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Wilsonian effective action of superstring theory

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Abstract: By integrating out the heavy fields in type II or heterotic string field theory one can construct the effective action for the light fields. This effective theory inherits all the algebraic structures of the parent theory and the effective action automatically satisfies the Batalin-Vilkovisky quantum master equation. This theory is manifestly ultraviolet finite, has only light fields as its explicit degrees of freedom, and the Feynman diagrams of this theory reproduce the exact scattering amplitudes of light states in string theory to any arbitrary order in perturbation theory. Furthermore in this theory the degrees of freedom of light fields above certain energy scale are also implicitly integrated out. This energy scale is determined by a particular parameter labelling a family of equivalent actions, and can be made arbitrarily low, leading to the interpretation of the effective action as the Wilsonian effective action.

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1 Introduction

The original motivation for string field theory was to develop tools for studying non-perturbative aspects of string theory. For open string field theory this goal has been partially realized in the form of non-trivial classical solutions that are not accessible to perturbation theory [1]. Similar success for closed strings has not been forthcoming despite some tantalizing numerical results [2]. Nevertheless superstring field theory can provide us with practical tools for addressing issues that arise within perturbation theory, e.g. in dealing with situations where the masses of external states get renormalized or the classical vacuum gets destabilized and new stable vacua arise near the original classical vacuum [3].

Superstring field theory has infinite number of fields with masses going all the way up to infinity. We may want to consider a situation where the only states that participate in the scattering are 'light' particles — those that are massless at string tree level — but we want to include the effect of higher derivative corrections as well as loop corrections where heavy string states propagate in the loop. In this case the most useful description will be provided by an effective action of light fields obtained by integrating out the heavy state contributions. This effective action will still be capable of reproducing the full string theory amplitudes involving light external states to any arbitrary order in perturbation theory. In particular, like the parent string field theory, this effective field theory will be free from ultraviolet (UV) divergences. Our goal in this paper will be to analyze the structure of such an effective action.

Our main result is that the interaction vertices of the effective field theory so constructed satisfy the same algebraic identities as those of the parent string field theory [4].

It then follows that the effective action satisfies the Batalin-Vilkovisky (BV) master equation and can be quantized using the BV formalism [5–9]. Furthermore following the analysis of [10, 11] one can show that in this effective action the modes of the light fields above a certain energy scale are also integrated out implicitly. This energy scale is determined by a parameter, known as the stub length, that the effective action inherits from the original action [12–14]. By varying this parameter we can change this energy scale, which also determines the effective UV cut-off of the theory. Therefore the effective action so constructed can be regarded as the Wilsonian effective action [15–17] of superstring field theory.

Given this effective action, one can use it to compute any quantity involving the light fields, e.g. S-matrix elements, using standard Feynman rules. The absence of UV divergences is made manifest by the exponential suppression of the vertices at large euclidean momenta provided one takes the ends of the integration contours over loop energies to be at $\pm i\infty$ [18]. Furthermore as long as the energies of the external states remain below the threshold of production of the heavy particles, the loop energy integration contours can be chosen to avoid the singularities in the vertices arising from the effect of integrating out the heavy particles.

It is also possible to construct the one particle irreducible (1PI) effective action of the light fields by also integrating out the light fields propagating in the loops. This satisfies a different set of properties which can also be derived using the same general procedure. Such an effective action can be useful for finding the correct vacuum, and computing the renormalized masses of light fields around this vacuum. For example the entire analysis of [3] could be carried out using such an effective action without having to deal with the heavy string states. The final result of course will remain unchanged.

The rest of the paper is organized as follows. In section 2 we review some of the algebraic structures that arise in the conventional superstring field theory, and derive some Ward identities for the off-shell amplitudes of this theory. In section 3 we integrate out the heavy fields of the theory and use manipulations similar to those in section 2 to derive certain identities involving the interaction vertices of this effective field theory. These identities are then used to show that the resulting effective action satisfies the BV master equation. We also describe the construction of the 1PI action of the light fields and show that this satisfies the classical master equation. In section 4 we describe the role of the stub length parameter in controlling the energy scale above which the modes of the light fields in the effective action are also implicitly integrated out. We conclude in section 5 with a discussion of our results and possible generalizations where, instead of integrating out all the heavy modes of the string, we integrate out the modes above a certain mass level and construct an effective theory of string fields below that mass level. In appendix A we describe explicitly the algorithm for constructing the vertices of the effective action using off-shell amplitudes of light states only. In appendix B we describe how a set of spurious fields which are present in the effective action do not play any role in the evaluation of Feynman diagrams.

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2 Algebraic structures in superstring field theory

We shall first review some algebraic structures that appear in the construction of superstring field theories [4] generalizing the corresponding results in closed bosonic string field theory [9, 12]. We shall denote by $\mathcal{H}_T$ the subspace of GSO even states in the matter ghost superconformal field theory (SCFT) satisfying

$$b_0^-|s\rangle = 0, \quad L_0^-|s\rangle = 0, \quad \text{for } |s\rangle \in \mathcal{H}_T\,, \tag{2.1}$$

where

$$b_0^\pm = b_0 \pm \bar{b}_0, \quad c_0^\pm = \frac{1}{2}(c_0 \pm \bar{c}_0), \quad L_0^\pm = L_0 \pm \bar{L}_0\,. \tag{2.2}$$

$L_n, \bar{L}_n$ are the total Virasoro generators, and $b_n, \bar{b}_n, c_n, \bar{c}_n$ are the modes of the usual $b, \bar{b}, c, \bar{c}$ ghost fields. We do not make any assumption about the background except that it is described by some arbitrary world-sheet SCFT. For heterotic string theory we shall denote by $\mathcal{H}_m$ the subspace of states in $\mathcal{H}_T$ carrying picture number m, whereas for type II string theories $\mathcal{H}_{m,n}$ will denote the subspace of states in $\mathcal{H}_T$ carrying left-moving picture number m and right-moving picture number n. We also define

$$\begin{aligned}
&\text{for heterotic :} \quad \widehat{\mathcal{H}}_T \equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-1/2}, \quad \widetilde{\mathcal{H}}_T \equiv \mathcal{H}_{-1} \oplus \mathcal{H}_{-3/2}\,, \\
&\text{for type II :} \begin{cases} \widehat{\mathcal{H}}_T \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1/2,-1} \oplus \mathcal{H}_{-1,-1/2} \oplus \mathcal{H}_{-1/2,-1/2} \\ \widetilde{\mathcal{H}}_T \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-3/2,-1} \oplus \mathcal{H}_{-1,-3/2} \oplus \mathcal{H}_{-3/2,-3/2}\,. \end{cases}
\end{aligned} \tag{2.3}$$

For both heterotic and type II string theories we take $|\varphi_r\rangle \in \widehat{\mathcal{H}}_T$, $|\varphi_r^c\rangle \in \widetilde{\mathcal{H}}_T$ to be appropriate basis states satisfying

$$\langle \varphi_r^c | c_0^- | \varphi_s \rangle = \delta_{rs}\,, \quad \langle \varphi_r | c_0^- | \varphi_s^c \rangle = \delta_{rs}\,. \tag{2.4}$$

The second relation follows from the first. Eq. (2.4) implies the completeness relation

$$\sum_r |\varphi_r\rangle\langle\varphi_r^c| c_0^- = \mathbf{1}, \quad \sum_r |\varphi_r^c\rangle\langle\varphi_r| c_0^- = \mathbf{1}\,, \tag{2.5}$$

acting on states in $\widehat{\mathcal{H}}_T$ and $\widetilde{\mathcal{H}}_T$ respectively. The basis states φ_r and φ_r^c will in general carry non-trivial grassmann parities which we shall denote by $(-1)^{\gamma_r}$ and $(-1)^{\gamma_r^c}$ respectively. In the NS sector of the heterotic theory and the NSNS and RR sector of type II theory, the grassmann parity of φ_r or φ_r^c is odd (even) if the ghost number of φ_r or φ_r^c is odd (even). In the R sector of the heterotic theory and the RNS and NSR sector of the type II theory, the grassmann parities of the states will have opposite correlation with the ghost number. It follows from the ghost number conservation rule and (2.4) that

$$(-1)^{\gamma_r + \gamma_r^c} = -1\,. \tag{2.6}$$

For heterotic string theories we shall denote by $\mathcal{G}$ the identity operator in the NS sector and the zero mode of the picture changing operator (PCO) in the R sector. For type II string theories $\mathcal{G}$ will be defined as the identity operator in the NSNS sector, zero mode of the left-moving picture number in the RNS sector, zero mode of the right-moving picture number in the NSR sector, and product of the zero modes of the left-moving and right-moving picture numbers in the RR sector. $\mathcal{G}$ satisfies

$$[\mathcal{G}, L_0^\pm] = 0, \qquad [\mathcal{G}, b_0^\pm] = 0\,, \qquad [\mathcal{G}, Q_B] = 0\,, \tag{2.7}$$

where Q_B is the BRST charge.

