

4D scattering amplitudes and asymptotic symmetries from 2D CFT

Clifford Cheung,^a Anton de la Fuente^b and Raman Sundrum^b

^aWalter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, CA 91125, U.S.A.

^bDepartment of Physics, University of Maryland, College Park, MD 20742, U.S.A.

E-mail: clifford.cheung@caltech.edu, delafuea@umd.edu, raman@umd.edu

ABSTRACT: We reformulate the scattering amplitudes of 4D flat space gauge theory and gravity in the language of a 2D CFT on the celestial sphere. The resulting CFT structure exhibits an OPE constructed from 4D collinear singularities, as well as infinite-dimensional Kac-Moody and Virasoro algebras encoding the asymptotic symmetries of 4D flat space. We derive these results by recasting 4D dynamics in terms of a convenient foliation of flat space into 3D Euclidean AdS and Lorentzian dS geometries. Tree-level scattering amplitudes take the form of Witten diagrams for a continuum of (A)dS modes, which are in turn equivalent to CFT correlators via the (A)dS/CFT dictionary. The Ward identities for the 2D conserved currents are dual to 4D soft theorems, while the bulk-boundary propagators of massless (A)dS modes are superpositions of the leading and subleading Weinberg soft factors of gauge theory and gravity. In general, the massless (A)dS modes are 3D Chern-Simons gauge fields describing the soft, single helicity sectors of 4D gauge theory and gravity. Consistent with the topological nature of Chern-Simons theory, Aharonov-Bohm effects record the “tracks” of hard particles in the soft radiation, leading to a simple characterization of gauge and gravitational memories. Soft particle exchanges between hard processes define the Kac-Moody level and Virasoro central charge, which are thereby related to the 4D gauge coupling and gravitational strength in units of an infrared cutoff. Finally, we discuss a toy model for black hole horizons via a restriction to the Rindler region.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Theory, Scattering Amplitudes

ARXIV EPRINT: [1609.00732](https://arxiv.org/abs/1609.00732)

Contents

1	Introduction	1
2	Setup	4
2.1	Bulk coordinates	4
2.1.1	Milne region	5
2.1.2	Rindler region	6
2.2	Boundary coordinates	6
2.3	Approach	8
3	Gauge theory	9
3.1	Mode expansion from $Milne_4$ to AdS_3	9
3.2	Scaling dimensions from AdS_3/CFT_2	11
3.3	Witten diagrams in AdS_3	12
3.3.1	Interaction vertices	12
3.3.2	Bulk-bulk propagator	12
3.3.3	Bulk-boundary propagator	13
3.4	Continuation from $Milne_4$ to $Mink_4$	15
3.5	$Mink_4$ scattering amplitudes as CFT_2 correlators	15
3.6	Conserved currents of CFT_2	16
3.6.1	$Mink_4$ soft theorems as CFT_2 Ward identities	17
3.6.2	Equivalence of $Milne_4$ and $Mink_4$ soft limits	19
3.7	Kac-Moody algebra of CFT_2	20
3.8	Chern-Simons theory and multiple soft emission	22
3.8.1	Abelian Chern-Simons theory	22
3.8.2	Non-Abelian Chern-Simons theory	25
3.8.3	Locating Chern-Simons theory in $Mink_4$	26
3.9	Wess-Zumino-Witten model and multiple soft emission	29
3.10	Relation to memory effects	29
3.10.1	Chern-Simons memory and the Aharonov-Bohm effect	30
3.10.2	Chern-Simons level from internal soft exchange	32
3.11	Toy model for a black hole horizon	33
4	Gravity	35
4.1	Stress tensor of CFT_2	36
4.1.1	Bulk-boundary propagator for AdS_3 graviton	36
4.1.2	Ward identity for CFT_2 stress tensor	37
4.1.3	Relationship to subleading soft theorems in $Mink_4$	38
4.2	Virasoro algebra of CFT_2	40
4.3	Chern-Simons theory and multiple soft emission	41
4.3.1	Equivalence to AdS_3 gravity	42

4.3.2	Virasoro central charge from internal soft exchange	43
4.4	Relation to asymptotic symmetries	44
4.4.1	From super-rotations to super-translations in Mink_4	44
4.4.2	Chern-Simons theory for super-translations?	45
5	Future directions	46

1 Introduction

The AdS/CFT correspondence [1–7] has revealed profound insights into the dualities equating theories with and without gravity. As an explicit formalism, it has also given teeth to the powerful notion of holography, fueling concrete progress on longstanding puzzles in an array of subjects, ranging from black hole physics to strongly coupled dynamics. Still, AdS/CFT professes the limits of its own applicability: the entire construction rests pivotally on the infrastructure of warped geometry.

In this paper, we explore a potential strategy for channeling the power of AdS/CFT into 4D Minkowski spacetime. This ambitious goal has a long history [8–16], typically with a focus on AdS/CFT in the limit of infinite AdS radius. Here we follow a different path, in line with the seminal work of [17, 18]. The crux of our approach is to foliate Minkowski spacetime into a family of warped 3D slices for which the methodology of AdS/CFT is applicable, recasting the dynamics of 4D flat space into the grammar of a 2D CFT.¹ We derive the central objects of this conjectured 2D CFT — namely the conserved currents and stress tensor — and show how the corresponding Kac-Moody and Virasoro algebras beautifully encode the asymptotic symmetries of 4D gauge theory [20–24] and gravity [25–27]. Our results give a unified explanation for the deep connections recently discovered [20–24, 28–32] between asymptotic symmetries and 4D soft theorems [33–37], allowing us to extend and understand these results further. As we will see, the 2D current algebras are dual to 3D Chern-Simons (CS) gauge fields that describe soft fields in 4D, and for which the phenomena of gauge [38–40] and gravitational “memories” [41–45] take the form of abelian and non-abelian Aharonov-Bohm effects [46–48].

Let us now discuss our results in more detail. In section 2.1, we set the stage by defining a convenient set of coordinates for 4D Minkowski spacetime (Mink_4). These coordinates are formally anchored to a fixed origin [17, 18, 49–51] intuitively representing the location of a hard scattering process. In turn, this choice naturally divides Mink_4 into two regions: the 4D Milne spacetimes (Milne_4) past and future time-like separated from the origin, and the 4D spherical Rindler spacetime (Rind_4) space-like separated from the origin. We then choose coordinates in which Milne_4 and Rind_4 are foliated into slices at a fixed proper distance from the origin, or equivalently at fixed Milne time and Rindler radius, respectively. Each Milne slice is equivalent to 3D Euclidean anti-de Sitter space (AdS_3). While this geometry is purely spatial from the 4D viewpoint, we will for notational convenience

¹See [19] and references therein for a handy review of 2D CFT.

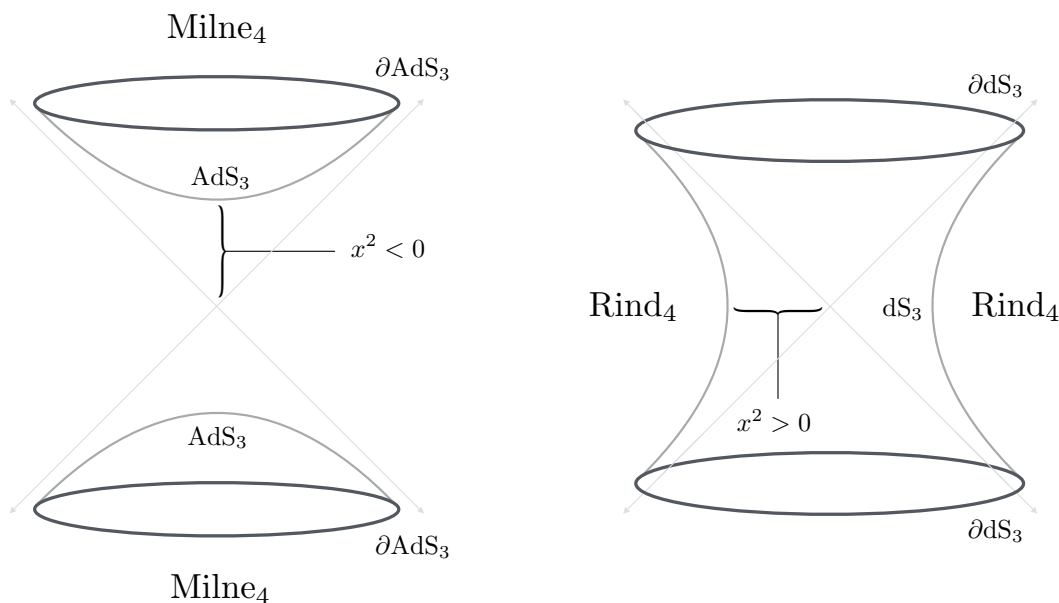


Figure 1. Minkowski space is divided into Milne and Rindler regions which are time-like and space-like separated from the origin, respectively. Each region is then foliated into a family of warped slices, each at a fixed proper distance from the origin.

refer to it as AdS_3 with the Euclidean signature implied. Similarly, each Rindler slice is equivalent to Lorentzian de Sitter (dS_3) spacetime.

In section 2.2, we show how the corresponding AdS_3 and dS_3 boundaries (∂AdS_3 and ∂dS_3) define a 2D celestial sphere at null infinity — the natural home of massless asymptotic states. By choosing the analog of Poincare patch coordinates on the warped slices, we find that the celestial sphere is labeled by complex variables (z, \bar{z}) that coincide with the projective spinor helicity variables frequently used in the study of scattering amplitudes. The geometry of our setup is depicted in figure 1, and our basic approach is outlined in section 2.3.

Armed with a foliation of Milne_4 into AdS_3 slices, we apply the $\text{AdS}_3/\text{CFT}_2$ dictionary, bearing in mind that the underlying spacetime is actually flat [17, 18]. To do so, in sections 3.1 and 3.2 we apply separation of variables to decompose all the degrees of freedom in Milne_4 into “harmonics” in Milne time, yielding a continuous spectrum of “massive” AdS_3 fields. Here the AdS_3 “mass” of each field is simply its Milne energy.² In section 3.3 we go on to show that the Witten diagrams of AdS_3 fields are precisely equal to flat space scattering amplitudes in Milne_4 , albeit with a modified prescription for LSZ reduction substituting AdS_3 bulk-boundary propagators for plane waves. In turn, the $\text{AdS}_3/\text{CFT}_2$ correspondence offers a formalism to recast these scattering amplitudes as correlators of a certain CFT_2 living on the celestial sphere. The operator product expansion corresponds to singularities in (z, \bar{z}) arising from collinear limits in the angular directions.

²This energy is in general not conserved in the “expanding Universe” defined by Milne spacetime, but it will be in a number of Weyl invariant theories of interest.

In section 3.4, we show how the AdS₃/CFT₂ dictionary in Milne₄ dovetails with the dS₃/CFT₂ dictionary [52–55] in Rind₄ by analytic continuation through the ambient Mink₄ embedding space. Here the mechanics of this continuation, as well as our calculations in general, are greatly simplified by employing the elegant embedding formalism of [56–60]. Notably, the appearance of dS₃ suggests that the underlying CFT₂ is *non-unitary*, as we see in detail. Putting it all together in section 3.5, we are then able to extend the mapping between 4D scattering amplitudes and 2D correlators to all of Minkowski spacetime.

A natural question now arises: which 4D scattering amplitudes are dual to the 2D correlators of conserved currents? For scattering amplitudes in the Milne region, the Witten diagrams for these correlators will involve massless AdS₃ fields. According to our decomposition into Milne harmonics, these massless modes have vanishing Milne energy, and thus correspond to the Milne soft limit of particles in the 4D scattering amplitude. In the case of gauge theory, we show in section 3.6 that the Milne soft limit coincides precisely with the usual soft limit taken with respect to Minkowski energy. As a result, the Ward identity for a conserved current in 2D is literally equal to the leading Weinberg soft theorem for gauge bosons in 4D, which we show explicitly for abelian gauge theory with matter as well as Yang-Mills (YM) theory. We thereby conclude that the conserved currents of the CFT₂ are dual to soft gauge bosons in Mink₄. It is attractive that the AdS₃/CFT₂ dictionary automatically guides us to identify 4D soft limits with 2D conserved currents. Afterwards, in section 3.7 we show how the existence of a 2D holomorphic conserved current relates to the presence of an infinite-dimensional Kac-Moody algebra.³

Next, we go on to construct the explicit AdS₃ dual of the CFT₂ for the current algebra subsector. In section 3.8, we show that soft gauge bosons of a single helicity comprise a 3D topological CS gauge theory in AdS₃ whose dual is the 2D chiral Wess-Zumino-Witten (WZW) model [63–66] discussed in section 3.9. As is well-known, this theory is a 2D CFT imbued with an infinite-dimensional Kac-Moody algebra. We show explicitly how hard particles in 4D decompose into massive 3D matter fields that source the CS gauge fields. Afterwards, we discuss the Kac-Moody level k_{CS} and its connection to internal exchange of soft gauge bosons. Our results suggest that the level is related to the 4D YM gauge coupling via $k_{\text{CS}} \sim 1/g_{\text{YM}}^2$.

We also show in section 3.10 how the topological nature of CS theories reflects the remarkable phenomenon of 4D gauge “memory” [38–40] in which soft fields record the passage of hard particles carrying conserved charges through specific angular regions on the celestial sphere. In our formulation, these memory effects are naturally encoded as abelian and non-abelian Aharonov-Bohm phases from the encircling of hard particle “tracks” by CS gauge fields.

Interestingly, ref. [67] proposed that gauge and gravitational memories have the potential to encode copious “soft hair” on black hole horizons, offering new avenues for understanding the information paradox, as reviewed in [68]. While black hole physics is not the primary focus of this work, our formalism does give a natural framework to study a toy

³Such a structure was observed long ago in amplitudes [61], serving as inspiration for the twistor string [62].

model for black hole horizons which we present in section 3.11. In particular, by excising the Milne regions of spacetime, we are left with a Rindler spacetime that describes a family of radially accelerating observers. We find that the CFT_2 structure extends to include the early and late time wavefunction at the Rindler horizon. In particular, the 2D conserved currents are dual to CS soft fields that record the insertion points of hard particles that puncture the horizon and that escape to null infinity.

In a parallel analysis for gravity, we show in section 4.1 that the Ward identity for the 2D stress tensor is an angular convolution of the subleading Weinberg soft theorem for gravitons in 4D. As for any CFT_2 , this theory is equipped with an infinite-dimensional Virasoro algebra that we discuss in section 4.2. Since the global $\text{SL}(2, \mathbb{C})$ subgroup is nothing but the 4D Lorentz group, these Virasoro symmetries are aptly identified as the “super-rotations” of the extended BMS algebra of asymptotic symmetries in 4D flat space [25–27]. We then consider the case of subleading soft gravitons and the CFT_2 stress tensor in section 4.3, arguing that the dual theory is simply AdS_3 gravity, which famously is equivalent to a CS theory in 3D [69, 70]. Afterwards, we go on to discuss the connections between 4D gravitational memory, and the Virasoro algebra. While the value of the Virasoro central charge c is subtle, our physical picture suggests that $c \sim m_{\text{Pl}}^2 L_{\text{IR}}^2$, where m_{Pl} is the 4D Planck scale and L_{IR} is an infrared cutoff. We then utilize the extended BMS algebra [71] to derive the CFT_2 Ward identity associated with “super-translations” [25, 26], and we confirm that they correspond to the leading Weinberg soft theorem for gravitons [28, 29].

Finally, let us pause to orient our results within the grander ambitions of constructing a holographic dual to flat space. Our central results rely crucially on the soft limit in 4D, wherein lie the hallmarks of 2D CFT. At the same time, a holographic dual to flat space will necessarily describe all 4D dynamics, including the soft regime. Hence, our results imply that the soft limit of any such dual will be described by a CFT. In this sense, the CFT structure derived in this paper should be interpreted as a stringent constraint on any holographic dual to flat space.

Note added: during the final stages of preparation for this paper, ref. [72] appeared, also deriving a 2D stress tensor for 4D single soft graviton emission.

2 Setup

As outlined in the introduction, our essential strategy is to import the holographic correspondence into flat space by reinterpreting Mink_4 as the embedding space for a family of AdS_3 slices [17, 18]. To accomplish this, we foliate Mink_4 into a set of warped geometries and mechanically invoke the $\text{AdS}_3/\text{CFT}_2$ dictionary, recasting its implications as old and new facts about flat space scattering amplitudes. We now define bulk and boundary coordinates natural to achieve this mapping.

2.1 Bulk coordinates

To begin, we define 4D Cartesian coordinates $x^\mu = (x^0, x^1, x^2, x^3)$ associated with the flat metric,

$$ds_{\text{Mink}_4}^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad (2.1)$$

and labeled by Greek indices (μ, ν, \dots) hereafter. As outlined in the introduction, it will be convenient to organize spacetime points in Minkowski space according to their proper distance from the origin. This partitions flat space into Milne and Rindler regions that are time-like and space-like separated from the origin.

2.1.1 Milne region

We foliate the 4D Milne region into hyperbolic slices of a fixed proper distance from the origin,

$$x^2 = -e^{2\tau}, \quad (2.2)$$

where τ is the Milne time coordinate. Together with the remaining spatial directions, τ defines a set of 4D hyperbolic Milne coordinates,

$$Y^I = (\tau, \rho, z, \bar{z}), \quad (2.3)$$

denoted by upper-case Latin indices (I, J, \dots) hereafter. The Milne coordinates Y^I are related to the Cartesian coordinates x^μ according to

$$\begin{aligned} x^0 &= \frac{e^\tau \rho}{2} \left(1 + \frac{1}{\rho^2} (1 + z\bar{z}) \right), & x^1 + ix^2 &= \frac{e^\tau z}{\rho}, \\ x^3 &= \frac{e^\tau \rho}{2} \left(1 - \frac{1}{\rho^2} (1 - z\bar{z}) \right), & x^1 - ix^2 &= \frac{e^\tau \bar{z}}{\rho}. \end{aligned} \quad (2.4)$$

The domain for each Milne coordinate is $\tau, \rho \in \mathbb{R}$ and $z, \bar{z} \in \mathbb{C}$. The regions $\rho > 0$ and $\rho < 0$ correspond to the two halves of Milne_4 — that is, the future and past Milne regions circumscribed by the future and past lightcones of the origin, respectively. So depending on the sign of ρ , the $\tau \rightarrow +\infty$ limit corresponds to either the asymptotic past or the asymptotic future. On the other hand, the $\tau \rightarrow -\infty$ limit corresponds to the $x^2 = 0$ boundary dividing the Milne and Rindler regions. In the context of a standalone Rindler spacetime, this boundary is known as the Rindler horizon.⁴ In the current setup, however, this horizon is a coordinate artifact simply because the underlying Minkowski space seamlessly joins the Milne and Rindler regions. Last but not least, (z, \bar{z}) denote complex stereographic coordinates on the celestial sphere. Note that the physical angles on the sky labeled by (z, \bar{z}) are antipodally identified for $\rho > 0$ and $\rho < 0$, due to the diametric mapping between celestial spheres in the asymptotic past and the asymptotic future.

By construction, the Milne coordinates are defined so that Milne_4 decomposes into a family of Euclidean AdS_3 geometries,

$$ds_{\text{Milne}_4}^2 = G_{IJ}(Y) dY^I dY^J = e^{2\tau} (-d\tau^2 + ds_{\text{AdS}_3}^2). \quad (2.5)$$

Each slice at fixed τ describes a 3D geometry equivalent to Euclidean AdS_3 spacetime in Poincare patch coordinates [4], so

$$ds_{\text{AdS}_3}^2 = g_{ij}(y) dy^i dy^j = \frac{1}{\rho^2} (d\rho^2 + dzd\bar{z}), \quad (2.6)$$

⁴More precisely, we are considering a spherical rather than the standard planar Rindler region reviewed in [73].

where lower-case Latin indices (i, j, \dots) denote AdS₃ coordinates,

$$y^i = (\rho, z, \bar{z}), \tag{2.7}$$

which are simply a restriction of the Milne coordinates, $Y^I = (\tau, y^i)$.

From eq. (2.6) it is obvious that ρ corresponds to the radial coordinate of AdS₃ and the $\rho \rightarrow 0$ limit defines the boundary ∂AdS_3 . Interpolating between the past and future Milne regions corresponds to an analytic continuation of the AdS₃ radius ρ to both positive and negative values.

2.1.2 Rindler region

A similar analysis applies to the 4D Rindler region, which we foliate with respect to

$$x^2 = e^{2\rho}, \tag{2.8}$$

where ρ is now the Rindler radial coordinate. Like before, we can define hyperbolic Rindler coordinates, $Y^I = (\rho, \tau, z, \bar{z})$, with the associated metric,

$$ds_{\text{Rind}_4}^2 = G_{IJ}(Y)dY^I dY^J = e^{2\rho} (d\rho^2 + ds_{\text{dS}_3}^2). \tag{2.9}$$

Splitting the Rindler coordinates by $Y^I = (\rho, y^i)$, we see that each slice at fixed ρ defines a Lorentzian dS₃ spacetime parameterized by $y^i = (\tau, z, \bar{z})$ and the corresponding metric,

$$ds_{\text{dS}_3}^2 = g_{ij}(y)dy^i dy^j = \frac{1}{\tau^2}(-d\tau^2 + dzd\bar{z}), \tag{2.10}$$

where τ is the conformal time of dS₃.

