinates in (2.7) to its complex conjugate. Therefore the sewing relation takes the form

$$z\bar{w} = e^{-s-i\theta}\,. \tag{6.2}$$

This is known as a cross handle. If the original Riemann surface had genus g, then this operation produces a non-orientable Klein surface with $2g + 2$ crosscaps. M and

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[8]The presence of the M dependent factor may be understood as follows. Since we have a factor of $2^{-b/2}$, and since the open string kinetic term involves a disc amplitude, it would be natural to multiply the open string kinetic term by a factor of $1/\sqrt{2}$. However we can remove this factor by scaling each open string field by $2^{1/4}$. This introduces the factor of $2^{M/4}$ in the definition of the interaction terms. There is no such factor for closed strings since the kinetic term is a genus 0 amplitude, and for this there is no additional factor of $1/2$.

b remain unchanged. It is easy to verify that (6.1) changes by a factor of $1/2$ under this operation. This again correctly accounts for the factor of $1/2$ in the projection operator.

If the original surface was non-orientable, with c crosscaps, then the effect of sewing two of its bulk punctures may be analyzed in a similar manner. We again have the projection operator $(1 + W)/2$ with 1 corresponding to sewing with a handle and W corresponding to sewing with a cross handle. Both operations increase the number of crosscaps by 2, leaving fixed $g(= 0)$, b and M. Under this (6.1) picks a factors of $1/2$, correctly accounting for the $1/2$ in the projection operator.

Next consider the effect of sewing a pair of boundary punctures of a Klein surface. Let us for definiteness consider the case where the two punctures lie on the same boundary. Again sewing introduces a factor of $(1 + W)/2$. For the term proportional to 1, the sewing reduces the number M of open string punctures by 2 and increases the number b of boundaries by 1. Under such changes, (6.1) picks up a factor of $1/2$, correctly accounting for the factor of $1/2$ in the projection operator. On the other hand if we pick the term proportional to W, then this reduces the number M of open string punctures by 2 and increases the number of crosscaps by 1, leaving the number of boundaries unchanged. Under such a change also (6.1) picks up the desired factor of $1/2$. Similar analysis can be done for the sewing of a pair of boundary punctures lying on two different boundaries.

It is easy to verify that (6.1) is also compatible with separating type degenerations. For example when we sew two oriented surfaces along bulk punctures, the total numbers for g, c, b and M all remain unchanged and therefore there is no change in (6.1). On the other hand in this case even though sewing introduces a factor of $(1 + W)/2$, both 1 and W produce the same set of oriented surfaces and therefore there is no factor of $1/2$. Similar agreement can be shown for sewing of boundary punctures. A description of different kinds of sewing that can arise in unoriented open closed string field theory can be found in [43] and reviewed in appendix B.

3. Finally, in defining $\{\widetilde{A}^c\}_D$ we must now include not only one point function of $c_o^- \widetilde{A}^c$ on the disc, but also on RP^2, together with an insertion of $\widehat{\mathcal{G}}$ as before. The latter is similar to one point function on the disc, but with the boundary condition on the disc replaced by a crosscap. In keeping with our discussion above we must accompany each crosscap and disc by a factor of $1/\sqrt{2}$ by including it in the definition of $\{\widetilde{A}^c\}_D$.

4. The choice of local coordinates at the punctures and PCO locations must be compatible with the orientifold projection.

5. In the presence of one or more crosscaps, only the total picture number is conserved but the holomorphic and anti-holomorphic picture numbers are not separately conserved. Therefore the corresponding $\widetilde{\mathcal{P}}_{0,b,c,N,M}$ will have multiple branches. We can jump between the branches by moving the PCOs to the crosscap via vertical segments

and converting holomorphic PCOs into anti-holomorphic PCOs or vice versa using the boundary condition on the crosscap.

With these few changes, the construction of $\mathcal{R}_{g,b,c,N,M}$ for unoriented open-closed string field theory proceeds in the same way as in the case of oriented strings, beginning with zero dimensional $\mathcal{R}_{g,b,c,N,M}$'s. Eqs. (2.11) and (2.3) are generalized to:

$$\{A_1^c \cdots A_N^c; A_1^o \cdots A_M^o\} \equiv \sum_{\substack{g,b,c \geq 0 \\ (g,b,c) \neq (0,1,0),(0,0,1) \text{ for } (N,M)=(1,0)}} (g_s)^{2g-2+b+c} 2^{-g-(b+c)/2+M/4}$$