Using the standard identification between the wave-function of the first quantized theory and fields in the second quantized theory, the fields in string field theory are represented as states in the SCFT. In the full BV quantized theory there are two sets of string fields: $|\Psi\rangle \in \widehat{\mathcal{H}}_T$ and $|\widetilde{\Psi}\rangle \in \widetilde{\mathcal{H}}_T$ without any further restriction. The action is given by[1]

$$S = \frac{1}{g_S^2}\left[-\frac{1}{2}\langle\widetilde{\Psi}|c_0^- Q_B \mathcal{G}|\widetilde{\Psi}\rangle + \langle\widetilde{\Psi}|c_0^- Q_B|\Psi\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\{\!\{\Psi^n\}\!\}\right], \tag{2.8}$$

where g_S is the string coupling, and $\{\!\{A_1 \cdots A_N\}\!\}$ is a multilinear function of $|A_1\rangle, \cdots |A_N\rangle \in \widehat{\mathcal{H}}_T$, constructed by first computing the correlation functions of these vertex operators together with PCO's and other ghost insertions on Riemann surfaces, and then integrating the result over certain *subspaces* of the moduli space of Riemann surfaces with punctures. The string fields $|\Psi\rangle$ and $|\widetilde{\Psi}\rangle$ are grassmann even i.e. the coefficient of a grassmann even (odd) basis state in the expansion of the string field is grassmann even (odd). In order to avoid cluttering up various formulæ due to sign factors, we shall multiply the grassmann odd vertex operators of the CFT by grassmann odd c-numbers and work with states A_i which are all grassmann even. Whenever needed we can always strip off these factors from both sides of an equation at the cost of picking up appropriate signs. With this convention $\{\!\{A_1 \cdots A_N\}\!\}$ is symmetric under exchange of external states and satisfies the relation,

$$\begin{aligned}
&\sum_{i=1}^{N}\{\!\{A_1 \cdots A_{i-1}(Q_B A_i)A_{i+1}\cdots A_N\}\!\} \\
&= -\frac{1}{2}\sum_{\substack{\ell,k\ge 0\\ \ell+k=N}}\ \sum_{\substack{\{i_a;a=1,\cdots \ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}}\{\!\{A_{i_1}\cdots A_{i_\ell}\varphi_s\}\!\}\{\!\{\varphi_r A_{j_1}\cdots A_{j_k}\}\!\}\langle\varphi_s^c|c_0^-\mathcal{G}|\varphi_r^c\rangle \\
&\quad -\frac{1}{2}\,g_S^2\,\{\!\{A_1\cdots A_N\varphi_s\varphi_r\}\!\}\,\langle\varphi_s^c|c_0^-\mathcal{G}|\varphi_r^c\rangle\,.
\end{aligned} \tag{2.9}$$

Using this relation one can show that the action (2.8) satisfies BV master equation [4] and can be quantized in the Siegel gauge $b_0^+|\Psi\rangle = 0$, $b_0^+|\widetilde{\Psi}\rangle = 0$.

We shall now describe the Siegel gauge propagator [3]. Since only the Ψ field appears in the interaction, the relevant propagator is the $\Psi - \Psi$ propagator. Instead of giving its expression directly we shall describe it by its operation of joining two Feynman diagrams. Let us suppose that $f(A_1, \cdots A_m, \varphi_s)$ denotes the contribution to the off-shell

[1] We shall work in the convention in which the path integral is carried out with the weight factor e^S. If we want to change this, e.g. use the weight factor e^{-S} or e^{iS}, we can achieve this by replacing g_S^2 by $-g_S^2$ or $-ig_S^2$ in all subsequent formulæ.

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amplitude[2] from a specific Feynman diagram with external states $A_1, \cdots A_m, \varphi_s \in \widehat{\mathcal{H}}_T$ and $g(B_1, \cdots B_n, \varphi_r)$ denotes the contribution from another Feynman diagram with external states $B_1, \cdots B_n, \varphi_r \in \widehat{\mathcal{H}}_T$. In both we use a normalization such that the contribution to $f(A_1, \cdots A_m, \varphi_s)$ from the elementary vertex is just $\{\!\{A_1 \cdots A_m \varphi_s\}\!\}$, and similarly for g. Generically f and g have no symmetry property since we are not considering sum over all diagrams. Now we can construct another Feynman diagram with external states $A_1, \cdots A_m, B_1, \cdots B_n$ by joining φ_s and φ_r by a propagator, and summing over s and r. Its contribution is given by[3]

$$-\, f(A_1, \cdots A_m, \varphi_s)\, g(B_1, \cdots B_n, \varphi_r)\, \langle \varphi_s^c | c_0^- b_0^+ (L_0^+)^{-1} \mathcal{G} | \varphi_r^c \rangle \,. \tag{2.10}$$

Note that f and/or g may have odd grassmann parity from the grassmann odd numbers hidden inside the A_i's, so one should be careful about their relative positioning. Similarly if $f(A_1, \cdots A_n, \varphi_s, \varphi_r)$ denotes a Feynman diagram with external states $A_1, \cdots A_n, \varphi_s, \varphi_r$ and if we consider a new Feynman diagram obtained by joining φ_s and φ_r by a propagator and summing over all choices of φ_s, φ_r, the new Feynman diagram is given by

$$-\,\frac{1}{2} g_S^2\, f(A_1, \cdots A_m, \varphi_s, \varphi_r) \langle \varphi_s^c | c_0^- b_0^+ (L_0^+)^{-1} \mathcal{G} | \varphi_r^c \rangle \,. \tag{2.11}$$

The power of g_S reflects that this operation increases the number of loops in the diagram by 1. The factor of 1/2 is a combinatorial factor.

Using this propagator and the elementary vertices encoded in the interaction term in (2.8) one can compute off-shell amplitudes of string field theory. Using standard manipulations these can be expressed as integrals over the moduli space of Riemann surfaces, with the integrand given by the correlation function of vertex operators of external states, PCO's and ghost fields on the Riemann surface. The correlation function depends on the choice of local holomorphic coordinates around the punctures and the PCO locations. String field theory provides us with these data. It also gives us a cell decomposition of the moduli space such that the contribution from each cell can be identified with a Feynman diagram of the string field theory. These choices are not unique, but are tightly constrained, and different choices give equivalent string field theories related by field redefinition [14]. Given a string field theory we can not only define the full off-shell amplitude, but consider other quantities commonly used in quantum field theories e.g. 1PI amplitudes obtained by summing over certain subset of diagrams. In the language of Riemann surface, this means that we only integrate over certain subspaces of the moduli space. We shall now describe some properties of these amplitudes.

Let $G(A_1, \cdots A_N)$ be the full off-shell truncated Green's function with external states $A_1, \cdots A_N$, obtained by summing over all Feynman diagrams with external states $A_1, \cdots A_N$, but dropping the tree level propagators of the external states. We impose

[2]Throughout this paper we shall mean by off-shell amplitude the truncated Green's function where the tree level propagators for external states are dropped.

[3]The expression for the propagator is somewhat different from the standard one (see e.g. [3]) where instead of a c_0^- the propagator contained a b_0^-. This difference can be traced to the inclusion of the c_0^- in the normalization (2.4) of the basis states.