2.2 Boundary coordinates

Given a hyperbolic foliation of Minkowski space, it is then natural to consider the spacetime boundary associated with each warped slice. To be concrete, let us focus here on Milne₄, although a similar story will apply to Rind₄.

Using the Milne coordinates in eq. (2.4), we express an arbitrary spacetime point in Milne₄ as

$$x^\mu = e^\tau \left(\frac{k^\mu}{\rho} + \rho q^\mu \right), \tag{2.11}$$

where we have defined the null vectors,

$$k^\mu = \frac{1}{2}(1 + z\bar{z}, z + \bar{z}, -iz + i\bar{z}, -1 + z\bar{z}) \quad \text{and} \quad q^\mu = \frac{1}{2}(1, 0, 0, 1). \tag{2.12}$$

In terms of the celestial sphere, k^μ is a vector pointing in the (z, \bar{z}) direction while q^μ is a reference vector pointing at complex infinity. Of course, while q^μ describes a certain physical angle on the sky, this is a coordinate artifact without any physical significance.

Given a null vector k^μ it is natural to define polarization vectors,

$$\begin{aligned} e^\mu &= \frac{1}{2}(\bar{z}, 1, -i, \bar{z}) \\ \bar{e}^\mu &= \frac{1}{2}(z, 1, i, z), \end{aligned} \tag{2.13}$$

where ϵ and $\bar{\epsilon}$ correspond to (+) and (−) helicity states, respectively. As usual, the helicity sum over products of polarization vectors yields a projector onto physical states,

$$\epsilon^\mu \bar{\epsilon}^\nu + \epsilon^\nu \bar{\epsilon}^\mu = \frac{1}{2} \left(\eta^{\mu\nu} - \frac{q^\mu k^\nu + q^\nu k^\mu}{qk} \right), \quad (2.14)$$

where $qk = -1/2$ is actually constant. Note also that the polarization vectors ϵ^μ and $\bar{\epsilon}^\mu$ and the reference vector q^μ are compactly expressed in terms derivatives of k^μ ,

$$\begin{aligned} \epsilon^\mu &= \partial_z k^\mu \\ \bar{\epsilon}^\mu &= \partial_{\bar{z}} k^\mu \\ q^\mu &= \partial_z \partial_{\bar{z}} k^\mu. \end{aligned} \quad (2.15)$$

The above expressions will be quite useful for manipulating expressions later on.

To go to the boundary of AdS_3 we take the limit of vanishing radial coordinate, $\rho \rightarrow 0$. According to eq. (2.11), any spacetime point at the boundary approaches a null vector,

$$x^\mu \stackrel{\rho \rightarrow 0}{=} \frac{e^\tau k^\mu}{\rho}, \quad (2.16)$$

so ∂AdS_3 is the natural arena for describing massless degrees of freedom. To appreciate the significance of this, recall that the in and out states of a scattering amplitude are inserted in the asymptotic past and future, defined by $\tau \rightarrow +\infty$. For massless particles, this implies that null trajectories at $\tau \rightarrow +\infty$ should approach $\rho \rightarrow 0$ so that asymptotic states originate at ∂AdS_3 in the far past or terminate at ∂AdS_3 in the far future. Said more precisely, ∂AdS_3 is none other than past and future null infinity restricted to the Milne region.⁵ Hence, ∂AdS_3 is a natural asymptotic boundary associated with the scattering of massless particles.

Finally, let us comment on the unexpected connection between our coordinates and the spinor helicity formalism commonly used in the study of scattering amplitudes. In particular, while the specific form of k^μ in eq. (2.12) was rigidly dictated by the choice of Poincare patch coordinates on AdS_3 , it also happens to be that

$$k^\mu = \lambda \sigma^\mu \bar{\lambda}, \quad (2.17)$$

where λ and $\bar{\lambda}$ are projective spinors,

$$\lambda = (z, 1) \quad \text{and} \quad \bar{\lambda} = (\bar{z}, 1), \quad (2.18)$$

in a normalization where $\text{tr}(\sigma^\mu \bar{\sigma}^\nu) = \eta^{\mu\nu}/2$. Here λ and $\bar{\lambda}$ are defined modulo rescaling, i.e. modulo the energy of the associated momentum. This projective property implies that the only invariant kinematic data stored in λ and $\bar{\lambda}$ is angular.

Meanwhile, the reference vector q^μ can also be expressed in spinor helicity form,

$$q^\mu = \eta \sigma^\mu \bar{\eta}, \quad (2.19)$$

⁵Past and future null infinity in the Rindler region is contained in the boundary of dS_3 .

where η and $\bar{\eta}$ are reference spinors,

$$\eta = (1, 0) \quad \text{and} \quad \bar{\eta} = (1, 0), \quad (2.20)$$

and the polarization vectors take the simple form,

$$\begin{aligned} \epsilon^\mu &= \eta \sigma^\mu \bar{\lambda} \\ \bar{\epsilon}^\mu &= \lambda \sigma^\mu \bar{\eta}. \end{aligned} \quad (2.21)$$

Thus, our hyperbolic foliation of Minkowski space has induced a coordinate system on the boundary that coincides with projective spinor helicity variables in a gauge specified by a particular set of reference spinors.

As usual, we can combine spinors into Lorentz invariant angle and square brackets,

$$\langle 12 \rangle = \lambda_{1\alpha} \lambda_{2\beta} \epsilon^{\alpha\beta} = z_1 - z_2 \quad \text{and} \quad [12] = \bar{\lambda}_{1\dot{\alpha}} \bar{\lambda}_{2\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} = \bar{z}_1 - \bar{z}_2. \quad (2.22)$$

Meanwhile, the invariant mass of two null vectors,

$$-(k_1 + k_2)^2 = \langle 12 \rangle [12] = |z_1 - z_2|^2, \quad (2.23)$$

is the natural distance between points on the celestial sphere.

As is familiar from the context of scattering amplitudes, expressions typically undergo drastic simplifications when expressed in terms of spinor helicity variables. For example, the celebrated Parke-Taylor formula for the color-stripped MHV amplitude in non-abelian gauge theory is

$$A_n^{\text{MHV}} = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} \sim \frac{(z_i - z_j)^4}{(z_1 - z_2)(z_2 - z_3) \dots (z_n - z_1)}. \quad (2.24)$$

Here the collinear singularities are manifest in the form of $z_i - z_{i+1}$ poles in the denominator. More generally, since projective spinors only carry angular information, they are useful for exposing the collinear behavior of expressions.

2.3 Approach

So far we have simply defined a convenient representation of 4D Minkowski space as Milne and Rindler regions foliated into warped 3D slices. While at last we appear poised to apply the AdS₃/CFT₂ dictionary, a naive ambiguity arises: Milne₄ reduces to a family of AdS₃ slices — to which should we apply the holographic correspondence? After all, each value of τ corresponds to a distinct AdS₃ geometry, each with a different curvature and position in Milne₄. Even stranger, the bulk dynamics of Mink₄ will in general intersect all foliations of both Milne₄ and Rind₄.

The resolution to this puzzle is rather straightforward — and ubiquitous in more conventional applications of AdS/CFT. Perhaps most familiar is the case of spacetimes with factorizable geometry, AdS \times \mathcal{M} , where \mathcal{M} is a compact manifold. In such circumstances, the appropriate course of action is to Kaluza-Klein (KK) reduce the degrees of freedom along the compact directions of \mathcal{M} . This generates a tower of KK modes in AdS to which

the standard AdS/CFT dictionary should then be applied. In a slightly more complicated scenario, the spacetime is a warped product of AdS and \mathcal{M} , where the AdS radius varies from point to point in \mathcal{M} . Here too, KK reduction to AdS — with some fiducial radius of curvature — can be performed, again resulting in a tower of KK modes.

Something very similar occurs in our setup because Milne_4 is simply a warped product of AdS_3 and \mathbb{R}_τ , the real line parameterizing Milne time. Here “KK reduction” corresponds to a decomposition of fields in Milne_4 into modes in Milne time τ which are in turn AdS_3 fields via separation of variables. Each mode is then interpreted as a separate particle residing in the dimensionally reduced AdS_3 . However, unlike the usual KK scenario, where the spectrum of particles is discrete, the non-compactness of \mathbb{R}_τ induces a continuous “spectrum” of AdS_3 modes. As we will see later, an effective “compactification” [22] occurs when we consider the soft limit, which is the analog of projecting onto zero modes in the standard Kaluza-Klein procedure.

In the subsequent sections we derive this mode decomposition for scalar and gauge theories in the Milne region. We consider theories that exhibit classical Weyl invariance, permitting Milne_4 to be recast as a nicely factorized geometry, $\text{AdS}_3 \times \mathbb{R}_\tau$, rather than a warped product. In this case the mode decomposition is especially simple because Milne energy is conserved. Note, however, that this is merely a technical convenience that is not essential for our main results. In particular, when we go on to consider the case of gravity, there will be no such Weyl invariance, but the reduction of Milne_4 down to AdS_3 modes is of course still possible.

Armed with a reduction of Milne_4 degrees of freedom down to AdS_3 , we then apply the $\text{AdS}_3/\text{CFT}_2$ dictionary to recast scattering amplitudes in the form of CFT_2 correlators. We then show how the embedding formalism offers a trivial continuation of these results from Milne_4 into Rind_4 and thus all of Mink_4 . Along the way, we will understand the 4D interpretation of familiar objects in the CFT_2 , including correlators, Ward identities, and current algebra.

3 Gauge theory

3.1 Mode expansion from Milne_4 to AdS_3

As a simple warmup, consider the case of a massless interacting scalar field in Minkowski space. For the sake of convenience, we focus on Weyl invariant theories, although as noted previously this is not a necessity. The simplest Weyl invariant action of a scalar is

$$S = \int_{\text{Milne}_4} d^4Y \sqrt{-G} \left(-\frac{1}{2} G^{IJ} \nabla_I \Phi \nabla_J \Phi - \frac{1}{12} R \Phi^2 - \frac{\lambda}{24} \Phi^4 \right), \quad (3.1)$$

for now restricting to the contribution to the action from Milne_4 . An identical analysis will apply to Rind_4 , and later we will discuss at length how to glue these regions together.

In eq. (3.1) the conformal coupling to the Ricci scalar has no dynamical effect in flat space because $R = 0$. Nevertheless, this interaction induces an improvement term in the stress tensor for the scalar that ensures Weyl invariance. The Weyl transformation is given

by

$$G_{IJ} \rightarrow \bar{G}_{IJ} = e^{-2\tau} G_{IJ}, \tag{3.2}$$

where the scalar transforms as

$$\Phi \rightarrow \bar{\Phi} = e^\tau \Phi. \tag{3.3}$$

Due to the classical Weyl invariance of the theory, the metric decomposes into a factorizable $\text{AdS}_3 \times \mathbb{R}_\tau$ geometry with the associated metric,

$$ds_{\text{AdS}_3 \times \mathbb{R}_\tau}^2 = \bar{G}_{IJ} dY^I dY^J = -d\tau^2 + ds_{\text{AdS}_3}^2, \tag{3.4}$$

where $ds_{\text{AdS}_3}^2$ is defined in eq. (2.6). Since the action is Weyl invariant we obtain

$$S = \int_{\text{AdS}_3} d^3y \sqrt{-\bar{G}} \int d\tau \left(-\frac{1}{2} \bar{G}^{IJ} \nabla_I \bar{\Phi} \nabla_J \bar{\Phi} - \frac{1}{12} \bar{R} \bar{\Phi}^2 - \frac{\lambda}{24} \bar{\Phi}^4 \right), \tag{3.5}$$

where $\bar{R} = -6$ is the curvature of the \bar{G}_{IJ} metric.

Given the factorizable geometry, it is natural to define a ‘‘Milne energy’’,

$$\omega = i\partial_\tau, \tag{3.6}$$

which is by construction a Casimir invariant under the AdS_3 isometries, or in the language of the dual CFT_2 , the global conformal group $\text{SL}(2, \mathbb{C})$. This $\text{SL}(2, \mathbb{C})$ is also the 4D Lorentz group acting on the Milne_4 embedding space of AdS_3 . By contrast, the usual Minkowski energy,

$$E = i\partial_0, \tag{3.7}$$

is of course not Lorentz invariant and thus not $\text{SL}(2, \mathbb{C})$ invariant, and so is less useful in identifying the underlying CFT_2 structure. Again, we emphasize here that the Weyl invariance of the scalar theory is an algebraic convenience that is not crucial for any of our final conclusions. When Weyl invariance is broken, then the Milne energy simply is not conserved.

We can now expand the scalar into harmonics in Milne time,

$$\phi(\omega) = \int d\tau e^{i\omega\tau} \bar{\Phi}(\tau), \tag{3.8}$$

where $\phi(\omega)$ are scalar fields in AdS_3 , analogous to the tower of KK modes that arise in conventional compactifications. In terms of these fields, the linearized action becomes

$$S_0 = \int_{\text{AdS}_3} d^3y \sqrt{g} \int d\omega \left(-\frac{1}{2} g^{ij} \nabla_i \phi(-\omega) \nabla_j \phi(\omega) + \frac{1}{2} (1 + \omega^2) \phi(-\omega) \phi(\omega) \right), \tag{3.9}$$

so a massless scalar field in Milne_4 decomposes into a tower of AdS_3 scalars with

$$m_\phi^2(\omega) = -(1 + \omega^2). \tag{3.10}$$

Curiously, the mass violates the 3D Breitenlohner-Freedman bound [74, 75] and is thus formally tachyonic in AdS_3 . In fact, as the Milne energy grows, the mass becomes more

tachyonic simply because we have mode expanded in a time-like direction. While such pathologies ordinarily imply an unbounded from below Hamiltonian, one should realize here that the AdS₃ theory is Euclidean and the true time direction actually lies outside the warped geometry.

Next, let us proceed to the case of 4D gauge theory. We consider the YM action,

$$S = -\frac{1}{2g_{\text{YM}}^2} \int_{\text{Milne}_4} d^4Y \sqrt{-G} \text{tr} (G^{IJ} G^{KL} F_{IK} F_{JL}), \quad (3.11)$$

again focusing on contributions from the Milne region. Here F_{IJ} is the Lie algebra-valued non-abelian gauge field strength. Under a Weyl transformation, the metric transforms according to eq. (3.2), while the gauge field is left invariant,

$$A_I \rightarrow A_I. \quad (3.12)$$

Due to the classical Weyl invariance of 4D YM theory, this transformation leaves the action unchanged, so

$$S = -\frac{1}{2g_{\text{YM}}^2} \int_{\text{AdS}_3} d^3y \sqrt{-\bar{G}} \int d\tau \text{tr} (\bar{G}^{IJ} \bar{G}^{KL} F_{IK} F_{JL}). \quad (3.13)$$

As before, the Weyl invariance of the action is a convenience whose main purpose is to simplify some of the algebra.

Decomposing the gauge field as $A_I = (A_\tau, A_i)$ and going to Milne temporal gauge, $A_\tau = 0$, we rewrite the linearized action as

$$S_0 = \frac{1}{g_{\text{YM}}^2} \int_{\text{AdS}_3} d^3y \sqrt{g} \int d\omega \text{tr} \left(-\frac{1}{2} g^{ij} g^{kl} f_{ik}(-\omega) f_{jl}(\omega) + \omega^2 \gamma^{ij} a_i(-\omega) a_j(\omega) \right), \quad (3.14)$$

where $f_{ij} = \partial_i a_j - \partial_j a_i$ is the linearized field strength associated with the Milne modes,

$$a_i(\omega) = \int d\tau e^{i\omega\tau} A_i(\tau). \quad (3.15)$$

From eq. (3.14) we see that the $a_i(\omega)$ are Proca vector fields in AdS₃ with mass

$$m_a^2(\omega) = -\omega^2. \quad (3.16)$$

The AdS₃ fields are formally tachyonic since we have mode expanded in the time-like Milne direction. In summary, we find that a massless vector in Milne₄ decomposes into a continuous tower of massive Proca vector fields in AdS₃.

3.2 Scaling dimensions from AdS₃/CFT₂

According to the standard holographic dictionary, each field in AdS₃ is dual to a CFT₂ primary operator with scaling dimension Δ dictated by the corresponding AdS₃ mass. From eq. (3.10) and eq. (3.16), we deduce that the scaling dimensions for scalar and vector primaries satisfy $\Delta_\phi(\Delta_\phi - 2) = m_\phi^2(\omega) = -(1+\omega)^2$ and $(\Delta_a - 1)^2 = m_a^2(\omega) = -\omega^2$. Both equations imply the following relationship between the scaling dimension and the Milne energy,

$$\Delta(\omega) = 1 \pm i\omega. \quad (3.17)$$

Since unitary CFTs and their Wick-rotated Euclidean versions have real scaling dimensions, the CFT encountered here is formally non-unitary. This is true despite the manifest unitarity of the underlying 4D dynamics.

3.3 Witten diagrams in AdS₃

With the mode decomposition just discussed, it is a tedious but straightforward exercise to derive an explicit action for the tower of AdS₃ modes descended from Milne₄. From this action we can then compute Witten diagrams in AdS₃. By the AdS₃/CFT₂ dictionary, these Witten diagrams are equivalent to correlators of a certain CFT₂. As we will argue here and in subsequent sections, these Witten diagrams and correlators are also equal to scattering amplitudes in Mink₄.

A priori, such a correspondence is quite natural. First of all, tree-level Witten diagrams and scattering amplitudes both describe a classical minimization problem — i.e. finding the saddle point of the action subject to a particular set of boundary conditions. Second, the CFT₂ resides on the ∂AdS_3 boundary, which at $\tau \rightarrow +\infty$ houses massless asymptotic in and out states.

In any case, we will derive an explicit mapping between the basic components of Witten diagrams and scattering amplitudes. The former are comprised of interaction vertices, bulk-bulk propagators, and bulk-boundary propagators, while the latter are comprised of interaction vertices, internal propagators, and a prescription for LSZ reduction. Let us analyze each of these elements in turn.

3.3.1 Interaction vertices

To compute the interaction vertices of the AdS₃ theory we simply express the interactions in Milne₄ in terms of the mode decomposition into massive AdS₃ fields. For example, the quartic self-interaction of the scalar field becomes

$$S_{\text{int}} = -\frac{\lambda}{24} \int_{\text{AdS}_3} d^3y \sqrt{g} \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \phi(\omega_1)\phi(\omega_2)\phi(\omega_3)\phi(\omega_4)\delta(\omega_1 + \omega_2 + \omega_3 + \omega_4), \tag{3.18}$$

so interactions in the bulk of Milne₄ translate into interactions among massive scalars in AdS₃. Due to the Weyl invariance of the original scalar theory, these interactions conserve Milne energy.

It is then clear that the interaction vertices of 3D Witten diagrams are equivalent to those of 4D flat space Feynman diagrams modulo a choice of coordinates — that is, Milne versus Minkowski coordinates, respectively. While these Witten diagram interactions typically involve complicated interactions among many AdS₃ fields, this is just a repackaging of standard Feynman vertices.

3.3.2 Bulk-bulk propagator

In this section we show that the bulk-bulk propagators of Milne harmonics in AdS₃ are simply a repackaging of Feynman propagators in Mink₄. To simplify our discussion, let us again revisit the case of the massless scalar field, although a parallel discussion holds for gauge theory but with the extra complication of gauge fixing.

Consider the Feynman propagator for a massless scalar field in flat space,

$$G(\tau, y, \tau', y')_{\text{Mink}_4} = \frac{i}{\square_{\text{Mink}_4}} = e^{-\tau'} \frac{i}{\square_{\text{AdS}_3} + 1 - \partial_\tau^2} e^{-\tau}, \tag{3.19}$$

where (τ, y) and (τ', y') are points in the Milne region. Here we have defined

$$\square_{\text{Mink}_4} = \nabla_I \nabla^I \quad \text{and} \quad \square_{\text{AdS}_3} = \nabla_i \nabla^i, \quad (3.20)$$

to be the d’Lambertian in Mink_4 and the Laplacian in AdS_3 , respectively. This expression is manifestly of the form of the AdS_3 propagator with $e^{-\tau}$ factors inserted to account for the non-trivial Weyl weight of the scalar field. Indeed, by applying the Weyl transformation and decomposing into Milne modes, we obtain the AdS_3 propagator for a scalar,

$$G(\omega, y, y')_{\text{AdS}_3} = \frac{i}{\square_{\text{AdS}_3} + 1 + \omega^2}, \quad (3.21)$$

which automatically satisfies the wave equation for a scalar in AdS_3 ,

$$(\nabla_i \nabla^i + 1 + \omega^2)G(\omega, y, y')_{\text{AdS}_3} = i\delta^3(y, y'). \quad (3.22)$$

Hence, the Feynman propagator is a particular convolution over a tower of AdS_3 propagators.

