Classical limit of diagonal form factors and HHL correlators

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ABSTRACT: We propose an expression for the classical limit of diagonal form factors in which we integrate the corresponding observable over the moduli space of classical solutions. In infinite volume the integral has to be regularized by proper subtractions and we present the one, which corresponds to the classical limit of the connected diagonal form factors. In finite volume the integral is finite and can be expressed in terms of the classical infinite volume diagonal form factors and subvolumes of the moduli space. We analyze carefully the periodicity properties of the finite volume moduli space and found a classical analogue of the Bethe-Yang equations. By applying the results to the heavy-heavy-light three point functions we can express their strong coupling limit in terms of the classical limit of the sine-Gordon diagonal form factors.

Contents

1 Introduction

Integrable two dimensional quantum field theories are very special as, in principle, they can be solved exactly by the bootstrap method. This method consists of two parts: the S-matrix bootstrap calculates the scattering matrix of the theory from global symmetries and from such physical requirements as crossing symmetry and unitarity [1, 2]. The second step is the form factor bootstrap, which uses the already calculated S-matrix to determine the matrix elements of local operators, from which the correlation functions can be built up [3–5]. This program has been pushed forward to many interesting theories including the sine-Gordon and sinh-Gordon theories [6, 7].

In the last decade there has been increasing interest and relevant progress in applying the bootstrap program for the AdS/CFT correspondence [8]. The S-matrix bootstrap was successfully implemented, which eventually lead to the complete description of the spectral problem. Recently the focus moved to the application of the form factor bootstrap. An axiomatic approach for world-sheet form factors was developed in [9, 10]. In [11] it was

suggested that finite volume diagonal form factors can be used to describe the Heavy-Heavy-Light (HHL) 3-point functions. This proposal has been tested both at weak [12] and strong coupling and for special operators [11]. Recently we also made a proposal, how the form factor axioms can be modified to describe the string field theory vertex, which corresponds to generic 3-point functions on the gauge theory side [13]. This was complemented by the hexagon approach [14], which were devised to calculate the 3-point functions directly by cutting the pant diagram into two hexagons. These hexagons were exactly calculated and the method was checked by comparing to weak coupling data [15, 16]. Later it was shown that in the diagonal limit the results reproduce the structure of the diagonal form factor proposal for HHL correlators [17, 18]. The HHH three point functions were also analyzed recently in [19].

In testing the HHL proposal at strong couplings a check for two particles was performed [11]. We observed that in this limit the 3-point function was related to the average of the light vertex operator over the moduli space of classical solutions. As the strong coupling limit of the model is related to the classical limit of the sine-Gordon theory it is natural to assume that the classical limit of form factors are just the average of the corresponding observable for the moduli space of classical solutions. The aim of our paper is to investigate this correspondence.

Interestingly, there were not many investigations on the classical limit of form factors. Goldstone and Jackiw [20, 21] showed that the classical kink solution is the Fourier transform of the form factor of the basic field between two moving kink states in the semi-classical limit, when the kink momentum is very small compared to its mass. Later Mussardo et al. extended the expression into a transparently relativistically covariant form and used its crossed version to determine the masses of boundstates [22]. In the diagonal limit these analyses dictate that the classical limit of the elementary fields' form factor between one-particle states should be the spatial integral of the static kink solution. Our work gives a meaning for this formula and generalizes the result for generic operators and for multi-particle states. Let us also mention that semiclassical finite volume form factors were analyzed in [23] in the conformal case. Here, in contrast, we focus on massive scattering theories. In such a theory, namely in the sinh-Gordon theory, Lukyanov analyzed the semi-classical expansion of the finite temperature expectation values of exponential fields [24]. These results are valid for any volume but only for the vacuum expectation value. Here we deal with asymptotically large volumes and expectation values in multiparticle states.

Our paper is organized as follows. In section 2 we give a brief heuristic introduction to the paper. In section 3 we present our proposal for the classical computation of multiparticle diagonal form factors in infinite volume. Then we move on in section 4 to describe the evaluation of finite volume expectation values in the classical limit and establish their link with the classical diagonal form factors of the previous section. In section 5 we briefly comment on the link with Heavy-Heavy-Light OPE coefficients and we close the paper with conclusions and two appendices.

2 Guide to the paper

Here we make a heuristic argument why the diagonal form factors should be evaluated in the classical limit by averaging the operators for the moduli space of classical solutions. Let us calculate the diagonal form factor by evaluating the path integral:

$$\langle p_1, \ldots, p_n | \mathcal{O}(\phi(x,t)) | p_n, \ldots, p_1 \rangle = \int_{\phi_{\text{in}}}^{\phi_{\text{out}}} d[\phi] \mathcal{O}(\phi(x,t)) e^{\frac{i}{\hbar}S[\phi]} \tag{2.1}$$

where the initial configuration, ϕ_{in}, is related to a multiparticle state with momenta $\{p_i\}$ prepared at $t \to -\infty$, while the final configuration, ϕ_{out}, is also a multiparticle state with the same momenta $\{p_i\}$ fixed at $t \to \infty$. As the particles' momenta are all different, for asymptotically large times particles form well-separated non-interacting one-particle states. There are many configurations with the prescribed momentum content, $\{p_i\}$, all of which can be obtained by shifting the trajectories of each of the asymptotic one-particle states, $\{x_i\}$. These shifts do not effect the scattering matrix, but modify the path integral and generate the moduli space of classical solutions. In the classical limit ($\hbar \to 0$) the path integral localizes exactly to this moduli space

$$\langle p_1, \ldots, p_n | \mathcal{O}(\phi(x,t)) | p_n, \ldots p_1 \rangle = \mathcal{N} \int_{\mathcal{M}} \prod dx_i \mathcal{O}(\phi_n(x, t, \{x_i\}, \{p_i\})) \tag{2.2}$$

where ϕ_n is the classical n-particle solution with momenta $\{p_i\}$ and shift parameters $\{x_i\}$, which form the classical moduli space $\mathcal{M}$ and the normalization is proportional to the action, which is constant on the moduli space: $\mathcal{N} \propto e^{\frac{i}{\hbar}S[\phi_n]}$.

The infinite volume moduli space is clearly noncompact and the relevant integral is infinite as it stands. This is in fact an exact counterpart of the divergences of the infinite volume form factor in the diagonal limit which arise due to disconnected pieces with smaller particle number (see section 3.1). The divergences in the classical integral (2.2) are indeed associated with fine tuning the moduli so as to follow the trajectories of a lower number of particles.[1] The structural similarity of the divergence structures of the quantum connected form factor and the classical integral (2.2) strongly suggests that there should be a choice of subtraction scheme in (2.2) which exactly reproduces the classical limit of diagonal form factors. The goal of the first part of this paper is indeed to explicitly propose such a scheme and thus to provide a classical formula for the connected n-particle diagonal form factor in an arbitrary integrable QFT. This is done in section 3.

In the case of a finite volume system, the moduli space is compact and the integral is finite. However, exact finite volume multiparticle solutions are exceedingly complicated to construct and are usually not known explicitly. Despite that, once we allow ourselves to neglect exponential e^{-mL} terms, we can construct approximate finite volume solutions by gluing together infinite volume solutions. This has been used in [11] for computing the HHL OPE coefficient for a two particle state. Here we give a formulation valid for any

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[1]E.g. for the case of two particles, there is a direction in moduli space so that the operator stays on top of one outgoing or ingoing soliton. This noncompact integration leads to a divergence associated with the single particle.

number of particles. Again we have to deal with a moduli space, but now it becomes a quotient of the infinite volume moduli space by some set of identifications Γ which are induced by the gluing procedure. This gluing procedure is not completely trivial as one has to take into account the classical time delays due to particle scattering. Using this procedure we may decompose the finite volume expectation value in terms of diagonal infinite volume form factors and coefficients involving (the classical limit of) Bethe ansatz Jacobian subdeterminants. This is a very nontrivial consistency check of our proposal for the classical formula for the connected diagonal form factor. All this is discussed in section 4 of the present paper.

3 Diagonal form factors and expectation values in infinite volume

In this section we summarize the definition of diagonal form factors. We propose formulas for their classical counterparts and check our ideas on the example of the sine-Gordon theory.

3.1 Diagonal form factors

Form factors are the matrix elements of local operators between asymptotic (initial or final) states:

$$\langle p_m, \ldots, p_1 | \mathcal{O}(x,t) | p'_1, \ldots, p'_n \rangle = e^{i\Delta E t - i\Delta P x} \langle p_m, \ldots, p_1 | \mathcal{O} | p'_1, \ldots, p'_n \rangle \qquad (3.1)$$

In an initial state particles are ordered as $p'_1 > \cdots > p'_n$, while in a final state oppositely. These two types of states are connected by the multiparticle scattering matrix, which factorizes into the product of two particle scatterings:[2]

$$|p_1, \ldots, p_n\rangle = \prod_{i<j} S(p_i, p_j) |p_n, \ldots, p_1\rangle \qquad (3.2)$$

The two particle scattering matrix satisfies unitarity $S(p_1, p_2) S(p_2, p_1) = 1$. The adjoint state is denoted by $|p_1, \ldots, p_n\rangle^\dagger = \langle p_n, \ldots, p_1 |$ and we choose the following normalization

$$\langle p_n, \ldots, p_1 | p'_1, \ldots, p'_n \rangle = \prod_{i=1}^{n} 2\pi E(p_i) \delta(p_i - p'_i) \qquad (3.3)$$

Both the initial and final states are eigenstates of the conserved charges including the momentum and the Hamiltonian

$$P|p_1, \ldots, p_n\rangle = \sum_{i=1}^{n} p_i |p_1, \ldots, p_n\rangle \quad ; \qquad H|p_1, \ldots, p_n\rangle = \sum_{i=1}^{n} E(p_i) |p_1, \ldots, p_n\rangle \qquad (3.4)$$

As the Hamiltonian generates time, while the momentum space evolution the space-time dependence of the matrix element can be easily determined (3.1), where Δ denotes the

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[2]We assume that we are either in a theory with one single particle type, or in a diagonally scattering subsector of a nondiagonal theory, otherwise, we have to decorate both the states and the scattering matrix with color labels.

