

Supersymmetric AdS_5 solutions of type IIB supergravity without D3 branes

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ABSTRACT: We analyse the most general bosonic supersymmetric solutions of type IIB supergravity whose metrics are warped products of five-dimensional anti-de Sitter space (AdS_5) with a five-dimensional Riemannian manifold M_5 , where the five-form flux vanishes, while all remaining fluxes are allowed to be non-vanishing consistent with $\text{SO}(4,2)$ symmetry. This completes the program of classifying all supersymmetric solutions of ten and eleven-dimensional supergravity with an AdS_5 factor. We investigate the supersymmetry conditions in some special cases, and demonstrate how these are satisfied by a solution originally found in [13], utilising the method of non-Abelian T-duality.

KEYWORDS: AdS-CFT Correspondence, Supergravity Models

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Contents

1	Introduction	1
2	The conditions for supersymmetry in $d = 5$	3
3	Bilinear equations	5
4	Introducing local coordinates	6
5	Complex M_4 and $P = 0$	9
6	An ansatz with $P \neq 0$	11
7	The solution of [13]	14
8	Conclusions	17
A	Bilinear definitions and the orthonormal frame	18
B	Algebraic analysis of (5.13) for $c \neq 0$	20
C	More details on the solution of [13]	23

1 Introduction

Via the AdS/CFT correspondence String or M-theory on a supersymmetric background containing an AdS_5 factor in the metric is expected to be dual to a four-dimensional superconformal field theory [1]. As such, there has been much interest in classifying supersymmetric AdS_5 solutions of IIA and IIB supergravity and M-theory. In [2] AdS_5 solutions of IIB with non-vanishing F_5 Ramond-Ramond (R-R) flux were classified. Whilst in [3] supersymmetric AdS_5 solutions of massive IIA were classified and new analytic solutions found. An analogous classification for M-theory was carried out in [4] and many new solutions were found.¹ In this work we plug a remaining gap in the classification of the IIB case. An alternative method for classifying supersymmetric supergravity solutions with an AdS factor in the metric, to that used in the above references and in this paper, was carried out in [8–10].

¹A later refinement of this work was carried out in [5] in which the additional conditions for $\mathcal{N} = 2$ supersymmetry were considered. It was later shown in [6] that the classification of [5] was the most general consistent with $\mathcal{N} = 2$ supersymmetry and an AdS_5 factor in M-theory. A later refinement of [2] was carried out in [7] to impose the additional condition of $\mathcal{N} = 2$ supersymmetry.

The case of vanishing self-dual five form, F_5 , was not considered in [2] and was implicitly assumed to be non-vanishing throughout. Attempts to set $F_5 = 0$ in the final equations of [2] run into inconsistencies as it involves dividing by zero. This case of vanishing F_5 corresponds to having no D3 branes in the theory and there is a close analogy between this and the no M2 branes case of AdS_4 in eleven-dimensional supergravity which was first classified in [11] and later extended in [12].

Completing this classification was motivated in part by the recent solutions found in [13]. Two new supersymmetric solutions of IIB supergravity were found with $F_5 = 0$ and are the first of their type. To obtain these solutions the authors begin with two well known AdS_5 Sasaki-Einstein solutions and perform a Non-Abelian T-duality (NATD) on an $\text{SU}(2)$ isometry to IIA followed by a T-duality along a remaining $\text{U}(1)$ to return to IIB. The supersymmetric solutions that are obtained have seed solutions $\text{AdS}_5 \times T^{(1,1)}$ and $\text{AdS}_5 \times Y^{p,q}$. Unfortunately these new solutions are singular and it was hoped that by completing this classification we would be able to find new non-singular solutions of this form. Finding non-singular AdS_5 solutions with vanishing F_5 remains an open problem.

In this paper we consider the most general bosonic supersymmetric solutions of type IIB supergravity with a warped metric of the form $\text{AdS}_5 \times M_5$, where M_5 is an internal manifold that admits a Riemannian metric. We set the self-dual five-form field strength, F_5 , to be vanishing but allow all other Neveu-Schwarz Neveu-Schwarz (NS-NS) and R-R fluxes to be non-vanishing and consistent with preserving the $\text{SO}(4,2)$ symmetry of AdS_5 . We use the well known method of analysing the G-structure determined by the Killing spinors as was employed, for example, in [2] (and references therein) from which some of this work is derived. We find that the internal manifold admits an identity structure which allows us to determine the metric in full generality. The geometry includes a hypersurface-orthogonal Killing vector which is a symmetry of the full solution and corresponds to the $\text{U}(1)$ R-symmetry in the putative dual superconformal field theory. Furthermore, analogous to the conclusion in [2], we find that supersymmetry implies that all the equations of motion and Bianchi identities are satisfied, though this does not follow immediately from their work.

The plan for the paper is as follows. In section 2 we present the conditions for preserving supersymmetry. In section 3 we present the torsion conditions and show that supersymmetry implies all the equations of motion and Bianchi identities. In section 4 we further the analysis by introducing local coordinates and reduce to a minimal set of necessary and sufficient conditions for a supersymmetric solution. In section 5 we consider a simple ansatz and find a singular solution, in section 6 we present a less simplified ansatz and reduce the solution to a single ODE to solve. In section 7 we show that the NATD-T-dual of $\text{AdS}_5 \times T^{(1,1)}$ solution found in [13] satisfies our equations. We conclude in section 8. We relegate some definitions and technical details to three appendices. The first contains the definitions of the bilinears and the calculation of the orthonormal frame used in the paper, the second contains algebraic analysis for the existence of non-singular solutions to the ansatz of section 5, whilst the third contains technical material used in section 7.

2 The conditions for supersymmetry in $d = 5$

We shall follow the conventions and notation of [2] for the type IIB supergravity field content, equations of motion, and supersymmetry variations. In addition to the ten-dimensional metric g_{MN} , the bosonic fields comprise the axion-dilaton $\tau = C^{(0)} + ie^{-\Phi}$, a complex three-form flux

$$G = ie^{\Phi/2}(\tau dB - dC^{(2)}), \tag{2.1}$$

where B and $C^{(2)}$, are the NS-NS and R-R two-form potentials, respectively, and a self-dual five-form $F_5 = *_{10}F_5$. Moreover, the axion and dilaton enter the equations of motion and supersymmetry variations through the following one-forms

$$P = \frac{i}{2}e^{\Phi} dC^{(0)} + \frac{1}{2} d\Phi, \tag{2.2}$$

$$Q = -\frac{1}{2}e^{\Phi} dC^{(0)}. \tag{2.3}$$

The covariant derivative D_M with respect to both local Lorentz transformations and local U(1) gauge transformations, is defined as

$$D_M = \nabla_M - iqQ_M, \tag{2.4}$$

where q is the charge of the field under the local U(1): P has charge 2, G has charge 1 and the Killing spinor ϵ has charge 1/2. We refer the reader to [2] for the equations of motion, Bianchi identities, and the supersymmetry variations for the gravitino ψ_M and dilatino λ .

We wish to characterise the most general class of bosonic supersymmetric solutions of type IIB supergravity with SO(4, 2) symmetry and *vanishing five-form flux*. Namely we require that

$$F_5 = 0, \tag{2.5}$$

which means that the solutions we study correspond to configurations *without D3 branes*. This is a slight difference to the analysis performed in [2], where it was (implicitly) assumed throughout that $F_5 \neq 0$. As pointed out in the introduction it is not possible to simply set $F_5 = 0$ in the final equations presented in [2]. Nevertheless much of the initial analysis conducted in their paper can be utilised and we shall indicate when this is possible and when it is not.

The $d = 10$ metric, in Einstein frame, takes the form of a warped product

$$ds_{10}^2 = e^{2\Delta} (ds_{\text{AdS}_5}^2 + ds_{M_5}^2), \tag{2.6}$$

where $ds_{\text{AdS}_5}^2$ is the metric on AdS₅ with Ricci tensor given by $R_{\mu\nu} = -4m^2(g_{\text{AdS}_5})_{\mu\nu}$ and $ds_{M_5}^2$ is the metric on a five-dimensional Riemannian internal space M_5 . In order to preserve the SO(4, 2) symmetry of the metric we require the fields to take values in; $\Delta \in \Omega^0(M_5, \mathbb{R})$, $P \in \Omega^1(M_5, \mathbb{C})$, $Q \in \Omega^1(M_5, \mathbb{R})$ and $G \in \Omega^3(M_5, \mathbb{C})$. Notice that with this ansatz the Bianchi identity for F_5 is trivially satisfied and it is therefore consistent to set $F_5 = 0$ without imposing any further conditions.

We will use the most general ansatz for the Killing spinor consistent with preserving minimal supersymmetry in AdS₅. This takes the form

$$\epsilon = e^{\Delta/2}(\psi \otimes \xi_1 \otimes \theta + \psi^c \otimes \xi_2^c \otimes \theta), \quad (2.7)$$

where we have rescaled the spinor by the factor $e^{\Delta/2}$ for later convenience. Here ψ is a Killing spinor on AdS₅ and ξ_i are two independent Spin(5) spinors on M_5 . Further discussion about the spinor ansatz and conventions can be found in appendix A of [2]. Requiring supersymmetry to be preserved yields the following conditions

$$D_m \xi_1 + \frac{1}{8} e^{-2\Delta} \gamma^{m_1 m_2} G_{m m_1 m_2} \xi_2 - \frac{i}{2} m \gamma_m \xi_1 = 0, \quad (2.8)$$

$$\bar{D}_m \xi_2 + \frac{1}{8} e^{-2\Delta} \gamma^{m_1 m_2} G_{m m_1 m_2}^* \xi_1 - \frac{i}{2} m \gamma_m \xi_2 = 0, \quad (2.9)$$

$$\gamma^m \partial_m \Delta \xi_1 + i m \xi_1 - \frac{1}{48} e^{-2\Delta} G_{m_1 \dots m_3} \gamma^{m_1 \dots m_3} \xi_2 = 0, \quad (2.10)$$

$$\gamma^m \partial_m \Delta \xi_2 + i m \xi_2 - \frac{1}{48} e^{-2\Delta} G_{m_1 \dots m_3}^* \gamma^{m_1 \dots m_3} \xi_1 = 0, \quad (2.11)$$

$$P_m \gamma^m \xi_2 + \frac{1}{24} e^{-2\Delta} \gamma^{m_1 \dots m_3} G_{m_1 \dots m_3} \xi_1 = 0, \quad (2.12)$$

$$P_m^* \gamma^m \xi_1 + \frac{1}{24} e^{-2\Delta} \gamma^{m_1 \dots m_3} G_{m_1 \dots m_3}^* \xi_2 = 0. \quad (2.13)$$

These can be obtained straightforwardly from the equations (3.3)–(3.8) in [2], by setting $f = 0$.²

Special cases. The possible stabilizer groups of the Spin(5) spinors ξ_i are the identity group or SU(2). Consequently M_5 may admit either an identity structure or an SU(2) structure.