$$\int_{\mathcal{R}_{g,b,c,N,M}} \Omega_{6g-6+3b+3c+2N+M}^{g,b,c,N,M}(A_1^c, \cdots, A_N^c; A_1^o, \cdots, A_M^o), \quad (6.3)$$

and,

$$[]_D = \frac{1}{\sqrt{2}} \mathbf{P} \,\widehat{\mathcal{G}} \left\{ e^{-\beta_b(L_0+\bar{L}_0)}|B\rangle + e^{-\beta_c(L_0+\bar{L}_0)}|C\rangle \right\}, \quad (6.4)$$

where β_b and β_c are positive constants and $|B\rangle$ and $|C\rangle$ are the boundary states for the disc and the crosscap. The form of the action, various identities described in section 1, the gauge transformation laws and the definition of the anti-bracket remains the same. We can also construct quantum BV master action by replacing $\mathcal{R}_{g,b,c,N,M}$'s by $\overline{\mathcal{R}}_{g,b,c,N,M}$'s that satisfy slightly different constraints as described in section 5.

Acknowledgments

A Signs of the terms in the 'main identity'

In this appendix we shall determine the signs of various terms on the right hand side of the main identity (1.17), following the strategy outlined at the end of section 2. We shall do the analysis iteratively in the dimension of $\mathcal{R}_{g,b,N,M}$ (or equivalently $\mathcal{M}_{g,b,N,M}$) associated with the interaction vertex, which we shall simply refer to as the dimension of the interaction vertex. Therefore we shall assume that (1.17) holds for vertices carrying dimension $\leq K$ and prove that it holds for vertices of dimension $K+1$.

In the following we shall analyze the behaviour of the amplitude near the boundary of the moduli space using the language of string field theory. This may give the impression that our argument is circular, i.e. we use string field theory to prove relations among string field theory interaction vertices. However the factorization property of the string amplitude
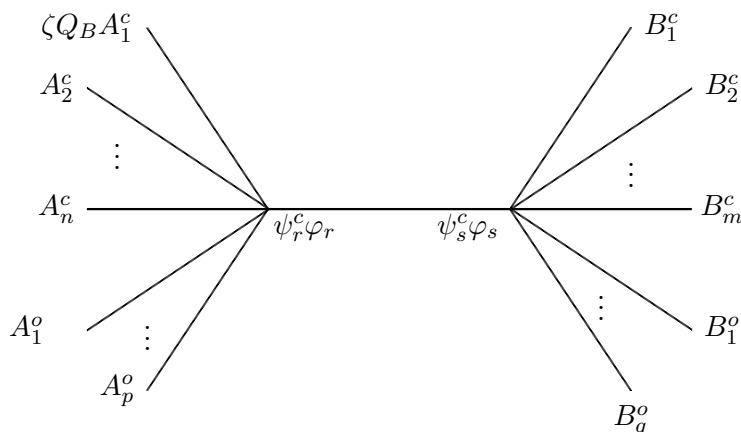
Figure 1. A Feynman diagram where a 1PI vertex with the external states $\zeta Q_B A_1^c, A_2^c \cdots,$ $A_n^c, A_1^o \cdots A_p^o$ and an internal closed string state and another 1PI vertex with the external states $B_1^c, \cdots, B_m^c, B_1^o, \cdots B_q^o$ and an internal closed string state are joined by a closed string propagator.

that we use is known to hold independently of string field theory, and tells us that the contribution to an amplitude near a separating type degeneration has the interpretation of the contribution from a Feynman diagram where two different diagrams are connected by an internal line. This has been shown in figure 1. Therefore our argument does not use any essential element of string field theory. We shall also use gauge invariance of the amplitude, that tells us that if in an amplitude we act ζQ_B in turn on each external state, the result vanishes. This property of the amplitude also follows from standard world-sheet analysis and does not rely on the existence of an underlying string field theory.

Let us consider an amplitude with external closed string states $A_1^c, \cdots A_n^c, B_1^c, \cdots B_m^c$ and external open string states $A_1^o, \cdots, A_p^o, B_1^o, \cdots, B_q^o$, with ζQ_B acting in turn on each external state. Gauge invariance requires this amplitude to vanish. However the contribution to this amplitude from the 1PI vertex does not vanish — rather it is given by the right hand side of the main identity (1.17).[9] Let us first focus on the particular contribution:

$$-g_s^2 \left\{ A_1^c \cdots A_n^c \, \mathcal{G} \zeta[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c; A_1^o \cdots A_p^o \right\}, \tag{A.1}$$

on the right hand side of (1.17). This is expected to get cancelled against the contribution from the Feynman diagram where a 1PI vertex with the external states $A_1^c, \cdots, A_n^c, A_1^o \cdots A_p^o$ and an internal closed string state and another 1PI vertex with the external states $B_1^c, \cdots, B_m^c, B_1^o, \cdots B_q^o$ and an internal closed string state are joined by a closed string propagator, and ζQ_B acts in turn on each of the $A_i^{c,o}$'s and $B_i^{c,o}$'s. Figure 1 shows one of these Feynman diagrams where ζQ_B acts on A_1^c. Our strategy will be to evaluate these Feynman diagrams explicitly and use this to test the sign in (A.1).