difference of the quantities on the two sides. In particular, the diagonal matrix element is independent of the insertion point and depends only on one set of momenta. This diagonal limit is not well defined, however, due to disconnected terms. Indeed, let us shift the momenta between the two sets of rapidities as $p_i' = p_i + \epsilon_i$ and investigate the $\epsilon_i \to 0$ limit. Crossing relation [3] allows one to put a particle with momentum p from the final state into an antiparticle with momentum $\bar{p}$ in the initial state as

$$\langle p_n, \ldots, p_2, p_1 | \mathcal{O} | p_1', p_2', \ldots p_n' \rangle = \langle p_n, \ldots, p_2 | \mathcal{O} | \bar{p}_1, p_1', \ldots, p_n' \rangle \tag{3.5}$$
$$+ \langle p_1 | p_1' \rangle \langle p_n, \ldots, p_2 | \mathcal{O} | p_2', \ldots, p_n' \rangle + \ldots$$

where we kept explicitly only the disconnected piece which survives in the diagonal limit. By crossing all particles and keeping only the relevant disconnected terms we can express the diagonal matrix element in terms of the "elementary" form factors — having vacuum in the adjoint state — as

$$\langle p_n, \ldots, p_1 | \mathcal{O} | p_1', \ldots p_n' \rangle = \langle 0 | \mathcal{O} | \bar{p}_n, \ldots, \bar{p}_1, p_1', \ldots, p_n' \rangle \tag{3.6}$$
$$+ \sum_i \langle p_i | p_i' \rangle \langle 0 | \mathcal{O} | \bar{p}_n, \ldots, \hat{\bar{p}}_i, \ldots, \bar{p}_1, p_1', \ldots, \hat{p}_i', \ldots, p_n' \rangle$$
$$+ \sum_{i,j} \langle p_i, p_j | p_i', p_j' \rangle \langle 0 | \mathcal{O} | \bar{p}_n, \ldots, \hat{\bar{p}}_i, \ldots, \hat{\bar{p}}_j, \ldots, \bar{p}_1, p_1', \ldots, \hat{p}_i', \ldots, \hat{p}_j' \ldots, p_n' \rangle + \ldots$$

where terms with hats are absent. In taking the diagonal limit $p_i' \to p_i$ we face two types of divergences. First, the states are normalized to delta functions (3.3). This can be cured either by subtracting the disconnected pieces or by putting the system into a finite volume. The second singularity type comes from taking the limit in the elementary form factor:

$$\langle 0 | \mathcal{O} | \bar{p}_n, \ldots, \bar{p}_1, p_1 + \epsilon_1, \ldots, p_n + \epsilon_n \rangle = \frac{\sum_{\{i_1, \ldots, i_n\}} a_{i_1 \ldots i_n} \epsilon_{i_1} \ldots \epsilon_{i_n}}{\epsilon_1 \ldots \epsilon_n} + \ldots \tag{3.7}$$

where we indicated the most singular terms. Clearly the expression depends on which way we take the diagonal limit. There are two typical definitions: the symmetric and the connected ones. In this paper we focus only on the connected evaluation,[3] which is defined as the finite, ϵ-independent, term in the expansion:

$$F_n(p_1, \ldots, p_n) = n! a_{1 \ldots n} \tag{3.8}$$

With this definition the diagonal matrix element, what we also call as the expectation value, can be formally written as:

$$\langle p_1, \ldots, p_n | \mathcal{O} | p_n, \ldots p_1 \rangle = \sum_{A \subseteq \{1, \ldots, n\}} \langle A | A \rangle F_{|\bar{A}|} \{\bar{A}\} \tag{3.9}$$
$$= F_n + \sum_i \langle i | i \rangle F_{n-1} \{1, \ldots, \hat{i}, \ldots n\}$$
$$+ \sum_{i,j} \langle i, j | j, i \rangle F_{n-2} \{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\} + \ldots$$

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[3]The other can be easily obtained by the kinematical singularity axiom of the form factors [25].

where $\bar{A}$ is the complement of A i.e. $\bar{A} = \{1, \ldots, n\} \setminus A$. We give a more concrete meaning of this formula by putting the system into a finite volume and carefully defining the innerproducts of the states. Alternatively, assuming that we can evaluate the expectation values, we can express the connected diagonal form factors recursively. We spell out the details for the 1 and 2-particle states: the 1-particle expectation value can be written as

$$\langle p|\mathcal{O}|p\rangle = F_1(p) + \langle p|p\rangle F_0 \tag{3.10}$$

or, alternatively, the connected diagonal form factor in terms of the expectation value reads as

$$F_0 = \langle 0|\mathcal{O}|0\rangle \quad ; \qquad F_1(p) = \langle p|\mathcal{O}|p\rangle - \langle p|p\rangle\langle 0|\mathcal{O}|0\rangle \tag{3.11}$$

The analogous relations for the two particle case are as follows

$$\langle p_2, p_1|\mathcal{O}|p_1, p_2\rangle = F_2(p_1, p_2) + \langle p_1|p_1\rangle F_1(p_2) + \langle p_2|p_2\rangle F_1(p_1) + \langle p_1, p_2|p_2, p_1\rangle F_0 \tag{3.12}$$

or alternatively

$$\begin{aligned} F_2(p_1, p_2) &= \langle p_2, p_1|\mathcal{O}|p_1, p_2\rangle - \langle p_1|p_1\rangle\langle p_2|\mathcal{O}|p_2\rangle - \langle p_2|p_2\rangle\langle p_1|\mathcal{O}|p_1\rangle + \langle p_1, p_2|p_2, p_1\rangle\langle 0|\mathcal{O}|0\rangle \\ &= \langle p_2, p_1|\mathcal{O}|p_1, p_2\rangle - \langle p_1|p_1\rangle F_1(p_2) - \langle p_2|p_2\rangle F_1(p_1) - \langle p_1, p_2|p_2, p_1\rangle F_0 \end{aligned} \tag{3.13}$$

We will see analogous relations in the classical limit.

3.2 Classical limit of diagonal form factors

In this subsection we propose an expression for the classical limit of the previously introduced diagonal form factors. In describing the limit we first note that the diagonal form factor can be thought of as the regularized quantum average of the operator $\mathcal{O}(\hat{\varphi}(x,t))$ in a given energy-momentum eigenstate. In the classical limit the operator will be replaced by the function of the field $\mathcal{O}(\varphi(x,t))$, while the state should correspond to a multiparticle solution with the same energy and momentum. Finite energy solutions in a classical integrable theory have multiparticle interpretations: the energy density is well concentrated around separated segments of straight lines. They are thought of as trajectories of particles, which interact locally, only when they get close to each other. Changing the initial location of a given particle leads to another solution with the same energy. Consequently, the space of n-particle solutions with a given energy has a moduli space isomorphic to $\mathbb{R}^n$. The quantum average of the operator $\mathcal{O}(\hat{\varphi}(x,t))$ should correspond in the classical limit to an average of the function $\mathcal{O}(\varphi(x,t))$ over this moduli space. The infinities, however, which appear for the expectation values in the quantum theory, are present also at the classical level, thus we need to introduce proper subtractions. Performing these subtractions we define a localized function, which we integrate over the moduli space of the classical solutions with a given energy. As the strong coupling limit of the HHL solutions can be mapped by the Pohlmeyer reduction to relativistic scattering theories we focus in this section on relativistic theories. We analyze the infinite volume multiparticle solutions first and then focus on the corresponding form factors. Sometimes it is useful to have explicit solutions

in mind. For this reason we provide explicit formulas for the sine-Gordon theory which is defined by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{\beta^2}(1 - \cos\beta\varphi) \qquad (3.14)$$

3.2.1 Classical solutions and their moduli space

We consider an integrable classical field theory, which admits finite energy localized solutions allowing for multiparticle interpretation. We introduce the moduli space of these solutions by proceeding in the particle number.

Vacuum. The vacuum solution is a translational invariant –constant– solution of the equation of motion, which we denote by φ_0. Its moduli space is discrete and usually consists of one single point. In the sine-Gordon case this point is $\varphi_0 = 0 \equiv \pm\frac{2\pi}{\beta}$.

1-particle. The simplest 1-particle solution is the static solution,[4] $\varphi_{\text{st}}(mx)$. The energy density of this solution, $\epsilon[\phi_{\text{st}}(mx)]$, is localized sharply around one point, which we choose to be the origin, $x = 0$. Shifting this point spans the moduli space of the static solutions. Each solution can be interpreted as a standing particle.

The moving 1-particle solution can be obtained by boosting the static solution:

$$\varphi_{\text{st}}(m\cosh\theta x - m\sinh\theta(t - t_1)) = \varphi_{\text{st}}(Ex - p(t - t_1)) = \varphi_{\text{st}}(E(x - x_1) - pt) \qquad (3.15)$$

By introducing the dimensionless variable $y = Ex - pt$ we can write the moving solution in the form

$$\varphi_1(x, t; y_1) \equiv \varphi_{\text{st}}(y - y_1) \qquad (3.16)$$

Due to translational invariance the shifted solution is also a solution and we parametrize the moduli space of the classical 1-particle solutions — of a given momentum — by $y_1 \in \mathbb{R}$, which is $y_1 = Ex_1$. We choose the sign of y_1, such that $y_1 \to \pm\infty$ shifts the particle's trajectory to $\pm\infty$. We assume that the theory has no internal symmetry, so the translation y_1, is the only continuous parameter of the moduli space. This moduli space is denoted by $\mathcal{M}^1 = \mathbb{R}$. The 1-particle solution with a given momentum p can be considered as a function of the space-time coordinates and the moduli parameter y_1 and we denoted this function by $\varphi_1(x, t; y_1)$, i.e. we do not write out explicitly the momentum dependence. As the energy density of the solution is concentrated around the zero of the argument of the static solution, we can think of this solution in terms of a particle's trajectory:

$$x(t) = v(t - t_1) = vt + x_1 \quad ; \qquad v = \tanh\theta \qquad (3.17)$$

In the sine-Gordon theory the solutions can be most conveniently expressed in terms of $\tan\frac{\beta\varphi}{4}$. In particular, the 1-particle solution, $\varphi_1(x, t; y_1) \equiv \varphi_1$, reads as

$$e_1 \equiv \tan\frac{\beta\varphi_1}{4} = e^{m\cosh\theta_1(x - x_1) - m\sinh\theta_1 t} = e^{y - y_1} \qquad (3.18)$$

It interpolates between 0 and $\frac{2\pi}{\beta}$ and is called the soliton. Anti-solitons can be described either by $-e_1$ or by e_1^{-1}. Actually $-e_1^{-1}$ is the soliton again.

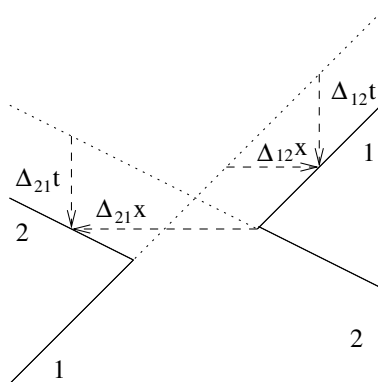[4]We introduced a mass parameter m to make x dimensionless.

Figure 1. Relativistic particle scattering process, in which particle 1 comes from the left and after scattering on particle 2 coming from the right it experiences a $\Delta_{12}x$ space-displacement and $\Delta_{12}t$ time-delays. Particle 2 has the analogous $\Delta_{21}x$ space-displacement and $\Delta_{21}t$ time delays.

2-particle. The 2-particle solution with momenta p_1 and p_2 denoted by $\varphi_2(x,t;y_1,y_2)$ generalizes the 1-particles solution as follows: the moduli space, $\mathcal{M}^2 = \mathbb{R}^2$, has two parameters y_1 and y_2, which are the respective shifts in the particles' trajectories, $y_i = E_i x_i$, such that $y_i \to \infty$ shifts particle i to $+\infty$. Upto a localized interaction domain, the solution is the composition of two 1-particle solutions. These 1-particle solutions, however are not the same before and after their interaction: there is a space displacement and a time delay. Focusing on the energy density we can interpret the 2-particle solution in terms of a collision process as follows. The particles travel freely

$$x_1(t) = v_1 t + x_1^- = v_1(t - t_1^-) \quad ; \quad x_2(t) = v_2 t + x_2^- = v_2(t - t_2^-) \tag{3.19}$$

before they interact, say at time $t = 0$. After the interaction they travel freely again as

$$x_1(t) = v_1 t + x_1^+ = v_1(t - t_1^+) \quad ; \quad x_2(t) = v_2 t + x_2^+ = v_2(t - t_2^+) \tag{3.20}$$

The result of the interaction is the experienced time delays or space displacements:

$$\Delta_{12}t = t_1^+ - t_1^- \quad ; \quad \Delta_{21}t = t_2^+ - t_2^- \quad ; \quad \Delta_{12}x = x_1^+ - x_1^- \quad ; \quad \Delta_{21}x = x_2^+ - x_2^- \tag{3.21}$$

We show the scattering process on the schematic figure 1, where, to be specific, we assumed that $p_1 > 0, p_2 < 0$ such that the space displacements have opposite signs: $\Delta_{12}x = -v_1 \Delta_{12}t > 0$ and $\Delta_{21}x = -v_2 \Delta_{21}t < 0$. These displacements are not the same for the two particles but can be related via the free movement of the center of energy:

$$\frac{E_1 x_1 + E_2 x_2}{E_1 + E_2} = \frac{E_1 v_1 + E_2 v_2}{E_1 + E_2}t + \frac{E_1 x_1^\pm + E_2 x_2^\pm}{E_1 + E_2} \tag{3.22}$$

