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AdS_4 backgrounds with $N > 16$ supersymmetries in 10 and 11 dimensions

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Abstract: We explore all warped $AdS_4 \times_w M^{D-4}$ backgrounds with the most general allowed fluxes that preserve more than 16 supersymmetries in $D = 10$- and 11-dimensional supergravities. After imposing the assumption that either the internal space M^{D-4} is compact without boundary or the isometry algebra of the background decomposes into that of AdS_4 and that of M^{D-4}, we find that there are no such backgrounds in IIB supergravity. Similarly in IIA supergravity, there is a unique such background with 24 supersymmetries locally isometric to $AdS_4 \times \mathbb{CP}^3$, and in $D = 11$ supergravity all such backgrounds are locally isometric to the maximally supersymmetric $AdS_4 \times S^7$ solution.

Keywords: AdS-CFT Correspondence, Flux compactifications, Supergravity Models, Superstring Vacua

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1 Introduction

AdS backgrounds in 10 and 11 dimensions that preserve N supersymmetries with $N > 16$ have found widespread applications both in supergravity compactifications and in AdS/CFT correspondence, for reviews see [1, 2] and references therein. One of the features of such backgrounds in AdS/CFT [3] is that the CFT R-symmetry group acts transitively on the internal space of the solution and this can be used to establish the dictionary between some of the operators of the CFT and spacetime Kaluza-Klein fields [4]. Therefore the question arises whether it is possible to find all such AdS solutions. Despite the progress that has been made during the years, a complete description of all AdS solutions that preserve $N > 16$ supersymmetries remains an open problem.

Recently however, there have been several developments which facilitate progress in this direction for a large class of warped flux AdS solutions. In [5–7], the Killing spinor equations (KSEs) of supergravity theories have been solved in all generality and the fractions of supersymmetry preserved by all warped flux AdS backgrounds have been identified. Furthermore global analysis techniques have also been introduced in the investigation of AdS backgrounds which can be used to a priori impose properties like the compactness of the internal space and the smoothness of the fields. Another key development is the proof of the homogeneity theorem [8] which for the special case of AdS backgrounds states that all such backgrounds that preserve $N > 16$ supersymmetries are Lorentzian homogeneous spaces.

So far it is known that the warped flux AdS_n, $n \geq 6$, backgrounds preserve either 16 or 32 supersymmetries and those that preserve 32 supersymmetries have been classified in [9]. In addition, it has been shown that there are no $N > 16$ AdS_5 backgrounds in $D = 11$ and (massive) IIA supergravities while in IIB supergravity all such backgrounds are locally isometric to the maximally supersymmetric $\mathrm{AdS}_5 \times S^5$ solution [10]. In particular the existence of a IIB AdS_5 solution that preserves 24 supersymmetries has been excluded. Moreover the $\mathrm{AdS}_n \times M^{D-n}$ solutions with M^{D-n} a symmetric coset space have been classified in [11–14]. Furthermore heterotic supergravity does not admit AdS solutions that preserve more than 8 supersymmetries [15].

The main task of this paper is to describe all warped AdS_4 backgrounds that admit the most general fluxes in 10 and 11 dimensions and preserve more than 16 supersymmetries. It has been shown in [5–7] that such backgrounds preserve $4k$ supersymmetries. Therefore, we shall investigate the backgrounds preserving 20, 24 and 28 as those with 32 supersymmetries have already been classified in [9]. In particular, we find that

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- IIB and massive IIA supergravity do not admit AdS_4 solutions with $N > 16$ supersymmetries.
- Standard IIA supergravity admits a unique solution up to an overall scale preserving 24 supersymmetries locally isometric to the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background of [16].
- All AdS_4 solutions of 11-dimensional supergravity that preserve $N > 16$ supersymmetries are locally isometric to the maximally supersymmetric $\mathrm{AdS}_4 \times S^7$ solution of [17, 18].

These results have been established under certain assumptions.[1] We begin with a spacetime which is a warped product $\mathrm{AdS}_4 \times_w M^{D-4}$, for $D = 10$ or 11, and allow for all fluxes which are invariant under the isometries of AdS_4. Then we shall assume that

1. either the solutions are smooth and M^{D-4} is compact without boundary
2. or that the even part of the Killing superalgebra of the background decomposes as a direct sum $\mathfrak{so}(3,2) \oplus \mathfrak{t}_0$, where $\mathfrak{so}(3,2)$ is the Lie algebra of isometries of AdS_4 and $\mathfrak{t}_0$ is the Lie algebra of the isometries of M^{D-4}.

It has been shown in [21] that for all AdS backgrounds, the first assumption implies the second. In addition for $N > 16$ AdS_4 backgrounds,[2] the second assumption implies the first. This is because $\mathfrak{t}_0$ is the Lie algebra of a compact group and all internal spaces are compact without boundaries. Smoothness also follows as a consequence of considering only invariant solutions.

The proof of the main statement of our paper is based first on the results of [5–7] that the number of supersymmetries preserved by AdS_4 backgrounds are $4k$ and so the solutions under consideration preserve 20, 24, 28 and 32 supersymmetries. Then the homogeneity theorem of [8] implies that all such backgrounds are Lorentzian homogeneous spaces. Moreover, it has been shown in [21] under the assumptions mentioned above that the Killing superalgebra of warped AdS_4 backgrounds that preserve $N = 4k$ supersymmetries is isomorphic to $\mathfrak{osp}(N/4|4)$, see also [22], and that the even subalgebra $\mathfrak{osp}(N/4|4)_0 = \mathfrak{so}(3,2) \oplus \mathfrak{so}(N/4)$ acts effectively on the spacetime with $\mathfrak{t}_0 = \mathfrak{so}(N/4)$ acting on the internal space. Thus together with the homogeneity theorem $\mathfrak{osp}(N/4|4)_0$ acts both *transitively* and *effectively* on the spacetime. Then we demonstrate in all cases that the warp factor A is constant. As a result all $N > 16$ AdS_4 backgrounds are product spaces $\mathrm{AdS}_4 \times M^{D-4}$. So the internal space M^{D-4} is a homogeneous space, $M^{D-4} = G/H$, and $\mathfrak{Lie}\, G = \mathfrak{so}(N/4)$. Therefore, we have demonstrated the following,

- The internal spaces of AdS_4 backgrounds that preserve $N > 16$ supersymmetries are homogeneous spaces that admit a transitive and effective action of a group G with $\mathfrak{Lie}\, G = \mathfrak{so}(N/4)$.

[1] Some assumptions are necessary to exclude the possibility that a warped AdS_4 background is not locally isometric to an AdS_n background with $n > 4$. This has been observed in [19] and explored in the context of KSEs in [20].

[2] In what follows, we use "$N > 16$ AdS backgrounds" instead of "AdS backgrounds that preserve $N > 16$ supersymmetries" for short.

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Having established this, one can use the classification of [23–26] to identify all the 6- and 7-dimensional homogeneous spaces that can occur as internal spaces for $N > 16$ AdS_4 backgrounds, see also tables[3] 1 and 3. Incidentally, this also means that if $N > 16$ backgrounds were to exist, the R-symmetry group of the dual CFT would have to act transitively on the internal space of the solution.

A direct observation of the classification of 6-dimensional homogeneous spaces G/H in table 1 reveals that those that can occur as internal spaces of AdS_4 backgrounds with $N > 16$ in 10 dimensions are

$$\begin{aligned} &\mathrm{Spin}(7)/\mathrm{Spin}(6)\ (N=28)\,, \qquad \mathrm{SU}(4)/S(\mathrm{U}(1)\times\mathrm{U}(3))\ (N=24)\,,\\ &\mathrm{Sp}(2)/\mathrm{U}(2)\qquad (N=20)\,, \qquad \mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{U}(1))\quad (N=20)\,, \end{aligned} \tag{1.1}$$

where N denotes the expected number of supersymmetries that can be preserved by the background and we always take G to be simply connected. Observe that there are no maximally supersymmetric AdS_4 solutions in 10-dimensional supergravities in agreement with the results of [9]. The proof of our result in IIB supergravity is based on a cohomological argument and does not use details of the 6-dimensional homogeneous spaces involved. However in (massive) IIA supergravity, one has to consider details of the geometry of these coset spaces. Solutions with strictly $N = 28$ and $N = 20$ supersymmetries are ruled out after a detailed analysis of the KSEs and dilaton field equation. In the standard IIA supergravity there is a solution with 24 supersymmetry and internal space locally isometric to the symmetric space $\mathrm{SU}(4)/S(\mathrm{U}(1)\times\mathrm{U}(3)) = \mathbb{CP}^3$. This solution has already been found in [16]. The homogeneous space $\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1)$, which is diffeomorphic to $\mathbb{CP}^3$, gives also a solution at a special region of the moduli space of parameters. This solution admits 24 supersymmetries and is locally isometric to that with internal space $\mathrm{SU}(4)/S(\mathrm{U}(1)\times\mathrm{U}(3))$.

The classification of 7-dimensional homogeneous spaces G/H in table 3 reveals that those that can occur as internal spaces of $N > 16$ AdS_4 backgrounds in 11 dimensions are

$$\begin{aligned} &\mathrm{Spin}(8)/\mathrm{Spin}(7)\ (N=32)\,, \quad \mathrm{Spin}(7)/G_2 \qquad (N=28)\,, \quad \mathrm{SU}(4)/\mathrm{SU}(3)\ (N=24)\,,\\ &\mathrm{Sp}(2)/\mathrm{Sp}(1)_{\mathrm{max}}\ (N=20)\,, \quad \mathrm{Sp}(2)/\Delta(\mathrm{Sp}(1))\ (N=20)\,,\\ &\mathrm{Sp}(2)/\mathrm{Sp}(1)\qquad (N=20)\,, \end{aligned} \tag{1.2}$$

where $\mathrm{Sp}(1)_{\mathrm{max}}$ and $\Delta(\mathrm{Sp}(1))$ denote the maximal and diagonal embeddings of $\mathrm{Sp}(1)$ in $\mathrm{Sp}(2)$, respectively, and G is chosen to be simply connected. It is known that there is a maximally supersymmetric solution $\mathrm{AdS}_4\times S^7$ with internal space $S^7 = \mathrm{Spin}(8)/\mathrm{Spin}(7)$ [17, 18]. After a detailed investigation of the geometry of the above homogeneous spaces, the solutions of the KSEs and the warp factor field equation, one can also show that the rest of the coset spaces do not give solutions with strictly 20, 24 and 28 supersymmetries. However as the homogeneous spaces $\mathrm{Spin}(7)/G_2$, $\mathrm{SU}(4)/\mathrm{SU}(3)$ and $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ are diffeomorphic

[3] These tables list the simply connected homogeneous spaces. This suffices for our purpose because we are investigating the geometry of the backgrounds up to local isometries. As $\mathfrak{so}(N/4)$ is simple the universal cover of G/H with $\mathfrak{Lie}(G) = \mathfrak{so}(N/4)$ is compact and homogeneous, see eg [27]. So the internal space can be identified with the universal cover $\tilde{G}/\tilde{H}$ of G/H for which $\tilde{G}$ can be chosen to be simply connected.

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to S^7, there is a region in the moduli space of their parameters which yields the maximally supersymmetric $\mathrm{AdS}_4 \times S^7$ solution.

The paper is organized as follows. In section 2, we show that there are no IIB $N > 16$ $\mathrm{AdS}_4 \times_w M^6$ solutions. In section 3, we show that there is an up to an over scale unique solution of IIA supersgravity that preserves 24 supersymmetries. In section 4, we demonstrate that all $N > 16$ AdS_4 backgrounds of 11-dimensional supergravity are locally isometric to the maximally supersymmetric $\mathrm{AdS}_4 \times S^7$ solution. In section 5 we state our conclusions. In appendix A, we explain our conventions, and in appendix B we summarize some aspects of the geometry of homogeneous spaces that is used throughout the paper. In appendices C, D and E, we present some formulae for the homogeneous spaces that admit a transitive action of a group with Lie algebra $\mathfrak{su}(k)$ or $\mathfrak{so}(5) = \mathfrak{sp}(2)$.

2 $N > 16$ $\mathrm{AdS}_4 \times_w M^6$ solutions in IIB

To investigate the IIB AdS_4 backgrounds, we shall use the approach and notation of [6] where Bianchi identities, field equations and KSEs are first solved along the AdS_4 subspace of $\mathrm{AdS}_4 \times_w M^6$ and then the remaining independent conditions along the internal space M^6 are identified. The bosonic fields of IIB supergravity are the metric, a complex 1-form field strength P, a complex 3-form field strength G and a real self-dual 5-form F. Imposing the symmetry of AdS_4 on the fields, one finds that the metric and form field strengths are given by

$$\begin{aligned} ds^2 &= 2du(dr + rh) + A^2(dz^2 + e^{2z/\ell}dx^2) + ds^2(M^6)\,, \\ G &= H, \quad P = \xi, \quad F = A^2 e^{z/\ell} du \wedge (dr + rh) \wedge dz \wedge dx \wedge Y + *_6 Y\,, \end{aligned} \tag{2.1}$$

where the metric has been written as a near-horizon geometry [30] with

$$h = -\frac{2}{\ell}dz - 2A^{-1}dA\,. \tag{2.2}$$

The warp factor A is a function on the internal manifold M^6, H is the complex 3-form on M^6, ξ is a complex 1-form on M^6 and Y is a real 1-form on M^6. The AdS_4 coordinates are (u, r, z, x) and we introduce the null-ortho-normal frame

$$\mathbf{e}^+ = du\,, \quad \mathbf{e}^- = dr + rh\,, \quad \mathbf{e}^z = A\,dz\,, \quad \mathbf{e}^x = Ae^{z/\ell}\,dx\,, \quad \mathbf{e}^i = \mathbf{e}^i_I\,dy^I\,, \tag{2.3}$$

where $ds^2(M^6) = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$. All gamma matrices are taken with respect to this null orthonormal frame.

The Bianchi identities along M^6 which are useful in the analysis that follows are

$$\begin{aligned} d(A^4 Y) &= 0, \qquad dH = iQ \wedge H - \xi \wedge \overline{H}, \\ \nabla^i Y_i &= -\frac{i}{288}\epsilon^{i_1 i_2 i_3 j_1 j_2 j_3} H_{i_1 i_2 i_3} \overline{H}_{j_1 j_2 j_3}\,, \\ dQ &= -i\xi \wedge \bar{\xi}\,, \end{aligned} \tag{2.4}$$

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where Q is the pull-back of the canonical connection of the upper-half plane on the space-time with respect to the dilaton and axion scalars of IIB supergravity. Similarly, the field equations of the warp factor is

$$A^{-1}\nabla^2 A = 4Y^2 + \frac{1}{48}H_{i_1i_2i_3}\overline{H}^{i_1i_2i_3} - \frac{3}{\ell^2}A^{-2} - 3A^{-2}(dA)^2\,, \tag{2.5}$$

and those of the scalar and 3-form fluxes are

$$\begin{aligned}\nabla^i\xi_i &= -3\partial^i \log A\,\xi_i + 2iQ^i\xi_i - \frac{1}{24}H^2\,,\\ \nabla^i H_{ijk} &= -3\partial^i \log A\, H_{ijk} + iQ^i H_{ijk} + \xi^i\overline{H}_{ijk}\,.\end{aligned} \tag{2.6}$$

The full set of Bianchi identities and field equations can be found in [6]. Note in particular that (2.5) implies that if A and the other fields are smooth, then A is nowhere vanishing on M^6.

2.1 The Killing spinors

After solving the KSEs along AdS_4, the Killing spinors of the background can be written as

$$\begin{aligned}\epsilon = \sigma_+ - \ell^{-1}x\Gamma_{xz}\tau_+ + e^{-\frac{z}{\ell}}\tau_+ + \sigma_- + e^{\frac{z}{\ell}}(\tau_- - \ell^{-1}x\Gamma_{xz}\sigma_-)\\ - \ell^{-1}uA^{-1}\Gamma_{+z}\sigma_- - \ell^{-1}rA^{-1}e^{-\frac{z}{\ell}}\Gamma_{-z}\,,\tau_+\,,\end{aligned} \tag{2.7}$$

where we have used the light-cone projections

$$\Gamma_\pm\sigma_\pm = 0\,, \quad \Gamma_\pm\tau_\pm = 0\,, \tag{2.8}$$

and $\sigma_\pm$ and $\tau_\pm$ are $\mathrm{Spin}(9,1)$ Weyl spinors depending only on the coordinates of M^6. The remaining independent KSEs are

$$\nabla_i^{(\pm)}\sigma_\pm = 0\,, \quad \nabla_i^{(\pm)}\tau_\pm = 0\,, \tag{2.9}$$

and

$$\left(\frac{1}{24}\not{H} + \not{G}C*\right)\sigma_\pm = 0\,, \quad \left(\frac{1}{24}\not{H} + \not{G}C*\right)\tau_\pm = 0\,, \tag{2.10}$$

as well as

$$\Xi^{(\pm)}\sigma_\pm = 0\,, \quad \left(\Xi^{(\pm)} \pm \frac{1}{\ell}\right)\tau_\pm = 0\,, \tag{2.11}$$

where

$$\nabla_i^{(\pm)} = \nabla_i \pm \frac{1}{2}\partial_i \log A - \frac{i}{2}Q_i \mp \frac{i}{2}\Gamma\!\!\!/ Y_i\Gamma_{xz} \pm \frac{i}{2}Y_i\Gamma_{xz} + \left(-\frac{1}{96}\Gamma\!\!\!/\not{H}_i + \frac{3}{32}\not{H}_i\right)C*\,, \tag{2.12}$$

$$\Xi^{(\pm)} = \mp\frac{1}{2\ell} - \frac{1}{2}\Gamma_z\not{\partial}A \pm \frac{i}{2}A\Gamma_x Y\!\!\!/ + \frac{1}{96}A\Gamma_z\not{H}C*\,, \tag{2.13}$$

and $C*$ is the charge conjugation matrix followed by standard complex conjugation. For some explanation of the notation see appendix A. (2.9) and (2.10) can be thought of as the

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naive restriction of gravitino and dilatino KSEs of IIB supergravity on M^6, respectively. (2.11) are algebraic and arise as integrability conditions of the integration of IIB KSEs over the AdS_4 subspace of the background. We do not assume that the Killing spinors factorize as Killing spinors on AdS_4 and Killing spinors on the internal manifold. It has been observed in [6] that if σ_+ is a Killing spinor, then

$$\tau_+ = \Gamma_{zx}\sigma_+ \,, \quad \sigma_- = A\Gamma_{-z}\sigma_+ \,, \quad \tau_- = A\Gamma_{-x}\sigma_+ \,, \tag{2.14}$$

are also Killing spinors. As a result AdS_4 solutions preserve $4k$ supersymmetries.

2.2 The non-existence of $N > 16$ AdS_4 solutions in IIB

2.2.1 Conditions on spinor bilinears

As it has already been mentioned, the two assumptions we have made in the introduction are equivalent for all IIB, (massive) IIA and 11-dimensional AdS_4 backgrounds that preserve $N > 16$ supersymmetries. Hence in what follows, we shall focus only on the restrictions on the geometry of the spacetime imposed by the first assumption which requires that the solutions are smooth and the internal space is compact without boundary.

To begin our analysis, a consequence of the homogeneity theorem [8] for solutions which preserve $N > 16$ supersymmetries is that the IIB scalars are constant which in turn implies that

$$\xi = 0 \,. \tag{2.15}$$

As Q is the pull-back of the canonical connection of the upper half plane with respect to the scalars and these are constant, $Q = 0$ as well.

Setting $\Lambda = \sigma_+ + \tau_+$ and after using the gravitino KSE (2.9), we find

$$\nabla_i \parallel \Lambda \parallel^2 = - \parallel \Lambda \parallel^2 A^{-1}\nabla_i A - iY_i\langle \Lambda, \Gamma_{xz}\Lambda\rangle + \frac{1}{48}\mathrm{Re}\langle \Lambda, \Gamma\!\!\!/ H_i C * \Lambda\rangle \,. \tag{2.16}$$

Next, observe that the algebraic KSE (2.11) implies

$$\frac{1}{48}\Gamma\!\!\!/ H C * \Lambda = \left(A^{-1}\Gamma^j\nabla_j A + i\Gamma^j\Gamma_{xz}Y_j\right)\Lambda + \ell^{-1}A^{-1}\Gamma_z(\sigma_+ - \tau_+) \,, \tag{2.17}$$

which, when substituted back into (2.16), yields

$$\nabla_i \parallel \Lambda \parallel^2 = 2\ell^{-1}A^{-1}\mathrm{Re}\langle \tau_+, \Gamma_{iz}\sigma_+\rangle \,. \tag{2.18}$$

However, the gravitino KSE (2.9) also implies that

$$\nabla^i \left(A\mathrm{Re}\langle \tau_+, \Gamma_{iz}\sigma_+\rangle\right) = 0 \,. \tag{2.19}$$

Thus, in conjunction with (2.18), we obtain

$$\nabla^2 \parallel \Lambda \parallel^2 + 2A^{-1}\nabla^i A\nabla_i \parallel \Lambda \parallel^2 = 0 \,. \tag{2.20}$$

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The Hopf maximum principle then implies that $\| \Lambda \|^2$ is constant, so (2.16) and (2.18) give the conditions

$$- \| \Lambda \|^2 A^{-1} \nabla_i A - i Y_i \langle \Lambda, \Gamma_{xz} \Lambda \rangle + \frac{1}{48} \mathrm{Re} \langle \Lambda, \Gamma \not{H}_i C * \Lambda \rangle = 0 \,, \tag{2.21}$$

and

$$\mathrm{Re} \langle \tau_+, \Gamma_{iz} \sigma_+ \rangle = 0 \,, \tag{2.22}$$

respectively. The above equation can be equivalently written as $\mathrm{Re} \langle \sigma_+, \Gamma_{ix} \sigma_+ \rangle = 0$.

The spinors σ_+ and τ_+ are linearly independent as it can be easily seen from (2.11). Moreover as a consequence of (2.22), they are orthogonal

$$\mathrm{Re} \langle \tau_+, \sigma_+ \rangle = 0 \,. \tag{2.23}$$

To see this take the real part of $\langle \tau_+, \Xi^{(+)} \sigma_+ \rangle - \langle \sigma_+, (\Xi^{(+)} + \ell^{-1}) \tau_+ \rangle = 0$. The conditions (2.19), (2.23) as well as the constancy of $\| \Lambda \|$ can also be derived from the assumption that the isometries of the background decompose into those of AdS_4 and those of the internal manifold [21].