Let us first consider the case of an SU(2) structure. This corresponds to setting one of the spinors to zero, without loss of generality, let us assume $\xi_2 = 0$. Then equation (2.10) reads

$$\gamma^m \partial_m \Delta \xi_1 = -i m \xi_1. \quad (2.14)$$

Following the use of Clifford algebra identities one can show easily that $\partial_n \Delta = 0$, and inserting this back into (2.14) we reach the contradiction $m \xi_1 = 0$. Whilst the $F_5 \neq 0$ case allowed for an SU(2) structure on M_5 , comprising the well known Sasaki-Einstein solutions, we conclude that there are no supersymmetric AdS₅ × M_5 solutions with $F_5 = 0$ in type IIB supergravity with M_5 admitting an SU(2) structure.³

Another interesting case to consider is $G = 0$. Such putative solutions would arise purely from D7 branes, and would be motivated by F-theory constructions. Setting $G = 0$ in equation (2.10) and (2.11) once again gives (2.14) and an analogous equation for ξ_2 which implies $\xi_1 = 0 = \xi_2$ and hence no supersymmetry is preserved. We therefore conclude that supersymmetric AdS₅ solutions of type IIB supergravity with vanishing five-form and three-form fluxes do not exist.

In the remainder of the paper we will assume that G is non-vanishing, and that both spinors ξ_i are not identically zero, thus giving a (local) identity structure on M_5 .

² f is the constant defined in [2] as $F_5 = f(\text{Vol}_{\text{AdS}_5} + \text{Vol}_5)$.

³In [3] it has also been shown that in type IIA supergravity there are no solutions of the form AdS₅ × M_5 with M_5 having an SU(2) structure either.

3 Bilinear equations

The identity structure is characterised by a set of one-forms, constructed as spinor bilinears, that can be used to define a canonical orthonormal frame on M_5 . In the analysis of the algebraic and differential conditions equivalent to the supersymmetry equations it is useful to consider also a number of scalar and two-form bilinears. We define these following the notation in [2] and we list them in appendix A. From the algebraic condition (3.25) in [2] we see that $F_5 = 0$ implies that $\sin \zeta = 0$;⁴ we can therefore import the bilinear equations from [2] where we set $\sin \zeta \equiv 0$ and $f \equiv 0$. The resulting differential conditions are⁵

$$e^{-4\Delta} d(e^{4\Delta} S) = 3imK, \quad (3.1)$$

$$e^{-6\Delta} D(e^{6\Delta} K_3) = P \wedge K_3^* - 4imW - e^{-2\Delta} * G, \quad (3.2)$$

$$e^{-4\Delta} d(e^{4\Delta} K_4) = -2mV, \quad (3.3)$$

$$e^{-8\Delta} d(e^{8\Delta} K_5) = -6mU, \quad (3.4)$$

while the algebraic conditions are

$$Z = 0 = \sin \zeta, \quad A = 1, \quad (3.5)$$

$$2i_{K_3} d\Delta = i_{K_3^*} P, \quad (3.6)$$

$$i_{K_5} d\Delta = 0 = i_{K_5} P, \quad (3.7)$$

$$(1 - |S|^2)e^{-2\Delta} * G = 2P \wedge K_3^* - (4d\Delta + 4imK_4) \wedge K_3 + 2 * (P \wedge K_3^* \wedge K_5 - 2d\Delta \wedge K_3 \wedge K_5). \quad (3.8)$$

Note that in [2] the differential condition on K_4 was implied by the remaining ones, because this one-form could be expressed as a linear combination of the other bilinears, as can be seen from (A.4), however this is no longer the case. Indeed, more generally, the orthonormal frame that we will use here, differs from the analogous one introduced in [2]. Using this orthonormal frame, presented in appendix A, we find that the metric takes the form

$$ds_{M_5}^2 = \frac{K_5^2}{|S|^2} + \frac{K_4^2}{1 - |S|^2} + \frac{K_3 \otimes K_3^*}{1 - |S|^2} + \frac{|S|^2}{1 - |S|^2} (\Im[S^{-1}K])^2. \quad (3.9)$$

This should be contrasted with the metric written in equation (3.53) of [2].

It is immediate from the analysis of [2] that K_5 defines a Killing vector. Moreover, here we will find that additionally K_5 is in fact a hypersurface-orthogonal Killing vector. This is most easily seen after we introduce local coordinates in the following section.

Analogously to [2], one can show K_5 is in fact a symmetry of the full solution, namely

$$\begin{aligned} \mathcal{L}_{K_5} \Delta &= \mathcal{L}_{K_5} \Phi = \mathcal{L}_{K_5} C^{(0)} = 0, \\ \mathcal{L}_{K_5} G &= 0. \end{aligned} \quad (3.10)$$

⁴Following the argument in appendix C of [2], and imposing $\sin \zeta = 0$, we find that it is not possible to have the spinors ξ_i non-vanishing and linearly dependent. We therefore restrict to the case of them being independent and admitting an identity structure.

⁵Here and in the rest of the paper $*$ denotes the Hodge star operator with respect to the five-dimensional metric $ds_{M_5}^2$.

In a putative dual $d = 4$ superconformal field theory this corresponds to having $U(1)$ R-symmetry and hence $\mathcal{N} = 1$ supersymmetry.

Let us now show that supersymmetry implies that all the equations of motion and Bianchi identities are satisfied. Most of the arguments presented in [2] to show that all the equations of motion and the P Bianchi identity are implied by supersymmetry can be used in our case, however, as alluded to in the introduction the argument showing that the Bianchi identity for G is satisfied is not valid if $F_5 = 0$. Below we present an argument that applies to both cases. Using the supersymmetry equations, we find

$$D(e^{6\Delta} X) = e^{6\Delta}(3im * X - e^{-2\Delta} SG + P \wedge Y), \tag{3.11}$$

$$e^{-6\Delta} \bar{D}(e^{6\Delta} Y) = 3im * Y + e^{-2\Delta} SG^* + P^* \wedge X, \tag{3.12}$$

$$e^{-6\Delta} D(e^{6\Delta} * X) = -e^{-2\Delta} G \wedge K + P \wedge *Y. \tag{3.13}$$

These equations are true even including a non-zero F_5 , as this drops out of the expressions. To recover the Bianchi identity for G one should take D of (3.11) and use (3.1), (3.12) and (3.13). As in [2], we conclude:

For the class of solutions with metric of the form (2.6), vanishing five-form flux and fluxes respecting $SO(4, 2)$ symmetry, all the equations of motion and Bianchi identities are implied by supersymmetry.

4 Introducing local coordinates

In this section we shall introduce local coordinates in which the set of BPS equations become more explicit. We begin by reducing on the Killing direction defined by K_5 , resulting in a 4-1 splitting of the metric. The transverse four-dimensional metric to the Killing direction admits an integrable almost product structure giving a further 3-1 splitting. The resulting BPS equations take a similar form to those presented in [2] in the $F_5 \neq 0$ case, but they are different. We shall conclude this section by introducing explicit coordinates on the remaining three-dimensional part of the metric, and obtaining expressions for the NS-NS and R-R two-form potentials.

We begin by choosing a local coordinate adapted to the Killing direction defined by K_5 . As a vector we have

$$K_5^\# = 3m \frac{\partial}{\partial \psi}, \tag{4.1}$$

and as a one-form

$$K_5 = \frac{|S|^2}{3m} (d\psi + \rho), \tag{4.2}$$

where ρ is a one-form with no $d\psi$ term. The factor of $3m$ is chosen for later convenience. The Lie derivative of S with respect to $K_5^\#$ is

$$\mathcal{L}_{K_5^\#} S = -3imS, \tag{4.3}$$

from which we find

$$S = -|S|e^{-i\psi} . \tag{4.4}$$

It is convenient to make the redefinitions

$$\mu = e^{-4\Delta} , \quad \eta = e^{4\Delta}|S| . \tag{4.5}$$

Then from (3.1) we have

$$K = \frac{\mu e^{-i\psi}}{3m}(\eta d\psi + i d\eta) , \tag{4.6}$$

and using the expression for K in appendix A we deduce that

$$K_5 = \frac{\eta^2 \mu^2}{3m} d\psi , \tag{4.7}$$

and is therefore a hypersurface-orthogonal Killing vector. Notice that the Killing vector is not fibered, $\rho = 0$, and this differs from [2]. Making the additional redefinitions

$$K_3 = \frac{\mu^{3/2}}{3m}\sigma , \quad K_4 = \frac{\mu}{3m}\beta , \tag{4.8}$$

the metric becomes

$$9m^2 ds^2 = \frac{1}{1 - \eta^2 \mu^2} (\mu^3 \sigma \otimes \sigma^* + \mu^2 \beta^2 + \mu^2 d\eta^2) + \eta^2 \mu^2 d\psi^2 . \tag{4.9}$$

Here β is a real one-form and σ is a complex one-form, and both have no leg along the Killing direction. We should now re-express the differential and algebraic conditions in terms of these redefined quantities. We find that (3.4) is automatically satisfied, whilst equation (3.3) becomes

$$d\beta = \frac{\mu^2}{3(1 - \eta^2 \mu^2)} [i\sigma^* \wedge \sigma - 2\eta d\eta \wedge \beta] . \tag{4.10}$$

Equation (3.2) becomes

$$D\sigma = \frac{1}{\eta^2 \mu^2 - 1} \left[(1 + \eta^2 \mu^2) P \wedge \sigma^* + \frac{4\mu^2 \eta}{3} d\eta \wedge \sigma + d \ln \mu \wedge \sigma + \frac{\eta^2 \mu^2}{3m} * (2P \wedge \sigma^* \wedge d\psi + d \ln \mu \wedge \sigma \wedge d\psi) \right] , \tag{4.11}$$

where we have used the expression for $*G$ given in (3.2). The remaining algebraic equations read

$$2i_{\sigma^*} P = -i_{\sigma} d \ln \mu , \tag{4.12}$$

$$\mathcal{L}_{\frac{\partial}{\partial \psi}} \mu = \mathcal{L}_{\frac{\partial}{\partial \psi}} \Phi = \mathcal{L}_{\frac{\partial}{\partial \psi}} C^0 = 0 . \tag{4.13}$$

These constitute the set of necessary and sufficient conditions that one needs to satisfy for supersymmetry.