Expressing this motion in terms of the quantities before and after the interaction leads to the relations

$$E_1 \Delta_{12}x + E_2 \Delta_{21}x = 0 \quad ; \quad p_1 \Delta_{12}t + p_2 \Delta_{21}t = 0 \tag{3.23}$$

where we used that $Ev = p$. As the 2-particle solution is the classical limit of a scattering process it is interesting to relate the appearing quantities to the S-matrix. The energy

derivative of the phase shift of the S-matrix is related in the semiclassical limit to the time delay as [26]:

$$(\partial_{E_1} p_1)\partial_{p_1}\delta(p_1, p_2) \to \Delta_{12}t \qquad ; \qquad S = e^{i\delta(p_1, p_2)} \qquad (3.24)$$

In particular, we can relate the time delays and space displacements to the classical limit, ϕ_{12}^c, of the quantity $\phi(p_1, p_2) = E_1\partial_{p_1}\delta(p_1, p_2)$ as

$$\phi(p_1, p_2) = E_1\partial_{p_1}\delta(p_1, p_2) \to \phi_{12}^c = E_1\frac{\partial E_1}{\partial p_1}\Delta_{12}t = p_1\Delta_{12}t = -E_1\Delta_{12}x = -\Delta_{12}y \quad (3.25)$$

i.e. the shift in the moduli space is nothing but the classical limit of $-\phi(p_1, p_2)$. The shift for the other particle is

$$-\phi(p_2, p_1) = E_2\partial_{p_2}\delta(p_1, p_2) \to -\phi_{21}^c = E_2\frac{\partial E_2}{\partial p_2}\Delta_{21}t = p_2\Delta_{21}t = -E_2\Delta_{21}x = -\Delta_{21}y$$
$$(3.26)$$

We can see from (3.23) that the shifts in the moduli parameters sum up to zero: $\Delta_{12}y + \Delta_{21}y = 0$. This motivates us to work with this moduli parameter and not with the space displacements or time delays.

In the sine-Gordon theory the 2-soliton solution, $\varphi_2(x, t, y_1, y_2) \equiv \varphi_2$, can be written in terms of the two 1-soliton solutions as [27]

$$\tan\frac{\beta\varphi_2}{4} \equiv e_{12} = \frac{e_1 + e_2}{1 - u_{12}^2 e_1 e_2} \quad ; \quad u_{12} = \tanh\frac{\theta_1 - \theta_2}{2} \quad ; \quad e_i = e^{m\cosh\theta_i x - m\sinh\theta_i t - y_i}$$
$$(3.27)$$

This solution, except for some local interaction domain, can be considered as two non-interacting one soliton solutions. The effect of the interaction is that the solitons experience some time delays. To calculate these time delays we analyze the solutions in the asymptotic limits. As the energy density is proportional to $\frac{1}{(e_{12}+e_{12}^{-1})^2}$ the nontrivial contributions come from the domains when e_i is not close either to 0 or to ∞. These are the places where the solitons are localized and agree with the zero of the exponent of e_i: $E_ix - p_it - y_i = 0$. Analyzing the $t \to -\infty$ limit we can see two nontrivial domains contributing. For $x < 0$ the quantity e_2 vanishes, while for $x > 0$ the other e_1 goes to infinity leading to

$$e_{12} = \begin{cases} e_1 & \text{for} \quad x < 0 \\ -\dfrac{1}{u_{12}^2 e_2} & \text{for} \quad x > 0 \end{cases} \qquad (3.28)$$

We can reparametrize the $x > 0$ soliton as

$$-\frac{1}{u_{12}^2 e_2} \to u_{12}^2 e_2 = e^{m\cosh\theta_2 x - m\sinh\theta_2 t + \phi_{12}^c} \quad ; \quad \phi_{12}^c = \log\left(\tanh^2\left(\frac{\theta_1 - \theta_2}{2}\right)\right) \quad (3.29)$$

In the $t \to \infty$ we found

$$e_{12} = \begin{cases} e_2 & \text{for} \quad x < 0 \\ -\dfrac{1}{u_{12}^2 e_1} & \text{for} \quad x > 0 \end{cases} \qquad (3.30)$$

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Parametrizing the particles' trajectories before and after the collision as $E_i x - p_i(t - t_i^\pm) = 0$ we can read off that before the collision $t_1^- = 0$ and $t_1^+ = p_1^{-1}\phi_{12}^c$, while after the collision $t_2^+ = 0$ and $t_2^- = p_2^{-1}\phi_{12}^c$. These lead to the following time delays:

$$\Delta_{12}t = t_1^+ - t_1^- = \frac{\phi_{12}^c}{m\sinh\theta_1} = \frac{\phi_{12}^c}{p_1} \quad ; \quad \Delta_{21}t = t_2^+ - t_2^- = 0 - \frac{\phi_{12}^c}{m\sinh\theta_2} = -\frac{\phi_{12}^c}{p_2} \quad (3.31)$$

Clearly the relation $p_1\Delta_{12}t + p_2\Delta_{21}t = 0$ is satisfied.

n-particle. The n particle solution with momenta $p_1, \ldots, p_n$ denoted by $\varphi_n(x, t; y_1, \ldots, y_n)$ depends on the space-time coordinates and on the moduli parameters $y_i \in \mathcal{M}^n = \mathbb{R}^n$, which are the respective translations of each individual particles. By shifting the leftmost particle to $y_1 \to -\infty$ the n particle solution reduces to the $n-1$ particle solution: $\varphi_n(x, t; \infty, y_2 \ldots, y_n) = \varphi_{n-1}(x, t; y_2, \ldots, y_n)$. By shifting the same particle to $y_1 \to \infty$ it scatters on each particle and suffers $\sum_{j=2}^n \Delta_{1j}y$ displacements. Additionally, it shifts the other particles by $\Delta_{j1}y$ leading to the solution $\varphi_n(x, t; \infty, y_2, \ldots, y_n) = \varphi_{n-1}(x, t; y_2 + \Delta_{21}y_2, \ldots, y_n + \Delta_{n1}y)$. In general, the n particle solution reduces to the $n-k$ particle solution, whenever the other k particles are translated to infinity.

In the sine-Gordon theory the n-soliton solution, $\varphi_n(x, t; y_1, \ldots, y_n) \equiv \varphi_n$, can be written as [27]

$$\tan\frac{\beta\varphi_n}{4} = \frac{\Im m(\tau)}{\Re e(\tau)} \quad ; \qquad \tau = \sum_{\mu_j=\{0,1\}} \prod_{j=1}^n (ie_j)^{\mu_j} \prod_{i<j} u_{ij}^{2\mu_i\mu_j} \qquad (3.32)$$

where

$$e_i = e^{m\cosh\theta_i x - m\sinh\theta_i t - y_i} \quad ; \qquad u_{ij} = \tanh\frac{\theta_i - \theta_j}{2} \qquad (3.33)$$

The classical time delay of the ith particle after passing through the jth particle is independent of the other particles and reads as

$$\Delta_{ij}t = \frac{\phi_{ij}^c}{p_i} \quad ; \qquad \phi_{ij}^c = \log\tanh^2\frac{\theta_i - \theta_j}{2} \qquad (3.34)$$

3.2.2 Classical form factors

As we mentioned before the quantum average of the operator $\mathcal{O}(\hat\varphi(x,t))$ should correspond in the classical limit to an average of the function $\mathcal{O}(\varphi(x,t))$ over the moduli space of classical solutions. Since the infinities which appear for the expectation values in the quantum theory are present also at the classical level we develop proper subtraction procedure. We proceed in the particle number. For reference we present the form factors of the trace of the energy-momentum tensor in the sine-Gordon theory

$$\Theta^c(\varphi) = \frac{m^2}{\beta^2}(1 - \cos\beta\varphi) = \frac{8m^2}{\beta^2\left(\tan\frac{\beta\varphi}{4} + \cot\frac{\beta\varphi}{4}\right)^2} \qquad (3.35)$$

Vacuum. The classical limit of the vacuum is the constant classical vacuum solution φ_0 and the classical limit of the vacuum expectation value of the operator $\mathcal{O}(\hat\varphi)$ is simply its value $\mathcal{O}[\varphi_0]$. If there are many vacua then the expression might depend on which vacuum we evaluate the operator.

In the sine-Gordon theory $\Theta(\varphi)$ is vanishing on the vacuum $\varphi_0 = 0 \equiv \frac{2\pi}{\beta}$.
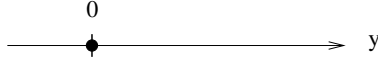
Figure 2. One particle moduli space $\mathcal{M}_1 = \mathbb{R}$. Black dot indicates the point, whose neighbourhood contributes to the 1-particle form factor.

1-particle. The classical limit of a 1-particle asymptotic state is the moving 1-particle solution. Its moduli space is $\mathcal{M}_1 = \mathbb{R}$ and the classical analogue of the quantum expectation value should be the average for the moduli parameter y_1:

$$\langle p|\mathcal{O}|p\rangle^c \to \int_{\mathcal{M}^1} dy_1 \, \mathcal{O}(\varphi_1(x,t,y_1)) \tag{3.36}$$

Similarly, however, to the quantum case (3.10) the expression is divergent if the operator has a vacuum expectation value. Analogy with the quantum case suggests to define the diagonal form factor after a proper subtraction (3.11): we define the classical 1-particle diagonal form factor of the operator $\mathcal{O}(\varphi)$ to be the integral

$$F_1^c = \int_{-\infty}^{\infty} dy_1 \, \{\mathcal{O}(\varphi_1(x,t,y_1)) - \mathcal{O}(\varphi_0)\} \tag{3.37}$$

As the 1-particle solution agrees with the vacuum solution away from the trajectory of the particle the function $\mathcal{O}(\varphi_1) - \mathcal{O}(\varphi_0)$ is well localized. Consequently, the integral has a finite support and gives a finite result. As the moduli parameter y_1 shifts the classical solution (both in space and in time) the integral is actually independent of the space-time coordinates (x,t). This fits very well to the picture of being the classical limit of the quantum diagonal form factor, which is also space-time independent. As this will be true also for multiparticle form factors we put $x = t = 0$ and omit to write out the space-time coordinates $\varphi_n(y_1,\dots,y_n) \equiv \varphi_n(0,0;y_1,\dots,y_n)$. To further simplify our formulas we analyze operators without vacuum expectation values. This can be easily arranged by redefining the operators as $\mathcal{O}(\varphi) \to (\mathcal{O}(\varphi) - \mathcal{O}(\varphi_0))$. These newly defined observables are then localized where the particles are localized. In particular, the 1-particle integral (3.37) collects its contribution from a small domain around $y_1 = 0$, which is indicated with a black dot in the moduli space.