We shall first compute the contribution to the amplitude from the Feynman diagram when ζQ_B acts on A_1^c, as shown in figure 1. Let us denote by $|\varphi_r\rangle$ the basis states in $\mathcal{H}^c$

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[9]As in [1] we shall follow the convention that the 1PI vertex contributes to the amplitude without any sign factor.

and by $|\tilde{\varphi}^r\rangle$ the conjugate basis in $\widetilde{\mathcal{H}}^c$, satisfying

$$\langle\tilde{\varphi}^r|c_0^-|\varphi_s\rangle = \delta_s^r = \langle\varphi_s|c_0^-|\tilde{\varphi}^r\rangle, \quad |\varphi_r\rangle\langle\tilde{\varphi}^r| = b_0^- = |\tilde{\varphi}^r\rangle\langle\varphi_r|, \tag{A.2}$$

so that the closed string field can be expanded as $\psi_r^c|\varphi_r\rangle$. Then the product of the two vertex factors in figure 1 can be expressed as:

$$\{(\zeta Q_B A_1^c)\cdots A_n^c \psi_r^c\varphi_r; A_1^o \cdots A_p^o\}\{\psi_s^c\varphi_s B_1^c \cdots B_m^c; B_1^o \cdots B_q^o\}. \tag{A.3}$$

We shall eventually get a propagator by contracting ψ_r^c and ψ_s^c. Therefore ψ_r^c and ψ_s^c must have same grassmann parity. Since the closed string field is grassmann even, $|\varphi_r\rangle$ and $|\varphi_s\rangle$ will also have the same grassmann parities as ψ_r^c and ψ_s^c. We shall denote this grassmann parity by $(-1)^{\varphi_r}$. We now pull the ψ_r^c from the $\psi_r^c|\varphi_r\rangle$ factor inside the first vertex and ψ_s^c from the $\psi_s^c|\varphi_s\rangle$ factor inside the second vertex to the left outside the respective $\{\cdots\}$'s. This does not generate any sign since the A_i^c's are grassmann even. ψ_r^c and ψ_s^c are now separated by $\{(\zeta Q_B A_1^c)\cdots A_n^c\varphi_r; A_1^o \cdots A_p^o\}$. This has grassmann parity $(-1)^{\varphi_r}$. Therefore the contraction of ψ_r^c and ψ_s^c gives a factor of $(-1)^{\varphi_r}\Delta_{rs}$ where Δ_{rs} is the propagator.

Δ_{rs} can be computed as follows. In the kinetic term, we expand $\langle\Psi^c|$ as $\psi_r^c\langle\varphi_r|$ and pull ψ_r^c to the left without picking any sign, and express $|\Psi^c\rangle$ as $\psi_s^c|\varphi_s\rangle$ and pull ψ_s^c to the left, picking a factor of $(-1)^{\varphi_r}$ since it has to pass through $\langle\varphi_r|c_0^-Q_B$ which has the same grassmann parity as ψ_s^c. Similar operation can be done for fields in $\widetilde{\mathcal{H}}^c$. Furthermore in the Siegel gauge we can replace Q_B by $(c_0L_0 + \bar{c}_0\bar{L}_0)$. Inverting the kinetic operator we now get a ψ_r-ψ_s propagator $\Delta_{rs} = -g_s^2(-1)^{\varphi_r}\langle\tilde{\varphi}^r|c_0^-b_0^+\mathcal{G}(L_0^+)^{-1}|\tilde{\varphi}^s\rangle$ [1].[10] After cancelling the $(-1)^{\varphi_r}$ factor in Δ_{rs} with the $(-1)^{\varphi_r}$ factor coming from the contraction of ψ_r and ψ_s across $\{(\zeta Q_B A_1^c)\cdots A_n^c\varphi_r; A_1^o \cdots A_p^o\}$, we arrive at the following expression for the amplitude shown in figure 1:

$$-g_s^2\{(\zeta Q_B A_1^c)\cdots A_n^c\varphi_r; A_1^o \cdots A_p^o\}\langle\tilde{\varphi}^r|c_0^-b_0^+\mathcal{G}(L_0^+)^{-1}|\tilde{\varphi}^s\rangle\{\varphi_s B_1^c \cdots B_m^c; B_1^o \cdots B_q^o\}. \tag{A.4}$$