2.2.2 The warp factor is constant and the 5-form flux vanishes

AdS_4 backgrounds preserving $4k$ supersymmetries admit k linearly independent Killing spinors σ_+. For every pair of such spinors σ_+^1 and σ_+^2 define the bilinear

$$W_i = A \, \mathrm{Re} \langle \sigma_+^1, \Gamma_{iz} \sigma_+^2 \rangle \,. \tag{2.24}$$

Then the gravitino KSE (2.9) implies that

$$\nabla_{(i} W_{j)} = 0 \,. \tag{2.25}$$

Therefore W is a Killing vector on M^6.

Next consider the algebraic KSE (2.11) and take the real part of $\langle \sigma_+^1, \Xi^{(+)} \sigma_+^2 \rangle - \langle \sigma_+^2, \Xi^{(+)} \sigma_+^1 \rangle = 0$ to find that

$$W^i \, \nabla_i A = 0 \,, \tag{2.26}$$

where we have used (2.22).

Similarly, taking the real part of the difference $\langle \sigma_+^1, \Gamma_{zx} \Xi^{(+)} \sigma_+^2 \rangle - \langle \sigma_+^2, \Gamma_{zx} \Xi^{(+)} \sigma_+^1 \rangle = 0$ and after using the condition (2.23), we find

$$i_W Y = 0 \,. \tag{2.27}$$

The conditions (2.26) and (2.27) are valid for all IIB AdS_4 backgrounds. However if the solution preserves more than 16 supersymmetries, an argument similar to that used for the proof of the homogeneity theorem in [8] implies that the Killing vectors W span the tangent spaces of M^6 at each point. As a result, we conclude that

$$dA = Y = 0 \,. \tag{2.28}$$

Therefore the warp factor A is constant and the 5-form flux F vanishes. So the background is a product $\mathrm{AdS}_4 \times M^6$, and as it has been explained in the introduction M^6 is one of the homogeneous spaces in (1.1).

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2.2.3 Proof of the main statement

To begin, it has been shown in [31] that all IIB AdS backgrounds that preserve $N \geq 28$ supersymmetries are locally isometric to the maximally supersymmetric ones. As there is not a maximally supersymmetric AdS_4 background in IIB, we conclude that there does not exist a AdS_4 solution which preserves $N \geq 28$ supersymmetries.

To investigate the $N = 20$ and $N = 24$ cases, substitute (2.28) into the Bianchi identities and field equations to find that H is harmonic and

$$H^2 = 0\,. \tag{2.29}$$

If H were real, this condition would have implied $H = 0$ and in turn would have led to a contradiction. This is because the field equation for the warp factor (2.5) cannot be satisfied. Thus we can already exclude the existence of such backgrounds.

Otherwise for solutions to exist, M^6 must be a compact, homogeneous, 6-dimensional Riemannian manifold whose de-Rham cohomology $H^3(M^6)$ has at least two generators and which admits a transitive and effective action of a group with Lie algebra isomorphic to either $\mathfrak{so}(6)$ or $\mathfrak{so}(5)$ for $N = 24$ and $N = 20$, respectively [21]. The homogeneous spaces that admit a transitive and effective action of $\mathfrak{so}(6)$ or $\mathfrak{so}(5) = \mathfrak{sp}(2)$ have already been listed in (1.1) and none of them satisfies these cohomology criteria. All compact homogeneous 6-manifolds have been classified in [25] and the complete list of the simply connected ones relevant here is given in table 1. Therefore, we conclude that there do not exist AdS_4 backgrounds preserving $N > 16$ supersymmetries in IIB supergravity.[4]

3 $N > 16$ $\mathrm{AdS}_4 \times_w M^6$ solutions in (massive) IIA

To begin, let us summarize the solution of Bianchi identities, field equations and KSEs for (massive) IIA $\mathrm{AdS}_4 \times_w M^6$ backgrounds as presented in [7] whose notation we follow. The bosonic fields of (massive) IIA supergravity are the metric, a 4-form field strength G, a 3-form field strength H, a 2-form field strength F, the dilaton Φ and the mass parameter S of massive IIA dressed with the dilaton. Imposing the symmetries of AdS_4 on the fields, one finds that

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + (\mathbf{e}^z)^2 + (\mathbf{e}^x)^2 + ds^2(M^6)\,, \\ G &= X\mathbf{e}^+ \wedge \mathbf{e}^- \wedge \mathbf{e}^z \wedge \mathbf{e}^x + Y\,, \quad H = H, \quad F = F, \quad \Phi = \Phi, \quad S = S, \end{aligned} \tag{3.1}$$

where $ds^2(M^6) = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$ and the frame $(\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^x, \mathbf{e}^z, \mathbf{e}^i)$ is defined as in (2.3). Note that the fields H, F, Φ and S do not have a component along AdS_4 and so we use the same symbol to denote them and their component along M^6. The warp factor A, S and X are functions of M^6, whereas Y, H and F are 4-form, 3-form and 2-form fluxes on M^6, respectively. The conditions imposed on the fields by the Bianchi identities and field

[4] Note that the possibility of IIB $\mathrm{AdS}_4 \times Z\backslash G/H$ backgrounds preserving $N > 16$ supersymmetry is also excluded, where Z is a discrete subgroup of G, as there are no IIB $\mathrm{AdS}_4 \times G/H$ local geometries that preserve $N > 16$ supersymmetries.

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	$M^6 = G/H$
(1)	$\frac{\mathrm{Spin}(7)}{\mathrm{Spin}(6)} = S^6$, symmetric space
(2)	$\frac{G_2}{\mathrm{SU}(3)}$ diffeomorphic to S^6
(3)	$\frac{\mathrm{SU}(4)}{S(\mathrm{U}(1)\times\mathrm{U}(3))} = \mathbb{CP}^3$, symmetric space
(4)	$\frac{\mathrm{Sp}(2)}{\mathrm{U}(2)}$, symmetric space
(5)	$\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ diffeomorphic to $\mathbb{CP}^3$
(6)	$\frac{\mathrm{SU}(3)}{T_{\max}}$ Wallach space
(7)	$\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\Delta(\mathrm{SU}(2))} \times \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\Delta(\mathrm{SU}(2))} = S^3 \times S^3$
(8)	$\mathrm{SU}(2) \times \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\Delta(\mathrm{SU}(2))}$ diffeomorphic to $S^3 \times S^3$
(9)	$\mathrm{SU}(2) \times \mathrm{SU}(2)$ diffeomorphic to $S^3 \times S^3$
(10)	$\frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)} = S^2 \times S^2 \times S^2$
(11)	$\frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(4)} = S^2 \times S^4$
(12)	$\frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(3)}{S(\mathrm{U}(1)\times\mathrm{U}(2))} = S^2 \times \mathbb{CP}^2$

Table 1. 6-dimensional compact, simply connected, homogeneous spaces.

equations after solving along the AdS_4 subspace can be found in [7]. Relevant to our analysis that follows are the Bianchi identities

$$\begin{aligned} &dH = 0, \qquad dS = Sd\Phi\,, \qquad dY = d\Phi \wedge Y + H \wedge F\,, \\ &dF = d\Phi \wedge F + SH\,, \quad d(A^4 X) = A^4 d\Phi\,, \end{aligned} \tag{3.2}$$

and the field equations for the fluxes

$$\begin{aligned} \nabla^2\Phi &= -4A^{-1}\partial^i A\,\partial_i\Phi + 2(d\Phi)^2 + \frac{5}{4}S^2 + \frac{3}{8}F^2 - \frac{1}{12}H^2 + \frac{1}{96}Y^2 - \frac{1}{4}X^2\,, \\ \nabla^k H_{ijk} &= -4A^{-1}\partial^k A\,H_{ijk} + 2\partial^k\Phi H_{ijk} + SF_{ij} + \frac{1}{2}F^{k\ell}Y_{ijk\ell}\,, \\ \nabla^j F_{ij} &= -4A^{-1}\partial^j A\,F_{ij} + \partial^j\Phi\,F_{ij} - \frac{1}{6}H^{jkl}Y_{ijkl}\,, \\ \nabla^\ell Y_{ijk\ell} &= -4A^{-1}\partial^\ell A\,Y_{ijk\ell} + \partial^\ell\Phi Y_{ijk\ell}\,, \end{aligned} \tag{3.3}$$

along M^6. Moreover, we shall use the field equation for the warp factor A and the Einstein field equation along M^6

$$\begin{aligned} \nabla^2 \log A &= -\frac{3}{\ell^2 A^2} - 4(d\log A)^2 + 2\,\partial_i \log A\partial^i\Phi + \frac{1}{96}Y^2 + \frac{1}{4}X^2 + \frac{1}{4}S^2 + \frac{1}{8}F^2\,, \\ R^{(6)}_{ij} &= 4\nabla_i\partial_j \log A + 4\partial_i \log A\,\partial_j \log A + \frac{1}{12}Y^2_{ij} - \frac{1}{96}Y^2\delta_{ij} + \frac{1}{4}X^2\delta_{ij} - \frac{1}{4}S^2\delta_{ij} \\ &\quad + \frac{1}{4}H^2_{ij} + \frac{1}{2}F^2_{ij} - \frac{1}{8}F^2\delta_{ij} - 2\nabla_i\nabla_j\Phi\,, \end{aligned} \tag{3.4}$$

where ∇ and $R^{(6)}_{ij}$ denote the Levi-Civita connection and the Ricci tensor of M^6, respectively.

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3.1 The Killing spinor equations

The solution of KSEs of (massive) IIA supergravity along the AdS_4 subspace can again be written as (2.7), where now $\sigma_\pm$ and $\tau_\pm$ are $\mathfrak{spin}(9,1)$ Majorana spinors that satisfy the lightcone projections $\Gamma_\pm\sigma_\pm = \Gamma_\pm\tau_\pm = 0$ and depend only on the coordinates of M^6. After the lightcone projections are imposed, $\sigma_\pm$ and $\tau_\pm$ have 16 independent components. These satisfy the gravitino KSEs

$$\nabla_i^{(\pm)}\sigma_\pm = 0\,, \quad \nabla_i^{(\pm)}\tau_\pm = 0\,, \tag{3.5}$$

the dilatino KSEs

$$\mathcal{A}^{(\pm)}\sigma_\pm = 0\,, \quad \mathcal{A}^{(\pm)}\tau_\pm = 0\,, \tag{3.6}$$

and the algebraic KSEs

$$\Xi^{(\pm)}\sigma_\pm = 0\,, \quad \left(\Xi^{(\pm)} \pm \frac{1}{\ell}\right)\tau_\pm = 0\,, \tag{3.7}$$

where

$$\begin{aligned}
\nabla_i^{(\pm)} &= \nabla_i \pm \frac{1}{2}\partial_i \log A + \frac{1}{8}\not{H}_i\Gamma_{11} + \frac{1}{8}S\Gamma_i + \frac{1}{16}\not{F}\Gamma_i\Gamma_{11} + \frac{1}{192}\not{Y}\Gamma_i \mp \frac{1}{8}X\Gamma_{zxi}\,,\\
\mathcal{A}^{(\pm)} &= \not{\partial}\Phi + \frac{1}{12}\not{H}\Gamma_{11} + \frac{5}{4}S + \frac{3}{8}\not{F}\Gamma_{11} + \frac{1}{96}\not{Y} \mp \frac{1}{4}X\Gamma_{zx}\,,\\
\Xi^{(\pm)} &= -\frac{1}{2\ell} + \frac{1}{2}\not{\partial}A\Gamma_z - \frac{1}{8}AS\Gamma_z - \frac{1}{16}A\not{F}\Gamma_z\Gamma_{11} - \frac{1}{192}A\not{Y}\Gamma_z \mp \frac{1}{8}AX\Gamma_x\,.
\end{aligned} \tag{3.8}$$

The first two equations arise from the naive restriction of the gravitino and dilatino KSEs of the theory on $\sigma_\pm$ and $\tau_\pm$, respectively, while the last algebraic equation is an integrability condition that arises from the integration of the IIA KSEs on AdS_4. As in the IIB case, the solutions of the above KSEs are related as in (2.14) and so such backgrounds preserve $4k$ supersymmetries.

3.2 AdS_4 solutions with $N > 16$ in IIA

3.2.1 Conditions on spinor bilinears

The methodology to establish conditions on the Killing spinor bilinears which follow from our assumption that either the solutions are smooth and the internal space is compact without boundary or that the even subalgebra of the Killing superalgebra decomposes as stated in the introduction is the same as that presented for IIB. However, the formulae are somewhat different. Setting $\Lambda = \sigma_+ + \tau_+$ and upon using the gravitino KSE (3.5), one finds

$$\nabla_i \parallel \Lambda \parallel^2 = -\nabla_i \log A \parallel \Lambda \parallel^2 - \frac{1}{4}S\langle\Lambda, \Gamma_i\Lambda\rangle - \frac{1}{8}\langle\Lambda, \not{F}_i\Gamma_{11}\Lambda\rangle - \frac{1}{96}\langle\Lambda, \not{Y}_i\Lambda\rangle\,. \tag{3.9}$$

After multiplying the algebraic KSE (3.7) with Γ_{iz} on the other hand, one gets

$$\begin{aligned}
\frac{1}{2\ell}\langle\Lambda, \Gamma_{iz}(\sigma_+ - \tau_+)\rangle = -\nabla_i A \parallel \Lambda \parallel^2 - \frac{A}{4}S\langle\Lambda, \Gamma_i\Lambda\rangle - \frac{A}{8}\langle\Lambda, \not{F}_i\Gamma_{11}\Lambda\rangle\\
- \frac{A}{96}\langle\Lambda, \not{Y}_i\Lambda\rangle\,.
\end{aligned} \tag{3.10}$$

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Using this, one can rewrite (3.9) as

$$\nabla_i \parallel \Lambda \parallel^2 = \frac{2}{\ell A}\langle \tau_+, \Gamma_{iz}\sigma_+\rangle \,. \tag{3.11}$$

On the other hand the gravitino KSE (3.5) gives

$$\nabla^i \left(A\langle \tau_+, \Gamma_{iz}\sigma_+\rangle\right) = 0 \,. \tag{3.12}$$

Therefore taking the divergence of (3.11), one finds

$$\nabla^2 \parallel \Lambda \parallel^2 + 2\nabla^i \log A \, \nabla_i \parallel \Lambda \parallel^2 = 0 \,. \tag{3.13}$$

An application of the Hopf maximum principle gives that $\parallel \Lambda \parallel^2$ is constant, which when inserted back into (3.9) and (3.11) yields

$$-\nabla_i \log A \parallel \Lambda \parallel^2 - \frac{1}{4} S\langle \Lambda, \Gamma_i \Lambda\rangle - \frac{1}{8}\langle \Lambda, \not{F} \Gamma_i \Gamma_{11} \Lambda\rangle - \frac{1}{96}\langle \Lambda, \not{Y} \Gamma_i \Lambda\rangle = 0 \,, \tag{3.14}$$

and

$$\langle \tau_+, \Gamma_{iz}\sigma_+\rangle = 0 \,, \tag{3.15}$$

respectively. The above condition can also be expressed as $\langle \sigma_+^1, \Gamma_{ix}\sigma_+^2\rangle = 0$ for any two solutions σ_+^1 and σ_+^2 of the KSEs.

As in IIB, the algebraic KSE (3.7) implies that $\langle \tau_+, \Xi^{(+)}\sigma_+\rangle - \langle \sigma_+, (\Xi^{(+)} + \ell^{-1})\tau_+\rangle = 0$. This together with (3.15) give that $\langle \sigma_+, \tau_+\rangle = 0$ and so the τ_+ and σ_+ Killing spinors are orthogonal.

3.2.2 The warp factor is constant

To begin, for every pair of solutions σ_+^1 and σ_+^2 of the KSEs we define the 1-form bilinear

$$W_i = A \, \mathrm{Im} \, \langle \sigma_+^1, \Gamma_{iz}\sigma_+^2\rangle \,. \tag{3.16}$$

Then the gravitino KSE (3.5) implies that

$$\nabla_{(i} W_{j)} = 0 \,, \tag{3.17}$$

therefore W is an Killing vector on M^6.

Next the difference $\langle \sigma_+^1, \Xi^{(+)}\sigma_+^2\rangle - \langle \sigma_+^2, \Xi^{(+)}\sigma_+^1\rangle = 0$ implies that

$$W^i \, \nabla_i A = 0 \,, \tag{3.18}$$

where we have used (3.15).

So far we have not used that the solutions preserve $N > 16$ supersymmetries. However if this is assumed, then (3.18) implies that the warp factor A is constant. This is a consequence of an adaptation of the homogeneity theorem on M^6. The homogeneity theorem also implies that Φ and S are constant. X is also constant as a consequence of the Bianchi

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identity (3.2). Therefore we have established that if the backgrounds preserve $N > 16$ supersymmetries, then

$$A = \text{const}\,, \quad \Phi = \text{const}\,, \quad S = \text{const}\,, \quad X = \text{const}\,. \tag{3.19}$$

As the warp factor is constant, all backgrounds that preserve $N > 16$ supersymmetries are products, $\mathrm{AdS}_4 \times M^6$. In addition as it has been explained in the introduction, M^6 is a homogeneous space admitting a transitive and effective action of a group G with Lie algebra $\mathfrak{so}(N/4)$. These homogeneous spaces have been listed in (1.1). In what follows, we shall explore all these 6-dimensional homogeneous spaces to search for IIA solutions that preserve $N > 16$ supersymmetries.

3.3 $N = 28$

There are no maximally supersymmetric AdS_4 backgrounds in (massive) IIA supergravity [9]. So the next case to be investigated is that with 28 supersymmetries. In such a case M^6 admits a transitive and effective action of a group with Lie algebra $\mathfrak{so}(7)$. Amongst the homogeneous spaces presented in (1.1), the only one with this property is $\mathrm{Spin}(7)/\mathrm{Spin}(6) = S^6$.

As $\mathrm{Spin}(7)/\mathrm{Spin}(6) = S^6$ is a symmetric space, all left-invariant forms are parallel with respect to the Levi-Civita connection and so represent classes in the de-Rham cohomology. As $H^2(S^6) = H^3(S^6) = H^4(S^6) = 0$, one concludes that $F = H = Y = 0$. Using this and (3.19), the dilatino KSE (3.6) implies that

$$\left(\frac{5}{4}S - \frac{1}{4}X\Gamma_{zx}\right)\sigma_+ = 0\,. \tag{3.20}$$

As it is the sum of two commuting terms one Hermitian and the other anti-Hermitian, the existence of solutions requires that both must vanish separately. As a result $S = X = 0$. Therefore all fluxes must vanish. This in turn leads to a contradiction as the field equation of the warp factor (3.4) cannot admit any solutions. Thus there are no (massive) IIA AdS_4 backgrounds preserving 28 supersymmetries.

3.4 $N = 24$

The internal space of AdS_4 backgrounds that preserve 24 supersymmetries admits a transitive and effective action of a group with Lie algebra $\mathfrak{so}(6) = \mathfrak{su}(4)$. The only space in (1.1) compatible with such an action is $\mathrm{SU}(4)/S(\mathrm{U}(1) \times \mathrm{U}(3)) = \mathbb{CP}^3$. Again this is a symmetric space and so all invariant forms are parallel with respect to the Levi-Civita connection. In turn they represent classes in the de-Rham cohomology. As $H^{\mathrm{odd}}(\mathbb{CP}^3) = 0$, this implies that $H = 0$.

It is well-known that this homogeneous space is a Kähler manifold and the left-invariant metric is given by the standard Fubini-Study metric on $\mathbb{CP}^3$. The even cohomology ring of $\mathbb{CP}^3$ is generated by the Kähler form ω. As a result the 2- and 4-form fluxes can be written as

$$F = \alpha\,\omega\,, \quad Y = \frac{1}{2}\,\beta\,\omega \wedge \omega\,, \tag{3.21}$$

for some real constants α and β to be determined.

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To determine α and β, let us first consider the dilatino KSE (3.6) which after imposing (3.19) reads

$$\left(\frac{5}{4}S+\frac{3}{8}\not{F}\Gamma_{11}+\frac{1}{96}\not{Y}-\frac{1}{4}X\Gamma_{zx}\right)\sigma_+=0\,. \tag{3.22}$$

The Hermitian and anti-Hermitian terms in this equation commute and so they can be separately imposed. Notice that the only non-trivial commutator to check is $[\not{F},\not{Y}]$ which vanishes because F is proportional to the Kähler form while Y is a (2,2)-form with respect to the associated complex structure. Thus we have

$$\left(\frac{3}{8}\not{F}\Gamma_{11}-\frac{1}{4}X\Gamma_{zx}\right)\sigma_+=0\,, \tag{3.23}$$

and

$$\left(\frac{5}{4}S+\frac{1}{96}\not{Y}\right)\sigma_+=0\,. \tag{3.24}$$

Inserting these into the algebraic KSE (3.7) simplifies to

$$(3S\Gamma_z-X\Gamma_x)\,\sigma_+=\frac{3}{\ell A}\sigma_+\,. \tag{3.25}$$

The integrability condition of this yields

$$X^2+9S^2=\frac{9}{\ell^2A^2}\,. \tag{3.26}$$

Next let us focus on (3.23) and (3.24). Choosing without loss of generality $\Gamma_{11}=\Gamma_{+-}\Gamma_{zx}\Gamma_{123456}$, (3.23) can be rewritten as

$$\alpha(\Gamma^{3456}+\Gamma^{1256}+\Gamma^{1234})\sigma_+=-\frac{X}{3}\sigma_+\,, \tag{3.27}$$

and similarly (3.24) as

$$\beta(\Gamma^{1234}+\Gamma^{1256}+\Gamma^{3456})\sigma_+=-5S\sigma_+\,, \tag{3.28}$$

where we have chosen an ortho-normal frame for which $\omega=\mathbf{e}^{12}+\mathbf{e}^{34}+\mathbf{e}^{56}$.