Siegel gauge condition on the internal states, but take the external states $A_1, \cdots A_N$ to be arbitrary elements of $\widehat{\mathcal{H}}_T$. Then $G(A_1, \cdots A_N)$ will be given by a sum of terms, each of which is given by a product of the propagators and vertices. Let us now consider the combination $\sum_{i=1}^N G(A_1, \cdots A_{i-1}, Q_B A_i,\ A_{i+1}, \cdots A_N)$. Since each A_i must come from some vertex $\{\!\{\cdots\}\!\}$ in a given Feynman diagram, the sum over i can be organized into subsets, where in a given subset Q_B acts on different external states of the same vertex. This can then be simplified using (2.9). This gives[4]

$$\begin{aligned}
&\sum_{i=1}^N G(A_1, \cdots A_{i-1}, Q_B A_i, A_{i+1} \cdots A_N) \qquad\qquad (2.12)\\
&= -\frac{1}{2} \sum_{\substack{\ell,k\geq 0\\ \ell+k=N}} \sum_{\substack{\{i_a;a=1,\cdots \ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}} G(A_{i_1}, \cdots A_{i_\ell}, \varphi_s) G(\varphi_r, A_{j_1} \cdots A_{j_k}) \langle \varphi_s^c | c_0^- \mathcal{G} | \varphi_r^c \rangle\\
&\quad -\frac{1}{2}\, g_S^2\, G(A_1, \cdots A_N, \varphi_s, \varphi_r)\, \langle \varphi_s^c | c_0^- \mathcal{G} | \varphi_r^c \rangle\\
&\quad -\frac{1}{2} \sum_{\substack{\ell,k\geq 0\\ \ell+k=N}} \sum_{\substack{\{i_a;a=1,\cdots \ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}} \Big[- G(A_{i_1}, \cdots A_{i_\ell}, Q_B\varphi_s) G(\varphi_r, A_{j_1} \cdots A_{j_k})\\
&\quad -(-1)^{\gamma_s} G(A_{i_1}, \cdots A_{i_\ell}, \varphi_s) G(Q_B\varphi_r, A_{j_1} \cdots A_{j_k}) \Big] \langle \varphi_s^c | c_0^- b_0^+ (L_0^+)^{-1} \mathcal{G} | \varphi_r^c \rangle\\
&\quad -\frac{g_S^2}{2} \Big[- G(A_1, \cdots A_N, Q_B\varphi_s, \varphi_r) - (-1)^{\gamma_s} G(A_1, \cdots A_N, \varphi_s, Q_B\varphi_r) \Big] \langle \varphi_s^c | c_0^- b_0^+ (L_0^+)^{-1} \mathcal{G} | \varphi_r^c \rangle .
\end{aligned}$$

The first two terms on the right hand side represent the contribution from the right hand side of (2.9) when we use (2.9) to simplify the contribution from individual vertices of the Feynman diagram. The other two terms on the right hand side come from the fact that while using (2.9) for a given vertex, we have to subtract the terms where Q_B acts on the legs of the vertex connected to internal propagators since on the left hand side of (2.12) Q_B only acts on the external states. The third term represents the contribution from a propagator that connects two disjoint Feynman diagrams, whereas the last term represents the contribution from a propagator that connects two external lines of a connected Feynman diagram. The overall minus signs in front of the third and the fourth terms come from having to move these from the left hand side of the equation, where they appear naturally, to the right hand side. The minus signs inside the square brackets come from the ones on the right hand sides of (2.10) and (2.11). The $(-1)^{\gamma_s}$ factors arise from having to move Q_B through φ_s. In the third term, we have included a factor of $1/2$ to compensate for

[4]To the best of our knowledge this form of the Ward identity has not been written down before even for closed bosonic string field theory. While its consequence, described in (2.15), has a standard form, we should add that this is not a convenient form for analyzing properties of on-shell amplitudes. The Green's function G has self energy insertions on the external legs and therefore diverges on-shell. One needs to work with its cousin Γ described in [3] where the full external propagators are removed and work with the Ward identities satisfied by Γ which take a different form. If there are tadpoles of light fields then G will be divergent even for off-shell external states and (2.12), (2.15) are formal. In this case one has to first construct the 1PI effective action, find its extremum and expand the action around this extremum to compute the off-shell amplitudes [3]. Nevertheless we present this analysis here since in section 3 we shall describe a similar analysis where all relevant quantities will be manifestly free from divergences.

the double counting associated with the $\{i_a\} \leftrightarrow \{j_b\}$ exchange. The 1/2 in the last factor arises from the right hand side of (2.11).

Using the completeness relation (2.5) we can now move Q_B inside the matrix element $\langle\varphi_s^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_r^c\rangle$ in the third and the fourth terms, e.g. we have

$$Q_B|\varphi_s\rangle\langle\varphi_s^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_r^c\rangle = Q_B b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_r^c\rangle = |\varphi_s\rangle\langle\varphi_s^c|c_0^- Q_B b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_r^c\rangle, \quad (2.13)$$

and

$$\begin{aligned}(-1)^{\gamma_s} Q_B|\varphi_r\rangle\langle\varphi_s^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_r^c\rangle &= Q_B|\varphi_r\rangle\langle\varphi_r^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_s^c\rangle = Q_B b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_s^c\rangle \\ = |\varphi_r\rangle\langle\varphi_r^c|c_0^- Q_B b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_s^c\rangle &= |\varphi_r\rangle\langle\varphi_s^c|c_0^- b_0^+ Q_B (L_0^+)^{-1}\mathcal{G}|\varphi_r^c\rangle \,. \end{aligned} \quad (2.14)$$

Using the relations $Q_B b_0^+ + b_0^+ Q_B = L_0^+$ and (2.7) one can now show that on the right hand side of (2.12) the third term cancels the first term and the fourth term cancels the second term. This gives us the Ward identity for the off-shell Green's function

$$\sum_{i=1}^{N} G(A_1, \cdots A_{i-1}, Q_B A_i, A_{i+1} \cdots A_N) = 0 \,. \quad (2.15)$$

As a simple example we can consider the tree level four point function, given by the sum of s, t and u-channel diagrams and the four point vertex. As a direct consequence of (2.9), the contribution of the 4-point vertex to the right hand side of (2.12) will be given by

$$\begin{aligned}&-G(A_1, A_2, \varphi_s)G(\varphi_r, A_3, A_4)\langle\varphi_s^c|c_0^-\mathcal{G}|\varphi_r^c\rangle - G(A_1, A_3, \varphi_s)G(\varphi_r, A_2, A_4)\langle\varphi_s^c|c_0^-\mathcal{G}|\varphi_r^c\rangle \\ &-G(A_1, A_4, \varphi_s)G(\varphi_r, A_2, A_3)]\langle\varphi_s^c|c_0^-\mathcal{G}|\varphi_r^c\rangle \,. \end{aligned} \quad (2.16)$$

The contribution from the s-channel diagram will be given by

$$\begin{aligned}&(G(A_1, A_2, Q_B\varphi_s)G(\varphi_r, A_3, A_4) \\ &+ (-1)^{\gamma_s} G(A_1, A_2, \varphi_s)G(Q_B\varphi_r, A_3, A_4))\langle\varphi_s^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_r^c\rangle \,. \end{aligned} \quad (2.17)$$

After moving Q_B inside the matrix element $\langle\varphi_s^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}|\varphi_r^c\rangle$ using the completeness relation, and using $Q_B b_0^+ + b_0^+ Q_B = L_0^+$, this cancels the first term in (2.16). Similarly the contribution from the t and u-channel diagrams cancel the second and third terms in (2.16).