To make these equations completely explicit, we can introduce the four remaining coordinates. It is a standard calculation (for example starting with (4.6)) to check that the four-dimensional metric transverse to the Killing direction has an integrable almost product structure. This allows one to introduce “splitting coordinates”, and gives a 3-1 splitting of the metric. In these coordinates the metric still takes the form presented in (4.9) however now the one-forms β and σ have no $d\eta$ term, though they are still in general functions of η . We may then split the five-dimensional exterior derivative as

$$d = d_3 + d\eta \frac{\partial}{\partial \eta} + d\psi \frac{\partial}{\partial \psi}, \quad (4.14)$$

where d_3 is the exterior derivative on the three-dimensional metric defined by the integrable almost product structure. Equation (4.10) now reads

$$d_3\beta = \frac{i\mu^2}{3(1-\eta^2\mu^2)}\sigma^* \wedge \sigma, \quad (4.15)$$

$$\partial_\eta\beta = -\frac{2\eta\mu^2}{3(1-\eta^2\mu^2)}\beta, \quad (4.16)$$

whilst (4.11) reads⁶

$$d_3\sigma - iQ_3 \wedge \sigma = \frac{1}{\eta^2\mu^2 - 1} \left[(1 + \eta^2\mu^2)P_3 \wedge \sigma^* + d_3 \ln \mu \wedge \sigma - 3m\eta\sqrt{1-\eta^2\mu^2} *_3 (2P_\eta\sigma^* + \partial_\eta \ln \mu \sigma) \right], \quad (4.17)$$

$$\partial_\eta\sigma - iQ_\eta\sigma = \frac{1}{\eta^2\mu^2 - 1} \left[(1 + \eta^2\mu^2)P_\eta\sigma^* + \frac{4\mu^2\eta}{3}\sigma + \partial_\eta \ln \mu \sigma - \frac{\mu^2\eta}{3m\sqrt{1-\eta^2\mu^2}} *_3 (2P_3 \wedge \sigma^* + d_3 \ln \mu \wedge \sigma) \right], \quad (4.18)$$

where we have used (4.13).

Thus for the most general, minimally supersymmetric AdS₅ solutions with vanishing five-form flux we need to solve the four differential equations (4.15)–(4.18) subject to the algebraic equation (4.12). We note that the integrability equation for (4.15) and (4.16) is automatically satisfied upon using (4.12), (4.17) and (4.18).

We may now introduce the three remaining coordinates along β and σ , which we will denote as x and y_i , with $i = 1, 2$. In particular, we write the three independent real one-forms as

$$\begin{aligned} \beta &= \gamma_x dx + \gamma_{y_1} dy_1 + \gamma_{y_2} dy_2, \\ \text{Re}[\sigma] &= \rho_x dx + \rho_{y_1} dy_1 + \rho_{y_2} dy_2, \\ \text{Im}[\sigma] &= \kappa_x dx + \kappa_{y_1} dy_1 + \kappa_{y_2} dy_2. \end{aligned} \quad (4.19)$$

⁶Here $*_3$ is the hodge star on the three-dimensional metric defined by the integrable almost product structure.

Notice that generically we cannot simplify further these expressions, and the equations (4.15)–(4.18) take the form of a very complicated set of coupled PDE’s. An explicit example of a rather generic solution will be presented later in section 7.

To obtain the explicit form of the NS-NS two-form B and the R-R two-form $C^{(2)}$ we can combine equations (3.11) and (3.12), to obtain

$$D(e^{6\Delta}(Y^* - X)) = -3ime^{6\Delta} * (Y^* + X) + e^{4\Delta}(S + S^*)G + e^{6\Delta}P \wedge (X^* - Y) . \quad (4.20)$$

It is then simple, but tedious, to extract the two two-forms B and $C^{(2)}$ from the real and imaginary parts of this equation, by using (3.1)–(3.4) and the results of appendix A. We find

$$B - \omega_B = \frac{e^{\Phi/2}\mu}{9m^2} \text{Re}[\sigma] \wedge d\psi , \quad (4.21)$$

$$C^{(2)} - \omega_C = C^{(0)} \frac{e^{\Phi/2}\mu}{9m^2} \text{Re}[\sigma] \wedge d\psi + \frac{e^{-\Phi/2}\mu}{9m^2} \text{Im}[\sigma] \wedge d\psi , \quad (4.22)$$

where ω_B and ω_C are undetermined closed two-forms. Analogous expressions relevant for the $F_5 \neq 0$ case were given in [14].

5 Complex M_4 and $P = 0$

Motivated by finding explicit solutions we set $P = 0$ in this section.⁷ Notice that setting $P = 0$ implies that μ is a function of η only.⁸ Setting $P = 0$ and $\mu = \mu(\eta)$ reduces the necessary and sufficient differential equations to

$$d_3\beta = \frac{2\mu^2}{3(1 - \eta^2\mu^2)} \text{Im}[\sigma] \wedge \text{Re}[\sigma] , \quad (5.3)$$

$$d_3\text{Re}[\sigma] = \frac{\mu\eta}{\eta^2\mu^2 - 1} \partial_\eta \ln \mu \beta \wedge \text{Im}[\sigma] , \quad (5.4)$$

$$d_3\text{Im}[\sigma] = -\frac{\mu\eta}{\eta^2\mu^2 - 1} \partial_\eta \ln \mu \beta \wedge \text{Re}[\sigma] , \quad (5.5)$$

⁷This condition imposes that the distinguished transverse four-dimensional foliation defined by the Killing vector ∂_ψ , which we call M_4 , has an integrable almost complex structure. Consider a holomorphic two-form constructed from the orthonormal frame of appendix A as

$$\begin{aligned} \Omega &\equiv (e^2 + ie^5) \wedge (e^4 - ie^3) \\ &= \frac{1}{2(\eta\mu - 1)} (e^{i\psi} X + e^{-i\psi} Y^* + 2W) . \end{aligned} \quad (5.1)$$

This then defines an almost complex structure on M_4 . In the second line we have expressed Ω in terms of the two-form bilinears. Imposing that this is integrable implies

$$P = g(e^4 + ie^3) + f(e^2 + ie^5) + h(e^4 - ie^3) , \quad (5.2)$$

where f, g, h are arbitrary complex functions (subject to satisfying the P equation of motion and Bianchi identity). Setting $P = 0$ solves this constraint therefore M_4 is complex in this case. It would have been more interesting to impose this more general form of P , however it is still a fairly complicated system of equations to solve and we were unable to do so.

⁸To see this use (4.12) to note that $d \ln \mu = f_{\kappa_4} K_4 + f_\eta d\eta$ for some real functions f_{κ_4} and f_η . Requiring that this is closed then implies that $f_{\kappa_4} = 0$.

and

$$\partial_\eta \beta = -\frac{2\eta\mu^2}{3(1-\eta^2\mu^2)}\beta, \quad (5.6)$$

$$\partial_\eta \text{Re}[\sigma] = \frac{1}{\eta^2\mu^2-1} \left(\frac{4\mu^2\eta}{3} + \partial_\eta \ln \mu \right) \text{Re}[\sigma], \quad (5.7)$$

$$\partial_\eta \text{Im}[\sigma] = \frac{1}{\eta^2\mu^2-1} \left(\frac{4\mu^2\eta}{3} + \partial_\eta \ln \mu \right) \text{Im}[\sigma]. \quad (5.8)$$

We see immediately that we may solve (5.6)–(5.8) as

$$\beta = \exp \left[\int \frac{2\mu^2\eta}{3(\eta^2\mu^2-1)} d\eta \right] \hat{\beta}, \quad (5.9)$$

$$\text{Re}[\sigma] = \exp \left[\int \frac{1}{\eta^2\mu^2-1} \left(\frac{4\mu^2\eta}{3} + \partial_\eta \ln \mu \right) d\eta \right] \hat{R}, \quad (5.10)$$

$$\text{Im}[\sigma] = \exp \left[\int \frac{1}{\eta^2\mu^2-1} \left(\frac{4\mu^2\eta}{3} + \partial_\eta \ln \mu \right) d\eta \right] \hat{I}, \quad (5.11)$$

where the hatted objects are η independent one-forms. We note that the above integrations may include arbitrary integration constants which we absorb into the η independent one-forms. Upon substituting these expressions into (5.3)–(5.5) one sees that the η dependence in (5.3) cancels automatically as it should. However the η dependence in (5.4) and (5.5) does not, we should have been suspicious if it cancelled as it would imply that μ could be any function of η , requiring that this expression is η independent gives us the defining differential equation for μ

$$\partial_\tau \left(\frac{\mu\eta}{\eta^2\mu^2-1} \partial_\eta \ln \mu \exp \left[\int \frac{2\mu^2\eta}{3(\eta^2\mu^2-1)} d\eta \right] \right) = 0. \quad (5.12)$$

We find a solution to the system of differential equations if we satisfy the second order non-linear differential equation

$$(3 + \eta^2\mu^2)\dot{\mu} + 6\eta^3\mu\dot{\mu}^2 + 3\eta(1 - \eta^2\mu^2)\ddot{\mu} = 0, \quad (5.13)$$

and the three differential equations

$$d_3 \hat{\beta} = \frac{2}{3} \hat{I} \wedge \hat{R}, \quad (5.14)$$

$$d\hat{R} = c\hat{\beta} \wedge \hat{I}, \quad (5.15)$$

$$d\hat{I} = -c\hat{\beta} \wedge \hat{R}. \quad (5.16)$$

Where c is a constant satisfying

$$c = \frac{\mu\eta}{\eta^2\mu^2-1} \partial_\eta \ln \mu \exp \left[\int \frac{2\mu^2\eta}{3(\eta^2\mu^2-1)} d\eta \right]. \quad (5.17)$$

Notice that c is non-zero if μ is non-constant and we shall distinguish between these two cases. For the $c = 0$ case we can write the solution in closed form and we will discuss it in the remainder of this section. However we are unable to write the $c \neq 0$ case in closed form and instead present algebraic analysis for the existence of non-singular solutions in appendix B.