To solve (3.27) and (3.28), we decompose σ_+ into eigenspaces of $J_1=\Gamma_{3456}$ and $J_2=\Gamma_{1256}$ and find that this leads to the relations

$$\alpha=-\frac{1}{3}X,\quad \beta=-5S\,, \tag{3.29}$$

for the eigenspaces $|+,+\rangle$, $|+,-\rangle$, $|-,+\rangle$, and

$$\alpha=\frac{1}{9}X,\quad \beta=\frac{5}{3}S\,, \tag{3.30}$$

for the eigenspace $|-,-\rangle$.

Before we proceed to investigate the KSEs further, let us focus on the field equations for the fluxes and the warp factor. Observe that $\alpha\neq0$. Indeed if $\alpha=0$, then the KSEs would

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have implied that $X = 0$. As $H = X = 0$, the dilaton field equation in (3.3) implies that all fluxes vanish. In such a case, the warp factor field equation in (3.4) cannot be satisfied.

Thus $\alpha \neq 0$. Then the field equation for the 3-form flux in (3.3) becomes $\alpha(S+4\beta) = 0$ and so this implies that $\beta = -1/4\, S$. This contradicts the results from KSEs in (3.29) and (3.30) above unless $\beta = S = 0$. Setting $S = Y = 0$ in the dilaton field equation in (3.3), it is easy to see that it is satisfied if and only if $\alpha = -1/3X$ and so σ_+ lies in the eigenspaces $|+,+\rangle$, $|+,-\rangle$ and $|-,+\rangle$. As $S = 0$, (3.26) implies that $X = \pm 3\ell^{-1}A^{-1}$ and so $\alpha = \mp\ell^{-1}A^{-1}$. The algebraic KSE (3.25) now reads $\Gamma_x\sigma_+ = \mp\sigma_+$. As $\alpha = -1/3X$, the common eigenspace of Γ_x, Γ_{3456} and Γ_{1256} on σ_+ spinors has dimension 6. Thus the number of supersymmetries that the background

$$\begin{aligned} ds^2 &= 2du(dr - 2\ell^{-1}rdz) + A^2(dz^2 + e^{2z/\ell}dx^2) + ds^2(\mathbb{CP}^3)\,, \\ G &= \pm 3\ell^{-1}Ae^{z/\ell}du\wedge dr\wedge dz\wedge dx\,, \qquad & H = S = 0\,, \\ F &= \mp\ell^{-1}A^{-1}\omega, & \Phi = \text{const}\,, \end{aligned} \tag{3.31}$$

with $R^{(6)}_{ij}\delta^{ij} = 24\ell^{-2}A^{-2}$, can preserve is 24.

To establish that (3.31) preserves 24 supersymmetries, it remains to investigate the gravitino KSE (3.5). As $\mathbb{CP}^3$ is simply connected it is sufficient to investigate the integrability condition

$$\left(\frac{1}{4}R_{ijmn}\Gamma^{mn} - \frac{1}{8}F_{im}\,F_{jn}\Gamma^{mn} - \frac{1}{12}X\,F_{ij}\Gamma_{zx}\Gamma_{11} - \frac{1}{72}X^2\Gamma_{ij}\right)\sigma_+ = 0\,, \tag{3.32}$$

of the gravitino KSE. The Riemann tensor of $\mathrm{SU}(4)/S(\mathrm{U}(1)\times\mathrm{U}(3))$ is

$$R_{ij,kl} = \frac{1}{4\ell^2A^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \frac{3}{4\ell^2A^2}(\omega_{ij}\omega_{kl} - \omega_{i[j}\omega_{kl]})\,. \tag{3.33}$$

Then a substitution of this and the rest of the fluxes into the integrability condition reveals that it is satisfied without further conditions. In a similar manner, one can check that the Einstein equation along M^6 is also satisfied. This is the IIA $N = 24$ solution of [16, 28].

3.5 $N = 20$

The internal space of AdS_4 backgrounds that preserve 20 supersymmetries admits an effective and transitive action of a group which has Lie algebra $\mathfrak{so}(5) = \mathfrak{sp}(2)$. An inspection of the homogeneous spaces in table 1 reveals that there are two candidate internal spaces namely the symmetric space $\mathrm{Sp}(2)/\mathrm{U}(2)$ and the homogeneous space $\mathrm{Sp}(2)/\mathrm{Sp}(1)\times\mathrm{U}(1)$. The symmetric space is the space of complex structures on $\mathbb{H}^2$ which are compatible with the quaternionic inner product while the homogeneous space is identified with the coset space of the sphere $\bar{x}x + \bar{y}y = 1$, $x, y \in \mathbb{H}$, with respect to the action $(x, y) \to (ax, ay)$, $a \in \mathrm{U}(1)$. The latter is diffeomorphic to $\mathbb{CP}^3$.

3.5.1 Sp(2)/U(2)

The geometry and algebraic properties of this symmetric space are described in appendix E. The most general left-invariant metric is

$$ds^2 = a\,\delta_{rs}\delta_{ab}\ell^{ra}\ell^{sb} = \delta_{rs}\delta_{ab}\mathbf{e}^{ra}\mathbf{e}^{sb}\,, \tag{3.34}$$

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where $a > 0$ is a constant and ℓ^{ra}, and $\mathbf{e}^{ra} = \sqrt{a}\,\ell^{ra}$ are the left-invariant and ortho-normal frames, respectively, and where $r, s = 1, 2, 3$ and $a, b = 4, 5$. The invariant forms are generated by the 2-form

$$\omega = \frac{1}{2}\delta_{rs}\epsilon_{ab}\,\mathbf{e}^{ra} \wedge \mathbf{e}^{sb}\,. \tag{3.35}$$

$\mathrm{Sp}(2)/\mathrm{U}(2)$ is a Kähler manifold with respect to the pair (ds^2, ω).

To continue we choose the metric on the internal manifold as (3.34) and the fluxes as in the $\mathrm{SU}(4)/S(\mathrm{U}(1)\times\mathrm{U}(3))$ case, i.e.

$$F = \alpha\,\omega\,, \quad Y = \frac{1}{2}\beta\,\omega\wedge\omega\,, \tag{3.36}$$

but now ω is given in (3.35), where α and β are constants. Since there are no invariant 3-forms on $\mathrm{Sp}(2)/\mathrm{U}(2)$, this implies $H = 0$. Performing a similar analysis to that in section 3.4, we find that $\beta = S = 0$, $\alpha = \mp\ell^{-1}A^{-1}$ and $X = \pm 3\ell^{-1}A^{-1}$, and σ_+ to satisfy the same Clifford algebra projections as in e.g. (3.27). This requires an appropriate re-labeling of the indices of the ortho-normal frame $\mathbf{e}^{ra}$ so that the left-invariant tensors take the same canonical form as those of $\mathrm{SU}(4)/S(\mathrm{U}(1)\times\mathrm{U}(3))$ expressed in terms of the ortho-normal frame $\mathbf{e}^i$. As a result, there are 24 spinors that solve the KSEs so far.

It remains to investigate the solutions of the gravitino KSE (3.5). As in the $\mathrm{SU}(4)/S(\mathrm{U}(1)\times\mathrm{U}(3))$ case in section 3.4, we shall investigate the integrability condition instead. This is again given as in (3.32). The curvature of the metric of this symmetric space is presented in (E.7). Using this the integrability condition (3.32) is written as

$$\begin{aligned}\Big[&\frac{1}{16a}(\delta^{cd}\Gamma_{rcsd} - \delta^{cd}\Gamma_{scrd})\delta_{ab} + \frac{1}{16a}\delta^{tu}(\Gamma_{taub} - \Gamma_{tbua})\delta_{rs} - \\ &\frac{1}{8}\ell^{-2}A^{-2}(\delta^{cd}\Gamma_{rcsd}\delta_{ab} - \Gamma_{sbra}) + \frac{1}{4}\ell^{-2}A^{-2}\delta_{rs}\epsilon_{ab}\Gamma_{zx}\Gamma_{11} \\ &- \frac{1}{8}\ell^{-2}A^{-2}\Gamma_{rasb}\Big]\sigma_+ = 0\,.\end{aligned} \tag{3.37}$$

Contracting with δ_{ab}, one finds that there are solutions which preserve more than 8 supersymmetries provided $a = \ell^2 A^2$. Then taking the trace of (3.37) with $\epsilon_{ab}\delta_{rs}$, we find that

$$\frac{1}{2}\not{\omega}\sigma_+ = -12\Gamma_{zx}\Gamma_{11}\sigma_+\,, \tag{3.38}$$

which is in contradiction to the condition (3.23) arising from the dilatino KSE. The symmetric space $\mathrm{Sp}(2)/\mathrm{U}(2)$ does not yield[5] AdS_4 solutions that preserve 20 supersymmetries.

3.5.2 $\mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{U}(1))$

The $\mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{U}(1))$ homogeneous space is described in appendix E. Introducing the left-invariant frame $\ell^A m_A = \ell^a W_a + \ell^{\underline{r}} T^{(+)}_{\underline{r}}$, the most general left-invariant metric is

$$ds^2 = a\,\delta_{ab}\ell^a\ell^b + b\,\delta_{\underline{rs}}\ell^{\underline{r}}\ell^{\underline{s}} = \delta_{ab}\mathbf{e}^a\mathbf{e}^b + \delta_{\underline{rs}}\mathbf{e}^{\underline{r}}\mathbf{e}^{\underline{s}}, \tag{3.39}$$

[5] $\mathrm{Sp}(2)/\mathrm{U}(2)$ can also be excluded as a solution because it is not a spin manifold [24].

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where we have introduced the ortho-normal frame $\mathbf{e}^a = \sqrt{a}\,\ell^a$, $\mathbf{e}^{\underline{r}} = \sqrt{b}\,\ell^{\underline{r}}$, and where $\underline{r} = 1, 2$ and $a, b = 1, \ldots, 4$. The invariant forms are generated by

$$I_3^{(+)} = \frac{1}{2}(I_3^{(+)})_{ab}\mathbf{e}^a \wedge \mathbf{e}^b\,, \quad \tilde{\omega} = \frac{1}{2}\epsilon_{\underline{rs}}\mathbf{e}^{\underline{r}} \wedge \mathbf{e}^{\underline{s}}\,, \quad \mathbf{e}^{\underline{r}} \wedge I_{\underline{r}}^{(+)}\,, \tag{3.40}$$

and their duals, where

$$I_{\underline{r}}^{(+)} = \frac{1}{2}(I_{\underline{r}}^{(+)})_{ab}\mathbf{e}^a \wedge \mathbf{e}^b\,. \tag{3.41}$$

The matrices $\left((I_r^{(\pm)})_{ab}\right)$ are a basis in the space of self-dual and anti-self dual 2-forms in $\mathbb{R}^4$ and are defined in (E.10). Imposing the Bianchi identities (3.2), one finds the relation

$$\frac{\alpha}{\sqrt{b}} - \frac{\beta\sqrt{b}}{2a} = S\,h\,, \tag{3.42}$$

and that the fluxes can be written as

$$\begin{aligned} F &= \alpha I_3^{(+)} + \beta\,\tilde{\omega}\,, \qquad H = h\,\epsilon_{\underline{rs}}\,\mathbf{e}^{\underline{r}} \wedge I_{\underline{s}}^{(+)}\,, \\ Y &= \gamma\,\tilde{\omega} \wedge I_3^{(+)} + \frac{1}{2}\,\delta\,I_3^{(+)} \wedge I_3^{(+)}\,, \end{aligned} \tag{3.43}$$

where α, β, h, γ and δ are constants.

The dilatino KSE (3.6) is the sum of hermitian and anti-hermitian Clifford algebra elements which commute and thus lead to the two independent conditions

$$\begin{aligned} \left(\frac{3}{8}\not{F}\Gamma_{11} - \frac{1}{4}X\Gamma_{zx}\right)\sigma_+ &= 0\,, \\ \left(\frac{5}{4}\,S + \frac{1}{12}\not{H}\Gamma_{11} + \frac{1}{96}\not{Y}\right)\sigma_+ &= 0\,. \end{aligned} \tag{3.44}$$

Using this to simplify the algebraic KSE (3.7), one finds

$$\left(\frac{1}{12}\not{H}\,\Gamma_{11}\Gamma_z + S\Gamma_z - \frac{X}{3}\Gamma_x\right)\sigma_+ = \frac{1}{\ell A}\,\sigma_+\,. \tag{3.45}$$

If we then insert the fluxes (3.43) into the above KSEs and set $J_1 = \Gamma^{24\underline{1}}\Gamma_{11}$, $J_2 = \Gamma^{13\underline{1}}\Gamma_{11}$ and $J_3 = \Gamma^{23\underline{2}}\Gamma_{11}$, we obtain

$$\begin{aligned} (\alpha(J_2J_3 - J_1J_3) + \beta J_1J_2)\,\sigma_+ + \frac{X}{3}\sigma_+ &= 0\,, \\ (5S + 2h(J_1 - J_2 - J_3 + J_1J_2J_3) + \gamma\,(J_2J_3 - J_1J_3) + \delta\,J_1J_2)\,\sigma_+ &= 0\,, \\ \left(\frac{1}{2}h(J_1 - J_2 - J_3 + J_1J_2J_3)\Gamma_z + S\Gamma_z - \frac{X}{3}\Gamma_x\right)\sigma_+ - \frac{1}{\ell A}\,\sigma_+ &= 0\,. \end{aligned} \tag{3.46}$$

As J_1, J_2, J_3 are commuting Hermitian Clifford algebra operators with eigenvalues ± 1, the KSE (3.45) can be decomposed along the common eigenspaces as described in table 2.

From the results of table 2, there are two possibilities to choose five σ_+ Killing spinors, namely those in eigenspaces (1) and (3) and those in eigenspaces (1) and (4). For both of these choices, the Bianchi identity (3.42) and the dilaton field equation give

$$\alpha = \beta = -\frac{X}{3}, \quad X = \pm\frac{3}{\ell A}, \quad b = 2a, \quad S = h = \gamma = \delta = 0\,. \tag{3.47}$$

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	$\vert J_1, J_2, J_3\rangle$	relations for the fluxes
(1)	$\vert+,+,+\rangle$, $\vert-,-,-\rangle$ $\vert+,+,-\rangle$, $\vert-,-,+\rangle$	$\beta = -\frac{X}{3}$, $5S + \delta = 0$ $(S\,\Gamma_z - \frac{X}{3}\Gamma_x)\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$
(2)	$\vert+,-,+\rangle$, $\vert-,+,-\rangle$	$2\alpha + \beta = \frac{X}{3}$, $5S - 2\gamma - \delta = 0$ $(S\,\Gamma_z - \frac{X}{3}\Gamma_x)\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$
(3)	$\vert+,-,-\rangle$	$2\alpha - \beta = -\frac{X}{3}$, $5S + 8h + 2\gamma - \delta = 0$ $((S+2h)\,\Gamma_z - \frac{X}{3}\Gamma_x)\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$
(4)	$\vert-,+,+\rangle$	$2\alpha - \beta = -\frac{X}{3}$, $5S - 8h + 2\gamma - \delta = 0$ $((S-2h)\,\Gamma_z - \frac{X}{3}\Gamma_x)\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$

Table 2. Decomposition of (3.46) KSE into eigenspaces.

In either case notice that these conditions imply the existence of six σ_+ Killing spinors as the conditions required for both $|+,-,-\rangle$ and $|-,+,+\rangle$ to be solutions are satisfied. So potentially this background can preserve $N = 24$ supersymmetries. To summarize, the independent conditions on the Killing spinors arising from those in (3.44) and those in table 2 are

$$\frac{1}{2}\left(I_3^{(+)} + \tilde{\not{\varphi}}\right)\sigma_+ = \sigma_+\,, \quad \Gamma_x\sigma_+ = -\frac{3}{\ell A X}\sigma_x\,. \tag{3.48}$$

These are the same conditions as those found in section 3.4 for $M^6 = \mathbb{CP}^3$.

It remains to investigate the gravitino KSE (3.5) or equivalently, as $\mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{U}(1))$ is simply connected, the corresponding integrability condition given again in (3.32). The curvature of the metric is given in (E.15). Moreover the Einstein equation (3.4) gives $a = \ell^2 A^2/2$. Using these and substituting the conditions (3.47) into the integrability condition, one can show that this is automatically satisfied provided that (3.48) holds. As a result, there are no AdS_4 backgrounds with internal space $\mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{U}(1)$ which preserve *strictly* 20 supersymmetries. However as shown above, there is a solution which preserves 24 supersymmetries for $b = 2a$. This is locally isometric to the $\mathrm{AdS}_4\times\mathbb{CP}^3$ solution found in section 3.4. Note that there are no $N > 24$ solutions as it can be seen by a direct computation or by observing that $\mathbb{CP}^3$ does not admit an effective and transitive action by the $\mathfrak{so}(N/4)$ subalgebra of the Killing superalgebra of such backgrounds. However there are $\mathrm{AdS}_4\times\mathrm{Sp}(2)/(\mathrm{Sp}(1)\times\mathrm{U}(1))$ solutions which preserve 4 supersymmetries [32].

4 $N > 16$ $\mathrm{AdS}_4\times_w M^7$ solutions in 11 dimensions

4.1 AdS_4 solutions in $D = 11$

Let us first summarize some of the properties of $\mathrm{AdS}_4\times_w M^7$ backgrounds in 11-dimensional supergravity as described in [5] that we shall use later. The bosonic fields are given as

$$\begin{aligned} ds^2 &= 2\mathbf{e}^+\mathbf{e}^- + (\mathbf{e}^z)^2 + (\mathbf{e}^x)^2 + ds^2(M^7)\,,\\ F &= X\,\mathbf{e}^+\wedge\mathbf{e}^-\wedge\mathbf{e}^z\wedge\mathbf{e}^x + Y\,, \end{aligned} \tag{4.1}$$

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where the null ortho-normal frame $(\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^z, \mathbf{e}^x, \mathbf{e}^i)$ is as in (2.3), but now $i, j = 1, \ldots, 7$, and the metric on the internal space M^7 is $ds^2(M^7) = \delta_{ij}\mathbf{e}^i\mathbf{e}^j$. X and Y are a function and 4-form on M^7, respectively.

The Bianchi identities of the 11-dimensional supergravity evaluated on the $\mathrm{AdS}_4 \times_w M^7$ background yield

$$dY = 0, \quad d(A^4 X) = 0\,. \tag{4.2}$$

Similarly, the field equations give

$$\nabla^k Y_{ki_1i_2i_3} + 4\nabla^k A\, Y_{ki_1i_2i_3} = -\frac{1}{24} X \epsilon_{i_1i_2i_3}{}^{k_1k_2k_3k_4} Y_{k_1k_2k_3k_4}\,, \tag{4.3}$$

$$\nabla^k \partial_k \log A = -\frac{3}{\ell^2 A^2} - 4\partial_k \log A\, \partial^k \log A + \frac{1}{3}X^2 + \frac{1}{144}Y^2\,, \tag{4.4}$$

and

$$R^{(7)}_{ij} - 4\nabla_i \partial_j \log A - 4\partial_i \log A \partial_j \log A = \frac{1}{12} Y^2_{ij} + \delta_{ij}\left(\frac{1}{6}X^2 - \frac{1}{144}Y^2\right), \tag{4.5}$$

where ∇ is the Levi-Civita connection on M^7.

4.2 The Killing spinors

The solution of the KSEs of $D = 11$ supergravity along the AdS_4 subspace of $\mathrm{AdS}_4 \times_w M^7$ given in [5] can be expressed as in (2.7) but now $\sigma_\pm$ and $\tau_\pm$ are $\mathfrak{spin}(10,1)$ Majorana spinors that depend on the coordinates of M^7. Again they satisfy the lightcone projections $\Gamma_\pm \sigma_\pm = \Gamma_\pm \tau_\pm = 0$. The remaining independent KSEs are

$$\nabla^{(\pm)}_i \sigma_\pm = 0\,, \quad \nabla^{(\pm)}_i \tau_\pm = 0\,, \tag{4.6}$$

and

$$\Xi^{(\pm)} \sigma_\pm = 0\,, \quad \left(\Xi^{(\pm)} \pm \frac{1}{\ell}\right)\tau_\pm = 0\,, \tag{4.7}$$

where

$$\nabla^{(\pm)}_i = \nabla_i \pm \frac{1}{2}\partial_i \log A - \frac{1}{288}\Gamma\!\!\!/ Y_i + \frac{1}{36} Y\!\!\!/_i \pm \frac{1}{12} X \Gamma_{izx}\,, \tag{4.8}$$

$$\Xi^{(\pm)} = \mp\frac{1}{2\ell} - \frac{1}{2}\Gamma_z \partial\!\!\!/ A + \frac{1}{288} A \Gamma_z Y\!\!\!/ \pm \frac{1}{6} A X \Gamma_x\,. \tag{4.9}$$

The former KSE is the restriction of the gravitino KSE on $\sigma_\pm$ and $\tau_\pm$ while the latter arises as an integrability condition as a result of integrating the gravitino KSE of 11-dimensional supergravity over the AdS_4 subspace of $\mathrm{AdS}_4 \times_w M^7$.

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4.3 AdS_4 solutions with $N > 16$ in 11 dimensions

4.3.1 Conditions on spinor bilinears

The conditions that arise from the assumption that M^7 be compact without boundary and the solutions be smooth are similar to those presented in the (massive) IIA case. In particular, one finds

$$\| \sigma_+ \| = \text{const}\,, \quad \langle \tau_+, \Gamma_{iz}\sigma_+ \rangle = 0\,, \quad \langle \sigma_+, \tau_+ \rangle = 0\,. \tag{4.10}$$

The proof follows the same steps as in the (massive) IIA case and so we shall not repeat it here.

4.3.2 The warp factor is constant

Using arguments similar to those presented in the (massive) IIA case, one finds that $W_i = A \operatorname{Im} \langle \sigma^1_+, \Gamma_{iz}\sigma^2_+ \rangle$ are Killing vectors on M^7 for any pair of Killing spinors σ^1_+ and σ^2_+ and that $i_W dA = 0$.

Next, let us suppose that the backgrounds preserve $N > 16$ supersymmetries. In such a case a similar argument to that presented for the proof of the homogeneity conjecture implies that the W vector fields span the tangent space of M^7 at every point and so A is constant. From the Bianchi identity (4.2) it then follows that X is constant as well. Thus we have established that

$$A = \text{const}\,, \quad X = \text{const}\,. \tag{4.11}$$

As a result, the space time is a product $AdS_4 \times M^7$, where M^7 is a homogeneous space. Further progress requires the investigation of individual homogeneous spaces of dimension 7 which have been classified in [26, 27] and they are presented in table 3. Requiring in addition that the homogeneous spaces which can occur as internal spaces of $N > 16$ AdS_4 backgrounds must admit an effective and transitive action of a group that has Lie algebra $\mathfrak{so}(N/4)$, one arrives at the homogeneous spaces presented in (1.2). In what follows, we shall investigate in detail the geometry of these homogeneous spaces to search for $N > 16$ AdS_4 backgrounds in 11-dimensional supergravity.