We now use (1.5) to express the last factor as $\langle\varphi_s|c_0^-|[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c\rangle$ and then perform the sum over s in (A.4) using the completeness relation to express the product of the last two factors as $\langle\tilde{\varphi}^r|c_0^-b_0^+\mathcal{G}(L_0^+)^{-1}|[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c\rangle$. On the other hand the first $\{\cdots\}$ factor in (A.4) can be expressed as

$$\langle\varphi_r|c_0^-[(\zeta Q_B A_1^c)\cdots A_n^c; A_1^o \cdots A_p^o]^c\rangle = \langle[(\zeta Q_B A_1^c)\cdots A_n^c; A_1^o \cdots A_p^o]^c|c_0^-|\varphi_r\rangle, \tag{A.5}$$

where we have used the fact that $[(\zeta Q_B A_1^c)\cdots A_n^c; A_1^o \cdots A_p^o]^c$ is grassmann odd. We can now perform the sum over r in (A.4) using completeness relation and express (A.4) as

$$-g_s^2\langle[(\zeta Q_B A_1^c)\cdots A_n^c; A_1^o \cdots A_p^o]^c|c_0^-b_0^+\mathcal{G}(L_0^+)^{-1}|[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c\rangle$$
$$= -g_s^2\langle b_0^+\mathcal{G}(L_0^+)^{-1}[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c|c_0^-|[(\zeta Q_B A_1^c)\cdots A_n^c; A_1^o \cdots A_p^o]^c\rangle$$
$$= -g_s^2\{(\zeta Q_B A_1^c)\cdots A_n^c b_0^+\mathcal{G}(L_0^+)^{-1}[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c; A_1^o \cdots A_p^o\}, \tag{A.6}$$

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[10]In computing the propagator, we only use the terms involving Q_B as kinetic operator and treat the quadratic and linear terms in $\{\cdots\}$ as interaction terms. This allows us to make contact with the world-sheet formulation, although in order to correctly compute the renormalized masses we have to work with the full quadratic term. In [1] the propagator was written as $-g_s^2\langle\tilde{\varphi}^s|c_0^-b_0^+\mathcal{G}(L_0^+)^{-1}|\tilde{\varphi}^r\rangle$. This can be shown to be equal to $-g_s^2(-1)^{\varphi_r}\langle\tilde{\varphi}^r|c_0^-b_0^+\mathcal{G}(L_0^+)^{-1}|\tilde{\varphi}^s\rangle$.

where in the first step we have used the fact that both factors of $[\cdots]^c$ are grassmann odd and in the last step we have used (1.5).

Similar expressions can be obtained when ζQ_B acts in turn on the other A_i^c's and A_i^o's. Using the main identity for lower number of external states, which we are allowed to use in a recursive proof, the effect of acting with ζQ_B in turn on the $B_i^{c,o}$'s can be represented by ζQ_B acting on $[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c$. There are additional terms corresponding to the right hand side of the main identity with $B_i^{c,o}$'s as external states but these would cancel against other Feynman diagrams with two propagators, where the right vertex of figure 1 is replaced by sub-diagram containing an additional propagator. This brings the sum of the Feynman diagrams of the form shown in figure 1 to a form where ζQ_B acts in turn on all the $A_i^{c,o}$'s and on $[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c$ in the right hand side of (A.6). We again use the main identity on this expression and throw away the right hand side which would cancel against contribution from Feynman diagrams where the left vertex in figure 1 will be replaced by sub-diagrams with an additional propagator. The left over term is given by $-\zeta Q_B$ acting on $b_0^+ \mathcal{G}(L_0^+)^{-1}$ in (A.6), producing just a factor of $-\mathcal{G}\zeta$. This brings the relevant contribution from the Feynman diagram with one closed string propagator to the form:

$$g_s^2 \{ A_1^c \cdots A_n^c \, \mathcal{G}\zeta[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^c; A_1^o \cdots A_p^o \}. \tag{A.7}$$

This cancels (A.1), confirming that the sign in (A.1) is correct.