Of course, the above statements are purely formal until the differential operator inverses are properly defined by an $i\epsilon$ prescription. The Minkowski propagator takes the usual $i\epsilon$ prescription,

$$G(\tau, y, \tau', y')_{\text{Mink}_4} = \frac{i}{\square_{\text{Mink}_4} + i\epsilon}, \quad (3.23)$$

which selects the Minkowski vacuum as the ground state of the theory. This is, however, not the natural vacuum of the Weyl-transformed geometry, $\text{AdS}_3 \times \mathbb{R}_\tau$, which is instead the *conformal vacuum* corresponding to the ground state with respect to the Milne Hamiltonian, i.e. τ translations. In order to match the propagator of the Minkowski vacuum we must choose the *thermal propagator* in $\text{AdS}_3 \times \mathbb{R}_\tau$ [73]. Thermality arises from the entanglement between the Milne and Rindler regions of Minkowski spacetime across the Rindler horizon $x^2 = 0$. With this prescription, Feynman propagators in Mink_4 can be matched directly to bulk-bulk propagators in AdS_3 . Note that thermality does not break the $\text{SL}(2, \mathbb{C})$ Lorentz symmetries, since these act only on the AdS_3 coordinates and not the Milne time or energy.

A similar story holds for gauge fields. Going to Milne temporal gauge, the Mink_4 gauge propagator can be expressed as a convolution over massive AdS_3 Proca propagators. These propagators satisfy the Proca wave equation sourced by a delta function,

$$(\nabla_k \nabla^k \delta_i^j - \nabla_i \nabla^j + \omega^2 \delta_i^j)G_{jl}(\omega, y, y')_{\text{AdS}_3} = i\delta_{il}\delta^3(y, y'), \quad (3.24)$$

where we have Fourier transformed to Milne harmonics.

3.3.3 Bulk-boundary propagator

We have now verified that the bulk interaction vertices and bulk-bulk propagators of Witten diagrams in AdS_3 are simply Feynman diagrammatic elements in the Milne_4 embedding space. The final step in matching Witten diagrams to scattering amplitudes is to match their respective boundary conditions. For Witten diagrams, the external lines are AdS_3 bulk-boundary propagators. For scattering amplitudes, the external lines are fixed by LSZ

reduction to be solutions of the Mink₄ free particle equations of motion — taken usually to be plane waves. Here we derive a concrete relationship between the bulk-boundary propagators and LSZ reduction.

To begin, let us compute the bulk-boundary propagator for primary fields of scaling dimension Δ . At this point it will be convenient to employ the elegant embedding formalism of [60], which derived formulas for the bulk-boundary propagator in terms of a flat embedding space of one higher dimension. Ordinarily, AdS is considered physical while the flat embedding space is an abstraction devised to simplify the bookkeeping of curved spacetime. Here the scenario is completely reversed: flat space is physical while AdS is the abstraction introduced in order to recast flat space dynamics into the language of CFT.

In the embedding formalism [60], the bulk-boundary propagator for a scalar primary is

$$K^\Delta = \frac{1}{(kx)^\Delta}. \tag{3.25}$$

Since we have lifted from AdS₃ to Mink₄, the right-hand side actually depends on 4D quantities. Specifically, the four-vector x^μ labels a point in Mink₄ while the four-vector k^μ labels a point (z, \bar{z}) on the boundary of AdS₃ according to eq. (2.12).

Already, we see an elegant subtlety that arises in the embedding formalism: each point in AdS₃ is recast as a point in Mink₄ with the implicit constraint $x^2 = -1$. In Milne coordinates, this corresponds to the constraint $\tau = 0$. We can, however, “lift” the bulk-boundary propagators from AdS₃ to Mink₄ by simply dropping this constraint, yielding a bulk-boundary propagator with an additional τ dependent factor, $e^{-\tau\Delta}$. Combined with an extra factor of e^τ for the Weyl weight of a scalar field, this generates a net phase $e^{-\tau(\Delta-1)} = e^{\mp i\omega\tau}$ from the definition of Δ in eq. (3.17). We immediately recognize this as the phase factor that accompanies the Fourier transform between τ dependent fields in AdS₃ \times \mathbb{R}_τ and ω dependent Milne harmonics. That is, the lifted propagators can be used to compute the boundary correlators of modes in AdS₃ in terms of boundary correlators of 4D states in AdS₃ \times \mathbb{R}_τ . The fact that the bulk-boundary propagators satisfy the free particle equations of motion in AdS₃ translates to the fact that the Weyl-transformed lifted propagators satisfy the free particle equations of motion in AdS₃ \times \mathbb{R}_τ via separation of variables. In turn, this implies that the embedding formalism bulk-boundary propagator in eq. (3.25) satisfies the equations of motion in Mink₄. This fact is straightforwardly checked by direct computation.

Next, consider the bulk-boundary propagator for a vector primary, K_i^Δ . This object is fundamentally a bi-vector since it characterizes propagation of a vector disturbance from the ∂AdS_3 boundary into the bulk of AdS₃. While the 3D bulk vector index is manifest, the 2D boundary vector index is suppressed — implicitly taken here to be either the z or \bar{z} component. As for the scalar, we can lift the AdS₃ bulk-boundary propagator to $K_I^\Delta = (K_\tau^\Delta, K_i^\Delta)$ where we assume Milne temporal gauge to set $K_\tau^\Delta = 0$. Going to Minkowski coordinates, we obtain

$$K_\mu^\Delta = \frac{\partial y^I}{\partial x^\mu} K_I^\Delta = \frac{1}{(kx)^\Delta} \left(\epsilon_\mu - \frac{\epsilon x}{kx} k_\mu \right), \tag{3.26}$$

where we have chosen the z component of the boundary vector. Here the dependence on boundary coordinates (z, \bar{z}) enters through k and ϵ according to eq. (2.12) and eq. (2.13).

Had we instead chosen the \bar{z} component of the boundary vector, we would have obtained the same expression as eq. (3.26) except with $\bar{\epsilon}$ instead of ϵ .

3.4 Continuation from Milne₄ to Mink₄

Until now, the ingredients of our discussion — interaction vertices, bulk-bulk propagators, and bulk-boundary propagators — have all been restricted to Milne region time-like separated from the origin. However, it is clear that scattering processes in general will also involve the Rindler region space-like separated from the origin. As we will see, this is not a problem because the Milne diagrammatic components — written in terms of flat space coordinates via the embedding formalism — can be trivially continued to the Rindler region and thus all of Minkowski spacetime.

To be concrete, recall the foliation of the Rindler region in eq. (2.8) and eq. (2.9). Each slice of constant ρ defines a Lorentzian dS₃ spacetime. In Rind₄, boundary correlators correspond to Witten diagrams of dS₃ fields descended from a mode decomposition with respect to the Rindler *momentum*, $\omega = i\partial_\rho$. Moreover, the lifted bulk-boundary propagators in Rind₄ are given precisely by eq. (3.25) and eq. (3.26), except continued to the full Mink₄ region for any value of x^2 . So the embedding formalism gives a perfect prescription for continuation from Milne to Rindler. One can also think of this as a simple analytic continuation of the original AdS₃ theory into dS₃, which shares the same $SL(2, \mathbb{C})$ Lorentz isometries.

This result implies that Mink₄ scattering amplitudes — properly LSZ-reduced on bulk-boundary propagators on both the Milne and Rindler regions — are equal to a 3D Witten diagrams for Milne and Rindler harmonics which splice together boundary correlators in AdS₃ and dS₃. Using these continued Witten diagrams, we can then define a set of CFT₂ correlators dual to scattering amplitudes through a hybrid of the AdS₃/CFT₂ and dS₃/CFT₂ [52–55] correspondences. Note that the smooth match between correspondences, given the Euclidean signature of AdS₃ and the Lorentzian signature of dS₃.

As a consequence, our proposed correspondence between Mink₄ and CFT₂ is subtle. While the Minkowski theory is unitary, the CFT₂ is not unitary in any familiar sense — a fact which is evident from the appearance of complex scaling dimensions in eq. (3.17). This is not a contradiction, since unlike the usual AdS₃/CFT₂ correspondence, the time direction and unitary evolution are emergent, as in the spirit of dS₃/CFT₂. The question of how flat space unitarity is encoded within a non-unitary CFT obviously deserves further study.

3.5 Mink₄ scattering amplitudes as CFT₂ correlators

Assembling the various diagrammatic ingredients, we see that Witten diagrams for the (A)dS₃ fields descended from the mode decomposition of Mink₄ are equal to 4D scattering amplitudes — albeit with a modified prescription for LSZ reduction in which the usual external wavepackets of fixed momentum are replaced with the lifted bulk-boundary propagators of eq. (3.25) and eq. (3.26). These alternative “wavepackets” may seem unfamiliar, but crucially, they can be expressed as superpositions of on-shell plane waves.

For the scalar field this is straightforward, since the bulk-boundary propagator in eq. (3.25) can be expressed as a Mellin transform of plane waves [17],

$$K^\Delta = \frac{1}{(kx + i\varepsilon)^\Delta} = \frac{i^{-\Delta}}{\Gamma(\Delta)} \int_0^\infty ds s^{\Delta-1} e^{iskx} e^{-\varepsilon s}, \quad (3.27)$$

where ε is an infinitesimal regulator. Here the right-hand side is manifestly a superposition of on-shell plane waves, e^{iskx} , since $k^2 = 0$.

Something similar happens for the gauge field since

$$K_\mu^\Delta = \left(\epsilon_\mu + \frac{k_\mu \partial_z}{\Delta} \right) \frac{1}{(kx)^\Delta}. \quad (3.28)$$

Using the simple observation that $k_\mu \partial_z(\cdot) = \partial_z(k_\mu \cdot) - \epsilon_\mu(\cdot)$, we see that eq. (3.27) and eq. (3.28) imply that K_μ^Δ is a superposition of on-shell plane waves, $\epsilon_\mu e^{iskx}$, up to a superposition of pure gauge transformations, $k_\mu e^{iskx}$.

In this way, we have shown that every Witten diagram can be written as a superposition of on-shell scattering amplitudes in Mink_4 , or equivalently as a single scattering amplitude with a modified LSZ-reduction to certain bulk-boundary wavepackets. By the (A)dS/CFT dictionary, this implies that the latter are equivalent to Euclidean correlators of a CFT_2 on the $\partial(\text{A})\text{dS}_3$ boundaries, which together form the entirety of past and future null infinity. Concretely, this implies the equivalence of correlators and scattering amplitudes,

$$\langle \mathcal{O}^{\Delta_1}(z_1, \bar{z}_1) \cdots \mathcal{O}^{\Delta_n}(z_n, \bar{z}_n) \rangle = A(K^{\Delta_1}(z_1, \bar{z}_1), \dots, K^{\Delta_n}(z_n, \bar{z}_n)) = \langle \text{out} | \text{in} \rangle, \quad (3.29)$$

where here we have restricted to scalar operators for simplicity, but the obvious generalization to higher spin applies. In eq. (3.29) the quantity A denotes a scattering amplitude with a modified LSZ-reduction replacing the usual plane waves with the lifted bulk-boundary propagators $K^{\Delta_i}(z_i, \bar{z}_i)$ corresponding to the boundary operators $\mathcal{O}^{\Delta_i}(z_i, \bar{z}_i)$. The associated scaling dimension of each operator is $\Delta_i = 1 + i\omega_i$, and if the bulk theory is conformally invariant in 4D, for example as in massless gauge theory at tree level, then $\sum_{i=1}^n \omega_i = 0$. The boundary operators are naturally divided into two types, \mathcal{O}_{in} and \mathcal{O}_{out} , depending on sign of the Minkowski energy $E > 0$ or $E < 0$, corresponding to scattering states that are incoming or outgoing, respectively. This equivalence of correlators and scattering amplitudes is depicted in figure 2.

3.6 Conserved currents of CFT_2

In eq. (3.29), we derived an explicit holographic correspondence between scattering amplitudes in Mink_4 and correlators of a certain CFT_2 . For gauge fields, the associated massive AdS_3 modes are dual to *non-conserved* currents in the CFT_2 while the massless AdS_3 modes are dual to *conserved* currents in the CFT_2 . Since the mass of an AdS_3 vector is proportional to its Milne energy by eq. (3.16), we can study the massless case by taking the limit of vanishing Milne energy $\omega = 0$, i.e. the Milne soft limit. For the dual vector primary operator, this corresponds to $\Delta = 1$, so the correlator reduces to the Ward identity for current conservation in the CFT_2 .

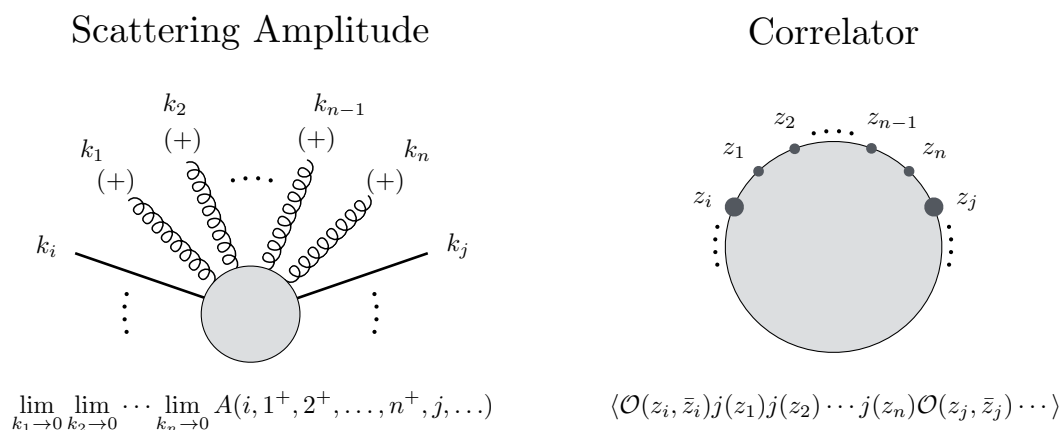


Figure 2. Equivalence of 4D scattering amplitudes and 2D correlators for the special case of multiple soft boson gauge emission and multiple conserved current insertion.

To start, consider the bulk-boundary propagator for a massless AdS_3 vector,

$$K_\mu = \frac{x^\rho f_{\rho\mu}}{(kx)^2}, \quad (3.30)$$

obtained by setting $\Delta = 1$ in eq. (3.26). Here we have defined linearized field strengths constructed from boundary data,

$$f_{\mu\nu} = k_\mu \epsilon_\nu - k_\nu \epsilon_\mu, \quad \text{and} \quad \bar{f}_{\mu\nu} = k_\mu \bar{\epsilon}_\nu - k_\nu \bar{\epsilon}_\mu. \quad (3.31)$$

Note that $x^\mu K_\mu = K_\tau = 0$ since we have chosen Milne temporal gauge. Remarkably, K_μ is actually a total derivative with respect to Mink_4 coordinates,

$$K_\mu = \partial_\mu \xi \quad \text{where} \quad \xi = \frac{\epsilon x}{kx}. \quad (3.32)$$

This fact dovetails beautifully with the results of [20, 21, 23, 24], which argued that there is physical significance to large gauge transformations that do not vanish at the boundary of Mink_4 . As we will see, concrete calculations are vastly simplified using the pure gauge form of K_μ .

3.6.1 Mink_4 soft theorems as CFT_2 Ward identities

Let us start with the simplest case of abelian gauge theory with arbitrary charged matter. We showed earlier that a Mink_4 scattering amplitude with a Milne soft gauge boson can be expressed as a Witten diagram for a massless AdS_3 vector field,

$$\langle j(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \int d^4x K_\mu(x) W^\mu(x). \quad (3.33)$$

Here the left-hand side is a correlator involving the $\Delta = 1$ conserved current of the CFT_2 and K_μ is the bulk-boundary propagator for the massless vector in AdS_3 . The function W^μ represents the remaining contributions to the Witten diagram from bulk interactions,

$$W^\mu(x) = \langle \text{out} | J^\mu(x) | \text{in} \rangle, \quad (3.34)$$

where J^μ is the gauge current operator of 4D Minkowski spacetime inserted between scattering states. Here the in and out states are defined according to the modified prescription for LSZ reduction shown in eq. (3.29).

Inserting the pure gauge form of K_μ in eq. (3.32) and integrating by parts, we obtain

$$\begin{aligned} \langle j(z)\mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle &= \int d^4x \partial_\mu \xi(x) \langle \text{out} | J^\mu(x) | \text{in} \rangle \\ &= - \int d^4x \xi(x) \partial_\mu \langle \text{out} | J^\mu(x) | \text{in} \rangle. \end{aligned} \quad (3.35)$$

By dropping total derivatives, we have implicitly assumed that W^μ describes a charge configuration that vanishes on the boundary. Naively, this stipulation is inconsistent if the bulk process involves charged external particles that propagate to the asymptotic boundary. However, this need not be a contradiction, provided W^μ is sourced by insertions of charged particles near but not quite on the boundary. Conservation of charge is effectively violated wherever the external particles are inserted, so

$$\partial_\mu \langle \text{out} | J^\mu(x) | \text{in} \rangle = - \sum_{i=1}^n q_i \delta^4(x - x_i) \langle \text{out} | \text{in} \rangle. \quad (3.36)$$

Here i runs over all the particles in the scattering process, q_i are their charges, and x_i are their insertion points near the ∂AdS_3 boundary. Crucially, we recall from eq. (2.11) that massless particles near the ∂AdS_3 boundary are located at positions x_i that are aligned with their associated on-shell momenta, k_i . This is simply the statement that the positions of asymptotic states on the celestial sphere point in the same directions as their momenta. In any case, the upshot is that as $\rho_i \rightarrow 0$, we can substitute $x_i \sim k_i$.

Plugging in eq. (3.32) and eq. (3.29), and replacing $x_i \sim k_i$, we can trivially integrate the delta function to obtain

$$\langle j(z)\mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \sum_{i=1}^n q_i \left(\frac{\epsilon k_i}{k k_i} \right) \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle, \quad (3.37)$$

which is exactly the Weinberg soft factor for soft gauge boson emission [33]. Here it was important that we identified $x_i \sim k_i$ so that the resulting Weinberg soft factor depends on the on-shell momenta, k_i . Later on, we will occasionally find it useful to switch back and forth between the position and momentum basis for the hard particles.

At the same time, this expression simplifies further because

$$\frac{\epsilon k_i}{k k_i} = \frac{1}{z - z_i}, \quad (3.38)$$

yielding the Ward identity for a 2D conserved current,

$$\langle j(z)\mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \sum_{i=1}^n \frac{q_i}{z - z_i} \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle. \quad (3.39)$$

So eq. (3.37) is simultaneously the soft theorem in Mink_4 , the Witten diagram for a massless vector in AdS_3 , and the Ward identity for a conserved current in the CFT_2 . From this

result we deduce that an insertion of the CFT conserved current is dual to a soft gauge boson emission.

The above analysis for abelian gauge theory is straightforwardly extended to the non-abelian case. The equation for approximate current conservation instead becomes

$$\partial_\mu \langle \text{out} | J^{a\mu}(x) | \text{in} \rangle = - \sum_{i=1}^n \delta^4(x - x_i) \langle \text{out} | T^a | \text{in} \rangle \quad (3.40)$$

so again plugging in $x_i \sim k_i$, we generalize eq. (3.39) to

$$\begin{aligned} \langle j(z)^a \mathcal{O}^{b_1}(z_1, \bar{z}_1) \cdots \mathcal{O}^{b_n}(z_n, \bar{z}_n) \rangle &= \sum_{i=1}^n f^{ab_i c_i} \left(\frac{\epsilon k_i}{k k_i} \right) \langle \mathcal{O}^{b_1}(z_1, \bar{z}_1) \cdots \mathcal{O}^{c_i}(z_i, \bar{z}_i) \cdots \mathcal{O}^{b_n}(z_n, \bar{z}_n) \rangle \\ &= \sum_{i=1}^n \frac{f^{ab_i c_i}}{z - z_i} \langle \mathcal{O}^{b_1}(z_1, \bar{z}_1) \cdots \mathcal{O}^{c_i}(z_i, \bar{z}_i) \cdots \mathcal{O}^{b_n}(z_n, \bar{z}_n) \rangle, \end{aligned} \quad (3.41)$$

which is the Mink_4 soft theorem and the CFT_2 Ward identity for non-abelian gauge theory.

The duality between soft gauge bosons and holomorphic currents has direct implications for scattering amplitudes. For example, consider the correlator for a sequence of holomorphic currents wedged between two operator insertions,

$$\langle \mathcal{O}^{a_i}(z_i, \bar{z}_i) j(z_1)^{a_1} \cdots j(z_n)^{a_n} \mathcal{O}^{a_j}(z_j, \bar{z}_j) \rangle. \quad (3.42)$$

Current conservation requires that this object be purely a holomorphic in the variables z_i . However, this expression can also be computed by sequential soft limits of an amplitude with two hard particles, yielding

$$\frac{1}{(z_i - z_1)(z_1 - z_2) \cdots (z_{n-1} - z_n)(z_n - z_j)}, \quad (3.43)$$

which is the color-stripped amplitude for multiple soft emission. To obtain this formula for the multiple leading soft limit it was important that sequential soft limits of single helicity gauge bosons commute when applied to color-stripped amplitudes. The resemblance of eq. (3.43) to the denominator of the Park-Taylor formula is not an accident: this form is required so that the only poles of the amplitude are collinear singularities.