4.4 $N = 28$, $\mathbf{Spin(7)}/G_2$

The maximally supersymmetric solutions have been classified before [9] where it has been shown that all are locally isometric to $AdS_4 \times S^7$ with $S^7 = \text{Spin}(8)/\text{Spin}(7)$. The only solution that may preserve $N = 28$ supersymmetries is associated with the homogeneous space $\text{Spin}(7)/G_2$, see (1.2). The Lie algebra $\mathfrak{spin}(7) = \mathfrak{so}(7)$ is again spanned by matrices M_{ij} as in (E.1) satisfying the commutation relations (E.2) where now $i, j = 1, 2, \ldots, 7$. Let us denote the generators of $\mathfrak{g}_2$ subalgebra of $\mathfrak{spin}(7)$ and those of the module $\mathfrak{m}$, $\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$, with G and A, respectively. These are defined as

$$G_{ij} = M_{ij} + \frac{1}{4} *_7 \varphi_{ij}{}^{kl} M_{kl}\,, \quad A_i = \varphi_i{}^{jk} M_{jk}\,, \tag{4.12}$$

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	$M^7 = G/H$
(1)	$\frac{\mathrm{Spin}(8)}{\mathrm{Spin}(7)} = S^7$, symmetric space
(2)	$\frac{\mathrm{Spin}(7)}{G_2} = S^7$
(3)	$\frac{\mathrm{SU}(4)}{\mathrm{SU}(3)}$ diffeomorphic to S^7
(4)	$\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)}$ diffeomorphic to S^7
(5)	$\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)_{\max}}$, Berger space
(6)	$\frac{\mathrm{Sp}(2)}{\Delta(\mathrm{Sp}(1))} = V_2(\mathbb{R}^5)$
(7)	$\frac{\mathrm{SU}(3)}{\Delta_{k,l}(\mathrm{U}(1))} = W^{k,l}$ k,l coprime, Aloff-Wallach space
(8)	$\frac{\mathrm{SU}(2)\times\mathrm{SU}(3)}{\Delta_{k,l}(\mathrm{U}(1))\cdot(1\times\mathrm{SU}(2))} = N^{k,l}$ k,l coprime
(9)	$\frac{\mathrm{SU}(2)^3}{\Delta_{p,q,r}(\mathrm{U}(1)^2)} = Q^{p,q,r}$ p,q,r coprime
(10)	$M^4 \times M^3$, $M^4 = \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(4)}$, $\frac{\mathrm{SU}(3)}{S(\mathrm{U}(1)\times\mathrm{U}(2))}$, $\frac{\mathrm{SU}(2)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}$ $M^3 = \mathrm{SU}(2)$, $\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\Delta(\mathrm{SU}(2))}$
(11)	$M^5 \times \frac{\mathrm{SU}(2)}{\mathrm{U}(1)}$, $M^5 = \frac{\mathrm{Spin}(6)}{\mathrm{Spin}(5)}$, $\frac{\mathrm{SU}(3)}{\mathrm{SU}(2)}$, $\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\Delta_{k,l}(\mathrm{U}(1))}$, $\frac{\mathrm{SU}(3)}{\mathrm{SO}(3)}$

Table 3. 7-dimensional compact, simply connected, homogeneous spaces.

where φ is the fundamental G_2 3-form, $*_7\varphi$ is its dual and $*_7$ is the duality operation along the 7-dimensional internal space. The non-vanishing components of φ and $*_7\varphi$ can be chosen as

$$\begin{aligned} &\varphi_{123} = \varphi_{147} = \varphi_{165} = \varphi_{246} = \varphi_{257} = \varphi_{354} = \varphi_{367} = 1\,, \\ &*_7\varphi_{1276} = *_7\varphi_{1245} = *_7\varphi_{1346} = *_7\varphi_{1357} = *_7\varphi_{2374} = *_7\varphi_{2356} = *_7\varphi_{4567} = 1\,, \end{aligned} \tag{4.13}$$

and we have raised the indices above using the flat metric. We have used the conventions for φ and $*_7\varphi$ of [29], where also several useful identities satisfied by φ and $*_7\varphi$ are presented. In particular observe that $\varphi_i{}^{jk}G_{jk} = 0$. The $\mathfrak{spin}(7)$ generators can be written as

$$M_{ij} = \frac{2}{3}\,G_{ij} + \frac{1}{6}\varphi_{ij}{}^{k}\,A_k\,, \tag{4.14}$$

and using this we obtain

$$\begin{aligned} [G_{ij}, G_{kl}] &= \frac{1}{2}(\delta_{il}G_{jk} + \delta_{jk}G_{il} - \delta_{ik}G_{jl} - \delta_{jl}G_{ik}) + \frac{1}{4}(*_7\varphi_{ij[k}{}^{m}G_{\ell]m} - *_7\varphi_{k\ell[i}{}^{m}G_{j]m})\,, \\ [A_i, G_{jk}] &= \frac{1}{2}(\delta_{ij}\,A_k - \delta_{ik}\,A_j) + \frac{1}{4}\,*_7\,\varphi_{ijk}{}^{l}A_l\,, \\ [A_i, A_j] &= \varphi_{ij}{}^{k}\,A_k - 4G_{ij}\,. \end{aligned} \tag{4.15}$$

Clearly, $\mathrm{Spin}(7)/G_2$ is a homogeneous space. As G_2 acts with the irreducible 7-dimensional representation on $\mathfrak{m}$, the left-invariant metric on $\mathrm{Spin}(7)/G_2$ is unique up to scale, therefore

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$\vert P_1, P_2, P_3\rangle$	relations for the fluxes
$\vert+,+,+\rangle$, $\vert+,+,-\rangle$, $\vert-,+,+\rangle$, $\vert+,-,-\rangle$ $\vert-,+,-\rangle$, $\vert-,-,+\rangle$, $\vert-,-,-\rangle$	$(-\frac{1}{6}\alpha\,\Gamma_z + \frac{1}{3}X\Gamma_x)\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$
$\vert+,-,+\rangle$	$(\frac{7}{6}\alpha\,\Gamma_z + \frac{1}{3}X\Gamma_x)\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$

Table 4. Decomposition of (4.19) KSE into eigenspaces.

we may choose an ortho-normal frame $\mathbf{e}^i$ such that

$$ds^2 = a\,\delta_{ij}\ell^i\ell^j = \delta_{ij}\mathbf{e}^i\mathbf{e}^j\,, \tag{4.16}$$

where $a > 0$ is a constant. The left-invariant forms are

$$\varphi = \frac{1}{3!}\,\varphi_{ijk}\,\mathbf{e}^i\wedge\mathbf{e}^j\wedge\mathbf{e}^k\,, \tag{4.17}$$

and its dual $*_7\varphi$. So the Y flux can be chosen as

$$Y = \alpha\;*_7\varphi\,,\quad \alpha = \text{const}\,. \tag{4.18}$$

Using this the algebraic KSE (4.7) can be expressed as

$$\left(\frac{1}{6}\alpha\,(P_1 - P_2 + P_3 - P_1\,P_2\,P_3 - P_2\,P_3 + P_1\,P_3 - P_1\,P_2)\,\Gamma_z + \frac{1}{3}X\,\Gamma_x\right)\sigma_+ = \frac{1}{\ell A}\sigma_+\,, \tag{4.19}$$

where $\{P_1, P_2, P_3\} = \{\Gamma^{1245}, \Gamma^{1267}, \Gamma^{1346}\}$ are mutually commuting, hermitian Clifford algebra operators with eigenvalues ±1. The solutions of the algebraic KSE on the eigenspaces of $\{P_1, P_2, P_3\}$ have been tabulated in table 4.

For backgrounds preserving $N > 16$ supersymmetries, one has to choose the first set of solutions in table 4 and so impose the condition

$$\frac{1}{36}\alpha^2 + \frac{1}{9}X^2 = \frac{1}{\ell^2A^2}\,. \tag{4.20}$$

However, the field equation for the warp factor A (4.4) gives

$$\frac{3}{\ell^2A^2} = \frac{1}{3}X^2 + \frac{7}{6}\alpha^2\,. \tag{4.21}$$

These two equations imply that $\alpha = 0$ and so $Y = 0$.

As $Y = 0$, the algebraic KSE is simplified to

$$\Gamma_x\sigma_+ = \frac{3}{\ell AX}\,\sigma_+\,, \tag{4.22}$$

and so σ_+ lies in one of the 8-dimensional eigenspaces of Γ_x provided that $X = \pm\frac{3}{\ell A}$. Thus instead of preserving 28 supersymmetries, the solution can be maximally supersymmetric.

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Indeed this is the case as we shall now demonstrate. The integrability condition of the gravitino KSE (4.6) becomes

$$\left(R_{ijk\ell}\,\Gamma^{k\ell}-\frac{1}{18}\,X^2\Gamma_{ij}\right)\sigma_+=0\,. \tag{4.23}$$

To investigate whether this can yield a new condition on σ_+, we find after a direct computation using the results of appendix B that the Riemann tensor in the ortho-normal frame is given by

$$R_{ijk\ell}=\frac{9}{4}a^{-1}(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk})\,. \tag{4.24}$$

So $S^7=\mathrm{Spin}(7)/G_2$ is equipped with the round metric. For supersymmetric solutions, one must set $a^{-1}=\frac{1}{81}X^2=\frac{1}{9\ell^2A^2}$. In such a case, the integrability condition of the gravitino KSE is automatically satisfied and so the solution preserves 32 supersymmetries. This solution is locally isometric to the maximally supersymmetric $\mathrm{AdS}_4\times S^7$ solution.

4.5 $\boldsymbol{N=24,\ \mathrm{SU}(4)/\mathrm{SU}(3)}$

As $\mathfrak{so}(6)=\mathfrak{su}(4)$, it follows from (1.2) that the internal space of an AdS_4 solution with 24 supersymmetries is the 7-dimensional homogeneous manifold $\mathrm{SU}(4)/\mathrm{SU}(3)$. The geometry of this homogeneous space is described in appendix C. The left-invariant metric can be rewritten as

$$ds^2=a\,\delta_{mn}\ell^m\ell^n+b\,(\ell^7)^2=\delta_{mn}\mathbf{e}^m\mathbf{e}^n+(\mathbf{e}^7)^2\,, \tag{4.25}$$

where we have introduced an ortho-normal frame $\mathbf{e}^m=\sqrt{a}\ell^m$, $\mathbf{e}^7=\sqrt{b}\ell^7$, and $m,n=1,\ldots,6$. The most general left-invariant 4-form flux Y can be chosen as

$$Y=\frac{1}{2}\,\alpha\,\omega\wedge\omega+\beta\,*_7(\mathrm{Re}\,\chi)+\gamma\,*_7(\mathrm{Im}\,\chi)\,, \tag{4.26}$$

where α,β,γ are constants and the left-invariant 4-forms are

$$\begin{aligned}
&*_7(\mathrm{Re}\,\chi)=\mathbf{e}^{1367}+\mathbf{e}^{1457}+\mathbf{e}^{2357}-\mathbf{e}^{2467}\,,\quad \omega=\mathbf{e}^{12}+\mathbf{e}^{34}+\mathbf{e}^{56}\,,\\
&*_7(\mathrm{Im}\,\chi)=-\mathbf{e}^{1357}+\mathbf{e}^{1467}+\mathbf{e}^{2367}+\mathbf{e}^{2457}\,,
\end{aligned} \tag{4.27}$$

expressed in terms of the ortho-normal frame. Having specified the fields, it remains to solve the KSEs. For this define the mutually commuting Clifford algebra operators

$$\begin{aligned}
&J_1=\cos\theta\,\Gamma^{1367}+\sin\theta\,\Gamma^{2457}\,,\quad J_2=\cos\theta\,\Gamma^{1457}+\sin\theta\,\Gamma^{2367}\,,\\
&J_3=\cos\theta\,\Gamma^{2357}+\sin\theta\,\Gamma^{1467}\,,
\end{aligned} \tag{4.28}$$

with eigenvalues ±1, where $\tan\theta=\gamma/\beta$. Then upon inserting Y into the algebraic KSE (4.7) and using the above Clifford algebra operators, we obtain

$$\begin{aligned}
\Big[&-\frac{\alpha}{6}(J_1J_2+J_1J_3+J_2J_3)\,\Gamma_z+\frac{\sqrt{\beta^2+\gamma^2}}{6}(J_1+J_2+J_3+J_1J_2J_3)\,\Gamma_z\\
&+\frac{1}{3}X\Gamma_x\Big]\sigma_+=\frac{1}{\ell A}\sigma_+\,.
\end{aligned} \tag{4.29}$$

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$\vert J_1, J_2, J_3\rangle$	relations for the fluxes
$\vert+,+,-\rangle$, $\vert+,-,+\rangle$, $\vert-,+,+\rangle$ $\vert+,-,-\rangle$, $\vert-,+,-\rangle$, $\vert-,-,+\rangle$	$(\frac{1}{6}\alpha\,\Gamma_z + \frac{1}{3}X\Gamma_x)\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$
$\vert+,+,+\rangle$	$[(-\frac{\alpha}{2} + \frac{2}{3}\sqrt{\beta^2+\gamma^2})\,\Gamma_z + \frac{1}{3}X\Gamma_x]\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$
$\vert-,-,-\rangle$	$[(-\frac{\alpha}{2} - \frac{2}{3}\sqrt{\beta^2+\gamma^2})\,\Gamma_z + \frac{1}{3}X\Gamma_x]\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$

Table 5. Decomposition of (4.29) KSE into eigenspaces.

The algebraic KSE (4.7) can then be decomposed into the eigenspaces of J_1, J_2 and J_3. The different relations on the fluxes for all possible sets of eigenvalues of these operators are listed in table 5.

The only possibility to obtain solutions with $N > 16$ supersymmetries is to choose the first set of eigenspinors in table 5. This leads to the integrability condition

$$\frac{\alpha^2}{36} + \frac{1}{9}X^2 = \frac{1}{\ell^2 A^2}\,, \tag{4.30}$$

from the remaining KSE. This together with the warp factor field equation (4.4)

$$\frac{1}{3}X^2 + \frac{1}{2}\alpha^2 + \frac{2}{3}(\beta^2+\gamma^2) = \frac{3}{\ell^2 A^2}\,, \tag{4.31}$$

implies

$$\frac{5}{4}\alpha^2 + 2(\beta^2+\gamma^2) = 0\,, \tag{4.32}$$

and so $\alpha = \beta = \gamma = 0$. Therefore $Y = 0$ and the solution is electric. As a result, the algebraic KSE (4.4) becomes

$$\Gamma_x\sigma_+ = \frac{3}{\ell A X}\sigma_+\,, \tag{4.33}$$

and so for $X = \pm 3\ell^{-1}A^{-1}$ it admits 8 linearly independent σ_+ solutions. So potentially, the background is maximally supersymmetric.

It remains to investigate the gravitino KSE. First of all, we observe that for $Y = 0$ the Einstein equation (4.5) along the internal space becomes

$$R_{ij} = \frac{1}{6}X^2\delta_{ij}\,. \tag{4.34}$$

Therefore, the internal space is Einstein. After some computation using the results in appendix C, one finds that the homogeneous space SU(4)/SU(3) is Einstein provided that $b = \frac{9}{4}a$. In that case, the curvature of the metric in the ortho-normal frame becomes

$$R_{ij,mn} = \frac{1}{4a}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\,, \tag{4.35}$$

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and so the internal space is locally isometric to the round 7-sphere. As expected from this, the integrability condition of the gravitino KSE (4.6)

$$\left(R_{ij,mn}\Gamma^{mn} - \frac{1}{18}X^2\Gamma_{ij}\right)\sigma_+ = 0\,, \tag{4.36}$$

has non-trivial solutions for $X^2 = 9a^{-1}$, i.e. $a = \ell^2 A^2$ and $b = \frac{9}{4}\ell^2 A^2$. With this identification of parameters, $\mathrm{AdS}_4 \times \mathrm{SU}(4)/\mathrm{SU}(3)$ is locally isometric to the maximally supersymmetric $\mathrm{AdS}_4 \times S^7$ background.

To summarize there are no AdS_4 solutions with internal space $\mathrm{SU}(4)/\mathrm{SU}(3)$ which preserve $16 < N < 32$ supersymmetries. However, for the choice of parameters for which $\mathrm{SU}(4)/\mathrm{SU}(3)$ is the round 7-sphere, the solution preserves 32 supersymmetries as expected.

4.6 $N = 20$

As mentioned in the introduction, the internal space of AdS_4 backgrounds that preserve 20 supersymmetries admits an effective and transitive action of a group which has Lie algebra $\mathfrak{so}(5) = \mathfrak{sp}(2)$. The field equation for Y (4.3) is

$$d *_7 Y = X\, Y\,. \tag{4.37}$$

As X is constant, note that for generic 4-forms Y this defines a nearly-parallel G_2-structure on M^7, see e.g. [33] for homogeneous G_2 structures. However, in what follows we shall not assume that Y is generic. In fact in many cases, it vanishes.

Amongst the 7-dimensional compact homogeneous spaces of (1.2), there are three candidate internal spaces. These are the Berger space $B^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)_{\mathrm{max}}$, $V_2(\mathbb{R}^5) = \mathrm{Sp}(2)/\Delta(\mathrm{Sp}(1))$, and $J^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, corresponding to the three inequivalent embeddings of $\mathrm{Sp}(1)$ into $\mathrm{Sp}(2)$. We will in the following examine each case separately, starting with the Berger space $\mathrm{Sp}(2)/\mathrm{Sp}(1)_{\mathrm{max}}$.

4.6.1 $\mathbf{Sp(2)/Sp(1)_{max}}$

The description of the Berger space $B^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)_{\mathrm{max}}$ as a homogeneous manifold is summarized in appendix D. B^7 is diffeomeorphic to the total space of an S^3 bundle over S^4 with Euler class ∓ 10 and first Pontryagin class ∓ 16 [34]. As a result $H^4(B^7, \mathbb{Z}) = \mathbb{Z}_{10}$ and B^7 is a rational homology 7-sphere. As $\mathfrak{sp}(2) = \mathfrak{so}(5)$ and $\mathfrak{sp}(1) = \mathfrak{so}(3)$, one writes $\mathfrak{so}(5) = \mathfrak{so}(3) \oplus \mathfrak{m}$ and the subalgebra $\mathfrak{so}(3)$ acts irreducibly on $\mathfrak{m}$ with the $\mathbf{7}$ representation. So B^7 admits a unique invariant metric up to a scale and it is Einstein. As the embedding of $\mathfrak{so}(3)$ into $\mathfrak{so}(7)$ factors through $\mathfrak{g}_2$, it also admits an invariant 3-form φ given in (4.13) which is unique up to a scale. Because there is a unique invariant 3-form φ, $d\varphi \propto *_7\varphi$ and B^7 is a nearly parallel G_2 manifold. Using these, we find that the invariant fields of the theory are

$$ds^2 = a\delta_{ij}\ell^i\ell^j = \delta_{ij}\mathbf{e}^i\mathbf{e}^j\,, \qquad Y = \frac{1}{4!}\alpha *_7 \varphi_{ijkm}\,\mathbf{e}^i \wedge \mathbf{e}^j \wedge \mathbf{e}^k \wedge \mathbf{e}^m\,, \tag{4.38}$$

where we have introduced the ortho-normal frame $\mathbf{e}^i = \sqrt{a}\,\ell^i$, $*_7\varphi$ is given in (4.13) and a, α are constants with $a > 0$.

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As the pair (ds^2, Y) exhibits the same algebraic relations as that of the $\mathrm{Spin}(7)/G_2$ case, the algebraic KSE (4.19) can be solved in the same way yielding the results of table 4. To find $N > 16$ AdS_4 solutions, one should consider the first set of eigenspinors of the table which in turn imply the relation (4.20) amongst the fluxes. This together with the field equation of the warp factor (4.21) leads again to the conclusion that $\alpha = 0$ and so $Y = 0$.

As a result of the analysis of the algebraic KSE, so far the background can admit up to 32 supersymmetries. It remains to investigate the solutions of the gravitino KSE. The curvature of B^7 is given by

$$R_{ij,km} = \frac{1}{10\,a}\delta_{k[i}\,\delta_{j]m} - \frac{1}{5a}\,*_7\,\varphi_{ijkm} + \frac{1}{a}\delta_{\alpha\beta}k^{\alpha}_{ij}k^{\beta}_{km}\,, \tag{4.39}$$

where k^α is given in appendix D. The integrability condition of the gravitino KSE for $Y = 0$ is given in (4.23). To solve this condition, we decompose the expression into the **7** and **14** representations of $\mathfrak{g}_2$ using the projectors

$$(P^{\mathbf{7}})^{ij}{}_{km} = \frac{1}{3}\left(\delta^i_{[k}\delta^j_{m]} - \frac{1}{2}\,*_7\,\varphi^{ij}{}_{km}\right), \quad (P^{\mathbf{14}})^{ij}{}_{km} = \frac{2}{3}\left(\delta^i_{[k}\delta^j_{m]} + \frac{1}{4}\,*_7\,\varphi^{ij}{}_{km}\right), \tag{4.40}$$

and noting that k^α as 2-forms are in the **14** representation. The integrability condition along the **7** representation gives $X^2 = \frac{81}{5}a^{-1}$ while along the **14** representation gives that the Killing spinors must be invariant under $\mathfrak{g}_2$. It turns out that there are two such σ_+ spinors however taking into account the remaining projection arising from the algebraic KSE, see (4.22), we deduce that the solution preserves 4 supersymmetries in total. This solution has already been derived in [23].