Next we consider the contribution:

$$- g_s \{ A_1^c \cdots A_n^c; \mathcal{G}\zeta[B_1^c \cdots B_m^c; B_1^o \cdots B_q^o]^o A_1^o \cdots A_p^o \}, \tag{A.8}$$

arising from the right hand side of (1.17). In the Ward identity for the amplitude, this is expected to get cancelled against the contribution from the sum of Feynman diagrams of the same type as the ones in figure 1, except that the internal line is now an open string instead of a closed string. To evaluate the contribution from the Feynman diagram we first introduce conjugate pair of basis states $|\phi_r\rangle \in \mathcal{H}^o, |\tilde{\phi}^r\rangle \in \widetilde{\mathcal{H}}^o$, satisfying

$$\langle \tilde{\phi}^r | \phi_s \rangle = \delta_s^r = \langle \phi_s | \tilde{\phi}^r \rangle, \quad |\phi_r\rangle\langle \tilde{\phi}^r| = 1 = |\tilde{\phi}^r\rangle\langle \phi_r|. \tag{A.9}$$

Let us first consider the case where ζQ_B acts on A_1^c. We proceed as in the earlier case, by first writing down the product of the two vertex factors:

$$\{(\zeta Q_B A_1^c)\cdots A_n^c; \psi_r^o \phi_r \, A_1^o \cdots A_p^o\}\{B_1^c \cdots B_m^c; \psi_s^o \phi_s \, B_1^o \cdots B_q^o\}. \tag{A.10}$$

Following the same approach as in the case of (A.3) we can prove that ϕ_r, ϕ_s have the same grassmann parity $(-1)^{\phi_r}$, the open string fields ψ_r^o and ψ_s^o multiplying them have grassmann parity $(-1)^{\phi_r+1}$, and $\{(\zeta Q_B A_1^c)\cdots A_n^c; \phi_r A_1^o \cdots A_p^o\}$ and $\{B_1^c \cdots B_m^c; \phi_s B_1^o \cdots B_q^o\}$ have grassmann parity $(-1)^{\phi_r+1}$. We can pull the ψ_r^o and ψ_s^o factors from the two amplitudes in (A.10) to the left outside $\{\cdots\}$ without picking any sign. The contraction of ψ_r^o with ψ_s^o produces a factor of $(-1)^{\phi_r+1}$ besides the propagator, since they are separated by $\{(\zeta Q_B A_1^c)\cdots A_n^c; \phi_r A_1^o \cdots A_p^o\}$. The propagator is obtained as follows. If we express $\langle \Psi^o|$ and $|\Psi^o\rangle$ (and similarly $\langle \widetilde{\Psi}^o|$) and $|\widetilde{\Psi}^o\rangle$) in the kinetic term as $\psi_r^o\langle\phi_r|$ and $\psi_s^o|\phi_s\rangle$ and then

pull ψ_r^o and ψ_s^o to the left, we get a factor of $(-1)^{\phi_r+1}$ from having to pass ψ_s^o through $\langle\phi_r|Q_B$. After replacing Q_B by $c_0 L_0$ in the Siegel gauge and inverting the kinetic operator, we get a ψ_r^o-ψ_s^o propagator of the form $-g_s(-1)^{\phi_r+1}\langle\tilde{\phi}^r|b_0\mathcal{G}(L_0)^{-1}|\tilde{\phi}^s\rangle$. Cancelling the $(-1)^{\phi_r+1}$ factor from the propagator with the $(-1)^{\phi_r+1}$ factor coming from the contraction, we arrive at the expression:

$$- g_s \{(\zeta Q_B A_1^c)\cdots A_n^c; \phi_r A_1^o \cdots A_p^o\}\langle\tilde{\phi}^r|b_0\mathcal{G}(L_0)^{-1}|\tilde{\phi}^s\rangle\{B_1^c\cdots B_m^c; \phi_s B_1^o\cdots B_q^o\}. \quad \text{(A.11)}$$

The rest of the analysis proceeds almost in the same way as for (A.4), with the $[\cdots]^c$'s replaced by $[\cdots]^o$, c_0^-'s removed, b_0^+ replaced by b_0 and φ_r, $\tilde{\varphi}_r$ replaced by ϕ_r, $\tilde{\phi}_r$. Using the fact that $[(\zeta Q_B A_1^c)\cdots A_n^c; A_1^o\cdots A_p^o]^o$ is grassmann even, we can verify that there are no extra signs compared to those that appeared in the earlier analysis. The final result takes the form analogous to (A.6)

$$- g_s\{(\zeta Q_B A_1^c)\cdots A_n^c; b_0\mathcal{G}(L_0)^{-1}[B_1^c\cdots B_m^c; B_1^o\cdots B_q^o]^o A_1^o\cdots A_p^o\}. \quad \text{(A.12)}$$