A singular solution. We look at the $c = 0$ solution of (5.13) which is equivalent to constant μ . For simplicity we set $\mu = 1$. We are now able to integrate (5.9)–(5.11); we find

$$\beta = (1 - \eta^2)^{1/3} \hat{\beta}, \quad \partial_\eta \hat{\beta} = 0, \quad (5.18)$$

$$\text{Re}[\sigma] = (1 - \eta^2)^{2/3} \hat{R}, \quad \partial_\eta \hat{R} = 0, \quad (5.19)$$

$$\text{Im}[\sigma] = (1 - \eta^2)^{2/3} \hat{I}, \quad \partial_\eta \hat{I} = 0. \quad (5.20)$$

We then need to solve

$$d_3 \hat{R} = 0 = d_3 \hat{I}, \quad (5.21)$$

$$d_3 \hat{\beta} = \frac{2}{3} \hat{I} \wedge \hat{R}. \quad (5.22)$$

As \hat{R} and \hat{I} are closed we may define coordinates y_1 and y_2 such that

$$\hat{R} = dy_2, \quad \hat{I} = dy_1. \quad (5.23)$$

A solution to (5.22) is

$$\hat{\beta} = \frac{2}{3} (dx + y_1 dy_2). \quad (5.24)$$

The metric is

$$9m^2 ds^2 = \eta^2 d\psi^2 + (1 - \eta^2)^{1/3} (dy_1^2 + dy_2^2) + \frac{4}{9(1 - \eta^2)^{1/3}} (y_1 dy_2 + dx)^2 + \frac{1}{1 - \eta^2} d\eta^2, \quad (5.25)$$

and we have

$$B = \frac{(1 - \eta^2)^{2/3}}{9m^2} dy_2 \wedge d\psi, \quad (5.26)$$

$$C^{(2)} = \frac{(1 - \eta^2)^{2/3}}{9m^2} dy_1 \wedge d\psi. \quad (5.27)$$

Note that the range of η should be either $\eta \in [0, 1]$ or $\eta \in [-1, 0]$. We find that the Ricci Scalar is given by $R = 28m^2$, whilst $R_{\mu\nu}R^{\mu\nu} = 336m^4$ however we find that $R_{\mu_1 \dots \mu_4} R^{\mu_1 \dots \mu_4}$ exhibits a singularity as $\eta \rightarrow \pm 1$ and therefore the solution is singular.

We note that for $F_5 \neq 0$ an analogous solution of the equations of [2] exists, which was missed previously, by setting $\phi, C^{(0)}$ and the warp factor to be constants. This solution is once again singular and the singularity appears first in the Ricci scalar, it has non-zero G and hence is also not Sasaki-Einstein. These solutions are unusual in the sense that the only other known solutions with constant warp factor are the Sasaki-Einstein solutions.

6 An ansatz with $P \neq 0$

The structure of the BPS equations suggests an ansatz in which the η coordinate plays a distinguished role, therefore we make an ansatz where everything depends non-trivially on

this coordinate only. This ansatz is also motivated by the existence of analogous solutions of other BPS systems. More concretely, we can attempt an ansatz precisely analogous to the one used in section 5 of [2] which led to an ODE for one function with a solution corresponding to the Pilch-Warner solution [15], however the analysis of section 5 suggests that we should relax the assumption $P = 0$.

In fact we take a more general ansatz than that considered in [2] by adding an $SO(2)$ rotation of σ by a η dependent phase θ . Namely, we consider

$$\beta = A(\eta)\tau_3, \tag{6.1}$$

$$\sigma = \frac{1}{\sqrt{\mu(\eta)}} e^{i\theta(\eta)} (C(\tau)\tau_2 - iB(\tau)\tau_1). \tag{6.2}$$

Where τ_a are the $SU(2)$ left-invariant one-forms satisfying $d\tau_1 = \tau_2 \wedge \tau_3$ and cyclic permutations. Here the η dependent functions A, B, C are all real valued functions of η only.

The part of the BPS system decoupled from the dilaton with respect to the case $\theta = 0$, is

$$\partial_\eta \log B = -\frac{1 + \eta^2 \mu^2}{2\mu\eta} \frac{B}{AC} - \frac{4\mu^2 \eta}{3(1 - \eta^2 \mu^2)} \tag{6.3}$$

$$\partial_\eta \log C = -\frac{1 + \eta^2 \mu^2}{2\mu\eta} \frac{C}{AB} - \frac{4\mu^2 \eta}{3(1 - \eta^2 \mu^2)} \tag{6.4}$$

$$\partial_\eta \log A = -\frac{2\eta\mu^2}{3(1 - \eta^2 \mu^2)} \tag{6.5}$$

$$A = -\frac{2\mu BC}{3(1 - \eta^2 \mu^2)} \tag{6.6}$$

$$\partial_\eta \mu = \frac{1 - \eta^2 \mu^2}{2\eta A} \left(\frac{C}{B} + \frac{B}{C} \right). \tag{6.7}$$

These are four differential equations plus one algebraic, for the four functions A, B, C, μ . However (6.5) is redundant and implied by the others, so it can be eliminated to give four equations for four functions, which is encouraging for the existence of solutions. This is a complicated system of ODEs. A possible strategy to solve it is to obtain an ODE of higher degree for one single function; as μ appears ‘‘most often’’ in the system the simplest equation to derive is one for μ . To this end we take two further derivatives of (6.7)

$$\dot{\mu} = \frac{1 - \eta^2 \mu^2}{2ABC\eta} (C^2 + B^2) = -\frac{3(1 - \eta^2 \mu^2)^2}{4\mu\eta B^2 C^2} (C^2 + B^2), \tag{6.8}$$

$$\ddot{\mu} = -\frac{3 + \eta^2 \mu^2}{3\eta(1 - \eta^2 \mu^2)} \dot{\mu} - \frac{1 + 3\eta^2 \mu^2}{\mu(1 - \eta^2 \mu^2)} (\dot{\mu})^2 - \frac{3(1 - \eta^2 \mu^2)(1 + \eta^2 \mu^2)}{\mu^2 \eta (B^2 + C^2)} \dot{\mu}, \tag{6.9}$$

and using the other equations we eventually arrive at the following third order equation⁹

$$\begin{aligned} \ddot{\mu} = & -\frac{1}{9\mu\eta^2(1 - \eta^2 \mu^2)^2(1 + \eta^2 \mu^2)} [3\eta(9 + 47\eta^2 \mu^2 + 31\eta^4 \mu^4 + 9\eta^6 \mu^6)\dot{\mu}^2 \\ & + 36\eta^4 \mu(2 + 3\eta^2 \mu^2 + 3\eta^4 \mu^4)\dot{\mu}^3 + 9\eta\mu(3 - 3\eta^2 \mu^2 + \eta^4 \mu^4 - \eta^6 \mu^6)\ddot{\mu} \\ & + \mu(9 + 15\eta^2 \mu^2 + 35\eta^4 \mu^4 + 5\eta^6 \mu^6)\dot{\mu} + 9\eta^2(3 + 5\eta^2 \mu^2 + \eta^4 \mu^4 - 9\eta^6 \mu^6)\ddot{\mu}\dot{\mu}] . \end{aligned} \tag{6.10}$$

⁹Note that $\sqrt{3}/\eta$ is a solution to this equation however it gives a metric with incorrect signature and so is discarded.

One can check that (5.13) actually implies this equation as it should, being the general equation for $P = 0$. (6.10) is clearly a necessary condition for a solution however it is not sufficient, notice that constant μ solves (6.10) however it does not solve (6.7) as B and C are necessarily non-zero. Once a solution is obtained we should be able to extract A, B, C from this data. In fact, we are able to integrate one combination of the equations. Dividing (6.3) by B^2 and (6.4) by C^2 and subtracting them we obtain

$$A^4 \left(\frac{1}{B^2} - \frac{1}{C^2} \right) = \tilde{k} \quad (6.11)$$

where \tilde{k} is an integration constant. Further using (6.6) we obtain

$$B^2 C^2 (C^2 - B^2) = k \frac{(1 - \eta^2 \mu^2)^4}{\mu^4} \quad (6.12)$$

where $k = (\frac{3}{2})^4 \tilde{k}$. It would have been nice to use this to find an equation of second order instead of third order, but we have not managed to do so. In any case, this constraint should be useful when doing regularity and numerical analysis as it gives some exact analytic control on the analysis. In particular, let us return to showing that once a solution for the third order equation is found, the complete solution can be reconstructed.

A solution μ of the third order equation depends generically on three integration constants. Given this, A can be integrated from (6.5), and contains another integration constant. We can then determine B and C by combining (6.7) with (6.11), where we regard A, μ and $\dot{\mu}$ as known functions and solve for B and C . We have

$$\begin{aligned} B^2 &= -\frac{2k}{9} \frac{(1 - \eta^2 \mu^2)^2}{\mu^2} \frac{1}{A^2} - \frac{3}{2} \frac{\eta \dot{\mu}}{\mu} A^2, \\ C^2 &= \frac{2k}{9} \frac{(1 - \eta^2 \mu^2)^2}{\mu^2} \frac{1}{A^2} - \frac{3}{2} \frac{\eta \dot{\mu}}{\mu} A^2. \end{aligned} \quad (6.13)$$

Notice these are algebraic equations, so no new integration constants are introduced, and we correctly have four integration constants, one for each function.