An identity similar to (2.15) can be derived for the 1PI amplitudes. Let $\{A_1 \cdots A_n\}$ denote the 1PI amplitude of the external states $A_1, \cdots A_n$. This will satisfy an identity similar to (2.12) with $G(A_1, \cdots A_n)$ replaced by $\{A_1 \cdots A_n\}$, and without the third term on the right hand side of (2.12). This is due to the fact that by definition, 1PI amplitudes do not include sum over Feynman diagrams in which a single propagator connects two other Feynman diagrams. Therefore the first term on the right hand side remains uncanceled and we arrive at the identity:

$$\begin{aligned}&\sum_{i=1}^{N}\{A_1 \cdots A_{i-1}(Q_B A_i)A_{i+1} \cdots A_N\} \\ &= -\frac{1}{2}\sum_{\substack{\ell,k\geq 0 \\ \ell+k=N}} \sum_{\substack{\{i_a;a=1,\cdots \ell\},\{j_b;b=1,\cdots k\} \\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}} \{A_{i_1} \cdots A_{i_\ell}\varphi_s\}\{\varphi_r A_{j_1} \cdots A_{j_k}\}\langle\varphi_s^c|c_0^-\mathcal{G}|\varphi_r^c\rangle \,. \end{aligned} \quad (2.18)$$

3 The effective action of light fields

In this section we shall analyze the effective action for light fields obtained by integrating out the heavy fields. By definition, the $L_0^+ - \alpha' k^2/2$ eigenvalue vanishes for the light states. We shall denote by P the projection operator into light states. P satisfies[5]

$$[P, b_0^\pm] = 0, \quad [P, L_0^\pm] = 0, \quad [P, \mathcal{G}] = 0, \quad [P, Q_B] = 0\,. \tag{3.1}$$

Consider a set of light off-shell states $a_1, \cdots a_N$. We denote by $\{\!\{a_1 \cdots a_N\}\!\}_e$ the total contribution to the amplitude with external states $a_1, \cdots a_N$ from *all* the Feynman diagrams of superstring field theory, but with the propagator factors appearing in (2.10), (2.11) replaced by $\langle\varphi_s^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}(1-P)|\varphi_r^c\rangle$. This removes the contributions of light fields from the propagator. Therefore $\{\!\{a_1 \cdots a_N\}\!\}_e$ can be regarded as the contribution to the off-shell amplitude due to the elementary N-point vertex of the effective theory, obtained by integrating out the heavy fields.[6] We can now repeat the argument leading to (2.12) with $G(\cdots)$ replaced by $\{\!\{\cdots\}\!\}_e$. On the left hand side of (2.12) and the first two terms on the right hand side of (2.12) we simply replace $G(\cdots)$ by $\{\!\{\cdots\}\!\}_e$, but in the last two terms of (2.12) the propagator factors will now have additional insertions of $(1-P)$ since this is the propagator used in the definition of $\{\!\{\cdots\}\!\}_e$. This gives

$$\begin{aligned}
&\sum_{i=1}^N \{\!\{a_1 \cdots a_{i-1}(Q_B a_i) a_{i+1} \cdots a_N\}\!\}_e \\
&= -\frac{1}{2} \sum_{\substack{\ell,k\geq 0\\ \ell+k=N}} \sum_{\substack{\{i_a;a=1,\cdots \ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}} \{\!\{a_{i_1}\cdots a_{i_\ell}\varphi_s\}\!\}_e \{\!\{\varphi_r a_{j_1}\cdots a_{j_k}\}\!\}_e \langle\varphi_s^c|c_0^-\mathcal{G}|\varphi_r^c\rangle \\
&\quad -\frac{1}{2}\, g_S^2 \{\!\{a_1\cdots a_N \varphi_s\varphi_r\}\!\}_e\, \langle\varphi_s^c|c_0^-\mathcal{G}|\varphi_r^c\rangle \\
&\quad -\frac{1}{2} \sum_{\substack{\ell,k\geq 0\\ \ell+k=N}} \sum_{\substack{\{i_a;a=1,\cdots \ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}} \Big[-\{\!\{a_{i_1}\cdots a_{i_\ell}(Q_B\varphi_s)\}\!\}_e \{\!\{\varphi_r a_{j_1}\cdots a_{j_k}\}\!\}_e \\
&\quad -(-1)^{\gamma_s}\{\!\{a_{i_1}\cdots a_{i_\ell}\varphi_s\}\!\}_e \{\!\{(Q_B\varphi_r) a_{j_1}\cdots a_{j_k}\}\!\}_e\Big] \langle\varphi_s^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}(1-P)|\varphi_r^c\rangle \\
&\quad -\frac{1}{2}\, g_S^2 \Big[-\{\!\{a_1\cdots a_N (Q_B\varphi_s)\varphi_r\}\!\}_e - (-1)^{\gamma_s}\{\!\{a_1\cdots a_N\varphi_s(Q_B\varphi_r)\}\!\}_e\Big] \\
&\quad \langle\varphi_s^c|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}(1-P)|\varphi_r^c\rangle\,.
\end{aligned} \tag{3.2}$$

[5]While P projects to states of mass level zero, it keeps all the momentum modes of the light fields. Therefore at this stage it will be premature to claim that integrating out modes outside the P invariant subspace will lead to the Wilsonian effective action. We shall see in the next section that by adjusting a parameter in string field theory we can effectively integrate out the high momentum modes of the light fields.

[6]Even in the presence of tadpoles of light fields, these amplitudes do not suffer from divergences of the kind mentioned in footnote 4.

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Now the third and the fourth terms on the right hand side cancel the first and the second terms only partially, leaving behind terms proportional to $\langle\varphi^c_s|c_0^-\mathcal{G}\,P|\varphi^c_r\rangle$:

$$\begin{aligned}
&\sum_{i=1}^{N}\{\!\!\{a_1\cdots a_{i-1}(Q_Ba_i)a_{i+1}\cdots a_N\}\!\!\}_e \\
&= -\frac{1}{2}\sum_{\substack{\ell,k\geq 0\\ \ell+k=N}}\ \sum_{\substack{\{i_a;a=1,\cdots\ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}}\{\!\!\{a_{i_1}\cdots a_{i_\ell}\varphi_s\}\!\!\}_e\{\!\!\{\varphi_r a_{j_1}\cdots a_{j_k}\}\!\!\}_e\langle\varphi^c_s|c_0^-\mathcal{G}\,P|\varphi^c_r\rangle \\
&\quad -\frac{1}{2}\,g_S^2\{\!\!\{a_1\cdots a_N\varphi_s\varphi_r\}\!\!\}_e\,\langle\varphi^c_s|c_0^-\mathcal{G}\,P|\varphi^c_r\rangle\,.
\end{aligned}\tag{3.3}$$

If we denote by $\{|\chi_\alpha\rangle\}$ and $\{|\chi^c_\alpha\rangle\}$ the basis states in $P\widehat{\mathcal{H}}_T$ and $P\widetilde{\mathcal{H}}_T$ respectively, satisfying

$$\langle\chi_\alpha|c_0^-|\chi^c_\beta\rangle=\delta_{\alpha\beta},\quad \langle\chi^c_\beta|c_0^-|\chi_\alpha\rangle=\delta_{\alpha\beta},\tag{3.4}$$

then we can express (3.3) as

$$\begin{aligned}
&\sum_{i=1}^{N}\{\!\!\{a_1\cdots a_{i-1}(Q_Ba_i)a_{i+1}\cdots a_N\}\!\!\}_e \\
&= -\frac{1}{2}\sum_{\substack{\ell,k\geq 0\\ \ell+k=N}}\ \sum_{\substack{\{i_a;a=1,\cdots\ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}}\{\!\!\{a_{i_1}\cdots a_{i_\ell}\chi_\alpha\}\!\!\}_e\{\!\!\{\chi_\beta a_{j_1}\cdots a_{j_k}\}\!\!\}_e\langle\chi^c_\alpha|c_0^-\mathcal{G}|\chi^c_\beta\rangle \\
&\quad -\frac{1}{2}\,g_S^2\,\{\!\!\{a_1\cdots a_N\chi_\alpha\chi_\beta\}\!\!\}_e\,\langle\chi^c_\alpha|c_0^-\mathcal{G}|\chi^c_\beta\rangle\,.
\end{aligned}\tag{3.5}$$

This may be considered as the key technical result of this paper.