In the sine-Gordon theory the one-particle connected diagonal form factor of Θ is

$$F_1^\Theta(\theta) = \frac{1}{4}(F_1^{T_{00}} - F_1^{T_{11}}) = \frac{M^2}{4}(\cosh^2\theta - \sinh^2\theta) = \frac{M^2}{4} \tag{3.38}$$

where M is the soliton mass. Let us calculate the classical form factor from (3.37):

$$F_1^c = \int_{-\infty}^{\infty} dy_1 \, \frac{m^2}{\beta^2}(1 - \cos\beta\varphi_1) = \frac{8m^2}{\beta^2}\int_0^{\infty} \frac{de_1}{e_1}\frac{1}{(e_1 + e_1^{-1})^2} = \frac{4m^2}{\beta^2} \tag{3.39}$$

which is consistent with the quantum formula as the classical limit of the soliton mass is $M^c = \frac{4m}{\beta}$.
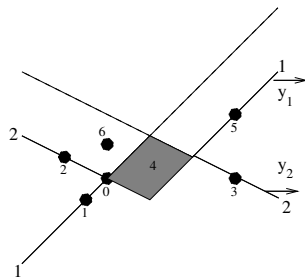
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Figure 3. The insertion of the operator as compared to the two particle scattering solutions. Special configurations for the operator are labeled from 0 to 6. 0 labels the location of the operator for $y_1 = 0$ and $y_2 = 0$ and it is assumed that we collect contributions in this case. 1: $y_2 > 0$ only contribution from particle 1 at $y_1 = 0$. 2: $y_1 > 0$ only contribution from particle 2 at $y_2 = 0$. 3: $y_1 < \phi_{12}^c$ only contribution from particle 2 at $y_2 = \phi_{12}^c$. 4: $\phi_{12}^c < y_1 < 0$ and $\phi_{12}^c < y_2 < 0$ two-particle contribution. 5: $y_2 < \phi_{12}^c$ only contribution from particle 1 at $y_1 = \phi_{12}^c$. 6: generic point, not mentioned above: no contribution at all.

2-particle. The moduli space of the two particle solution, $\mathcal{M}_2 = \mathbb{R}^2$, contains separate shifts in each particle's locations y_1 and y_2. The classical analogue of the quantum average should correspond to the integral

$$\langle p_2, p_1 | \mathcal{O} | p_1, p_2 \rangle^c \to \int_{\mathcal{M}^2} dy_1 dy_2 \mathcal{O}(\varphi_2(y_1, y_2)) \tag{3.40}$$

However, as the quantum formula (3.13) suggests the integral is infinite whenever the one particle form factor is nonzero. Indeed, for operators without vacuum expectation value, the contributions come from the trajectories of the particles, which form the scattering process on figure 1. Let us analyze this two particle scattering picture and insert the operator at the origin to see the effects of the various shifts in y_i. Technically it is simpler to draw the particle trajectories unchanged and shift the operator in the opposite way, see figure 3. The characteristic quantity in the process is the space-time displacements, which translate to the moduli parameter as $\Delta_{12}y = E_1 \Delta_{12}x = -\phi_{12}^c = -\Delta_{21}y = -E_2 \Delta_{21}x$.

The translation of figure 3 into the moduli space tells the domain where the particles are located or, equivalently, the domain where the function $\mathcal{O}(\varphi_{12}(y_1, y_2))$ is non-vanishing and the integral (3.40) collects its contributions from. See the left of figure 4. Near the 1-particle lines the other particle is far away and the solution can be approximated with a 1-particle solution, which depends only on one moduli parameter. The integral for the other moduli parameter will then give infinite contribution. To define a finite quantity we have to subtract the contributions of the infinite one particle lines. These one particle lines are not the same before and after the interactions, i.e. they are shifted by $\Delta_{12}y = -\phi_{12}^c$. The interaction domain is localized within a square of size $\Delta_{12}y$ and in subtracting the one particle lines we have an ambiguity in choosing the end and the start of the shifted semi-infinite lines. Different choices lead to different form factors and we present here only the one, which corresponds to the classical limit of the connected form factors, see the right of figure 4. From the subtraction point of view we consider the interaction to be point like at $y_1 = y_2 = 0$. For $y_1 < 0$ we shift particle 1 to $-\infty$, while for $y_1 > 0$ we shift it to $+\infty$
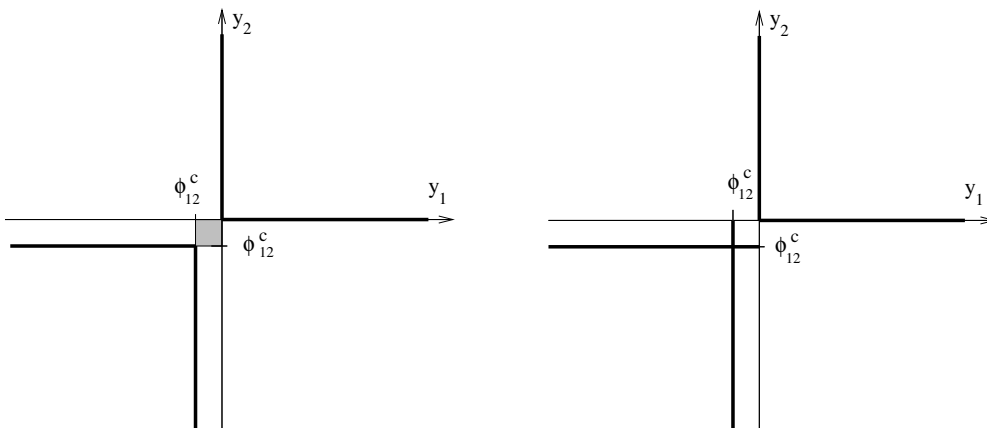
Figure 4. Domains in the moduli space $\mathcal{M}_2$ where the function $\mathcal{O}(\varphi_{12}(y_1, y_2))$ takes non-vanishing contributions are indicated on the left. Subtracted one particle contributions are indicated on the right.

and subtract the obtained contributions. We repeat the same for particle 2 and arrive at the definition of the classical two particle diagonal form factor:

$$F_2^c(p_1, p_2) = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \Big[\mathcal{O}[\varphi_2(y_1, y_2)] - \Theta(-y_1)\mathcal{O}[\varphi_2(-\infty, y_2)] \tag{3.41}$$

$$-\Theta(y_1)\mathcal{O}[\varphi_2(\infty, y_2)] - \Theta(-y_2)\mathcal{O}[\varphi_2(y_1, -\infty)] - \Theta(y_2)\mathcal{O}[\varphi_2(y_1, \infty)] \Big]$$

This integrand is localized in both moduli parameter in a finite domain around the origin denoted by the shadowed region on the left of figure 4, i.e. on $\phi_{12}^c < y_1 < 0$ and $\phi_{12}^c < y_2 < 0$.

In the sine-Gordon theory the 2-particle connected form factor of Θ is

$$F_2^\Theta(\theta_1 - \theta_2) = \frac{1}{4}(F_2^{T_{00}} - F_2^{T_{11}}) = \frac{M^2}{4} 2\phi_{12}(\cosh\theta_1 \cosh\theta_2 - \sinh\theta_1 \sinh\theta_2)$$

$$= \frac{M^2}{2}\phi_{12}\cosh(\theta_1 - \theta_2) \tag{3.42}$$

We can compare the classical limit of this expression with our definition, which reads as

$$F_2^c(\theta_1, \theta_2) = \frac{8m^2}{\beta^2} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \left[\frac{1}{(e_{12} + e_{12}^{-1})^2} - \Theta(-y_1)\frac{1}{(u_{12}^2 e_2 + (u_{12}^2 e_2)^{-1})^2} \right. \tag{3.43}$$

$$\left. -\Theta(y_1)\frac{1}{(e_2 + e_2^{-1})^2} - \Theta(-y_2)\frac{1}{(u_{12}^2 e_2 + (u_{12}^2 e_2)^{-1})^2} - \Theta(y_2)\frac{1}{(e_1 + e_1^{-1})^2} \right]$$

Alternatively we can change the integration variables for e_1 and e_2:

$$F_2^c(\theta_1, \theta_2) = \frac{8m^2}{\beta^2} \int_0^{\infty} \frac{de_1}{e_1} \int_0^{\infty} \frac{de_2}{e_2} \left[\frac{1}{(e_{12} + e_{12}^{-1})^2} - \Theta(1 - e_1)\frac{1}{(u_{12}^2 e_2 + (u_{12}^2 e_2)^{-1})^2} \right.$$

$$-\Theta(e_1 - 1)\frac{1}{(e_2 + e_2^{-1})^2} - \Theta(1 - e_2)\frac{1}{(u_{12}^2 e_2 + (u_{12}^2 e_2)^{-1})^2}$$

$$\left. -\Theta(e_2 - 1)\frac{1}{(e_1 + e_1^{-1})^2} \right] \tag{3.44}$$

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Since $e_{12} = \frac{e_1 + e_2}{1 - u_{12}^2 e_1 e_2}$ the integral depends only on u_{12}^2. We managed to perform this integral and obtained

$$F_2^c(\theta_1, \theta_2) = -\frac{4m^2}{\beta^2} \frac{u_{12}^2 + 1}{u_{12}^2 - 1} \log u_{12}^4 = \frac{8m^2}{\beta^2} \cosh(\theta_1 - \theta_2) \log \tanh^2 \frac{\theta_1 - \theta_2}{2} \qquad (3.45)$$

which is the classical limit of the connected diagonal form factor.

For the application of the HHL three point functions we calculate the classical form factors of the operators

$$\mathcal{O}_k(\varphi) = e^{ik\beta\varphi} - 1 \qquad (3.46)$$

in the sine-Gordon theory. As these operators do not have any vacuum expectation value the 1-particle form factor is obtained as

$$F_1^{\mathcal{O}_k} = \int_{-\infty}^{\infty} dy_1 \, (e^{ik\beta\varphi_1} - 1) = \int_0^{\infty} \frac{de_1}{e_1} \left\{ \left(\frac{2i - e_1 + e_1^{-1}}{e_1 + e_1^{-1}} \right)^{2k} - 1 \right\} \qquad (3.47)$$

Performing the integral we found

$$F_1^{\mathcal{O}_k} = \left\{ -4, -\frac{16}{3}, -\frac{92}{15}, -\frac{704}{105} \right\} \quad ; \qquad k = 1, 2, 3, 4 \qquad (3.48)$$

for the first few cases. In the following we focus on the two particle form factors and evaluate the general formula

$$F_{2,c}^{\mathcal{O}_k} = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \Big[\mathcal{O}_k[\varphi_2(y_1, y_2)] - \Theta(-y_1) \mathcal{O}_k[\varphi_2(-\infty, y_2)] \qquad (3.49)$$

$$-\Theta(y_1) \mathcal{O}_k[\varphi_2(\infty, y_2)] - \Theta(-y_2) \mathcal{O}_k[\varphi_2(y_1, -\infty)] - \Theta(y_2) \mathcal{O}_k[\varphi_2(y_1, \infty)] \Big]$$

First, using the definition of e_{12}, we can rewrite the operator as

$$\mathcal{O}_k(\varphi) = e^{ik\beta\varphi} - 1 = \left(\frac{2i - e_{12} + e_{12}^{-1}}{e_{12} + e_{12}^{-1}} \right)^{2k} - 1 \qquad (3.50)$$

The integrand can alternatively be reformulated as

$$\mathcal{O}_k(y_1, y_2) - \frac{1}{1 + e^{y_1}} \mathcal{O}_k(-\infty, y_2) - \frac{1}{1 + e^{y_2}} \mathcal{O}_k(y_1, -\infty)$$

$$- \frac{e^{y_1}}{1 + e^{y_1}} \mathcal{O}_k(\infty, y_2) - \frac{e^{y_2}}{1 + e^{y_2}} \mathcal{O}_k(y_1, \infty) \qquad (3.51)$$

since the difference integrates to zero. In the following we change variables from (y_1, y_2) to (e_1, e_2). Clearly the integral depends only on $u_{12} = \tanh \frac{\theta_1 - \theta_2}{2}$, what we abbreviate by u in the following. We performed the two integrals one after the other and obtained the following result:

$$\begin{pmatrix} F_2^{\mathcal{O}_1}(\theta_1, \theta_2) \\ F_2^{\mathcal{O}_2}(\theta_1, \theta_2) \\ F_2^{\mathcal{O}_3}(\theta_1, \theta_2) \\ F_2^{\mathcal{O}_4}(\theta_1, \theta_2) \end{pmatrix} = \begin{pmatrix} \frac{16(u^2+1)}{u^2-1} \\ \frac{64(u^2+1)^3}{3(u^2-1)^3} \\ \frac{16(23u^{10}+155u^8+590u^6+590u^4+155u^2+23)}{15(u^2-1)^5} \\ \frac{256(u^2+1)^3(11u^8+100u^6+738u^4+100u^2+11)}{105(u^2-1)^7} \end{pmatrix} \log u \quad ; \quad u = \tanh \frac{\theta_1 - \theta_2}{2}$$

$$(3.52)$$

The higher formulas get heavy after this point and but it would be nice to have a compact expression for them.