3.6.2 Equivalence of Milne_4 and Mink_4 soft limits

We have shown that the Ward identities of for 2D conserved currents are the same as the Weinberg soft theorems for 4D gauge theory [33]. However, an astute reader will realize that the Weinberg soft theorems correspond to the limit of small Minkowski energy, $E = i\partial_0$ while our construction has centered on the Milne energy, $\omega = i\partial_\tau$ since it is an $\text{SL}(2, \mathbb{C})$ Lorentz invariant quantity. Naively this is discrepant, but as we will now show, the Milne and Minkowski soft limits, $E \rightarrow 0$ and $\omega \rightarrow 0$, are one and the same.

To see why, we compute a correlator for a non-conserved current $j^\Delta(z)$ and take the limit towards $\Delta \rightarrow 1$ or equivalently, the Milne soft limit $\omega \rightarrow 0$. The correlator to start is

$$\langle j^\Delta(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \int d^4x K_\mu^\Delta(x) W^\mu(x). \quad (3.44)$$

Here W_μ is defined as in eq. (3.34) and for K_μ^Δ we plug in eq. (3.27) and eq. (3.28) to obtain

$$\langle j^\Delta(z)\mathcal{O}(z_1, \bar{z}_1)\cdots\mathcal{O}(z_n, \bar{z}_n)\rangle = \frac{i^{-\Delta}}{\Gamma(\Delta)} \left(\epsilon_\mu + \frac{k_\mu \partial_z}{\Delta} \right) \int_0^\infty ds s^{\Delta-1} \langle \text{out} | \tilde{J}^\mu(sk) | \text{in} \rangle, \quad (3.45)$$

where \tilde{J}^μ is the Fourier transform of J^μ . At this point we recognize \tilde{J}^μ as a Feynman diagram with an injection of momentum sk . Notice that the integration variable s has taken the role of the Minkowski energy of the inserted momentum. The 4D Ward identity for on-shell gauge theory amplitudes is

$$\langle \text{out} | k_\mu \tilde{J}^\mu(sk) | \text{in} \rangle = 0, \quad (3.46)$$

whenever \tilde{J}^μ is evaluated at on-shell kinematics. Again using $k_\mu \partial_z(\cdot) = \partial_z(k_\mu \cdot) - \epsilon_\mu(\cdot)$, we are then permitted to reshuffle derivatives in eq. (3.45), where the first term on the right-hand side of this substitution vanishes by the Ward identity. Doing so, we arrive at our final expression for the correlator,

$$\langle j^\Delta(z)\mathcal{O}(z_1, \bar{z}_1)\cdots\mathcal{O}(z_n, \bar{z}_n)\rangle = \frac{i^{-\Delta}(\Delta-1)}{\Gamma(\Delta+1)} \int_0^\infty ds s^{\Delta-1} \langle \text{out} | \epsilon_\mu \tilde{J}^\mu(sk) | \text{in} \rangle. \quad (3.47)$$

Since \tilde{J}^μ is evaluated at the on-shell momentum sk and dotted into the on-shell polarization ϵ , we again verify that the correlator is a superposition of on-shell scattering amplitudes.

Returning to eq. (3.47), we take the $\Delta \rightarrow 1$ limit that corresponds to the Milne soft limit $\omega \rightarrow 0$ that defines a massless vector in AdS_3 . However, this limit requires care because the integral over s is dominated near $s = 0$ from infrared divergence in the amplitude. In particular, the Weinberg soft theorem says that

$$\langle \text{out} | \epsilon_\mu \tilde{J}^\mu(sk) | \text{in} \rangle \stackrel{s \rightarrow 0}{\simeq} \frac{1}{s} \sum_{i=1}^n q_i \left(\frac{\epsilon k_i}{k k_i} \right) \langle \text{out} | \text{in} \rangle + \text{regular in } s. \quad (3.48)$$

However, this $1/s$ singularity is regulated by oscillatory contributions coming from the $s^{\Delta-1}$ factor in the integrand, so

$$\int_0^\infty \frac{ds}{s} s^{i\omega}(\cdot) = -\frac{i}{\omega}(\cdot) + \text{regular in } \omega. \quad (3.49)$$

The singularity in ω is cancelled by the prefactor in eq. (3.47), which is proportional to ω in this limit. Combining all terms, we then find that eq. (3.47) simplifies to the Weinberg soft factor in eq. (3.39), just as advertised. Hence, we learn that the Milne soft limit $\omega \rightarrow 0$ and the Minkowski soft limit $E \rightarrow 0$ coincide, both generating the Weinberg soft theorem.

3.7 Kac-Moody algebra of CFT_2

The existence of a holomorphic conserved current $j(z)$ signals an infinite-dimensional symmetry algebra encoded in the CFT_2 [19]. Since $\partial_{\bar{z}} j(z) = 0$, we can Laurent expand the holomorphic current in the usual fashion,

$$j(z) = \sum_{m=-\infty}^{\infty} \frac{j_m}{z^{m+1}}, \quad (3.50)$$

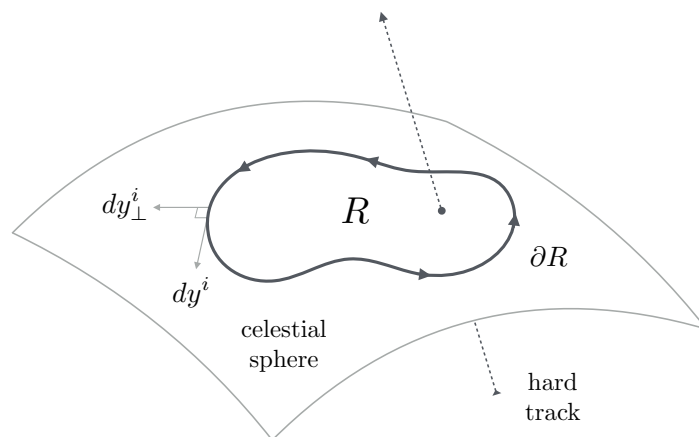


Figure 3. The celestial sphere houses a region R whose boundary ∂R encircles the trajectory of a hard particle. The single helicity Aharonov-Bohm phase around ∂R is simultaneously *i)* the cumulative charge of hard tracks threading R , *ii)* the integrated velocity kick experienced by test charges along ∂R , i.e. the electromagnetic memory effect, and *iii)* the Ward identity for the holomorphic conserved current of the 2D CFT. Here dy^i is the infinitesimal vector tangent to ∂R while dy^i_\perp is the infinitesimal vector orthogonal to ∂R but still on the celestial sphere.

yielding the infinitely many charges j_m of an abelian Kac-Moody algebra. Furthermore, a generalized “soft charge” can be defined with respect to a contour ∂R in the z coordinate bounding a 2D “patch” R on the celestial sphere. Such a patch is depicted in figure 3. We can associate to this patch an arbitrary holomorphic function $\lambda(z)$ to define the soft charge,

$$j_{R,\lambda} = \oint_{\partial R} dz \lambda(z) j(z). \tag{3.51}$$

By the Ward identity for the 2D conserved current in eq. (3.39) and Cauchy’s theorem, this quantity counts number of charged particles in the scattering amplitude threading the region R ,

$$\langle j_{R,\lambda} \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \sum_{i \in R} q_i \lambda(z_i) \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle. \tag{3.52}$$

This is an angle dependent charge conservation equation, where the left-hand side is the correlator of the “soft charge” and the right-hand side consists of the sum over hard particle charges within some angular acceptance.

Since $j(z)$ is a holomorphic current, $\partial_{\bar{z}}$ acting on its correlators should vanish everywhere except at the insertion points of operators. This is verified by applying $\partial_{\bar{z}}$ to eq. (3.39), yielding

$$\partial_{\bar{z}} \langle j(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = 2\pi \sum_{i=1}^n q_i \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle, \tag{3.53}$$

where we have used the identity from complex analysis,

$$\partial_{\bar{z}} \left(\frac{1}{z} \right) = 2\pi \delta^2(z, \bar{z}). \tag{3.54}$$

According to eq. (3.53), global charge conservation then requires that $\sum_{i=1}^n q_i = 0$, so the sum of all charges is zero.

3.8 Chern-Simons theory and multiple soft emission

In the previous sections we verified that soft gauge bosons in Mink_4 correspond to massless vectors in AdS_3 dual to conserved currents in a CFT_2 . At the same time, we noted that the associated bulk-boundary propagators are pure gauge, suggesting an underlying AdS_3 theory with no propagating degrees of freedom. As this is the calling card of a topological gauge theory, CS theory is the natural candidate to describe the massless vectors of AdS_3 . In this section we argue that this is precisely the case. We stress that the purely topological character is restricted to just the soft gauge sector of the 4D theory, dual to the 2D current algebra of the CFT_2 . More generally, the KK reduced AdS_3 description is a CS gauge theory *coupled to non-topological matter*. These degrees of freedom correspond to all 4D fields that carry finite Milne energy ω .

3.8.1 Abelian Chern-Simons theory

To begin, let us revisit the lifted bulk-boundary propagator K_μ as a solution to the classical field equations for a gauge field. Since the bulk-boundary propagator is pure gauge, its associated field strength vanishes everywhere, including on any AdS_3 slice,

$$\partial_i K_j - \partial_j K_i = 0. \tag{3.55}$$

Rather trivially, this coincides with the equation of motion for an abelian CS gauge field A_i , whose field strength satisfies

$$F_{ij} = 0, \tag{3.56}$$

indicating the absence of propagating degrees of freedom expected in a topological theory. Hence, far from sources, the bulk-boundary propagator K_i is a solution to the equations of motion for a CS gauge field A_i , whose action is

$$S_{\text{CS}} = \int_{\text{AdS}_3} d^3y A_i F_{jk} \varepsilon^{ijk}. \tag{3.57}$$

Since the CS theory is topological, the bulk spacetime, AdS_3 , is not so important, but the boundary, ∂AdS_3 , is crucial. In fact, we must fix specific boundary conditions for the CS gauge theory. Because the CS theory has a first order equation of motion, we can either specify A_z on ∂AdS_3 or $A_{\bar{z}}$ on ∂AdS_3 , but not both [64]. As we will soon see, these choices correspond to the soft (+) or (−) helicity sectors of the 4D gauge theory, respectively.

It is instructive to see how this CS theory arises from the Milne soft limit, starting from the regime of finite Milne energy $\omega \neq 0$, where the scaling dimension is $\Delta = 1 \pm i\omega$ according to eq. (3.17). As before, we interpret the lifted bulk-boundary propagator as a classical gauge field solution, $A_\mu = K_\mu^\Delta$. It is easily checked that the associated field strength $F_{\mu\nu} \neq 0$ for $\Delta \neq 1$ and therefore is not pure gauge. However, the field strength satisfies the self-dual equation,

$$F_{\mu\nu} = i\tilde{F}_{\mu\nu}, \tag{3.58}$$

where the Hodge dual field is

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \quad (3.59)$$

We now recall that the self-dual condition in eq. (3.58) simply indicates that the electric and magnetic fields are phase shifted, consistent with a polarized electromagnetic wave. Thus the self-dual condition restricts to the gauge field to the (+) helicity sector. Had we began instead with the complex conjugate bulk-boundary propagator, we would have obtained the anti-self-dual condition that defines the (−) helicity sector.

Note that for real gauge fields, self-duality is of course only possible in Euclidean signature. However, we are in Lorentzian signature, so the self-dual condition implicitly entails a formal complexification of the gauge fields.

In Milne coordinates, the self-dual condition becomes

$$F_{ij} = \frac{1}{2}i\epsilon_{ijk}\partial_\tau A_k, \quad (3.60)$$

where we have dropped a term using the temporal Milne gauge condition $A_\tau = 0$. Fourier transforming to Milne harmonics, we see that the right-hand side is proportional to the Milne energy, $i\partial_\tau = \omega$. Eq. (3.60) is then none other than the Proca-CS equation of motion for a gauge field of mass $i\omega$. To revert to the case of a $\Delta = 1$ conserved current, we take the corresponding limit of vanishing Milne energy $\omega = 0$, in which case the right-hand side vanishes, reproducing our expression from eq. (3.56).

From the above analysis we conclude that the Witten diagrams corresponding to correlators of conserved CFT₂ currents $j(z)$ and $\bar{j}(\bar{z})$ are computed with AdS₃ CS gauge fields describing soft gauge bosons of a single helicity in Mink₄. Importantly, our discussion thus far has centered on the abelian field equations, which automatically linearize so as to factorize the (+) and (−) helicity sectors. In these theories the (+) and (−) helicity gauge bosons do not couple directly, so the corresponding CFT₂ has both a conserved holomorphic current $j(z)$ and a conserved anti-holomorphic current $\bar{j}(\bar{z})$.

Up until now we have focused solely on soft sector of the gauge theory, neglecting all hard quanta that appear in the form of hard charged matter or hard gauge bosons. However, this relates to a possible point of confusion, which is that the self-dual solutions just described are only solutions of the *source free* equations of motion. Naively, in the presence of sources, this self-duality will be spoiled. This is, however, not actually a problem once we remember that the bulk-boundary propagators are by definition solutions to the source free, homogeneous equations of motion. This is obvious because $K_{\mu\nu}$ is simply a function of its end points and not any particular property of a current. We can see this diagrammatically in figure 4, which shows how the bulk-boundary propagator undergoes self-dual, free propagation before making contact with a hard source.

Indeed, from this picture it is straightforward to see how the CS field interacts with hard sources. Recall the Witten diagram corresponding to soft gauge boson emission,

$$\int d^4x K_\mu(x)W^\mu(x) = \int d^3y \sqrt{g} K_i(y) \int d\tau W^i(\tau, y), \quad (3.61)$$

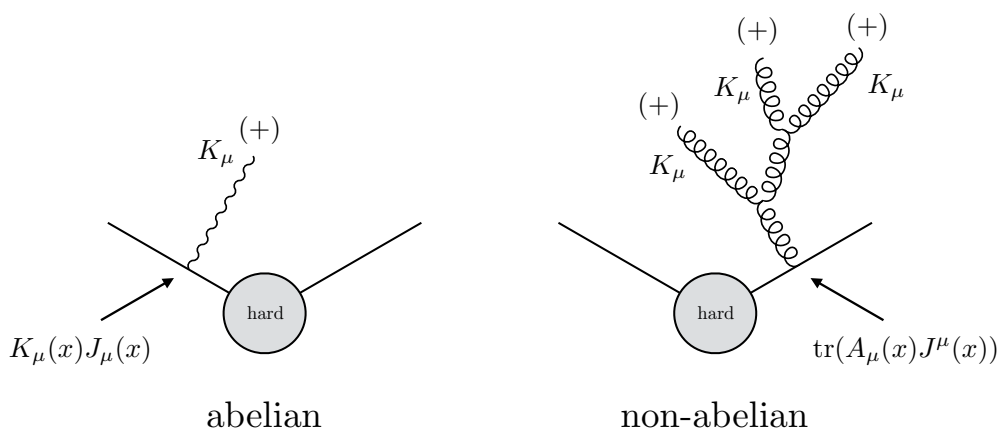


Figure 4. Single emission of an abelian gauge boson and multiple emission of non-abelian gauge bosons. In both cases, external legs connect to bulk-boundary propagators K_μ . In the non-abelian case, these soft emissions accumulate into a soft branch described by the field A_μ .

where in eq. (3.34), W^i denotes remainder of the Witten diagram,

$$W^i(\tau, y) = \langle \text{out} | J^i(\tau, y) | \text{in} \rangle, \tag{3.62}$$

computed from the matrix elements of current J^i . On the right-hand side we have used the fact that in Milne coordinates, the bulk-boundary propagator is τ independent since it corresponds to a Milne zero mode. Hence, we see that if K_i is to be interpreted as a classical configuration of a CS gauge field A_i in AdS_3 , then it couples to a Milne time-integrated version of W^μ , given by

$$W_{\text{eff}}^i(y) = \int d\tau W^i(\tau, y) = \langle \text{out} | J_{\text{eff}}^i(y) | \text{in} \rangle, \tag{3.63}$$

where we have defined a Milne time-integrated current,

$$J_{\text{eff}}^i(y) = \int d\tau J^i(\tau, y). \tag{3.64}$$

If we think of the full current J^μ as physically representing an array of hard particle world lines and interactions in Mink_4 , then the Milne time-integrated current J_{eff}^i is a static record of the hard “tracks” defined by these trajectories throughout all of time. It is to these hard tracks in AdS_3 to which the CS gauge field A_i couples. See figure 4 for a schematic depicting the absorption of an abelian soft gauge boson by a hard track.

Altogether, we see that this describes a purely AdS_3 description derived from a “KK reduction” of hard particles in Mink_4 into massive AdS_3 fields coupled covariantly to the CS gauge field, A_i . The corresponding action is then

$$S_{\text{CS}} = \int_{\text{AdS}_3} d^3y A_i F_{jk} \varepsilon^{ijk} + \int_{\text{AdS}_3} d^3y \sqrt{g} A_i J_{\text{eff}}^i. \tag{3.65}$$

Massive AdS_3 fields contribute to 3D Witten diagrams which are equivalent to the matrix elements of the 4D current J^μ projected down to the zero mode J_{eff}^i in order to couple to

the CS gauge field describing the Milne soft mode. So the couplings of the bulk-boundary propagator match to a CS action given by eq. (3.65).

3.8.2 Non-Abelian Chern-Simons theory

For non-abelian gauge theories the story is more complicated because (+) and (−) helicity gauge bosons interact directly and non-linearly. Nevertheless, a similar story applies. To understand why, consider tree-level non-abelian gauge theory, subject to a restriction of the external states to be a single helicity, say (+). Let us not even take the soft limit — instead, consider both soft and hard (+) particles for the purpose of this discussion.

By definition, the bulk-boundary propagators K_μ^Δ satisfy the self-dual condition in eq. (3.58). The non-abelian subtlety arises because multiple external soft gauge bosons will in general interact and merge into soft “branches” which then attach to the hard bulk current, as depicted in figure 4. Mathematically, each soft branch can be described by a non-abelian gauge field $A_\mu(x)$, defined from the corresponding Feynman diagram for that particular tree of soft gauge bosons. So explicitly, $A_\mu(x)$ is some integral over products of soft interaction vertices and bulk-boundary and bulk-bulk propagators. Here x is the bulk point at which the soft branch connects to the hard diagram, again as indicated in figure 4. So by definition, $A_\mu(x)$ is comprised solely of soft elements.

In this way we see that the soft branch field A_μ is just the perturbative expansion of a classical solution to the non-abelian YM equations of motion, where the free limit reverts to a superposition of K_μ^Δ bulk-boundary propagators for the external lines. Since all external lines are taken to be (+) helicity, this linear superposition is self-dual. In turn, this implies that the full non-linear soft branch A_μ is also non-linearly self-dual, since self-dual configurations continue to be self-dual upon non-linear classical evolution. This follows since the self-dual equations are first order and thus guarantee satisfaction of the second order YM field equations. As before, one can naively worry about violation of the self-dual condition by sources. However, there is again no obstruction because the soft branch field is a solution to the source free non-linear equations of motion, independent of the hard source. A schematic of our physical picture is shown in figure 5.

We thereby conclude that the soft branch field A_μ satisfies the 4D non-abelian self-dual equations, given by eq. (3.58) where A_μ is the matrix valued gauge field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$ is the full non-linear field strength. Moreover, in Milne temporal gauge, the non-abelian CS equations of motion are given by eq. (3.60) where the left-hand side contains the full non-linear field strengths. By momentum conservation, if the external legs of the soft branch are Milne soft, then so too is A_μ , so the right-hand side of eq. (3.60) is zero. Thus, we verify that the soft branch field satisfies the non-abelian CS equation of motion.

Finally, let us discuss the interactions of the non-abelian CS fields with the hard process. The analysis is same as for the abelian case, except the hard process couples to non-abelian soft branches rooted in a multiplicity of soft external gauge bosons, rather than a single abelian bulk-boundary propagator. In particular, the Witten diagram for

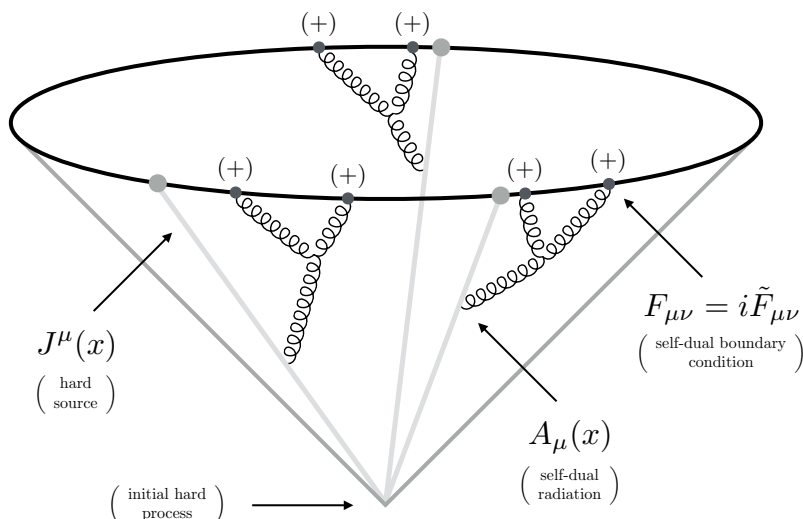


Figure 5. Schematic depicting soft, single helicity non-abelian gauge bosons coupling to hard sources. Each soft branch is initiated by a set of (+) helicity soft gauge bosons, so the corresponding field configuration is self-dual.

multiple soft emission is

$$\int d^4x \operatorname{tr}(A_\mu(x)W^\mu(x)) = \int d^3y \sqrt{g} \operatorname{tr}(A_i(y)W_{\text{eff}}^i(y)), \quad (3.66)$$

where A_μ is the soft branch field and W_μ again characterizes the hard current. Here we have defined a Milne time-integrated current W_{eff}^i as in eq. (3.63), only for a matrix valued current. Reminiscent of KK reduction, we see that the hard particles in Minkowski space couple to the soft branch field only through a zero mode projection of the hard current.