Given the identity (3.5) one can now construct the string field theory action satisfying BV master equation in a straightforward manner following the procedure described in [4] and reviewed in section 2. We introduce two sets of grassmann even string fields, $\Phi\in P\widehat{\mathcal{H}}_T$ and $\widetilde\Phi\in P\widetilde{\mathcal{H}}_T$, i.e. both containing only light states. The effective master action is given by

$$S_e=\frac{1}{g_S^2}\left[-\frac{1}{2}\langle\widetilde\Phi|c_0^-Q_B\mathcal{G}|\widetilde\Phi\rangle+\langle\widetilde\Phi|c_0^-Q_B|\Phi\rangle+\sum_{n=1}^{\infty}\frac{1}{n!}\{\!\!\{\Phi^n\}\!\!\}_e\right].\tag{3.6}$$

Even though we used Siegel gauge to integrate out the heavy fields, there is no restriction on $|\Phi\rangle$ and $|\widetilde\Phi\rangle$ to satisfy the Siegel gauge condition. This is in the spirit of the BV formalism where the master action is constructed *before gauge fixing*. In order to show that (3.6) satisfies the BV master equation we proceed as follows [4]:

1. We define $\widehat{\mathcal{H}}_+$ and $\widetilde{\mathcal{H}}_+$ to be the subspaces of $\widehat{\mathcal{H}}_T$ and $\widetilde{\mathcal{H}}_T$ respectively containing states of ghost numbers ≥ 3, and $\widehat{\mathcal{H}}_-$ and $\widetilde{\mathcal{H}}_-$ to be the subspaces of $\widehat{\mathcal{H}}_T$ and $\widetilde{\mathcal{H}}_T$ respectively containing states of ghost numbers ≤ 2. We organize the basis states $\{|\chi_\alpha\rangle\}$ into $\{|\widehat\chi^-_\alpha\rangle\}$ and $\{|\widehat\chi_+^\alpha\rangle\}$ of $P\widehat{\mathcal{H}}_-$ and $P\widehat{\mathcal{H}}_+$ respectively, and the basis states $\{|\chi^c_\alpha\rangle\}$ into basis states $\{|\widetilde\chi^-_\alpha\rangle\}$ and $\{|\widetilde\chi_+^\alpha\rangle\}$ of $P\widetilde{\mathcal{H}}_-$, and $P\widetilde{\mathcal{H}}_+$ respectively, satisfying orthonormality conditions[7]

$$\langle\widehat\chi^-_\alpha|c_0^-|\widetilde\chi_+^\beta\rangle=\delta_\alpha{}^\beta=\langle\widetilde\chi_+^\beta|c_0^-|\widehat\chi^-_\alpha\rangle,\quad \langle\widetilde\chi^-_\alpha|c_0^-|\widehat\chi_+^\beta\rangle=\delta_\alpha{}^\beta=\langle\widehat\chi_+^\beta|c_0^-|\widetilde\chi^-_\alpha\rangle\,,\tag{3.7}$$

[7]By an abuse of notation we are using the same indices α to label the new basis even though the label runs over a smaller set for the new basis.

and the completeness relations

$$\sum_\alpha |\widehat{\chi}_\alpha^-\rangle\langle\widetilde{\chi}_+^\alpha|c_0^- + \sum_\alpha |\widehat{\chi}_+^\alpha\rangle\langle\widetilde{\chi}_\alpha^-|c_0^- = \mathbf{1}, \quad \sum_\alpha |\widetilde{\chi}_\alpha^-\rangle\langle\widehat{\chi}_+^\alpha|c_0^- + \sum_\alpha |\widetilde{\chi}_+^\alpha\rangle\langle\widehat{\chi}_\alpha^-|c_0^- = \mathbf{1}\,, \tag{3.8}$$

acting on states in $P\widehat{\mathcal{H}}_T$ and $P\widetilde{\mathcal{H}}_T$ respectively.

2. The light string fields $|\Phi\rangle$ and $|\widetilde{\Phi}\rangle$ are expanded as

$$\begin{aligned} |\widetilde{\Phi}\rangle &= \sum_\alpha |\widetilde{\chi}_\alpha^-\rangle\widetilde{\phi}^\alpha + \sum_\alpha (-1)^{\gamma_\alpha^*+1}|\widetilde{\chi}_+^\alpha\rangle\phi_\alpha^*\,, \\ |\Phi\rangle - \frac{1}{2}\mathcal{G}|\widetilde{\Phi}\rangle &= \sum_\alpha |\widehat{\chi}_\alpha^-\rangle\phi^\alpha + \sum_\alpha (-1)^{\widetilde{\gamma}_\alpha^*+1}|\widehat{\chi}_+^\alpha\rangle\widetilde{\phi}_\alpha^*\,. \end{aligned} \tag{3.9}$$

Here γ_α^*, γ_α, $\widetilde{\gamma}_\alpha^*$ and $\widetilde{\gamma}_\alpha$ label the grassmann parities of ϕ_α^*, ϕ^α, $\widetilde{\phi}_\alpha^*$ and $\widetilde{\phi}^\alpha$ respectively. We shall identify the variables $\{\phi^\alpha, \widetilde{\phi}^\alpha\}$ as 'fields' and the variables $\{\phi_\alpha^*, \widetilde{\phi}_\alpha^*\}$ as the conjugate 'anti-fields' in the BV quantization of the theory.

3. Given two functions F and G of all the fields and anti-fields, we now define their anti-bracket as:

$$\{F,G\} = \frac{\partial_R F}{\partial\phi^\alpha}\frac{\partial_L G}{\partial\phi_\alpha^*} + \frac{\partial_R F}{\partial\widetilde{\phi}^\alpha}\frac{\partial_L G}{\partial\widetilde{\phi}_\alpha^*} - \frac{\partial_R F}{\partial\phi_\alpha^*}\frac{\partial_L G}{\partial\phi^\alpha} - \frac{\partial_R F}{\partial\widetilde{\phi}_\alpha^*}\frac{\partial_L G}{\delta\widetilde{\phi}^\alpha}\,, \tag{3.10}$$

where the subscripts R and L of ∂ denote left and right derivatives respectively. We also define

$$\Delta F \equiv \frac{\partial_R}{\partial\phi^\alpha}\frac{\partial_L F}{\partial\phi_\alpha^*} + \frac{\partial_R}{\partial\widetilde{\phi}^\alpha}\frac{\partial_L F}{\partial\widetilde{\phi}_\alpha^*}\,. \tag{3.11}$$

4. Using (3.8), (3.9) and (3.10) one gets, after some algebra,

$$g_S^4\{S_e,S_e\} = -2\sum_n \frac{1}{(n-1)!}\{\!\!\{\Phi^{n-1}Q_B\Phi\}\!\!\}_e - \sum_{m,n}\frac{1}{m!n!}\{\!\!\{\chi_\beta\Phi^m\}\!\!\}_e\{\!\!\{\chi_\alpha\Phi^n\}\!\!\}_e\,\langle\chi_\beta^c|c_0^-\mathcal{G}|\chi_\alpha^c\rangle. \tag{3.12}$$

Here $|\chi_\alpha\rangle$'s denote the original choice of basis states in $\widehat{\mathcal{H}}_T$ before splitting it into $\widehat{\mathcal{H}}_\pm$.

5. On the other hand using (3.8), (3.9) and (3.11) we get

$$\Delta S_e = -\frac{1}{2\,g_S^2}\sum_n \frac{1}{n!}\{\!\!\{\Phi^n\chi_\beta\chi_\alpha\}\!\!\}_e\langle\chi_\beta^c|c_0^-\mathcal{G}|\chi_\alpha^c\rangle\,. \tag{3.13}$$

6. Using the identity (3.5), and eqs. (3.12), (3.13) one can show that the action S_e given in (3.6) satisfies the quantum BV master equation

$$\frac{1}{2}\{S_e,S_e\} + \Delta S_e = 0\,. \tag{3.14}$$

One can choose Siegel gauge $b_0^+|\Phi\rangle = 0 = b_0^+|\widetilde{\Phi}\rangle$ and derive the Feynman rules in a straightforward manner following a procedure identical to the one used for the full string

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field theory. One finds that the field Φ describes an interacting field, while the degrees of freedom associated with $\widetilde{\Phi}$ describe decoupled free fields. By analyzing the amplitudes with external Φ legs one arrives at the conclusion that when two Feynman diagrams are joined by a Φ propagator one has an expression analogous to (2.10)

$$- f(a_1,\cdots a_m,\chi_\alpha)\, g(b_1,\cdots b_n,\chi_\beta)\, \langle\chi^c_\alpha|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}|\chi^c_\beta\rangle\,. \tag{3.15}$$