The remaining θ dependent part of the system leads to the following equations

$$\cos 2\theta \partial_\eta \Phi + \sin 2\theta e^\Phi \partial_\eta C^{(0)} = \frac{1 - \eta^2 \mu^2}{2\mu\eta A} \left(\frac{C}{B} - \frac{B}{C} \right), \quad (6.14)$$

$$\cos 2\theta e^\Phi \partial_\eta C^{(0)} - \sin 2\theta \partial_\eta \Phi = 0, \quad (6.15)$$

$$\partial_\eta \theta = -\frac{1}{2} e^\Phi \partial_\eta C^{(0)}. \quad (6.16)$$

Interestingly, this decoupled set of equations can be completely integrated (assuming $\theta \neq 0$), namely we have

$$\partial_\eta \log \cot 2\theta = \frac{1 - \eta^2 \mu^2}{2\mu\eta A} \left(\frac{C}{B} - \frac{B}{C} \right), \quad (6.17)$$

$$e^{-(\Phi - \Phi_0)} = \sin 2\theta, \quad (6.18)$$

$$C^{(0)} = -e^{\Phi_0} \cos 2\theta + C_0^{(0)}, \quad (6.19)$$

where Φ_0 and $C_0^{(0)}$ are two integration constants.

We have the third order equation, or equivalently a coupled system of first order equations. Once a solution is found, the phase θ can be determined by integrating (6.17), and finally the dilaton and axion are determined algebraically in terms of θ .

Note that, for the purposes of studying (numerically) a system of first order equations, it may be convenient to consider the functions μ , A , and then to pick one, say B . C is then determined algebraically, and the $\partial_\eta C$ equation is then implied. This system reads

$$\dot{A} = -\frac{2\eta\mu^2}{3(1-\eta^2\mu^2)}A, \quad (6.20)$$

$$\dot{B} = \left(\frac{1+\eta^2\mu^2}{3\eta(1-\eta^2\mu^2)} \frac{B^2}{A^2} - \frac{4\mu^2\eta}{3(1-\eta^2\mu^2)} \right) B, \quad (6.21)$$

$$\dot{\mu} = \frac{1}{3\eta} \left(\frac{B^2}{A^2} + \frac{9(1-\eta^2\mu^2)^2}{4\mu^2 B^2} \right) \mu. \quad (6.22)$$

Finding solutions to this ansatz is dependent on solving the third order non-linear differential equation (6.10). Our preliminary studies were inconclusive and we leave the numerical study of (6.10) as an open problem.

7 The solution of [13]

Part of the motivation for completing this work was to clarify the geometry underlying the two supersymmetric solutions in [13] which circumvented the classification of [2]. In this final section we show that the supersymmetric NATD-T dual of the $\text{AdS}_5 \times \text{T}^{(1,1)}$ solution in [13] satisfies our classification. We were unable to directly solve the equations of the classification to recover the solution (due to the complexity of the equations), as was done in [2] for the Pilch-Warner solution. We instead bypassed this problem by finding the Killing spinors from which we constructed the geometry by way of the spinor bilinears. We first begin this section by writing down the solution found in [13].

We use the coordinates $x_1 = \rho \sin \chi$, $x_2 = \rho \cos \chi$ and for simplicity set $\alpha' = 1$. The $d=10$ metric in string frame¹⁰ is

$$\begin{aligned} ds^2 = & ds^2(\text{AdS}_5) + L^2\lambda_1^2 d\theta_1^2 + \frac{1}{L^2PQ} \left((L^4\lambda^2\lambda_1^2 + x_1^2) dx_1 + x_1x_2 dx_2 \right)^2 \\ & + \frac{L^2\lambda_1^2}{P} dx_2^2 + \frac{1}{L^2WQ} (Q d\phi_1 - \lambda^2x_1x_2 \cos \theta_1 dx_1 - \lambda^2(L^4\lambda_1^4 + x_2^2) \cos \theta_1 dx_2)^2 \\ & + \frac{L^2\lambda^2\lambda_1^4x_1^2 \sin^2 \theta_1}{W} d\xi^2, \end{aligned} \quad (7.1)$$

where

$$Q = L^4\lambda^2\lambda_1^4 + \lambda_1^2x_1^2 + \lambda^2x_2^2, \quad W = \lambda_1^2Q \sin^2 \theta_1 + \lambda^2\lambda_1^2x_1^2 \cos^2 \theta_1, \quad P = L^4\lambda^2\lambda_1^2 + x_1^2.$$

The constants λ and λ_1 take the values $1/3$ and $1/\sqrt{6}$ respectively and L is the radius of AdS_5 . The dilaton is

$$e^{-2\Phi} = L^4W, \quad (7.2)$$

¹⁰Recall that the classification is in Einstein frame.

whilst the NS-NS two-form is given by¹¹

$$B = -\frac{\lambda_1^2 x_1}{W} \left(\lambda^2 x_1 \cos \theta_1 d\phi_1 + \lambda^2 x_2 \sin^2 \theta_1 dx_1 - x_1 (\lambda^2 \cos^2 \theta_1 + \lambda_1^2 \sin^2 \theta_1) dx_2 \right) \wedge d\xi. \quad (7.3)$$

The non-zero RR-fluxes¹² are

$$F_1 = 4L^4 \lambda \lambda_1^4 \sin \theta_1 d\theta_1, \quad (7.4)$$

$$F_3 = \frac{4L^4 \lambda \lambda_1^6 x_1 \sin \theta_1}{W} d\theta_1 \wedge d\xi \\ \wedge [\lambda^2 x_2 \sin^2 \theta_1 dx_1 - x_1 (\lambda^2 \cos^2 \theta_1 + \lambda_1^2 \sin^2 \theta_1) dx_2 + \lambda^2 x_1 \cos \theta_1 d\phi_1], \quad (7.5)$$

and of course their hodge duals. In the notation of this classification the corresponding elements are

$$m = \frac{1}{L}, \quad (7.6)$$

$$\eta = L^2 \lambda_1^2 x_1 \sin \theta_1, \quad (7.7)$$

$$\mu = \frac{1}{L^2 \sqrt{W}} = e^\Phi, \quad (7.8)$$

$$d\psi = -d\xi, \quad (7.9)$$

$$\beta = (-x_1 \cos \theta_1 dx_1 - x_2 \cos \theta_1 dx_2 + L^4 \lambda_1^4 \sin \theta_1 d\theta_1 + x_2 d\phi_1), \quad (7.10)$$

$$\sigma = \frac{L \lambda_1^2}{W^{1/4}} [x_1 x_2 \sin^2 \theta_1 dx_1 + (L^4 \lambda_1^4 + x_2^2) \sin^2 \theta_1 dx_2 + x_1^2 \cos \theta_1 d\phi_1 \\ + iL^2 \sqrt{W} (\cos \theta_1 dx_2 + x_2 \sin \theta_1 d\theta_1 - d\phi_1)]. \quad (7.11)$$

Further details on the derivation of this dictionary is presented in appendix C. One may check that (7.1) takes the form of (3.9) with these identifications. For the explicit form of the NS-NS two form we find

$$B = \frac{e^{\Phi/2} \mu}{9m^2} \text{Re}[\sigma] \wedge d\psi - dx_2 \wedge d\psi, \quad (7.12)$$

whilst $C^{(2)}$ is not given in [13] for us to compare with, however it is trivial to show that F_3 agrees with that derived from the general expressions (4.21) and (4.22).

We have checked that this solution satisfies all the conditions of the classification, as an illustrative example we present the solution of (4.10). First define the function $E = (L^4 \lambda_1^4 + x_2^2) \sin^2 \theta_1 + x_1^2 \cos^2 \theta_1$. A short calculation gives

$$i\sigma^* \wedge \sigma - 2\eta d\eta \wedge \beta = 2L^{16} \lambda_1^4 E [dx_2 \wedge d\phi_1 + x_1 \sin \theta_1 d\theta_1 \wedge dx_1 + x_2 \sin \theta_1 d\theta_1 \wedge dx_2], \quad (7.13)$$

¹¹We correct a minor typographical error here by adding the $\cos \theta_1$ term in front of $d\phi_1$.

¹²These are the ones that appear in the equations of motion, $F_n = dC_{n-1} - C_{n-3} \wedge dB$.

whilst

$$\frac{3(1 - \eta^2 \mu^2)}{\mu^2} d\beta = L^{16} \lambda \lambda_1^2 E [dx_2 \wedge d\phi_1 + x_1 \sin \theta_1 d\theta_1 \wedge dx_1 + x_2 \sin \theta_1 d\theta_1 \wedge dx_2] . \tag{7.14}$$

Upon substituting the values of the constants, λ and λ_1 we find that they are equal. The equation for σ follows similarly but is vastly more complicated than the one illustrated above and for this reason we do not present it.

In section 4 we saw that the integrable almost product structure implied that the one-forms β and σ had no $d\tau$ term, we would like to verify this. To do so we must write the one-forms in the form (4.19). To this end, we make the change of coordinates

$$x = \phi_1 , \tag{7.15}$$

$$y_1 = \frac{1}{2}(x_1^2 + x_2^2) + L^4 \lambda_1^4 \ln(\cos \theta_1) , \tag{7.16}$$

$$y_2 = \ln \left(\frac{x_2}{\cos \theta_1} \right) , \tag{7.17}$$

$$\eta = L^2 \lambda_1^2 x_1 \sin \theta_1 . \tag{7.18}$$

In these coordinates the coefficients for the one-forms, in the notation of (4.19), are

$$\gamma_x = x_2 , \quad \gamma_{y_1} = -\cos \theta_1 , \quad \gamma_{y_2} = 0 , \tag{7.19}$$

$$\rho_x = \frac{L \lambda_1^2 x_1^2 \cos \theta_1}{W^{1/4}} , \quad \rho_{y_1} = \frac{L \lambda_1^2 x_2 \sin^2 \theta_1}{W^{1/4}} , \quad \rho_{y_2} = \frac{L^5 \lambda_1^6 x_2 \sin^2 \theta_1}{W^{1/4}} , \tag{7.20}$$

$$\kappa_x = -L^3 \lambda_1^2 W^{1/4} , \quad \kappa_{y_1} = 0 , \quad \kappa_{y_2} = L^3 \lambda_1^2 x_2 \cos \theta_1 W^{1/4} . \tag{7.21}$$

It is clear that this satisfies the integrable almost product structure. We have again checked that with these new coordinates the equations of the classification are satisfied and once again the equations to solve are very complicated. We had hoped this solution would have motivated further ansatz, unfortunately this was not the case. Interestingly this solution has an additional Killing vector, ∂_x , to what the classification implies. Imposing this extra Killing direction does not give much in the way of simplification of the equations and so this ansatz was swiftly dropped in favour of the ones we have presented.