Finally, the Witten diagram associated with a non-abelian gauge field is also eq. (3.61), only with matrix valued gauge fields and a color trace. In turn, this implies that multiple soft gauge boson emissions are dictated by a non-abelian CS action,

$$S_{\text{CS}} = \int_{\text{AdS}_3} d^3y \operatorname{tr} \left(A_i F_{jk} + \frac{2}{3} A_i A_j A_k \right) \varepsilon^{ijk} + \int_{\text{AdS}_3} d^3y \sqrt{g} \operatorname{tr} (A_i J_{\text{eff}}^i), \quad (3.67)$$

where as before, we have defined J_{eff}^i to be the Milne time-integrated “tracks” of the hard particles in the scattering process.

As in the abelian case, the first order nature of the CS theory requires that we specify a boundary condition for A_z or $A_{\bar{z}}$ but not both. We see that this corresponds to keeping a single helicity in the soft limit. This explains the proposal of [22] to restrict to single helicity soft limits because of the non-commutation of opposite helicity soft limits in non-abelian gauge theory. This contrasts with abelian gauge theory, where both helicities can be described simultaneously because they do not interact with each other directly.

3.8.3 Locating Chern-Simons theory in Mink_4

In the previous sections we constructed abelian and non-abelian CS theories characterizing multiple emissions of soft, single helicity gauge bosons. The CS gauge fields interact with a

Milne time-integrated current describing the tracks of hard particles. While the underlying 3D spacetime is the AdS₃ obtained by dimensional reduction, it will be illuminating to understand where the CS gauge field is precisely “located” in 4D spacetime. To see this we now consider a slightly different but more intuitive derivation.

For simplicity, consider the case of abelian gauge theory, where we solve the field equations in the presence of a current. This differs from our earlier approach, where soft branches were described by a gauge field satisfying the source free equations of motion. Here we instead start with the current source and then compute the resulting gauge field configuration, in line with the usual approach taken in classical electrodynamics.

In Milne temporal gauge and working in ω frequency space, the gauge field generated by a particular current is

$$A_i(\omega, y) = \int d^3y' \sqrt{g} G_{ij}(\omega, y, y') J^j(\omega, y'), \quad (3.68)$$

where the right-hand side is the current convolved with a Proca propagator satisfying eq. (3.24) for a vector in AdS₃ of “mass squared” equal to $-\omega^2$.

The key observation is that the Proca wave equation for a vector in AdS₃ factorizes [76] into

$$\nabla_k \nabla^k \delta_i^j - \nabla_i \nabla^j + \omega^2 \delta_i^j = \Pi_i^{+k} \Pi_k^{-j} = \Pi_i^{-k} \Pi_k^{+j}, \quad (3.69)$$

where each projection operator is

$$\Pi_i^{\pm j} = \omega \delta_i^j \pm \epsilon_i^{jk} \nabla_k. \quad (3.70)$$

From the form of eq. (3.69), it is clear that any functions that are annihilated by $\Pi_i^{\pm j}$ will also be annihilated by the wave equation. Eq. (3.69) then implies that the Proca propagator is

$$G_{ij} = \frac{G_{ij}^+ + G_{ij}^-}{2\omega}, \quad (3.71)$$

where G_{ij}^{\pm} separately satisfy the first order wave equations,

$$\Pi_i^{\pm j} G_{jk}^{\pm} = i \delta_{ik} \delta^3(y, y'). \quad (3.72)$$

These are nothing more than the equations of motion for a pair of Proca-CS fields of “mass” $\pm i\omega$, so G_{ij}^{\pm} denote the corresponding Proca-CS propagators. We thus find that an off-shell Proca vector in AdS₃ can be described by a pair of off-shell Proca-CS fields.

Now let us return to eq. (3.68), taking the matrix element of this equation between 4D in and out states. Using eq. (3.62), this sends the current J^i to the quantity W^i , which characterizes hard particles sources. Eq. (3.68) then becomes

$$\langle \text{out} | A_i(\omega, y) | \text{in} \rangle = \int d^3y' \sqrt{g} G_{ij}(\omega, y, y') W^j(\omega, y'). \quad (3.73)$$

Because the bulk-bulk propagator splits into halves as discussed before, so too does the resulting gauge field A_i ,

$$\langle \text{out} | A_i^{\pm}(\omega, y) | \text{in} \rangle = \frac{1}{2\omega} \int d^3y' \sqrt{g} G_{ij}^{\pm}(\omega, y, y') W^j(\omega, y'). \quad (3.74)$$

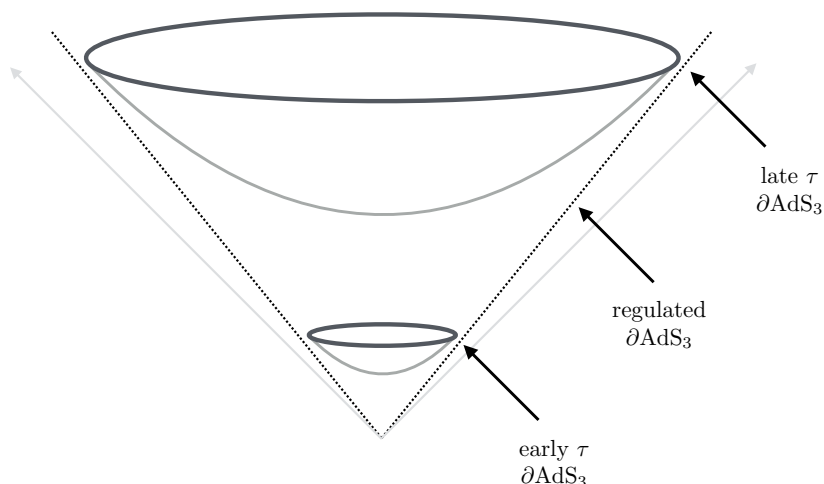


Figure 6. Depiction of the regulated boundary ∂AdS_3 . At late Milne times $\tau \rightarrow +\infty$, the boundary of the corresponding AdS_3 slice approaches null infinity. Meanwhile, at early Milne times $\tau \rightarrow -\infty$, this boundary approaches the origin.

For values of y in the bulk of AdS_3 , the physical significance of these off-shell Proca-CS fields is not completely transparent. However, as y approaches the boundary of AdS_3 , the gauge field A_i becomes radiation-dominated and A_i^\pm should be interpreted as the two helicities of on-shell electromagnetic radiation. Since the bulk-boundary propagator is just the boundary limit of the bulk-bulk propagator, we see that we have just been computing the same Witten diagram for gauge boson emission discussed in our earlier derivation of the CS structure. In any case, away from the boundary, the Proca-CS field of eq. (3.74) can be understood as an off-shell extension of the helicity decomposition for general y .

To explicitly construct the CS gauge field we simply take the Milne soft limit of the Proca-CS field. However, from eq. (3.74) it is clear that this limit only exists if we first multiply by ω . Thus, the CS gauge fields must correspond to the modified limit,

$$-i \lim_{\omega \rightarrow 0} \omega A_i^\pm(\omega, y) = \int d\tau \partial_\tau A_i^\pm(\tau, y) = A_i^\pm(\tau \rightarrow +\infty, y) - A_i^\pm(\tau \rightarrow -\infty, y). \quad (3.75)$$

We can better understand this result by “regulating” the boundary of AdS_3 in the standard way used in the context of AdS/CFT in global coordinates. In terms of the 4D embedding, this prescription corresponds to an infinitesimal “narrowing” of the lightcone bounding the Milne wedge, as depicted in figure 6. We then see that within the regulated Milne region, $\tau \rightarrow -\infty$ is the Minkowski origin, so the soft field is trivial there.

We thereby conclude that from the 4D perspective, the CS gauge field is the *single helicity and Milne late* soft field at null infinity. At the same time, the charged matter of the CS theory is captured by J_{eff}^i , which characterizes the hard-particle tracks in the bulk of Minkowski, cumulative over all time. We will see later how this relates to the phenomenon of electromagnetic memories.

3.9 Wess-Zumino-Witten model and multiple soft emission

We can now invoke the established AdS₃/CFT₂ dictionary to relate our CS theory in AdS₃ to the chiral half of a rational CFT₂ known as the WZW model [63–66]. The WZW model is simply a 2D non-linear sigma model defined by the action,

$$S_{\text{WZW}} = \int_{\partial\text{AdS}_3} d^2z \text{tr}(\partial_{\bar{z}}U^{-1}\partial_zU) + \frac{1}{3} \int_{\text{AdS}_3} d^3y \text{tr}(\partial_iUU^{-1}\partial_jUU^{-1}\partial_kUU^{-1})\varepsilon^{ijk}, \quad (3.76)$$

where the second term is topological. As is well-known [19], the WZW model enjoys both a holomorphic current $j^a(z)$ and an anti-holomorphic current $\bar{j}^a(\bar{z})$. Laurent expanding $j^a(z)$ gives

$$j^a(z) = \sum_{m=-\infty}^{\infty} \frac{j_m^a}{z^{m+1}}, \quad (3.77)$$

where j_m^a are the generators of an infinite-dimensional non-abelian Kac-Moody algebra. Notably, CS theory is equivalent to just the holomorphic sector of the WZW model [63–66], matching with the fact that it only describes a single helicity of the 4D gauge theory.

The operator product expansion for a conserved current in the WZW model is

$$j^a(z)\mathcal{O}^b(z') \sim \frac{f^{abc}\mathcal{O}^c(z')}{z-z'}, \quad (3.78)$$

which from eq. (3.41) is plainly equivalent to the Weinberg soft factor for a non-abelian gauge theory. As z denotes stereographic coordinates on the celestial sphere, we see that the operator product expansion is dual to an expansion in the collinear singularities of scattering amplitudes.

Let us comment on an innocuous but perhaps important fact about the WZW model, which is that the stress tensor is directly related to the holomorphic current via the Sugawara construction [19],

$$t(z) \sim \sum_a j^a(z)j^a(z). \quad (3.79)$$

The existence of a stress tensor for the current algebra alone reinforces the fact that the soft sector described by the WZW model is itself a consistent sub-CFT₂ within the full CFT₂ describing both soft and hard particles. In the sub-CFT₂, the hard particles are only visible as soft color sources represented by Wilson lines along the hard tracks. However, the dynamics of the hard particles themselves require additional structure which will add additional contributions to the stress tensor beyond the Sugawara construction.

At the same time, the Sugawara construction implies a connection between the 2D stress tensor and double collinear gauge boson emission. This structure is highly suggestive given the known link between the self-dual sectors of gauge theory and gravity [77] which manifests the so-called BCJ double copy [78].

3.10 Relation to memory effects

We have argued that 4D scattering amplitudes for soft gauge boson emission are described by 3D CS gauge theory with matter. To the soft sector, the hard particles appear as Wilson

line color sources along their tracks. These Wilson lines, together with insertions of the 2D conserved current operators formulate a classic CS calculation that reproduces the known 4D soft factors.

A corollary of this CS structure is an intrinsic topological character to 4D soft emission. Consistent with this picture, we will see how the soft sector elegantly exhibits the physics of the abelian and non-abelian AB effects — the hallmark of CS physics [47, 48, 63]. For simplicity, we focus for now on the abelian case, identifying these AB effects and relating them to the previously identified phenomenon of electromagnetic memories [38–40].

3.10.1 Chern-Simons memory and the Aharonov-Bohm effect

To begin, let us consider the contour-integrated form of the CFT₂ Ward identity for the holomorphic current derived in eq. (3.39),

$$\oint_{\partial R} dz \langle j(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \sum_{i \in R} q_i \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle, \quad (3.80)$$

where again R is a 2D patch on the celestial sphere near the boundary of AdS₃. This region is depicted in figure 3. From our earlier discussion, we saw that the current algebra of the CFT₂ is dual to a CS theory describing soft, single helicity gauge bosons in Mink₄ at late Milne time $\tau \rightarrow +\infty$. By the standard AdS₃/CFT₂ grammar applied to CS,

$$j(z) \sim A_z(\tau \rightarrow +\infty, \rho \rightarrow 0, z), \quad (3.81)$$

where \sim denotes the holographic duality and we implicitly take the limit $\tau \rightarrow +\infty$ before $\rho \rightarrow 0$. Here A_z is simply z component of the soft gauge field on ∂AdS_3 at late Milne times. Plugging back into eq. (3.80) we obtain

$$\oint_{\partial R} dz A_z(\tau \rightarrow +\infty, \rho \rightarrow 0, z) = \sum_{i \in R} q_i, \quad (3.82)$$

which is implicitly evaluated inside a correlator with additional hard operators, as in eq. (3.80). To avoid unnecessary notational clutter, this will also be true of the rest of the expressions in this section.

The above result has the form of a 3D AB phase for the CS gauge field at $\tau \rightarrow +\infty$. From our earlier analysis, we saw that the equation of motion for the CS gauge field in the presence of hard sources is

$$F_{ij} = \epsilon_{ijk} J_{\text{eff}}^k, \quad (3.83)$$

where $J_{\text{eff}}^i = \int d\tau J^i$ is the Milne time-integrated current. Integrating this equation over the region R , we find that

$$\int_R F = \int_R *J_{\text{eff}} = \int d\tau \int_R *J = \sum_{i \in R} q_i. \quad (3.84)$$

Therefore, the AB phase around the loop ∂R is equal to the the field strength flux through R , which is in turn equal to the Milne time-integrated charge flux through R . In this way

the hard particles in the scattering process will induce AB phases in the CS gauge field characterizing soft, single helicity emissions.

It is important to realize that the AB phase under discussion is *not* literally the standard 4D AB effect, but rather a 3D “chiral” version restricted to single helicity radiation. In particular, the complex contour integral performed in eq. (3.82) only picks out the (+) helicity component,

$$\oint_{\partial R} dz A_z(\tau \rightarrow +\infty, y) = \oint_{\partial R} dy^i A_i^+(\tau \rightarrow +\infty, y). \quad (3.85)$$

The restriction to a single helicity is crucial — without it we would have

$$\oint_{\partial R} dy^i A_i(\tau \rightarrow +\infty, y) = \oint_{\partial R} dy^i A_i^+(\tau \rightarrow +\infty, y) + A_i^-(\tau \rightarrow +\infty, y) = \int_R F^+ + F^- = 0. \quad (3.86)$$

The last expression is the integral over R of the total field strength, including both (+) and (−) contributions. Since R lies on the celestial sphere, the integral runs over the *radial* magnetic flux at null infinity which vanishes due to the transversality of asymptotic electromagnetic radiation.

The physical interpretation of eq. (3.82) becomes more transparent if we realize that the restriction to (+) helicity modes indirectly relates the components of the gauge field tangent to the contour to the components *normal* to the contour but still tangent to the celestial sphere, as shown in figure 3. Applying this also to the (−) helicity components together with the cancellation in eq. (3.86), we find

$$\oint_{\partial R} dy^i A_i^\pm(\tau \rightarrow +\infty, y) = \frac{1}{2} \oint_{\partial R} dy_\perp^i A_i(\tau \rightarrow +\infty, y), \quad (3.87)$$

where dy_\perp^i is the vector perpendicular to dy^i but tangent to the celestial sphere. Crucially, the A_i on the right-hand side of the above equation is not restricted by helicity, so $A_i = A_i^+ + A_i^-$.

A simple physical interpretation of the above result follows if we consider a scattering process for a set of electrically neutral in states scattering into charged out states. In this case we are permitted to restrict to the regulated Milne region of figure 6, in which case $A_i(\tau \rightarrow -\infty, y) = 0$ as an initial condition. The above equation then becomes

$$\oint_{\partial R} dy^i A_i^\pm(\tau \rightarrow +\infty, y) = \frac{1}{2} \oint_{\partial R} dy_\perp^i \int d\tau E_i(\tau, y), \quad (3.88)$$

where $E_i = \partial_\tau A_i$ is the electric field. A massive probe charge undergoes acceleration proportional to the local electric field, so the “memory field” $\int d\tau E_i$ is literally equal to the cumulative “velocity kick” received by the probe from the soft radiation in the scattering process. This is nothing but the electromagnetic memory effect [38].

We have shown that the AB phase is simply a contour of the memory field $\int d\tau E_i$. Consequently, one might then wonder if the AB phase somehow contains less information than the original memory field. In fact, it is actually possible to construct the memory field from the AB phase since the naively missing information is given by the curl of the

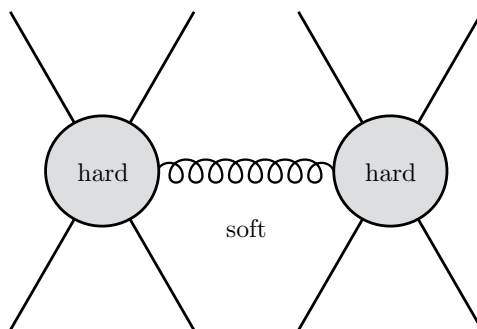


Figure 7. Internal exchange of a soft gauge boson between two hard processes. The associated Feynman diagram comes with a factor of $1/g_{\text{YM}}^2$ while the associated Witten diagram comes with k_{CS} , suggesting that $k_{\text{CS}} \sim 1/g_{\text{YM}}^2$.

electric field. However, by the classical field equations, this is proportional to $\partial_\tau B_i$, which vanishes in the Milne soft limit. We thereby conclude that the AB phase and by extension the CS gauge field is equivalent to the memory field.

3.10.2 Chern-Simons level from internal soft exchange

We have just seen how the AB effect in the 3D CS description for 4D soft emission encodes a velocity kick for charged particles that embodies the electromagnetic memory effect. While electromagnetic memory is most simply measured with massive charged probes, an alternative approach would be to configure a secondary hard process comprised massless charged particles that measure the soft emission from an initial scattering. In the CS theory, this corresponds to diagrams composed of disjoint charged currents connected only by the exchange of an *internal* CS gauge line, as depicted in figure 7. This requires a new element, as thus far we have only matched the external CS lines to external soft emission lines in 4D.

Obviously, the exchange of an internal CS gauge field in AdS_3 is dual to a Mink_4 scattering amplitude with an internal soft gauge boson exchange. Such an amplitude describes two hard processes connected by a soft internal gauge boson, so it only occurs at very special kinematics. Since this particle travels a great distance before it is reabsorbed, it can be assigned a helicity. The external soft emission and absorption processes studied earlier are then just sewn together as factorization channels of this composite process.

Internal gauge exchange in CS is also important in because it encodes the CS level, k_{CS} , reflecting quantum fluctuations of the gauge field. When the Lie algebra is normalized independently of the couplings of the gauge theory, the CS action reads

$$S_{\text{CS}} = \frac{k_{\text{CS}}}{4\pi} \int_{\text{AdS}_3} d^3y \text{tr} \left(A_i F_{jk} + \frac{2}{3} A_i A_j A_k \right) \varepsilon^{ijk}, \tag{3.89}$$

while the action for YM theory in 4D is

$$S_{\text{YM}} = -\frac{1}{2g_{\text{YM}}^2} \int_{\text{Mink}_4} d^4x \text{tr} (F_{\mu\nu} F^{\mu\nu}). \tag{3.90}$$

Notably, the solutions to the classical CS and YM equations of motion *do not depend* on k_{CS} nor g_{YM} since these are prefactors of the action, and thus drop out of the homogenous field equations. Said another way, at tree level these couplings can be reabsorbed into the definition of \hbar . Hence, the gauge field describing the soft external branches depicted in figure 5 are actually independent of these parameters. On the other hand, these variables do enter into diagrams with internal CS gauge lines, or equivalently Mink_4 processes with intermediate soft gauge boson exchange. In CS perturbation theory [79, 80], the former comes with a factor of $1/k_{\text{CS}}$ and the latter, with a factor of g_{YM}^2 . Therefore, we conclude that

$$k_{\text{CS}} \sim \frac{1}{g_{\text{YM}}^2}, \tag{3.91}$$

in agreement with [20] but not [22], which argued for a vanishing Kac-Moody level.

This result can also be obtained from the following heuristic derivation. Substituting the self-dual constraint, $F_{\mu\nu} = i\tilde{F}_{\mu\nu}$, into the YM action in the regulated Milne region of figure 6, we find that

$$S_{\text{YM}} = -\frac{i}{2g_{\text{YM}}^2} \int_{\text{Mink}_4} d^4x \text{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) = \frac{i}{2g_{\text{YM}}^2} \int_{\text{Mink}_4} d^4x \partial_\sigma \text{tr} \left(A_\mu F_{\nu\rho} + \frac{2}{3} A_\mu A_\nu A_\rho \right) \varepsilon^{\mu\nu\rho\sigma}, \tag{3.92}$$

which is a total derivative. In principle, this total derivative will integrate to all the boundaries of the regulated Milne_4 . However, due to our choice of Milne temporal gauge $A_\tau = 0$ and the Milne soft limit $F_{\tau i} = 0$, the only boundary that contributes is at late τ . Thus, we again obtain the non-abelian CS action in eq. (3.89) where A_i is the gauge field at $\tau \rightarrow +\infty$. Matching this to the CS action, we verify eq. (3.91).