On the other hand when two legs of a connected Feynman diagram are joined by a Φ propagator, we have the analog of (2.11)

$$-\frac{1}{2}\, g_S^2\, f(a_1,\cdots a_m,\chi_\alpha,\chi_\beta)\langle\chi^c_\alpha|c_0^- b_0^+ (L_0^+)^{-1}\mathcal{G}|\chi^c_\beta\rangle\,. \tag{3.16}$$

By writing down an analog of (2.12) in the effective field theory one can show that the full amplitude constructed by summing over all Feynman diagrams of light fields with $\{\!\!\{a_1\cdots a_N\}\!\!\}_e$ as elementary vertices satisfies the same identity as (2.15). This is of course a reflection of the fact that the full amplitude constructed from the effective action and the original string field theory action are identical.[8]

We can also introduce the notion of 1PI amplitudes $\{a_1\cdots a_N\}_e$ built from the elementary vertices described above. By proceeding in the same way as from (2.9) to (2.18), one can show that $\{a_1\cdots a_N\}_e$ satisfies the identity

$$\begin{aligned}
&\sum_{i=1}^N \{a_1\cdots a_{i-1}(Q_B a_i)a_{i+1}\cdots a_N\}_e \\
&= -\frac{1}{2}\sum_{\substack{\ell,k\geq 0\\ \ell+k=N}}\ \sum_{\substack{\{i_a;a=1,\cdots \ell\},\{j_b;b=1,\cdots k\}\\ \{i_a\}\cup\{j_b\}=\{1,\cdots N\}}} \{a_{i_1}\cdots a_{i_\ell}\chi_\alpha\}_e\{\chi_\beta a_{j_1}\cdots a_{j_k}\}_e\langle\chi^c_\beta|c_0^-\mathcal{G}|\chi^c_\alpha\rangle\,.
\end{aligned} \tag{3.17}$$

Using this one can construct 1PI action for light fields:

$$S_e^{1PI} = \frac{1}{g_S^2}\left[-\frac{1}{2}\langle\widetilde{\Phi}|c_0^- Q_B\mathcal{G}|\widetilde{\Phi}\rangle + \langle\widetilde{\Phi}|c_0^- Q_B|\Phi\rangle + \sum_{n=1}^\infty \frac{1}{n!}\{\Phi^n\}_e\right]. \tag{3.18}$$

This satisfies the classical master equation

$$\{S_e^{1PI}, S_e^{1PI}\} = 0\,. \tag{3.19}$$

Furthermore by restricting the fields $|\Phi\rangle$ and $|\widetilde{\Phi}\rangle$ to carry ghost number 2, we arrive at the gauge invariant 1PI action of space-time matter fields with gauge transformation laws

$$\delta|\widetilde{\Phi}\rangle = Q_B|\widetilde{\Lambda}\rangle + \sum_n \frac{1}{n!}[\Phi^n\Lambda]_e\,,\quad \delta|\Phi\rangle = Q_B|\Lambda\rangle + \sum_n \frac{1}{n!}\mathcal{G}\,[\Phi^n\Lambda]_e\,. \tag{3.20}$$

Here the gauge transformation parameters $|\Lambda\rangle$ and $|\widetilde{\Lambda}\rangle$ are states of $P\widehat{\mathcal{H}}_T$ and $P\widetilde{\mathcal{H}}_T$ respectively, carrying ghost number 1. For $|a_i\rangle\in P\widehat{\mathcal{H}}_T$, $[a_1\cdots a_N]_e$ describes a state in $P\widetilde{\mathcal{H}}_T$ defined via

$$\langle a_0|c_0^-|[a_1\cdots a_N]_e\rangle = \{a_0 a_1\cdots a_N\}_e\,,\quad \forall |a_0\rangle\in P\widehat{\mathcal{H}}_T\,. \tag{3.21}$$

[8] As mentioned in footnote 4, if there are tadpoles of light states then the full amplitude is divergent. In such cases one has to first construct the 1PI effective action, find the extremum of this action and then compute the Green's functions by expanding the action around the new background.

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The 1PI action so constructed is useful for analyzing various properties of light fields. For example the quantum corrected vacuum can be found by identifying the translationally invariant extremum of the action. The renormalized masses can be computed directly by solving the linearized equations of motion around the extremum of the action.[9]

Before concluding this section, we shall give some physical insight into the definition of the vertex $\{\!\{a_1 \cdots a_N\}\!\}_e$. We have defined it as the result of building up the off-shell amplitude of light states by summing over Feynman diagrams of superstring field theory after subtracting off the contribution due to light states from the propagators. This would be a complicated procedure involving piecing together amplitudes of heavy states with the help of (2.10), (2.11) with additional insertion of $(1-P)$ factors in the propagators. A simpler definition involves proceeding from the other end, where we begin with the full off-shell amplitude of light states and carry out appropriate subtraction involving light intermediate states. A systematic procedure for doing this has been described in appendix A. The full amplitude has simple interpretation as the integrals of correlation functions of vertex operators of light states over the full moduli space of Riemann surfaces. The subtraction terms also involve products of light state propagators and correlation functions of off-shell vertex operators of light states over lower genus Riemann surfaces. Therefore with this definition one never has to worry about computing off-shell amplitudes of heavy states. The analysis nevertheless requires some data from string field theory since the definition of the off-shell amplitude, as well as the association of subspaces of the moduli space with the Feynman diagrams, requires data from string field theory. We shall elaborate on this in section 5. A similar procedure can be adapted for defining the 1PI vertices $\{a_1 \cdots a_N\}_e$. In this case the subtraction of the contribution of light states has to be applied only on those propagators which connect two Feynman diagrams that are otherwise disjoint.

4 The effective action as Wilsonian effective action

In this section we shall argue following [10, 11] that the effective action derived in section 3 is actually a Wilsonian effective action in which the high momentum modes of the light fields are also effectively integrated out. For this we shall first illustrate this mechanism in the context of a quantum field theory.

Consider an ordinary quantum field theory, possibly containing multiple fields of different masses. A Feynman diagram of this theory is given by the integral over loop momenta of an integrand that is given by the product of propagators and vertices. Let us represent the denominator factor of each propagator as an integral over a Schwinger parameter as follows:

$$(k^2+m^2)^{-1} = \int_0^\infty ds\, e^{-s(k^2+m^2)}\,. \tag{4.1}$$

Now let us replace the right hand side by

$$\int_\Lambda^\infty ds\, e^{-s(k^2+m^2)} + \int_0^\Lambda ds\, e^{-s(k^2+m^2)} = e^{-\Lambda(k^2+m^2)}(k^2+m^2)^{-1} + \int_0^\Lambda ds\, e^{-s(k^2+m^2)}\,. \tag{4.2}$$

[9] Even though in ten dimensions the masses of light states do not get renormalized due to gauge invariance, this is not always true in lower dimensions, e.g. in the situation analyzed in [3].

The first term contains a pole at $k^2 + m^2 = 0$. Let us redefine the vertices by absorbing a factor of $e^{-\Lambda(k^2+m^2)/2}$ in each of the two vertices that the propagator connects to. If we do this for each propagator and pick the first term in each propagator then the result will have an interpretation of Feynman diagram in a new theory in which each vertex is scaled by a factor of $e^{-\Lambda\sum_i(k_i^2+m_i^2)/2}$, with the sum running over all the external legs of the vertex. This still leaves us with the second term in (4.2) which needs to be taken into account. Since this does not have any pole at $k^2 + m^2 = 0$, its contribution may be represented by including a further additive contribution to the vertices. This leads to a new set of vertices (and the standard propagator) which give identical contribution to the S-matrix as the original vertices.

As a simple example we can consider a tree level four point amplitude corresponding to the 's-channel diagram'. Representing the propagator as in (4.2), we can regard the contribution to the four point function from the second term in (4.2) as coming from a new additive contribution to the four point vertex. On the other hand the $e^{-\Lambda(k^2+m^2)}$ factor in the first term can be absorbed into a multiplicative factor of $e^{-\Lambda(k^2+m^2)/2}$ in each of the two three point vertices. This procedure can be carried out for all tree and loop amplitudes.