4.6.2 $\mathbf{Sp(2)/\Delta(Sp(1))}$

The decomposition of the Lie algebra $\mathfrak{sp}(2) = \mathfrak{so}(5)$ suitable to describe this homogeneous space can be found in appendix E. Writing $\ell^A m_A = \ell^{ra}M_{ra} + \ell^7 T_7$ for the left-invariant frame, $r = 1, 2, 3$ and $a = 4, 5$, the most general left-invariant metric is

$$ds^2 = \delta_{rs}g_{ab}\ell^{ra}\ell^{sb} + a_4(\ell^7)^2\,, \tag{4.41}$$

where g_{ab} is a positive definite symmetric 2×2-matrix, $a > 0$ a constant, and the left-invariant forms are generated by

$$\ell^7 = \ell^7\,, \quad \frac{1}{2}\delta_{rs}\epsilon_{ab}\ell^{ra}\wedge\ell^{sb}\,, \quad \frac{1}{3!}\epsilon_{rst}\ell^{ra}\wedge\ell^{sb}\wedge\ell^{tc}\,. \tag{4.42}$$

To simplify the analysis of the geometry that follows, we note that without loss of generality the matrix (g_{ab}) can chosen to be diagonal. To see this, perform an orthogonal transformation $O \in \mathrm{SO}(2)$ to bring (g_{ab}) into a diagonal form. Such a transformation can be compensated with a frame rotation

$$\ell^{ra} \to O^a{}_b\ell^{rb}\,. \tag{4.43}$$

Demanding that $\ell^A m_A$ is invariant implies that M_{ra} has to transform as $M_{ra} \to O^b{}_a M_{rb}$. However, it is straight forward to observe that such a transformation is an automorphism of

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$\mathfrak{so}(5)$ that preserves the decomposition (E.5), i.e. the structure constants of the Lie algebra remain the same. As a result, we can diagonalize the metric and at the same time use the same structure constants to calculate the geometric quantities of the homogeneous space. Under these orthogonal transformations the first two left-invariant forms are invariant while there is a change of basis in the space of left-invariant 3-forms.

To continue take[6] $(g_{ab}) = \mathrm{diag}(a_1, a_2)$. Then introduce the ortho-normal frame $\mathbf{e}^7 = \sqrt{a_4}\,\ell^7, \mathbf{e}^{r4} = \sqrt{a_1}\,\ell^{r4}$ and $\mathbf{e}^{r5} = \sqrt{a_2}\,\ell^{r5}$. In this frame the most general left-invariant metric and Y flux can be written as

$$\begin{aligned} ds^2 &= \delta_{ab}\delta_{rs}\mathbf{e}^{ra}\mathbf{e}^{sb} + (\mathbf{e}^7)^2\,, \\ Y &= \beta_1\,\mathbf{e}^7\wedge\chi_{444} + \beta_2\,\mathbf{e}^7\wedge\chi_{445} + \beta_3\,\mathbf{e}^7\wedge\chi_{455} + \beta_4\,\mathbf{e}^7\wedge\chi_{555} + \beta_5\,\psi\,, \end{aligned} \tag{4.44}$$

where $\beta_1, \beta_2, \cdots, \beta_5$ are constants,

$$\chi_{abc} = \frac{1}{3!}\epsilon_{rst}\mathbf{e}^{ra}\wedge\mathbf{e}^{sb}\wedge\mathbf{e}^{tc}\,, \quad \psi = \frac{1}{2}\omega\wedge\omega\,, \tag{4.45}$$

and

$$\omega = \frac{1}{2}\delta_{rs}\epsilon_{ab}\mathbf{e}^{ra}\wedge\mathbf{e}^{sb}\,. \tag{4.46}$$

The Bianchi identity for Y is automatically satisfied. On the other hand the field equation for Y in (4.3) yields the conditions

$$\begin{gathered} \frac{\beta_3}{2}\sqrt{\frac{a_2}{a_4a_1}} - \beta_1 X = 0\,, \quad -\beta_2\sqrt{\frac{a_2}{a_4a_1}} + \frac{3\beta_4}{2}\sqrt{\frac{a_1}{a_4a_2}} - \beta_2 X = 0\,, \\ \frac{3\beta_1}{2}\sqrt{\frac{a_2}{a_4a_1}} - \beta_3\sqrt{\frac{a_1}{a_2a_4}} - \beta_3 X = 0\,, \qquad \frac{\beta_2}{2}\sqrt{\frac{a_1}{a_4a_2}} - \beta_4 X = 0\,, \\ \beta_5\Big(X + \sqrt{\frac{a_4}{a_1a_2}}\Big) = 0\,, \end{gathered} \tag{4.47}$$

where we have chosen the top form on M^7 as $d\mathrm{vol} = \mathbf{e}^7\wedge\chi_{444}\wedge\chi_{555}$.

Before we proceed to investigate the various cases which arise from solving the linear system (4.47), let us consider first the case in which F is electric, i.e. it is proportional to the volume form of AdS_4. In such a case $\beta_1 = \cdots = \beta_5 = 0$. The algebraic KSE then gives

$$\frac{1}{3}X\Gamma_x\sigma_+ = \frac{1}{\ell A}\sigma_+\,, \tag{4.48}$$

and the field equations along M^7 imply that

$$R_{ij} = \frac{1}{6}X^2\delta_{ij}\,, \tag{4.49}$$

and so M^7 is Einstein. The Einstein condition on the metric of M^7 requires that

$$a_1 = a_2\,, \quad a_4 = \frac{3}{2}a_1\,. \tag{4.50}$$

[6] We have performed the analysis that follows also without taking (g_{ab}) to be diagonal producing the same conclusions.

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To investigate whether there are solutions preserving 20 supersymmetries, it remains to consider the integrability condition of the gravitino KSE (4.36). Indeed using the expressions (E.18) and (E.19) for the curvature of this homogeneous space, the integrability condition along the directions 7 and ra gives $X^2 = (27/8)a_1^{-1}$ while along the ra and sb directions requires additional projections. For example after taking the trace with δ^{ab} and setting $r = 1$ and $s = 2$, the condition is $\Gamma^{1245}\sigma_+ = \sigma_+$ which leads to solutions that preserve 16 supersymmetries or less, where the gamma matrices are in the ortho-normal frame and $\Gamma^{r4} = \Gamma^r, \Gamma^{r5} = \Gamma^{3+r}$. Hence there are no $N > 16$ AdS_4 solutions.

Next let us turn to investigate the solutions of the linear system (4.47). The last condition implies that

$$\text{either} \quad \beta_5 = 0 \quad \text{or} \quad X = -\sqrt{\frac{a_4}{a_1 a_2}}\,. \tag{4.51}$$

To continue consider first the case that $\beta_5 \neq 0$.

$\beta_5 \neq 0$. Substituting the second equation in (4.51) into the linear system (4.47), one finds that

$$\begin{aligned} &\beta_3\frac{a_2}{2} + \beta_1 a_4 = 0\,, \quad (a_4 - a_1)\beta_3 + \frac{3}{2}a_2\beta_1 = 0\,,\\ &\beta_2\frac{a_1}{2} + a_4\beta_4 = 0\,, \quad (a_4 - a_2)\beta_2 + \frac{3}{2}a_1\beta_4 = 0\,. \end{aligned} \tag{4.52}$$

Now there are several cases to consider. First suppose that the parameters of the metric a_1, a_2, a_4 are such that the only solutions of the linear system above are $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$. In such case $Y = \beta_5\psi$ and Y has the same algebraic properties as that of the SU(4)/SU(3) case with $\beta = \gamma = 0$ and $\alpha = \beta_5$. As a result, the algebraic KSE together with the Einstein equation for the warp factor imply that $\beta_5 = 0$ as well and so $Y = 0$. This violates our assumption that $\beta_5 \neq 0$. In any case, the 4-form flux F is electric which we have already investigated above and have found that such a configuration does not admit solutions with $N > 16$ supersymmetries.

Next suppose that the parameters of the metric are chosen such that

$$\text{either} \quad \beta_1 = \beta_3 = 0\,, \quad \text{or} \quad \beta_2 = \beta_4 = 0\,. \tag{4.53}$$

These two cases are symmetric so it suffices to consider one of the two. Suppose that $\beta_2 = \beta_4 = 0$ and $\beta_1, \beta_3 \neq 0$. In such a case

$$\frac{3}{4}a_2^2 - a_4(a_4 - a_1) = 0\,, \tag{4.54}$$

with $\frac{3}{4}a_1^2 - a_4(a_4 - a_2) \neq 0$. Setting $P_1 = \Gamma^{7156}, P_2 = \Gamma^{7345}$ and $P_3 = \Gamma^{7264}$, the algebraic KSE can be written as

$$\begin{aligned} \Big[\frac{1}{18}\Big(&- 3\beta_1 P_1P_2P_3 + \beta_3(P_1 + P_2 + P_3) - 3\beta_5(P_1P_2 + P_1P_3 + P_2P_3)\Big)\Gamma_z\\ &+ \frac{1}{3}X\Gamma_x\Big]\sigma_+ = \frac{1}{\ell A}\sigma_+\,. \end{aligned} \tag{4.55}$$

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$\|P_1, P_2, P_3\rangle$	relations for the fluxes
$\|+,+,+\rangle$	$[\frac{1}{6}(-\beta_1+\beta_3-3\beta_5)\Gamma_z+\frac{1}{3}X\Gamma_x]\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$
$\|+,+,-\rangle$, $\|+,-,+\rangle$, $\|-,+,+\rangle$	$[\frac{1}{18}(3\beta_1+\beta_3+3\beta_5)\Gamma_z+\frac{1}{3}X\Gamma_x]\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$
$\|-,-,+\rangle$, $\|-,+,-\rangle$, $\|+,-,-\rangle$	$[\frac{1}{18}(-3\beta_1-\beta_3+3\beta_5)\Gamma_z+\frac{1}{3}X\Gamma_x]\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$
$\|-,-,-\rangle$	$[\frac{1}{6}(\beta_1-\beta_3-3\beta_5)\Gamma_z+\frac{1}{3}X\Gamma_x]\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$

Table 6. Decomposition of (4.55) KSE into eigenspaces.

As P_1, P_2, P_3 are commuting and have eigenvalues ±1, the above algebraic equation decomposes into eigenspaces as tabulated in table 6.

To find solutions with 20 supersymmetries or more, we can either choose one of the two eigenspaces with 3 linearly independent eigenspinors and both eigenspaces with a single eigenspinor or both eigenspaces with 3 linearly independent eigenspinors. In the former case the algebraic KSE will admit 20 Killing spinors and in the latter 24 Killing spinors.

Let us first consider the case with 20 Killing spinors. In such a case, we find that

$$\beta_1=\beta_3\,,\quad \beta_1=3\beta_5\,, \tag{4.56}$$

and

$$\frac{1}{36}\beta_1^2+\frac{1}{9}X^2=\frac{1}{\ell^2A^2}\,, \tag{4.57}$$

where we have considered the second eigenspace with 3 eigenspinors in table 6. The case where the first such eigenspace with 3 eigenspinors is chosen can be treated in a similar way. The condition (4.57) follows as an integrability condition to the remaining algebraic KSE involving Γ_z and Γ_x. On the other hand, the field equation of the warp factor (4.4) implies that

$$\frac{7}{54}\beta_1^2+\frac{1}{9}X^2=\frac{1}{\ell^2A^2}\,, \tag{4.58}$$

which together with (4.57) gives $\beta_1=0$ and so $Y=0$. The solution cannot preserve $N>16$ supersymmetries.

Next consider the case with 24 Killing spinors. In this case, we find that

$$3\beta_1=-\beta_3\,, \tag{4.59}$$

and the integrability of the remaining algebraic KSE gives

$$\frac{1}{36}\beta_5^2+\frac{1}{9}X^2=\frac{1}{\ell^2A^2}\,. \tag{4.60}$$

On the other hand the field equation of the warp factor (4.4) gives

$$\frac{1}{9}X^2+\frac{2}{9}\beta_1^2+\frac{1}{6}\beta_5^2=\frac{1}{\ell^2A^2}\,. \tag{4.61}$$

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$\vert P_1, P_2, P_3\rangle$	relations for the fluxes
$\vert+,+,+\rangle$	$[\frac{1}{18}(4\alpha - 9\beta_5)\Gamma_z + \frac{1}{3}X\Gamma_x]\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$
$\vert+,+,-\rangle, \vert+,-,+\rangle, \vert-,+,+\rangle$ $\vert-,-,+\rangle, \vert-,+,-\rangle, \vert+,-,-\rangle$	$[\frac{1}{6}\beta_5\Gamma_z + \frac{1}{3}X\Gamma_x]\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$
$\vert-,-,-\rangle$	$[\frac{1}{18}(-4\alpha - 9\beta_5)\Gamma_z + \frac{1}{3}X\Gamma_x]\vert\cdot\rangle = \frac{1}{\ell A}\vert\cdot\rangle$

Table 7. Decomposition of (4.66) KSE into eigenspaces.

Comparing this with (4.60), one finds that the β's vanish and so $Y = 0$. Thus there are no solutions with $N > 16$ for either β_1, β_3 or β_2, β_4 non-vanishing.

It remains to investigate the case that all $\beta_1, \ldots, \beta_5 \neq 0$. This requires that the determinant of the coefficients of the linear system (4.52) must vanish, i.e.

$$\frac{3}{4}a_2^2 - a_4(a_4 - a_1) = 0\,, \quad \frac{3}{4}a_1^2 - a_4(a_4 - a_2) = 0\,. \tag{4.62}$$

Taking the difference of the two equations, we find that

$$\text{either} \quad a_1 = a_2\,, \quad \text{or} \quad a_4 = \frac{3}{4}(a_1 + a_2)\,. \tag{4.63}$$

Substituting a_4 above into (4.62), we find that $a_1 = a_2$. So without loss of generality, we set $a_1 = a_2 = a$. Then the linear system (4.52) can be solved to yield

$$\beta_3 = -3\beta_1\,, \quad \beta_2 = -3\beta_4\,. \tag{4.64}$$

Setting

$$\begin{aligned} P_1 &= \cos\theta\Gamma^{7156} + \sin\theta\Gamma^{7234}\,, \quad P_2 = \cos\theta\Gamma^{7345} + \sin\theta\Gamma^{7126}\,, \\ P_3 &= \cos\theta\Gamma^{7264} + \sin\theta\Gamma^{7315}\,, \end{aligned} \tag{4.65}$$

the algebraic KSE (4.7) can be rewritten as

$$\begin{aligned} \Big[\frac{1}{18}\big(\alpha P_1P_2P_3 + \alpha(P_1 + P_2 + P_3) - 3\beta_5(P_1P_2 + P_1P_3 + P_2P_3)\big)\Gamma_z \\ + \frac{1}{3}X\Gamma_x\Big]\sigma_+ = \frac{1}{\ell A}\sigma_+\,, \end{aligned} \tag{4.66}$$

where $\tan\theta = \beta_3/\beta_2$ and $\alpha = \sqrt{\beta_2^2 + \beta_3^2}$. As these Clifford algebra operations commute and have eigenvalues ± 1, the restrictions of this equation to the eigenspaces of P_1, P_2 and P_3 are given in table 7.

To find solutions with 20 supersymmetries, one needs to consider the eigenspace in table 7 with 6 eigenspinors. In such a case the integrability of the remaining KSE requires that

$$\frac{1}{36}\beta_5^2 + \frac{1}{9}X^2 = \frac{1}{\ell^2 A^2}\,. \tag{4.67}$$

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Comparing this with the field equation of the warp factor

$$\frac{1}{9}X^2+\frac{1}{6}\beta_5^2+\frac{1}{18}(\beta_1^2+\beta_4^2)+\frac{1}{54}(\beta_2^2+\beta_3^2)=\frac{1}{\ell^2A^2}\,, \tag{4.68}$$

we find that all β's must vanish and so $Y=0$. Thus the flux F is electric and as we have demonstrated such background does not admit $N>16$ AdS_4 supersymmetries.

$\boldsymbol{\beta_5=0}$. Since the backgrounds with electric flux F cannot preserve $N>16$ supersymmetries, we have to assume that at least one of the pairs (β_1,β_3) and (β_2,β_4) do not vanish. If either the pair (β_1,β_3) or (β_2,β_4) is non-vanishing, the investigation of the algebraic KSE proceeds as in the previous case with $\beta_5\neq0$. In particular, we find that the algebraic KSE (4.7) together with the field equation for the warp factor imply that all β's vanish and the flux F is electric. So there are no solutions preserving $N>16$ supersymmetries.

It remains to investigate the case that $\beta_1,\beta_2,\beta_3,\beta_4\neq0$. If this is the case, the determinant of the linear system (4.47) must vanish which in turn implies that

$$-\frac{3}{4}\frac{a_2}{a_1a_4}+X\left(X+\sqrt{\frac{a_1}{a_2a_4}}\right)=0\,,\qquad -\frac{3}{4}\frac{a_1}{a_2a_4}+X\left(X+\sqrt{\frac{a_2}{a_1a_4}}\right)=0\,. \tag{4.69}$$

The solution of these equations is

$$\text{either}\quad a_1=a_2\,,\quad \text{or}\quad X=-\frac{3}{4}\frac{a_1+a_2}{\sqrt{a_1a_2a_4}}\,. \tag{4.70}$$

Substituting the latter equation into (4.69), one again finds that $a_1=a_2$. So without loss of generality we take $a_1=a_2$ in which case

$$\text{either}\quad X=\frac{1}{2\sqrt{a_4}}\,,\quad \text{or}\quad X=-\frac{3}{2\sqrt{a_4}}\,. \tag{4.71}$$

For the latter case, the linear system (4.47) gives

$$\beta_3=-3\beta_1\,,\qquad \beta_2=-3\beta_4\,. \tag{4.72}$$

After setting $\beta_5=0$, the investigation of the algebraic KSE can be carried out as that described in table 7. As a result after comparing with the field equation for the warp factor, the β's vanish and F is electric. Thus there are no solutions preserving $N>16$ supersymmetries.

It remains to investigate the case that $X=1/(2\sqrt{a_4})$ in (4.71). In this case, the linear system (4.47) gives

$$\beta_1=\beta_3\,,\qquad \beta_2=\beta_4\,. \tag{4.73}$$

Using the P_1,P_2 and P_3 as in (4.65), the algebraic KSE (4.7) becomes

$$\left[\frac{1}{18}\big(-3\alpha P_1P_2P_3+\alpha(P_1+P_2+P_3)\big)\Gamma_z+\frac{1}{3}X\Gamma_x\right]\sigma_+=\frac{1}{\ell A}\sigma_+\,, \tag{4.74}$$

and the solutions in the eigenspaces of P_1,P_2 and P_3 are described in table 8. To preserve $N>16$ supersymmetries, one has to consider either one of the eigenspaces with 3 eigenspinors and the eigenspace with 2 eigenspinors or both of the eigenspaces with 3 eigenspinors. In either case, one finds that all β's vanish and so $Y=0$. Then F is electric and such solutions do not preserve $N>16$ supersymmetries. Therefore we conclude that the homogenous space $\mathrm{Sp}(2)/\Delta(\mathrm{Sp}(1))$ does not give rise to AdS_4 backgrounds with $N>16$.

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$\lvert P_1, P_2, P_3\rangle$	relations for the fluxes
$\lvert+,+,+\rangle$, $\lvert-,-,-\rangle$	$\frac{1}{3}X\Gamma_x\lvert\cdot\rangle = \frac{1}{\ell A}\lvert\cdot\rangle$
$\lvert+,+,-\rangle$, $\lvert+,-,+\rangle$, $\lvert-,+,+\rangle$	$[\frac{2}{9}\alpha\Gamma_z + \frac{1}{3}X\Gamma_x]\lvert\cdot\rangle = \frac{1}{\ell A}\lvert\cdot\rangle$
$\lvert-,-,+\rangle$, $\lvert-,+,-\rangle$, $\lvert+,-,-\rangle$	$[-\frac{2}{9}\alpha\Gamma_z + \frac{1}{3}X\Gamma_x]\lvert\cdot\rangle = \frac{1}{\ell A}\lvert\cdot\rangle$

Table 8. Decomposition of (4.74) KSE into eigenspaces.

4.6.3 Sp(2)/Sp(1)

The geometry of this homogeneous space is described in appendix E where the definition of the generators of the algebra and expressions for the curvature and invariant forms can be found. A left-invariant frame is $\ell^A m_A = \ell^a W_a + \ell^r T_r^{(+)}$, where $a = 1, \ldots, 4$ and $r = 1, 2, 3$. Then the most general left-invariant metric is

$$ds^2 = a\delta_{ab}\ell^a\ell^b + g_{rs}\ell^r\ell^s\,, \tag{4.75}$$

where $a > 0$ is a constant and (g_{rs}) is any constant 3×3 positive definite symmetric matrix.

To simplify the computations that follow, it is convenient to use the covariant properties of the decomposition of $\mathfrak{sp}(2) = \mathfrak{so}(5)$ as described in (E.9) to restrict the number of parameter in the metric. In particular, observe that the decomposition (E.9) remains invariant under the transformation of the generators

$$T_r^{(+)} \to O_r{}^s T_s^{(+)}\,, \quad W_a \to U_a{}^b W_b\,, \quad T_r^{(-)} \to T_r^{(-)}\,, \tag{4.76}$$

where $O \in \mathrm{SO}(3)$ and $U \in \mathrm{Spin}(3) \subset \mathrm{SO}(4)$ defined as

$$O_r{}^s I_s^{(+)} = U I_r^{(+)} U^{-1}\,, \tag{4.77}$$

as $I_r^{(+)}$ are the gamma matrices of the Majorana spinor representation of $\mathfrak{so}(3)$ on $\mathbb{R}^4 = \mathbb{C}^2 \oplus \bar{\mathbb{C}}^2$. Furthermore notice that $U I_r^{(-)} U^{-1} = I_r^{(-)}$ as U is generated by $I_r^{(+)}$ which commute with all $I_s^{(-)}$. The orthogonal rotations O act on the matrix (g_{rs}) as $g \to OgO^{-1}$. As (O, U) is an automorphism of $\mathfrak{so}(5)$ which leaves the decomposition (E.9) invariant, we can use O to put the matrix (g_{rs}) into diagonal form. So from now on without loss of generality, we set $(g_{rs}) = \mathrm{diag}(b_1, b_2, b_3)$ with $b_1, b_2, b_3 > 0$, see also [35].