n-particle. In the case of n-particles the quantum average should go to the classical moduli average:

$$\langle p_n, \ldots, p_1 | \mathcal{O} | p_1, \ldots, p_n \rangle^c \to \int_{\mathcal{M}^n} dy_1 \ldots dy_n \mathcal{O}(\varphi_n(y_1, \ldots, y_n)) \tag{3.53}$$

To regulate this expression we have to subtract successively the lower particle number contributions in the spirit of (3.9). The subtraction, which corresponds to the classical limit of the diagonal connected form factor reads as:

$$
\begin{aligned}
F_n^c(p_1, \ldots, p_n) = \prod_i \int_{-\infty}^{\infty} dy_i \Bigg\{ &\mathcal{O}[\varphi_n(y_1, \ldots, y_n)] - \sum_{i,\epsilon_i} \Theta(\epsilon_i y_i) \mathcal{O}[\varphi_n(y_1, \ldots, \epsilon_i \infty, \ldots, y_n)] \\
&+ \sum_{i,j,\epsilon_i,\epsilon_j} \Theta(\epsilon_i y_i, \epsilon_j y_j) \mathcal{O}[\varphi_n(y_1, \ldots, \epsilon_i \infty, \ldots, \epsilon_j \infty, \ldots, y_n)] + \ldots \\
&+ (-1)^k \sum_{\{i_k,\epsilon_k\}} \Theta(\epsilon_{i_1} y_{i_1}, \ldots, \epsilon_{i_k} y_{i_k}) \mathcal{O}[\varphi_n(y_1, \ldots, \epsilon_{i_1} \infty, \ldots, \epsilon_{i_k} \infty, \ldots, y_n)] + \ldots \Bigg\} \\
&\equiv \prod_i \int_{-\infty}^{\infty} dy_i \, \mathcal{O}[\varphi_n(y_1, \ldots, y_n)]_c
\end{aligned} \tag{3.54}
$$

where $\Theta(\epsilon_{i_1} y_{i_1}, \ldots, \epsilon_{i_k} y_{i_k}) = \prod_{j=1}^{k} \Theta(\epsilon_{i_j} y_{i_j})$. We can also express each term in terms of the lower order connected terms. This unifies the signs as:

$$
\begin{aligned}
F_n^c(p_1, \ldots, p_n) = \int_{\mathcal{M}^n} d\vec{y} \Bigg\{ &\mathcal{O}[\varphi_n(y_1, \ldots, y_n)] - \sum_{i,\epsilon_i} \Theta(\epsilon_i y_i) \mathcal{O}[\varphi_n(y_1, \ldots, \epsilon_i \infty, \ldots, y_n)]_c \\
&- \sum_{i,j,\epsilon_i,\epsilon_j} \Theta(\epsilon_i y_i, \epsilon_j y_j) \mathcal{O}[\varphi_n(y_1, \ldots, \epsilon_i \infty, \ldots, \epsilon_j \infty, \ldots, y_n)]_c + \ldots \\
&- \sum_{\{i_k,\epsilon_k\}} \Theta(\epsilon_{i_1} y_{i_1}, \ldots, \epsilon_{i_k} y_{i_k}) \mathcal{O}[\varphi_n(y_1, \ldots, \epsilon_{i_1} \infty, \ldots, \epsilon_{i_k} \infty, \ldots, y_n)]_c + \ldots \Bigg\}
\end{aligned} \tag{3.55}
$$

where we denoted the integration for the moduli space as $\int_{\mathcal{M}^n} d\vec{y} = \prod_i \int_{-\infty}^{\infty} dy_i$.

4 Diagonal form factors and expectation values in finite volume

In this section we generalize the previous analysis for finite volume. We assume that the volume L is asymptotically large and neglect all exponentially small vacuum polarization effects. We start by recalling the available results for the quantum theory and then develop the classical finite volume form factors in parallel with section 2.

4.1 Finite volume diagonal form factors

We analyze a quantum field theory in a large volume L and focus on the leading (polynomial) finite size correction of the expectation values. In this approximation the finite

and infinite volume form factors differ only by the normalization of states [28]. The finite volume states $|p_1, \ldots, p_n\rangle_L$ are eigenstates of energy and momentum with the eigenvalues

$$P|p_1, \ldots, p_n\rangle_L = \sum_{k=1}^n p_k |p_1, \ldots, p_n\rangle_L \quad ; \quad H|p_1, \ldots, p_n\rangle_L = \sum_{k=1}^n E(p_k)|p_1, \ldots, p_n\rangle_L \quad (4.1)$$

which are formally the same as the ones in infinite volume. The basic difference is that a finite volume state is symmetric in the momenta, and the momenta are quantized in a volume-dependent way by the Bethe-Yang equation

$$e^{ip_k L} \prod_{j:j \neq k} S(p_k, p_j) = 1 \quad ; \quad k = 1, \ldots, N \quad (4.2)$$

In practice, we take the logarithm of this equation

$$\Phi_k = p_k L - i \sum_{j:j \neq k} \log S(p_k, p_j) = 2\pi I_k \quad (4.3)$$

and use the quantization numbers $\{I_k\}$ to label finite volume states $|p_1, \ldots, p_n\rangle_L \equiv |I_1, \ldots, I_n\rangle$. Due to the discreteness of the finite volume spectrum the states are normalized to Kronecker δ-functions:

$$\langle J_m, \ldots, J_1 | I_1, \ldots, I_n \rangle = \delta_{n,m} \delta_{I_1 J_1} \ldots \delta_{I_n J_n} \quad (4.4)$$

in contrast to the infinite volume states which are normalized to Dirac δ functions. Both the finite and infinite volume states form complete bases and we can relate them for large volumes by comparing the resolution of the identity. For large volumes the momentum eigenstates are very dense and we can change variables $\{p_i\} \rightarrow \{I_i\}$ via eq. (4.3) leading to the relation

$$|p_1, \ldots, p_n\rangle_L = \mathcal{N}|p_1, \ldots, p_n\rangle \quad ; \quad \mathcal{N}^{-1} = \sqrt{\prod_{i<j} S(p_i, p_j) \rho_n(p_1, \ldots, p_n)} \quad (4.5)$$

Here the density of states is defined by the Jacobian:

$$\rho_n(p_1, \ldots, p_n) = \det[\Phi_{ij}] \quad ; \quad \Phi_{ij} = E(p_i) \frac{\partial \Phi_j}{\partial p_i} = \left(E(p_i) L + \sum_{k=1}^n \phi_{ik} \right) \delta_{ij} - \phi_{ij} \quad (4.6)$$

We also included the multiparticle S-matrix to compensate the order dependence of the infinite volume state. We denoted the derivative of the phase of the S-matrix with respect to the first argument as

$$\phi_{jk} = \phi(p_j, p_k) = -iE(p_j) \frac{\partial}{\partial p_j} \log S(p_j, p_k) \quad (4.7)$$

The derivative w.r.t. to the second argument is related to ϕ_{jk} by unitarity: $-iE(p_k)\frac{\partial}{\partial p_k} \log S(p_j, p_k) = -\phi_{kj}$.

Using the finite volume norm of states Saleur suggested an expression for the finite volume expectation value in terms of the infinite volume connected diagonal form factors [29]:[5]

$$
\begin{aligned}
{}_L\langle p_n, \ldots, p_1 | \mathcal{O} | p_1, \ldots p_n\rangle_L & \\
= \frac{1}{\rho\{1, \ldots, n\}} \sum_A \bar{\rho}\{A\} F_{|\bar{A}|}\{\bar{A}\} & \\
= \frac{F_n + \sum_i \bar{\rho}\{i\} F_{n-1}\{1, \ldots, \hat{i}, \ldots n\} + \sum_{i,j} \bar{\rho}\{i,j\} F_{n-2}\{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\} + \ldots}{\rho\{1, \ldots, n\}} &
\end{aligned}
\tag{4.8}
$$

where

$$
\bar{\rho}\{i_1, \ldots, i_m\} = \det_{jk}\left[\Phi_{i_j i_k}\right]
\tag{4.9}
$$

is the determinant of the minor of the Jacobi matrix involving the set of labels $\{i_1, \ldots, i_m\}$. In particular, for one and two particles we have

$$
{}_L\langle p | \mathcal{O} | p \rangle_L = \frac{1}{\rho_1(p)}(F_1(p) + \rho_1(p) F_0) \quad ; \qquad \rho_1(p) = EL
\tag{4.10}
$$

$$
{}_L\langle p_2, p_1 | \mathcal{O} | p_1, p_2 \rangle_L = \frac{F_2(p_1, p_2) + \bar{\rho}_1(p_1) F_1(p_2) + \bar{\rho}_1(p_2) F_1(p_1) + \rho_2(p_1, p_2) F_0}{\rho_2(p_1, p_2)}
\tag{4.11}
$$

where

$$
\rho_2(p_1, p_2) = L^2 E_1 E_2 + L(\phi_{12} E_2 + \phi_{21} E_1) ; \quad \bar{\rho}_1(p_1) = E_1 L + \phi_{12} ; \quad \bar{\rho}_1(p_2) = E_2 L + \phi_{21}
\tag{4.12}
$$

and $E_i = E(p_i)$.

The expression (4.8) for the finite volume expectation values are valid upto exponentially small corrections. It incorporates all polynomial correction in L^{-1}, which come from two sources. Its explicit dependence sits in the norm of the states, while implicitly it depends on L via the momenta, which satisfy the Bethe-Yang equation (4.3). Observe that this expression is finite and provides a regularization of the analogous infinite volume formula (3.10).

4.2 Classical limit of expectation values

Recall that the expectation value can be thought of as the quantum average of the operator $\mathcal{O}(\hat{\varphi}(x,t))$ in a finite volume energy-momentum eigenstate. The classical analogue of this formula should be in which we integrate the function $\mathcal{O}(\varphi(x,t))$ over the moduli space of the classical finite volume solutions with the same energy and momentum. Similarly how the finiteness of the volume regularized the quantum average, the classical integral is finite, too. The quantum formula (4.8) expresses this finite average in terms of the infinite volume diagonal form factors and the sub-densities $\bar{\rho}_k$. In an analogous way we express the classical average in terms of the classical diagonal form factors and the classical limit of the sub-densities $\bar{\rho}_k$. We start by constructing the finite volume multiparticle solutions and by determining their moduli space. We then rewrite the classical average in terms of the classical diagonal form factors.

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[5]Similar formula was proposed for symmetric diagonal form factors in [25] and proved later in [30].
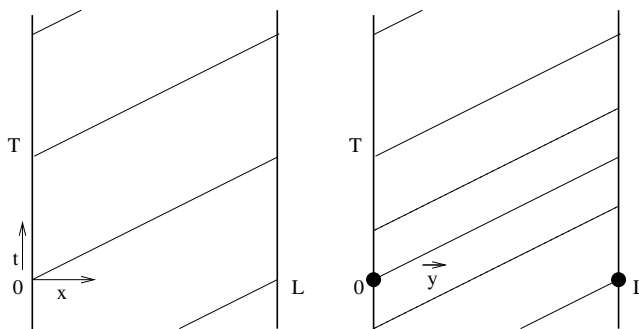
Figure 5. The 1-particle trajectory in finite volume is $x(t) = v_1 t + x_1 - nL$, where x being understood modulo L. The moduli parameter shifts the solution both in space and in time and its periodicity is $Y_1 = E_1 L$.