3.11 Toy model for a black hole horizon

As recently discussed [67], it is interesting to understand in what sense asymptotic symmetries and the memory effect constitute a new kind of “hair” in the presence of black hole horizons. While this paper has focused on uncovering a CFT_2 structure underlying Mink_4 scattering amplitudes, our strategy incidentally offers a baby version of the black hole problem in the form of the Rindler horizon, say as seen by radially accelerating observers in the Rindler region. For such observers we can excise all of Mink_4 spacetime that lies behind a “stretched” Rindler horizon, excluding the Milne regions altogether, as depicted in figure 8.

The physical observables relevant to the remaining Rindler region are *thermal* correlators⁶ which encode the wavefunction describing the particles emitted to or from null infinity together with the stretched horizon. First, let us remind the reader of the Rind_4 coordinates in eq. (2.9), where each hyperbolic slice at fixed Rindler radius ρ defines a dS_3 spacetime labeled by conformal time τ . As discussed earlier, the roles of ρ and τ in the Rindler region are swapped relative to the Milne region. So for any dS_3 slice, the corresponding ∂dS_3 boundary is defined by the end of time limit $\tau \rightarrow 0$. Meanwhile, $\rho \rightarrow +\infty$ corresponds to null infinity, while $\rho = \rho_{\text{stretch}}$ for large and negative ρ_{stretch} defines the stretched horizon.

⁶Here “thermal” is with respect to dS_3 time in static patch coordinates, as experienced by a Rindler observer.

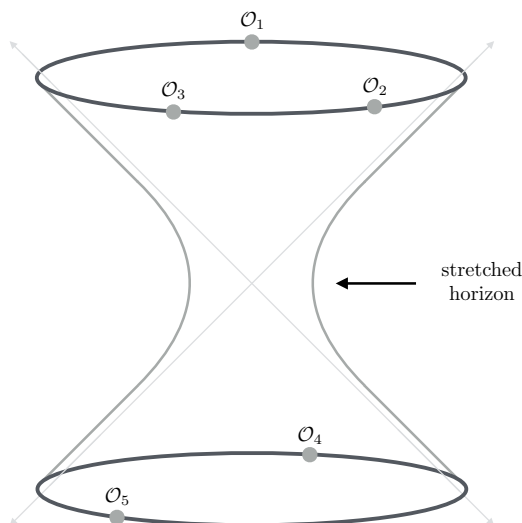


Figure 8. The stretched horizon in Rindler spacetime. The dots denote operator insertions at early and late times on the stretched horizon.

Therefore a correlator in Rind_4 has the form of a Mink_4 correlator, $\langle \text{out} | \mathcal{O}_1 \cdots \mathcal{O}_n | \text{in} \rangle$. Here the in and out states label particles emitted from and to null infinity in the far past or future, respectively. Meanwhile, the operators \mathcal{O}_i denote insertions of particle fields on the stretched horizon at early or late times. These operators are generic probes of the wavefunction of the stretched horizon. Despite the fact that we have restricted physical spacetime to the Rindler region outside the stretched horizon, we must compute this correlator using Minkowski Feynman diagrams in order to match the *thermal* Rindler correlators.

Such diagrams will now consist of four ingredients: the three already discussed — interaction vertices, propagators, and LSZ wave packets — together with additional propagators running from the \mathcal{O}_i inserted in the far past or future of the stretched horizon to interactions in the bulk of the Rindler region. Since the stretched horizon at fixed $\rho = \rho_{\text{stretch}}$ has Lorentzian dS_3 geometry, these additional propagators describe a bulk point in dS_3 and a boundary point on ∂dS_3 , so they are *bulk-boundary* propagators from this perspective. Therefore by the close analogy with our Milne manipulations, we see that the Rind_4 correlators are boundary correlators in a dS_3 theory which can be reinterpreted as dual to CFT_2 correlators. In the standard dS_3/CFT_2 picture, the CFT_2 is dual to the late time wavefunction of the Universe [54]. So the CFT_2 describing the Rindler region is dual to the late time wavefunction of Rindler, up to and including the stretched horizon and given initial conditions for the wavefunction at early times.

In this context, let us analyze the physics of the CS gauge field and electromagnetic memory in the Rindler region. By the exact analog of eq. (3.75), we can locate the CS field in Rind_4 by taking the limit of soft Rindler momentum, so the CS field corresponding to (+) helicity is

$$A_i(y) = A_i^+(\rho \rightarrow +\infty, y) - A_i^+(\rho = \rho_{\text{stretch}}, y). \tag{3.93}$$

The second term represents the component of the soft “memory” field that remembers the hard charges that fall into the Rindler horizon. We see this explicitly because, retaining this component, the analog of our AB phase associated with a region R on null infinity of the Milne region in eq. (3.82), now reads in Rindler as

$$\oint_{\partial R} dz A_z(\tau \rightarrow 0, z) = \sum_{i \in R} q_i, \tag{3.94}$$

where again the above expression is implicitly evaluated within a 2D correlator. Here A_z is given by the two terms in eq. (3.93). We thereby conclude that the AB phase measures hard charges passing through an angular region R , regardless of whether those hard charges are falling into the horizon or are headed out to null infinity. If one measures the charges heading out to null infinity, the CS field will encode information on where exactly the hard charges entered the horizon. This in some sense offers a sharper form of “hair” [67] compared to the usual asymptotic electric field of a black hole, which remembers the charge that has fallen into the horizon but without regard to the angle of entry.

4 Gravity

We have described the emergence of CFT structure in gauge theory amplitudes, but of course the hallmark of a true CFT is a 2D stress tensor. The Sugawara construction yields a stress tensor constructed from the 2D holomorphic currents dual to soft gauge fields, but this can only be a component of the full stress tensor since it does not account for hard particle dynamics. As usual in AdS/CFT, to find the full stress tensor we must consider gravity, to which we now turn. Our aim will be to reframe many of the important aspects of 4D gravity in terms of the language of 2D CFT.

We will follow the same basic strategy for gravity as for scalar and gauge theory, moving briskly through those aspects which are closely analogous and focusing on those which introduce major new considerations. The most important such consideration is that gravity in asymptotically flat space is not Weyl invariant, since the 4D Einstein-Hilbert action,

$$S_{\text{EH}} = \frac{m_{\text{Pl}}^2}{2} \int_{\text{Mink}_4} d^4x \sqrt{-g} R, \tag{4.1}$$

depends on the dimensionful Planck mass, m_{Pl} . For the sake of exposition, we will often restrict to the Milne region for explicit calculations, bearing in mind that we can straightforwardly continue into the Rindler region and thus all of Minkowski space via the embedding formalism.

In any case, while the dynamics cannot be mapped into a factorizable geometry like $\text{AdS}_3 \times \mathbb{R}_\tau$, this is merely a technical inconvenience. As in gauge theory, one can nevertheless apply a decomposition into AdS_3 and $d\text{S}_3$ modes, resulting in 3D Witten diagrams equivalent to 4D scattering amplitudes with a particular prescription for LSZ reduction onto bulk-boundary propagators.

4.1 Stress tensor of CFT₂

In this section, we derive a 2D stress tensor corresponding to soft gravitons in 4D. We will show that the Ward identity for the 2D stress tensor is a particular angular convolution of the *subleading* soft factor for graviton emission [37]. Notably, the subleading soft factor differs from the leading factor in that it depends on the angular momentum of each external leg rather than the momentum.

The pursuit of a 2D stress tensor will naturally lead us to the Virasoro algebra, which directly manifests the super-rotation [27] asymptotic symmetries of 4D Minkowski space. Commuting these with ordinary translations, we then derive the BMS super-translations [25, 26]. This approach is anti-historical, but more natural from the holographic approach taken here.

4.1.1 Bulk-boundary propagator for AdS₃ graviton

By the AdS₃/CFT₂ dictionary, the 2D stress tensor is a tensor primary operator of scaling dimension $\Delta = 2$ dual to a massless tensor field, h_{ij} , in AdS₃. Again using the embedding formalism, we write down the bulk-boundary propagator for h_{ij} lifted from 3D to 4D via

$$K_{\mu\nu} = N(x^2) \times \frac{x^\rho x^\sigma f_{\rho\mu} f_{\sigma\nu}}{(kx)^4}. \tag{4.2}$$

Here, the normalization $N(x)$ parameterizes an inherent ambiguity in the lift, arising because AdS₃ lives on the constrained surface $x^2 = -1$. For gauge theory we sidestepped this ambiguity, since the underlying Weyl invariance implied that the dynamics are independent of the scale set by the constrained surface. However, there is no such invariance of 4D gravity due to the dimensionful gravitational constant, so we must find an alternate way to identify $N(x)$.

Of course, $N(x)$ should be chosen so $K_{\mu\nu}$ is a solution of the linearized Einstein's equations in Mink₄. Imposing this condition leaves two possibilities: either $N(x) = 1$ or $N(x) = x^2$. A priori, either solution is reasonable, but as will see, the latter is the correct choice. The reason for this is that in standard AdS₃ gravity the Virasoro symmetries arise as asymptotic symmetries of AdS₃ encoded in solutions to the 3D Einstein's equations. Famously, all such solutions are pure gauge [81, 82], and are thus diffeomorphisms of AdS₃ itself. At the linearized level this is reflected in the fact that the bulk-boundary propagator for $\Delta = 2$ on AdS₃ is a linearized 3D diffeomorphism about AdS₃. In order to recast the Virasoro symmetries as asymptotic symmetries of Mink₄, we should look for a lift of the AdS₃ bulk-boundary propagator that yields a pure linearized large diffeomorphism in Mink₄.

A straightforward calculation shows that for the bulk-boundary propagator $K_{\mu\nu}$ is not a pure 4D diffeomorphism for $N(x) = 1$, but is for $N(x) = x^2$. This is reasonable since the Milne₄ is a warped product of AdS₃ and \mathbb{R}_τ associated with a warp factor $x^2 = e^{2\tau}$, which we now see is crucial to lift 3D diffeomorphisms into a 4D diffeomorphisms. Fixing $N(x) = x^2$, our final expression for the lifted bulk-boundary propagator for $\Delta = 2$ is

$$K_{\mu\nu} = x^2 \times \frac{x^\rho x^\sigma f_{\rho\mu} f_{\sigma\nu}}{(kx)^4}. \tag{4.3}$$

Since this is a pure diffeomorphism, it can be written as

$$K_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad \text{where} \quad \xi_\mu = \frac{1}{3} \partial_z^3 (x^\rho \bar{f}_{\rho\mu} \log kx), \quad (4.4)$$

where $\bar{f}_{\mu\nu}$ is defined in eq. (3.31). This form for $K_{\mu\nu}$ will be quite useful for explicit calculations.

Applying the logic of AdS/CFT, the bulk-boundary propagator for h_{ij} corresponds to the insertion of a local CFT₂ stress tensor $t(z)$ or its complex conjugate $\bar{t}(\bar{z})$. In the subsequent sections, we will see how the bulk-boundary propagator $K_{\mu\nu}$ relates to single and multiple soft graviton emission in 4D.

Finally, let us comment on the curious fact the bulk-boundary propagator for gravity is proportional to the square of the bulk-boundary propagator for gauge theory, so

$$K_{\mu\nu} = x^2 K_\mu K_\nu. \quad (4.5)$$

The simplicity of this is remarkable, given the known (gauge)² = gravity relations that arise from the KLT [83] relations and the closely related BCJ [78] relations. Given also the connection between BCJ and the soft limit [84], it is likely that the above equation is not an accident, and is perhaps a sign of some deeper underlying construction.

4.1.2 Ward identity for CFT₂ stress tensor

Given the central role of the 2D stress tensor $t(z)$, it is natural to ask about the 4D dual of this quantity. Repeating our strategy for gauge theory, we now calculate the Ward identity for the 2D stress tensor using AdS/CFT. To do so, we compute a correlator of the stress tensor via the associated Witten diagram,

$$\langle t(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \int d^4x K_{\mu\nu}(x) W^{\mu\nu}. \quad (4.6)$$

Here $K_{\mu\nu}$ is the bulk-boundary propagator in eq. (4.3) and $W^{\mu\nu}$ parameterizes the remainder of the Witten diagram,

$$W^{\mu\nu} = \langle \text{out} | T^{\mu\nu}(x) | \text{in} \rangle, \quad (4.7)$$

computed as an insertion of the 4D stress tensor operator $T^{\mu\nu}$ inserted between in and out states.

Substituting the pure gauge form of the bulk-boundary propagator in eq. (4.4), we obtain

$$\begin{aligned} \langle t(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle &= \int d^4x [\partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x)] \langle \text{out} | T^{\mu\nu}(x) | \text{in} \rangle \\ &= -\frac{1}{3} \partial_z^3 \left(\int d^4x \bar{f}_{\mu\nu} \log kx \partial_\rho \langle \text{out} | x^\mu T^{\rho\nu}(x) - x^\nu T^{\rho\mu}(x) | \text{in} \rangle \right), \end{aligned} \quad (4.8)$$

where in the second line we have shuffled around terms and performed an integration by parts, dropping boundary terms. Importantly, the expression sandwiched between in

and out states is the relativistic angular momentum tensor. This quantity is conserved everywhere except at insertions associated with the external legs, so

$$\partial_\rho \langle \text{out} | x^\mu T^{\rho\nu}(x) - x^\nu T^{\rho\mu}(x) | \text{in} \rangle = - \sum_{i=1}^n J_i^{\mu\nu} \delta^4(x - x_i), \quad (4.9)$$

where $J_i^{\mu\nu}$ is the angular momentum of each external particle and x_i is its insertion point near the boundary. As before, we substitute the position of the external particles inserted near the boundary $\rho_i \rightarrow 0$ with their corresponding momenta, so $x_i \sim k_i$. As a result, the expression for the Ward identity will involve manifestly on-shell quantities. Plugging this substitution into the Ward identity, we obtain

$$\langle t(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \frac{1}{3} \partial_z^3 \left(\sum_{i=1}^n \log k k_i \bar{f}_{\mu\nu} J_i^{\mu\nu} \right) \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle. \quad (4.10)$$

In the above equation, the angular momentum generator is implicitly defined in momentum basis, so e.g. it acts on a hard scalar leg as

$$J_i^{\mu\nu} = k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}}. \quad (4.11)$$

The analogous expression for hard legs with spin has a simple representation in terms of spinor helicity variables. From eq. (4.10) we see directly the connection between the stress tensor in the CFT_2 and rotations acting on the boundary of Mink_4 . This is not accidental, and as we will see later is a hint of the super-rotation asymptotic symmetries of 4D flat space.

To compare this to the usual 2D stress tensor Ward identity, it is actually convenient to briefly revert to position space for the hard particles. To do so we send $k_i \sim x_i$ in eq. (4.10) and eq. (4.11) and go to Milne coordinates. Taking the $\rho_i \rightarrow 0$ limit, we find

$$\langle t(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle \sim \sum_{i=1}^n \left[\frac{h_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right] \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle, \quad (4.12)$$

where the conformal weight is

$$h_i = \left. \frac{\partial}{\partial \log \rho_i^2} \right|_{\rho_i \rightarrow 0}, \quad (4.13)$$

for a 2D scalar operator dual to a hard 4D scalar particle. Up to an overall constant normalization, eq. (4.12) is none other than the Ward identity for the stress tensor of the CFT_2 . Of course, this analysis can be extended straightforwardly to include hard particles with spin.

4.1.3 Relationship to subleading soft theorems in Mink_4

Next, we derive the explicit relationship between the Ward identity for the 2D stress tensor and the soft graviton theorems. To do so, it will be convenient to introduce an auxiliary operator $\tilde{t}(z, \bar{z})$ which is a $\Delta = 0$ tensor primary operator of the 2D CFT. Note that we

do not assign *independent* physical import to this $\Delta = 0$ operator, which is why we refer to it as auxiliary.

From the embedding formalism, the bulk-boundary propagator for $\tilde{t}(z, \bar{z})$ is

$$\tilde{K}_{\mu\nu} = \frac{x^\rho x^\sigma f_{\rho\mu} f_{\sigma\nu}}{(kx)^2}. \quad (4.14)$$

Importantly, this bulk-boundary propagator is a pure linearized diffeomorphism equal to

$$\tilde{K}_{\mu\nu} = \partial_\mu \tilde{\xi}_\nu + \partial_\nu \tilde{\xi}_\mu \quad \text{where} \quad \tilde{\xi}_\mu = \frac{1}{2} \partial_z (x^\rho f_{\rho\mu} \log kx). \quad (4.15)$$

Repeating our steps from before, calculate an arbitrary correlator involving $\tilde{t}(z, \bar{z})$,

$$\begin{aligned} \langle \tilde{t}(z, \bar{z}) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle &\sim \partial_z \left(\sum_{i=1}^n \log k k_i f_{\mu\nu} J_i^{\mu\nu} \right) \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle \\ &= \sum_{i=1}^n \frac{\epsilon k_i}{k k_i} f_{\mu\nu} J_i^{\mu\nu} \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle, \end{aligned} \quad (4.16)$$

where the right-hand side is literally the subleading graviton soft factor [37]. While interesting, this observation is only useful because $\tilde{t}(z, \bar{z})$ happens to be directly related to $t(z)$ by a handy integral transform in (z, \bar{z}) . Indeed, by comparing the definitions of ξ_μ and $\tilde{\xi}_\mu$ in eq. (4.4) and eq. (4.15), respectively, we see that these quantities are related by the differential equation,

$$\partial_{\bar{z}} \xi_\mu \sim \partial_z^3 \tilde{\xi}_\mu^*, \quad (4.17)$$

dropping unimportant numerical prefactors. Notably, the above equation is equivalent to the CFT₂ equation $\partial_z t(z) \sim \partial_{\bar{z}}^3 t^\dagger(z, \bar{z})$, which when evaluated inside a correlator yields zero on both sides except for delta function support at the insertion points of hard operators. In fact, we can verify this fact by applying $\partial_{\bar{z}}^3$ directly to eq. (4.16). Since $\partial_{\bar{z}}^3 f_{\mu\nu} = 0$, this implies that at least one $\partial_{\bar{z}}$ derivative will act on $\epsilon k_i / k k_i = 1/(z - z_i)$, producing a delta function $\delta^2(z - z_i)$ from the identity in eq. (3.54). This is a non-trivial check that the structure of the subleading graviton soft theorem ensures conservation of the CFT₂ stress tensor.

In any case, we would like to solve the differential equation in eq. (4.17) by constructing a formal anti-derivative,

$$\partial_{\bar{z}}^{-1} = \frac{1}{2\pi} \int d^2 z' \frac{1}{z - z'}, \quad (4.18)$$

which satisfies $\partial_{\bar{z}} \partial_{\bar{z}}^{-1} = 1$ as a result of eq. (3.54). Solving eq. (4.17) then yields

$$\xi_\mu(z, \bar{z}) \sim \partial_z^3 \int d^2 z' \frac{1}{z - z'} \tilde{\xi}_\mu^*(z', \bar{z}'), \quad (4.19)$$

suppressing all τ and ρ dependence. Inserting this relation into the Ward identity for the stress tensor, we obtain our final expression,

$$\begin{aligned} \langle t(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle &\sim \partial_z^3 \int d^2 z' \frac{1}{z - z'} \langle \tilde{t}(z, \bar{z})^\dagger \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle \\ &\sim \partial_z^3 \int d^2 z' \frac{1}{z - z'} \sum_{i=1}^n \frac{\bar{\epsilon}' k_i}{k' k_i} \bar{f}'_{\mu\nu} J_i^{\mu\nu} \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle, \end{aligned} \quad (4.20)$$

where k' , ϵ' , and \bar{f}' are functions of (z', \bar{z}') . This result says that the Ward identity for the 2D stress tensor is proportional to a particular angular integral over the subleading soft graviton factor. Physically, this corresponds to a particular superposition of soft graviton emission in all directions (z', \bar{z}') .

Let us pause to discuss the peculiar integral structure of eq. (4.20). Naively, it is odd that the CFT_2 stress tensor should be expressed as a non-local function in (z, \bar{z}) but this was actually essential to maintain consistency between the 2D and 4D pictures. To see why, recall from eq. (4.12) that the canonical form of the 2D stress tensor Ward identity has manifest double and single poles in z . In turn, this OPE corresponds to collinear singularities in 4D, but graviton scattering amplitudes are famously free of such collinear singularities. Hence, the only way to square these apparently inconsistent statements is if the 2D stress tensor is actually a *non-local* function of the graviton scattering amplitude in (z, \bar{z}) , as eq. (4.20) clearly is. Only then is it possible for the singularity structure of the 2D stress tensor Ward identity to arise consistently from the analytic properties of graviton amplitudes.

4.2 Virasoro algebra of CFT_2

The Virasoro algebra places immense constraints on the structure of correlators in the CFT_2 . It is obviously of great interest to understand the implications of these constraints on the dual scattering amplitudes in Mink_4 . As we will see, the corresponding infinite-dimensional Virasoro algebra in 2D has a direct connection to the asymptotic symmetries of 4D flat space [27, 30].