The left-invariant 4-forms are generated by

$$\psi = \frac{1}{4!}\epsilon_{abcd}\ell^a \wedge \ell^b \wedge \ell^c \wedge \ell^d\,, \quad \rho_{rs} = \frac{1}{2}\epsilon_{rpq}\ell^p \wedge \ell^q \wedge I_s^{(+)}\,, \tag{4.78}$$

where

$$I_s^{(+)} = \frac{1}{2}(I_s^{(+)})_{ab}\,\ell^a \wedge \ell^b\,. \tag{4.79}$$

Therefore the 4-form flux Y can be chosen as

$$Y = \alpha\psi + \beta^{rs}\rho_{rs}\,, \tag{4.80}$$

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where α and β^{rs} are constants. Then it is straightforward to find that the Bianchi identity $dY = 0$ implies that

$$\beta^{rs} = \beta^{sr} \,. \tag{4.81}$$

Furthermore define $\sigma = \frac{1}{3!}\epsilon_{rst}\ell^r \wedge \ell^s \wedge \ell^t$ and choose as top form $d\mathrm{vol} = a^2\sqrt{b_1b_2b_3}\,\sigma \wedge \psi$. Then the field equation for Y, $d *_7 Y = XY$, gives the linear system

$$\begin{aligned}
&\sum_{r=1}^{3} b_r\beta^{rr} = \sqrt{b_1b_2b_3}\,X\alpha\,, \quad \frac{\alpha}{2}\frac{\sqrt{b_1b_2b_3}}{a^2} - \frac{1}{3}\frac{\sum_{r=1}^{3} b_r\beta^{rr}}{\sqrt{b_1b_2b_3}} = \frac{X}{3}\beta \\
&\left(b_r\beta^{rs} + \beta^{rs}b_s - \frac{2}{3}\delta^{rs}\sum_{t=1}^{3} b_t\beta^{tt}\right) = \sqrt{b_1b_2b_3}X\left(\beta^{rs} - \frac{1}{3}\delta^{rs}\beta\right),
\end{aligned} \tag{4.82}$$

where there is no summation over the indices r and s on the left-hand side of the last equation and $\beta = \delta_{rs}\beta^{rs}$.

Before we proceed to investigate the solutions of the linear system, notice that if $\beta^{rs} = 0$, then $\alpha = 0$ and so F is electric. The supersymmetry preserved by these solutions will be investigated later. As we shall demonstrate such solutions cannot preserve more than 16 supersymmetries.

Furthermore writing $Y = \alpha\psi + Y_\beta$, where $Y_\beta = \beta^{rs}\rho_{rs}$, the field equation of the warp factor in (4.4) can be written as

$$\frac{1}{9}X^2 + \frac{1}{18}\alpha^2 a^{-4} + \frac{1}{432}(Y_\beta)^2 = \frac{1}{\ell^2 A^2}\,. \tag{4.83}$$

As we shall demonstrate, the compatibility of this field equation with the algebraic KSE rules out the existence of $N > 16$ backgrounds.

Returning to the solutions of (4.82), let us focus on β^{rs} with $r \neq s$. There are several cases to consider.

Either $\beta^{rs} \neq 0$ for all $r \neq s$ or $\beta^{rs} = 0$ for all $r \neq s$. If β^{rs}, $r \neq s$, are all non-vanishing, the last equation in (4.82) implies that

$$b_1 = b_2 = b_3\,, \quad X = 2\frac{b_1}{\sqrt{b_1b_2b_3}}\,. \tag{4.84}$$

As a result, the metric is invariant under SO(3) and this can be used to bring β^{rs} into diagonal form. Of course (β^{rs}) is also diagonal if $\beta^{rs} = 0$ for all $r \neq s$.

So without loss of generality, we can assume that (β^{rs}) is diagonal. Setting

$$J_1 = \Gamma^{6714}\,, \quad J_2 = \Gamma^{6723}\,, \quad J_3 = \Gamma^{7524}\,, \tag{4.85}$$

where all gamma matrices are in the ortho-normal basis and $\{\Gamma^i\} = \{\Gamma^a, \Gamma^{4+r}\}$, the algebraic KSE can be written as

$$\begin{aligned}
\Big(\frac{1}{6}\Big[&-\alpha a^{-2}J_1J_2 + \frac{a^{-1}}{\sqrt{b_1b_2b_3}}\big(\sqrt{b_1}\beta^{11}(J_1+J_2) + \sqrt{b_2}\beta^{22}J_3(1+J_1J_2) \\
&+ \sqrt{b_3}\beta^{33}J_3(J_1+J_2)\big)\Big]\Gamma_z + \frac{1}{3}X\Gamma_x\Big)\sigma_+ = \frac{1}{\ell A}\sigma_+\,.
\end{aligned} \tag{4.86}$$

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$\vert J_1, J_2, J_3\rangle$	relations for the fluxes
$\vert +,+,\pm\rangle$	$\big(\frac{1}{6}[-\alpha a^{-2}+2\frac{a^{-1}}{\sqrt{b_1b_2b_3}}(\sqrt{b_1}\beta^{11}\pm\sqrt{b_2}\beta^{22}$ $\pm\sqrt{b_3}\beta^{33})]\Gamma_z+\frac{1}{3}X\Gamma_x\big)\vert\cdot\rangle=\frac{1}{\ell A}\vert\cdot\rangle$
$\vert +,-,\pm\rangle$ $\vert -,+,\pm\rangle$	$\big(\frac{1}{6}\alpha a^{-2}\Gamma_z+\frac{1}{3}X\Gamma_x\big)\vert\cdot\rangle=\frac{1}{\ell A}\vert\cdot\rangle$
$\vert -,-,\pm\rangle$	$\big(\frac{1}{6}[-\alpha a^{-2}+2\frac{a^{-1}}{\sqrt{b_1b_2b_3}}(-\sqrt{b_1}\beta^{11}\pm\sqrt{b_2}\beta^{22}$ $\mp\sqrt{b_3}\beta^{33})]\Gamma_z+\frac{1}{3}X\Gamma_x\big)\vert\cdot\rangle=\frac{1}{\ell A}\vert\cdot\rangle$

Table 9. Decomposition of (4.86) KSE into eigenspaces.

The decomposition of the algebraic KSE into the eigenspaces of the commuting Clifford algebra operators J_1, J_2, J_3 is illustrated in table 9.

To construct $N > 16$ solutions, we have to include the eigenspace with four eigenspinors. The integrability condition of the remaining KSE described in table 9 gives

$$\frac{1}{36}\alpha^2 a^{-4}+\frac{1}{9}X^2=\frac{1}{\ell^2A^2}\,. \tag{4.87}$$

Comparing (4.87) with the field equation for the warp factor (4.83), we find that $\alpha=\beta^{rs}=0$. Therefore $Y=0$ and so F is electric.

$\beta^{12},\beta^{13}\neq 0$ and $\beta^{23}=0$. As the other two cases for which either $\beta^{13}=0$ or $\beta^{12}=0$ with the rest of the components non-vanishing can be treated in a similar way, we take without loss of generality that $\beta^{23}=0$ and $\beta^{12},\beta^{13}\neq 0$. In such a case, the last condition in (4.82) gives

$$X=\frac{b_1+b_2}{\sqrt{b_1b_2b_3}}\,,\quad b_2=b_3\,. \tag{4.88}$$

The metric is invariant under an $\mathrm{SO}(2)\subset\mathrm{SO}(3)$ symmetry which acts with the vector representation on the vector (β^{12},β^{13}) and leaves the form of (β^{rs}) invariant. As a result up to an $\mathrm{SO}(2)$ transformation, we can set $\beta^{13}=0$ as well. Furthermore, if $b_1\neq b_2$, the diagonal terms in the last condition in (4.82) give

$$\beta^{11}=-\beta^{22}=-\beta^{33}\,. \tag{4.89}$$

On the other hand if $b_1=b_2$ the analysis reduces to that of the previous case. Therefore for $b_1\neq b_2$, Y can be written as

$$Y=\alpha\psi+\beta^{11}(\rho_{11}-\rho_{22}-\rho_{33})+\beta^{12}(\rho_{12}+\rho_{21})\,. \tag{4.90}$$

Introducing the Clifford algebra operators

$$J_1=\cos\theta\Gamma^{6714}+\sin\theta\Gamma^{6724}\,,\quad J_2=\cos\theta\Gamma^{5724}-\sin\theta\Gamma^{5714}\,,\quad J_3=\Gamma^{1234}\,, \tag{4.91}$$

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$\|J_1, J_2, J_3\rangle$	relations for the fluxes
$\|\pm,+,-\rangle$	$\Big(\frac{1}{6}[-\alpha a^{-2}+\frac{2a^{-1}}{\sqrt{b_1}b_2}(\pm\sqrt{b_2}\beta^{11}$ $+\sqrt{(\beta^{11})^2+(\beta^{12})^2}(\pm\sqrt{b_1}+\sqrt{b_2}))]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$
$\|+,\pm,+\rangle$ $\|-,\pm,+\rangle$	$\left(\frac{1}{6}\alpha a^{-2}\Gamma_z+\frac{1}{3}X\Gamma_x\right)\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$
$\|\pm,-,-\rangle$	$\Big(\frac{1}{6}[-\alpha a^{-2}+\frac{2a^{-1}}{\sqrt{b_1}b_2}(\mp\sqrt{b_2}\beta^{11}$ $+\sqrt{(\beta^{11})^2+(\beta^{12})^2}(\pm\sqrt{b_1}-\sqrt{b_2}))]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$

Table 10. Decomposition of (4.92) KSE into eigenspaces.

where $\tan\theta=\beta^{12}/\beta^{11}$, the algebraic KSE can be written as

$$\Big(\frac{1}{6}\Big[\alpha a^{-2}J_3+\frac{a^{-1}}{\sqrt{b_1}b_2}(\sqrt{b_2}\beta^{11}J_1J_2(1-J_3)$$
$$+\sqrt{(\beta^{11})^2+(\beta^{12})^2}(\sqrt{b_1}J_1+\sqrt{b_2}J_2)(1-J_3))\Big]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\sigma_+=\frac{1}{\ell A}\sigma_+\,. \tag{4.92}$$

The decomposition of the algebraic KSE into the eigenspaces of the commuting Clifford algebra operators J_1, J_2, J_3 is illustrated in table 10.

To construct solutions preserving more than 16 supersymmetries, we have to include the eigenspace with four eigenspinors leading again to the integrability condition (4.87). Comparing again with the field equations of the warp factor (4.83), we deduce that F is electric.

$\beta^{13}=\beta^{23}=0$ but $\beta^{12}\neq 0$. All three cases for which only one of the three off-diagonal components of (β^{rs}) is non-zero can be treated symmetrically. So without loss of generality, one can take $\beta^{13}=\beta^{23}=0$ but $\beta^{12}\neq 0$. In this case, the last equation in (4.82) has four branches of solutions depending on the choice of the b_1, b_2 and b_3 components of the metric.

1. $b_1=b_2=b_3=b$
   The last equation in (4.82) then implies $X=2/\sqrt{b}$ and the aforementioned residual SO(3) symmetry can be used to put β^{rs} to be diagonal.

2. $b_1=b_2$, $b_2\neq b_3$
   The last equation in (4.82) then implies $X=2/\sqrt{b_3}$ and $\beta^{33}=0$. The aforementioned residual SO(2) symmetry can be used to put β^{rs} to be diagonal.

3. $b_2\neq b_3$, $b_1+b_2=2b_3$
   The last equation in (4.82) then implies $X=(b_1+b_2)/\sqrt{b_1b_2b_3}$ and $\beta^{11}=\beta^{22}=0$. In such a case, Y reads

$$Y=\alpha\psi+\beta^{33}\rho_{33}+\beta^{12}(\rho_{12}+\rho_{21})\,. \tag{4.93}$$

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$\|J_1, J_2, J_3\rangle$	relations for the fluxes
$\|\pm,+,-\rangle$	$\Big(\frac{1}{6}[-\alpha a^{-2}+\frac{2a^{-1}}{\sqrt{b_1b_2b_3}}(\pm\sqrt{b_3}\beta^{33} -\beta^{12}(\pm\sqrt{b_2}-\sqrt{b_1}))]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$
$\|+,\pm,+\rangle$ $\|-,\pm,+\rangle$	$\left(\frac{1}{6}\alpha a^{-2}\Gamma_z+\frac{1}{3}X\Gamma_x\right)\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$
$\|\pm,-,-\rangle$	$\Big(\frac{1}{6}[-\alpha a^{-2}+\frac{2a^{-1}}{\sqrt{b_1b_2b_3}}(\mp\sqrt{b_3}\beta^{33} -\beta^{12}(\pm\sqrt{b_2}+\sqrt{b_1}))]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\|\cdot\rangle=\frac{1}{\ell A}\|\cdot\rangle$

Table 11. Decomposition of (4.95) KSE into eigenspaces.

Choosing

$$J_1=\Gamma^{1457}\,,\quad J_2=\Gamma^{2467}\,,\quad J_3=\Gamma^{1234}\,, \tag{4.94}$$

the algebraic KSE can be written as

$$\Big(\frac{1}{6}\Big[\alpha a^{-2}J_3+\frac{a^{-1}}{\sqrt{b_1b_2b_3}}\Big(\sqrt{b_3}\beta^{33}J_1J_2(1-J_3) -\beta^{12}(\sqrt{b_2}J_1-\sqrt{b_1}J_2)(1-J_3)\Big)\Big]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\sigma_+=\frac{1}{\ell A}\sigma_+\,. \tag{4.95}$$

The decomposition of the algebraic KSE into the eigenspaces of J_1, J_2, J_3 is illustrated in table 11. Again the eigenspace with four eigenspinors has to be included in the construction of $N > 16$ backgrounds. As a result, this leads to the integrability condition (4.87) which together with the warp factor field equation (4.83) imply that F is electric.

4. $b_1\neq b_2$, $b_1+b_2\neq 2b_3$
The last equation in (4.82) then implies

$$X=(b_1+b_2)/\sqrt{b_1b_2b_3}\,,\quad \beta^{11}=-\beta^{22}=\beta^{33}(2b_3-b_1-b_2)/(b_1-b_2)\,. \tag{4.96}$$

In such a case, Y reads

$$Y=\alpha\psi+\beta^{11}\left(\rho_{11}-\rho_{22}+\frac{b_1-b_2}{2b_3-b_1-b_2}\rho_{33}\right)+\beta^{12}(\rho_{12}+\rho_{21})\,. \tag{4.97}$$

With the choice of commuting Clifford algebra operators as in (4.91), the algebraic KSE can be written as

$$\Big(\frac{1}{6}\Big[\alpha a^{-2}J_3+\frac{a^{-1}}{\sqrt{b_1b_2b_3}}\Big(\frac{(b_1-b_2)\sqrt{b_3}}{b_1+b_2-2b_3}\beta^{11}J_1J_2(1-J_3) +\sqrt{(\beta^{11})^2+(\beta^{12})^2}(\sqrt{b_1}J_1+\sqrt{b_2}J_2)(1-J_3)\Big)\Big]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\sigma_+=\frac{1}{\ell A}\sigma_+\,. \tag{4.98}$$

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$\vert J_1, J_2, J_3\rangle$	relations for the fluxes
$\vert\pm,+,-\rangle$	$\Big(\frac{1}{6}[-\alpha a^{-2}+\frac{2a^{-1}}{\sqrt{b_1 b_2 b_3}}(\pm\frac{(b_1-b_2)\sqrt{b_3}}{b_1+b_2-2b_3}\beta^{11}$ $+\sqrt{(\beta^{11})^2+(\beta^{12})^2}(\pm\sqrt{b_1}+\sqrt{b_2}))]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\vert\cdot\rangle=\frac{1}{\ell A}\vert\cdot\rangle$
$\vert+,\pm,+\rangle$ $\vert-,\pm,+\rangle$	$\left(\frac{1}{6}\alpha a^{-2}\Gamma_z+\frac{1}{3}X\Gamma_x\right)\vert\cdot\rangle=\frac{1}{\ell A}\vert\cdot\rangle$
$\vert\pm,-,-\rangle$	$\Big(\frac{1}{6}[-\alpha a^{-2}+\frac{2a^{-1}}{\sqrt{b_1}b_2}(\mp\frac{(b_1-b_2)\sqrt{b_3}}{b_1+b_2-2b_3}\beta^{11}$ $+\sqrt{(\beta^{11})^2+(\beta^{12})^2}(\pm\sqrt{b_1}-\sqrt{b_2}))]\Gamma_z+\frac{1}{3}X\Gamma_x\Big)\vert\cdot\rangle=\frac{1}{\ell A}\vert\cdot\rangle$

Table 12. Decomposition of (4.98) KSE into eigenspaces.

The decomposition of the algebraic KSE into the eigenspaces of J_1, J_2, J_3 is illustrated in table 12.

To construct $N > 16$ solutions, we again have to include the eigenspace with four eigenspinors which leads to the integrability condition (4.87). Comparing with the warp factor field equation (4.83), we again deduce that F is electric.

It remains to investigate the number of supersymmetries preserved by the solutions for which F is electric. For this, one has to investigate the integrability condition of the gravitino KSE (4.36). Using the expression for the curvature of metric in (E.25)–(E.28) and requiring that the solution preserves $N > 16$, we find that

$$\delta^{ca}\delta^{db}(I_r^{(-)})_{ab}\left(R_{cd,mn}\Gamma^{mn}-\frac{1}{18}X^2\Gamma_{cd}\right)\sigma_+=0\,, \tag{4.99}$$

implies that

$$a-\frac{1}{8}\delta^{rs}g_{rs}-\frac{1}{18}a^2\,X^2=0\,. \tag{4.100}$$

Next requiring again that $N > 16$, one finds that the condition

$$\delta^{ca}\delta^{db}(I_r^{(+)})_{ab}\left(R_{cd,mn}\Gamma^{mn}-\frac{1}{18}X^2\Gamma_{cd}\right)\sigma_+=0\,. \tag{4.101}$$

gives that

$$\begin{aligned}&\delta^{pq}g_{pq}\epsilon_{rst}-\frac{1}{2}a^{-1}\epsilon_t{}^{pq}g_{pr}g_{qs}-2g_{tp}\epsilon^p{}_{rs}=0\,,\\ &-\frac{3}{4}g_{rs}+\frac{1}{8}\delta^{pq}g_{pq}\delta_{rs}+a\delta_{rs}-\frac{1}{18}a^2\,X^2\delta_{rs}=0\,.\end{aligned} \tag{4.102}$$

Substituting (4.100) into the second equation in (4.102), one finds after a bit of analysis that

$$b_1=b_2=b_3\,. \tag{4.103}$$

Setting $b=b_1=b_2=b_3$ and substituting this back into (4.100) and (4.102), one deduces that

$$2a=b\,,\quad X^2=9b^{-1}\,. \tag{4.104}$$

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As $X^2 = 9\ell^{-2}A^{-2}$, we have $b = \ell^2 A^2$ and $a = (1/2)\ell^2 A^2$. The rest of the integrability condition is satisfied without further conditions. So every solutions that preserves $N > 16$ supersymmetries is maximally supersymmetric and so locally isometric to $\mathrm{AdS}_4 \times S^7$.

One can confirm this result by investigating the Einstein equation (4.5). As all solutions with electric F are Einstein $R^{(7)}_{ij} = (1/6)X^2\delta_{ij}$, it suffices to identify the left-invariant metrics on $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ that are Einstein. There are two Einstein metrics [35, 36] on $\mathrm{Sp}(2)/\mathrm{Sp}(1)$ given by

$$X^2 = 9b^{-1}\,, \quad 2a = b\,, \quad b_1 = b_2 = b_3 = b\,, \tag{4.105}$$

and

$$X^2 = \frac{81}{25}b^{-1}\,, \quad 2a = 5b\,, \quad b_1 = b_2 = b_3 = b\,, \tag{4.106}$$

where the first one is the round metric on S^7, see also [37]. The second one does not give $N > 16$ supersymmetric solutions.

5 Conclusions

We have classified up to local isometries all warped AdS_4 backgrounds with the most general allowed fluxes in 10- and 11-dimensional supergravities that preserve $N > 16$ supersymmetries. We have demonstrated that up to an overall scale, the only solutions that arise are the maximally supersymmetric solution $\mathrm{AdS}_4 \times S^7$ of 11-dimensional supergravity [17, 18] and the $N = 24$ solution $\mathrm{AdS}_4 \times \mathbb{CP}^3$ of IIA supergravity [16]. These two solutions are related via dimensional reduction along the fibre of the Hopf fibration $S^1 \to S^7 \to \mathbb{CP}^3$.

The assumption we have made to prove these results is that either the solutions are smooth and the internal space is compact without boundary or that the even part $\mathfrak{g}_0$ of the Killing superalgebra of the backgrounds decomposes as $\mathfrak{g}_0 = \mathfrak{so}(3,2) \oplus \mathfrak{t}_0$. In fact these two assumptions are equivalent for $N > 16$ AdS_4 backgrounds. It may be possible to weaken these assumptions but they cannot be removed altogether. This is because in such a case additional solutions will exist. For example the maximally supersymmetric $\mathrm{AdS}_7 \times S^4$ solution of 11-dimensional supergravity [38] can be re-interpreted as a maximally supersymmetric warped AdS_4 solution. However in such case the "internal" 7-dimensional manifold M^7 is not compact and the even subalgebra of the Killing superalgebra $\mathfrak{g}_0$ does not decompose as $\mathfrak{so}(3,2) \oplus \mathfrak{t}_0$.

We have identified all AdS_4 backgrounds up to a local isometry. Therefore, we have specified all the local geometries of the internal spaces G/H of these solutions. However the possibility remains that there are more solutions which arise via additional discrete identifications $Z\backslash G/H$, where Z is a discrete subgroup of $Z \subset G$. The $\mathrm{AdS}_4 \times Z\backslash G/H$ solutions will preserve at most as many supersymmetries as the $\mathrm{AdS}_4 \times G/H$ solutions. As in IIB and massive IIA supergravities there are no $N > 16$ $\mathrm{AdS}_4 \times G/H$ solutions, there are no $N > 16$ $\mathrm{AdS}_4 \times Z\backslash G/H$ solutions either. In IIA theory, the possibility remains that there can be $\mathrm{AdS}_4 \times Z\backslash\mathbb{CP}^3$ solutions with 24 and 20 supersymmetries. In $D = 11$ supergravity as $\mathrm{AdS}_4 \times S^7$ preserves 32 supersymmetries, there may be $\mathrm{AdS}_4 \times Z\backslash S^7$ solutions

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preserving 28, 24 and 20 supersymmetries. Such solutions have been used in the context of AdS/CFT in [39]. A systematic investigation of all possible $N > 16$ $\mathrm{AdS}_4 \times Z\backslash G/H$ backgrounds will involve the identification of all discrete subgroups of G. The relevant groups here are SU(4) and Spin(8), see e.g. [46] for an exposition of discrete subgroups of SU(4) and references therein.