4.2.1 Classical solutions and their moduli space

The main difference between the infinite and finite volume solutions is that the moduli space of the latter is finite. Let us analyze it with increasing particle numbers.

Vacuum. The vacuum solution is automatically periodic and doesn't have any moduli parameter.

1-particle. The finite volume one particle solution is usually very complicated and incorporates exponentially small finite size corrections. As we focus only on the polynomial correction in L^{-1} the exact solution can be approximated by the infinite volume solution. In this approximation the particles can be considered pointlike and we merely continue the particle's trajectory periodically as shown on figure 5.

The periodicity of this solution in time is $T_1 = \frac{L}{v_1}$ and within each time period the finite volume 1-particle solution is

$$\varphi_1(x, t, y_1)_L = \varphi_{\text{st}}(E_1 x - y_1 - p_1(t - nT_1)) \tag{4.13}$$

with some appropriately chosen n. The time/space periodicity translates into the y- periodicity on the moduli space as:

$$y_1 \equiv y_1 + Y_1 \quad ; \qquad Y_1 = p_1 T_1 = E_1 L \equiv \rho_1^c \tag{4.14}$$

Denoting the shift vector $y_1 \to y_1 + Y_1$, by $\Delta_1 y = Y_1$, the finite volume moduli space is the factor space

$$\mathcal{M}_L^1 = \frac{\mathcal{M}^1}{\Delta_1 y} \tag{4.15}$$

which can be chosen to be the interval $[0, Y_1]$. Clearly this moduli space is finite.

2-particle. The exact finite volume two particle solution is usually very complicated, but we can easily construct a good approximate solution from the infinite volume two particle solution as follows: we take two free particles which travel as $x_i = v_i t + x_i^-$ and are well separated. (In a large volume it is always possible). This can be approximated
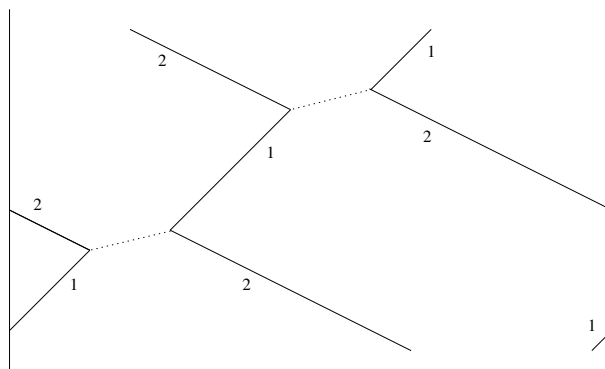
Figure 6. Approximate finite volume 2-particle solution: near the interaction pont we use the infinite volume 2-particle-, while away of them, the periodically continued infinite volume 1-particle solutions.

by two 1-particle solutions. When the particles get close to each other we replace this solution with the infinite volume 2-particle solution. After the collision process, modeled by the two particle solution, the particles are far away form each other and the 1-particle approximation is correct again. However, the two trajectories are now shifted as $x_1 = v_1 t + x_1^- + \Delta_{12} x$ and $x_2 = v_2 t + x_2^- + \Delta_{21} x$. When any of these "outgoing" particles reaches the periodicity border, 0 or L, it will come back from the other side and together with the other particle form a separated two particle initial state similar we started with. We then repeat the previous scattering process and by following this procedure we build up an approximate finite volume 2-particle solution: near the interaction pont we use the infinite volume 2-particle-, while away from them, the infinite volume 1-particle solutions as show on figure 6. We denote this solution as $\varphi_2(x, t; y_1, y_2)_L$ where y_1 and y_2 are related to the original coordinates (x_1^-, x_2^-) of the particles.

Next we should understand the structure of the finite volume moduli space. To parametrize this space we use the y_1 and y_2 shifts of the individual particles' locations as we used in the infinite volume case. We search for such transformations on y_1, y_2 which leave the finite volume solution invariant. We are going to factor out with these transformations. In the 1-particle case we simply moved the particle around the volume, which lead to the periodicity. This is similar how we move a particle in the BY equation at the quantum level. Let us mimic this transformation for the two particle case. If we shift particle 1 to the right then it passes through particle 2 and comes back on the other side. Due to the interaction the periodicity for particle 1 is shorthand by the space-displacement as $L_1 = L - \Delta_{12} x$. The analogue periodicity in the moduli space is

$$Y_1 = E_1 L - E_1 \Delta_{12} x = E_1 L + \phi_{12}^c \tag{4.16}$$

This shift, however, does not leave the two particle configuration invariant, because it is not the periodicity of the classical solution (trajectory). The reason is that having passed through particle 2 it suffered a $\Delta_{21} x$ displacement thus for the full periodicity we have to move back particle 2 by $-\Delta_{21} x$. Consequently, the full periodicity is the simultaneous

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shifts on the plane

$$(y_1, y_2) \rightarrow (y_1 + Y_1, y_2 - \phi_{21}^c) = (y_1, y_2) + (\Delta_1 y_1, \Delta_1 y_2) \quad ; \qquad \Delta_1 y = (E_1 L + \phi_{12}^c, -\phi_{12}^c) \tag{4.17}$$

Similarly we can move also particle 2 into the right direction around the circle. First we leave on the right and appear on the left and then pass through particle 1. As $\Delta_{21} x$ is negative the effective periodicity is shorthand to be $L - |\Delta_{21} x|$ or in the moduli space to

$$Y_2 = E_2 L + E_2 \Delta_{21} x = E_2 L + \phi_{12}^c \tag{4.18}$$

Now passing particle 2 from the left through particle 1 the displacement of particle 1 is $-\Delta_{12} x$ which we compensate by adding $\Delta_{12} x$. The full periodicity shift in the moduli space is then

$$(y_1, y_2) \rightarrow (y_1 - \phi_{12}^c, y_2 + Y_2) = (y_1, y_2) + (\Delta_2 y_1, \Delta_2 y_2) \quad ; \qquad \Delta_2 y = (-\phi_{12}^c, E_2 L + \phi_{12}^c) \tag{4.19}$$

The finite volume moduli space is obtained by factoring out the infinite volume moduli space by the two shift transformations

$$\mathcal{M}_L^2 = \frac{\mathcal{M}^2}{\{\Delta_1 y, \Delta_2 y\}} \tag{4.20}$$

The volume of this moduli space is finite,

$$\text{Vol}_2 = \rho_2^c = \det[\Delta_1 y, \Delta_2 y] = L^2 E_1 E_2 + L(\phi_{12}^c E_2 + \phi_{12}^c E_1) \tag{4.21}$$

and is nothing but the classical limit, ρ^c, of the density of states (4.12).

n-particle. The finite volume approximate n-particle solution is constructed as follows: we start with n separated straight lines at $t = 0$ with trajectories $x_i = v_i t + x_-^i$. The corresponding n-particle solution is approximated by the sum of the one-particle solutions. Whenever k particles' lines approach each other (within the interaction distance $\Delta_{ij} x$) we replace the sum of the k one particle solution with the infinite volume k-particle solution. We do this construction on the cylinder (i.e. in a periodic way). We denote this approximate finite volume solution by $\varphi_n(x, t; y_1, \ldots, y_n)_L$.

In order to determine the moduli space we analyze the symmetry of the configuration. Let us move the ith particle around the cylinder. When we pass particle j we use the two particle scattering, so the ith particle suffers a $\Delta_{ij} x$, while the jth particle a $\Delta_{ji} x$ displacement. In the moduli parameter we multiply x_i by E_i: $y_i = E_i x_i$. The simultaneous transformation (shifts) in the moduli space which leaves the configuration invariant is

$$i^{\text{th}} : \quad (y_1, \ldots, y_i, \ldots, y_n) \rightarrow (y_1 + \Delta_i y_1, \ldots y_i + \Delta_i y_i, \ldots, y_n + \Delta_i y_n) \tag{4.22}$$

$$\Delta_i y = (-\phi_{i1}^c, \ldots, Y_i, \ldots, -\phi_{in}^c) \quad ; \qquad Y_i = L E_i + \sum_{j:j \neq i} \phi_{ij}^c \tag{4.23}$$

The finite volume moduli space is the infinite volume moduli space factored out by all the n shift vectors

$$\mathcal{M}_L^n = \frac{\mathcal{M}^n}{\{\Delta_1 y, \ldots, \Delta_n y\}} \tag{4.24}$$

Figure 7. Finite volume moduli space of 1-particle solutions. It is periodic with period Y_0. Black dots indicate the region where the 1-particle average (4.26) collects its contribution.

The volume of the phase space is the classical limit of ρ_n:

$$\text{Vol}_n = \rho_n^c = \det[\Delta_1 y, \ldots, \Delta_n y] \tag{4.25}$$

4.2.2 Classical averages

Similarly to the infinite volume case the quantum average of the operator $\mathcal{O}(\hat{\varphi}(x,t))$ corresponds in the classical limit to the average of the function $\mathcal{O}(\varphi(x,t))$ over the finite volume moduli space of classical solutions. We express these finite quantities in terms of the infinite volume form factors and finite subvolumes of the moduli space.

Vacuum. As the vacuum solution is the same in finite and infinite volumes the corresponding form factor is also the same $\mathcal{O}(\varphi_0)$. To simplify formulas we assume in the following that the observable, $\mathcal{O}$, does not have any vacuum value.

1-particle. The classical expectation value of the function $\mathcal{O}(\varphi)$ in a 1-particle state with momentum p is its average over the moduli space of the finite volume solution

$$_L\langle p|\mathcal{O}|p\rangle_L^c = \frac{1}{Y_1}\int_0^{Y_1} dy_1\, \mathcal{O}(\varphi_1(x,t,y_1)_L) \tag{4.26}$$

The main difference compared to the infinite volume expression is that it is finite by itself. In the following we express this quantity in terms of the infinite volume classical form factor. Clearly the expectation value is independent of the space-time coordinates (x,t) thus we insert the operator at the origin $(0,0)$, where the 1-particle solution is passing by. For operators without vacuum expectation value the integral collects contributions around the origin (denoted by a black circle on figure 7).