It is clear that this procedure removes some of the contribution from the composite Feynman diagrams and transfers it to the definition of the elementary vertices. This is precisely the notion of integrating out certain set of degrees of freedom to generate a new but equivalent theory. It is also easy to see which degrees of freedom are being integrated out. For this let us consider the case of massless fields and euclidean theory so that k^2 is positive. In this case most of the contribution of the modes with $k^2 \gg \Lambda^{-1}$ is in the second term in (4.2) — the contribution of these modes to the first term is exponentially suppressed. Therefore including the contribution from the second term in the vertices corresponds to integrating out the modes carrying momentum $\gg \Lambda^{-1/2}$. This is precisely the notion of Wilsonian effective action.

Let us now return to string field theory. It turns out that string field theories automatically come with the parameter Λ [12]. This corresponds to the freedom of rescaling the local holomorphic coordinates around the punctures by a constant. This rescaling automatically multiplies the off-shell amplitudes (and hence also the elementary vertices) by a factor of $e^{-\Lambda\sum_i(k_i^2+m_i^2)/2}$ where the $(k_i^2 + m_i^2)$ factor arises from the conformal weight of the vertex operator. However the rescaling of the local holomorphic coordinates also needs to be accompanied by a change in the cell decomposition of the moduli space — transferring some part of the moduli space corresponding to composite Feynman diagrams to elementary vertices — in order to preserve the relations (2.9). Therefore the operation of rescaling the local holomorphic coordinates in the definition of off-shell amplitudes — known in the string field theory literature as the operation of adding stubs — is precisely the operation of rearranging the contribution from different Feynman diagrams in a quantum field theory using (4.2). As argued before, this operation is equivalent to integrating out the high momentum modes. Therefore we see that the effective action we have obtained automatically represents Wilsonian effective action in which by controlling the stub length we can adjust the energy scale above which the modes of the light fields are integrated out.

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This also determines the effective UV cut-off of the theory, encoded in the exponential suppression factor in the vertices.[10]

In the BV formalism the operation of adding stubs is generated by a symplectic transformation of the fields and does not change the physical theory [14]. In particular this operation generates a field redefinition of the 1PI effective action and leaves the S-matrix unchanged [20, 21].

5 Discussion

Construction of superstring field theory has two parts. The first part involves associating, to each elementary vertex of the field theory with a certain number of external lines, a certain subspace of the moduli space of punctured Riemann surfaces, and specifying, for each Riemann surface belonging to the subspace, a choice of local holomorphic coordinates around the punctures and PCO locations. The number of punctures match the number of external lines and the genus of the Riemann surface determines the power of g_S by which the contribution from the vertex is multiplied. Generically a vertex with a given number of external lines receives contribution from all genera. The choice of this data is not unique, but neither is this completely arbitrary as it has to satisfy some stringent constraints that will be described below. Given this data one can draw all possible Feynman diagrams for a given amplitude, and for each of these Feynman diagrams associate a subspace of the moduli space of Riemann surfaces using the following algorithm. When two elementary vertices are joined by a Feynman propagator to generate a new Feynman diagram as in (2.10), this corresponds to gluing the corresponding Riemann surfaces via the plumbing fixture relation

$$zw = e^{-s+i\theta}, \quad 0 \le s < \infty, \quad 0 \le \theta < 2\pi \,, \tag{5.1}$$

where z and w are the local holomorphic coordinates around the punctures which are joined by the propagator. This not only generates a family of Riemann surfaces corresponding to the Feynman diagram but also specifies the local coordinates around the punctures and the PCO locations on each member of this new family.[11] A similar interpretation can be given for the case where two lines coming out of a single vertex are joined by a propagator as in (2.11). Repeating this procedure one can generate a family of Riemann surfaces corresponding to every Feynman diagram, together with choice of local holomorphic coordinates around the punctures and PCO locations for each member of this family. The consistency of the original choice guarantees that the family of Riemann surfaces corresponding to all Feynman diagrams covers the full moduli space of Riemann surfaces in a one to one fashion and furthermore that the choice of local holomorphic coordinates around the punctures and

[10] Note however that in a given scattering amplitude with some fixed total center of mass energy E carried by the incoming particles, the loop energy integration contours veer away from the imaginary axis by a distance of order E in order to go around the poles of the propagators [18, 19]. Since the vertices grow exponentially for real energy, a large stub length does not help us in reducing the range of loop energy integration below E.

[11] When two punctures associated with Ramond sector states are joined this way, one has an additional insertion of the zero mode of the PCO around one of the punctures.

the PCO locations are continuous across the boundaries that separate the subspaces of the moduli space associated with different Feynman diagrams.

In the second step one focusses on superstring theory in some specific background associated with a choice of the superconformal field theory of matter and ghost fields and computes the amplitude associated with a Feynman diagram by inserting the vertex operators at the punctures using the predefined local holomorphic coordinates and integrating the correlation function over the corresponding subspace of the moduli space of Riemann surfaces. Based on this one arrives at the definition of $\{\!\{A_1 \cdots A_N\}\!\}$ and the BV master action described in section 2.

The result of our analysis can be interpreted by saying that these data can be used to construct other useful quantities in string theory, e.g. Wilsonian effective action and 1PI effective action for light fields. For the interaction vertices of the Wilsonian effective action we take the expression for the full off-shell amplitude of light states and subtract from each Feynman diagram the contribution to the propagator from the light states following the procedure described in appendix A. For the 1PI effective action we carry out a similar procedure, but perform the subtraction only from the propagators connecting two disjoint Feynman diagrams. For amplitudes with light fields as external states, these actions contain the full information of the original string field theory.

The construction described in this paper can be generalized to other situations by replacing the projection operator P to light states by some other projection operator satisfying the identities given in (3.1). In particular we could choose P to be a projection operator into another mass level, or into a set of mass levels. Projecting into a fixed mass level may be useful, e.g. for computing the renormalized masses of the states at that mass level, while projecting into a set of mass levels can be useful for computing S-matrix elements where the external states are chosen from that set. This will automatically implement the procedure described in [20–23] for integrating out states at other mass levels while computing the renormalized mass at a given level. Note however that if the theory has tadpoles of mass level zero fields requiring a shift in the vacuum under quantum corrections then it is not possible to integrate out the degrees of freedom associated with the mass level zero states. The projection operator P must always include the mass level zero states so that the corresponding 1PI action can be used to determine the correct vacuum before proceeding to compute the renormalized masses. Even in the absence of tadpoles, an effective action in which the fields of lower mass levels are integrated out will necessarily be complex, reflecting that the scattering could produce the states that have been integrated out. For this reason, for computing a given scattering amplitude, it is best to keep all the fields up to the mass level that can be produced in the final state, and integrate out all fields above this mass level. This will have the advantage of having to deal with only a finite number of fields, but all the standard manipulations can be carried out with this action. For example the proof of unitarity given in [24] can be repeated with this formulation of the theory without any change.

Given the complexity of string field theory — having infinite number of fields — one could wonder whether what we have learned from string field theory can be reformulated as some suitable modification of the world-sheet description of scattering amplitudes. The effective action just described can be regarded as the answer to this question. The data required for writing down the action involves computation of correlation functions on the Riemann surface of only states below a certain mass level, with certain subtractions that also involves correlation functions of the same states. Nevertheless the action so constructed is free from all divergences. Therefore in this formalism all the divergences of the S-matrix appear as usual infrared divergences of a quantum field theory and can be dealt with using standard tools of quantum field theory.

One could also consider other projection operators satisfying (3.1). As an example, consider toroidal compactification of type II string theory and take P to be the projection operator on to states for which the left and right-moving oscillators are forced to be in their GSO invariant ground state. By integrating out all other fields following the procedure described in this paper, one would arrive at the double field theory action envisaged in [25]. Such a field theory may be useful for describing scattering of states in the kinematic regime in which the P non-invariant states are not produced in the final state.

It is tempting to conjecture that the form of the effective action we have found extends beyond conventional string theory, e.g. the Wilsonian effective action of the eleven dimensional M-theory may be given by a formula similar to (3.6) with the vertices satisfying the identities (3.5), even though they are not given as integrals over the moduli space of Riemann surfaces. M-theory is known to exist [26, 27] but there is no systematic method for computing scattering amplitudes beyond the leading order due to the absence of a dimensionless expansion parameter. The artificial dimensionless parameter given by the ratio of the cut-off scale and the Planck scale could serve the purpose of an expansion parameter, with higher loop contributions being suppressed by powers of this parameter as long as the momenta of external states remain below the cut-off scale. From this perspective, construction of M-theory reduces to the problem of finding solutions to (3.5) subject to the boundary condition at low momentum provided by the eleven dimensional supergravity.