It is clear from our results on AdS_4 backgrounds that supersymmetric AdS solutions which preserve $N > 16$ supersymmetries in 10- and 11-dimensions are severely restricted. Consequently there are few gravitational duals for superconformal theories with a large number of supersymmetries which have distinct local geometries. For example, the superconformal theories of [40–42] have gravitational duals which are locally isometric to the $\mathrm{AdS}_5 \times S^5$ maximally supersymmetric background as there are no distinct local AdS_5 geometries that preserve strictly 24 supersymmetries [10]. In general our results also suggest that there may not be a large number of backgrounds that preserve $N > 16$ supersymmetries in 10- and 11-dimensional supergravities. So it is likely that all these solutions can be found in the future.

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A Notation and conventions

Our conventions for forms are as follows. Let ω be a k-form, then

$$\omega = \frac{1}{k!}\omega_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \,, \quad \omega^2_{ij} = \omega_{i\ell_1\ldots\ell_{k-1}}\omega_j{}^{\ell_1\ldots\ell_{k-1}} \,, \quad \omega^2 = \omega_{i_1\ldots i_k}\omega^{i_1\ldots i_k} \,. \tag{A.1}$$

We also define

$$\not{\omega} = \omega_{i_1\ldots i_k}\Gamma^{i_1\ldots i_k} \,, \quad \not{\omega}_{i_1} = \omega_{i_1 i_2\ldots i_k}\Gamma^{i_2\ldots i_k} \,, \quad \not{\Gamma\omega}_{i_1} = \Gamma_{i_1}{}^{i_2\ldots i_{k+1}}\omega_{i_2\ldots i_{k+1}} \,, \tag{A.2}$$

where the Γ_i are the Dirac gamma matrices.

The inner product $\langle\cdot,\cdot\rangle$ we use on the space of spinors is that for which space-like gamma matrices are Hermitian while time-like gamma matrices are anti-hermitian, i.e. the Dirac spin-invariant inner product is $\langle\Gamma_0\cdot,\cdot\rangle$. The norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is taken with respect to $\langle\cdot,\cdot\rangle$, which is positive definite. For more details on our conventions see [5–7].

B Homogeneous and symmetric spaces

In the following section we shall collect some useful properties of homogeneous spaces which have facilitated our analysis of AdS_4 backgrounds. A more detailed review can be found in e.g. [43, 44].

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Consider the left coset space $M = G/H$, where G is a compact connected semisimple Lie group G which acts effectively from the left on $M = G/H$ and H is a closed Lie subgroup of G. Let us denote the Lie algebras of G and H with $\mathfrak{g}$ and $\mathfrak{h}$, respectively. As there is always an invariant inner product on $\mathfrak{g}$, it can be used to take the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ and so

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \,. \tag{B.1}$$

Denote the generators of $\mathfrak{h}$ with h_α, $\alpha = 1, 2, \ldots, \dim \mathfrak{h}$ and a basis in $\mathfrak{m}$ as m_A, $A = 1, \ldots, \dim \mathfrak{g} - \dim \mathfrak{h}$. In this basis, the brackets of the Lie algebra $\mathfrak{g}$ take the following form

$$\begin{aligned} [h_\alpha, h_\beta] &= f_{\alpha\beta}{}^{\gamma}\, h_\gamma \,, \quad [h_\alpha, m_A] = f_{\alpha A}{}^{B}\, m_B \,, \\ [m_A, m_B] &= f_{AB}{}^{C}\, m_C + f_{AB}{}^{\alpha}\, h_\alpha \,. \end{aligned} \tag{B.2}$$

If $f_{AB}{}^{C} = 0$, that is $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, the space is symmetric.

Let $g : U \subset G/H \to G$ be a local section of the coset. The decomposition of the Maurer-Cartan form in components along $\mathfrak{h}$ and $\mathfrak{m}$ is

$$g^{-1} dg = \ell^A\, m_A + \Omega^\alpha\, h_\alpha \,, \tag{B.3}$$

which defines a local left-invariant frame ℓ^A and a canonical left-invariant connection Ω^α on G/H. The curvature and torsion of the canonical connection are

$$\begin{aligned} R^\alpha &\equiv d\Omega^\alpha + \frac{1}{2} f_{\beta\gamma}{}^{\alpha} \Omega^\beta \wedge \Omega^\gamma = -\frac{1}{2} f_{BC}{}^{\alpha} \ell^B \wedge \ell^C \,, \\ T^A &\equiv d\ell^A + f_{\beta C}{}^{A} \Omega^\beta \wedge \ell^C = -\frac{1}{2} f_{BC}{}^{A} \ell^B \wedge \ell^C \,, \end{aligned} \tag{B.4}$$

respectively, where the equalities follow after taking the exterior derivative of (B.3) and using (B.2). If G/H is symmetric, then the torsion vanishes.

A left-invariant p-form ω on G/H can be written as

$$\omega = \frac{1}{p!}\, \omega_{A_1 \ldots A_p}\, \ell^{A_1} \wedge \ldots \wedge \ell^{A_p} \,, \tag{B.5}$$

where the components $\omega_{A_1 \ldots A_p}$ are constant and satisfy

$$f_{\alpha [A_1}{}^{B}\, \omega_{A_2 \ldots A_p] B} = 0 \,. \tag{B.6}$$

The latter condition is required for invariance under the right action of H on G. All left-invariant forms are parallel with respect to the canonical connection.

It remains to describe the metrics of G/H which are left-invariant. These are written as

$$ds^2 = g_{AB}\, \ell^A \ell^B \,, \tag{B.7}$$

where the components g_{AB} are constant and satisfy

$$f_{\alpha A}{}^{C}\, g_{BC} + f_{\alpha B}{}^{C}\, g_{AC} = 0 \,. \tag{B.8}$$

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For symmetric spaces, the canonical connection coincides with the Levi-Civita connection of invariant metrics. So all non-vanishing left-invariant forms are harmonic and represent non-trivial elements in the de Rham cohomology of G/H. However if G/H is strictly homogeneous this is not the case since the canonical connection has non-vanishing torsion.

Suppose G/H is homogeneous and equipped with an invariant metric g. To describe the results of the paper, it is required to find the Levi-Civita connection of g and its curvature. Let Φ be the Levi-Civita connection in the left-invariant frame. As the difference of two connections is a tensor, we set

$$\Phi^A{}_B = \Omega^\alpha f_{\alpha B}{}^A + \ell^C Q_{C,}{}^A{}_B \,. \tag{B.9}$$

As Φ is metric and torsion free, we have

$$\begin{aligned} \Phi_{AB} + \Phi_{BA} &= 0 \,, \\ d\ell^A + \Phi^A{}_B \wedge \ell^B &= 0 \,. \end{aligned} \tag{B.10}$$

These equations can be solved for Q to find that

$$\Phi^A{}_B = \Omega^\alpha \, f_{\alpha B}{}^A + \frac{1}{2} \left(g^{AD} \, f_{DB}{}^E \, g_{CE} + g^{AD} \, f_{DC}{}^E \, g_{BE} + f_{CB}{}^A \right) \, \ell^C \,. \tag{B.11}$$

In turn the Riemann curvature 2-form $R^A{}_B$ is

$$R^A{}_B = \frac{1}{2} \left(Q_{C,}{}^A{}_E \, Q_{D,}{}^E{}_B - Q_{D,}{}^A{}_E \, Q_{C,}{}^E{}_B - Q_{E,}{}^A{}_B \, f_{CD}{}^E - f_{CD}{}^\alpha \, f_{\alpha B}{}^A \right) \, \ell^C \wedge \ell^D \,. \tag{B.12}$$

This is required for the investigation of the gravitino KSE. Note that the expression for $\Phi^A{}_B$ is considerably simplified whenever the coset space is naturally reductive because the structure constants $f_{ABC} = f_{AB}{}^E \, g_{CE}$ are then skew symmetric.

C $\mathfrak{su}(k)$

Here we shall collect some formulae that are useful in understanding the homogeneous spaces that admit a transitive action of a group with Lie algebra $\mathfrak{su}(k)$. A basis over the reals of anti-hermitian $k \times k$ traceless complex matrices is

$$(M_{ab})^c{}_d = \frac{1}{2} (\delta_a{}^c \delta_{bd} - \delta_b{}^c \delta_{ad}) \,, \quad (N_{ab})^c{}_d = \frac{\nu(ab)}{2} i \left(\delta_a{}^c \delta_{bd} + \delta_b{}^c \delta_{ad} - \frac{2}{k} \delta_{ab} \delta_c{}^d \right) , \tag{C.1}$$

where $\nu(ab)$ is a normalization factor and $a, b, c, d = 1, \ldots, k$. The trace of these matrices yields an invariant inner product on $\mathfrak{su}(k)$. In particular the non-vanishing traces are

$$\begin{aligned} \mathrm{tr}(M_{ab} M_{a'b'}) &= -\frac{1}{2} (\delta_{aa'} \delta_{bb'} - \delta_{ab'} \delta_{ba'}) \,, \\ \mathrm{tr}(N_{ab} N_{a'b'}) &= -\frac{\nu(ab)\nu(a'b')}{2} \left(\delta_{aa'} \delta_{bb'} + \delta_{ab'} \delta_{ba'} - \frac{2}{k} \delta_{ab} \delta_{a'b'} \right) . \end{aligned} \tag{C.2}$$

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It is customary to choose the normalization factors ν such that all generators have the same length. In such a case, they will depend on k. However in what follows, it is more convenient to choose $\nu = 1$. The Lie brackets of $\mathfrak{su}(k)$ are

$$\begin{aligned}
[M_{ab}, M_{a'b'}] &= \frac{1}{2}(\delta_{ba'}M_{ab'} + \delta_{ab'}M_{ba'} - \delta_{aa'}M_{bb'} - \delta_{bb'}M_{aa'})\,,\\
[M_{ab}, N_{a'b'}] &= \frac{1}{2}(\delta_{ba'}N_{ab'} - \delta_{ab'}N_{ba'} - \delta_{aa'}N_{bb'} + \delta_{bb'}N_{aa'})\,,\\
[N_{ab}, N_{a'b'}] &= -\frac{1}{2}(\delta_{ba'}M_{ab'} + \delta_{ab'}M_{ba'} + \delta_{aa'}M_{bb'} + \delta_{bb'}M_{aa'})\,.
\end{aligned} \tag{C.3}$$

We shall proceed to describe the homogeneous spaces in (1.1) and (1.2) that admit a transitive $\mathrm{SU}(k)$ action.

C.1 $M^k = \mathbb{CP}^{k-1} = \mathbf{SU}(k)/S(\mathbf{U}(k) \times \mathbf{U}(1))$

To describe the $\mathbb{CP}^{k-1}$ homogeneous space, we set

$$\mathfrak{h} = \mathfrak{s}(\mathfrak{u}(k-1) \oplus \mathfrak{u}(1)) = \mathbb{R}\langle M_{rs}, N_{rs}, N_{kk})\rangle\,, \quad \mathfrak{m} = \mathbb{R}\langle M_{rk}, N_{sk}\rangle\,, \tag{C.4}$$

where $r, s = 1, \ldots, k-1$. The brackets of the Lie subalgebra $\mathfrak{s}(\mathfrak{u}(k-1) \oplus \mathfrak{u}(1))$ can be read off from those in (C.3) while those involving elements of $\mathfrak{m}$ are

$$[M_{rk}, M_{sk}] = -\frac{1}{2}M_{rs}\,, \quad [M_{rk}, N_{sk}] = \frac{1}{2}N_{rs} - \frac{1}{2}\delta_{rs}N_{kk}\,, \quad [N_{rk}, N_{sk}] = -\frac{1}{2}M_{rs}\,, \tag{C.5}$$

and

$$\begin{aligned}
[M_{rs}, M_{tk}] &= \frac{1}{2}(\delta_{ts}M_{rk} - \delta_{tr}M_{sk})\,, & [M_{rs}, N_{tk}] &= \frac{1}{2}(\delta_{ts}N_{rk} - \delta_{tr}N_{sk})\,,\\
[N_{rs}, M_{tk}] &= \frac{1}{2}(\delta_{ts}N_{rk} + \delta_{tr}N_{sk})\,, & [N_{rs}, N_{tk}] &= -\frac{1}{2}(\delta_{ts}M_{rk} + \delta_{tr}M_{sk})\,,\\
[N_{kk}, M_{sk}] &= -N_{rk}\,, & [N_{kk}, N_{rk}] &= M_{rk}\,.
\end{aligned} \tag{C.6}$$

The left-invariant frame is $\ell^A m_A = \ell^r M_{rk} + \ell^{\tilde{r}} N_{rk}$. The most general left-invariant metric can be expressed as

$$ds^2 = a\,(\delta_{rs}\ell^r\ell^s + \delta_{\tilde{r}\tilde{s}}\ell^{\tilde{r}}\ell^{\tilde{s}})\,, \tag{C.7}$$

where $a > 0$ is a constant. The left-invariant forms of $\mathbb{CP}^{k-1}$ are generated by the (Kähler) 2-form

$$\omega = a\,\delta_{r\tilde{s}}\ell^r \wedge \ell^{\tilde{s}}\,. \tag{C.8}$$

The non-vanishing components of the curvature of the metric in the ortho-normal frame are

$$\begin{aligned}
R_{rs,pq} &= -\frac{1}{4a}\left(\delta_{rq}\delta_{sp} - \frac{1}{a}\delta_{rp}\delta_{sq}\right), & R_{rs,\tilde{p}\tilde{q}} &= -\frac{1}{4a}\left(\delta_{r\tilde{q}}\delta_{s\tilde{p}} - \frac{1}{a}\delta_{r\tilde{p}}\delta_{s\tilde{q}}\right),\\
R_{r\tilde{s},p\tilde{q}} &= \frac{1}{4a}\left(\delta_{r\tilde{q}}\delta_{\tilde{s}p} + \delta_{rp}\delta_{\tilde{s}\tilde{q}}\right) + \frac{1}{2a}\delta_{r\tilde{s}}\delta_{p\tilde{q}}\,, & R_{\tilde{r}\tilde{s},\tilde{p}\tilde{q}} &= -\frac{1}{4a}\left(\delta_{\tilde{r}\tilde{q}}\delta_{\tilde{s}\tilde{p}} - \delta_{\tilde{r}\tilde{p}}\delta_{\tilde{s}\tilde{q}}\right).
\end{aligned} \tag{C.9}$$

This expression of the curvature matches that in (3.33) for $\mathbb{CP}^3$ up to an overall scale.

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C.2 $M^k = \mathrm{SU}(k)/\mathrm{SU}(k-1)$

Next let us turn to the $\mathrm{SU}(k)/\mathrm{SU}(k-1)$ homogeneous space. The embedding of $\mathfrak{su}(k-1) = \mathbb{R}\langle M^{(k-1)}_{rs}, N^{(k-1)}_{rs}\rangle$, where $r,s = 1,\ldots,k-1$, into $\mathfrak{su}(k) = \mathbb{R}\langle M^{(k)}_{ab}, N^{(k)}_{ab}\rangle$ is given by

$$M^{(k-1)}_{rs} = M^{(k)}_{rs}\,, \quad N^{(k-1)}_{rs} = N^{(k)}_{rs} + \frac{1}{k-1}\delta_{rs}N^{(k)}_{kk}\,. \tag{C.10}$$

As $\mathfrak{m} = \mathbb{R}\langle M^{(k)}_{rk}, N^{(k)}_{sk}, N^{(k)}_{kk}\rangle$, the (non-vanishing) commutators involving elements of $\mathfrak{m}$ are

$$\begin{aligned}
[M^{(k)}_{rk}, M^{(k-1)}_{sk}] &= -\frac{1}{2}M^{(k)}_{rs}\,, \quad [M^{(k)}_{rk}, N^{(k)}_{sk}] = \frac{1}{2}N^{(k-1)}_{rs} - \frac{k}{2(k-1)}\delta_{rs}N^{(k)}_{kk}\,,\\
[N^{(k)}_{rk}, N^{(k)}_{sk}] &= -\frac{1}{2}M^{(k-1)}_{rs}\,,
\end{aligned} \tag{C.11}$$

and

$$\begin{aligned}
[M^{(k-1)}_{rs}, M^{(k)}_{tk}] &= \frac{1}{2}(\delta_{ts}M^{(k)}_{rk} - \delta_{tr}M^{(k)}_{sk})\,, \qquad [M^{(k-1)}_{rs}, N^{(k)}_{tk}] = \frac{1}{2}(\delta_{ts}N^{(k)}_{rk} - \delta_{tr}N^{(k)}_{sk})\,,\\
[N^{(k-1)}_{rs}, M^{(k)}_{tk}] &= -\frac{1}{k-1}\delta_{rs}N^{(k)}_{tk} + \frac{1}{2}(\delta_{ts}N^{(k)}_{rk} + \delta_{tr}N^{(k)}_{sk})\,,\\
[N^{(k-1)}_{rs}, N^{(k)}_{tk}] &= \frac{1}{k-1}\delta_{rs}M^{(k)}_{tk} - \frac{1}{2}(\delta_{ts}M^{(k)}_{rk} + \delta_{tr}M^{(k)}_{sk})\,,\\
[N^{(k)}_{kk}, M^{(k)}_{rk}] &= -N^{(k)}_{rk}\,, \qquad [N^{(k)}_{kk}, N^{(k)}_{rk}] = M^{(k)}_{rk}\,.
\end{aligned} \tag{C.12}$$

Setting $\ell^A m_A = \hat{\ell}^r M^{(k)}_{rk} + \hat{\ell}^{\tilde{r}} N^{(k)}_{rk} + \hat{\ell}^0 N^{(k)}_{kk}$ for the left-invariant frame, a direct computation reveals that the most general invariant metric is

$$ds^2 = a\,(\delta_{rs}\hat{\ell}^r\hat{\ell}^s + \delta_{\tilde{r}\tilde{s}}\hat{\ell}^{\tilde{r}}\hat{\ell}^{\tilde{s}}) + b(\hat{\ell}^0)^2\,, \tag{C.13}$$

where $a, b > 0$ are constants. Moreover the left-invariant 2- and 3-forms for $k = 4$ are generated by

$$\hat{\omega} = \delta_{r\tilde{s}}\hat{\ell}^r \wedge \hat{\ell}^{\tilde{s}}\,, \quad \hat{\ell}^0 \wedge \hat{\omega}\,, \quad \mathrm{Re}\,\hat{\chi}\,, \quad \mathrm{Im}\,\hat{\chi}\,, \tag{C.14}$$

and their duals, where

$$\hat{\chi} = \frac{1}{3!}\,\epsilon_{rst}(\hat{\ell}^r + i\hat{\ell}^{\tilde{r}}) \wedge (\hat{\ell}^s + i\hat{\ell}^{\tilde{s}}) \wedge (\hat{\ell}^t + i\hat{\ell}^{\tilde{t}})\,, \tag{C.15}$$

is the holomorphic (3,0)-form.

However for convenience, we re-label the indices of the left-invariant frame as $\ell^{2r-1} = \hat{\ell}^r$, $\ell^{2r} = \hat{\ell}^{\tilde{r}}$, $\ell^7 = \hat{\ell}^0$, $r = 1,2,3$ in which case the left-invariant metric can be rewritten as

$$ds^2 = a\,\delta_{mn}\ell^m\ell^n + b\,(\ell^7)^2 = \delta_{mn}\mathbf{e}^m\mathbf{e}^n + (\mathbf{e}^7)^2\,, \tag{C.16}$$

where we have introduced an ortho-normal frame $\mathbf{e}^m = \sqrt{a}\,\ell^m, \mathbf{e}^7 = \sqrt{b}\,\ell^7$, and $m,n = 1,\ldots,6$. Note also that up to an overall scale, the left-invariant 2- and 3-forms can be re-written in terms of the ortho-normal frame. In particular, we have

$$\omega = \mathbf{e}^{12} + \mathbf{e}^{34} + \mathbf{e}^{56}\,, \quad \mathbf{e}^7 \wedge \omega\,, \quad \mathrm{Re}\,\chi\,, \quad \mathrm{Im}\,\chi\,, \tag{C.17}$$

where

$$\chi = (\mathbf{e}^1 + i\mathbf{e}^2) \wedge (\mathbf{e}^3 + i\mathbf{e}^4) \wedge (\mathbf{e}^5 + i\mathbf{e}^6)\,. \tag{C.18}$$

We shall use this ortho-normal basis to solve the KSEs for this internal space.

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D The Berger space $B^7 = \mathrm{Sp}(2)/\mathrm{Sp}(1)_{\mathrm{max}}$

To describe the geometry of the Berger space B^7, one identifies the vector representation **5** of $\mathfrak{so}(5) = \mathfrak{sp}(2)$ with the symmetric trace-less representation of $\mathfrak{so}(3) = \mathfrak{sp}(1)$ and then decomposes the adjoint representation of $\mathfrak{so}(5)$ in $\mathfrak{so}(3)$ representations as $\mathbf{10} = \mathbf{3} \oplus \mathbf{7}$, where **7** is the symmetric traceless representation of $\mathfrak{so}(3)$ constructed with three copies of the vector representation. As a result $\mathfrak{so}(5) = \mathfrak{so}(3) \oplus \mathfrak{m}$, where $\mathfrak{so}(3)$ and $\mathfrak{m}$ are identified with the 3-dimensional and 7-dimensional representations, respectively.