What is the action of the Virasoro generators on scattering amplitudes? To answer this, we revisit the 2D stress tensor Ward identity in eq. (4.10). Expanding the derivatives in z , we obtain

$$\begin{aligned} \langle t(z)\mathcal{O}(z_1, \bar{z}_1)\cdots\mathcal{O}(z_n, \bar{z}_n)\rangle &= \\ &= \sum_{i=1}^n \left(\frac{2}{3} \left(\frac{\epsilon k_i}{k k_i} \right)^3 - \left(\frac{\epsilon k_i}{k k_i} \right)^2 \partial_z + \left(\frac{\epsilon k_i}{k k_i} \right) \partial_z^2 \right) \bar{f}_{\mu\nu} J_i^{\mu\nu} \langle \mathcal{O}(z_1, \bar{z}_1)\cdots\mathcal{O}(z_n, \bar{z}_n)\rangle. \end{aligned} \tag{4.21}$$

For simplicity, consider the limit in which the soft graviton is collinear to a hard external leg located at z' on the celestial sphere. A Laurent expansion of this expression around $z = z'$ yields

$$\langle t(z)\mathcal{O}(z', \bar{z}')\cdots\rangle = \left(\frac{2\bar{f}'_{\mu\nu}/3}{(z-z')^3} - \frac{\partial_{z'}\bar{f}'_{\mu\nu}/3}{(z-z')^2} + \frac{\partial_{z'}^2\bar{f}'_{\mu\nu}/3}{z-z'} + \dots \right) J'^{\mu\nu} \langle \mathcal{O}(z', \bar{z}')\cdots\rangle, \tag{4.22}$$

where all primed quantities are evaluated at $z = z'$ and we have used eq. (3.38). Here the ellipses denote non-singular contributions which originate from the other hard legs in the process.

We can now compare eq. (4.22) directly to definition of the Virasoro generators,

$$t(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} = \dots + \frac{L_1}{z^3} + \frac{L_0}{z^2} + \frac{L_{-1}}{z} + \dots, \tag{4.23}$$

only Laurent expanded around $z = z'$. Matching terms by eye, we ascertain the identities of the $SL(2, \mathbb{C})$ Virasoro generators,

$$\begin{aligned} L_1 &\sim -i(K_2 + iJ_2) - (K_1 + iJ_1) \\ L_0 &\sim K_3 + iJ_3 \\ L_{-1} &\sim -i(K_2 + iJ_2) + (K_1 + iJ_1), \end{aligned} \tag{4.24}$$

up to a constant normalization factor. Here K_3 and J_3 denote the generators of $J'^{\mu\nu}$ corresponding to boosts and rotations around the axis of the hard particle, while $K_{1,2}$ and $J_{1,2}$ are those for the transverse directions. Since these generators only act on the collinear hard particle, they are effectively local Lorentz transformations. Thus, the identification of the full Virasoro algebra as the algebra of super-rotations is indeed appropriate.

This result offers a physical interpretation for the action of $t(z)$ on scattering amplitudes. The passage of collinear emitted soft gravitons induces a Lorentz transformation that acts locally on a hard leg. Operationally, this “jiggles” the hard particle in a way that displaces it relative to the direction of its original trajectory. This local Lorentz transformation has the same effect as a net displacement of the detectors residing at the boundary of spacetime.

4.3 Chern-Simons theory and multiple soft emission

To understand multiple soft emissions in gravity, we proceed in parallel with our analysis for non-abelian gauge theory. Our aim is to describe the dynamics of multiple external soft gravitons that interact and merge in the gravitational analog of figure 4. As before, we can parameterize the dynamics of the entire soft branch with a graviton field $H_{\mu\nu}(x)$ at the juncture x with the hard process characterized by $T_{\mu\nu}(x)$. In the limit of vanishing gravitational coupling, $H_{\mu\nu}$ will approach a superposition of independent soft gravitons, each described by the bulk-boundary propagator $K_{\mu\nu}$ from eq. (4.3). Hence, the branch structure of soft gravitons is rooted in external legs connected through these bulk-boundary propagators. Said another way, the soft branch is simply the solution to the *non-linear* sourceless Einstein’s equations with free-field approximation given by $K_{\mu\nu}$.

Now consider a closely analogous situation for 3D Witten diagrams, where an AdS_3 branch field $h_{ij}(y)$ similarly characterizes the web of soft gravitons merging before making contact with a hard source at y . Here h_{ij} can be treated as a perturbation of the background AdS_3 metric g_{ij} defined in eq. (2.6). The full metric in 3D is then

$$\tilde{g}_{ij} = g_{ij} + h_{ij}, \tag{4.25}$$

where g_{ij} is the background AdS_3 metric from eq. (2.6). Eq. (4.25) is a solution to Einstein’s equations in AdS_3 whose free field asymptotics near ∂AdS_3 are given by bulk-boundary propagators. Since all solutions to AdS_3 gravity are pure diffeomorphisms of AdS_3 [81, 82], h_{ij} corresponds to precisely such a non-linear diffeomorphism.

Next, using the same prescription as for bulk-boundary propagators, we can lift this diffeomorphism from 3D to 4D. In particular, we have that $H_{ij} = e^{2\tau} h_{ij}$, where H_{ij} are the non-zero components of the 4D branch field H_{IJ} in Milne temporal gauge. The $x^2 = e^{2\tau}$

warp factor is the same one required in the bulk-boundary propagator for the 2D stress tensor. Since H_{IJ} is a 4D diffeomorphism around flat space, we find

$$(\eta_{\mu\nu} + H_{\mu\nu}(x))dx^\mu dx^\nu = e^{2\tau}(-d\tau^2 + \tilde{g}_{ij}(y)dy^i dy^j). \quad (4.26)$$

In conclusion, at the fully non-linear level, multiple subleading soft emissions are described by a branch $H_{\mu\nu}$ that encodes large diffeomorphisms of the AdS_3 metric.

Since these soft perturbations of the metric are Milne zero modes, they couple to hard particle tracks according to

$$\int d^4x H_{\mu\nu}(x)T^{\mu\nu}(x) = \int d^3y \sqrt{g} h_{ij}(y) \int d\tau e^{6\tau} T^{ij}(\tau, y) = \int d^3y \sqrt{g} h_{ij}(y) T_{\text{eff}}^{ij}(y), \quad (4.27)$$

where in the last line we have defined

$$T_{\text{eff}}^{ij}(y) = \int d\tau e^{6\tau} T^{ij}(\tau, y), \quad (4.28)$$

the Milne time-integrated stress tensor in a warped version of eq. (4.3).

4.3.1 Equivalence to AdS_3 gravity

Similar to the case of gauge theory, we have seen that 4D soft graviton modes correspond to solutions of 3D gravity which are pure diffeomorphisms. It is then expected that the resulting theory is topological, which is reasonable because gravity in AdS_3 is famously equivalent to a CS theory, at least perturbatively [69]. In particular, one can define a non-abelian CS gauge field, $A_i^{\pm a} = \epsilon^{abc} \Omega_{ibc} \pm i e_i^a$, where e is the dreibein, Ω is the spin connection, and the index $a = 1, 2, 3$ runs through the local tangent space. The gauge group of the CS theory is $\text{SL}(2, \mathbb{C})$, corresponding to the global isometries of AdS_3 , or equivalently, the Lorentz group in Mink_4 . Concretely, $A_i^{\pm a}$ corresponds to the Lorentz generators $J^a \pm iK^a$, where J^a and K^a are rotations and boosts, respectively.

Via the embedding formalism, $A_i^{a\pm}$ is associated with (+) and (−) helicity soft gravitons. Moreover, since the commutator $[J^a + iK^a, J^b - iK^b] = 0$ vanishes, the $\text{SL}(2, \mathbb{C})$ gauge group factorizes, so there is no intrinsic reason why we must restrict to a single helicity like we did for non-abelian CS theory. At the level of the dual CFT_2 we are then permitted to compute mixed correlators involving both the holomorphic and anti-holomorphic stress tensor, $t(z)$ and $\bar{t}(\bar{z})$.

With the non-abelian structure clarified, we can Laurent expand the holomorphic stress tensor into the infinite set of non-abelian Virasoro charges. Relatedly, the CS structure of the subleading soft amplitudes again implies that the dynamics of soft gravitons is governed by a non-abelian analog of the AB effect, where the CS graviton field is the now the field encoding memory effects. Unlike for electromagnetic memories, we have not as yet matched this kind of AB effect in detail with the “spin memory” effects discussed already in the literature [45].

A final note on the rigor of our conclusions here: what we have shown thus far is that LSZ reduction onto $\Delta = 2$ bulk-boundary propagators gives a consistent picture for multiple subleading soft emissions. We have not yet proven that the scattering amplitudes

of plane waves have the requisite commutativity amongst multiple subleading soft limits required for simultaneous LSZ reduction onto multiple bulk-boundary propagators. But we expect that the AdS₃ gravity picture should identify any obstructions to multiple soft limits, as it did in non-abelian CS gauge theory for mixed soft helicities. While no such obstructions appear here, it would still be interesting to compute explicitly the commutativity properties of subleading graviton soft limits for these amplitudes in Minkowski space.

4.3.2 Virasoro central charge from internal soft exchange

The Virasoro central charge, c , is arguably the most important quantity in a 2D CFT [85]. In theories with semi-classical AdS₃ duals, c is given by the AdS₃ Planck scale in units of the AdS₃ length. However, much like the gauge coupling in YM theory, the Planck scale enters simply as an overall factor in the gravity action, so it drops out of the homogeneous Einstein’s equations. So at tree level, the soft branches characterizing multiple graviton emission are insensitive to the Planck scale and thus c .

To make sense of c , we must then consider the gravitational analog of figure 7, which depicts a set of two hard processes exchanging a soft internal graviton. We interpret one process as a “measurement apparatus” for the subleading soft graviton emission of the other. Notably, the corresponding AdS₃ Witten diagram is suppressed by $1/c$, while the Mink₄ scattering amplitude goes as $1/m_{\text{Pl}}^2$. However, unlike before when we matched the CS level to the gauge coupling, here there is a dimensional mismatch between c and m_{Pl}^2 . This means that an infrared length scale L_{IR} does not decouple from the process. One can think of L_{IR} as a formal scale separating “hard” from “soft”. We thereby conclude that the Virasoro central charge scales as

$$c \sim m_{\text{Pl}}^2 L_{\text{IR}}^2. \tag{4.29}$$

Just this type of infrared sensitivity is present in the spin-memory effect described in [45].

We can see this more directly by writing the 4D Einstein-Hilbert action in eq. (4.1) in terms of the 3D metric \tilde{g}_{ij} characterizing a soft branch in AdS₃, as shown in eq. (4.25). Since \tilde{g}_{ij} is related by a diffeomorphism to the pure AdS₃ background metric, the resulting action should just be proportional to 3D gravity with a cosmological constant. The simple τ dependence of the action straightforwardly factors, yielding

$$S_{\text{EH}} = \frac{m_{\text{Pl}}^2}{2} \int_{\text{Milne}_4} d^4 Y \sqrt{-G} R = \frac{m_{\text{Pl}}^2}{2} \int_{-\infty}^{\tau_{\text{late}}} d\tau e^{2\tau} \int_{\text{AdS}_3} d^3 y \sqrt{\tilde{g}} (\tilde{R} + 1), \tag{4.30}$$

where we have taken “unit” dimensionally reduced AdS₃ radius of curvature, in keeping with the normalization of our other formulas, and where τ_{late} relates to L_{IR} by

$$L_{\text{IR}} \sim e^{\tau_{\text{late}}}. \tag{4.31}$$

Although we are not carefully treating the physics underlying τ_{late} here, we can nevertheless estimate the central charge from this rough scaling,

$$c \sim \frac{m_{\text{Pl}}^2}{2} \int_{-\infty}^{\tau_{\text{late}}} d\tau e^{2\tau} \sim m_{\text{Pl}}^2 e^{2\tau_{\text{late}}} \sim m_{\text{Pl}}^2 L_{\text{IR}}^2. \tag{4.32}$$

We leave a formal analysis of the Virasoro central charge for future work.

4.4 Relation to asymptotic symmetries

4.4.1 From super-rotations to super-translations in Mink₄

Let us now discuss the relation between our results and the asymptotic symmetries of Mink₄. While there is an expansive literature on this subject, we will be quite brief here. Long ago, BMS [25, 26] discovered the existence of an infinite-dimensional symmetry of asymptotically flat space corresponding to super-translations at null infinity. Physically, these super-translations are diffeomorphisms of retarded time that depend on angles on the celestial sphere.

More recently, [27] argued that the super-translation algebra can be further extended to include super-rotations encoding an underlying Virasoro algebra. From their analysis of large diffeomorphisms, they proposed an extended BMS algebra [71],

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} \\ [P_{mn}, P_{rs}] &= 0 \\ [L_m, P_{rs}] &= \left(\frac{m+1}{2} - r\right) P_{m+r,s}, \end{aligned} \tag{4.33}$$

dropping for the moment the Virasoro central charge. Here the Virasoro generators L_m correspond to the super-rotations while the generators P_{mn} correspond to super-translations. The Poincare sub-algebra is

$$L_{-1}, L_0, L_1, \quad \bar{L}_{-1}, \bar{L}_0, \bar{L}_1, \quad P_{00}, P_{01}, P_{10}, P_{11}, \tag{4.34}$$

where the four super-translation generators are nothing more than the four components of the usual momentum generator $P_{\alpha\dot{\alpha}}$ in the spinor basis where $\alpha, \dot{\alpha} = 0, 1$. Refs. [28–30] later showed that the super-translations and super-rotations, at least at the level of single soft emission, arise from the leading and subleading Weinberg soft theorems.

Here we will use eq. (4.33) as a guide for constructing super-translations as a combination of super-rotations and ordinary translations. While ordinary translations are quite obscure in Milne and Rindler coordinates, they are of course still a symmetry of flat space, so they should also be global symmetries of the CFT. Since the 2D stress tensor is comprised of super-rotation generators, we can commute it with regular translations to obtain

$$[t(z), P_{00}] = \sum_{m=-\infty}^{\infty} \frac{1}{z^{m+2}} [L_m, P_{00}] = \frac{1}{2} \sum_{m=-\infty}^{\infty} \frac{m+1}{z^{m+2}} P_{m0} = -\frac{\partial_z j(z)}{2}. \tag{4.35}$$

In analogy with the 2D CFT for gauge theory, we have defined a super-translation current,

$$j(z) = \sum_{m=-\infty}^{\infty} \frac{P_{m0}}{z^{m+1}}, \tag{4.36}$$

which is holomorphically conserved, so $\partial_{\bar{z}} j(z) = 0$.

We can use this result to determine the Ward identity for $j(z)$. From our formula for the 2D stress tensor Ward identity in eq. (4.10), we already see an explicit connection

to super-rotations through the angular momentum operators $J_i^{\mu\nu}$ acting on the hard legs. Now taking the commutator of eq. (4.10) with P_{00} , we obtain

$$\langle [t(z), P_{00}] \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle \sim \partial_z^3 \left(\sum_{i=1}^n \log k k_i \bar{f}_{\mu\nu} [J_i^{\mu\nu}, k_i^\rho] q_\rho \right) \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle. \quad (4.37)$$

Here we have used that $P_{00} = qP$ to go from explicit spinor index notation to a more covariant form. We can evaluate this expression using the fact that global translations $P^\mu = \sum_{i=1}^n k_i^\mu$ have a non-vanishing commutator with the angular momentum generators acting on the hard legs,

$$[J_i^{\mu\nu}, k_i^\rho] = k_i^\mu \eta^{\nu\rho} - k_i^\nu \eta^{\mu\rho}. \quad (4.38)$$

Applying these relations, eq. (4.39) simplifies to

$$\sim \partial_z \left(\sum_{i=1}^n \frac{\epsilon k_i}{k k_i} q k_i \right) \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle = \partial_z \left(\sum_{i=1}^n \frac{q k_i}{z - z_i} \right) \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle. \quad (4.39)$$

Comparing with eq. (4.35), we see that the Ward identity for the super-translation current is

$$\langle j(z) \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle \sim \sum_{i=1}^n \frac{q k_i}{z - z_i} \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle. \quad (4.40)$$

Hence, we deduce that the charge associated with the super-translation Ward identity is the physical momentum in the q direction.

4.4.2 Chern-Simons theory for super-translations?

We have shown how 4D super-translations can be obtained from the 2D stress tensor $t(z)$ via the commutation relations of the extended BMS algebra. Furthermore, we saw that correlators of $t(z)$ correspond to a particular angular convolution of the subleading graviton soft theorem. Given the underlying connection of $j(z)$ to super-translations, it is then quite natural for $j(z)$ to relate to the *leading* graviton soft theorem. As we will see, this is indeed the case.

To understand why, we revisit the auxiliary tensor primary $\tilde{t}(z, \bar{z})$ defined in eq. (4.16), whose correlators are literally equal to the 4D subleading soft graviton factor. In particular, let us consider the CFT₂ operator, $[\tilde{t}(z, \bar{z}), P_{00}]$, defined by the commutator of this auxiliary tensor and regular translations.

It is simple to see that the bulk-boundary propagator associated with the operator $[\tilde{t}(z, \bar{z}), P_{00}]$ is a pure diffeomorphism. In particular, since $P_{00} = qP$ the bulk-boundary propagator for $[\tilde{t}(z, \bar{z}), P_{00}]$ is by definition just the derivative of the bulk-boundary propagator of $\tilde{t}(z, \bar{z})$ in the q direction. Concretely, this implies that the bulk-boundary propagator for $[\tilde{t}(z, \bar{z}), P_{00}]$ is simply $q^\rho \partial_\rho \tilde{K}_{\mu\nu}$, where $\tilde{K}_{\mu\nu}$ is the bulk-boundary propagator for $\tilde{t}(z, \bar{z})$. Since the latter is a pure diffeomorphism, so too is the former. As we will see, this happens for a reason: this commutator is directly related to the holomorphic current for super-translations, $j(z)$.

Using our now standard methodology, let us compute the correlator for this commutator,

$$\langle [\tilde{t}(z, \bar{z}), P_{00}] \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle \sim \sum_{i=1}^n \frac{\epsilon k_i}{k_i k_i} f_{\mu\nu}[J_i^{\mu\nu}, k_i^\rho] q_\rho \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle, \quad (4.41)$$

again using that $P_{00} = qP$ and $P^\mu = \sum_{i=1}^n k_i^\mu$. The above correlator simplifies to

$$\sim \sum_{i=1}^n \frac{(\epsilon k_i)^2}{k_i k_i} qk \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle \sim \sum_{i=1}^n \frac{\bar{z} - \bar{z}_i}{z - z_i} qk_i \langle \mathcal{O}(z_1, \bar{z}_1) \cdots \mathcal{O}(z_n, \bar{z}_n) \rangle, \quad (4.42)$$

where $qk = -1/2$ since k is projective but qk_i tracks the physical momentum of the hard particle in the q direction. As advertised, the right-hand side of this expression is precisely the leading Weinberg soft graviton factor [33] in our variables.

Comparing with eq. (4.40), we deduce that the holomorphic super-translation current is

$$j(z) = \partial_{\bar{z}} [\tilde{t}(z, \bar{z}), P_{00}]. \quad (4.43)$$

Since the bulk-boundary propagator for $[\tilde{t}(z, \bar{z}), P_{00}]$ is a pure diffeomorphism, so too is the one for $j(z)$. This suggests that there should again be a “bulk” topological description of the holomorphic 2D super-translation current, sensitive to the passage of hard particles.

While this result is encouraging, there are several reasons why such a topological description of super-translations cannot be a straightforward CS theory. First of all, from eq. (4.43), we see that $j(z)$ is not a primary operator, as would be the case for the dual of a CS gauge field, and is instead descendant from a commutator of $\tilde{t}(z, \bar{z})$. Relatedly, the global subgroup of super-translations, i.e. ordinary translations, transform under the $SL(2, \mathbb{C})$ Lorentz group, unlike the global subgroup of a Kac-Moody algebra dual to a CS theory, which is $SL(2, \mathbb{C})$ invariant. In any case, it would be very interesting to determine a bulk topological description for super-translations, if indeed one exists.

5 Future directions

A central result of this work is a recasting of 4D scattering amplitudes and their soft limits as correlators of a 2D CFT. In particular, we showed that soft fields in 4D gauge theory and gravity have a description in terms of 3D CS theory en route to a mapping onto 2D conserved currents via AdS_3/CFT_2 . Remarkably, a number of physically significant aspects of 4D — soft theorems, asymptotic symmetries, and memory effects — are elegantly encoded as 2D Ward identities, their associated Kac-Moody and Virasoro symmetries, and 3D Aharonov-Bohm type effects. Of course, the results presented here are but a first step in exploring the possible implications of AdS/CFT for flat space, and more generally, CS theories for describing soft gauge and gravitational phenomena. Many questions remain, offering numerous avenues for future work that we now discuss.

First and foremost, we would like to better understand the role of unitarity in the 2D CFT, which cannot itself be unitary nor even a Wick rotation of a unitary CFT. Rather, since time is emergent, so too must be unitarity, which will then be non-manifest in the

2D description. On the other hand, starting from unitary 4D scattering amplitudes the 2D correlators must still somehow encode unitarity. However, what we really seek is some *independent principle* within the CFT guaranteeing 4D unitarity.