We note that this solution, like our one, is singular [13]. The Ricci tensor blows up as $\theta_1 \rightarrow 0$ or π whilst $x_1 \rightarrow 0$. Furthermore the dilaton also blows up at these points. Computing the invariants $R_{\mu\nu} R^{\mu\nu}$ and $R_{\mu_1 \dots \mu_4} R^{\mu_1 \dots \mu_4}$ we also find that these are singular at these points but only these points. This solution therefore exhibits two singular points.

Though the solution is singular it would still be interesting to interpret this solution's field theory dual and also its brane realisation. A method was proposed in [16] where they considered the type IIA non-Abelian T dual of $\text{AdS}_5 \times S^5$ and propose a a D4/NS5 brane set-up and a linear quiver to describe its dual SCFT.

In [13] they also present another supersymmetric type IIB solution with $F_5 = 0$, namely the NATD-T dual of the $\text{AdS}_5 \times Y^{p,q}$ solution. This solution will also satisfy the classification presented here however we have not checked the details.

8 Conclusions

This work has plugged the remaining gap in the classification of all AdS₅ supersymmetric solutions of type IIB supergravity. Together with [2–4] our work concludes the classification of all supersymmetric AdS₅ solutions of $d = 10$ and $d = 11$ supergravity. We find that the geometry of M_5 is different to that of the $F_5 \neq 0$ case. It should be possible to interpret these results in terms of the “Exceptional Sasaki-Einstein (ESE) geometry” of [17].¹³ It would be interesting to see how the ESE structure is interpreted in terms of the bilinears. A similar analysis was carried out in [17] for the case of $F_5 \neq 0$.

One of the motivations for doing this work was to find new non-singular supersymmetric solutions relevant for AdS/CFT. From [13] we knew that singular supersymmetric solutions did exist, however the only solution we found was once again singular. In particular from the analysis performed in section 5 and appendix B we conclude that there are no non-singular solutions with $P = 0$. Contrast this with the $F_5 \neq 0$ case [2] where one finds the infinitely many Sasaki-Einstein solutions and the Pilch-Warner solution (which has $P = 0$), whilst in type IIA [3] one finds infinitely many massive IIA solutions and recovers previously known massless solutions such as the Maldacena-Núñez solution. Moreover in eleven dimensions many new solutions were found [4]. It is therefore disappointing that we have been unable to find new non-singular solutions.

However there are solution generating techniques one may use to find new solutions with $F_5 = 0$ (and also $F_5 \neq 0$). As pointed out in [13] if one begins with a Sasaki-Einstein solution with at least $SU(2) \times U(1) \times U(1)$ and follows their procedure for applying the Non-Abelian T-duality followed by the T-duality one obtains solutions with $F_5 = 0$, whether they are supersymmetric and non-singular is case dependent. Moreover one may obtain solutions with $F_5 = 0$ by T-dualising a IIA solution whose F_4 flux has a leg over the direction that is being dualised over for all components, once again supersymmetry and regularity is case dependent.

An interesting class of solutions are those which can be represented in both IIA, IIB and possibly also in eleven-dimensional supergravity. It may be fruitful to compare the supersymmetry conditions of this classification with the different cases, [4] and [3]. More concretely if we assume ∂_x is a Killing vector we may T-dualise over it to type IIA where we are then able to compare this classification with [3]. Uplifting to 11d allows us to compare with [4].

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¹³We thank Daniel Waldram for clarifications on this point.

A Bilinear definitions and the orthonormal frame

We define all the bilinears appearing in the paper. The scalar bilinears are

$$\begin{aligned}
 A &\equiv \frac{1}{2}(\bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2), \\
 A \sin \zeta &\equiv \frac{1}{2}(\bar{\xi}_1 \xi_1 - \bar{\xi}_2 \xi_2), \\
 S &\equiv \bar{\xi}_2^c \xi_1, \\
 Z &\equiv \bar{\xi}_2 \xi_1.
 \end{aligned} \tag{A.1}$$

The vector bilinears are

$$\begin{aligned}
 K^m &\equiv \bar{\xi}_1^c \gamma^m \xi_2, \\
 K_3^m &\equiv \bar{\xi}_2 \gamma^m \xi_1, \\
 K_4^m &\equiv \frac{1}{2}(\bar{\xi}_1 \gamma^m \xi_1 - \bar{\xi}_2 \gamma^m \xi_2), \\
 K_5^m &\equiv \frac{1}{2}(\bar{\xi}_1 \gamma^m \xi_1 + \bar{\xi}_2 \gamma^m \xi_2).
 \end{aligned} \tag{A.2}$$

The two-form bilinears are

$$\begin{aligned}
 W_{mn} &\equiv -\bar{\xi}_2 \gamma_{mn} \xi_1, \\
 V_{mn} &\equiv -\frac{i}{2}(\bar{\xi}_1 \gamma_{mn} \xi_1 - \bar{\xi}_2 \gamma_{mn} \xi_2), \\
 U_{mn} &\equiv -\frac{i}{2}(\bar{\xi}_1 \gamma_{mn} \xi_1 + \bar{\xi}_2 \gamma_{mn} \xi_2), \\
 X_{mn} &\equiv \bar{\xi}_1^c \gamma_{mn} \xi_1, \\
 Y_{mn} &\equiv \bar{\xi}_2^c \gamma_{mn} \xi_2,
 \end{aligned} \tag{A.3}$$

One finds that they satisfy the following algebraic relations

$$K_5 = \sin \zeta K_4 + \text{Re} [Z^* K_3] - \text{Re} [S^* K], \tag{A.4}$$

$$0 = \sin \zeta V - U - \frac{i}{2} K^* \wedge K + \text{Re} [iZ^* W], \tag{A.5}$$

$$S^* X = (1 + \sin \zeta) W - (K_4 + K_5) \wedge K_3, \tag{A.6}$$

$$S^* Y = (1 - \sin \zeta) W^* - (K_4 - K_5) \wedge K_3^*. \tag{A.7}$$

These relations may be computed by making use of Fierz identities, however we find it simpler to compute these by using an orthonormal frame which we shall construct below.

Following [2] we take the basis of gamma matrices of Cliff(5) to be

$$\begin{aligned}\gamma^1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I \\ \gamma^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I \\ \gamma^a &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \tau^a\end{aligned}\tag{A.8}$$

where $\tau^a = -i\sigma^a$ and σ^a are the Pauli matrices. In this basis the charge conjugation intertwiner is given by $C = I \otimes \tau^2$. We label the corresponding basis by e^i . We decompose the spinors ξ_i as $s_i \otimes \theta_i$ where s_i are spinors of Cliff(3) and θ_i spinors of Cliff(2). At the moment the basis is completely arbitrary which allows us to impose that the two vectors K_4 and K_5 lie in the $(e^1 - e^2)$ plane and in particular K_5 to be parallel with e^1 . We find

$$s_1 = \sqrt{2} \begin{pmatrix} \cos \theta \cos \phi \\ -\sin \theta \sin \phi \end{pmatrix}, \quad s_2 = \sqrt{2} \begin{pmatrix} \sin \theta \cos \phi \\ \cos \theta \sin \phi \end{pmatrix}\tag{A.9}$$

where we have set $\bar{\theta}_i \theta_i = 1$ and added suitable normalization to enforce $A = 1$. We can now write the scalar and vector bilinears as functions of θ, ϕ, θ_i . Requiring $\sin \zeta = 0$ implies that $\cos 2\theta = 0$ otherwise $\cos 2\phi = 0$ which then implies $K_5 = 0$. Choosing K_3 to lie in the $(e^3 - e^4)$ plane one can choose:

$$\theta_1 = \begin{pmatrix} e^{i\alpha} \\ 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 \\ e^{i\alpha} \end{pmatrix}\tag{A.10}$$

from which we obtain the final form of the vector bilinears

$$\begin{aligned}K_5 &= \cos 2\phi e^1, \quad K_4 = -\sin 2\phi e^2, \quad K_3 = \sin 2\phi (e^4 - ie^3), \\ K &= e^{2i\alpha} e^1 - i \sin 2\phi e^{2i\alpha} e^5,\end{aligned}\tag{A.11}$$

and the one non-trivial scalar bilinear

$$S = -e^{2i\alpha} \cos 2\phi.\tag{A.12}$$

The two-forms in terms of this orthonormal basis are

$$\begin{aligned}U &= -\sin 2\phi e^{15}, \quad V = e^{34} - \cos 2\phi e^{25}, \quad W = (i \cos 2\phi e^5 - e^2) \wedge (e^4 - ie^3), \\ X &= e^{2i\alpha} (\sin 2\phi e^1 + \cos 2\phi e^2 - ie^5) \wedge (e^4 - ie^3), \\ Y &= e^{2i\alpha} (-\sin 2\phi e^1 + \cos 2\phi e^2 + ie^5) \wedge (e^4 + ie^3).\end{aligned}\tag{A.13}$$

B Algebraic analysis of (5.13) for $c \neq 0$

For $c \neq 0$ equations (5.14)–(5.16) have solution:

$$\hat{\beta} = \frac{1}{c} \tau_3, \tag{B.1}$$

$$\hat{R} = \sqrt{\frac{3}{2|c|}} \tau_2, \tag{B.2}$$

$$\hat{I} = \sqrt{\frac{3}{2|c|}} \tau_1, \tag{B.3}$$

where τ_i are the $SU(2)$ left invariant one-forms if $c > 0$ and the $SL(2, \mathbb{R})$ left invariant one-forms if $c < 0$.¹⁴ The metric becomes

$$9m^2 ds^2 = \eta^2 \mu^2 d\psi^2 + \frac{\mu^2}{1 - \eta^2 \mu^2} d\eta^2 + \mu^2 (1 - \eta^2 \mu^2) \left(\frac{1}{\eta^2 \dot{\mu}^2} \tau_3^2 + \frac{3}{2|\eta \dot{\mu}| \mu} (\tau_1^2 + \tau_2^2) \right) \tag{B.4}$$

We have managed to find a solution to the differential equation (5.13) when $c \neq 0$, namely $\mu = \sqrt{3}/\eta$. Unfortunately this is not an admissible solution as it gives a metric with the wrong signature which can be clearly seen from the above.