This decomposition can be implemented as follows. Consider the basis W_{ab}, $a, b, c, d = 1, \ldots, 5$,

$$(W_{ab})^c{}_d = \delta^c_a \delta_{bd} - \delta^c_b \delta_{ad} \,, \tag{D.1}$$

in $\mathfrak{so}(5)$ leading to the commutators

$$[W_{ab}, W_{a'b'}] = (\delta_{ba'} W_{ab'} + \delta_{ab'} W_{ba'} - \delta_{aa'} W_{bb'} - \delta_{bb'} W_{aa'}) \,. \tag{D.2}$$

Then re-write each basis element using the **5** representation $\mathfrak{so}(3)$ as $W_{rs,tu}$, where $r, s, t, u = 1, 2, 3$. Decomposing this into $\mathfrak{so}(3)$ representations, one finds that

$$\begin{aligned} W_{rs,tu} = {} & O_{ru}\delta_{st} + O_{su}\delta_{rt} + O_{rt}\delta_{su} + O_{st}\delta_{ru} \\ & + \epsilon^p{}_{st} S_{pru} + \epsilon^p{}_{rt} S_{psu} + \epsilon^p{}_{su} S_{prt} + \epsilon^p{}_{ru} S_{pst} \,, \end{aligned} \tag{D.3}$$

where $O \in \mathfrak{so}(3)$ and $S \in \mathfrak{m}$. Using this one can proceed to describe the homogeneous space B^7. However, this decomposition does not automatically reveal the G_2 structure which is necessary in the analysis of the supersymmetric solutions. Instead, we shall follow an adaptation [34] of the description in [23] and [45, appendix A.1]. For this use the inner product

$$\langle W_{ab}, W_{a'b'} \rangle = -\frac{1}{2} \mathrm{tr}(W_{ab} W_{a'b'}) \,, \tag{D.4}$$

which is $\mathfrak{so}(5)$ invariant and the basis W_{ab}, $a < b$, is ortho-normal. In this basis, the structure constants of $\mathfrak{so}(5)$ are skew-symmetric. Then identify the $\mathfrak{so}(3)$ subalgebra of $\mathfrak{so}(5)$ with the span of the ortho-normal vectors

$$\begin{aligned} & h_1 = \frac{1}{\sqrt{5}}(-W_{12} - W_{34} + \sqrt{3} W_{35}) \,, \quad h_2 = \frac{1}{\sqrt{5}}(-W_{13} + W_{24} + \sqrt{3} W_{25}) \,, \\ & h_3 = \frac{1}{\sqrt{5}}(-2W_{14} + W_{23}) \,. \end{aligned} \tag{D.5}$$

We choose the subspace $\mathfrak{m}$ to be orthogonal to $\mathfrak{so}(3)$ and an ortho-normal basis in $\mathfrak{m}$ introduced as

$$\begin{aligned} & m_1 = \frac{1}{2\sqrt{5}}(4W_{12} - W_{34} + \sqrt{3} W_{35}) \,, && m_2 = \frac{1}{2\sqrt{5}}(4W_{13} + W_{24} + \sqrt{3} W_{25}) \,, \\ & m_3 = \frac{1}{\sqrt{5}}(-W_{14} - 2W_{23}) \,, && m_4 = \frac{1}{2}(\sqrt{3} W_{34} + W_{35}) \,, \quad m_5 = \frac{1}{2}(\sqrt{3} W_{24} - W_{25}) \,, \\ & m_6 = W_{15} \,, && m_7 = W_{45} \,. \end{aligned} \tag{D.6}$$

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Then it is straightforward to show that

$$[h_\alpha, h_\beta] = \frac{1}{\sqrt{5}}\epsilon_{\alpha\beta}{}^{\gamma} h_\gamma\,, \quad [h_\alpha, m_i] = k_{\alpha i}{}^{j} m_j\,, \quad [m_i, m_j] = \frac{1}{\sqrt{5}}\varphi_{ij}{}^{k} m_k + k_{ij}{}^{\alpha} h_\alpha\,, \tag{D.7}$$

where φ is given in (4.13), the indices are raised and lowered with the flat metric and

$$\begin{aligned}
k^1 &= -\frac{3}{2\sqrt{5}} m_2 \wedge m_3 - \frac{\sqrt{3}}{2} m_2 \wedge m_6 - \frac{\sqrt{3}}{2} m_3 \wedge m_5 + \frac{2}{\sqrt{5}} m_4 \wedge m_7 + \frac{1}{2\sqrt{5}} m_5 \wedge m_6\,,\\
k^2 &= \frac{3}{2\sqrt{5}} m_1 \wedge m_3 - \frac{\sqrt{3}}{2} m_1 \wedge m_6 - \frac{\sqrt{3}}{2} m_3 \wedge m_4 - \frac{1}{2\sqrt{5}} m_4 \wedge m_6 + \frac{2}{\sqrt{5}} m_5 \wedge m_7\,,\\
k^3 &= -\frac{3}{2\sqrt{5}} m_1 \wedge m_2 - \frac{\sqrt{3}}{2} m_1 \wedge m_5 - \frac{\sqrt{3}}{2} m_2 \wedge m_4 + \frac{1}{2\sqrt{5}} m_4 \wedge m_5 + \frac{2}{\sqrt{5}} m_6 \wedge m_7\,.
\end{aligned}$$

So $f_{ij}{}^{k} = \frac{1}{\sqrt{5}}\varphi_{ij}{}^{k}$ and the Jacobi identities imply that φ is invariant under the representation of $\mathfrak{so}(3)$ on $\mathfrak{m}$. Therefore the embedding of $\mathfrak{so}(3)$ in $\mathfrak{so}(7)$ defined by (k^1, k^2, k^3) factors through $\mathfrak{g}_2$. This is useful in the analysis of the gravitino KSE.

E $\mathfrak{so}(5) = \mathfrak{sp}(2)$

To describe the various homogeneous spaces that we are using which admit a transitive action of a group with Lie algebra $\mathfrak{so}(5) = \mathfrak{sp}(2)$, choose a basis in $\mathfrak{so}(5)$ as

$$(M_{\tilde{a}\tilde{b}})_{\tilde{c}\tilde{d}} = \frac{1}{2}(\delta_{\tilde{a}\tilde{c}}\,\delta_{\tilde{b}\tilde{d}} - \delta_{\tilde{a}\tilde{d}}\,\delta_{\tilde{b}\tilde{c}}), \tag{E.1}$$

in $\mathfrak{sp}(2) = \mathfrak{so}(5)$, where $M_{\tilde{a}\tilde{b}}, \tilde{a}, \tilde{b} = 1, \ldots, 5$. The commutators are

$$[M_{\tilde{a}\tilde{b}}, M_{\tilde{a}'\tilde{b}'}] = \frac{1}{2}(\delta_{\tilde{a}\tilde{b}'} M_{\tilde{b}\tilde{a}'} + \delta_{\tilde{b}\tilde{a}'} M_{\tilde{a}\tilde{b}'} - \delta_{\tilde{b}\tilde{b}'} M_{\tilde{a}\tilde{a}'} - \delta_{\tilde{a}\tilde{a}'} M_{\tilde{b}\tilde{b}'})\,. \tag{E.2}$$

In what follows, we shall describe various decompositions $\mathfrak{so}(5) = \mathfrak{h} \oplus \mathfrak{m}$ for different choices of a subalgebra $\mathfrak{h}$ and summarize some of their algebraic and geometric properties that we are using in this work.

E.1 $M^6 = \mathbf{Sp(2)/U(2)}$

The subalgebra $\mathfrak{h}$ and $\mathfrak{m}$ are spanned as

$$\mathfrak{u}(2) = \mathfrak{u}(2) \equiv \mathbb{R}\,\langle T_r, T_7\rangle = \mathbb{R}\,\langle \frac{1}{2}\epsilon_r{}^{st} M_{st}, M_{45}\rangle\,, \tag{E.3}$$

and

$$\mathfrak{m} = \mathbb{R}\,\langle M_{ra}\rangle = \mathbb{R}\,\langle M_{r4}, M_{r5},\rangle\,, \tag{E.4}$$

respectively, where $r, s, t = 1, 2, 3$ and $a, b, c, \ldots = 4, 5$. In this basis the non-vanishing commutators are

$$\begin{aligned}
[T_r, T_s] &= -\frac{1}{2}\epsilon_{rs}{}^{t} T_t\,, \quad [T_r, M_{sa}] = -\frac{1}{2}\epsilon_{rs}{}^{t} M_{ta}\,, \quad [T_7, M_{ra}] = -\frac{1}{2}\epsilon_{ab} M_{rb}\,,\\
[M_{ra}, M_{sb}] &= -\frac{1}{2}\delta_{ab}\epsilon_{rs}{}^{t} T_t - \frac{1}{2}\delta_{rs}\epsilon_{ab} T_7\,.
\end{aligned} \tag{E.5}$$

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Clearly this is a symmetric coset space admitting an invariant metric

$$ds^2 = a\,\delta_{rs}\delta_{ab}\ell^{ra}\ell^{sb} = \delta_{rs}\delta_{ab}\mathbf{e}^{ra}\mathbf{e}^{sb}\,, \tag{E.6}$$

where $a > 0$ is a constant, and ℓ^{ra} and $\mathbf{e}^{ra} = \sqrt{a}\,\ell^{ra}$ are the left-invariant and ortho-normal frames, respectively. The curvature of the symmetric space in the ortho-normal frame is

$$R_{ra\,sb,tc\,ud} = \frac{1}{4a}(\delta_{rt}\delta_{su} - \delta_{ru}\delta_{st})\delta_{ab}\delta_{cd} + \frac{1}{4a}\delta_{rs}\delta_{tu}\epsilon_{ab}\epsilon_{cd}\,, \tag{E.7}$$

which is instrumental in the investigation of the gravitino KSE in section 3.5.1.

E.2 $M^6 = \mathbf{Sp(2)/(Sp(1) \times U(1))}$

Viewing the elements of Sp(2) as quaternionic 2×2 matrices, $\mathrm{Sp}(1)\times\mathrm{U}(1) \subset \mathrm{Sp}(1)\times\mathrm{Sp}(1)$ is embedded in Sp(2) along the diagonal. To describe this embedding choose a basis in $\mathfrak{sp}(2) = \mathfrak{so}(5)$ as in (E.1) and set

$$T^{(\pm)}_r = \frac{1}{2}\varepsilon^{rst}M^{st} \pm M^{r4}\,, \quad , W_a = \sqrt{2}M_{a5}\,, \tag{E.8}$$

where $r = 1, 2, 3$ and now $a = 1, \ldots 4$. In terms of this basis, the non-vanishing commutators of $\mathfrak{sp}(2)$ are

$$\begin{aligned}[T^{(\pm)}_r, T^{(\pm)}_s] &= -\epsilon_{rs}{}^t T^{(\pm)}_t\,, \quad [T^{(\pm)}_r, W_a] = \frac{1}{2}(I^{(\pm)}_r)^b{}_a W_b\,,\\ [W_a, W_b] &= -\frac{1}{2}\big((I^{(+)}_r)_{ab}T^{(+)}_r + (I^{(-)}_r)_{ab}T^{(-)}_r\big)\,,\end{aligned} \tag{E.9}$$

where

$$(I^{(\pm)}_r)^4{}_s = \mp\delta_{rs}\,, \quad (I^{(\pm)}_r)^s{}_4 = \pm\delta^s{}_r\,, \quad (I^{(\pm)}_r)^s{}_t = \epsilon_{rst}\,. \tag{E.10}$$

Observe that $(I^{(\pm)}_r)$ are bases in the spaces of (anti-)self-dual forms in $\mathbb{R}^4$ and that

$$I^{(\pm)}_r I^{(\pm)}_s = -\delta_{rs}\mathbf{1} - \epsilon_{rst}I^{(\pm)}_t\,. \tag{E.11}$$

The subalgebra $\mathfrak{h}$ and $\mathfrak{m}$ are spanned as

$$\mathfrak{h} = \mathfrak{sp}(1)\oplus\mathfrak{u}(1) = \mathbb{R}\langle T^{(-)}_r, T^{(+)}_3\rangle\,, \tag{E.12}$$

and

$$\mathfrak{m} = \mathbb{R}\langle W_a, T^{(+)}_1, T^{(+)}_2\rangle\,, \tag{E.13}$$

respectively. Introducing the left-invariant frame, $\ell^A m_A = \ell^a W_a + \ell^{\underline{r}}T^{(+)}_{\underline{r}}$, where $\underline{r} = 1, 2$, the left-invariant metric can be written as

$$ds^2 = a\,\delta_{ab}\ell^a\ell^b + b\,\delta_{\underline{rs}}\ell^{\underline{r}}\ell^{\underline{s}} = \delta_{ab}\mathbf{e}^a\mathbf{e}^b + \delta_{\underline{rs}}\mathbf{e}^{\underline{r}}\mathbf{e}^{\underline{s}}\,, \tag{E.14}$$

where $a, b > 0$ and we have introduced the ortho-normal frame $\mathbf{e}^a = \sqrt{a}\,\ell^a$, $\mathbf{e}^{\underline{r}} = \sqrt{b}\,\ell^{\underline{r}}$.

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The curvature of this metric in the ortho-normal frame is

$$
\begin{aligned}
R_{ab,cd} &= \left(\frac{1}{2a} - \frac{3b}{16a^2}\right)(\delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc}) + \frac{3b}{16a^2}\left((I_3^{(+)})_{ab}(I_3^{(+)})_{cd} - (I_3^{(+)})_{a[b}(I_3^{(+)})_{cd]}\right),\\
R_{a\underline{r},b\underline{s}} &= \frac{b}{16a^2}\delta_{ab}\delta_{\underline{rs}} + \left(\frac{1}{4a} - \frac{b}{16a^2}\right)\epsilon_{\underline{rs}}(I_3^{(+)})_{ab}\,,\\
R_{ab,\underline{rs}} &= \left(\frac{1}{2a} - \frac{b}{8a^2}\right)\epsilon_{\underline{rs}}(I_3^{(+)})_{ab}, \quad R_{\underline{rs},\underline{tu}} = \frac{1}{b}\epsilon_{\underline{rs}}\epsilon_{\underline{tu}}\,.
\end{aligned}
\tag{E.15}
$$

We shall use these expressions in the investigation of the gravitino KSE in section 3.5.2.

E.3 $M^7 = \mathbf{Sp(2)/\Delta(Sp(1))}$

The decomposition of the Lie algebra $\mathfrak{sp}(2) = \mathfrak{so}(5)$ suitable for the description of this homogeneous space is as in (E.3) but now $\mathfrak{h}$ and $\mathfrak{m}$ are spanned as

$$
\mathfrak{h} = \mathbb{R}\langle T_r\rangle\,, \quad \mathfrak{m} = \mathbb{R}\langle M_{ra}, T_7\rangle\,,
\tag{E.16}
$$

respectively, where $r = 1,2,3$ and $a = 4,5$. Introducing the left-invariant frame as $\ell^A t_A = \ell^{ra}M_{ra} + \ell^7 T_7$, the left-invariant metric is

$$
ds^2 = \delta_{rs}g_{ab}\ell^{ra}\ell^{sb} + a_4(\ell^7)^2\,,
\tag{E.17}
$$

where (g_{ab}) is a symmetric constant positive definite 2×2 matrix and $a_4 > 0$ is a constant. The curvature of this metric in the left-invariant frame is

$$
\begin{aligned}
R_{pc\,qd,}{}^{ra}{}_{sb} = &-\frac{1}{16}a_4^{-1}\delta^r_p\delta_{qs}g^{ae}((\Delta g)_{ec} - a_4\epsilon_{ec})((\Delta g)_{db} + a_4\epsilon_{db})\\
&+\frac{1}{16}a_4^{-1}\delta^r_q\delta_{ps}g^{ae}((\Delta g)_{ed} - a_4\epsilon_{ed})((\Delta g)_{cb} + a_4\epsilon_{cb})\\
&+\frac{1}{8}\epsilon_{cd}\delta_{pq}\delta^r_s g^{ae}\epsilon_{eb}(\delta^{t_1t_2}g_{t_1t_2} - a_4) - \frac{1}{4}\delta_{cd}\delta^a_b(\delta_{ps}\delta^r_q - \delta_{qs}\delta^r_p)\,,
\end{aligned}
\tag{E.18}
$$

and

$$
\begin{aligned}
R_{7\,ar,}{}^{7}{}_{bs} = &\frac{1}{16}a_4^{-1}((\Delta g)_{ad} + a_4\epsilon_{ad})g^{de}\epsilon_{eb}(\delta^{t_1t_2}g_{t_1t_2} - a_4)\delta_{rs}\\
&-\frac{1}{8}a_4^{-1}\epsilon_a{}^d((\Delta g)_{db} + a_4\epsilon_{db})\delta_{rs}\,,
\end{aligned}
\tag{E.19}
$$

where

$$
(\Delta g)_{ab} = \epsilon_a{}^d g_{db} + \epsilon_b{}^d g_{da}\,,
\tag{E.20}
$$

(g^{ab}) is the inverse matrix of (g_{ab}) and the indices of ϵ are raised and lowered with δ_{ab}. The Ricci tensor again in the left-invariant frame is

$$
\begin{aligned}
R_{ra\,sb} = &\left[\frac{a_4^{-1}}{16}g^{dc}(\Delta g)_{da}(\Delta g)_{cb} - \frac{1}{16}g^{dc}(\Delta g)_{cb}\epsilon_{da} + \frac{1}{16}g^{cd}\epsilon_{ca}\epsilon_{db}(\delta^{t_1t_2}g_{t_1t_2} - 2a_4)\right.\\
&\left.+\frac{a_4^{-1}}{16}(\Delta g)_{ad}g^{dc}\epsilon_{cb}\,\delta^{t_1t_2}g_{t_1t_2} - \frac{a_4^{-1}}{8}\epsilon_a{}^d(\Delta g)_{db} + \frac{5}{8}\delta_{ab}\right]\delta_{rs}\,,
\end{aligned}
\tag{E.21}
$$

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and

$$R_{77} = -\frac{3}{8}\frac{a_4}{\det g}(\delta^{t_1t_2}g_{t_1t_2} - a_4) + \frac{3}{8}a_4\delta_{ab}g^{ab} - \frac{3}{8}\epsilon_a{}^d(\Delta g)_{db}g^{ab}\,. \tag{E.22}$$

It is straightforward to compute the Ricci tensor for (g_{ab}) diagonal. This concludes the summary of the geometry for this homogeneous space.

E.4 $M^7 = \mathbf{Sp(2)/Sp(1)}$

The decomposition of the Lie algebra $\mathfrak{sp}(2) = \mathfrak{so}(5)$ suitable for the description of this homogeneous space is as in (E.9), where in this case

$$\mathfrak{so}(3) = \mathbb{R}\langle T_r^{(-)}\rangle\,, \quad \mathfrak{m} = \mathbb{R}\langle W_a, T_r^{(+)}\rangle\,, \tag{E.23}$$

and where $r = 1,2,3$ and $a = 1,\ldots,4$. Introducing the left-invariant frame as $\ell^A m_A = \ell^a W_a + \ell^r T_r^{(+)}$, the most general left-invariant metric is

$$ds^2 = a\delta_{ab}\ell^a\ell^b + g_{rs}\ell^r\ell^s\,, \tag{E.24}$$

where $a > 0$ is a constant and (g_{rs}) is any constant 3×3 positive definite symmetric matrix. The non-vanishing components of the curvature tensor of this metric in the left-invariant frame is

$$\begin{aligned}
R_{cd,}{}^a{}_b &= \frac{a^{-1}}{16}\Big[\delta^{ae}\delta^{pr}\delta^{qs}(I_p^{(+)})_{ec}g_{rs}(I_q^{(+)})_{db} - (d,c)\Big] - \frac{a^{-1}}{8}\delta^{ae}\delta^{pr}\delta^{qs}(I_p^{(+)})_{eb}g_{rs}(I_q^{(+)})_{cd} \\
&\quad + \frac{1}{2}(\delta^a_c\delta_{db} - \delta^a_d\delta_{cb})\,, \qquad\qquad (E.25)\\
R_{rs}{}^a{}_b &= \frac{a^{-1}}{4}\delta^{pq}g_{pq}\epsilon_{rs}{}^t(I_t^{(+)})^a{}_b - \frac{a^{-2}}{8}\epsilon^{pqt}(I_t^{(+)})^a{}_b g_{pr}g_{qs} - \frac{a^{-1}}{2}\epsilon_{rs}{}^t\delta^{pq}(I_p^{(+)})^a{}_b g_{qt}\,, \qquad (E.26)\\
R_{ra}{}^s{}_b &= \frac{1}{8}\big[g^{sm}\epsilon_{mr}{}^n g_{np}\delta^{pt} + g^{sm}\epsilon_m{}^{tn}g_{nr}\big](I_t^{(+)})_{ab} + \frac{1}{8}\epsilon_r{}^{sp}(I_p^{(+)})_{ab} + \frac{a^{-1}}{16}\delta_{ab}\delta^{sm}g_{mr} \\
&\quad + \frac{a^{-1}}{16}\epsilon^{smn}g_{mr}(I_n^{(+)})_{ab}\,, \qquad\qquad (E.27)
\end{aligned}$$

and

$$R_{rs,pq} = g_{pl}R_{rs}{}^l{}_q = \epsilon_{rs}{}^m\epsilon_{pq}{}^n X_{mn}\,, \tag{E.28}$$

where

$$\begin{aligned}
X_{mn} &= \frac{1}{2}\delta_{mk}\delta_{nl}g^{kl}(\delta^{q_1q_2}\delta^{p_1p_2}g_{q_1p_1}g_{q_2p_2}) \\
&\quad - 2g_{mn} + \delta_{mn}\delta^{pq}g_{pq} - \frac{1}{4}\delta_{mk}\delta_{nl}g^{kl}(\delta^{q_1q_2}g_{q_1q_2})^2\,,
\end{aligned} \tag{E.29}$$

and the matrix (g^{rs}) is the inverse of (g_{rs}). The Ricci tensor in the left-invariant frame is

$$\begin{aligned}
R_{ab} &= -\frac{a^{-1}}{8}\delta^{pq}g_{pq}\delta_{ab} + \frac{3}{2}\delta_{ab}\,, \\
R_{rs} &= \frac{1}{4}a^{-2}\delta^{mn}g_{mr}g_{ns} + (\delta_{rs}\delta_{pq}g^{pq} - \delta_{rp}\delta_{sq}g^{pq})\delta^{mn}X_{mn} + \delta_{rp}g^{pm}X_{ms} + \delta_{sp}g^{pm}X_{mr} \\
&\quad - \delta_{pq}g^{pq}X_{rs} - \delta_{rs}g^{pq}X_{pq}\,.
\end{aligned} \tag{E.30}$$

It is straightforward to find the Ricci tensor for (g_{rs}) diagonal. This homogeneous space admits two Einstein metrics one of which is the round sphere metric on S^7. This will be explored further in the investigation of the gravitino KSE in section 4.6.3.

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