The finite volume expectation value in terms of the infinite volume form factor can be written as

$$_L\langle p|\mathcal{O}|p\rangle_L^c = \frac{1}{Y_1}\int_{-\frac{Y_1}{2}}^{\frac{Y_1}{2}} dy_1\, \mathcal{O}[\varphi_1(y_1)_L] = \frac{1}{Y_1}\int_{-\infty}^{\infty} dy_1\, \mathcal{O}[\varphi_1(y_1)] = \frac{F_1^c}{\rho_1^c} \tag{4.27}$$

where we used the fact that the contribution comes from a local region around the origin and extended the domain of integration to infinity. We also used that in our approximation the infinite and the finite volume solutions are the same. The difference between the two expressions in (4.27) is exponentially small and can be neglected. This finite volume classical average is exactly the classical limit of the quantum finite volume expectation value (4.10).
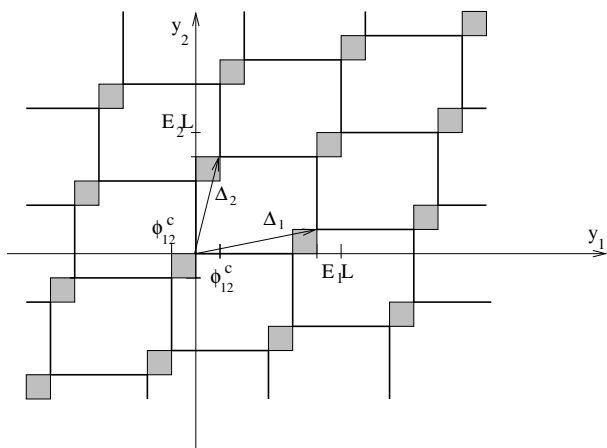
Figure 8. Moduli space indicating the domains where the integral (4.28) collects its contributions. The picture is periodic with the shifts $\{\Delta_1 y, \Delta_2 y\}$ to be factored out.
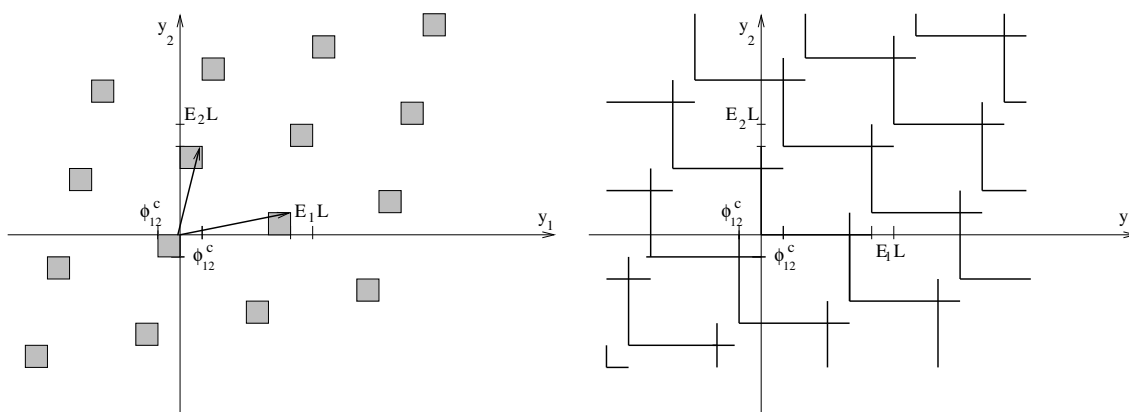

Figure 9. 2-particle and 1-particle contributions of (4.28) in the moduli space. We indicated the shift vectors $\{\Delta_1 y, \Delta_2 y\}$ on the left figure explicitly.

2-particle. The classical 2-particle expectation value is defined by averaging the observable over the moduli space

$$
{}_L\langle p_2, p_1 | \mathcal{O} | p_1, p_2 \rangle_L^c = \frac{1}{\text{Vol}_2} \int_{\mathcal{M}_L^2} dy_1 dy_2 \mathcal{O}(\varphi_2(y_1, y_2)_L) \tag{4.28}
$$

The integral collects completely well-defined finite contributions from the domain indicated on figure 8.

This figure is the finite volume analogue of figure 4. Similarly to the infinite volume case let us separate the 2-particle and the 1-particle contributions. It is indicated on figure 9.

In order to express the average in terms of the form factor we subtract from $\mathcal{O}(\varphi_2(y_1, y_2)_L)$ the one particle contributions and add them back. We should be careful with the subtraction as it has to be done in a way, which respects the shift symmetries

of the finite volume moduli space, $\{\Delta_1 y, \Delta_2 y\}$:

$$
\mathcal{O}(\varphi_2(y_1, y_2))_c = \mathcal{O}(\varphi_2(y_1, y_2)) - \Theta(y_1)\Theta(Y_1 - y_1)\mathcal{O}(\varphi_2(\infty, y_2))
$$
$$
-\Theta(y_2)\Theta(Y_2 - y_2)\mathcal{O}(\varphi_2(y_1, \infty)) \tag{4.29}
$$

This formula is valid in the fundamental domain, and should be extended periodically with the shifts $\{\Delta_1 y, \Delta_2 y\}$. Although two subtracted pieces seem missing as compared to the infinite volume expression, by shifting this function with the appropriate moduli transformations the missing pieces can be recovered as

$$
- \Theta(-y_1)\mathcal{O}[\varphi_2(-\infty, y_2)] - \Theta(-y_2)\mathcal{O}[\varphi_2(y_1, -\infty)] \tag{4.30}
$$

With these subtractions the classical finite volume expectation value is

$$
{}_L\langle p_2, p_1|\mathcal{O}|p_1, p_2\rangle_L^c = \frac{1}{\text{Vol}_2} \int_{\mathcal{M}_L^2} dy_1 dy_2 \Big\{ \mathcal{O}(\varphi_2(y_1, y_2))_c + \Theta(y_1)\Theta(Y_1 - y_1)\mathcal{O}(\varphi_2(\infty, y_2))
$$
$$
+\Theta(y_2)\Theta(Y_2 - y_2)\mathcal{O}(\varphi_2(y_1, \infty)) \Big\} \tag{4.31}
$$

As both the 2-particle and the 1-particle integrands are localized we can extend the integration domains appropriately to infinity. The integrand in the subtracted/added back pieces factorize in y_1 and y_2. As the two particle solutions reduce to the 1-particle solutions when a particle shifted to infinity $\varphi_2(\infty, y_2) = \varphi_1(y_2)$ the integration for y_2 give the infinite volume diagonal form factor $F_1^c(p_2)$, while the integration for y_1 with the Θ gives only the respective volume Y_1. Putting everything together we obtain

$$
{}_L\langle p_2, p_1|\mathcal{O}|p_1, p_2\rangle_L^c = \frac{1}{\rho_2^c} \left(F_2^c(p_1, p_2) + Y_1 F_1^c(p_2) + Y_2 F_1^c(p_1) \right) \tag{4.32}
$$

which is exactly the classical limit of formula (4.12).

n-particle. The n-particle classical finite volume averages can be defined as

$$
{}_L\langle p_n, \ldots, p_1|\mathcal{O}|p_1, \ldots, p_2\rangle_L^c = \frac{1}{\text{Vol}_n} \int_{\mathcal{M}_L^n} dy_1 \ldots dy_n \mathcal{O}(\varphi_n(y_1, \ldots, y_n)_L) \tag{4.33}
$$

This can be expressed in terms of the infinite volume connected integrands as

$$
{}_L\langle p_n, \ldots, p_1|\mathcal{O}|p_1, \ldots, p_2\rangle_L^c = \frac{1}{\text{Vol}_n} \int_{\mathcal{M}_L^n} d\vec{y} \Big\{ \mathcal{O}(\varphi_{1\ldots n}(y_1, \ldots, y_n)_c) \tag{4.34}
$$
$$
+ \sum_i \Theta\{y_i\} \mathcal{O}(\varphi_n(y_1, \ldots, \infty, \ldots, y_n))_c
$$
$$
+ \sum_{i,j} \Theta\{y_i, y_j\} \mathcal{O}(\varphi_n(y_1, \ldots, \infty, \ldots, \infty, \ldots, y_n))_c + \ldots
$$
$$
+ \sum_{\{i_k\}} \Theta\{y_{i_1}, \ldots, y_{i_k}\} \mathcal{O}(\varphi_n(\{y_{i_1}, \ldots, y_{i_k}\} \to \infty))_c + \ldots \Big\}
$$

The contributions of the lower order terms are such that after implementing the various shifts the infinite volume subtractions are locally restored. In particular, it implies that

the various Θ terms are the characteristic functions of the orthogonal projections of the moduli space to the relevant set of variables. For one coordinate it is

$$\Theta\{y_i\} = \Theta(y_i)\Theta(Y_i - y_i) = \begin{cases} 1 & \text{if } y_i = \alpha_i \Delta_i y_i \quad \text{for some } \alpha_i \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{4.35}$$

For two coordinates it reads as

$$\Theta\{y_i, y_j\} = \begin{cases} 1 & \text{if } y_i = \alpha_i \Delta_i y_i + \alpha_j \Delta_j y_i \text{ and } y_j = \alpha_i \Delta_i y_j + \alpha_j \Delta_j y_j \\ & \qquad\qquad\qquad\qquad\qquad \text{for some } \alpha_i, \alpha_j \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{4.36}$$

while in general as

$$\Theta\{y_{i_1}, \ldots, y_{i_k}\} = \begin{cases} 1 & \text{if for all } a = 1, \ldots, k : y_{i_a} = \sum_{j=1}^{k} \alpha_j \Delta_{i_j} y_{i_a} \\ & \qquad\qquad\qquad\qquad \text{for some for some } \alpha_{i_1}, \ldots, \alpha_{i_k} \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{4.37}$$

By performing the integral the integrand factorizes into the classical connected form factors in one set of variables and the various classical densities in the complementer set of variables leading to the formula, in which the average of the observable $\mathcal{O}$ over the moduli space of the classical n- particle solution can be written as

$$_L\langle p_n, \ldots, p_1 | \mathcal{O} | p_1, \ldots, p_2 \rangle^c_L = \frac{F^c_n(p_1, \ldots, p_n) + \sum_i \rho^c_1(p_i) F^c_{n-1}(p_1, \ldots, \hat{p}_i, \ldots, p_n) + \ldots}{\rho^c_n} \tag{4.38}$$

which is the classical analogue of formula (4.8).

5 Some comments on HHL correlation functions

As indicated in the introduction, it was the computation of Heavy-Heavy-Light correlation functions that was our main motivation for developing the formalism of classical computation of finite volume expectation values and diagonal form factors. In [11] we conjectured an identification between OPE coefficients for 'symmetric' operators i.e. when the two heavy operators are conjugate to each other, and diagonal form factors/finite volume expectation values.

Indeed, in the case where the heavy operator has charges only on the S^5, the 2-point correlation function of the heavy state is

$$x_{\tau_0}(\tau) = R \tanh \kappa(\tau - \tau_0) \qquad z_{\tau_0}(\tau) = \frac{R}{\cosh \kappa(\tau - \tau_0)} \qquad \text{and} \quad X^I_{\{y^I\}}(\sigma, \tau) \tag{5.1}$$

where the solution on the S^5 also depends on its own set of moduli $\{y^I\}$ (n moduli for an n-particle state). We see that there is an additional moduli τ_0 which is the relative time

shift between the AdS geodesic and the solution on the S^5. The modified prescription for HHL correlators proposed in [11] is

$$C_{\text{HHL}} = const \cdot \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} d\tau_0 \int_{\text{moduli}/\Gamma} \int d\tau d\sigma V_L \left[x_{\tau_0}(\tau), z_{\tau_0}(\tau), X_{\{y^I\}}^I(\sigma, \tau) \right] \quad (5.2)$$

where we restricted ourselves to the case of conjugate heavy operators. Here we implicitly assume that the contribution of the heavy vertex operators in the diagonal case will not have any moduli dependence and thus will not contribute to the above expression. Once we deal with the τ_0 integral, which is usually trivial, the remaining integral over the moduli space of finite volume classical solutions has exactly the same structure as the integral appearing in the computation of finite volume classical expectation values discussed extensively in the previous section. Thus one can adopt the decomposition into diagonal form factors obtained above also to this case.[6] We conjectured that such general decomposition extends also beyond the classical case. This has recently been verified at weak coupling and in the hexagon approach in [12, 17, 18].

In order to illustrate this formula, let us apply it to an interesting class of scalar operators including both supergravity and massive short string excitations for which the vertex operators are known explicitly in the classical limit. This family was introduced in [31] and the vertex operators are

$$\left(\frac{x^2 + z^2}{z} \right)^{-\Delta_L} \left[\partial X^K \bar{\partial} X^K \right]^r \quad (5.3)$$

Since the AdS part factorizes, the τ_0 integral can be easily carried out an one is left with

$$C_{\text{HHL}} \propto \int_{\text{moduli}/\Gamma} \int d\tau d\sigma \left[\partial X^K \bar{\partial} X^K \right]^r \quad (5.4)$$

Now, specializing to the heavy solution to be contained in the $S^2 \subset S^5$, we can use Pohlmeyer reduction formula to identify

$$\partial X^K \bar{\partial} X^K = \cos \beta \phi \quad (5.5)$$

where ϕ is a sine-Gordon field. Thus one can reduce the computation of this class of HHL correlators to diagonal form factors of the operators $e^{ik\beta\phi}$ in the sine-Gordon theory, for which we gave some explicit expressions in the previous sections. Note that the full expression for the finite volume expectation value will be different as the Bethe Ansatz factors will be different from the ones in sine-Gordon theory.