We now present some algebraic analysis on the existence of regular solutions to (5.13), considering first the case $c > 0$ and then the case $c < 0$. We must find the range of the coordinate η and show that the metric is regular for all values of η in this range. To do so we find values of η for which the metric shrinks, equivalently some function of the metric becomes zero, yet the metric remains non-singular. Upon using (5.13) and its first derivative in η , we find that the Ricci scalar is given in the two cases by

$$R_{c>0} = \frac{m^2}{2\mu^4} (56\mu^4 + 240\dot{\mu}\eta\mu^3 + \dot{\mu}^2(9 + 171\eta^2\mu^2)), \tag{B.5}$$

$$R_{c<0} = R_{c>0} + \frac{24m^2\eta\dot{\mu}}{\mu(1 - \eta^2\mu^2)}. \tag{B.6}$$

$c > 0$ analysis. We first consider the case where the function $1 - \eta^2 \mu^2$ vanishes, let this point be η_0 . Near to η_0 we may write

$$1 - \eta^2 \mu^2 \simeq \gamma (\eta - \eta_0)^{2\alpha} \tag{B.7}$$

for some constants α and γ . Making the change of coordinate,¹⁵

$$r = \frac{(\eta - \eta_0)^{1-\alpha}}{\gamma^{(1/2)}(1-\alpha)} \tag{B.8}$$

we have

$$\frac{d\eta^2}{1 - \eta^2 \mu^2} = dr^2, \tag{B.9}$$

¹⁴The left invariant $SU(2)$ one-forms satisfy $d\tau_1 = \tau_2 \wedge \tau_3$ and cyclic permutations, whilst the $SL(2, \mathbb{R})$ one-forms satisfy $d\tau_1 = \tau_2 \wedge \tau_3$, $d\tau_2 = \tau_3 \wedge \tau_1$, $d\tau_3 = \tau_1 \wedge \tau_2$.

¹⁵Note that we have implicitly assumed $\alpha \neq 1$ here. However for $\alpha = 1$ one finds that the Ricci-scalar has a singularity as $\eta \rightarrow \eta_0$.

and

$$1 - (\eta\mu)^2 = \gamma^{\frac{1}{1-\alpha}} ((1-\alpha)r)^{\frac{2\alpha}{1-\alpha}}. \quad (\text{B.10})$$

Requiring that the latter expression is proportional to r^2 , as it should be for a regular solution, we find $\alpha = 1/2$. Near to η_0 the metric takes the form

$$9m^2 ds^2 = \mu^2 \left[\eta^2 d\psi^2 + dr^2 + \frac{r^2\gamma^2}{4} \left(\frac{1}{\eta^2\dot{\mu}^2}\tau_3^2 + \frac{3}{2|\eta\dot{\mu}|\mu}(\tau_1^2 + \tau_2^2) \right) \right]. \quad (\text{B.11})$$

For regularity we require that the metric looks locally like $S^1 \times \mathbb{R}^4$. For this to occur we require the factors in front of the left invariant one-forms to be equal and the overall factor to be $r^2/4$. Using the expression for μ near η_0 we find $\eta_0 \leq 0$ and $\gamma = -\frac{2}{3\eta_0}$, we have implicitly assumed that we are away from $\eta_0 = 0$ to obtain γ . Notice however that if we are at $\eta = 0$ then the solution will not be regular as μ is then necessarily unbounded in order to satisfy $1 - \eta_0^2\mu^2 = 0$. We find that for any η_0 strictly negative with μ satisfying $1 - \eta^2\mu^2 = 0$ at η_0 this will define an endpoint of the range of η and the metric will be regular at this point.

We may ask whether it is possible for there to be two such values of η , for which $1 - \eta^2\mu^2 = 0$ away from $\eta = 0$. Assume that η_1 and η_2 are two such values, and that there is no point $\eta_3 \in (\eta_1, \eta_2)$ such that $1 - \eta_3^2\mu(\eta_3)^2 = 0$, otherwise we have not chosen our range for η correctly. Without loss of generality and with the previous analysis in mind set $\eta_1 < \eta_2 < 0$. Near to η_a , $a = 1, 2$, we have $\dot{\mu}(\eta) = -2/(3\eta^2)$. Therefore for $\eta = \eta_1 + \epsilon_1$, with ϵ_1 a small positive number, $\mu(\eta_1 + \epsilon_1) < -1/(\eta_1 + \epsilon_1)$ however near to η_2 we have, for ϵ_2 a small positive number, $\mu(\eta_2 - \epsilon_2) > -1/(\eta_2 - \epsilon_2)$. With the additional and not unreasonable assumption that μ is continuous we must have that at some point $\eta_3 \in (\eta_1, \eta_2)$ that $1 - (\eta_3\mu(\eta_3))^2 = 0$ and hence we reach a contradiction as we assumed no η_3 existed. We conclude that no two such points exist.

Assume now that $\eta = 0$ is a regular boundary solution. For regularity it is necessary that μ takes a finite value at $\eta = 0$ or that it diverges as $O(1/\eta)$. A regular solution occurs if the last bracketed term in (B.4) is finite in the limit as η goes to 0 or it goes to zero as η^2 and has the metric of a three-sphere. If we consider these cases then $\mu \propto \log \eta$ or $\mu \propto \eta^{-\alpha}$, $\alpha > 1$ as $\eta \rightarrow 0$. However one now finds that the full $d = 10$ metric has singular Ricci scalar as $\eta \rightarrow 0$ in both cases. Moreover if we expand (5.13) about $\tau = 0$ we find that the only solution with this asymptotic behaviour is the true solution that gives the incorrect signature. This suggests that $\eta = 0$ is not a boundary condition that gives a non-singular metric.

The remaining possibilities are $\mu(\eta_0) = 0$ for some η_0 , that $\dot{\mu}(\eta_0) = 0$ or that $\eta \rightarrow -\infty$. We first look at the $\dot{\mu}(\eta_0) = 0$ case. Equation (5.13) implies that either $\eta_0 = 0$, $1 - \eta_0^2\mu^2(\eta_0) = 0$ or $\ddot{\mu}(\eta_0) = 0$ at η_0 . We can rule out both the first and second choices from our previous analysis, leaving us to conclude that $\ddot{\mu}(\eta_0) = 0$. We then find that all the derivatives of μ vanish at this point by taking further derivatives of (5.13) and evaluating at η_0 . Assuming, not unreasonably, that μ is analytic at this point we conclude that μ is a constant everywhere violating $c \neq 0$.

We next consider the possibility that $\mu(\eta_0) = 0$. Then, near to η_0 , we may write

$$\mu = \gamma(\eta - \eta_0)^\alpha, \quad (\text{B.12})$$

with $\alpha > 0$ and the metric takes the form

$$9m^2 ds^2 = \mu^2 \left[\eta^2 d\psi^2 + d\eta^2 + \frac{1}{\alpha^2 \beta^2 \eta_0^2 (\eta - \eta_0)^{2(\alpha-1)}} \tau_3^2 + \frac{3}{2\alpha \beta^2 \eta_0 (\eta - \eta_0)^{2\alpha-1}} (\tau_1^2 + \tau_2^2) \right]. \quad (\text{B.13})$$

One can see immediately that this is not regular for any $\alpha > 0$ and η_0 as the Ricci scalar diverges. For μ diverging at η_0 one still requires $1 - \eta^2 \mu^2 > 0$, for a metric with the correct signature, and therefore $\eta_0 = 0$ which was covered in a previous case.

Finally we study the possibility that $\eta \rightarrow -\infty$. It is best if we make the change of coordinate $\eta = -1/r$. With this change of coordinate the metric takes the form

$$9m^2 ds^2 = \mu^2 \left[\frac{1}{r^2} d\psi^2 + \frac{r^4}{1 - \frac{\mu^2}{r^2}} dr^2 + \frac{r^2 - \mu^2}{r^2} \left(\frac{1}{r^2 \mu'^2} \tau_3^2 + \frac{3}{2r \mu' \mu} (\tau_1^2 + \tau_2^2) \right) \right] \quad (\text{B.14})$$

We still require that $1 - \mu^2/r^2 > 0$ and so for small r , μ must take the form

$$\mu = a_1 r + a_2 r^2 + \dots, \quad (\text{B.15})$$

with $|a_1| < 1$. From looking at the last term in (B.4) we see that we need $\mu = ar$. With a further coordinate transformation $s = r^4/4$ the metric takes the form

$$9m^2 ds^2 = a^2 \left(d\psi^2 + \frac{ds^2}{1 - a^2} + \frac{1 - a^2}{a^2} \left(\tau_3^2 + \frac{3}{2} (\tau_1^2 + \tau_2^2) \right) \right). \quad (\text{B.16})$$

The metric takes the form of $S^1 \times \mathbb{R} \times S^3$ where the S^3 is squashed. Note however as $r \rightarrow 0$ we have the form of μ in this limit and inserting this into (5.13) we find that $a = \sqrt{3}$ and hence the metric has the wrong signature. This suggests that there are no non-singular solutions for $c > 0$ and we turn our attention to $c < 0$ in the following subsection.

$c < 0$ analysis. We now consider the case of $c < 0$, recall that now τ_i are the left invariant $\text{SL}(2, \mathbb{R})$ one-forms. Most of the arguments from the $c > 0$ case are still applicable and we shall make use of these when possible. Note that the possibility of $1 - \eta^2 \mu^2 = 0$ at η_0 will no longer give a non-singular metric as before. This can be seen directly from the Ricci scalar in equation (B.3).