We now sum over terms where ζQ_B acts in turn on all the external states. Each of these terms can be analyzed in the same way, leading to expressions similar to (A.12) with the position of ζQ_B shifted. Using the main identity for vertices of lower dimension and throwing away terms that would cancel with Feynman diagrams with two propagators, we can, as in the case of (A.6), express the sum of these terms as the negative of the term where ζQ_B acts on $b_0\mathcal{G}(L_0)^{-1}$, producing just a factor of $\mathcal{G}\zeta$. This brings the boundary contribution to the form:

$$g_s\{A_1^c\cdots A_n^c; \mathcal{G}\zeta[B_1^c\cdots B_m^c; B_1^o\cdots B_q^o]^o A_1^o\cdots A_p^o\}. \quad \text{(A.13)}$$

This cancels (A.8), confirming that the sign in (A.8) is correct.

Finally we turn to the contribution

$$- g_s^2\{\zeta[A_c^1\cdots A_n^c B_1^c\cdots B_m^c; A_1^o\cdots A_m^o B_1^o\cdots B_q^o]^c\}_D, \quad \text{(A.14)}$$

arising on the right hand side of (1.17). The Feynman diagram that cancels it is the sum of the diagrams of the form shown in figure 2, with ζQ_B acting on each external state in turn. This amplitude can be analyzed in the same way as (A.3) and is given by an expression similar to (A.4):

$$- g_s^2\{\zeta Q_B A_c^1\cdots A_n^c B_1^c\cdots B_m^c \varphi_r; A_1^o\cdots A_m^o B_1^o\cdots B_q^o\}\langle\tilde{\varphi}^r|c_0^- b_0^+ (L_0^+)^{-1}|\varphi_s\rangle\{\tilde{\varphi}^s\}_D, \quad \text{(A.15)}$$

together with similar terms where ζQ_B acts in turn on the other external states. Note that there is no $\mathcal{G}$ insertion in the propagator since we have to use the ψ_r-$\tilde{\psi}_s$ propagator [1]. We now carry out manipulations similar to those for (A.4) to express (A.15) in the form given in (A.6):

$$- g_s^2 \{\zeta Q_B A_c^1\cdots A_n^c B_1^c\cdots B_m^c b_0^+ (L_0^+)^{-1}[\ ]_D; A_1^o\cdots A_m^o B_1^o\cdots B_q^o\}. \quad \text{(A.16)}$$

The vertex appearing in (A.16) has one less dimension compared to the original vertex, since we replace a boundary by a closed string puncture where $b_0^+ (L_0^+)^{-1}[\ ]_D$ is inserted.
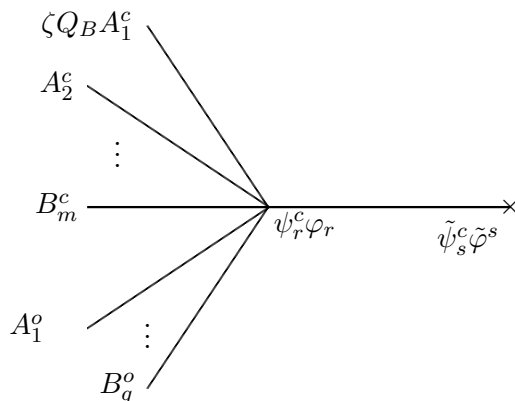
Figure 2. A Feynman diagram where a 1PI vertex with the external states $\zeta Q_B A_1^c, A_2^c \cdots,$ $A_n^c, B_1^c, \cdots, B_m^c, A_1^o \cdots A_p^o, B_1^o, \cdots B_q^o$ and an internal closed string state in $\mathcal{H}^c$ and another one point 1PI vertex with an internal closed string state in $\widetilde{\mathcal{H}}^c$ are joined by a closed string propagator.

Therefore we can use the main identity (1.17). Throwing away the terms on the right hand side of (1.17) that would cancel against Feynman diagrams with two propagators, and using BRST invariance of $[\ ]_D$, we can express the sum of Feynman diagrams of the type shown in figure 2 as

$$g_s^2 \{ A_c^1 \cdots A_n^c B_1^c \cdots B_m^c \zeta[\ ]_D ; A_1^o \cdots A_m^o B_1^o \cdots B_q^o \}$$
$$= g_s^2 \{ \zeta[A_c^1 \cdots A_n^c B_1^c \cdots B_m^c ; A_1^o \cdots A_m^o B_1^o \cdots B_q^o]^c \}_D \,, \qquad (A.17)$$

where in the last step we used (1.8). This cancels (A.14), confirming that the sign in (A.14) is correct.