6 Conclusions

In the present paper we proposed a scheme for performing computations in the classical limit of two classes of observables in integrable field theories: diagonal form factors in infinite volume and finite volume expectation values of local operators. A key ingredient

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[6]In appendix B we discuss a minor subtlety which is nevertheless harmless.

of the proposal is an integration over the moduli space of classical multiparticle solutions which correspond to a single multiparticle quantum state. The integration over the infinite volume moduli space is divergent which in fact mimics the structure of divergences in infinite volume form factors in the diagonal limit. The main contribution of this part of the paper is to provide a concrete prescription for subtraction terms which lead to the infinite volume connected diagonal form factor which is a perfectly finite quantity.

In the case of finite volume expectation values, although the moduli space has finite volume, it has a nontrivial periodicity structure due to the time delays characteristic of soliton scattering. We show that the relevant integral can be naturally evaluated in terms of the classical diagonal form factors identified in the first part of the paper and volume factors which turn out to be equivalent to subdeterminants of the Bethe ansatz equations. In this way the known expression of finite volume expectation values in terms of diagonal form factors are explicitly realized in terms of the proposed classical expressions. This is a very nontrivial consistency check of the proposed expressions.

The relevance of the obtained results is twofold. On the one hand, the algorithm for the classical evaluation of diagonal form factors may be an important and useful crosscheck of the full quantum expressions, as these are in fact extremely complicated, as they would arise from a diagonal singular limit of a $2n$-particle form factor. On the other hand, within the AdS/CFT correspondence the evaluation of Heavy-Heavy-Light OPE coefficients reduces, as advocated in [11], to an integral over the moduli space of the finite volume solution. This led to the conjecture spelled out in [11] that the HHL OPE coefficients of 'symmetric' operators are related to diagonal form factors through finite volume expectation values of the appropriate part of the worldsheet vertex operator. The contribution of the present paper in this respect is to provide a framework which works for any number of particles.

There are many interesting directions of future research. It would be particularly interesting to determine the exact finite volume multiparticle solutions for an integrable QFT. Then one could analyze the moduli space of these solutions and map its periodicity properties. A proper geometric quantization of this moduli space should lead to the Bethe-Yang equations. In the paper we provided explicit expressions for the classical limit of diagonal connected form factors with two particles for the exponential operators in the sine-Gordon theory. It would be challenging to evaluate the classical limit of the complicated quantum expression including multiple contour integrals to check our proposal. We calculated the explicit expressions for low powers of the exponential operators directly. It would be nice to find a closed expression for generic powers and to extend the results for higher multiparticle states. Work is in progress into these directions. The semiclassical finite volume form factors analyzed in [23] for the conformal case also revealed a connection with moduli space and Bethe-Ansatz equations. It would be very interesting to elaborate the connection between our results and [23] in order to find a unified description.

Acknowledgments

A Normalizations

In this appendix we comment on the normalizations of the states and form factors. Clearly the normalization of multiparticle states affect the form factor of all operators in a uniform way.

In the paper we chose the normalization

$$\langle p_n, \ldots, p_1 | p_1', \ldots, p_n' \rangle = \prod_{i=1}^{n} 2\pi E(p_i) \delta(p_i - p_i') \tag{A.1}$$

which is very natural from the relativistic point of view as it is invariant under Lorentz transformations. It is nothing but δ normalization in rapidity space. In non-relativistic theories we could alternatively normalize to δ functions in momentum space

$$_x\langle p_n, \ldots, p_1 | p_1', \ldots, p_n' \rangle_x = \prod_{i=1}^{n} 2\pi \delta(p_i - p_i') \tag{A.2}$$

which is indicated by a subscript x as $2\pi\delta(p) = \int e^{ipx} dx$. The diagonal matrix elements can be easily related in the two normalizations

$$\langle p_1, \ldots, p_n | \mathcal{O} | p_n, \ldots p_1 \rangle = \prod_i E(p_i) \, _x\langle p_1, \ldots, p_n | \mathcal{O} | p_n, \ldots p_1 \rangle_x \tag{A.3}$$

just as form factors

$$F(p_1, \ldots, p_n) = \prod_i E(p_i) F^x(p_1, \ldots, p_n) \tag{A.4}$$

Changing the normalization of states will also change the density of states to

$$\rho_n^x(p_1, \ldots, p_n) = \det\left[\Phi_{ij}^x\right] \quad ; \quad \Phi_{ij}^x = \frac{\partial \Phi_j}{\partial p_i} = \left(L + \sum_{k=1}^{n} \phi_{ik}^x\right)\delta_{ij} - \phi_{ij}^x \tag{A.5}$$

where

$$\phi_{jk}^x = \phi^x(p_j, p_k) = -i\frac{\partial}{\partial p_j} \log S(p_j, p_k) \tag{A.6}$$

The finite volume expectation value is related to the Kronecker normalized states and is thus normalization-independent:

$$_L\langle p_n, \ldots, p_1 | \mathcal{O} | p_1, \ldots p_n \rangle_L$$

$$= \frac{1}{\rho^x\{1, \ldots, n\}} \sum_A \bar\rho^x\{A\} F_{|\bar A|}^x \{\bar A\} \tag{A.7}$$

$$= \frac{F_n^x + \sum_i \bar\rho^x\{i\} F_{n-1}^x\{1, \ldots, \hat{i}, \ldots n\} + \sum_{i,j} \bar\rho^x\{i,j\} F_{n-2}^x\{1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n\} + \ldots}{\rho^x\{1, \ldots, n\}}$$

As in the classical limit

$$\phi^x(p_j, p_k) \to \Delta_{jk} x \tag{A.8}$$

it is more natural to think of the moduli space in terms of shifts of the x-coordinate of the multiparticle solution. This coordinate is dual to the momentum coordinate and it is easy to see from the normalization change

$$y_i = E_i x_i \quad ; \qquad \int dy_i = E_i \int dx_i \tag{A.9}$$

that the classical infinite volume form factor can be obtained as

$$
\begin{aligned}
F_n^{x,c}(p_1,\ldots,p_n) = \prod_i \int_{-\infty}^{\infty} dx_i \Big\{ &\mathcal{O}[\varphi_n(x_1,\ldots,x_n)] - \sum_{i,\epsilon_i} \Theta(\epsilon_i x_i) \mathcal{O}[\varphi_n(x_1,\ldots,\epsilon_i\infty,\ldots,x_n)]_c \\
&- \sum_{i,j,\epsilon_i,\epsilon_j} \Theta(\epsilon_i x_i)\Theta(\epsilon_j x_j) \mathcal{O}[\varphi_n(x_1,\ldots,\epsilon_i\infty,\ldots,\epsilon_j\infty,\ldots,x_n)]_c + \ldots \\
&- \sum_{\{i_k,\epsilon_k\}} \prod_{j=1}^{k} \Theta(\epsilon_{i_j} x_{i_j}) \mathcal{O}[\varphi_n(x_1,\ldots,\epsilon_{i_1}\infty,\ldots,\epsilon_{i_k}\infty,\ldots,x_n)]_c + \ldots \Big\}
\end{aligned}
\tag{A.10}
$$

What is nice about this normalization is that the classical analogue of the Bethe-Yang equation has a direct geometric meaning. Indeed, moving particle i around the volume the x-space moduli parameters change as

$$i^{th}: \quad (x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n) \to (x_1 + \Delta_i x_1,,\ldots x_i + \Delta_i x_i,\ldots,x_n + \Delta_i x_n) \tag{A.11}$$

$$\Delta_i x = \left(-\Delta_{1i}x,\ldots L - \sum_{j:j\neq i} \Delta_{ij}x,\ldots,-\Delta_{ni}x \right) \tag{A.12}$$

The volume of the coordinate-moduli space is indeed the classical limit of the density of states

$$\mathcal{M}_L^{n,x} = \frac{\mathcal{M}^n}{\{\Delta_1 x,\ldots,\Delta_n x\}} \quad ; \qquad \mathrm{Vol}_n^x = \rho_n^{x,c} = \det[\Delta_1 x,\ldots,\Delta_n x] \tag{A.13}$$

B Connections to HHL 3-point functions

In our previous paper we conjectured that the HHL three point functions can be described by finite volume diagonal averages. In the strong coupling (classical) limit we suggested a new way of calculating these 3-point functions by integrating the light vertex operator for the moduli space of classical solutions. We explicitly checked and connected these proposals by evaluating the two magnon matrix element of the dilaton vertex operator.

Our analysis for relativistic theories implies that the diagonal finite volume matrix elements in the classical limit correspond to the integral of the classical observable for the moduli space of classical solutions. This, when applied to the HHL 3-point functions would imply the conjecture for multiparticle state, however there is a caveat. Namely the AdS/CFT correspondence is not described by a relativistic theory. Only its classical limit can be mapped via the Pohlmeyer reduction to a relativistic theory. In this map one also introduce a kind of gauge transformation, which changes the effective size of the system

and it is not quite clear that the quantum-classical correspondence applies. In the following we analyze the strong coupling (classical) limit of the quantum formulas and show that it is consistent with the relativistic classical expressions.

We first recall that the strong coupling limit of the scattering matrix is:

$$-i \log S(p_1, p_2) = -g\left(\cos\frac{p_1}{2} - \cos\frac{p_2}{2}\right) \log \frac{\sin^2(\frac{p_1-p_2}{4})}{\sin^2(\frac{p_1+p_2}{4})} \tag{B.1}$$

where g is the coupling constant, which goes to infinity. It is related to the classical expression, which can be obtained by integrating the time delay, by a gauge transformation and normalization [32]:

$$-ig \log S^c(p_1, p_2) = -i \log S(p_1, p_2) - g p_1 E_2 \tag{B.2}$$

In the finite volume formulas we need to calculate the density of states, which is then expressed in terms of

$$-i\frac{\partial \log S(p_k, p_j)}{\partial p_k} = g\Delta_{kj} x + g E_j \tag{B.3}$$

via

$$\Phi_{ij} = \frac{\partial \Phi_j}{\partial p_i} = g\left(-\Delta_{1j} x - E_j, \ldots, g^{-1}L + \sum_{k:k\neq j}(\Delta_{kj} x + E_k), \ldots, -\Delta_{nj} x - E_j\right) \tag{B.4}$$

Introducing

$$\tilde{L} = g^{-1}L + \sum_i E_i \tag{B.5}$$

we can simply write

$$g^{-1}\Phi_{ij} = \left(-\Delta_{1j} x, \ldots, \tilde{L} + \sum_{j\neq i}\Delta_{ij} x, \ldots, -\Delta_{Ni} x\right) - E_j(1, \ldots, 1) \tag{B.6}$$

The determinant of Φ_{ij} is the classical limit of the quantum density, which we would like to relate to $\rho^{x,c}$. The key observation is that

$$L^{-1}\det[\Phi_{ij}] = g^n \tilde{L}^{-1}\rho^{x,c} \tag{B.7}$$

This can be shown by simultaneous transformations on both matrices. First, by subtracting the first column from each we get rid off the extra E_j terms everywhere except the first column, such that the rest of the matrices coincides. In the second step we add each row to the first. As a result, the first row will be zero except the first element, which is $\tilde{L} - \sum_i E_i = g^{-1}L$ for $\det[\Phi_{ij}]$, while $\tilde{L}$ for ρ^x.

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