Assume that $\eta = 0$ is a boundary condition. In the previous argument for $\eta = 0$ in the $c > 0$ case, we did not reference the particular form of the metric until computing the Ricci scalar of the full $d = 10$ metric, once again this diverges as $\eta \rightarrow 0$ and this suggests that $\eta = 0$ is not a regular boundary condition. The argument that forbid non-singular solutions with $\dot{\mu}(\eta_0) = 0$ still applies in the $c < 0$ case and so this is also not possible. Moreover we cannot have $\mu(\eta_0) = 0$ for the same reasons as in the $c < 0$ case as the Ricci scalar diverges. Note that the final possibility for a boundary value is $\eta \rightarrow \pm\infty$. As $\eta = 0$ gives a singular point for the manifold we cannot have the range to be $\eta \in (-\infty, \infty)$ and therefore there are no two points for η to take a value in. If one completes the analysis for $\eta \rightarrow \pm\infty$ one again finds that the manifold is singular at these points.

From the analysis of this and the previous subsection we conclude that no non-singular analytic solutions with $P = 0$ exist with $c > 0$ and $c < 0$.

C More details on the solution of [13]

In this appendix we present details about the derivation of (7.6)–(7.11). We make no claims that all the work in this appendix is original, only the final expressions (7.6)–(7.11). As pointed out in the text we were unable to solve the equations of the classification in order to recover this solution, in hindsight this was to be expected as it solves very non-trivial equations compared to the ansatz we have considered. Instead we found the Killing spinor of the NATD-T solution and from it constructed the spinor bilinears which allowed us to recover the solution. One may solve the Killing spinor equations directly for the NATD-T solution however this is very difficult and may be avoided. Instead one can use the Killing spinors of $T^{(1,1)}$, which are relatively simple to find, and transform them under the corresponding NATD and T dualities. It is this method that we present below.

The Buscher rules [18] give the transformation of the NS-NS sector under T-duality whilst [19] first gave the transformation of the RR-fluxes. The transformation of the Killing spinors was found in [20]. It is also well known how the geometry changes under NATD, see [21] for the transformation of the NS-NS sector, though we shall follow the conventions in [22]. The transformation of the RR-fluxes was found in [23] whilst in [24] it was found how a Killing spinor transforms under NATD. We shall briefly present the transformation of the Killing spinors under both NATD and T-duality for the ease of the reader.

Under a NATD or T-duality there is some ambiguity with the transformation of the vielbeins. Left and right movers of the world-sheet have different transformation properties and therefore define two different frame fields. These two frames must be equivalent as they define the same geometry and so are related by a Lorentz transformation of the form:

$$\hat{e}_+ = \Lambda \hat{e}_- . \tag{C.1}$$

This Lorentz transformation induces an action on spinors by the matrix Ω which satisfies

$$\Omega^{-1} \Gamma^a \Omega = \Lambda^a_b \gamma^b . \tag{C.2}$$

Type IIB supersymmetry is parametrised by two $d = 10$ Majorana-Weyl spinors of the same chirality whilst type IIA is parametrised by two $d = 10$ Majorana-Weyl spinors of opposite chirality. We shall denote these two spinors generically as χ_1 and χ_2 , their chiralities are unimportant for the calculation and so we do not distinguish their chiralities. Under a NATD or T-duality

$$\chi_1 \rightarrow \chi_1 \quad \chi_2 \rightarrow \Omega^{-1} \chi_2 . \tag{C.3}$$

where for a T-duality along a Killing vector, ∂_x , Ω takes the form

$$\Omega_{U(1)}^{-1} = -\frac{1}{\sqrt{G_{xx}}} \Gamma_{11} \Gamma_x , \tag{C.4}$$

where x is a curved index on Γ_x . Under a NATD, with respect to an $SU(2)$ isometry along the flat directions 1, 2 and 3, Ω takes the form

$$\Omega_{SU(2)}^{-1} = -\frac{\Gamma^{(11)}}{\sqrt{1 + \zeta^2}} (\Gamma^{123} + \zeta_a \Gamma^a) , \tag{C.5}$$

where for our purposes

$$\zeta^1 = \frac{x_1 \cos \xi}{L^2 \lambda_1 \lambda}, \quad \zeta^2 = \frac{x_1 \sin \xi}{L^2 \lambda_1 \lambda}, \quad \zeta^3 = \frac{x_2}{L^2 \lambda_1^2}. \quad (\text{C.6})$$

Note that both Ω 's defined above are unitary in our basis.

To begin we solve the Killing spinor equations of the Klebanov-Witten solution, $T^{(1,1)}$, in the canonical vielbein basis for performing the NATD

$$\begin{aligned} e^{\theta_1} &= L\lambda_1 d\theta_1, & e^{\phi_1} &= L\lambda_1 \sin \theta_1 d\phi_1, \\ e^{1,2} &= L\lambda_1 \tau_{1,2}, & e^3 &= L\lambda(\tau_3 + \cos \theta_1 d\phi_1), \end{aligned} \quad (\text{C.7})$$

where τ_i are the left invariant $SU(2)$ one-forms. With this basis, the Killing spinors are

$$\chi_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \chi_2 = \frac{1}{2} \begin{pmatrix} i \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad (\text{C.8})$$

where the choice of normalization is for later convenience. From these two spinors we may construct ξ_1 and ξ_2 as used in the classification

$$\xi_1 = \chi_1 + i\chi_2, \quad \xi_2 = \chi_1 - i\chi_2, \quad (\text{C.9})$$

note that it is the χ 's that transform as (C.3) and not the ξ 's. Under the NATD the Killing spinors become

$$\chi_1 \rightarrow \chi_1, \quad \chi_2 \rightarrow \Omega_{SU(2)}^{-1} \chi_2, \quad (\text{C.10})$$

whilst the vielbeins that change are¹⁶

$$\begin{aligned} \hat{e}^1 &= -\frac{\lambda_1}{LQ} \left[((L^4 \lambda_1^2 \lambda^2 + x_1^2) \cos \xi + L^2 \lambda^2 x_2 \sin \xi) dx_1 \right. \\ &\quad \left. + x_1 (x_2 \cos \xi - L^2 \lambda_1^2 \sin \xi) (dx_2 + L^2 \lambda^2 (d\xi + \cos \theta_1 d\phi_1)) \right] \\ \hat{e}^2 &= -\frac{\lambda_1}{LQ} \left[((L^4 \lambda^2 \lambda_1^2 + x_1^2) \sin \xi - L^2 \lambda^2 x_2 \cos \xi) dx_1 \right. \\ &\quad \left. + x_1 (L^2 \lambda_1^2 \cos \xi + x_2 \sin \xi) (dx_2 + L^2 \lambda^2 (d\xi + \cos \theta_1 d\phi_1)) \right] \\ \hat{e}^3 &= -\frac{\lambda}{LQ} [x_1 x_2 dx_1 + (L^4 \lambda_1^4 + x_2^2) dx_2 - L^2 \lambda_1^2 x_1^2 (d\xi + \cos \theta_1 d\phi_1)]. \end{aligned} \quad (\text{C.11})$$

One now has all the information to perform the T-duality. After both dualities the $T^{(1,1)}$ spinors become

$$\chi_1 \xrightarrow{\text{NATD}} \chi_1 \xrightarrow{\text{T}} \chi_1, \quad \chi_2 \xrightarrow{\text{NATD}} \Omega_{SU(2)}^{-1} \chi_2 \xrightarrow{\text{T}} \Omega_{U(1)}^{-1} \Omega_{SU(2)}^{-1} \chi_2. \quad (\text{C.12})$$

¹⁶Notice that we have rotated \hat{e}^1 and \hat{e}^2 with respect to those presented in appendix 6 of [13]. We have also added some extra factors of λ and λ_1 which we found to be missing.

One may now compute all the spinor bilinears. One finds for the scalar bilinears

$$A = 1, \quad (\text{C.13})$$

$$\sin \zeta = 0, \quad (\text{C.14})$$

$$Z = 0, \quad (\text{C.15})$$

$$S = -\frac{\lambda_1^2 x_1 \sin \theta e^{i\xi}}{\sqrt{W}}. \quad (\text{C.16})$$

From S one finds

$$\xi = -\psi, \quad \eta\mu = \frac{\lambda_1^2 x_1 \sin \theta}{\sqrt{W}}. \quad (\text{C.17})$$

Moreover one sees that the warp factor arises from putting the $d = 10$ metric into Einstein frame and therefore we have the identification $e^{2\Delta} = \mu^{-1/2} = e^{-\Phi/2}$. From this we find

$$\eta = L^2 \lambda_1^2 x_1 \sin \theta_1. \quad (\text{C.18})$$

One is able to find the one-form bilinears K_5 and K from this information by using (4.6) and (4.7) and we may use this as a check for the result defined directly from the Killing spinors. Computing the one-form bilinears from the Killing spinors one finds

$$K = \frac{L\lambda\lambda_1^2 e^{i\xi}}{\sqrt{W}} (i(\sin \theta dx_1 + x_1 \cos \theta d\theta) - x_1 \sin \theta d\xi), \quad (\text{C.19})$$

$$K_5 = -\frac{L\lambda\lambda_1^4 x_1^2 \sin^2 \theta}{W} d\xi, \quad (\text{C.20})$$

$$K_4 = \frac{\lambda(-x_1 \cos \theta_1 dx_1 - x_2 \cos \theta_1 dx_2 + L^4 \lambda_1^4 \sin \theta_1 d\theta_1 + x_2 d\phi_1)}{L\sqrt{W}}, \quad (\text{C.21})$$

$$K_3 = \frac{\lambda\lambda_1^2}{LW} [x_1 x_2 \sin^2 \theta_1 dx_1 + (L^4 \lambda_1^4 + x_2^2) \sin^2 \theta_1 dx_2 + x_1^2 \cos \theta_1 d\phi_1 + iL^2 \sqrt{W} (\cos \theta_1 dx_2 + x_2 \sin \theta_1 d\theta_1 - d\phi_1)]. \quad (\text{C.22})$$

Finally, using the redefinitions used in the classification (4.7) and (4.8), one recovers (7.7)–(7.11). The change of coordinates (7.15)–(7.18) follows from noticing that ϕ can be identified with x and then observing that certain combinations of dx_i and $d\theta_1$ appear only. From these combinations by adding suitable functions and requiring that they are closed one recovers the change of coordinates presented.

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