B Review of non-orientable surfaces

In this appendix we shall review some properties of non-orientable surfaces following [42–44].

A 2 dimensional surface is non-orientable if we cannot assign an orientation to the surface uniquely, i.e. there exist closed curves such that the tangent bundle, parallel transported along the curve, comes back with opposite orientation. A non-orientable 2 dimensional surface can be described using the 'crosscap' — a disc whose diametrically opposite points have been identified. By attaching an arbitrary number of crosscaps to a sphere with holes, we can generate an arbitrary non-orientable surface in 2 dimensions. Examples of non-orientable surfaces which appear at tree and one loop level in unoriented string theory are real projective plane (RP^2), Mobius strip and Klein bottle. The RP^2 is a sphere with one crosscap, the Mobius strip is a disc with one crosscap whereas Klein bottle is a sphere with two crosscaps. Therefore to construct RP^2, we remove a disc from the sphere and identify the diametrically opposite points of the resulting hole, whereas to construct the Mobius strip, we remove a disc from the interior of the disc and identify the diametrically opposite points of the resulting hole. A Klein bottle can be obtained by removing two discs from the sphere and identifying the diametrically opposite points of each of the resulting hole.

We now recall some general statements about two dimensional surfaces. Any compact orientable 2 dimensional manifold is topologically equivalent to a sphere with g handles and b boundaries. On the other hand, any compact non-orientable 2 dimensional manifold is topologically equivalent to a sphere with c crosscaps and b boundaries. Surfaces with both handles and crosscaps are redundant, since, in the presence of crosscaps, a handle can be replaced by two crosscaps:

$$\text{handle } + \text{ crosscap } = 3 \, \text{crosscaps} \,. \tag{B.1}$$

Therefore for non orientable surfaces, the number of boundaries and crosscaps specify the surface topologically. The Euler number of a general surface, having g handles (or c crosscaps) and b boundaries is given by

$$\begin{aligned} \text{Orientable surfaces}: \quad &\chi = 2 - 2g - b \\ \text{Non orientable surfaces}: \quad &\chi = 2 - c - b \,. \end{aligned} \tag{B.2}$$

These determine the power of the string coupling constant g_s in a given amplitude. Often, one combines the two formula in (B.2) into a single one to write $\chi = 2 - 2g - c - b$ with the understanding that we choose $g = 0$ for the non-orientable surfaces and $c = 0$ for the oriented surfaces. In the same convention, the dimension of moduli space of an arbitrary two dimensional surface with N bulk punctures and M boundary punctures is given by

$$\dim\left(\mathcal{M}_{g,b,c,N,M}\right) = 6g - 6 + 3b + 3c + 2N + M \,. \tag{B.3}$$

Also for getting a non-zero correlation function, the required value of the total picture number on a surface is given by

$$4g - 4 + 2b + 2c \,. \tag{B.4}$$

This, together with the picture numbers carried by the string states, dictates the number of PCOs one needs to insert on the surface.

The conformal killing groups of sphere and disc without punctures are SL(2,$\mathbb{C}$) and SL(2,$\mathbb{R}$) respectively. There are 3 complex conformal Killing vectors (CKVs) on the sphere and 3 real CKVs on the disc. Moreover, the volume of these conformal killing groups is infinite. This implies that the 1 and 2 point sphere amplitudes do not give any contribution to the 1PI effective action.[11] Similarly, the 1 and 2 point disc amplitudes for external open strings also do not give any contribution to the 1PI effective action. However, the amplitude of 1 closed string on the disc does not vanish since the resulting surface — disc with one bulk puncture — has only one real CKV, generating a finite volume U(1) group. The conformal killing group of RP^2 is SU(2) which has finite volume. Hence, we can also have non zero 1-point function of closed strings on RP^2.

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[11]It has been argued in a recent paper [45] that the two point function on the sphere does not vanish. This, however, represents the standard forward contribution to the S-matrix present in any quantum field theory, including string field theory, and is needed for unitarity of the theory [6]. Therefore this does not require us to add a new term in the action. The same comment holds for two point function on the disc.