Another open question relates to the role of 4D massive particles. The foliation approach taken here is in principle consistent with such a generalization, but there will surely be new subtleties. Certainly with massive particles, the Weyl invariance used to simplify even the free particle analysis will be lost, and a more general complex set of scaling dimensions will arise. Relatedly, massive particles will not actually reach null infinity, but must “sensed” sub-asymptotically.

More involved will be an extension of our results to loop level, where our foliation approach should apply. With loops, it is likely that the CS description for soft gauge boson modes will have a level which depends on the infrared scale separating “hard” from “soft”, due to the running of the gauge coupling. An obvious exception is if the gauge coupling is at an infrared fixed point, in which case there may be a non-perturbative level free of infrared scale dependence. It would be interesting to understand whether the usual level quantization of CS theory implies that only 4D gauge theories with suitably quantized gauge couplings have a non-perturbative CS soft limit.

Furthermore, it was shown in [86] and [87] that the subleading soft theorems of gauge theory and gravity are valid at tree level but are corrected at one-loop and higher. Interestingly, these corrections appear to be critically tied to infrared divergences [88]. This is naively quite disturbing because we saw that the subleading soft theorem for gravity is at the root of the Ward identities for the 2D stress tensor.⁷ However, more carefully examined, there need be no actual conflict. The Ward identity for the 2D stress tensor is *related but not equal* to the subleading graviton soft theorem, which is corrected at one loop. In fact, the complicated angular convolution in eq. (4.20) implies a highly non-trivial prescription for LSZ reduction that must be applied to the amplitude from the start. It is possible that at loop level, the 2D stress tensor continues to exist with some modified relationship to the Minkowski soft limit. In any case, it is of utmost importance to study the robustness of our picture at loop level.

A distinct but related question is to what extent the subleading soft theorems for gauge theory and the subsubleading soft theorems for gravity — which are known to be universal at tree level — might arise within the structure of the CFT₂. For example, from the CFT perspective, new *non-conserved* vector currents should robustly arise from taking the conserved limit of non-conserved tensor operators [89], which are AdS/CFT dual to the KK “graviphoton” of the effective compactification implied by the soft limit.

The OPE is a central feature of any CFT, which in the present context corresponds to the structure of 4D collinear singularities. This suggests that the CFT structure may facilitate some constructive method for building scattering amplitudes from collinear data. This is reminiscent of the BCFW recursion relations, which when reduced down to three-particle amplitudes effectively does this. On the other hand, the importance of self-dual

⁷The Ward identities for holomorphic conserved currents on the other hand arise from the leading gauge and gravity soft theorems which are *not* loop-corrected, and are therefore unthreatened.

configurations and the appearance of natural reference spinors η and $\bar{\eta}$ throughout the discussion might naturally connect with CSW rule constructions for scattering amplitudes. The focus on soft limits and collinear singularities also suggests connections with soft-collinear effective theory [90, 91], which may well be important for a loop-level formulation of asymptotic symmetries and the ideas presented in this paper.

There is also the question of whether our results can shed new light on the information paradox. As proposed in [67], soft “hair” could offer an intriguing caveat to the usual picture of black hole information loss. Nevertheless, stated purely in terms of soft radiation and gauge and gravitational memories, it is unclear how such a classical effect can resolve the paradox. On the other hand, our results connect these effects to Aharonov-Bohm effects on the celestial sphere, which may offer a more quantum mechanical approach to this problem. Also deserving of further study is our toy model for black hole horizons coming from the Rindler horizon of Minkowski spacetime. In our picture, the restriction to the Rindler region revealed an extension of the CFT structure onto the past and future boundary of the horizon — effectively the dS/CFT dual of the past and future wavefunction of the horizon. Here, the CFT gives a description of this horizon, extending the notion of asymptotic symmetries in its presence. It would be interesting if these features, especially those related to topological structure of memories, extended to real black holes in less symmetric spacetimes.

Finally, it would be worthwhile to see if the foliation approach followed here can be applied to spacetimes other than Mink_4 , for example AdS_4 , to uncover new symmetries and topological features emerging in special limits.

Acknowledgments

C.C. is supported by a Sloan Research Fellowship and a DOE Early Career Award under Grant No. DE-SC0010255. A.D. and R.S. are supported in part by the NSF under Grant No. PHY-1315155 and by the Maryland Center for Fundamental Physics. R.S. would also like to thank the Gordon and Betty Moore Foundation for the award of a Moore Distinguished Scholar Fellowship to visit Caltech, as well as the hospitality of the Walter Burke Institute for Theoretical Physics, where a substantial part of this work was completed. The authors are grateful to Nima Arkani-Hamed, Ricardo Caldeira Costa, Liam Fitzpatrick, Ted Jacobson, Dan Kapec, Jared Kaplan, Juan Maldacena, Ira Rothstein, and Anthony Speranza for useful discussions and comments.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](https://arxiv.org/abs/hep-th/9711200)] [[INSPIRE](https://inspirehep.net/literature/47588)].

- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large- N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [[hep-th/9905111](#)] [[INSPIRE](#)].
- [5] J. Polchinski, *Introduction to gauge/gravity duality*, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2010)*, (2010), pg. 3 [[arXiv:1010.6134](#)] [[INSPIRE](#)].
- [6] R. Sundrum, *From fixed points to the fifth dimension*, *Phys. Rev. D* **86** (2012) 085025 [[arXiv:1106.4501](#)] [[INSPIRE](#)].
- [7] J. Penedones, *TASI lectures on AdS/CFT*, [arXiv:1608.04948](#) [[INSPIRE](#)].
- [8] J. Polchinski, *S matrices from AdS space-time*, [hep-th/9901076](#) [[INSPIRE](#)].
- [9] L. Susskind, *Holography in the flat space limit*, [hep-th/9901079](#) [[INSPIRE](#)].
- [10] M. Gary, S.B. Giddings and J. Penedones, *Local bulk S-matrix elements and CFT singularities*, *Phys. Rev. D* **80** (2009) 085005 [[arXiv:0903.4437](#)] [[INSPIRE](#)].
- [11] J. Penedones, *Writing CFT correlation functions as AdS scattering amplitudes*, *JHEP* **03** (2011) 025 [[arXiv:1011.1485](#)] [[INSPIRE](#)].
- [12] A.L. Fitzpatrick and J. Kaplan, *Scattering states in AdS/CFT*, [arXiv:1104.2597](#) [[INSPIRE](#)].
- [13] A.L. Fitzpatrick and J. Kaplan, *Analyticity and the holographic S-matrix*, *JHEP* **10** (2012) 127 [[arXiv:1111.6972](#)] [[INSPIRE](#)].
- [14] A.L. Fitzpatrick and J. Kaplan, *Unitarity and the holographic S-matrix*, *JHEP* **10** (2012) 032 [[arXiv:1112.4845](#)] [[INSPIRE](#)].
- [15] M.F. Paulos, J. Penedones, J. Toledo, B.C. van Rees and P. Vieira, *The S-matrix bootstrap I: QFT in AdS*, [arXiv:1607.06109](#) [[INSPIRE](#)].
- [16] M.F. Paulos, J. Penedones, J. Toledo, B.C. van Rees and P. Vieira, *The S-matrix bootstrap II: two dimensional amplitudes*, [arXiv:1607.06110](#) [[INSPIRE](#)].
- [17] J. de Boer and S.N. Solodukhin, *A holographic reduction of Minkowski space-time*, *Nucl. Phys. B* **665** (2003) 545 [[hep-th/0303006](#)] [[INSPIRE](#)].
- [18] S.N. Solodukhin, *Reconstructing Minkowski space-time*, *IRMA Lect. Math. Theor. Phys.* **8** (2005) 123 [[hep-th/0405252](#)] [[INSPIRE](#)].
- [19] P.D. Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*, *Grad. Texts Contemp. Phys.*, Springer, New York U.S.A. (1997) [[INSPIRE](#)].
- [20] A. Strominger, *Asymptotic symmetries of Yang-Mills theory*, *JHEP* **07** (2014) 151 [[arXiv:1308.0589](#)] [[INSPIRE](#)].
- [21] T. He, P. Mitra, A.P. Porfyriadis and A. Strominger, *New symmetries of massless QED*, *JHEP* **10** (2014) 112 [[arXiv:1407.3789](#)] [[INSPIRE](#)].
- [22] T. He, P. Mitra and A. Strominger, *2D Kac-Moody symmetry of 4D Yang-Mills theory*, *JHEP* **10** (2016) 137 [[arXiv:1503.02663](#)] [[INSPIRE](#)].

- [23] D. Kapec, M. Pate and A. Strominger, *New symmetries of QED*, [arXiv:1506.02906](#) [[INSPIRE](#)].
- [24] A. Strominger, *Magnetic corrections to the soft photon theorem*, *Phys. Rev. Lett.* **116** (2016) 031602 [[arXiv:1509.00543](#)] [[INSPIRE](#)].
- [25] H. Bondi, M.G.J. van der Burg and A.W.K. Metzner, *Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems*, *Proc. Roy. Soc. Lond. A* **269** (1962) 21 [[INSPIRE](#)].
- [26] R.K. Sachs, *Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times*, *Proc. Roy. Soc. Lond. A* **270** (1962) 103 [[INSPIRE](#)].
- [27] G. Barnich and C. Troessaert, *Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited*, *Phys. Rev. Lett.* **105** (2010) 111103 [[arXiv:0909.2617](#)] [[INSPIRE](#)].
- [28] A. Strominger, *On BMS invariance of gravitational scattering*, *JHEP* **07** (2014) 152 [[arXiv:1312.2229](#)] [[INSPIRE](#)].
- [29] T. He, V. Lysov, P. Mitra and A. Strominger, *BMS supertranslations and Weinberg's soft graviton theorem*, *JHEP* **05** (2015) 151 [[arXiv:1401.7026](#)] [[INSPIRE](#)].
- [30] D. Kapec, V. Lysov, S. Pasterski and A. Strominger, *Semiclassical Virasoro symmetry of the quantum gravity S-matrix*, *JHEP* **08** (2014) 058 [[arXiv:1406.3312](#)] [[INSPIRE](#)].
- [31] V. Lysov, S. Pasterski and A. Strominger, *Low's subleading soft theorem as a symmetry of QED*, *Phys. Rev. Lett.* **113** (2014) 111601 [[arXiv:1407.3814](#)] [[INSPIRE](#)].
- [32] T.T. Dumitrescu, T. He, P. Mitra and A. Strominger, *Infinite-dimensional fermionic symmetry in supersymmetric gauge theories*, [arXiv:1511.07429](#) [[INSPIRE](#)].
- [33] S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516 [[INSPIRE](#)].
- [34] F.E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, *Phys. Rev.* **110** (1958) 974 [[INSPIRE](#)].
- [35] T.H. Burnett and N.M. Kroll, *Extension of the low soft photon theorem*, *Phys. Rev. Lett.* **20** (1968) 86 [[INSPIRE](#)].
- [36] C.D. White, *Factorization properties of soft graviton amplitudes*, *JHEP* **05** (2011) 060 [[arXiv:1103.2981](#)] [[INSPIRE](#)].
- [37] F. Cachazo and A. Strominger, *Evidence for a new soft graviton theorem*, [arXiv:1404.4091](#) [[INSPIRE](#)].
- [38] L. Bieri and D. Garfinkle, *An electromagnetic analogue of gravitational wave memory*, *Class. Quant. Grav.* **30** (2013) 195009 [[arXiv:1307.5098](#)] [[INSPIRE](#)].
- [39] S. Pasterski, *Asymptotic symmetries and electromagnetic memory*, [arXiv:1505.00716](#) [[INSPIRE](#)].
- [40] L. Susskind, *Electromagnetic memory*, [arXiv:1507.02584](#) [[INSPIRE](#)].
- [41] Y. Zeldovich and A. Polnarev, *Radiation of gravitational waves by a cluster of superdense stars*, *Sov. Astron.* **18** (1974) 17.
- [42] V.B. Braginsky and K.S. Thorne, *Gravitational-wave bursts with memory and experimental prospects*, *Nature* **327** (1987) 123.
- [43] D. Christodoulou, *Nonlinear nature of gravitation and gravitational wave experiments*, *Phys. Rev. Lett.* **67** (1991) 1486 [[INSPIRE](#)].

- [44] A. Strominger and A. Zhiboedov, *Gravitational memory, BMS supertranslations and soft theorems*, *JHEP* **01** (2016) 086 [[arXiv:1411.5745](#)] [[INSPIRE](#)].
- [45] S. Pasterski, A. Strominger and A. Zhiboedov, *New gravitational memories*, *JHEP* **12** (2016) 053 [[arXiv:1502.06120](#)] [[INSPIRE](#)].
- [46] Y. Aharonov and D. Bohm, *Significance of electromagnetic potentials in the quantum theory*, *Phys. Rev.* **115** (1959) 485 [[INSPIRE](#)].
- [47] G.W. Moore and N. Read, *Nonabelions in the fractional quantum Hall effect*, *Nucl. Phys. B* **360** (1991) 362 [[INSPIRE](#)].
- [48] X.G. Wen, *Non-Abelian statistics in the fractional quantum Hall states*, *Phys. Rev. Lett.* **66** (1991) 802 [[INSPIRE](#)].
- [49] Y.-T. Chien, M.D. Schwartz, D. Simmons-Duffin and I.W. Stewart, *Jet physics from static charges in AdS*, *Phys. Rev. D* **85** (2012) 045010 [[arXiv:1109.6010](#)] [[INSPIRE](#)].
- [50] M. Campiglia and A. Laddha, *Asymptotic symmetries of QED and Weinberg's soft photon theorem*, *JHEP* **07** (2015) 115 [[arXiv:1505.05346](#)] [[INSPIRE](#)].
- [51] R.N.C. Costa, *Holographic reconstruction and renormalization in asymptotically Ricci-flat spacetimes*, *JHEP* **11** (2012) 046 [[arXiv:1206.3142](#)] [[INSPIRE](#)].
- [52] A. Strominger, *The dS/CFT correspondence*, *JHEP* **10** (2001) 034 [[hep-th/0106113](#)] [[INSPIRE](#)].
- [53] E. Witten, *Quantum gravity in de Sitter space*, in *Strings 2001: International Conference*, Mumbai India January 5–10 2001 [[hep-th/0106109](#)] [[INSPIRE](#)].
- [54] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013 [[astro-ph/0210603](#)] [[INSPIRE](#)].
- [55] D. Anninos, T. Hartman and A. Strominger, *Higher spin realization of the dS/CFT correspondence*, *Class. Quant. Grav.* **34** (2017) 015009 [[arXiv:1108.5735](#)] [[INSPIRE](#)].
- [56] P.A.M. Dirac, *Wave equations in conformal space*, *Annals Math.* **37** (1936) 429 [[INSPIRE](#)].
- [57] L. Cornalba, M.S. Costa and J. Penedones, *Deep inelastic scattering in conformal QCD*, *JHEP* **03** (2010) 133 [[arXiv:0911.0043](#)] [[INSPIRE](#)].
- [58] S. Weinberg, *Six-dimensional methods for four-dimensional conformal field theories*, *Phys. Rev. D* **82** (2010) 045031 [[arXiv:1006.3480](#)] [[INSPIRE](#)].
- [59] M.S. Costa, J. Penedones, D. Poland and S. Rychkov, *Spinning conformal correlators*, *JHEP* **11** (2011) 071 [[arXiv:1107.3554](#)] [[INSPIRE](#)].
- [60] M.S. Costa, V. Gonçalves and J. Penedones, *Spinning AdS propagators*, *JHEP* **09** (2014) 064 [[arXiv:1404.5625](#)] [[INSPIRE](#)].
- [61] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett. B* **214** (1988) 215 [[INSPIRE](#)].
- [62] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [[hep-th/0312171](#)] [[INSPIRE](#)].
- [63] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [[INSPIRE](#)].
- [64] S. Elitzur, G.W. Moore, A. Schwimmer and N. Seiberg, *Remarks on the canonical quantization of the Chern-Simons-Witten theory*, *Nucl. Phys. B* **326** (1989) 108 [[INSPIRE](#)].

- [65] E. Witten, *On holomorphic factorization of WZW and coset models*, *Commun. Math. Phys.* **144** (1992) 189 [[INSPIRE](#)].
- [66] S. Gukov, E. Martinec, G.W. Moore and A. Strominger, *Chern-Simons gauge theory and the AdS_3/CFT_2 correspondence*, [hep-th/0403225](#) [[INSPIRE](#)].
- [67] S.W. Hawking, M.J. Perry and A. Strominger, *Soft hair on black holes*, *Phys. Rev. Lett.* **116** (2016) 231301 [[arXiv:1601.00921](#)] [[INSPIRE](#)].
- [68] D. Harlow, *Jerusalem lectures on black holes and quantum information*, *Rev. Mod. Phys.* **88** (2016) 15002 [[arXiv:1409.1231](#)] [[INSPIRE](#)].
- [69] E. Witten, *$(2+1)$ -dimensional gravity as an exactly soluble system*, *Nucl. Phys. B* **311** (1988) 46 [[INSPIRE](#)].
- [70] E. Witten, *Three-dimensional gravity revisited*, [arXiv:0706.3359](#) [[INSPIRE](#)].
- [71] G. Barnich and C. Troessaert, *BMS charge algebra*, *JHEP* **12** (2011) 105 [[arXiv:1106.0213](#)] [[INSPIRE](#)].
- [72] D. Kapec, P. Mitra, A.-M. Raclariu and A. Strominger, *A 2D stress tensor for 4D gravity*, [arXiv:1609.00282](#) [[INSPIRE](#)].
- [73] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, *Camb. Monogr. Math. Phys.*, Cambridge Univ. Press, Cambridge U.K. (1984) [[INSPIRE](#)].
- [74] P. Breitenlohner and D.Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauged extended supergravity*, *Phys. Lett. B* **115** (1982) 197 [[INSPIRE](#)].
- [75] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Annals Phys.* **144** (1982) 249 [[INSPIRE](#)].
- [76] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, *Selfduality in odd dimensions*, *Phys. Lett. B* **136** (1984) 38 [*Addendum ibid.* **B 137** (1984) 443] [[INSPIRE](#)].
- [77] R. Monteiro and D. O’Connell, *The kinematic algebra from the self-dual sector*, *JHEP* **07** (2011) 007 [[arXiv:1105.2565](#)] [[INSPIRE](#)].
- [78] Z. Bern, J.J.M. Carrasco and H. Johansson, *New relations for gauge-theory amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [[arXiv:0805.3993](#)] [[INSPIRE](#)].
- [79] S. Axelrod and I.M. Singer, *Chern-Simons perturbation theory*, in *Differential geometric methods in theoretical physics. Proceedings, 20th International Conference*, New York U.S.A. June 3–7 1991, pg. 3 [[hep-th/9110056](#)] [[INSPIRE](#)].
- [80] S. Axelrod and I.M. Singer, *Chern-Simons perturbation theory. II*, *J. Diff. Geom.* **39** (1994) 173 [[hep-th/9304087](#)] [[INSPIRE](#)].
- [81] H. Leutwyler, *A $(2+1)$ -dimensional model for the quantum theory of gravity*, *Nuovo Cim. A* **42** (1966) 159.
- [82] A. Staruszkiewicz, *Gravitation theory in three-dimensional space*, *Acta Phys. Polon.* **24** (1963) 735 [[INSPIRE](#)].
- [83] H. Kawai, D.C. Lewellen and S.-H. Henry Tye, *A relation between tree amplitudes of closed and open strings*, *Nucl. Phys. B* **269** (1986) 1 [[INSPIRE](#)].
- [84] S. Oxburgh and C.D. White, *BCJ duality and the double copy in the soft limit*, *JHEP* **02** (2013) 127 [[arXiv:1210.1110](#)] [[INSPIRE](#)].

- [85] J. Cardy, *The ubiquitous ‘c’: from the Stefan-Boltzmann law to quantum information*, *J. Stat. Mech.* **10** (2010) P10004 [[arXiv:1008.2331](#)] [[INSPIRE](#)].
- [86] Z. Bern, S. Davies and J. Nohle, *On loop corrections to subleading soft behavior of gluons and gravitons*, *Phys. Rev. D* **90** (2014) 085015 [[arXiv:1405.1015](#)] [[INSPIRE](#)].
- [87] S. He, Y.-T. Huang and C. Wen, *Loop corrections to soft theorems in gauge theories and gravity*, *JHEP* **12** (2014) 115 [[arXiv:1405.1410](#)] [[INSPIRE](#)].
- [88] Z. Bern, S. Davies, P. Di Vecchia and J. Nohle, *Low-energy behavior of gluons and gravitons from gauge invariance*, *Phys. Rev. D* **90** (2014) 084035 [[arXiv:1406.6987](#)] [[INSPIRE](#)].
- [89] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, *Class. Quant. Grav.* **30** (2013) 104003 [[arXiv:1204.3882](#)] [[INSPIRE](#)].
- [90] I. Stewart and C. Bauer, *The soft-collinear effective theory*, https://courses.edx.org/c4x/MITx/8.EFTx/asset/notes_sctnotes.pdf.
- [91] A.J. Larkoski, D. Neill and I.W. Stewart, *Soft theorems from effective field theory*, *JHEP* **06** (2015) 077 [[arXiv:1412.3108](#)] [[INSPIRE](#)].