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A Construction of the interaction vertices of the effective action

In this appendix we shall give a systematic procedure for constructing the interaction vertices $\{\!\{a_1 \cdots a_N\}\!\}_e$ that go into the definition of the Wilsonian effective action. For this we shall need to introduce some notation. The original superstring field theory provides us with a cell decomposition of the moduli space of Riemann surfaces with punctures, with every cell corresponding to a particular Feynman diagram. It also specifies the choice of local holomorphic coordinates at the punctures and the PCO locations for every Riemann surface. This allows us to express the off-shell amplitude with external states $a_1, \cdots a_N$

as $\int I$ where the integral runs over the moduli space and the integrand I is expressed in terms of appropriate correlations functions on the Riemann surface. The contribution to $\{\!\!\{a_1 \cdots a_N\}\!\!\}_e$ will be given by a similar integral but with different integrand, taking different forms inside different cells of the moduli space. We shall now describe how to construct the integrand.

First consider the cell C associated with the Feynman diagram that has a single vertex with N external legs and no propagators. The contribution of this cell to $\{\!\!\{a_1 \cdots a_N\}\!\!\}_e$ is given by $\int_C I$ where I is the same integrand that appears in the expression for the full off-shell amplitude.

Now suppose we have a cell C_1 corresponding to a Feynman diagram in which we have a single propagator. This could either connect two legs of a single vertex or two vertices. Riemann surfaces in this cell have a natural relation to the Riemann surfaces corresponding to the vertex (pair of vertices) that remains when the propagator is removed. The former is obtained by sewing two parts of the latter Riemann surface (pair of Riemann surfaces) using the plumbing fixture relation

$$z\,w = e^{-s+i\theta}, \qquad 0 \le s < \infty, \quad 0 \le \theta < 2\pi\,. \tag{A.1}$$

Here z and w are the local holomorphic coordinates around the extra punctures that arise due to the removal of the propagator. The moduli of the original Riemann surface labelling points inside the cell C_1 can be labelled by the moduli of the cell $\widetilde{C}_1$ corresponding to the vertex (pair of vertices), and the variables (s, θ).

Now take the original Feynman diagram and replace the propagator by the propagator of the light states. This contribution will be expressed in terms of the product of the propagator of the light states given by $(k^2)^{-1}\langle \chi^c_\alpha | c_0^- b_0^+ \mathcal{G} | \chi^c_\beta\rangle$ and the contribution to the off-shell amplitude from the constituent vertex (pair of vertices). The latter can be expressed as an integral over $\widetilde{C}_1$, while the $(k^2)^{-1}$ term in the propagator can be expressed as[12]

$$\frac{\alpha'}{4\pi}\int_0^\infty ds \int_0^{2\pi} d\theta\, e^{-s\,\alpha' k^2/2}\,. \tag{A.2}$$

When the propagator connects two vertices, k is fixed in terms of the external momenta using momentum conservation. When the propagator connects two legs of a single vertex, we carry out the integration over momenta k treating it as a gaussian integral. In either case, since the coordinates of $\widetilde{C}_1$ together with (s, θ) give the coordinates of C_1, we see that the net contribution has been expressed as an integral over the cell C_1. The integrand of course is different from the integrand I that appeared in the original off-shell amplitude. Let us call this integrand I_1. Then the contribution to $\{\!\!\{a_1 \cdots a_N\}\!\!\}_e$ from the cell C_1 is given by $\int_{C_1}(I - I_1)$. The subtraction term removes the contribution of light states from the propagator and renders the integral free from divergences in the $s \to \infty$ limit.

Next consider the case of a Feynman diagram with two propagators, which we shall label by 1 and 2. We define I to be the integrand appearing in the original amplitude,

[12] The $\alpha'/2$ factor multiplying k^2 has been fixed using the fact that k^2 appears in the expression for L_0^+ in the combination $\alpha' k^2/2$.

I_r for $r = 1,2$ to be the integrand that appears when we replace the r-th propagator by the light state propagator and use (A.2), and I_{12} to be the integrand that appears when we replace both the propagators by light state propagators and use (A.2). The contribution to $\{\!\{a_1 \cdots a_N\}\!\}_e$ from the cell C_{12} associated with this Feynman diagram is given by $\int_{C_{12}}(I - I_1 - I_2 + I_{12})$, with the additive factor of I_{12} compensating for the over subtraction that we have made by subtracting I_1 and I_2 from I.

The general procedure is now clear. Contribution to $\{\!\{a_1 \cdots a_N\}\!\}_e$ from a cell $C_{12\cdots k}$ associated with a Feynman diagram with k propagators labelled by $1, \cdots k$ is given by

$$\int_{C_{12\cdots k}} \sum_{\ell=0}^{k} (-1)^\ell \sum_{\{r_1,\cdots r_\ell\}\subseteq\{1,2,\cdots k\}} I_{r_1\cdots r_\ell}\,, \tag{A.3}$$

where $I_{r_1\cdots r_\ell}$ is the integrand that we get by replacing the propagators $r_1, \cdots r_\ell$ by light state propagators and then applying (A.2).

The procedure described above gives an expression for $\{\!\{a_1 \cdots a_N\}\!\}_e$ that is free from all divergences as long as the energies of external states $a_1, \cdots a_N$ are below the threshold of production of heavy states. Above this threshold we shall get divergences from the region where one or more Schwinger parameters s appearing in (A.2) become large (see e.g. [19]). As described in section 5, this may be avoided by working with a different effective action in which only states above the threshold are integrated out. In this case we carry out the algorithm described above by including in the subtraction terms not just the light state propagators but the projection of the full propagator to states up to a fixed mass level.

B Removing spurious fields

Since the effective action has only the light fields as its degrees of freedom, one would expect the number of fields to be finite. This is indeed true in the NS sector since there are no bosonic zero modes which can act on a light state to create infinite number of light states. However in the R sector we have zero modes of γ in the picture number $-1/2$ sector and zero modes of β in the picture number $-3/2$ sector. Therefore we can create infinite number of states at mass level zero by acting with these oscillators. We shall now show that only a finite number of these states appear in the computation of Feynman diagrams.

Let us focus on the right-moving Ramond sector state, the analysis for the left-moving Ramond sector will be identical. We drop all reference to the left-moving sector and also the momentum labels and the space-time spinor index coming from the right-moving sector. The only exception will be the zero modes of the b, c, $\bar{b}$, $\bar{c}$ ghosts since for these the condition (2.1) couples the two sectors. Denoting by $|p\rangle$ the Ramond ground state of picture number p in this convention, the unwanted modes of $|\Phi\rangle$ are the coefficients of $(\gamma_0)^n c_1|-1/2\rangle$ and $(\gamma_0)^n c_0^+ c_1|-1/2\rangle$ for $n \geq 1$. Their duals, in the sense described in (3.4), are $c_0^+(\beta_0)^n c_1|-3/2\rangle$ and $(\beta_0)^n c_1|-3/2\rangle$ respectively. Now (3.15), (3.16) show that the propagator involves $b_0^+\mathcal{G}$ acting on the dual basis state. The b_0^+ annihilates the state $(\beta_0)^n c_1|-3/2\rangle$. On the other hand acting on $c_0^+(\beta_0)^n c_1|-3/2\rangle$, $b_0^+\mathcal{G}$ will generate $\mathcal{G}(\beta_0)^n c_1|-3/2\rangle$. This is a state carrying right-moving picture number $-1/2$ and ghost number $(1-n)$. It is easy to see that this vanishes for $n \geq 1$ — there is simply no candidate state with the right ghost number and picture number at mass level zero. This shows that the fields associated with states of the form described above do not appear as intermediate states in the Feynman diagrams. Similar arguments together with (3.12), (3.13) show that these fields also do not contribute to $\{S_e, S_e\}$ or ΔS_e. Therefore for all practical computation we can drop these fields and their dual, and work with a finite number of fields.

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