The parametrization of the world-sheet with coordinates (σ, τ) defines an orientation to the world-sheet locally. The orientation can be reversed by making the following transformation

$$\Omega: \qquad \sigma \to \sigma' = \ell - \sigma\,, \qquad \tau \to \tau' = \tau \tag{B.5}$$

The parameter ℓ, describing the length of the string in σ coordinates, is usually chosen to be π for open strings and 2π for the closed strings. In complex coordinates $z = e^{\tau + i\sigma}$, the operator Ω acts as

$$\begin{aligned}
\text{Closed strings}: \quad & z \to \bar{z}\,, & \bar{z} \to z\,, \\
\text{Open strings}: \quad & z \to -\bar{z}\,, & \bar{z} \to -z\,.
\end{aligned} \tag{B.6}$$

The operator Ω is called world-sheet parity operator. Since acting twice with Ω gives us the original orientation, we have $\Omega^2 = 1$ and hence the eigenvalues of Ω are ± 1.

Orientifold operation typically corresponds to taking a projection by the operator Ω possibly accompanied by another symmetry transformation acting on the world-sheet superconformal field theory. It could for example involve reversing the directions of certain space-time coordinates. Let us denote by W this combined operation. We can take $W^2 = 1$.[12] In the theory obtained by quotienting the original theory by W, we keep only those states that carry W eigenvalue $+1$. This can be achieved using the projection operator P:

$$P \;=\; \frac{1 + W}{2}\,. \tag{B.7}$$

Note that the propagator also has GSO projection in all superstring field theories and $L_0 = \bar{L}_0$ projection in the closed string sector, but we do not display this explicitly.

Let us now review how non-orientable surfaces appear in the unoriented theories. Consider a loop diagram. In the intermediate state, we need to sum over all the states. Since we only want to keep the $W = 1$ states, the intermediate states must include the projection operator $(1 + W)/2$. This corresponds to cutting the world-sheet describing the propagation of the intermediate state, inserting the projection operator P and then sewing back the cut-edges. Then the 1 part of the projector corresponds to gluing the cut edges with the same orientation — via the sewing relations (2.7) for closed strings and (2.8) for the open strings. However, the W part of the projector corresponds to first reversing the orientation of one edge and then gluing it with the other edge. The corresponding sewing relations are $z\bar{w} = q \equiv e^{-s - i\theta}$ for closed strings and $z\bar{w} = e^{-s}$ for open strings. This produces a non-orientable surface even if the original surface was oriented. Note that both, the oriented and the non-orientable surface produced this way, come with weight half.

With this understanding we can now describe the effect of different types of sewing in the unoriented theories. One general result that one can infer from this is that sewing two

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[12]If W^2 is not identity, then it must be given by some symmetry U of the world-sheet theory that does not involve world-sheet parity transformation. We can first define a theory where we take the quotient of the world-sheet theory by U. This is an ordinary orbifold superconformal field theory describing oriented strings. In this theory U acts as identity operator, and therefore $W^2 = 1$. The desired orientifold is now given by the W quotient of the orbifold theory.

Sewing type	degeneration type	Boundary sewn	Sewing relation	change in topology
closed-closed	separating	-	$zw = q, \ z\bar{w} = q$	-
closed-closed	non separating	-	$zw = q$	$g \to g+1$ $c \to c+2$
closed-closed	non separating	-	$z\bar{w} = q$	$c \to 2g+2$ $c \to c+2$
open-open	separating	different	$zw - e^{-s}, \ z\bar{w} = e^{-s}$	$b \to b-1$
open-open	non separating	same	$zw = -e^{-s}$	$b \to b+1$
open-open	non separating	same	$z\bar{w} = e^{-s}$	$c \to 2g+1$ $c \to c+1$
open-open	non separating	different	$zw = -e^{-s}$	$g \to g+1, \ b \to b-1$ $c \to c+2, \ b \to b-1$
open-open	non separating	different	$z\bar{w} = e^{-s}$	$c \to 2g+2, \ b \to b-1$ $c \to c+2, \ b \to b-1$

Table 1. Relation between the topology of the sewed surface to the topology of the surface(s) before sewing. The last column of some of the rows have two entries. The top entry refers to the case where the original surface before sewing is oriented, while the bottom entry refers to the case where the original surface was non-orientable.

different surfaces never produces a non-orientable surface (unless one of the sewed surface itself is non-orientable). The reason for this is that while sewing two surfaces Σ with Σ', we can consider the sewing of a fixed surface Σ with all possible surfaces Σ' having the same topology. For sewing with W, we just change $w \to \bar{w}$ without changing the topology of Σ'. Hence, the new surface obtained by $w \to \bar{w}$ is already included in the original list of surfaces Σ'. So, by sewing with W, we do not generate any new surface which had not been already generated by the sewing with 1. In contrast, for non-separating type degeneration sewing with W produces non-orientable surfaces from oriented surfaces for reasons explained earlier. We reproduce in table 1 the results described in [43] for different type of sewing and the associated degeneration of the sewed surface.

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